3264 & All That
Intersection Theory in Algebraic Geometry

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Introduction

"Es gibt nach des Verf. Erfarhrung kein besseres Mittel, Geometrie zu lernen, als das Studium des Schubertschen 'Kalküls der abzählenden Geometrie'."
(There is, in the author's experience, no better means of learning geometry than the study of Schubert's "Calculus of Enumerative Geometry.")
–B. L. van der Waerden (in a Zentralblatt review of an introduction to enumerative geometry by Hendrik de Vries).

0.1 Why you want to read this book

Algebraic geometry is one of the central subjects of mathematics. All but the most analytic of number theorists speak our language, as do mathematical physicists, homotopy theorists, complex analysts, symplectic geometers, representation theorists.... How else could you get between such apparently disparate fields as topology and number theory in one hop, except via algebraic geometry?

And intersection theory is at the heart of algebraic geometry. From the very beginnings of the subject, the fact that the number of solutions to a system of polynomial equations is, in many circumstances, constant as we vary the coefficients of those polynomials has fascinated algebraic geometers. The distant extensions of this idea still drive the field forward.

At the outset of the 19th century, it was to extend "preservation of number" that algebraic geometers made two important choices: to work over the complex numbers rather than the real numbers, and to work in projective space rather than affine space. (With these choices the two points of intersection of a line and an ellipse have somewhere to go as the ellipse moves away from the real points of the line, and the same for the point of intersection of two lines as the lines become parallel.) Over the course of the century, geometers refined the art of counting solutions to geometric problems—introducing the central notion of a parameter space; proposing the notions of an equivalence relation on cycles and a product on the equivalence classes and using these in many subtle calculations. These constructions were fundamental to the developing study of algebraic curves and surfaces.

In a different field, it was the search for a mathematically precise way of describing intersections that underlay Poincaré's study of what became algebraic topology. We owe Poincaré duality and a great deal more in algebraic topology directly to this search. His inability to work with continuous spaces (now called manifolds) led him to develop the idea of a simplicial complex, too.

Despite the lack of precise foundations, nineteenth century enumerative geometry rose to impressive heights: for example Schubert, whose *Kalkül der abzählenden Geometrie* (Schubert [1979]) represents the summit of intersection theory at the time of its writing, calculated the number of twisted cubics tangent to 12 quadrics—and got the right answer (5,819,539,783,680). Imagine landing a jumbo jet blindfolded!

At the outset of the 20th century, Hilbert made finding rigorous foundations for Schubert calculus one of his celebrated Problems, and the quest to put intersection theory on a sound footing drove much of algebraic geometry for the following century; the search for a definition of multiplicity fueled the subject of commutative algebra in work of van der Waerden, Zariski, Samuel, Weil and Serre. This progress culminated, towards the end of the century, in the work of Fulton and MacPherson and then in Fulton's landmark book *Intersection Theory* (Fulton [1984]), which both greatly extended the range of intersection theory and put the subject on a precise and rigorous foundation.

The development of intersection theory is far from finished. Today the focus is on things like virtual fundamental cycles, quantum intersection rings, Gromov-Witten theory and the extension of intersection theory from schemes to stacks.

A central part of a central subject of mathematics—of course you want to read this book!

0.2 Why we wrote this book

Given the centrality of the subject, it is not surprising how much of algebraic geometry one encounters in learning enumerative geometry. And that's how this book came to be written, and why: like van der Waerden, we found that intersection theory makes for a great "second course" in algebraic geometry, weaving together threads from all over the subject. Moreover, the new ideas one encounters in this setting have a context in which they're not merely more abstract definitions for the student to memorize, but tools that help answer concrete questions.

0.3 What's with the title?

The number in the title of this book is a reference to the solution of a classic problem in enumerative geometry: the determination, by Chasles, of the number of smooth conic plane curves tangent to five given general conics. The problem is emblematic of the dual nature of the subject. On the one hand, the number itself is of little significance: life would not be materially different if there were more or fewer. But the fact that the problem is well-posed—that there is a Zariski open subset of the space of 5-tuples of conics $(C_1, \dots, C_5)$ for which the number of conics tangent to all five is constant, and that we can in fact determine that number—is at the heart of algebraic geometry. And the insights developed in the pursuit of a rigorous derivation of the number—the recognition of the need for, and introduction of, a new parameter space for plane conics, and the understanding of why intersection products are well defined for this space—are landmarks in the development of algebraic geometry.

The rest of the title is from "1066 & All That" by W. C. Sellar and R. J. Yeatman, a parody of English history textbooks; in many ways the number 3264 of conics tangent to five general conics is as emblematic of enumerative geometry as the date 1066 of the Battle of Hastings is of English history.

0.4 What's in this book

> We are dealing here with a fundamental and almost paradoxical difficulty. Stated briefly, it is that learning is sequential but knowledge is not. A branch of mathematics... consists of an intricate network network of interrelated facts, each of which contributes to the understanding of those around it. When confronted with this network for the first time, we are forced to follow a particular path, which involves a somewhat arbitrary ordering of the facts.
> –Robert Osserman.

Where to begin? To start with the technical underpinnings of a subject risks losing the reader before the point of all that preliminary work is made clear; but to defer the logical foundations carries its own dangers—as the unproved assertions mount up, the reader may well feel adrift.

Intersection theory poses a particular challenge in this regard, since the development of its foundations is so demanding. It is possible, however, to state fairly simply and precisely the main foundational results of the subject, at least in the limited context of intersections on smooth projective varieties. The reader who is willing to take these results on faith for a little while, and accept this restriction, can then be shown "what the subject is good for," in the form of examples and applications. This is the path we've chosen in this book, as we'll now describe.

0.4.1 Overture

In the first chapter, the "Overture," we introduce rational equivalence, the Chow ring, the pull-back and push-forward maps and Chern classes—the "Dogma" of the subject. We follow this with a range of simple examples to give the reader a sense of the themes to come: the computation of Chow rings of affine and projective spaces, their products and blowups. To illustrate how intersection theory is used in algebraic geometry, we examine loci of various types of singular cubic plane curves, thought of as subvarieties of the projective space $\mathbb{P}^9$ parametrizing plane cubics. Finally, we introduce briefly the notion of *Chern classes*, a fundamental tool that will be taken up fully in Chapter 7.

We should mention here that there are many other possible cycle theories that allow us to carry out the basic constructions of intersection theory. These, and the relations among them, are discussed in Appendix **??**.

0.4.2 Grassmannians

The intersection rings of the Grassmannians are archtypal examples of intersection theory. Chapters 2 and 3 are devoted to them and the geometry that underlies them. Here we introduce the Schubert cycles, which form a basis for the Chow ring, and use them to solve a number of geometric problems, illustrating again how intersection theory is used to solve enumerative problems.

0.4.3 What's under the hood: foundational results

The next two chapters, 4 and 5, are devoted to proofs and substantial discussion of the results stated in the Overture. Chapter 4 introduces the foundational ideas of commutative algebra, centered around finite extensions of one-dimensional rings, that are necessary in establishing the properties of the pushforward map.

We have chosen to base our development of the theory on the *Moving Lemma*, a central result that allows us to define intersection products in a relatively intuitive way, albeit in the limited context of intersections on smooth projective varieties. Chapter 5 introduces the beautiful geometry, first described by Severi and refined by Chow and others, of its proof.

This is not the only way to develop intersection products. The "deformation to the normal cone" (explained in Chapter 15), the foundational idea used in Fulton [1984], has many advantages, and leads ultimately to a sharper and more general theory. As the student becomes an expert the transition to the new point of view will be natural. But we felt that, in its more limited sphere, the approach via the Moving Lemma gives a direct intuition for why the basic results are true that is difficult to gain from the newer approach. One might compare the situation to that encountered in learning algebraic geometry. No one would doubt that the theory of schemes is a more natural and supple language for the subject; but a beginner may be better served by learning the more concrete case of varieties first.

0.4.4 Chern classes

We then come to a watershed in the subject. In a brief interlude (Chapter 6) we review the basics of the theory of vector bundles (also known as locally free sheaves) and their direct images, leading up to the theorem on cohomology and base change. With these notions in hand, Chapter 7 takes up in earnest a notion that is at the center of modern intersection theory, and indeed of modern algebraic geometry: Chern classes. As with the development of intersection theory we define these in the classical style, as degeneracy loci of collections of sections. This interpretation provides useful intuition and is basic to many applications of the theory.

0.4.5 Applications I: using the tools

We illustrate the use of Chern classes by taking up two classical problems: Chapter 8 deals with the question of how many lines lie on a hypersurface (for example there are exactly 27 lines on each smooth cubic surface), and Chapter 9 looks at the singular hypersurfaces in a one-dimensional family (for example, what is the degree of the discriminant of a polynomial in several variables). Using the basic technique of *linearization*, these problems can be translated into problems of computing Chern classes. These and the next few chapters are organized around geometric problems involving constructions of useful vector bundles and the calculation of their Chern classes.

0.4.6 Parameter spaces

Chapter 10 deals with an area in which intersection theory has had a profound influence on modern algebraic geometry: *parameter spaces* and their compactifications. This is illustrated with the five-conic problem; there is also a discussion of the modern example of Kontsevich spaces, and an application of those.

0.4.7 Applications II: further developments

The remainder of the book introduces a series of increasingly advanced topics. Chapters 11, 12 and 13 deal with a situation ubiquitous in the subject, the intersection theory on projective bundles, and its applications to subjects such as projective duality and the enumerative geometry of contact conditions.

Chern classes are defined in terms of the loci where collections of sections of a vector bundle become dependent. These can be interpreted as loci where maps from trivial vector bundles drop rank. The Porteous formula, proved and applied in Chapter 14, generalizes this, expressing the classes of the loci where a map between two general vector bundles has a given rank or less in terms of the Chern classes of the two bundles involved.

0.4.8 Advanced topics

Next, we come to some of the developments of the modern theory of intersections. In Chapter 15, we introduce the notion of “excess” intersections and the *excess intersection formula*, one of the subjects that was particularly mysterious in the nineteenth century but that was elucidated by Fulton and MacPherson. This theory makes it possible to describe the intersection class of two cycles even if their intersection has “too large” dimension. Central to this development is the idea of *deformation to the normal cone*, a construction fundamental to the work of Fulton and Macpherson; we use this to prove the famous “key formula” comparing intersections of cycles in a subvariety $Z \subset X$ to the intersections of those cycles in X, and use this in turn to give a description of the Chow ring of a blow-up.

Chapter 16 contains an account of Riemann-Roch formulas, leading up to a description of Grothendieck's version. The chapter concludes with a number of examples and applications showing how Grothendieck's formula can be used.

0.4.9 Applications III: the Brill-Noether theorem

The last chapter of the book, Chapter **??**, explains an application of enumerative geometry to a problem that is central in the study of algebraic curves and their moduli spaces: the existence of special linear series on curves. We give the proof of this theorem by Kempf and Kleiman-Laksov, which draws upon many of the ideas and techniques of the book, plus one new one: the use of topological cohomology in the context of intersection theory. This is also a wonderful illustration of the way in which enumerative geometry can be the essential ingredient in the proof of a purely qualitative result.

0.4.10 Relation of this book to "Intersection Theory"

Fulton's book on intersection theory (Fulton [1984]) is a great work. It sets up a rigorous framework for intersection theory in a generality significantly extending and refining what was known before and laying out an enormous number of applications. It is a work that can serve as an encyclopedic reference to the subject.

By contrast, the present volume is intended as a textbook in algebraic geometry, a second course in which the classical side of intersection theory is a starting point for exploring many topics in geometry. We introduce the intersection product at the outset, and focus on basic examples. We use concrete problems to motivate the introduction of new tools from all over algebraic geometry. Our book is not a substitute for Fulton's: it has a different aim. We hope that it will provide the reader with intuition and motivation that will make reading Fulton's book easier.

One point that should be made explicit is that this trade-off—making the restrictive hypothesis of smoothness, in exchange for relative ease of intuition and immediate application—does obscure some important aspects of cycle theory. By way of analogy, imagine that, as a topologist, you could define and work with homology—but only for compact, oriented manifolds. Even with this restriction, you could develop a pretty good sense of what homology groups represent, and be able to carry out most of the standard applications of homology; if you were primarily interested in applications to manifolds, this would be enough. You might even think you understood cohomology and cup products as well: after all, in this limited context (rational) cohomology is isomorphic to homology, and via this isomorphism the cup product is simply the intersection of cycles. But this would be a mistake: to gain a real understanding of homology and cohomology, you have to develop both in a broader context. In the same way, to really understand the cycle theory of algebraic varieties, you have to read Fulton.

0.4.11 Keynote problems

To highlight the sort of problems we'll learn to solve, and to motivate the material we present, we'll begin each chapter with some *keynote questions*.

0.5 Prerequisites, Notation, Conventions

0.5.1 What you need to know before starting

When it comes to prerequisites, there are two distinct questions: what you should know to start reading this book; and what you should be prepared to learn along the way.

Of these, the second is by far the more important. In the course of developing and applying intersection theory we introduce many key techniques of algebraic geometry, such as deformation theory, specialization methods, characteristic classes, commutative and homological algebra and topological methods. That's not to say that you need to know these things going in. Just the opposite, in fact: reading this book should be viewed as an occasion to learn them.

So what do you need before starting?

(a) An undergraduate course in classical algebraic geometry or its equivalent, comprising the elementary theory of affine and projective varieties. *An Invitation to Algebraic Geometry* (Smith et al. [2000]) contains almost everything required. Other books that cover this material include *Undergraduate Algebraic Geometry* (Reid [1988a]), *Introduction to Algebraic Geometry* (Hassett [2007]) and, at a somewhat more advanced level *Algebraic Geometry I: Complex Projective Varieties* (Mumford [1976]), *Basic Algebraic Geometry, Volume I* (Shafarevich [1974]) and *Algebraic Geometry: A First Course* (Harris [1992]). The last three include much more than we'll use here.

(b) An acquaintance with the language of schemes. This would be amply covered by the first three chapters of *The Geometry of Schemes* (Eisenbud and Harris [2000]).

(c) An acquaintance with coherent sheaves and their cohomology. For this, *Faisceax Algébriques Cohérents* (Serre [1955]) remains an excellent source (it's written in the language of varieties, but applies nearly word for word to projective schemes over a field, the context in which this book is written).

In particular, *Algebraic Geometry* (Hartshorne [1977]) contains much more than you need to know to get started.

0.5.2 Language

Throughout this book, a *scheme* X will be a scheme of finite type over an algebraically closed field K, and a *variety* will be a reduced, irreducible scheme. If X is a variety we write $K(X)$ for the field of rational functions on X. A *sheaf* on X will be a coherent sheaf unless otherwise noted.

By a *point* we mean a closed point. Recall that a *locally closed* subscheme U of a scheme X is a scheme that is an open subset of a closed subscheme of X. We generally use the term "subscheme" (without any modifier) to mean a closed subscheme, and similarly for "subvariety."

A consequence of the finite type hypothesis is that to any subscheme Y of X has a *primary decomposition*: locally, we can write the ideal of Y as an irredundant intersection of primary ideals with distinct associated primes. We can correspondingly write Y globally as an irredundant union of closed subschemes Y_i whose supports are distinct subvarieties of X. In this expression, the subschemes Y_i whose supports are maximal—corresponding to the minimal primes in the primary decomposition—are uniquely determined by Y; they are called the *irreducible components* of Y. The remaining subschemes are called *embedded components*; they are not determined by Y, though their supports are.

If a family of objects is parametrized by a scheme B, by a "general" member of the family we will mean one belonging to an open dense subset of B. By a "very general" member we will mean one belonging to the complement of a countable union of proper subvarieties of B.

By the *projectivization* $\mathbb{P}V$ of a vector space V we'll mean the scheme $\mathrm{Proj}(\mathrm{Sym}^* V^*)$; this is the space whose closed points correspond to one-dimensional subspaces of V.

If X and $Y \subset \mathbb{P}^n$ are subvarieties of projective space, we define the *join* of X and Y, denoted $\overline{XY}$, to be the closure of the union of lines meeting X and Y at distinct points. If $X = \Gamma \subset \mathbb{P}^n$ is a linear space, this is just the cone over Y with vertex Γ; if X and Y are both linear subspaces, this is simply their span.

There is a one-to-one correspondence between vector bundles on a scheme X and locally free sheaves on X. We will use the terms interchangeably, generally preferring "line bundle" and "vector bundle" to "invertible sheaf" and "locally free sheaf".

By a *linear system*, or *linear series*, on a scheme X we will mean a pair $(\mathcal{L}, V)$ where $\mathcal{L}$ is a line bundle on X and $V \subset H^0(\mathcal{L})$ a vector space of sections. Associating to a section $\sigma \in H^0(\mathcal{L})$ its zero locus $V(\sigma)$, we can also think of a linear system as a family $\mathcal{D} = \{V(\sigma) \mid \sigma \in V\}$ of subschemes parametrized by the projective space $\mathbb{P}V$; in this setting, we will sometimes refer to the linear system $\mathcal{D}$. By the *dimension* of the linear series we mean the dimension of the projective space $\mathbb{P}V$ parametrizing it; that is, $\dim V - 1$.

We write $\mathcal{O}_{X,Y}$ for the local ring of X along Y, and more generally, if $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules then we write $\mathcal{F}_Y$ for the corresponding $\mathcal{O}_{X,Y}$-module.

We can identify the Zariski tangent space to affine space $\mathbb{A}^n$ with $\mathbb{A}^n$ itself. If $X \subset \mathbb{A}^n$ is a subscheme, by the *affine tangent space* to X at a point p we will mean the affine linear subspace $p + T_pX \subset \mathbb{A}^n$. If $X \subset \mathbb{P}^n$ is a subscheme, by the *projective tangent space* to X at $p \in X$, denoted $\mathbb{T}_pX \subset \mathbb{P}^n$, we will mean the closure in $\mathbb{P}^n$ of the affine tangent space to $X \cap \mathbb{A}^n$ for any open subset $\mathbb{A}^n \subset \mathbb{P}^n$ containing p. Concretely, if X is the zero locus of polynomials F_α, the projective tangent space is the common zero locus of the linear forms

$$L_\alpha(Z) = \frac{\partial F_\alpha}{\partial Z_0}(p)Z_0 + \cdots + \frac{\partial F_\alpha}{\partial Z_n}(p)Z_n.$$

By a "one-parameter family" we will always mean a family $X \to B$ with B smooth and one-dimensional (an open subset of a smooth curve, or spec of a DVR or power series ring in one variable), with marked point $0 \in B$. In this context, "with parameter t" means t is a local coordinate on the curve, or a generator of the maximal ideal of the DVR or power series ring.

0.5.3 Basic results on dimension and smoothness

There are a number of theorems in algebraic geometry that we'll use repeatedly; we give the statements and references here.

To start with, we will often use the following basic results of commutative algebra:

Theorem 0.1 (Krull's Principal Ideal Theorem)**.** *An ideal generated by n elements in a Noetherian ring has codimension $\leq n$.*

See Eisenbud [1995] Theorem 10.2 for a discussion and proof. We will also use the following important extension of the Principal Ideal Theorem:

Theorem 0.2 (Generalized Principle Ideal Theorem)**.** *If $f : Y \to X$ is a morphism of varieties, and X is smooth, then for any subvariety $A \subset X$,*

$$\operatorname{codim} f^{-1}A \leq \operatorname{codim} A.$$

In particular, if A, B are subvarieties of X, and C is an irreducible component of $A \cap B$, then $\operatorname{codim} C \leq \operatorname{codim} A + \operatorname{codim} B$.

The proof of this result can be reduced to the case of an intersection of two subvarieties, one of which is locally a complete intersection, by expressing the inverse image $f^{-1}A$ as an intersection with the graph $\Gamma_f \subset X \times Y$ of f. In this form it follows from Krull's Theorem. The result hold in greater generality; see Serre [2000] Theorem V.3. Smoothness is necessary for this (Exercise 1.103).

Theorem 0.3 (Chinese Remainder Theorem)**.** *A module of finite length over a commutative ring is the direct sum of its localizations at finitely many maximal ideals.*

Theorem 0.4 (Jordan-Hölder Theorem)**.** *A module M of finite length over a commutative local ring R with maximal ideal $\mathfrak{m}$ has a maximal sequence of submodules (called its composition series) $M \supsetneq \mathfrak{m}M \cdots \supsetneq \mathfrak{m}^k M = 0$ whose length k is called the length of M.*

For a discussion and proof see Eisenbud [1995] Chapter 2, and especially Theorem 2.13.

Theorem 0.5 (Bertini)**.** *If $\mathcal{D}$ is a linear system on a variety X in characteristic 0, the general member of $\mathcal{D}$ is smooth outside the base locus of $\mathcal{D}$ and the singular locus of X*

This is the form in which we'll usually apply Bertini. But there is another version that is equivalent in characteristic 0 but allows for an extension to positive characteristic:

Theorem 0.6 (Bertini)**.** *If $f : X \to \mathbb{P}^n$ is any generically separated morphism from a smooth, quasiprojective variety X to projective space, then the preimage $f^{-1}(H)$ of a general hyperplane $H \subset \mathbb{P}^n$ is smooth.*

> We are all familiar with the after-the-fact tone—weary, self-justificatory, aggrieved, apologetic—shared by ship captains appearing before boards of inquiry to explain how they came to run their vessels aground, and by authors composing forewords.
> –John Lanchester

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Overture

Keynote Questions

(a) Let F_0, F_1 and $F_2 \in \mathbb{C}[X, Y, Z]$ be three general homogeneous cubic polynomials in three variables. How many linear combinations $t_0F_0 + t_1F_1 + t_2F_2$ (up to scalars) factor as a product of a linear and a quadratic polynomial? (Answer on page 48)

(b) Let F_0, F_1, F_2 and $F_3 \in \mathbb{C}[X, Y, Z]$ be four general homogeneous cubic polynomials in three variables. How many linear combinations $t_0F_0 + t_1F_1 + t_2F_2 + t_3F_3$ (again, up to scalars) factor as a product of three linear polynomials? (Answer on page 48)

(c) If A, B, C are general homogeneous quadratic polynomials in 3 variables, for how many triples $t = (t_0, t_1, t_2)$ is $(A(t), B(t), C(t))$ proportional to (t_0, t_1, t_2)? (Answer on page 40)

(d) Let $S \subset \mathbb{P}^3$ be a smooth cubic surface, and $\{H_\lambda \subset \mathbb{P}^3\}_{l \in \mathbb{P}^1}$ a general pencil of planes (that is, the set of planes containing a fixed general line $L \subset \mathbb{P}^3$). For how many values of λ is the intersection $H_\lambda \cap S$ a singular curve? (Answer on page 34)

(e) Let $L \subset \mathbb{P}^3$ be a line, and let S and $T \subset \mathbb{P}^3$ be surfaces of degrees s and t containing L. Suppose that the intersection $S \cap T$ is the union of L and a smooth curve C. What are the degree and genus of C? (Answer on page 55)

(f) Let $S \subset \mathbb{P}^3$ be a smooth quartic surface over the complex numbers. What is the Euler characteristic of S, viewed as a smooth 4-manifold? (Answer on page 64)

The machinery necessary for intersection theory is beautiful and subtle, and it will take us quite a few pages to develop. Nevertheless, a few facts suffice to begin using the theory. In this chapter we will explain some of these facts without worrying about how they are proved, and give some simple examples to illustrate their application. Our goal is to give the reader a road-map for reading the rest of the book, and to provide the box of tools that are used most commonly in applications—the "dogma" of intersection theory.

We begin by describing the definition of the Chow ring of a smooth projective variety and explaining some of the properties that make it useful. The second section is a series of progressively more interesting examples of computations of Chow rings of familiar varieties, with easy applications. Following this, we see an example of a different kind: we use facts about the Chow ring to describe some geometrically interesting loci in the projective space of singular cubic plane curves.

In the next two sections, we introduce the *canonical class*, a distinguished element of the Chow ring of any smooth variety, and show how to calculate it in simple cases; we then briefly describe intersection theory on surfaces, a setting in which intersection theory takes a particularly simple and useful form.

In the final section of this chapter we turn to another major pillar of the subject, the theory of Chern classes. This is followed by some simple applications.

Very little is proven in the theory sections of this chapter; but in the sections of examples we have tried to give fairly complete expositions. Proofs of the basic facts about Chow rings stated here are given in Chapters 4 and 5. The theory of Chern classes is developed in earnest starting in Chapter 7.

1.1 The Chow groups and their ring structure

We start with the Chow groups. These form what a sort of homology theory for varieties—abelian groups associated to a geometric object that are described as a group of cycles modulo an equivalence relation. In the case of a smooth variety, the Chow groups form a graded ring, the Chow ring. This is analogous to the ring structure on the homology of smooth compact manifolds that can be "imported", using Poincaré duality, from the natural ring structure on cohomology.

Throughout this Chapter, and indeed throughout this book, we will work over an algebraically closed ground field K. Virtually everything we do could be formulated more generally, and occasionally we comment on how one would do this.

1.1.1 Cycles

Let X be any algebraic variety (or, more generally, scheme). The *group of cycles* on X, denoted $Z(X)$, is the free abelian group generated by the set of subvarieties (reduced irreducible subschemes) of X. The group $Z(X)$ is graded by dimension: we write $Z_k(X)$ for the group of cycles that are formal linear combinations of subvarieties of dimension k (these are called *k-cycles*), so that $Z(X) = \oplus_k Z_k(X)$. A cycle $Z = \sum n_i Y_i$ is *effective* if the coefficients n_i are all non-negative. A *divisor* (sometimes called a *Weil divisor*) is an $(n-1)$-cycle on a pure n-dimensional scheme. It follows from the definition that $Z(X) = Z(X_{\text{red}})$; that is, $Z(X)$ is insensitive to whatever non-reduced structure X may have.

We can associate to any closed subscheme $Y \subset X$ an effective cycle: since our schemes are Noetherian, the local ring of a generic point of an irreducible component of any scheme has finite length as a module over itself. Thus if $Y \subset X$ is a subscheme, and $Y_1, \dots, Y_s$ are the irreducible components of the reduced scheme Y_{red}, then each local ring $\mathcal{O}_{Y,Y_i}$ has a finite composition series of length, say l_i. We define the cycle Z associated to Y to be the formal combination $\sum l_i Y_i$, which is well-defined by the Jordan-Hölder Theorem 0.4. When we want to emphasize the distinction between subscheme and cycle, we will write $\langle Y \rangle$ for the cycle associated to the subscheme Y.

1.1.2 Rational equivalence

Next we will define *rational equivalence* between cycles, and also the Chow group, which is the group of cycles mod rational equivalence. Speaking informally we say that two cycles $A_0, A_1 \in Z(X)$ are rationally equivalent if there is a rationally parametrized family of cycles interpolating between them—that is, a cycle on $\mathbb{P}^1 \times X$ whose restrictions to two fibers $\{t_0\} \times X$ and $\{t_1\} \times X$ are A_0 and A_1. Here is the formal definition:

Definition 1.1. Let $Rat(X) \subset Z(X)$ be the subgroup generated by differences of the form

$$\big\langle \Phi \cap (\{t_0\} \times X) \big\rangle - \big\langle \Phi \cap (\{t_1\} \times X) \big\rangle,$$

where t_0 and $t_1 \in \mathbb{P}^1$ and Φ is a subvariety of $\mathbb{P}^1 \times X$ not contained in any fiber $\{t\} \times X$. Two cycles are *rationally equivalent* if their difference is in

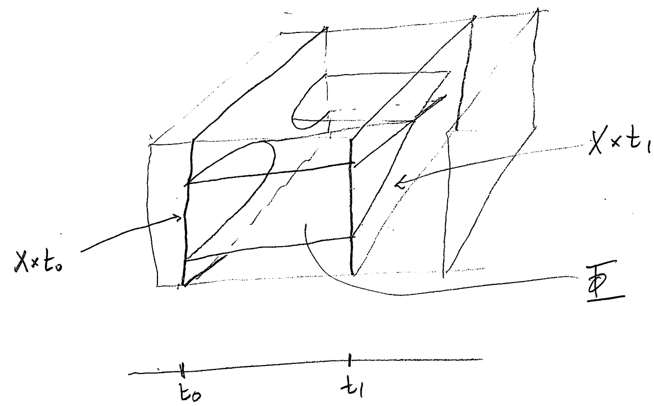


FIGURE 1.1. Rational equivalence between a hyperbola and the union of two lines in $\mathbb{P}^2$

$Rat(X)$, and two subschemes are rationally equivalent if their associated cycles are rationally equivalent—see Figure 1.1. The *Chow group* of X is the quotient $A(X) = Z(X)/Rat(X)$, the *group of rational equivalence classes of cycles on* X. If $A \in Z(X)$ is a cycle, we write $[Y] \in A(X)$ for its equivalence class; and if $Y \subset X$ is a subscheme, we will abuse notation slightly and denote simply by $[Y]$ the class of the cycle $\langle Y \rangle$ associated to Y.

There is another useful way to characterize the group $Rat(X)$ of cycles rationally equivalent to 0. If X is an affine variety and $f \in \mathcal{O}_X$ is a function on X other than 0, then by Krull's Principal Ideal Theorem 0.1 the isolated components of the subscheme defined by f are all of codimension 1, so the cycle defined by this subscheme is a divisor; we call it $Div(f)$, the *divisor of* f. If Y is any irreducible codimension 1 subscheme of X we write $\mathrm{ord}_Y(f)$ for the order of vanishing of f along Y, so we have

$$Div(f) = \sum_Y ord_Y(f)\langle Y \rangle.$$

If $\alpha = f/g$ is a rational function on X, we define the divisor $Div(f/g) = Div(f) - Div(g)$—see Figure 1.2. We will also denote by $Div_0(\alpha)$ and $Div_\infty(\alpha)$ the positive and negative parts of $Div(\alpha)$—in other words, the divisor of zeros of α and the divisor of poles of α respectively. We will show after Proposition 4.9 that these divisors are independent of the choice of representation f/g of α. (Note that $Div_0(\alpha)$ need not be the same thing as $Div(f)$ for any representation $\alpha = f/g$ of α: for example, on the cone $Q = V(XZ - Y^2) \subset \mathbb{P}^3$, the rational function $\alpha = X/Y$ has divisor $L - M$, where L is the line $X = Y = 0$ and M the line $X = Z = 0$; but as the

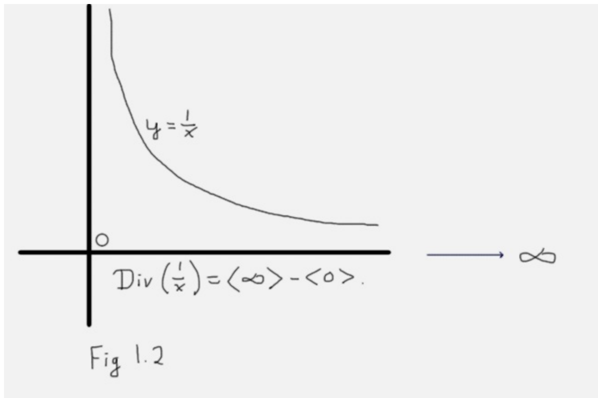


FIGURE 1.2. Graph of the rational function y/x on $\mathbb{P}^1$, showing $[x=0]-[y=0]=0$ in $A(\mathbb{P}^1)$

reader can check, α can't be written in any neighborhood of the vertex $(0,0,0,1)$ of Q as a ratio $\alpha = f/g$ with $Div(f) = L$ and $Div(g) = M$.)

If X is a variety then the field of rational functions on X is the same as the field of rational functions on any open affine subset U of X, so if α is a rational function on X then we get a divisor $Div(\alpha|_U)$ on each open subset $U \subset X$ by restricting α. These agree on overlaps, and thus define a divisor $Div(\alpha)$ on X itself. We will see that the association $\alpha \mapsto Div(\alpha)$ is a homomorphism from the multiplicative group of nonzero rational functions to the additive group of divisors on X.

Proposition 1.2. *The group $Rat(X) \subset Z(X)$ is generated by all divisors of rational functions on all subvarieties of X.*

See Corollary 4.18 for the proof.

Since the divisor of a rational function on an m-dimensional subvariety of X is a sum of cycles of dimension $m-1$, the Chow group is graded by dimension,

$$A(X) = \bigoplus_k A_k(X)$$

and $A_k(X)$ is called the *group of rational equivalence classes of k-cycles.*

Example 1.3. It follows from Proposition 1.2 that two 0-cycles on a curve C (by which we mean a 1-dimensional variety) are rationally equivalent if and only if they differ by the divisor of a rational function. In particular, the cycles associated to two points on C will be rationally equivalent if and only if C is birational to $\mathbb{P}^1$, the isomorphism being given by a rational function that defines the rational equivalence.

Example 1.4. We will soon see that two subvarieties of $\mathbb{P}^n$ are rationally equivalent if and only if they have the same dimension and degree—in this case rational equivalence is a rather loose relationship.

When X is equidimensional, the *codimension* of a subvariety $Y \subset X$ is $\dim X - \dim Y$. When X is also smooth and equidimensional, we will write $A^c(X)$ for the group $A_{\dim X - c}$, and think of it as the group of codimension c cycles, mod rational equivalence. (It would occasionally be convenient to adopt the same notation when X is singular, but this would conflict with established convention—see the discussion in Section 5.5 below.)

1.1.3 First Chern class of a line bundle

One way that rational equivalence classes of cycles arise is from line bundles. If $\mathcal{L}$ is a line bundle on a variety X and σ is a rational section, then on an open affine set U of a covering of X we may write σ in the form f_U/g_U and define $Div(\sigma)|_U = Div(f) - Div(g)$. Again, this patches on overlaps, and thus defines a divisor on X, which is a *Cartier divisor* (see Hartshorne [1977] ****Section number****). Moreover, if τ is another rational section of $\mathcal{L}$ then $\alpha = \sigma/\tau$ is a well defined rational function, so

$$Div(\sigma) - Div(\tau) = Div(\alpha) \equiv 0 \bmod Rat(X).$$

Thus for any line bundle $\mathcal{L}$ on a quasiprojective scheme X we may define the *first Chern class*

$$c_1(\mathcal{L}) \in A(X)$$

to be the rational equivalence class of the divisor σ for any rational section σ. Note that there is no distinguished cycle in the equivalence class. As a first example, we see that $c_1(\mathcal{O}_{\mathbb{P}^n}(d))$ is the class of any hypersurface of degree d; in the notation of Section 1.2.2 it is $d\zeta$, where ζ is the class of a hyperplane.

Recall that $\mathrm{Pic}(X)$ is by definition the group of isomorphism classes of line bundles $\mathcal{L}$ on X, with addition law $[\mathcal{L}] + [\mathcal{L}'] = [\mathcal{L} \otimes \mathcal{L}']$.

Proposition 1.5. *If X is a variety of dimension n then c_1 is a group homomorphism*

$$c_1 : \mathrm{Pic}(X) \to A_{n-1}(X),$$

If X is smooth then c_1 is an isomorphism.

If $Y \subset X$ is a divisor in a smooth variety X, then the ideal sheaf of Y is a line bundle denoted $\mathcal{O}_X(-Y)$, and the inverse of the map c_1 above takes Y to $\mathcal{O}_X(Y) := \mathcal{O}_X(-Y)^{-1}$.

In Section 1.5 we will describe Chern classes for vector bundles more generally. They are one of the most powerful tools of intersection theory.

Proof of Proposition 1.5. To see that c_1 is a group homomorphism, suppose that $\mathcal{L}$ and $\mathcal{L}'$ are line bundles on X. If σ and σ' are rational sections of $\mathcal{L}$ and $\mathcal{L}'$ respectively, then $\sigma \otimes \sigma'$ is a rational section of $\mathcal{L} \otimes \mathcal{L}'$ whose divisor is $Div(\sigma) + Div(\sigma')$.

Now assume that X is smooth and projective. Since the local rings of X are unique factorization domains, every codimension 1 subvariety is a Cartier divisor, so to any divisor we can associate a unique line bundle and a rational section. Forgetting the section, we get a line bundle, and thus a map from the group of divisors to $\mathrm{Pic}(X)$. By Proposition 1.2, rationally equivalent divisors differ by the divisor of a rational function and thus correspond to different rational sections of the same bundle. It follows that the map on divisors induces a map on $A^1(X)$, inverse to the map c_1. $\square$

In case X is singular the map $c_1 : \mathrm{Pic}(X) \to A_{n-1}(X)$ is in general neither injective or surjective. For example, if X is an irreducible plane cubic with a node, then $c_1 : \mathrm{Pic}(X) \to A_1(X)$ is not a monomorphism (Exercise 1.60). On the other hand, if $X \subset \mathbb{P}^3$ a quadric cone with vertex p then $A_1(X) = \mathbb{Z}$, generated by the class of a line, and the image of $c_1 : \mathrm{Pic}(X) \to A_1(X)$ is $2\mathbb{Z}$ (Exercise 1.61).

1.1.4 First results on the Chow group

Proposition 1.2 makes it obvious that if $Y \subset X$ is a closed subscheme then the identification of the cycles on $\mathbb{P}^1 \times Y$ as cycles on $\mathbb{P}^1 \times X$ induces a map $Rat(Y) \to Rat(X)$, and thus a map $A(Y) \to A(X)$. Further, the intersection of a subvariety of X with the open set $U = X \setminus Y$ is a subvariety of U (possibly empty), so there is restriction homomorphism $Z(X) \to Z(U)$. It turns out that this induces a homomorphism of Chow groups. From this we get the fourth part of the following useful proposition; see Proposition 4.12 for the proof.

Proposition 1.6. *Let X be a scheme.*

(a) $A(X) = A(X_{\mathrm{red}})$.

(b) *If X is irreducible of dimension k, then $A_k(X) \cong \mathbb{Z}$, with generator $[X]$, called the* fundamental class *of X. More generally, if the irreducible components of X are $X_1, \dots, X_m$ then the classes $[X_i]$ generate a free abelian subgroup of rank m in $A(X)$.*

(c) *(Mayer-Vietoris) If X_1, X_2 are closed subschemes of X, then there is a right exact sequence*

$$A(X_1 \cap X_2) \longrightarrow A(X_1) \oplus A(X_2) \longrightarrow A(X_1 \cup X_2) \longrightarrow 0.$$

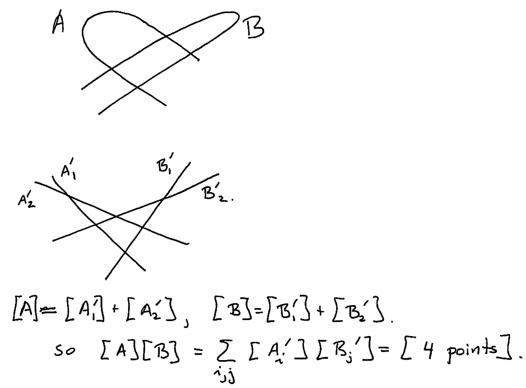


FIGURE 1.3. Two conics meet in four points. [[SILVIO the curves should be conics.]]

(d) *(Excision) If $Y \subset X$ is a closed subscheme and $U = Y \setminus X$ is its complement, then the the inclusion and restriction maps of cycles give a right exact sequence*

$$A(Y) \longrightarrow A(X) \longrightarrow A(U) \longrightarrow 0.$$

For example, if X is the spectrum of an artinian ring, then X has no proper subvarieties except for its irreducible components, so Parts (a) and (b) imply that $A(X) = A_0(X)$ is the free abelian group on the components of X.

1.1.5 Intersection products

One of the earliest results about intersections is the theorem of Bézout (1776): If curves $D, E \subset \mathbb{P}^2$, of degrees d and e, intersect transversely, then they intersect in exactly de points. Two things about this result are striking: first, the cardinality of the intersection does not depend on the choice of curves, beyond knowing their degrees and that they meet transversely. Given this invariance, the theorem follows from the obvious fact that a union of d general lines meets a union of e general lines in de points (Figure 1.3).

Second, the answer, de, is a product, suggesting that some sort of ring structure is present. A great deal of the development of algebraic geometry over the past 200 years is bound up in the attempt to understand, generalize and apply these ideas.

We will show that, indeed, cycles mod rational equivalence on any smooth quasi-projective variety form a ring of this sort. Moreover, the ring structure is determined by one geometrically natural condition, which we now explain.

We say that subvarieties A, B of X intersect *transversely* at a point p if A, B and X are all smooth at p and the tangent spaces to A and B at p together span the tangent space to X; that is,

$$T_pA + T_pB = T_pX.$$

We say that A and B are *generically transverse*, or that they intersect *generically transversely* if they meet transversely at a general point of each component C of $A \cap B$. If X is smooth, this is equivalent to saying that $\operatorname{codim} C = \operatorname{codim} A + \operatorname{codim} B$ and C is generically reduced. Note that if $\operatorname{codim} A + \operatorname{codim} B > \dim X$, then then A and B are generically transverse if and only if they are disjoint.

Theorem-Definition 1.7. If X is a smooth quasiprojective variety then there is a unique product structure on $A(X)$ satisfying the condition:

(*) If subvarieties A, B of X are generically transverse then

$$[A][B] = [A \cap B].$$

This structure makes $A(X)$ into an associative, commutative ring, graded by codimension, called the *Chow ring* of X.

To prove Theorem 1.7 we must somehow compare an arbitrary pair of cycles with a pair of cycles that meet generically transversely. The key to our development will be the *Moving Lemma*—see Figure 1.4:

Theorem 1.8 (Moving Lemma)**.** *Let X be a smooth quasiprojective variety.*

(a) *For every $\alpha, \beta \in A(X)$ there are cycles $A = \sum m_iA_i$ and $B = \sum n_jB_j$ in $Z(X)$ that represent α and β such that A_i intersects B_j generically transversely for each i and j; and*

(b) *The class*

$$\sum_{i,j} m_in_j[A_i \cap B_j]$$

in $A(X)$ is independent of the choice of such A, B.

The hypothesis of smoothness in Theorem 1.8 cannot be avoided: we'll see in Section 5.5 examples of singular X and classes α such that the support of any cycle representing the class α must contain the singular locus of X (the simplest such example would be to take $X \subset \mathbb{P}^3$ a quadric cone, and α the class of a line of X), so that it may simply not be possible to find

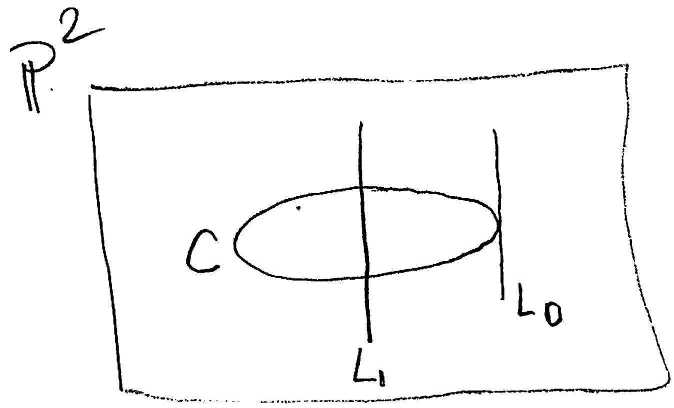


FIGURE 1.4. The cycle L_0 can be moved to the rationally equivalent cycle L_1, which is transverse to the given subvariety C.

representatives of two given classes $\alpha, \beta \in A(X)$ intersecting generically transversely. Smoothness is also an essential hypothesis in Theorem 1.7; we'll also see in Section 5.5 examples of varieties X where no intersection product satisfying the basic condition (*) of Theorem 1.7 can be defined.

The news is not uniformly negative, however: intersection products *can* be defined on singular varieties if we impose some restrictions on the classes involved. For example, we can intersect the class of an arbitrary cycle A with the first Chern class of a line bundle, as we'll see in Proposition 1.10. See Appendix **??** for further discussion.

When a sufficiently large group of automorphisms acts on X the Moving Lemma is easy to prove: we can use an automorphism of X to move one of the cycles we would like to intersect, as in the following special case of a result of Kleiman (see Theorem 5.15):

Theorem 1.9 (Kleiman's theorem in characteristic 0)**.** *Suppose that an algebraic group G acts transitively on a variety X over a field of characteristic 0. If A and $B \subset X$ are any subvarieties, then there is an open dense set of $g \in G$ such that gA is generically transverse to B. Moreover, if G is affine, then $gA \sim A$ for any $g \in G$.*

If the algebraic group G acting on X is affine, then this gives us the first half of the Moving Lemma. Kleiman showed that the same conclusion holds in positive characteristic under the stronger hypothesis that G acts transitively on nonzero tangent vectors to X (Theorem 5.15), but this stronger hypothesis fails in many settings in which we will be interested, such as products of projective spaces and Grassmannians, and the conclusion fails as well (Kleiman [1974]).

Another case when the Moving Lemma is easy is when the class of the cycle to be moved has the form $c_1(\mathcal{L})$ for some line bundle $\mathcal{L}$. In this case we don't need smoothness. We also get a useful formula for the product of any class with $c_1(\mathcal{L})$:

Proposition 1.10. *Suppose that X is a quasiprojective variety, and $\mathcal{L}$ is a line bundle on X. If $Y_1, \dots, Y_n$ are any subvarieties of X then there is a cycle in the class of $c_1(\mathcal{L})$ that is generically transverse to each Y_i. If X is smooth, then*

$$c_1(\mathcal{L})\left[\sum_i m_i Y_i\right] = \sum_i m_i c_1(\mathcal{L}|_{Y_i}).$$

We have abused our notation: the class $c_1(\mathcal{L}|_{Y_i})$ on the right hand side of the formula is actually a class in $A(Y_i)$, so to be precise we should have written $j_*(c_1(\mathcal{L}|_{Y_i})$, where $j : Y_i \to X$ is the inclusion. This imprecision points to an important theoretical fact: the intersection is actually well-defined, not just on X, but as a class on Y_i. This is the beginning of the theory of the "refined intersection product" defined in Fulton [1984].

We imposed the hypothesis of smoothness in Proposition 1.10 because we have only discussed products in this context. In fact, the formula could be used to define an action of a class of the form $c_1(\mathcal{L})$ on $A(X)$ for any scheme X. This is the point of view taken by Fulton.

Sketch of Proof of Proposition 1.10. Since X is quasiprojective there is an ample bundle $\mathcal{L}'$ on X. For a sufficiently large integer n both the line bundles $\mathcal{L}'^{\otimes n}$ and $\mathcal{L}'^{\otimes n} \otimes \mathcal{L}$ are very ample, so by Bertini's Theorem there are sections $\sigma \in H^0(\mathcal{L}'^{\otimes n})$ and $\tau \in H^0(\mathcal{L}'^{\otimes n} \otimes \mathcal{L})$ whose zero loci $Div(\sigma)$ and $Div(\tau)$ are generically transverse to each Y_i. The class $c_1(\mathcal{L})$ is rationally equivalent to the cycle $Div(\sigma) - Div(\tau)$, proving the first assertion. Moreover, $c_1(\mathcal{L})[Y_i] = [Div(\sigma) \cap Y_i] - [Div(\tau) \cap Y_i]$ by definition. Since $Div(\sigma) \cap Y_i = Div(\sigma|_{Y_i})$, and similarly for τ, we are done. □

Multiplicities. We can easily extend the range of cases where intersection products have a simple geometric interpretation: if the intersection of subvarieties A and $B \subset X$ of codimensions a and b on a smooth variety X is *dimensionally proper*—that is, every component Z of $A \cap B$ has codimension $a+b$—then for each component Z of $A \cap B$ there is an *intersection multiplicity* $m_Z(A, B)$, such that the intersection cycle

$$AB = \sum m_Z(A, B) \cdot Z$$

has class $[A][B] \in A^{a+b}(X)$—see Figure 1.4. The key facts about the intersection multiplicity are:

(a) $m_Z(A, B) > 0$ for any component Z of $A \cap B$, with $m_Z(A, B) = 1$ if and only if A and B intersect transversely at a general point of Z;

(b) In case A and B are local complete intersection subvarieties of X (or, more generally, Cohen-Macaulay) at a general point of Z, then $m_Z(A, B)$ is the multiplicity of the component of the scheme $A \cap B$ supported on Z; and

(c) $m_Z(A, B)$ depends only on the local structure of A and B at a general point of Z.

Note that as a consequence of (b), if A and $B \subset X$ are Cohen-Macaulay and intersect in the expected codimension, we have simply

$$[A][B] = [A \cap B].$$

(A scheme is said to be Cohen-Macaulay if each of its local rings is Cohen-Macaulay. For a treatment of Cohen-Macaulay rings see Eisenbud [1995] Chapter 18.)

In Section 5.2 we will explain one way to define the multiplicity, and prove statements (a), (b), (c) in the case where both A and B are Cohen-Macaulay—see Theorem 5.10. In Example 5.13 we will see a case where the intersection multiplicity is not given by the mulitiplicities of the components of the intersection scheme, showing that the Cohen-Macaulay condition is necessary.

1.1.6 Functoriality

The key to working with Chow groups is to understand how they behave with respect to morphisms between varieties. To know what to expect, think of the analogous situation with homology and cohomology. A smooth projective variety of (complex) dimension n over the complex numbers is a compact oriented $2n$-manifold, so $H_{2m}(X)$ can be identified canonically with $H^{2n-2m}(X)$ (singular homology and cohomology). If we think of $A(X)$ as being analogous to $H_*(X)$, then we should expect $A_m(X)$ to be a covariant functor from smooth projective varieties to groups, via some sort of pushforward maps preserving dimension. If we think of $A(X)$ as analogous to $H^*(X)$, then we should expect $A(X)$ to be a contravariant functor from smooth projective varieties to rings, via some sort of pullback maps preserving codimension. Both these expectations are realized.

Proper pushforward. If $f : Y \to X$ is a proper map of schemes, then the image of a subvariety $A \subset Y$ is a subvariety $f(A) \subset X$. One might at first guess that the pushforward could be defined by sending the class of A to the class of $f(A)$, and this would not be far off the mark. But this would not preserve rational equivalence (an example is pictured in Figure 1.5). Rather, we must take multiplicities into account.

If A is a subvariety and $\dim A = \dim f(A)$ then $f|_A : A \to f(A)$ is *generically finite*, in the sense that the field of rational functions $K(A)$ is a finite extention of the field $K(f(A))$ (this follows because they are both finitely generated fields, of the same transcendence degree ($= \dim A$) over the ground field). Geometrically the condition can be expressed by saying that for a generic point $x \in f(A)$, the preimage $y := f|_A^{-1}(x)$ in A is a finite scheme. In this case the degree $n := [K(A) : K(f(A))]$ of the extension of rational function fields is equal to the degree of y over x for a dense open subset of $x \in f(A)$, and this common value n is called the *degree* of the covering of $f(A)$ by A. Under these circumstances, we must count $f(A)$ with multiplicity n in the pushforward cycle:

Definition 1.11 (Pushforward for cycles)**.** Let $f : Y \to X$ be a proper map of schemes, and let $A \subset X$ be a subvariety.

(a) If $f(A)$ has strictly lower dimension than A, then we define $f_*\langle A\rangle = 0$.

(b) If $\dim f(A) = \dim A$ and $f|_A$ has degree n, then we define $f_*\langle A\rangle = n \cdot \langle f(A)\rangle$.

(c) We extend f to all cycles on Y by linearity; that is, for any collection of subvarieties $A_i \subset Y$ we set $f_*(\sum m_i\langle A_i\rangle) = \sum m_i f_*\langle A_i\rangle$.

With this definition, the pushforward of of cycles preserves rational equivalence (see 4.15):

Theorem 1.12. *If $f : Y \to X$ is a proper map of schemes, then the map $f_* : Z(Y) \to Z(X)$ defined above induces a map of groups $f_* : A_k(Y) \to A_k(X)$ for each k.*

It is often hard to prove that a given class in $A(X)$ is nonzero, but the fact that the pushforward map is well-defined gives us a start:

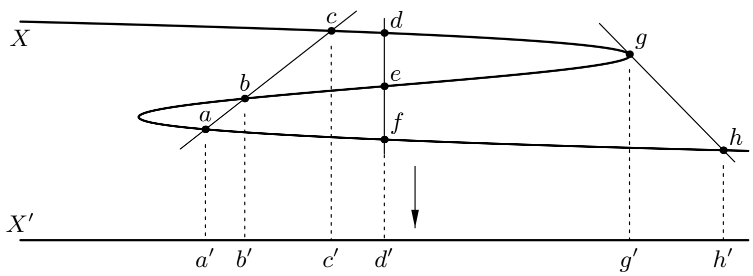
Proposition 1.13. *If X is proper over* $\operatorname{Spec} K$*, then there is a map* $\deg :$ *$A_0(X) \to \mathbb{Z}$ taking the class $[p]$ of each closed point $p \in X$ to 1.*

As stated, Proposition 1.13 uses our standing hypothesis that the ground field is algebraically closed. Without this restriction we would have to count each (closed) point by the degree of its residue field extension over the ground field.

We will typically use this proposition together with the intersection product: if A is a k-dimensional subvariety of a smooth projective variety X and B is a k-codimensional subvariety of X such that $A \cap B$ is finite and nonempty, then the map

$$A_k(X) \to \mathbb{Z} : \ [Z] \mapsto \deg\bigl([Z][B]\bigr)$$

$$a + b + c \sim d + e + f \sim 2g + h$$



$$a' + b' + c' \sim 3d' \sim 2g' + h'$$

FIGURE 1.5. Pushforwards of equivalent cycles are equivalent

sends $[A]$ to a nonzero integer. Thus no integer multiple $m[A]$ of the class A could be zero.

Flat and projective pullback. We next turn to the pullback. Let $f : Y \to X$ be a morphism, and $A \subset X$ a subvariety of codimension c. A good pullback map $f^* : A(X) \to A(Y)$ on cycles should preserve rational equivalence, and in the nicest case, when $f^{-1}(A)$ is generically reduced of codimension c, it should be geometric, in the sense that $f^*([A]) = [f^{-1}(A)]$. It turns out that this requirement determines f^*, at least for maps of smooth varieties such that the corresponding extension of rational function fields is separable (*generically separable* maps). This is a consequence of a refinement of the Moving Lemma (see Lemma 5.28 and Theorem 5.8):

Theorem 1.14. *Let $f : Y \to X$ be a generically separable morphism of smooth quasiprojective varieties.*

(a) *Given any class $\alpha \in A^c(X)$, there exists a cycle $A = \sum n_i A_i \in Z^c(X)$ representing α such that $f^{-1}(A_i)$ is generically reduced of codimension c in Y for all i; and*

(b) *The class $\sum n_i[f^{-1}(A_i)] \in A^c(Y)$ is independent of the choice of such A.*

In these circumstances, we can define the pullback $f^*(\alpha)$ to be the class $\sum n_i[f^{-1}(A_i)] \in A^c(Y)$:

Theorem 1.15. *Let $f : Y \to X$ be a generically separable map of smooth quasiprojective varieties.*

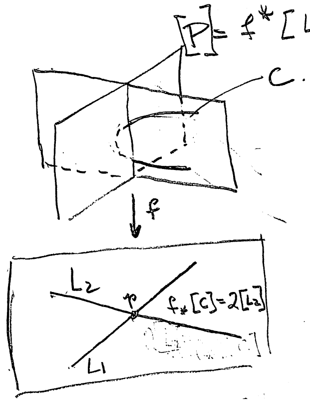


FIGURE 1.6. $2[p] = (f_*[P])[C] = f_*([f^*L_1][C]) = [L_1]f_*[C] = [L_1][2L_2]$

(a) *There is a unique way of defining a map of groups $f^* : A^c(X) \to A^c(X)$ such that $f^*([A]) = [f^{-1}(A)]$ whenever $f^{-1}(A)$ is generically reduced of codimension c. The map f^* is a ring homomorphism, and makes A into a contravariant functor from the category of smooth projective varieties to the category of graded rings.*

(b) *(Push-Pull formula) The map $f_* : A(Y) \to A(X)$ is a map of graded modules over the graded ring $A(X)$. More explicitly, if $\alpha \in A^k(X)$ and $\beta \in A_l(Y)$ then then*

$$f_*(f^*\alpha \cdot \beta) = \alpha \cdot f_*\beta \in A_{l-k}(X).$$

The last statement of this theorem is the result of applying appropriate multiplicities to the set-theoretic equality $f(f^{-1}(A) \cap B) = A \cap f(B)$ (see the proof of Theorem 5.8); see Figure 1.6.

One simple case of a projective morphism is the inclusion map from a closed subvariety $\iota : Y \subset X$. When X and Y are smooth, our definition of intersections and pullbacks makes it clear that if A is any subvariety of X, then $[A][Y]$ is represented by the same cycle as $\iota^*([A])$—except that these are considered as classes in different varieties! More precisely, we can write

$$[A][Y] = \iota_*(\iota^*[A]).$$

In this case the extra content of Theorem 1.15 is that this cycle is well defined as a cycle on Y, not only as a cycle on X. Fulton [1984] Section 8.1 shows that it is even well-defined as a "refined intersection class" on $X \cap Y$ and, more generally, he proves the existence of such a refined version of the pullback under a proper, locally complete intersection morphism (of which a map of smooth projective varieties is an example).

We can also define a good pull-back map for flat morphisms. The flat case is simpler than the projective case for two reasons: first, the preimage of a subvariety of codimension k is always of codimension k; and second, rational functions on the image pull back to rational functions on the source. We will use the flat case as a stepping stone in the proof of the projective case, and also to analyze maps of affine space bundles.

Theorem 1.16. *Let $\pi : Y \to X$ be a flat map of schemes. The map π^* defined on cycles by*

$$\pi^*(\langle A\rangle) := \langle \pi^{-1}(A)\rangle \quad \textit{for every subvariety } A \subset X$$

preserves rational equivalence, and thus induces a map of Chow groups preserving the grading by codimension. If X and Y are smooth and quasiprojective, then $\pi^* : A(X) \to A(Y)$ is a ring homomorphism.

1.2 Chow ring examples

In this section we will use the techniques described above to compute the Chow groups of a few spaces, and to compute the classes of some interesting cycles. We will generally give proofs that are complete modulo the three foundational Theorems 1.7, 1.12 and 1.15 above.

Though we are assuming that the ground field is algebraically closed, the results here could easily be extended to the general case by counting closed points with multiplicities equal to the degrees of their residue class field extensions.

1.2.1 Varieties built from affine spaces

We start with a basic result about open subsets of affine space:

Proposition 1.17. *If $U \subset \mathbb{A}^n$ is an open set then $A(U) = A_n(U) = \mathbb{Z}\cdot[U]$.*

Proof. By part (b) of Proposition 1.6, $A_n(U) = \mathbb{Z} \cdot [U]$, so it suffices to show that the class $[Y]$ of any subvariety $Y \subsetneq U$ is zero. By part (d) of 1.6, it is enough to do this in the case $U = \mathbb{A}^n$.

Let $Y \subset \mathbb{A}^n$ be a proper subvariety, and choose $z = z_1, \ldots, z_n$ be coordinates on $\mathbb{A}^n$ so that the origin does not lie in Y. We let

$$\begin{aligned} W^\circ &= \{(t, z) \mid \frac{z}{t} \in Y\} \\ &= V(f(z/t) \mid f(z) \text{ vanishes on } Y\}) \subset (\mathbb{A}^1 \setminus 0) \times \mathbb{A}^n, \end{aligned}$$

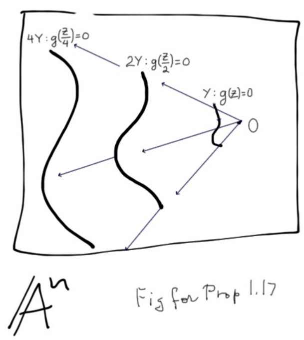


FIGURE 1.7. Scalar Multiplication gives a rational equivalence between an affine variety not containing the origin and the empty set.

and let $W \subset \mathbb{P}^1 \times \mathbb{A}^n$ be the closure of W° in $\mathbb{P}^1 \times \mathbb{A}^n$. Note that W°, being the image of $(\mathbb{A}^1 \setminus 0) \times Y$, is irreducible, and hence so is W.

The fiber of W over the point $t = 1$ is just Y. On the other hand, since the origin in $\mathbb{A}^n$ does not lie in Y there is some polynomial $g(z)$ that vanishes on Y and has a nonzero constant term c. The function $G(t, z) = g(z/t)$ on $(\mathbb{A}^1 \setminus 0) \times \mathbb{A}^n$ then extends to a regular function on $(\mathbb{P}^1 \setminus 0) \times \mathbb{A}^n$, with constant value c on the fiber $\infty \times \mathbb{A}^n$. Thus the fiber of W over $t = \infty \in \mathbb{P}^1$ is empty, establishing the equivalence $Y \sim 0$. See Figure 1.7 $\square$

See Section 4.1.1 for a more systematic treatment of this idea.

Affine stratifications. In general we will work with very partial knowledge of the the Chow groups of a variety, but when X admits an *affine stratification*—a special kind of decomposition into a union of affine spaces—we can know them completely. This will help us compute the Chow groups of projective space, Grassmannians, and many other interesting rational varieties.

We say that a scheme X is *stratified* by a finite collection of irreducible, locally closed subschemes U_i if X is a disjoint union of the U_i and, in addition, if $\overline{U_i}$ meets U_j, then $\overline{U_i}$ contains U_j. The sets U_i are called the *strata* of the stratification, while the closures $Y_i := \overline{U_i}$ are called the *closed strata*. The stratification can be recovered from the closed strata Y_i: we simply take

$$U_i = Y_i \setminus \bigcup_{Y_j \subsetneq Y_i} Y_j.$$

Definition 1.18. We say that a stratification of X with strata U_i is:

- *affine* if each open stratum is isomorphic to some $\mathbb{A}^k$; and
- *quasi-affine* if each U_i is isomorphic to an open subset of some $\mathbb{A}^k$.

For example, a complete flag of subspaces $\mathbb{P}^0 \subset \mathbb{P}^1 \subset \cdots \subset \mathbb{P}^n$ gives an affine stratification of projective space; the closed strata are just the $\mathbb{P}^i$ and the open strata are affine spaces $U_i = \mathbb{P}^i \setminus \mathbb{P}^{i-1} \cong \mathbb{A}^i$.

Proposition 1.19. *If a scheme X has a quasi-affine stratification then $A(X)$ is generated by the classes of the closed strata.*

In general, the classes of the strata need not be independent (for example, the affine line, with $A(\mathbb{A}^1) = \mathbb{Z}$, also has a quasi-affine stratification consisting of a single point and its complement.) But a much stronger assertion holds for affine stratifications: Totaro [to appear] proves that in this case the classes of the closed strata are independent in $A(X)$, so that $A(X)$ is the free abelian group generated by its closed strata. Amazingly, no elementary proof of this simple assertion is known. However, using the intersection product it is easy to prove the result in the cases we will encounter.

Proof of Proposition 1.19. We will do induction on the number of strata U_i. If this number is 1 then the assertion is that of Proposition 1.17.

Let U_0 be a minimal stratum. Since the closure of U_0 is a union of strata, U_0 must already be closed. It follows that $U := X \setminus U_0$ is stratified by the strata other than U_0. By induction, $A(U)$ is generated by the classes of the closures of these strata, and by Proposition 1.17. $A(U_0)$ is generated by $[U_0]$. By part d) of Proposition 1.6 the sequence

$$\mathbb{Z} \cdot [U_0] = A(U_0) \longrightarrow A(X) \longrightarrow A(X \setminus U_0) \longrightarrow 0.$$

is right exact. Since the classes in $A(U)$ of the closed strata in U come from the classes of (the same) closed strata in X, it follows that $A(X)$ is generated by the classes of the closed strata. □

1.2.2 The Chow ring of $\mathbb{P}^n$

Theorem 1.20. *The Chow ring of $\mathbb{P}^n$ is*

$$A(\mathbb{P}^n) = \mathbb{Z}[\zeta]/(\zeta^{n+1}),$$

where $\zeta \in A^1(\mathbb{P}^n)$ is the rational equivalence class of a hyperplane; more generally, the class of a variety of codimension k and degree d is $d\zeta^k$.

In particular, the Theorem implies that $A^m(\mathbb{P}^n) \cong \mathbb{Z}$ for $0 \leq m \leq n$, generated by the class of an $(n-m)$-plane. The natural proof, given below, uses the intersection product. A proof that does not use the intersection product is given in Chapter 4 (Theorem 4.4).

We note that since the sections of the *tautological line bundle* $\mathcal{O}_{\mathbb{P}^n}(1)$ correspond to linear forms on $\mathbb{P}^n$, the class ζ could also be described as the first Chern class, $\zeta = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$.

Proof. It follows from Proposition 1.19 that $A^k(\mathbb{P}^n)$ is generated by the class of any $(n-k)$-plane $L \subset \mathbb{P}^n$. Using Proposition 1.13 we get $A^n(\mathbb{P}^n) = \mathbb{Z}$. Since a general $(n-k)$ plane L intersects a general k-plane M transversely in one point, multiplication by $[M]$ induces a surjective map $A^k(\mathbb{P}^n) \to A^n(\mathbb{P}^n) = \mathbb{Z}$, so $A^k(\mathbb{P}^n) = \mathbb{Z}$ for all k.

An $(n-k)$-plane L in $\mathbb{P}^n$ is the transverse intersection of k hyperplanes so

$$[L] = \zeta^k,$$

where $\zeta \in A^1(\mathbb{P}^n)$ is the class of a hyperplane. Finally, since a subvariety $X \subset \mathbb{P}^n$ of dimension $n-k$ and degree d intersects a general k-plane transversely in d points, we have $\deg([X]\zeta^{(n-k)}) = d$. Since $\deg(\zeta^n) = 1$, we conclude that $[X] = d\zeta^k$. □

A nice qualitative result follows from Theorem 1.20:

Corollary 1.21. *Any map from $\mathbb{P}^n$ to a quasi-projective variety of dimension strictly less than n is constant.*

Proof. Let $\varphi : \mathbb{P}^n \to X$ be the map, which we may assume is surjective. If φ is not constant, then the preimage of a general hyperplane section of X will be disjoint from the preimage of a general point of X. But if $0 < \dim X < n$, then the preimage of a hyperplane section of X has dimension $n-1$ and the preimage of a point has dimension > 0. Since any two such subvarieties of $\mathbb{P}^n$ must meet, this is a contradiction. □

Theorem 1.20 implies the analogue of Poincaré duality for $A(\mathbb{P}^n)$: $A_k(\mathbb{P}^n)$ is dual to $A^k(\mathbb{P}^n)$ via the intersection product. The reader should be aware that in cases where the Chow groups and the homology groups are different, Poincaré duality generally does *not* hold for the Chow ring; for example, when X is a variety, $A_{\dim X}(X) \cong \mathbb{Z}$, but $A_0(X)$ need not even be finitely generated.

As an immediate consequence of Theorem 1.20, we get a general form of of Bézout's Theorem.

Theorem 1.22 (Bézout's Theorem)**.** *If $Z_1, \dots, Z_k \subset \mathbb{P}^r$ are subvarieties of codimensions $c_1, \dots, c_k$ with $\sum c_i \le r$ and the Z_i intersect generically* *transversely, then*

$$\deg(Z_1 \cap \cdots \cap Z_k) = \prod \deg(Z_i).$$

In particular, two subvarieties $X, Y \subset \mathbb{P}^r$ having complementary dimension and intersecting transversely will intersect in exactly $\deg(X) \cdot \deg(Y)$ points. $\square$

Using multiplicities we can extend this formula to the case where the varieties intersect properly (that is, the Z_i all have codimension equal to $\sum c_i$). If we assume as well that the varieties are generically Cohen-Macaulay along their intersections, then the multiplicities are equal to the lengths of the components of the intersection scheme, and thus are encoded in the intersection cycle. These things are satisfied, for example when two plane curves meet along a finite set of points, the classical case treated by Bézout's Theorem. More generally, we can treat a dimensionally proper intersection of hypersurfaces in any projective space:

Corollary 1.23. *If c hypersurfaces $Z_1, \ldots, Z_c \subset \mathbb{P}^r$ meet in a scheme of codimension c with irreducible (but not necessarily reduced) components $C_1, \ldots, C_t$, then*

$$\sum [C_j] = \prod [Z_i]$$

and in particular

$$\sum \deg C_j = \prod \deg Z_i.$$

Degrees of Veronese varieties. Let

$$\nu = \nu_{n,d} : \mathbb{P}^n \to \mathbb{P}^N \quad \text{where} \quad N = \binom{n+d}{n} - 1$$

be the *Veronese map*

$$[Z_0, \ldots, Z_n] \mapsto [\ldots, Z^I, \ldots]$$

where Z^I ranges over over monomials of degree d in $n+1$ variables. The image $\Phi = \Phi_{n,d} \subset \mathbb{P}^N$ of the Veronese map $\nu = \nu_{n,d}$ is called the *d-th Veronese variety of* $\mathbb{P}^n$, as is any subvariety of $\mathbb{P}^N$ projectively equivalent to it. This variety may be characterized, up to automorphisms of the target $\mathbb{P}^N$, as the map associated to the complete linear system $|\mathcal{O}_{\mathbb{P}^n}(d)|$; in other words, by the property that the preimages $\nu^{-1}(H) \subset \mathbb{P}^n$ of hyperplanes $H \subset \mathbb{P}^N$ comprise all hypersurfaces of degree d in $\mathbb{P}^n$.

In characteristic 0 there is another attractive description: Writing $\mathbb{P}^n = \mathbb{P}V$, where V is an $(n+1)$-dimensional vector space, $\nu_{n,d}$ is projectively equivalent to the map taking $\mathbb{P}V \to \mathbb{P}\operatorname{Sym}^d V$ by $[v] \mapsto [v^d]$; for if the coordinates of v are $v_0, \ldots, v_n$ then the coordinates of v^d are

$$\frac{d!}{\prod_i d_i!}(v_0^{d_0} \cdots v_n^{d_n}).$$

If the characteristic is zero then the coefficients are nonzero, so we may rescale by an automorphism of $\mathbb{P}^n$ to get the standard Veronese map above.

We can use Theorem 1.22 to compute the degrees of a Veronese variety:

Proposition 1.24. *The degree of $\Phi_{n,d}$ is d^n.*

Proof. The degree of Φ is the cardinality of its intersection with n general hyperplanes $H_1, \dots, H_n \subset \mathbb{P}^N$; since the map ν is one-to-one, this is in turn the cardinality of the intersection $f^{-1}(H_1) \cap \cdots \cap f^{-1}(H_n) \subset \mathbb{P}^n$. The preimages of the hyperplanes H_i are n general hypersurfaces of degree d in $\mathbb{P}^n$. By Bézout's Theorem, the cardinality of their intersection is d^n. $\square$

Degree of the dual of a hypersurface. The same idea allows us to compute the degree of the the *dual variety* of a smooth hypersurface $X \subset \mathbb{P}^n$, that is, the set of points $X^* \subset \mathbb{P}^{n*}$ corresponding to hyperplanes of $\mathbb{P}^n$ that are tangent to X. (In Chapter 12 we will generalize this notion substantially, discussing the duals of varieties of higher codimension and singular varieties as well.)

The set X^* is a variety because it is the image of X under the morphism $\mathcal{G}_X : X \to \mathbb{P}^{n*}$ sending a point $p \in X$ to its tangent hyperplane $\mathbb{T}_p X$; in coordinates, if X is the zero locus of the homogeneous polynomial $F(Z_0, \dots, Z_n)$ then $\mathcal{G}$ is given by the formula

$$\mathcal{G}_X : p \mapsto \Big[\frac{\partial F}{\partial Z_0}, \dots, \frac{\partial F}{\partial Z_n}\Big].$$

Since X is smooth, the partials of F have no common zeros on X, so this is indeed a morphism.

Moreover, the fact that the partials of F have no common zeros says that if $d > 1$ the map $\mathcal{G}_X$ cannot have positive-dimensional fibers: if $\mathcal{G}_X$ were constant on a complete curve $C \subset X$, the restrictions to C of the partials of F would be scalar multiples of each other and so would have a common zero. Thus if $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree $d \geq 2$, the dual variety $X^* \subset \mathbb{P}^{n*}$ will again be a hypersurface, though not usually smooth. (If $d = 1$, the map $\mathcal{G}_X$ is constant and X^* a point.) The smoothness hypothesis is necessary here; for example, the dual Q^* of th quadric cone $Q = V(XZ - Y^2) \subset \mathbb{P}^3$ is a conic curve in $\mathbb{P}^{3*}$.

Even more is true in characteristic 0: we will see as a consequence of Corollary 12.20 that the map $\mathcal{G}_X$ is birational onto its image. We can use this to deduce the degree of the dual:

Proposition 1.25. *If $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree $d > 1$ over a field of characteristic 0, then the dual of X is a hypersurface of degree $d(d-1)^{n-1}$.*

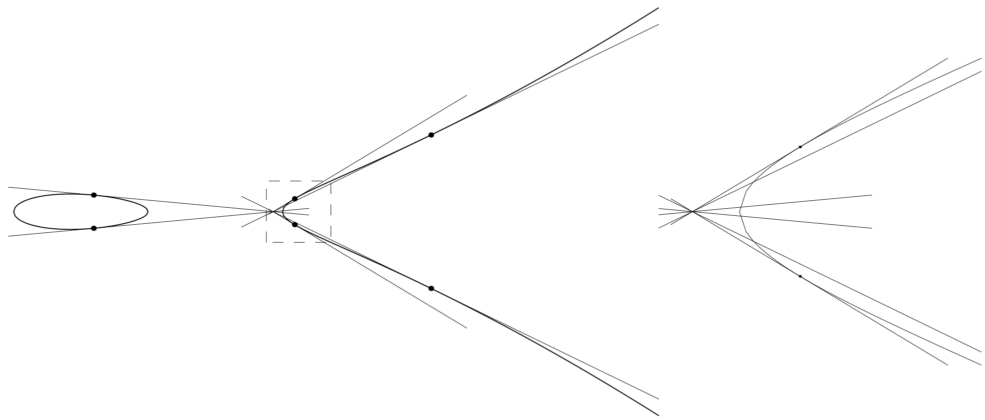


FIGURE 1.8. Six of the lines through a general point are tangent to a smooth plane cubic (but often not all the lines are defined over $\mathbb{R}$). ****show the enlarged "box" around the figure on the right.****

Proof. This is a straightforward application of Bézout's Theorem 1.22 and Bertini's Theorem 0.5. The degree of the dual variety $X^* \subset \mathbb{P}^{n*}$ is the number of points of intersection of X^* and $n-1$ general hyperplanes. Since by Corollary 12.20 the map $\mathcal{G}_X : X \to X^* \subset \mathbb{P}^{n*}$ is birational, this is the same as the number of points of intersection of the preimages $\mathcal{G}_X^{-1}(H_i)$ of $n-1$ general hyperplanes $H_i \subset \mathbb{P}^{n*}$. Since $\mathcal{G}_X$ is given by the partial derivatives of the defining equation F of X, the preimages of these hyperplanes are the intersections of X with the hypersurfaces $Z_i \subset \mathbb{P}^n$ of degree $d-1$ in $\mathbb{P}^n$ given by general linear combinations of these partial derivatives. Inasmuch as the partials of F have no common zeros, Bertini tells us that the hypersurfaces given by $n-1$ general linear combinations will intersect transversely with X, so by Bézout the number of these points of intersection is the product $d(d-1)^{n-1}$ of their degrees. $\square$

For example, suppose that X is a smooth cubic curve in $\mathbb{P}^2$. By the formula above the degree of X^* is 6. Since a general hyperplane in $\mathbb{P}^{2*}$ corresponds to the set of lines through a general point $p \in \mathbb{P}^2$, there will be exactly six lines in $\mathbb{P}^2$ through p tangent to X, as shown in Figure 1.8.

Note also that this gives us the answer to Keynote Question (d): the number of singular plane sections of a cubic surface $S \subset \mathbb{P}^3$ in a pencil is $3 \cdot 2^2 = 12$.

1.2.3 Products of projective spaces

Though the Chow ring of a smooth variety behaves like cohomology in many ways, there are important differences. For example the cohomology ring of the product of two spaces is given by the Künneth formula: ($H^*(X \times Y) = H^*(X) \otimes H^*(Y)$), but in general there is no Künneth formula for the Chow rings of products of varieties. Even for a product of two smooth curves C and D of genera $g, h \geq 1$ we have no algorithm for calculating $A^1(C \times D)$, and no idea at all what $A^2(C \times D)$ looks like, beyond the fact that it can't be in any sense finite-dimensional (Mumford [1962]).

However the Chow ring of the product of a variety with a projective space does obey the Künneth formula, as we'll prove in a more general context in Theorem 11.9. For the moment we will content ourselves with the special case where both factors are projective spaces:

Theorem 1.26. *The Chow ring of $\mathbb{P}^r \times \mathbb{P}^s$ is given by the formula*

$$A(\mathbb{P}^r \times \mathbb{P}^s) \cong A(\mathbb{P}^r) \otimes A(\mathbb{P}^s).$$

Equivalently, if α and $\beta \in A^1(\mathbb{P}^r \times \mathbb{P}^s)$ denote the pullbacks, via the projection maps, of the hyperplane classes on $\mathbb{P}^r$ and $\mathbb{P}^s$ respectively, then

$$A(\mathbb{P}^r \times \mathbb{P}^s) \cong \mathbb{Z}[\alpha, \beta]/(\alpha^{r+1}, \beta^{s+1}).$$

Moreover, the class of the hypersurface defined by a bihomogeneous form of bidegree (d, e) on $\mathbb{P}^r \times \mathbb{P}^s$ is $d\alpha + e\beta$.

The Künneth formula also holds for products of any two varieties with affine stratifications; for this result, which currently has no easy proof, see Totaro [to appear].

Proof. We can proceed exactly as in Theorem 1.20. We may construct an affine stratification of $\mathbb{P}^r \times \mathbb{P}^s$ by choosing flags of subspaces

$$\Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_{r-1} \subset \Lambda_r = \mathbb{P}^r$$

and

$$\Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_{s-1} \subset \Gamma_s = \mathbb{P}^s$$

with $\dim \Lambda_i$ and $\dim \Gamma_i = i$, and taking the closed strata to be

$$\Xi_{a,b} = \Lambda_{r-a} \times \Gamma_{s-b} \subset \mathbb{P}^r \times \mathbb{P}^s.$$

The open strata

$$\tilde{\Xi}_{a,b} := \Xi_{a,b} \setminus \left(\Xi_{a-1,b} \ \cup \ \Xi_{a,b-1}\right)$$

of this stratification are affine spaces. Invoking Proposition 1.19, we conclude that the Chow groups of $\mathbb{P}^r \times \mathbb{P}^s$ are generated by the classes $\varphi_{a,b} =$

$[\Xi_{a,b}] \in A^{a+b}(\mathbb{P}^r \times \mathbb{P}^s)$. Since $\Xi_{a,b}$ is the transverse intersection of the pullbacks of a hyperplanes in $\mathbb{P}^r$ and b hyperplanes in $\mathbb{P}^s$ we have

$$\varphi_{a,b} = \alpha^a \beta^b,$$

and in particular $\alpha^{r+1} = \beta^{s+1} = 0$. This shows that $A(\mathbb{P}^r \times \mathbb{P}^s)$ is a homomorphic image of

$$\mathbb{Z}[\alpha, \beta]/(\alpha^{r+1}, \beta^{s+1}) = \mathbb{Z}[\alpha]/(\alpha^{r+1}) \otimes_{\mathbb{Z}} \mathbb{Z}[\beta]/(\beta^{s+1}).$$

On the other hand, $\Xi_{r,s}$ is a single point, so $\deg \varphi_{r,s} = 1$. The pairing

$$A^{p+q}(\mathbb{P}^r \times \mathbb{P}^s) \times A^{r+s-p-q}(\mathbb{P}^r \times \mathbb{P}^s) \to \mathbb{Z}; \quad ([X], [Y]) \to \deg([X][Y])$$

sends $(\alpha^p\beta^q, \alpha^m\beta^n)$ to 1 if $p + m = r$ and $q + n = s$, because in this case the intersection is transverse and consists of one point, and to 0 otherwise, since then the intersection is empty. This shows that the monomials of bidegree (p, q), for $0 \le p \le r$ and $0 \le q \le s$, are linearly independent over $\mathbb{Z}$, proving the first statement.

If $F(X, Y)$ is a bihomogeneous polynomial of bidegree (d, e), then since $F(X, Y)/X_0^d Y_0^e$ is a rational function on $\mathbb{P}^r \times \mathbb{P}^s$, the class of the hypersurface X defined by $F = 0$ is d times the class of the hypersurface $X_0 = 0$ plus e times the class of the hypersurface $Y_0 = 0$. In particular, $[X] = d\alpha + e\beta$. □

Degrees of Segre varieties. The Segre variety $\Sigma_{k,s}$ is by definition the image of the product $\mathbb{P}^r \times \mathbb{P}^s$ in $\mathbb{P}^{(r+1)(s+1)-1}$ under the map

$$\sigma_{r,s} : \big([X_0, \dots, X_r], [Y_0, \dots, Y_s]\big) \mapsto [\dots, X_i Y_j, \dots].$$

The map $\sigma_{r,s}$ is an immersion because on each open set where one of the X_i and one of the Y_j are equal to 1 the rest of the coordinates can be recovered from the products.

If V and W are vector spaces of dimensions $r+1$ and $s+1$ we may write $\sigma_{r,s}$ without bases by the formula

$$\begin{aligned} \sigma_{r,s} : \mathbb{P}V \times \mathbb{P}W &\to \mathbb{P}(V \otimes W) \\ (v, w) &\mapsto v \otimes w. \end{aligned}$$

For example, the map $\sigma_{1,1}$ is defined by the four forms $a = X_0Y_0, b = X_0Y_1, c = X_1Y_0, d = X_1Y_1$, and these satisfy the equation $ac - bd = 0$; thus the Segre variety $\Sigma_{1,1}$ is the nonsingular quadric in $\mathbb{P}^3$.

Proposition 1.27. *The degree of the Segre embedding of $\mathbb{P}^r \times \mathbb{P}^l$ is*

$$\deg \Sigma_{r,l} = \binom{r+l}{r}.$$

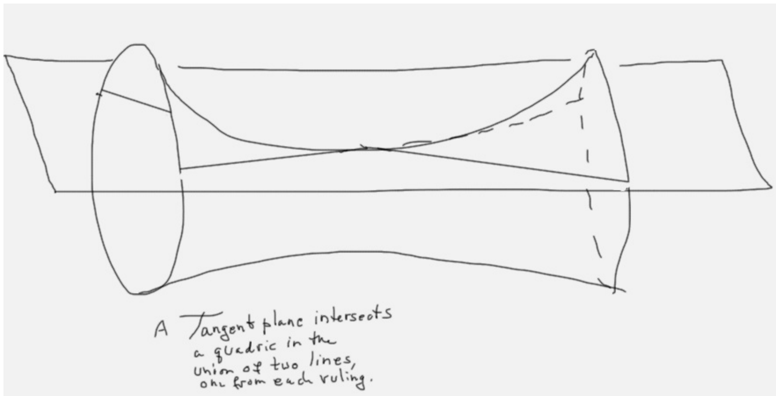


FIGURE 1.9. A tangent plane to a quadric in $\mathbb{P}^3$ meets the quadric in two lines, one from each ruling.

Proof. The degree of $\Sigma_{r,s}$ is the number of points in which it meets the intersection of $r+s$ hypersurfaces in $\mathbb{P}^{(r+1)(s+1)-1}$. Since $\sigma_{r,s}$ is an embedding, we may compute this number by pulling back these hypersurfaces to $\mathbb{P}^r \times \mathbb{P}^s$ and computing in the Chow ring of $\mathbb{P}^r \times \mathbb{P}^s$. Thus $\deg \Sigma_{r,l} = \deg(\alpha+\beta)^{r+l}$, which gives the desired formula because $(\alpha+\beta)^{r+l} = \binom{r+l}{r}\alpha^r\beta^l$. $\square$

For instance the Segre variety $\mathbb{P}^1 \times \mathbb{P}^r \subset \mathbb{P}^{2r+1}$ has degree $r+1$. These varieties are among those called *rational normal scrolls* (see for example Harris [1992]). The simplest of these is the smooth quadric surface $Q \subset \mathbb{P}^3$, which is the Segre image of $\mathbb{P}^1 \times \mathbb{P}^1$; the pullbacks α and β of the point classes via the two projections are the classes of the lines of the two rulings of Q, and we have $\zeta = \alpha+\beta$, where ζ is the hyperplane class on $\mathbb{P}^3$ restricted to Q—a fact that is readily apparent if we look at the intersection of Q with any tangent plane.

This discussion can be readily generalized to arbitrary products of projective spaces, as in Exercise 1.62.

The class of the diagonal. Next we will find the class δ of the diagonal $\Delta \subset \mathbb{P}^r \times \mathbb{P}^r$ in the Chow group $A^r(\mathbb{P}^r \times \mathbb{P}^r)$, and more generally the class γ_f of the graph of a map $f : \mathbb{P}^r \to \mathbb{P}^s$. Apart from the applications of such a formula, this will introduce the *method of undetermined coefficients*, which we'll use many times in the course of this book. (Another approach to this problem, via specialization, is given in Exercise 1.63.)

By Theorem 1.26 we have

$$A(\mathbb{P}^r \times \mathbb{P}^r) = \mathbb{Z}[\alpha,\beta]/(\alpha^{r+1}, \beta^{r+1}).$$

where α and $\beta \in A^1(\mathbb{P}^r \times \mathbb{P}^r)$ are the pullbacks, via the two projection maps, of the hyperplane class in $A^1(\mathbb{P}^r)$.

To find the class $\delta = [\Delta]$ of the diagonal, we first observe that it is necessarily expressible as a linear combination

$$\delta = c_0\alpha^r + c_1\alpha^{r-1}\beta + c_2\alpha^{r-2}\beta^2 + \cdots + c_r\beta^r$$

for some $c_0, \ldots, c_r \in \mathbb{Z}$. We can determine the coefficients c_i by taking the product of both sides of this expression with various classes of complementary codimension: specifically, if we intersect both sides with the class $\alpha^i\beta^{r-i}$ and take degrees we have

$$c_i = \deg(\delta \cdot \alpha^i\beta^{r-i}).$$

We can evaluate the product $(\delta \cdot \alpha^i\beta^{r-i})$ directly: if Λ and Γ are general linear subspaces of codimension i and $r-i$ respectively, then $[\Lambda \times \Gamma] = \alpha^i\beta^{r-i}$. Moreover,

$$(\Lambda \times \Gamma) \cap \Delta \cong \Lambda \cap \Gamma$$

is a reduced point, so

$$\begin{aligned} c_i &= \deg(\delta \cdot \alpha^i\beta^{r-i}) \\ &= \#\big(\Delta \cap (\Lambda \times \Gamma)\big) \\ &= \#(\Lambda \cap \Gamma) \\ &= 1. \end{aligned}$$

Thus

$$\delta = \alpha^r + \alpha^{r-1}\beta + \cdots + \alpha\beta^{r-1} + \beta^r.$$

See Figure 1.10. (This formula and its derivation will be familiar to anyone who's had a course in algebraic topology. As partisans we can't resist pointing out that algebraic geometry had it first!)

The class of a graph. Let $f : \mathbb{P}^r \to \mathbb{P}^s$ be a morphism given by $(s+1)$ homogeneous polynomials F_i of degree d that have no common zeros:

$$f : [X_0, \ldots, X_r] \mapsto \big[F_0(X), F_1(X), \ldots, F_s(X)\big].$$

By Corollary 1.21, we must have $s \geq r$. Let $\Gamma_f \subset \mathbb{P}^r \times \mathbb{P}^s$ be the graph of f. What is its class $\gamma_f = [\Gamma_f] \in A^r(\mathbb{P}^r \times \mathbb{P}^s)$?

We will assume that the characteristic of the ground field K is 0 so that we can apply Theorem 1.9 (though in fact the formula we'll derive is valid in all characteristics).

As before, we can write

$$\gamma_f = c_0\alpha^r\beta^{s-r} + c_1\alpha^{r-1}\beta^{s-r+1} + c_2\alpha^{r-2}\beta^{s-r+2} + \cdots + c_r\beta^s$$

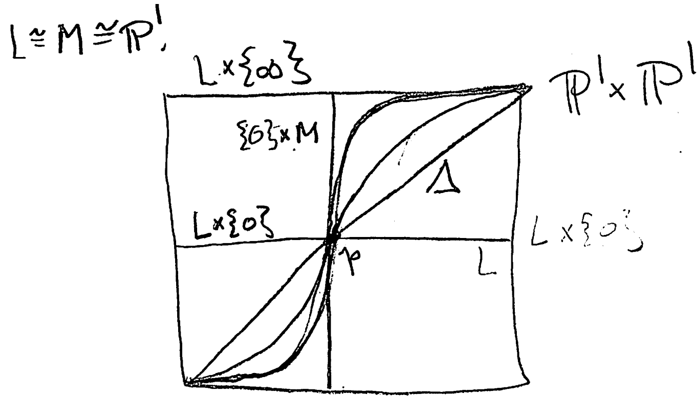


FIGURE 1.10. $[\Delta][L \times \{0\} = 1 = [\Delta][\{0\} \times M]$ so $[\Delta] = [\{0\} \times M] + [L \times \{0\}]$, as one also sees from the degeneration in the figure.

for some $c_0, \dots, c_r \in \mathbb{Z}$, and as before we can determine the coefficients c_i in this expression by intersecting both sides with a cycle of complementary dimension:

$$c_i = (\gamma_f \cdot \alpha^i \beta^{r-i}) = \#\big(\Gamma_f \cap (\Lambda \times \Phi)\big)$$

for general linear subspaces $\Lambda \cong \mathbb{P}^{r-i}$ and $\Phi \cong \mathbb{P}^{s-r+i} \subset \mathbb{P}^s$. By Theorem 1.9 the intersection $\Gamma_f \cap (\Lambda \times \Phi)$ is generically transverse.

Finally, $\Gamma_f \cap (\Lambda \times \Phi)$ is the zero locus, in Λ, of $r - i$ general linear combinations of the polynomials $F_0, \dots, F_r$. By Bertini's Theorem 0.5, the corresponding hypersurfaces will intersect transversely, and by Bézout's Theorem the intersection will consist of d^{r-i} points. Thus we arrive at the formula:

Proposition 1.28. *If $f : \mathbb{P}^r \to \mathbb{P}^s$ is a regular map given by polynomials of degree d on $\mathbb{P}^r$, the class γ_f of the graph of f is given by*

$$\gamma_f = \sum_{i=0}^{r} d^i \alpha^i \beta^{s-i} \in A^s(\mathbb{P}^r \times \mathbb{P}^s).$$

Using this formula, we can answer a general form of Keynote Question (c). We'll again assume characteristic 0. A sequence $F_0, \dots, F_r$ of general homogeneous polynomials of degree d in $r + 1$ variables defines a map $f : \mathbb{P}^r \to \mathbb{P}^r$, and we can count the fixed points $\{t = (t_0, \dots, t_r) \in \mathbb{P}^r \mid f(t) = t\}$. Since the F_i are general, we could take them to be general translates under $GL_{r+1} \times GL_{r+1}$ of arbitrary polynomials so the cardinality of this set is the degree of the intersection of the graph γ_f of f with the diagonal $\Delta \subset \mathbb{P}^r \times \mathbb{P}^r$. This is

$$\begin{aligned}\delta \cdot \gamma_f &= \deg\bigl((\alpha^r + \alpha^{r-1}\beta + \cdots + \beta^r) \cdot (d^r\alpha^r + d^{r-1}\alpha^{r-1}\beta + \cdots + \beta^r)\bigr)\\ &= d^r + d^{r-1} + \cdots + d + 1;\end{aligned}$$

in particular, if A, B, C are general quadratic forms then there are exactly 7 points $t = (t_0, t_1, t_2) \in \mathbb{P}^2$ such that $(A(t), B(t), C(t)) = (t_0, t_1, t_2)$.

Note that in case $d = 1$ and $s = r$, Proposition 1.28 implies that a general $(r+1)\times(r+1)$ matrix has $r+1$ eigenvalues. It also follows that an arbitrary matrix has at least one eigenvalue.

1.2.4 The blowup of $\mathbb{P}^n$ at a point

The *blowup* of $\mathbb{P}^n$ at a point p is a morphism $\pi : B \to \mathbb{P}^n$ where $B \subset \mathbb{P}^n \times \mathbb{P}^{n-1}$ is the closure of the graph of the projection $\pi_p : \mathbb{P}^n \setminus \{p\} \to \mathbb{P}^{n-1}$ from p, and π is the projection on the first factor.

$$\begin{tikzcd} & B \arrow[dl, "\pi"'] \arrow[dr, "\alpha"] & \\ \mathbb{P}^n \arrow[rr, dotted, "\pi_p"] & & \mathbb{P}^{n-1} \end{tikzcd}$$

Since the graph of the projection is isomorphic to the domain $\mathbb{P}^n \setminus \{p\}$, B is irreducible, and it is not hard to write explict equations for it, and to show that it is smooth as well; see for example Section IV.2 of Eisenbud and Harris [2000].

If $n = 1$ the map π_p extends to an everywhere regular map, so the blowup is isomorphic to $\mathbb{P}^1$ (of course $\mathbb{P}^1 \times \mathbb{P}^0$ is also isomorphic to $\mathbb{P}^1$.) For this reason we will restrict ourselves to the case $n \geq 2$.

The *exceptional divisor* $E \subset B$ is defined to be $\pi^{-1}(p)$, the preimage of p in B, which, as a subset of $\mathbb{P}^n \times \mathbb{P}^{n-1}$, is $\{p\} \times \mathbb{P}^{n-1}$. Some other obvious divisors on B are the preimages of the hyperplanes of $\mathbb{P}^n$. If the hyperplane $H \subset \mathbb{P}^n$ contains p, then its preimage is the sum of two irreducible divisors, E and H'; the latter is called the *strict transform* of H. More generally, if $Z \subset \mathbb{P}^n$ is any subvariety, we define the strict transform of Z to be the closure in B of the preimage $\pi^{-1}(Z \setminus \{p\})$. See Figure 1.11.

To compute the Chow ring of B, we start from a stratification of B, using the geometry of the projection map $\alpha : B \to \mathbb{P}^{n-1}$ to the second factor. We do this by first choosing a stratification of the target $\mathbb{P}^{n-1}$, and taking the preimages in B of these strata. Then we choose a divisor Λ that maps isomorphically by α to $\mathbb{P}^{n-1}$—a *section* of α—and take, as additional strata, the intersections of these preimages with Λ.

We will choose as our section the preimage $\Lambda = \pi^{-1}(\Lambda')$ of a hyperplane $\Lambda' \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n$ not containing the point p. (There are other possible choices of a section, such as the exceptional divisor $E \subset B$; see Exercise 15.5.3).)

To carry this out, let

$$\Gamma'_0 \subset \Gamma'_1 \subset \cdots \subset \Gamma'_{n-2} \subset \Gamma'_{n-1} = \mathbb{P}^{n-1}.$$

be a flag of linear subspaces and for $k = 1, 2, \ldots, n$ let

$$\Gamma_k = \alpha^{-1}(\Gamma'_{k-1}) \subset B.$$

Since the fibers of $\alpha : B \to \mathbb{P}^{n-1}$ are projective lines, the dimension of Γ_k is k. Next, for $k = 0, 1, \ldots, n-1$ we set

$$\Lambda_k = \Gamma_{k+1} \cap \Lambda,$$

so that Λ_k is the preimage of Γ'_k under the isomorphism $\alpha|_\Lambda : \Lambda \to \mathbb{P}^{n-1}$.

The subvarieties $\Gamma_1, \ldots, \Gamma_n, \Lambda_0, \ldots, \Lambda_{n-1}$ are the closed strata of a stratification of B, with inclusion relations

$$\begin{tikzcd}
\Lambda_0 \arrow[r, hook] \arrow[dr, hook] & \Lambda_1 \arrow[r, hook] \arrow[dr, hook] & \Lambda_2 \arrow[r, hook] & \cdots \arrow[r, hook] & \Lambda_{n-2} \arrow[r, hook] \arrow[dr, hook] & \Lambda_{n-1} \arrow[dr, hook] & \\
& \Gamma_1 \arrow[r, hook] & \Gamma_2 \arrow[r, hook] & \cdots \arrow[r, hook] & \Gamma_{n-2} \arrow[r, hook] & \Gamma_{n-1} \arrow[r, hook] & \Gamma_n = B.
\end{tikzcd}$$

To visualize this, take the standard picture of the blowup of $\mathbb{P}^n$ at a point (shown in Fig. 1.11 in case $n = 2$). If we rotate this picture and "untwist" it, we realize the blow-up as the total space of a $\mathbb{P}^1$-bundle over $\mathbb{P}^{n-1}$, via the projection map α, as in Fig. 1.12.

As we'll soon see, this is an affine stratification, so that the classes of the closed strata generate the Chow group $A(B)$. (In fact, the open strata are isomorphic to affine spaces, so that it follows from Totaro [to appear] that they generate $A(B)$ freely; we will verify this independently when we determine the intersection products.) We describe the Chow ring of B as follows:

Proposition 1.29. *Let B be the blowup of $\mathbb{P}^n$ at a point, with $n \geq 2$. With notation as above, the Chow ring $A(B)$ is the free abelian group on the generators $[\Lambda_k] = [\Lambda_{n-1}]^{n-k}$ for $k = 0, \ldots, n-1$ and $[\Gamma_k] = [\Gamma_{n-1}]^{n-k}$ for $k = 1, \ldots, n$. The class of the exceptional divisor E is $[\Lambda_{n-1}] - [\Gamma_{n-1}]$. If we set $\lambda = [\Lambda_{n-1}]$ and $e = [E]$, then*

$$A(B) \cong \frac{\mathbb{Z}[\lambda, e]}{(\lambda e, \lambda^n + (-1)^n e^n)}$$

as rings.

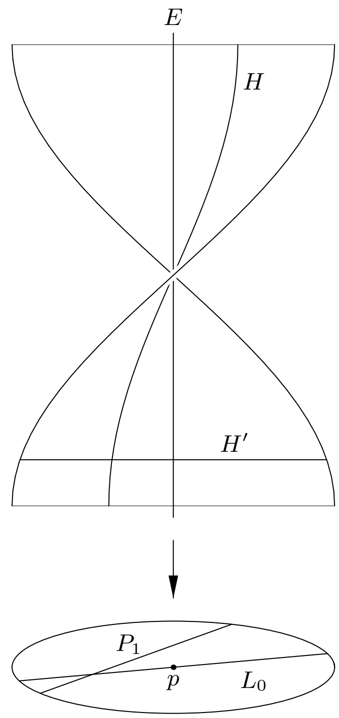


FIGURE 1.11. Blowup of $\mathbb{P}^2$****Relabel: P_1 should be Λ'; H should be Λ; L_0 should be L and H' should be $\tilde{L}$.****

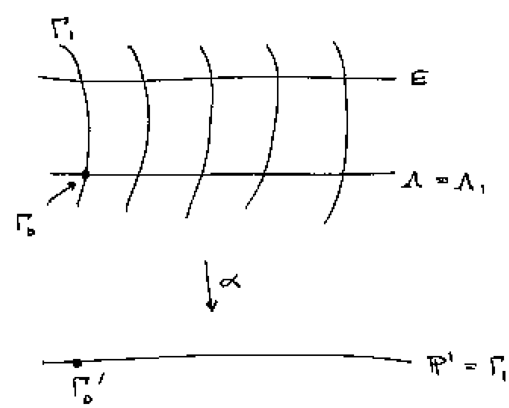


FIGURE 1.12. Blowup of $\mathbb{P}^2$ as $\mathbb{P}^1$-bundle

Proof. We start by verifying that the open strata $\tilde{\Gamma}_1, \dots, \tilde{\Gamma}_n, \tilde{\Lambda}_0, \dots, \tilde{\Lambda}_{n-1}$ of the stratification of B with closed strata Γ_k, Λ_k are isomorphic to affine spaces. This is immediate for the strata $\tilde{\Lambda}_k$. For the strata $\tilde{\Gamma}_k$, we choose coordinates $(x_0, \dots, x_n)$ on $\mathbb{P}^n$ so that $p = (1, 0, \dots, 0)$ and $\Lambda' \subset \mathbb{P}^n$ is the hyperplane $x_0 = 0$. By definition, $B \subset \mathbb{P}^n \times \mathbb{P}^{n-1}$ is the locus

$$B = \{((x_0, \dots, x_n), (y_1, \dots, y_n)) \mid x_i y_j = x_j y_i \ \forall i, j \geq 1\}.$$

Say the $(k-1)$-plane $\Gamma'_{k-1} \subset \mathbb{P}^{n-1}$ is given by $y_1 = \cdots = y_{n-k} = 0$. We can write the open stratum $\tilde{\Gamma}_k = \alpha^{-1}(\Gamma'_{k-1} \setminus \Gamma'_{k-2}) \cap (B \setminus \Lambda)$ as the locus

$$\tilde{\Gamma}_k = \{((1, 0, \dots, 0, \lambda, \lambda y_{n-k+2}, \dots, \lambda y_n), (0, \dots, 0, 1, y_{n-k+2}, \dots, y_n))\};$$

the functions $\lambda, y_{n-k+2}, \dots, y_n$ then give an isomorphism of $\tilde{\Gamma}_k$ with $\mathbb{A}^k$.

It follows that the classes

$$\lambda_k = [\Lambda_k] \quad \text{and} \quad \gamma_k = [\Gamma_k] \in A_k(B)$$

generate the Chow groups of B; we now want to compute their intersection products.

Since Λ_k is the the preimage of a k-plane in $\mathbb{P}^n$ not containing p, and any two such are linearly equivalent in $\mathbb{P}^n$, the class of the pullback of any k-plane in $\mathbb{P}^n$ not containing p is also equal to λ_k. Similarly, the class of the proper transform of any k-plane in $\mathbb{P}^n$ containing p is γ_k. Having all these representative cycles for the classes λ_k and γ_k makes it easy to determine their intersection products.

For example, a general k-plane in $\mathbb{P}^n$ intersects a general l-plane transversely in a general $(k+l-n)$-plane; thus

$$\lambda_k \lambda_l = \lambda_{k+l-n} \quad \text{for all } k + l \geq n.$$

Similarly, the intersection of a general k-plane in $\mathbb{P}^n$ containing p with a general l-plane in not containing p is a general $(k+l-n)$-plane not containing p, so that

$$\gamma_k \lambda_l = \lambda_{k+l-n} \quad \text{for all } k + l \geq n,$$

and likewise

$$\gamma_k \gamma_l = \gamma_{k+l-n} \quad \text{for all } k + l \geq n + 1.$$

Note the restriction $k+l \geq n+1$ on the last set of products: in case $k+l = n$, the proper transforms of a general k-plane through p and a general l-plane through p are disjoint.

This determines the Chow ring of B. Since the pairing $A_k(B) \times A_{n-k}(B) \to A_0(B) \cong \mathbb{Z}$ is given by

$$\lambda_k \lambda_{n-k} = \lambda_k \gamma_{n-k} = \gamma_k \lambda_{n-k} = 1 \quad \text{and} \quad \gamma_k \gamma_{n-k} = 0,$$

it is nondegenerate, and the classes $\lambda_0, \dots, \lambda_{n-1}$ and $\gamma_1, \dots, \gamma_n$ freely generate $A(B)$.

It follows that we can express the class of the exceptional divisor E in terms of the generators Λ_{n-1} and Γ_{n-1} of $A_{n-1}(B)$. The most geometric way to do this is to observe that Λ'_{n-1} is linearly equivalent in $\mathbb{P}^n$ to a hyperplane $\Sigma \subset \mathbb{P}^n$ containing p so the pullback of Σ is linearly equivalent to the union of the exceptional divisor E and a divisor D. Since D projects to a hyperplane of $\mathbb{P}^{n-1}$, it is contained in the preimage Γ of such a hyperplane. Since Γ is a $\mathbb{P}^1$ bundle over its image it is irreducible, and comparing dimensions, we see that $D = \Gamma$. Since any two hyperplanes in $\mathbb{P}^{n-1}$ are rationally equivalent, so are their pullbacks to B; thus $\Lambda_{n-1} \sim D + E \sim \Gamma_{n-1} + E$, or $[E] = \lambda_{n-1} - \gamma_{n-1}$.

We now turn to the ring structure, and let $\lambda = [\Lambda_{n-1}]$ and $e = [E] = \lambda - \gamma_{n-1}$. Since $\Lambda_{n-1} \cap E = \varnothing$ we have

$$\lambda e = 0.$$

Also,

$$\lambda_k = \lambda^{n-k}, \quad \text{for } k = 0, \ldots, n-1$$

and since $\gamma_{n-1} = \lambda - e$,

$$\gamma_k = \gamma_{n-1}^{n-k} = (\lambda - e)^{n-k} = \lambda^{n-k} + (-1)^{n-k} e^{n-k}, \quad \text{for } k = 1, \ldots, n.$$

It follows that λ and e generate $A(B)$ as a ring. In addition to the relation $\lambda e = 0$, they satisfy the relation

$$0 = \gamma_{n-1}^n = (\lambda - e)^n = \lambda^n + (-1)^n e^n.$$

Thus the Chow ring is a homomorphic image of the ring

$$A' := \mathbb{Z}[\lambda, e]/(\lambda e, \lambda^{n-1} - (-1)^n e^n).$$

It is clear that every homogeneous element of degree m in A' is a $\mathbb{Z}$-linear combination of e^m and λ^m. Since $A^m(B)$ is a free $\mathbb{Z}$-module of rank 2, this implies that the map $A' \twoheadrightarrow A$ is an isomorphism. $\square$

We have computed the intersection products of the Λ_k and Γ_k by taking representatives that meet transversely (and the possibility of doing this motivated our choice of Λ as a cross section of α above). Since E is the only irreducible variety in the class $[E]$ we cannot give a representative for e^2 quite as easily. But as we have seen, $E \sim \langle \Lambda_{n-1} \rangle - \langle \Gamma_{n-1} \rangle$ and both Λ_{n-1} and Γ_{n-1} are transverse to E (this illustrates the conclusion of the Moving Lemma!) It follows that

$$e^2 = [E \cap (\Lambda - \Gamma)] = -[E \cap \Gamma_{n-1}].$$

Since E projects isomorphically to $\mathbb{P}^{n-1}$ and Γ projects to a hyperplane in $\mathbb{P}^{n-1}$ we see that $E \cap \Gamma_{n-1}$ is a hyperplane in E; that is, $[E]^2$ is the *negative* of the class of a hyperplane in E.

The Chow ring of the blow-up of $\mathbb{P}^3$ along a line is worked out in Exercises 1.74-1.76. More generally, we will see how to describe the Chow ring of a general projective bundle in Chapter 11, and the Chow ring of a more general blow-up in Chapter 15.

1.2.5 Loci of singular plane cubics

This section represents an important shift in viewpoint, from studying the Chow rings of common and useful algebraic varieties to studying Chow rings of parameter spaces. It is a hallmark of algebraic geometry that the set of varieties with given numerical invariants—and more generally, schemes, morphisms, bundles and other geometric objects—is often itself an algebraic variety, sometimes called a *parameter space.* Applying intersection theory to the study of such a parameter space, we learn something about the geometry of the objects parametrized, and about geometrically characterized classes of these objects. This gets us into the subject of *enumerative geometry*, and was one of the principal motivations for the development of intersection theory in the 19th century.

By way of illustration we will focus on the family of curves of degree 3 in $\mathbb{P}^2$: plane cubics. Plane cubics are parametrized by the set of homogeneous cubic polynomials $F(X, Y, Z)$ in three variables, modulo scalars; that is, by $\mathbb{P}^9$. *For simplicity, we will assume throughout this section that the characteristic of the ground field is zero.*

There is a continuous family of isomorphism classes of smooth plane cubics, parametrized naturally by the affine line (see Hartshorne [1977]); but there are only a finite number of isomorphism classes of singular plane cubics:

- irreducible plane cubics with a node;
- irreducible plane cubics with a cusp;
- plane cubics consisting of a smooth conic and a line meeting it transversely;
- plane cubics consisting of a smooth conic and a line tangent to it;
- plane cubics consisting of three non-concurrent lines ("triangles");
- plane cubics consisting of three concurrent lines ("asterisks");
- cubics consisting of a double line and a line; and finally
- cubics consisting of a triple line.

These are illustrated in Figures 1.13–1.15) where the arrows represent specialization (that is, A→B indicates that the class B is contained in the closure of the class A.)

The locus in $\mathbb{P}^9$ of curves of each type is an orbit of PGL_3 and a locally closed subset of $\mathbb{P}^9$; their closures give a stratification of $\mathbb{P}^9$. What are these

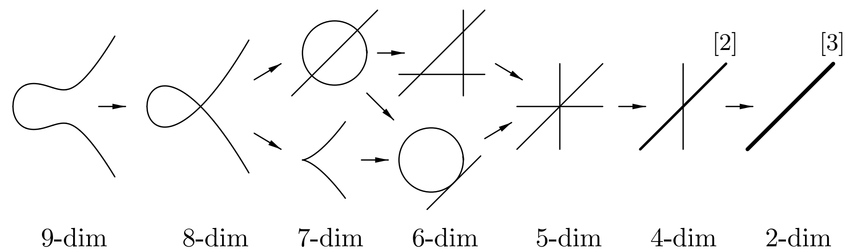


FIGURE 1.13. Hierarchy of singular plane cubic curves.

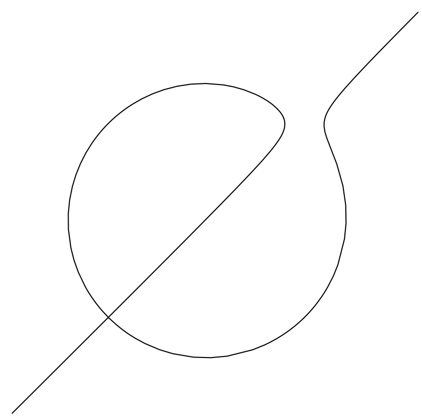


FIGURE 1.14. Nodal cubic about to become the union of a conic and a transverse line: $(y^2-x^2(x+1))+100(x-y)((x-\frac{1}{2})^2+(y-\frac{1}{2})^2-\frac{1}{2})$

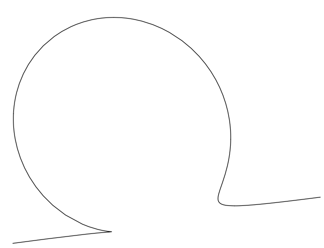


FIGURE 1.15. Cuspidal cubic about to become the union of a conic and a tangent line: $y^2-x^3+7y(x^2+(y-1)^2-1)$

loci like? What are their dimensions? Which lie in the closures of which others? Where is each one smooth and singular? What are their tangent spaces and tangent cones? What are their degrees as subvarieties of $\mathbb{P}^9$?

Some of these questions are easy to answer. For example, the dimensions are given in Figure 1.13, and the reader can verify them as an exercise. The specialization relationships indicated in the chart are also easy, because to establish that one orbit lies in the closure of another, it suffices to exhibit a one-parameter family with an open set of parameter values corresponding to one type and a point corresponding to the other. The non-inclusion relations are subtler—why is a triangle not a specialization of a cuspidal cubic?—but can also be proven by focussing on the singularities of the curves. The tangent spaces require more work; we'll give some examples in Exercises 1.78-1.79, in the context of establishing a transversality statement, and we'll see more of these, as well as some tangent cones, in the discussion in Section 9.8.3.

While we can answer almost any question of this sort about plane cubics, the answers to analogous questions for higher degree curves or hypersurfaces of higher dimension, for example about the stratification by singularity type, remain mysterious. Even questions about the dimension and irreducibility of these loci are mostly open; they are a topic of active research. See Greuel et al. [2007] for an introduction to this area.

For example, it is known that the locus of plane curves of degree d having exactly δ nodes is irreducible of codimension δ in the projective space $\mathbb{P}^N$ of all plane curves of degree d (see for example Harris and Morrison [1998]), and its degree has also been determined (Caporaso and Harris [1998]). But we don't know the answers to the analogous questions for plane curves with δ nodes and κ cusps; and when it comes to more complicated singularities even existence questions are open. (For example, for $d > 6$ it's not known whether there exists a rational plane curve $C \subset \mathbb{P}^2$ of degree d whose singularities consist of just one double point (that is, a point p such that a general line through p meets C with multiplicity just 2.)

In the rest of this section we will determine the degrees of the loci of reducible cubics, triangles and asterisks. In the exercises we indicate how to compute the degrees of the other loci of plane cubics, except for the loci of irreducible cubics with a node, and of irreducible cubics with a cusp; these will be computed in Section 9.4.2 and Section 13.4 respectively.

Reducible cubics. Let $\Gamma \subset \mathbb{P}^9$ be the locus of reducible cubics and/or non-reduced cubics. We can describe Γ as the image of the map

$$\tau : \mathbb{P}^2 \times \mathbb{P}^5 \to \mathbb{P}^9$$

from the product of the space $\mathbb{P}^2$ of homogeneous linear forms and the space $\mathbb{P}^5$ of homogeneous quadratic polynomials to $\mathbb{P}^9$, given simply by multiplication: $([F],[G]) \mapsto [FG]$. Inasmuch as the coefficients of the product FG are bilinear in the coefficients of F and G, the pullback $\tau^*(\zeta)$ of the hyperplane class $\zeta \in A^1(\mathbb{P}^9)$ is the sum

$$\tau^*(\zeta) = \alpha + \beta$$

where α and β are the pullbacks to $\mathbb{P}^2 \times \mathbb{P}^5$ of the hyperplane classes on $\mathbb{P}^2$ and $\mathbb{P}^5$.

By unique factorization of polynomials, the map τ is birational onto its image; it follows that the degree of Γ is given by

$$\begin{aligned} \deg(\Gamma) &= \deg\bigl(\tau^*(\zeta)^7\bigr) \\ &= \deg\bigl((\alpha+\beta)^7\bigr) \\ &= 21 \end{aligned}$$

and this is the answer to Keynote Question (a).

Another way to calculate the degree of Γ is described in Exercises 1.78–1.80.

Triangles. A similar analysis gives the answer to Keynote Question (b)—how many cubics in a three-dimensional linear system factor completely, as a product of three linear forms. Here, the key object is the locus $\Sigma \subset \mathbb{P}^9$ of such totally reducible cubics, which we may call *triangles*; the keynote question asks us for the number of points of intersection of Σ with a general 3-plane, which by Bertini's Theorem is just the degree of Σ.

Since Σ is the image of the map

$$\begin{aligned} \mu : \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 &\to \mathbb{P}^9 \\ ([L_1],[L_2],[L_3]) &\mapsto [L_1 L_2 L_3], \end{aligned}$$

we can proceed as before, with the one difference that the map is now no longer birational, but rather is generically 6-to-1. Thus if α_1, α_2 and $\alpha_3 \in A^1(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$ are the pullbacks of the hyperplane class, so that

$$\mu^*(\zeta) = \alpha_1 + \alpha_2 + \alpha_3,$$

we get

$$\begin{aligned} \deg(\Sigma) &= \frac{1}{6} \deg\,(\alpha_1+\alpha_2+\alpha_3)^6 \\ &= \frac{1}{6}\binom{6}{2,2,2} \\ &= 15. \end{aligned}$$

This is the answer to Keynote Question (b): in a general three-dimensional linear system of cubics, there will be exactly 15 triangles.

Asterisks. By an *asterisk*, we mean a cubic consisting of the sum of three concurrent lines. To see that this locus is indeed a subvariety of $\mathbb{P}^9$ and to calculate its degree, let

$$\mu : \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^9$$

be as above, and consider the subset

$$\Phi = \{(L_1, L_2, L_3) : L_1 \cap L_2 \cap L_3 \neq \varnothing\} \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2;$$

the locus $A \subset \mathbb{P}^9$ of asterisks is then the image $\mu(\Phi)$ of Φ under the map μ. If we write the line L_i as the zero locus of the linear form

$$a_{i,1}X + a_{i,2}Y + a_{i,3}Z$$

then the condition that $L_1 \cap L_2 \cap L_3 \neq \varnothing$ is equivalent to the vanishing

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = 0.$$

This is a homogeneous trilinear form on $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$, from which we see that Φ is a closed subset of $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$, and A is a closed subset of $\mathbb{P}^9$. Moreover, we see that the class of Φ is

$$[\Phi] = \alpha_1 + \alpha_2 + \alpha_3 \in A^1(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2),$$

so that the pullback via μ of 5 general hyperplanes in $\mathbb{P}^9$ will intersect Φ in

$$\deg\Big([\Phi]\,(\alpha_1 + \alpha_2 + \alpha_3)^5\Big) = \deg\,(\alpha_1 + \alpha_2 + \alpha_3)^6 = \binom{6}{2,2,2} = 90$$

points. Since the map $\mu|_\Phi : \Phi \to A$ has degree 6, it follows that the degree of the locus $A \subset \mathbb{P}^9$ of asterisks is 15.

1.3 The canonical class and the adjunction formula

Let X be a smooth n-dimensional variety. By the *canonical bundle* ω_X of X we mean the top exterior power $\wedge^n \Omega_X$ of the cotangent bundle Ω_X of X; this is the line bundle whose sections are regular n-forms. By the *canonical class* we'll mean the first Chern class $c_1(\omega_X) \in A^1(X)$ of this line bundle. Perhaps reflecting the German language history of the subject, this class is commonly denoted K_X.

The canonical class is probably the single most important indicator of the behavior of X, geometrically, topologically and arithmetically. For example, the only topological invariant of a smooth projective curve X over the complex field $\mathbb{C}$ is its genus $g = g(X)$, and we have

$$\deg(K_X) = 2g - 2.$$

Further the geometry over $\mathbb{C}$ and the arithmetic over $\mathbb{Q}$ of X is fundamentally different in the cases where $\deg K_X$ is negative, zero or positive, corresponding to $g = 0, 1$ or $g \geq 2$.

Example 1.30 ($\mathbb{P}^n$)**.** We can easily determine the canonical class of projective space. To do this, we have only to write down a rational n-form ω on $\mathbb{P}^n$ and determine its divisors of zeros and poles. For example, if $X_0, \ldots, X_n$ are homogenenous coordinates on $\mathbb{P}^n$ and

$$x_i = \frac{X_i}{X_0}, \quad i = 1, \ldots, n$$

affine coordinates on the open set $U \cong \mathbb{A}^n \subset \mathbb{P}^n$ given by $X_0 \neq 0$, we may take ω to be the rational n-form given in U by

$$\omega = dx_1 \wedge \cdots \wedge dx_n.$$

The form ω is regular and nonzero in U, so we have only to determine its order of zero or pole along the hyperplane $H = V(X_0)$ at infinity. To this end, let $U' \subset \mathbb{P}^n$ be the open set $X_n \neq 0$, and take affine coordinates $y_0, \ldots, y_{n-1}$ on U', with $y_i = X_i/X_n$. We have

$$x_i = \begin{cases} \frac{y_i}{y_0}, & \text{for } i = 1, \ldots, n-1; \text{ and} \\ \frac{1}{y_0}, & \text{for } i = n, \end{cases}$$

so that

$$dx_i = \begin{cases} \frac{1}{y_0} dy_i - \frac{y_i}{y_0^2} dy_0, & \text{for } i = 1, \ldots, n-1; \text{ and} \\ \frac{-1}{y_0^2} dy_0, & \text{for } i = n. \end{cases}$$

Taking wedge products, we see that

$$\omega = dx_1 \wedge \cdots \wedge dx_n = \frac{(-1)^n}{y_0^{n+1}} dy_0 \wedge \cdots \wedge dy_{n-1},$$

whence

$$Div(\omega) = -(n+1)H,$$

so

$$K_{\mathbb{P}^n} = -(n+1)\zeta,$$

where $\zeta \in A^1(\mathbb{P}^n)$ is the hyperplane class.

1.3.1 Adjunction

Let X again be a smooth variety of dimension n, and suppose that $Y \subset X$ is a smooth $(n-1)$-dimensional subvariety. There is a natural way to relate the canonical class of Y to that of X: if we compare the tangent bundle

$\mathcal{T}_Y$ of Y with the restriction $\mathcal{T}_X|_Y$ to Y of the tangent bundle $\mathcal{T}_X$ of X, we get an exact sequence

$$0 \to \mathcal{T}_Y \to \mathcal{T}_X|_Y \to \mathcal{N}_{Y/X} \to 0,$$

where $\mathcal{N}_{Y/X}$ is, by definition, the normal bundle of Y in X. Taking exterior powers, this gives an equality of line bundles

$$(\wedge^n \mathcal{T}_X)|_Y \cong \wedge^{n-1}\mathcal{T}_Y \otimes \mathcal{N}_{Y/X}$$

and dualizing we have

$$\omega_Y \cong \omega_X|_Y \otimes \mathcal{N}^*_{Y/X}.$$

Moreover, we can compute the normal bundle in another way. There is an exact sequence

$$0 \longrightarrow I_{Y/X}/I^2_{Y/X} \xrightarrow{\delta} \Omega_X|_Y \longrightarrow \Omega_Y \longrightarrow 0,$$

where the map δ sends the germ of a function to the germ of its differential (see Eisenbud [1995] Theorem ****). This identifies the dual of the normal bundle of Y in X with the locally free sheaf $I_{Y/X}/I^2_{Y/X}$. In the case of primary interest to us, when Y is a cartier divisor in X, the ideal sheaf $I_{Y/X}$ of Y in X is the line bundle $\mathcal{O}_X(-Y)$, and the sheaf $I_{Y/X}/I^2_{Y/X} = \mathcal{O}_Y \otimes I_{Y/X}$ is its restriction to Y, denoted $\mathcal{O}_Y(-Y)$. Combining this with the previous expression, we have what is commonly called the *adjunction formula*:

Proposition 1.31 (Adjunction Formula)**.** *If $Y \subset X$ is a smooth, $(n-1)$-dimensional subvariety of a smooth n-dimensional variety, then*

$$\omega_Y = \omega_X(Y)|_Y;$$

In particular, if Y is a curve then the degree of K_Y is given by an intersection product:

$$\deg K_Y = \deg((K_X + [Y])[Y]).$$

1.3.2 Canonical classes of hypersurfaces and complete intersections

We can combine the adjunction formula with the calculation in Example 1.30 to calculate the canonical class of hypersurfaces and more generally complete intersections. To start, let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree d. We have

$$\omega_X = \omega_{\mathbb{P}^n}(X)|_X = \mathcal{O}_X(d-n-1),$$

and f

$$K_X = (d-n-1)\zeta$$

where $\zeta = c_1(\mathcal{O}_X(1)) \in A^1(X)$ is the hyperplane class.

More generally, suppose

$$X = Z_1 \cap \cdots \cap Z_k$$

is a smooth complete intersection of hypersurfaces $Z_1, \ldots, Z_k$ of degrees $d_1, \ldots, d_k$. Applying adjunction repeatedly to the partial intersections $Z_1 \cap \cdots \cap Z_i$, we see that

$$\omega_X = \mathcal{O}_X(-n - 1 + \sum d_i)$$

and so

$$K_X = (-n - 1 + \sum d_i)\zeta.$$

This argument is not complete, because even though X is assumed smooth, the partial intersections $Z_1 \cap \cdots \cap Z_i$ may not be. One way to complete it is to extend the definition of the canonical bundle to possibly singular complete intersections—the adjunction formula is true in this greater generality. Alternatively, if we order the hypersurfaces $Z_i = V(F_i)$ so that $d_1 \geq \cdots \geq d_k$ and replace F_i by a linear combination

$$F_i' = F_i + \sum_{j=i+1}^{k} G_j F_j$$

with G_j general of degree $d_i - d_j$, the hypersurfaces $Z_i' = V(F_i')$ will have intersection X, and by Bertini's Theorem 0.5, the partial intersections will be smooth.

1.4 Curves on surfaces

Aside from enumerative problems, intersection products appeared in algebraic geometry as a central tool in the theory of surfaces developed mostly by the Italians in the early twentieth century. In this section we describe some of the basic ideas. This will serve to illustrate the use of intersection products in a simple setting, and also provide us with formulas that will be useful throughout the book. A different treatment of some of this material is in the last chapter of Hartshorne [1977]; and much more can be found, for example, in Beauville's beautiful little book on algebraic surfaces Beauville [1996].

Throughout this section we'll use some classical notation: if S is a smooth surface and $\alpha, \beta \in A^1(S)$ then we will write $\alpha \cdot \beta$ for the degree $\deg(\alpha\beta)$ of their product $\alpha\beta \in A^2(S)$, and we refer to this as the *intersection number* of the two classes. Further, if $C \subset S$ is a curve we'll abuse notation and write C for the class $[C] \in A^1(S)$. Thus, for example, if C and $D \subset S$ are two curves, we'll write $C \cdot D$ in place of $\deg([C] \cdot [D])$ and we'll write

C^2 for $\deg([C]^2)$. The reader should not be misled by this (potentially misleading) notation into thinking that $A^2(S) = \mathbb{Z}$!—indeed, for many smooth projective surfaces S the group $A^2(S)$ is not even finite-dimensional in any reasonable sense (Mumford [1962]).

1.4.1 The genus formula

One of the first formulas in which intersection products appeared was the *genus formula*, a straightforward rearrangement of the adjunction formual that describes the genus of a smooth curve on a smooth projective surface (we'll generalize it to some singular curves in Section 1.4.4): If $C \subset S$ is a smooth curve of genus g on a smooth surface, then

$$K_C = (K_S + C)|_C;$$

since the degree of the canonical class on C is $2g - 2$, this yields

$$g = \frac{C^2 + K_S \cdot C}{2} + 1. \tag{1.1}$$

Example 1.32 (Plane curves)**.** By way of examples, consider first a smooth curve $C \subset \mathbb{P}^2$ of degree d. If we let $\zeta \in A^1(\mathbb{P}^2)$ be the class of a line, we have $C \sim d\zeta$ and $K_{\mathbb{P}^2} \sim -3\zeta$, so the genus of C is

$$g = \frac{-3d + d^2}{2} + 1 = \frac{(d-1)(d-2)}{2}.$$

Thus we recover, for example, the well-known facts that lines and smooth conics have genus 0 while smooth cubics have genus 1.

Example 1.33 (Curves on a quadric)**.** Now suppose that $Q \subset \mathbb{P}^3$ is a smooth quadric surface, and that $C \subset Q$ is a smooth curve of bidegree (d, e)—that is, a curve linearly equivalent to d times a line of one ruling plus e times a line of the other (equivalently, in terms of the isomorphism $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, a curve given by a bihomogeneous polynomial of bidegree (d, e)). Let α and $\beta \in A^1(Q)$ be the classes of the lines of the two rulings of Q, as in the discussion in Section 1.2.3 above, and let $\zeta = \alpha + \beta$ be the hyperplane class. Applying adjunction to $Q \subset \mathbb{P}^3$, we have

$$K_Q = (K_{\mathbb{P}^3} + Q)|_Q = -2\zeta = -2\alpha - 2\beta.$$

Thus, by the genus formula,

$$\begin{aligned} g &= \frac{(d\alpha + e\beta)^2 - 2(\alpha + \beta)(d\alpha + e\beta)}{2} + 1 \\ &= \frac{2de - 2d - 2e}{2} + 1 \\ &= (d-1)(e-1). \end{aligned}$$

1.4.2 The self-intersection of a curve on a surface

We can sometimes use the genus formula to determine the self-intersection of a curve on a surface. For example, suppose that $S \subset \mathbb{P}^3$ is a smooth surface of degree d and $L \subset S$ is a line. Letting $\zeta \in A^1(S)$ denote the plane class and applying adjunction to $S \subset \mathbb{P}^3$, we have $K_S \sim (d-4)\zeta$, so that $L \cdot K_S = d - 4$; since the genus of L is zero, the genus formula yields

$$0 = \frac{L^2 + d - 4}{2} + 1$$

or

$$L^2 = 2 - d.$$

The cases $d = 1$ (a line on a plane) and $d = 2$ are probably familiar already; in case $d \geq 3$ the formula implies the qualitative statement that *a smooth surface $S \subset \mathbb{P}^3$ of degree 3 or more can contain only finitely many lines.* (See Exercise 1.97 below for a sketch of a proof; see also Exercise 1.96 for an alternative derivation of $L^2 = 2 - d$.)

1.4.3 Linked curves in $\mathbb{P}^3$

Another application of the genus formula yields a classical relation between what are called *linked curves* in $\mathbb{P}^3$.

Let S and $T \subset \mathbb{P}^3$ be smooth surfaces of degrees s and t, and suppose that the scheme-theoretic intersection $S \cap T$ consists of the union of two smooth curves C and D with no common components. Let the degrees of C and D be c and d, and let their genera be g and h respectively. By Bézout's theorem, we have

$$c + d = st,$$

so that the degree of C determines the degree of D. What is much less obvious is that the degree and genus of C determine the degree and genus of D. Here is one way to derive the formula.

To start, we use the genus formula as above to determine the self-intersection of C on S: since $K_S \sim (s-4)\zeta$, we have

$$g = \frac{C^2 + K_S \cdot C}{2} + 1 = \frac{C^2 + (s-4)c}{2} + 1$$

and hence

$$C^2 = 2g - 2 - (s-4)c$$

(generalizing our formula above for the self-intersection of a line). Next, since $C + D \sim t\zeta \in A^1(S)$, we can write the intersection number of C and D on S as

$$C \cdot D = C(t\zeta - C) = tc - \bigl(2g - 2 - (s-4)c\bigr) = (s+t-4)c - (2g-2).$$

This in turn allows us to determine the self-intersection of D on S:

$$D^2 = D(t\zeta - C) = td - \big((s+t-4)c - (2g-2)\big).$$

Applying the genus formula to D we arrive at

$$\begin{aligned} h &= \frac{D^2 + K_S \cdot D}{2} + 1 \\ &= \frac{td - \big((s+t-4)c - (2g-2)\big) + (s-4)d}{2} + 1. \end{aligned}$$

Simplifying, this says

$$h - g = \frac{s+t-4}{2}(d-c); \tag{1.2}$$

in English, *the difference in the genera of C and D is proportional to the difference in their degrees, with ratio $(s+t-4)/2$.*

The answer to Keynote Question (e) is a special case of this: if $L \subset \mathbb{P}^3$ is a line, and S and T general surfaces of degrees s and t containing L, then writing $S \cap T = L \cup C$, we see that C is a curve of degree $st - 1$ and genus

$$h = \frac{(s+t-4)(st-2)}{2}.$$

As is often the case with enumerative formulas, this is just the beginning of a whole theory. The theory of Liaison describes the relationship between the geometry of linked curves such as C and D above. The theory in general is far more broadly applicable (the curves C and D need only be Cohen-Macaulay, and we need no hypotheses at all on S and T) and ultimately provides a complete answer to the question of when two given curves $C, D \subset \mathbb{P}^3$ can be connected by a series of curves $C = C_0, C_1, \ldots, C_{n-1}, C_n = D$ with C_i and C_{i+1} linked as above. We will see a typical application of the notion of linkage in Exercise 1.99 below; for the general theory see Peskine and Szpiro [1974].

1.4.4 The blow-up of a surface

Blowing up is a basic operation that associates to any scheme X and subscheme Y a morphism $\pi : Bl_Y(X) \to X$. The general operation is described and characterized in Eisenbud and Harris [2000]Chapter ****. The blow-up of a point on a surface plays an important role in the theory of surfaces, and we will now explain a little of this theory. Locally, such blow-ups look like the blowup of $\mathbb{P}^2$ at a point, which was treated inSection 1.2.4.

To fix notation we let $p \in S$ be a point in a smooth projective surface, and write $\pi : \tilde{S} \to S$ for the blow up. We write $E = \pi^{-1}(p) \subset \tilde{S}$ for the preimage of p, called the *exceptional divisor*, and we write $e \in A^1(\tilde{S})$ for its class. We will use the following definitions and facts:

- $\pi : \tilde{S} \to S$ is birational, and if $q \in E \subset \tilde{S}$ is any point of the exceptional divisor, then there are generators z, w for the maximal ideal ideal of $\mathcal{O}_{\tilde{S},q}$ and generators x, y for the maximal ideal of $\mathcal{O}_{S,p}$ such that $\pi^*x = zw$ and $\pi^*y = w$, and E is defined locally by the equation $w = 0$. In particular, $\tilde{S}$ is smooth, and E is a Cartier divisor.
- If C is a smooth curve through p, then the *proper transform* $\tilde{C}$ of C, which is by definition the closure of $\pi^{-1}(C) \cap (\tilde{S} \setminus E)$ in $\tilde{S}$, meets E transversely in one point.
- More generally, if C has an *ordinary m-fold point at p* then $\tilde{C}$ meets E transversely in m distinct points. Here we say that C has an ordinary m-fold point at p if the completion of the local ring of C at p has the form
$$\hat{\mathcal{O}}_{C,p} \cong K[[x, y]]/(\prod_{i=1}^{m} /(x - \lambda_i y)$$
for some distinct $\lambda_1, \ldots, \lambda_m \in K$, which says geometrically that C has m branches at p with distinct tangents.

In Chapter 15, we'll describe the Chow ring of an arbitrary blow-up of a smooth variety along a smooth subvariety. This general formula is beyond our means at this point, but we can already describe the case of $A(\tilde{S})$.

Proposition 1.34. (a) *As abelian groups, $A(\tilde{S}) = A(S) \oplus \mathbb{Z}e$.*

(b) *For any $\alpha, \beta \in A_1(S)$, $\pi^*\alpha \cdot \pi^*\beta = \pi^*(\alpha\beta)$;*

(c) *For any $\alpha \in A_1(S)$, $e \cdot \pi^*\alpha = 0$; and*

(d) *$e^2 = -[q]$ for any point $q \in E$ (in particular, $\deg(e^2) = -1$).*

Proof. We first show that π_* and π^* are inverse isomorphisms between $A^2(S)$ and $A^2(\tilde{S})$. By the Moving Lemma, if $\alpha \in A_0(S)$ is any class, we can write $\alpha = [A]$ for some $A \in Z_0(S)$ with support disjoint from p; thus $\pi_*\pi^*\alpha = \alpha$. Likewise, if $\alpha \in A_0(\tilde{S})$ is any class, we can write $\alpha = [A]$ for some $A \in Z_0(S)$ with support disjoint from E; thus $\pi^*\pi_*\alpha = \alpha$.

We next turn to A^1. If $\alpha \in A_1(S)$ is any class, we can write $\alpha = [A]$ for some $A \in Z_1(S)$ with support disjoint from p; thus $\pi_*\pi^*\alpha = \alpha$. On the other hand, the kernal of the pushforward map $\pi_* : Z_1(\tilde{S}) \to Z_1(S)$ is just the subgroup generated by e, the class of E. Thus we have an exact sequence
$$0 \to \langle e \rangle \to A_1(\tilde{S}) \to A_1(S) \to 0,$$
with $\pi^* : A_1(S) \to A_1(\tilde{S})$ splitting the sequence.

Thus it remains to show that the class e is not torsion in $A_1(\tilde{S})$. This follows from the formula $e^2 = -1$, which we will prove independently below.

Part (b) of the proposition simply recalls the fact that π^* is a ring homomorphism. For part (c) we use the push-pull formula:

$$\pi_*(e \cdot \pi^*\alpha) = \pi_* e \cdot \alpha = 0.$$

For part (d), let $C \subset S$ be any curve smooth at p, so that the proper transform $\tilde{C} \subset \tilde{S}$ of C will intersect E transversely at one point q. We have then

$$\pi^*[C] = [\tilde{C}] + e$$

and intersecting both sides with the class e yields

$$0 = [q] + e^2,$$

so the self-intersection number of e is $\deg e^2 = -1$. □

Canonical class of a blowup. We can express the canonical class of $\tilde{S}$ in terms of the canonical class of S as follows.

Proposition 1.35. *With notation as above,*

$$K_{\tilde{S}} = \pi^* K_S + e.$$

Proof. We must show that if ω is a rational 2-form on S, regular and nonzero at p, then the pullback $\pi^*\omega$ vanishes simply along E. Let $q \in E \subset \tilde{S}$, and let (z, w) be generators of the maximal ideal of $\mathcal{O}_{\tilde{S},q}$ such that there are generators (x, y) for the maximal ideal of $\mathcal{O}_{S,p}$ with

$$\pi^* x = zw \quad \text{and} \quad \pi^* y = w.$$

It follows that

$$\pi^* dx = zdw + wdz \quad \text{and} \quad \pi^* dy = dw.$$

Thus

$$\pi^*(dx \wedge dy) = w(dz \wedge dw).$$

Since the local equation of E at q is $w = 0$, this shows that $\pi^* dx$ vanishes simply along E as required. □

The genus formula with singularities. It will be useful in a number of situations to have a version of the genus formula (1.1) that works for singular curves $C \subset S$. To start with the simplest case, suppose that $C \subset S$ is a curve smooth away from a point $p \in C$ of multiplicity m. Assume moreover that p is an ordinary m-fold point, so that in particular the proper transform $\tilde{C}$ is smooth. We could of course invoke the genus formula on $\tilde{S}$, but we want to give a formula for the genus g of $\tilde{C}$ in terms of intersection numbers on S itself.

As divisors

$$\pi^* C = \tilde{C} + mE,$$

so that

$$[\tilde{C}] = \pi^*[C] - me.$$

From Proposition 1.35 we have

$$K_{\tilde{S}} = \pi^* K_S + e,$$

and putting this together with the genus formula for $\tilde{C} \subset \tilde{S}$ and Proposition 1.34, we have

$$\begin{aligned} g &= \frac{\tilde{C}^2 + K_{\tilde{S}} \cdot \tilde{C}}{2} + 1 \\ &= \frac{(\pi^* C - me)^2 + (\pi^* K_S + e)(\pi^* C - me)}{2} + 1 \\ &= \frac{C^2 + K_S \cdot C}{2} + 1 - \binom{m}{2}. \end{aligned}$$

More generally, if $C \subset S$ has singular points $p_1, \ldots, p_\delta$ of multiplicity $m_1, \ldots, m_\delta$, and the proper transform $\tilde{C}$ of C in the blow-up $Bl_{\{p_1,\ldots,p_\delta\}}$ of S at the points p_i is smooth, we have

$$g = \frac{C^2 + K_S \cdot C}{2} + 1 - \sum \binom{m_i}{2}.$$

We can extend this further, to general singular curves $C \subset S$, by using iterated blow-ups, or by generalizing the adjunction formula, using the fact that any curve on a smooth surface has a canonical bundle (see for example Hartshorne [1977], Theorem III.7.11).

1.5 Chern classes

The theory of Chern classes is one of the most powerful tools in intersection theory, and will be one of the central techniques of this book starting with Chapter 7. In Section 1.1.3 we defined the first Chern class $c_1(\mathcal{L})$ of a line bundle $\mathcal{L}$ on a variety X by setting

$$c_1(\mathcal{L}) = [Div(\sigma)] \in A_{\dim X - 1}(X)$$

for any rational section σ of $\mathcal{L}$. More generally, Chern classes $c_i(\mathcal{E}) \in A^i(X)$ are defined for any vector bundle $\mathcal{E}$.

Many classes of interesting loci turn out to be the Chern classes of bundles. There are very strong techniques for computing Chern classes, so it is often the case that reducing an enumerative problem to a Chern class computation makes the solution easy. In this section we will explain the beginning of this theory. Detailed proofs of the results, and much more, will be found in Chapter 7 and subsequent chapters.

1.5.1 Chern classes as degeneracy loci

Although the theory of Chern classes is quite general (and we will give a formula that can serve as a general definition in Chapter 11), we will restrict ourselves here to bundles on smooth projective varieties. In the case of a line bundle $\mathcal{L}$ that we have already treated, one can reduce the definition of $c_1\mathcal{L}$ to the case where $\mathcal{L}$ has a section by using the formula $c_1\mathcal{L} = c_1(\mathcal{L}(m)) - c_1\mathcal{O}_X(m)$. A similar reduction (with a slightly more complicated formula) works for vector bundles, so in this section we will give the definitions only for a bundle $\mathcal{E}$ generated by its global sections.

Proposition-Definition 1.36. Suppose that $\mathcal{E}$ is a vector bundle of rank r on a projective variety X, and let $\sigma_0, \dots, \sigma_{r-i} \in H^0(\mathcal{E})$ be a collection of global sections of $\mathcal{E}$, and let $\sigma : \mathcal{O}_X^{r-i+1} \to \mathcal{E}$ be the map sending the j-th basis element to σ_j.

There is a unique closed subscheme $Y \subset X$ such that, if $U \subset X$ is any open subset over which $\mathcal{E}$ is trivial, the scheme $Y \cap U$ is defined by the ideal of $(r-i+1) \times (r-i+1)$ minors of the $(r-i+1) \times r$ matrix,

$$\mathcal{O}_U^{r-i+1} \to \mathcal{E}|_U \cong \mathcal{O}_U^r$$

representing σ.

If $\mathcal{E}$ is generated by its global sections, then the degeneracy subscheme of $r-i+1$ general global sections of $\mathcal{E}$ has pure codimension i in X, and we define $c_i\mathcal{E}$ to be the class of this subscheme, which is independent of the choice of the general sections.

We shall return to such degeneracy subschemes, also called *degeneracy loci*, in Chapters 12 and 14, and we shall show (Corollary 14.2) that the degeneracy scheme of any set of $r-i+1$ sections of $\mathcal{E}$ has class equal to $c_i\mathcal{E}$ so long as it has codimension i.

For example, supposing always that $\mathcal{E}$ is a vector bundle of rank r on X, and is generated by global sections:

- $c_0\mathcal{E}$: Where are no sections dependent? By our definition, this is the locus defined by the $(r+1)\times(r+1)$ minors of a matrix of size $r\times(r+1)$. Since any such minor would be zero, $c_0\mathcal{E} = [X]$. Since this is the identity element in the Chow ring, we usually write $c_0\mathcal{E} = 1$.
- $c_1\mathcal{E}$: Here we are dealing with a map $\sigma : \mathcal{O}_X^r \to \mathcal{E}$ defined by r sections. The $r \times r$ minor an $r \times r$ matrix is just the determinant, which can be globally represented by the r-th exterior power of σ,

$$\mathcal{O}_X = \wedge^r(\mathcal{O}_X^r \xrightarrow{\wedge^r\sigma} \wedge^r \mathcal{E}.$$

Of course the image of $1 \in \mathcal{O}_X$ is then a global section of the line bundle $\wedge^r\mathcal{E}$, so $c_1\mathcal{E} = c_1(\wedge^r\mathcal{E})$, and the definition reduces to that for line bundles.

- $c_r\mathcal{E}$: This is the "degeneracy locus of 1 section". The section is locally given by r functions $f_1, \dots f_r$, and the definition simpy says that $c_r\mathcal{E}$ is the class of the subscheme locally defined as $V(f_1, \dots, f_r)$.
- $c_i\mathcal{E}$ for $i > r$: Where are $r - i + 1 \leq 0$ sections dependent? Nowhere, of course! So these classes are all 0.

A useful consequence of the definition is that if $X' \subset X$ is any subvariety and $\mathcal{E}$ is a vector bundle on X, then the pullback of $c_i\mathcal{E}$ to X' is equal to $c_i(\mathcal{E}|_{X'})$. This is immediate for bundles generated by their global sections, and reduces easily to that case.

1.5.2 Whitney's formula and the splitting principle

Two results help enormously in the computation of Chern classes: the Whitney formula and the Splitting Principle.

The Whitney formula (Theorem 1.37) is best expressed in terms of the *total Chern class* of a bundle $\mathcal{E}$ is defined as

$$c(\mathcal{E}) := c_0(\mathcal{E}) + c_1(\mathcal{E}) + c_2(\mathcal{E}) + \cdots = 1 + c_1(\mathcal{E}) + c_2(\mathcal{E}) + \cdots \in A(X).$$

Whitney's formula says that the total Chern class is multiplicative on short exact sequences:

Theorem 1.37 (Whitney's formula)**.** *If*

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

is a short exact sequence of vector bundles on X then

$$c(\mathcal{E}) = c(\mathcal{E}')c(\mathcal{E}'').$$

Example 1.38. Sums of Line Bundles If $\mathcal{E}$ is the direct sum of line bundles $\mathcal{L}_i$, or even has a filtration whose quotients are the $\mathcal{L}_i$, then Whitney's Formula says that $c(\mathcal{E}) = \prod c(\mathcal{L}_i) = \prod(1 + c_1\mathcal{L}_i)$, so $c_i\mathcal{E}$ is the result of applying the i-th elementary symmetric function to the classes $c_1\mathcal{L}_i$.

Whitney's Formula can often be used in conjunction with the Splitting Principle (Lemma 7.8), which may be thought of thus:

Informal splitting principle. *Any identity among the Chern classes of bundles that is true for bundles that are direct sums of line bundles is true in general.*

This is a consequence of a construction: for any variety X and bundle $\mathcal{E}$ on X, we will (in Chapter 7) construct a map $\pi : Y \to X$ (as a flag bundle over X) such that the pullback $\pi^*(A(X) \to A(Y)$ is monomorphism, and $\pi^*\mathcal{E}$ has a filtration by subbundles, whose successive quotients are line bundles $\mathcal{L}_i$. By Whitney's Formula this implies

$$c(\pi^*\mathcal{E}) = c(\oplus\pi^*\mathcal{L}_i).$$

An important consequence of Whitney's Formula and the Splitting Principle is that the Chern classes of a bundle vanish above the rank, something we saw already in the case of line bundles with enough sections.

Corollary 1.39. *If $\mathcal{E}$ is a vector bundle of rank r, then $c_i\mathcal{E} = 0$ for $i > r$.*

Proof. The result is true—by definition—for line bundles. If $\mathcal{E}$ split as $\oplus_{i=1}^r \mathcal{L}_i$ for line bundles $\mathcal{L}_i$, then by Whitney's Formula $c(\mathcal{E}) = \prod_{i=1}^r (1 + c_1\mathcal{L}_i)$, which has no terms of degree $> r$. $\square$

We will now illustrate the use of Whitney's Formula and the Splitting Principle with a few useful computations.

Example 1.40. Duals If $\mathcal{E} = \oplus\mathcal{L}_i$ then $c(\mathcal{E}^*) = \prod(1+c_1\mathcal{L}_i^*) = \prod(1-c_1\mathcal{L}_i$, Since $c_1(\mathcal{L}^*) = -c_1(\mathcal{L})$ when $\mathcal{L}$ is a line bundle. Given this, Whitney's Formula shows that $c_i(\mathcal{E}^*) = (-1)^i c_i(\mathcal{E})$. By the splitting principle, this last formula holds for any bundle.

Example 1.41. Symmetric squares Suppose that $\mathcal{E}$ is a bundle of rank 2. If $\mathcal{E}$ split as a direct sum $\mathcal{E} = \mathcal{L} \oplus \mathcal{M}$ of line bundles $\mathcal{L}$ and $\mathcal{M}$ with Chern classes $c_1(\mathcal{L}) = \alpha$ and $c_1(\mathcal{M}) = \beta$ then, by Whitney's formula, $c(E) = (1+\alpha)(1+\beta)$, whence

$$c_1(\mathcal{E}) = \alpha + \beta \quad \text{and} \quad c_2(\mathcal{E}) = \alpha\beta.$$

Further, we would have

$$\operatorname{Sym}^2 \mathcal{E} = \mathcal{L}^{\otimes 2} \oplus (\mathcal{L} \otimes \mathcal{M}) \oplus \mathcal{M}^{\otimes 2},$$

from which we would deduce

$$\begin{aligned} c(\operatorname{Sym}^2 \mathcal{E}) &= (1+2\alpha)(1+\alpha+\beta)(1+2\beta) \\ &= 1 + 2(\alpha+\beta) + (2\alpha^2 + 8\alpha\beta + 2\beta^2) + 4\alpha\beta(\alpha+\beta). \end{aligned}$$

This expression may be rewritten in a way that involves only the Chern classes of $\mathcal{E}$: as the reader may immediately check, it is equal to

$$1 + 2c_1(\mathcal{E}) + \big(2c_1(\mathcal{E})^2 + 4c_2(\mathcal{E})\big) + 4c_1(\mathcal{E})c_2(\mathcal{E}).$$

By the splitting principle, this is a valid expression for $c(\operatorname{Sym}^2 \mathcal{E})$ whether or not $\mathcal{E}$ actually splits.

In principle we could use the same method to give formulas for the Chern classes of any symmetric or exterior power, or even of any multilinear functor.

1.5.3 Chern classes of varieties

The most important vector bundles on a smooth variety X are its tangent and cotangent bundles $\mathcal{T}_X$ and Ω_X. Their Chern classes are so important in geometry that they are usually just called the *Chern classes of* X. We have already seen one example of these in the definition of the canonical class of X: this was defined to be the first Chern class of the top exterior power $\omega_X = \wedge^{\dim X}\Omega_X$, which is just the first Chern class $c_1(\Omega_X)$ of the cotangent bundle of X.

Example 1.42 ($\mathbb{P}^n$)**.** The *Euler sequence*

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \to \mathcal{T}_{\mathbb{P}^n} \to 0$$

(described in Section 2.2.4) makes it easy to compute the Chern classes of the tangent and cotangent bundles of $\mathbb{P}^n$: applying the Whitney formula, we see that

$$c(\mathcal{T}_{\mathbb{P}^n}) = \frac{c(\mathcal{O}_{\mathbb{P}^n}(1)^{n+1})}{c(\mathcal{O}_{\mathbb{P}^n})} = c(\mathcal{O}_{\mathbb{P}^n}(1)^{n+1}) = (1+\zeta)^{n+1} \in A(\mathbb{P}^n),$$

where $\zeta = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ is the class of a hyperplane. To compute the Chern classes of $\Omega_{\mathbb{P}^n} = \mathcal{T}^*_{\mathbb{P}^n}$ we could use the general formula for the Chern classes of a dual, or we could dualize the sequence above; either way, we obtain

$$c(\Omega_{\mathbb{P}^n}) = (1-\zeta)^{n+1}.$$

In particular, we see again that the canonical class of $\mathbb{P}^n$ is given by $K_{\mathbb{P}^n} = c_1(\Omega_{\mathbb{P}^n}) = -(n+1)\zeta$, as we established in Example 1.30

We saw in Section 1.3.1 how to relate the tangent bundles of a smooth variety X and a smooth divisor $Y \subset X$, and applied this in Section 1.3.2 to find the canonical classes of hypersurfaces and complete intersections in $\mathbb{P}^n$. We can do the same for the Chern classes in general:

Example 1.43 (Chern classes of hypersurfaces)**.** Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree d. We saw in Section 1.3.1 that we have an exact sequence of vector bundles

$$0 \to \mathcal{T}_X \to \mathcal{T}_{\mathbb{P}^n}|_X \to \mathcal{N} \to 0,$$

where $\mathcal{N} = \mathcal{N}_{X/\mathbb{P}^n}$ is the normal bundle of X in $\mathbb{P}^n$. Since $\mathcal{N} \cong \mathcal{O}_X(d)$, we have $c(\mathcal{N}) = 1 + d\zeta$, and thus

$$c(\mathcal{T}_X) = \frac{(1+\zeta)^{n+1}}{1+d\zeta} = (1+\zeta)^{n+1}(1 - d\zeta + d^2\zeta^2 - \dots).$$

We can generalize this calculation to complete intersections:

Example 1.44 (Chern classes of complete intersections)**.** Suppose that $X = Z_1 \cap \cdots \cap Z_k \subset \mathbb{P}^n$ is the complete intersection of k hypersurfaces of degrees $d_1, \ldots, d_k$ defined by forms F_i of degrees d_i. The relations among the F_i are generated by the Koszul relations $F_j F_i - F_i F_j = 0$. This means that if we restrict to Y, where the F_i vanish, we get

$$\mathcal{I}_{Y/X}/\mathcal{I}_{Y/X}^2 = \mathcal{I}_{Y/X}|Y = \oplus_i \mathcal{O}_Y(-d_i),$$

so the normal bundle $\mathcal{N} = \mathcal{N}_{X/\mathbb{P}^n}$ of X in $\mathbb{P}^n$ is a direct sum $\mathcal{N} = \oplus \mathcal{O}_X(d_i)$. Applying the Whitney formula, we get

$$c(\mathcal{T}_X) = \frac{(1+\zeta)^{n+1}}{\prod(1+d_i\zeta)}.$$

1.5.4 The topological Euler characteristic

Here is an application involving the top Chern class of the tangent bundle, $c_{\dim X}\mathcal{T}_X$, derived from the Poincaré-Hopf Index Theorem. Recall that the *topological Euler characteristic* of a manifold M is by definition $\chi_{\mathrm{top}}(M) := \sum_i (-1)^i \dim_{\mathbb{Q}} H^i(M;\mathbb{Q})$, where $H^i(M;\mathbb{Q})$ refers to the singular cohomology group. (When M is a smooth projective variety over $\mathbb{C}$, we will mean by $\chi_{\mathrm{top}}(M)$ the topological Euler characteristic with respect to the classical, or analytic, topology.)

Theorem 1.45 (Poincaré-Hopf Theorem)**.** *If M is a smooth compact orientable manifold, and σ is a vector field with isolated zeros, then*

$$\chi_{\mathrm{top}}(M) = \sum_{x|\sigma(x)=0} \mathrm{index}_x(\sigma).$$

A beautiful account of this classic result can be found in Milnor [1997]. Now suppose that M is a smooth complex projective variety. If the tangent bundle $\mathcal{T}_X$ is generated by global sections then it will have a section σ that vanishes at only finitely many points, and vanishes simply there. Since this section is represented locally by complex analytic functions, its index at each of its zeros will be 1, and we may replace the sum in the Poincaré-Hopf Theorem by the number of its zeros. This is, by definition, the top Chern class of $\mathcal{T}_X$. An elementary topological argument, given in Appendix **??******refers to an unfinished Appendix********, shows that this is true more generally:

Theorem 1.46. *If X is a smooth projective variety then*

$$\chi_{\mathrm{top}}(X) = \deg\big(c_{\dim X}(\mathcal{T}_X)\big).$$

Example 1.47 (Euler characteristic of $\mathbb{P}^n$)**.** Since $c(\mathcal{T}_{\mathbb{P}^n}) = (1+\zeta)^{n+1}$, where $\zeta = c_1(\mathcal{O}_{\mathbb{P}^n}(1)$ is the class of a hyperplane, we deduce that

$$\chi_{\mathrm{top}}(\mathbb{P}^n) = \deg\big(c_n(\mathcal{T}_{\mathbb{P}^n})\big) = n+1.$$

Of course this is immediate from the fact that $H^{2i}(\mathbb{P}^n, \mathbb{Q}) = \mathbb{Q}$ for $i = 0, \ldots, n$ while $H^{2i+1}(\mathbb{P}^n, \mathbb{Q}) = 0$ for all i.

Example 1.48 (Euler characteristic of a hypersurface)**.** Now let X be a smooth hypersurface of degree d in $\mathbb{P}^n$. From the exact sequence

$$0 \to \mathcal{T}_X \to \mathcal{T}_{\mathbb{P}^n}|_X \to \mathcal{N}_{X/\mathbb{P}^n} \to 0.$$

The normal bundle of X is $\mathcal{N}_{X/\mathbb{P}^n} = \mathcal{O}_X(d)$ so $c(\mathcal{N}_{X/\mathbb{P}^n}) = 1 + d\zeta_X$, where ζ_X is the hyperplane class on X. By Whitney's formula

$$c(\mathcal{T}_X) = \frac{(1+\zeta_X)^{n+1}}{(1+d\zeta_X)} = ((1+\zeta_X)^{n+1})(1 - d\zeta_X + d^2(\zeta_X)^2 + \cdots).$$

Taking the component of degree $\dim X = n-1$, we get

$$c_{n-1}(\mathcal{T}_X) = \sum_{i=0}^{n-1} (-1)^i \binom{n+1}{n-1-i} d^i \zeta_X^{n-1}.$$

Since the degree of ζ_X^{n-1} is the number of points of intersection of $n-1$ general hyperplanes on the $(n-1)$ dimensional variety X, we have $\zeta_X^{n-1} = d$. Thus, finally,

$$\chi_{\mathrm{top}}(X) = \deg(c_{n-1}(\mathcal{T}_X)) = \sum_{i=0}^{n-1} (-1)^i \binom{n+1}{n-1-i} d^{i+1}.$$

Responding to Keynote Question (f), this shows in particular that if X is a smooth quartic surface then

$$\chi_{\mathrm{top}}(X) = \binom{4}{2} \cdot 4 - \binom{4}{1} \cdot 16 + \binom{4}{0} \cdot 64 = 24.$$

1.5.5 The 27 lines on a cubic surface

To explain how Chern classes can be used to linearize otherwise difficult problems, we explain the Chern class approach to a famous classical result:

Theorem 1.49. *Each smooth cubic surface in $\mathbb{P}^3$ contains exactly 27 distinct lines.*

One way to prove Theorem 1.49 is to show that any smooth cubic surface can be realized as the blowup of $\mathbb{P}^2$ in 6 fairly general points, and to analyze the geometry of such a blowup in detail (see for example Manin [1986] or Reid [1988b]). The Chern class approach that we will now indicate has the advantage of applying equally to related results where no such analysis is available. For example, the Chern class method will also show that a general quintic threefold in $\mathbb{P}^4$ contains exactly 2875 lines (a computation that played an important role in the discovery of mirror symmetry; see for example Morrison [1993]), or even that a general hypersurface of degree 37 in $\mathbb{P}^{20}$ contains exactly

$$479849240965383456367278060519107076039364076181726998 5515$$

lines, even though we have no description of the geometry of these hypersurfaces analogous to the one mentioned above for the cubic surface.

Sketch of the Chern class approach. The first ingredient in the Chern class approach is to make the set of lines in $\mathbb{P}^3$ into an algebraic variety, the Grassmannian $\mathbb{G}(1,3)$, and to determine its Chow ring $A(\mathbb{G}(1,3))$ (this will be done in in Chapter 2).

We can *linearize* the problem using the observation that, if we give a particular line L in $\mathbb{P}^3$, then the condition that L lie on X can be expressed as 4 linear conditions on the coefficients of F: To see this, note that restricting a cubic form on $\mathbb{P}^3$ to a line $L \cong \mathbb{P}^1$ is a linear map from the 20-dimensional space of cubic forms on $\mathbb{P}^3$ onto the 4-dimensional vector space V_L of cubic forms on L, and the condition $L \subset X$ is that F maps to 0 in V_L.

The next step in this approach is to show that, as the line L varies over $\mathbb{G}(1,3)$, the 4-dimensional spaces $V_L = H^0(\mathcal{O}_L(3))$ of cubic forms on the varying L's "fit together" to form a vector bundle $\mathcal{V}$ of rank 4 on $\mathbb{G}(1,3)$. A cubic form F on $\mathbb{P}^3$, through its restriction to each V_L, defines an algebraic global section σ_F of this vector bundle. Thus the number of lines contained in the cubic surface X is the number of zeros of the section σ_F, and if this number is finite then at least the scheme of zeros will have class $c_4(\mathcal{V})$.

It may seem at this point that all we have done is to give our ignorance a fancy name. But this is where tools like the Whitney formula and the splitting principle come in. Briefly: just as the vector spaces $H^0(\mathcal{O}_L(3))$ fit together to form a vector bundle of rank 4, the spaces $H^0(\mathcal{O}_L(1))$ fit together to form a simpler bundle of rank 2, usually denoted $\mathcal{S}^*$, and the bundle $\mathcal{V}$ is simply the symmetric cube $\operatorname{Sym}^3 \mathcal{S}^*$ of $\mathcal{S}^*$. Being a simpler bundle, it's not hard to calculate the Chern classes $c_i(\mathcal{S}^*)$ (this will be done in Chapter 7); and then the Whitney formula and the splitting principle allow us to express the Chern classes of $\mathcal{V}$ in terms of those of $\mathcal{S}^*$, as in Example 1.41. Putting these together, we'll arrive in Chapter 8 at the formula

$$\deg c_4(\mathcal{V}) = 27.$$

Of course to prove Theorem 1.49 one still has to show that the number of lines on any smooth cubic surface is finite, and that the zeros all occur with multiplicity 1. $\square$

1.5.6 The Chern character

The *Grothendieck group* $K(X)$ of vector bundles on a variety X is defined as the free abelian group $\oplus\mathbb{Z}[A]$ on the isomorphism classes of vector bundles A on X modulo relations $[A] + [C] = [B]$ for every short exact sequence of vector bundles $0 \to A \to B \to C \to 0$. This group has a natural ring structure, where the product is given by tensor product.

Whitney's Formula tells us that the total Chern class converts direct sums of vector bundles into products in the Chow ring, and thus defines a function from the Grothendieck ring $K(X)$ to the multiplicative group of units in $A(X)$. If we are willing to allow rational coefficients, we can make a different combination of the Chern classes, called the *Chern character*, that is actually a ring homomorphism

$$\mathrm{Ch} : K(X) \otimes \mathbb{Q} \to A(X) \otimes \mathbb{Q}$$

To understand the definition, first consider the case of line bundles $\mathcal{L}$ and $\mathcal{M}$. The first Chern class of the tensor product $c(\mathcal{L}\otimes\mathcal{M})$ is $c_1\mathcal{L}+c_1\mathcal{M}$.Thus the simplest function of the Chern classes that takes the Chern classes of a tensor product of line bundles to a product in the Chow ring is the exponential,

$$\mathrm{Ch}([\mathcal{L}]) = e^{c_1\mathcal{L}} = 1 + c_1\mathcal{L} + \frac{(c_1\mathcal{L})^2}{2} + \frac{(c_1\mathcal{L})^3}{6} + \cdots .$$

Note that this apparently infinite sum is actually finite, since $c_1\mathcal{L}$ is nilpotent in $A(X)$.

If $\mathcal{E} = \oplus\mathcal{L}_i$ is a direct sum of line bundles, then for Ch to preserve sums we must define

$$\mathrm{Ch}(\mathcal{E}) = \sum_i e^{c_1\mathcal{L}_i}.$$

The coefficients of this power series are of course symmetric in the elements $c_1\mathcal{L}$, and thus can be expressed in terms of the elementary symmetric functions of these quantities—that is, in terms of the Chern classes of $\mathcal{E}$. We define $\mathrm{Ch}(\mathcal{E})$ in general by using these expressions, and get

$$\mathrm{Ch}(\mathcal{E}) = \mathrm{rank}(\mathcal{E}) + c_1(\mathcal{E}) + \frac{c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})}{2} + \cdots \in A(X) \otimes \mathbb{Q}.$$

The splitting principle implies that this formula does indeed give a ring homomorphism. We will see the Chern character in use when we come to describe the Grothendieck-Hirzebruch-Riemann-Roch formulas in Chapter 16.

The Chern character allows us to answer another very natural question: how much information about a vector bundle is contained in its Chern classes?

Theorem 1.50 (Grothendieck)**.** *If X is a smooth projective variety, then the map*

$$\mathrm{Ch} : K(X) \otimes \mathbb{Q} \to A(X) \otimes \mathbb{Q}$$

is an isomorphism of rings.

See Borel and Serre [1958] for a proof.

1.6 Exercises

Exercise 1.51. Let $\nu = \nu_{2,2} : \mathbb{P}^2 \to \mathbb{P}^5$ be the quadratic Veronese map. If $C \subset \mathbb{P}^2$ is a plane curve of degree d, show that the image $\nu(C)$ has degree $2d$. (In particular, this means that the Veronese surface $S \subset \mathbb{P}^5$ contains only curves of even degree!)

Exercise 1.52. More generally, let $\nu = \nu_{n,d} : \mathbb{P}^n \to \mathbb{P}^N$ be the degree d Veronese map. If $X \subset \mathbb{P}^n$ is a variety of dimension k and degree e, show that the image $\nu(X)$ has degree $d^k e$.

Exercise 1.53. Let $\sigma = \sigma_{r,s} : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^{(r+1)(s+1)-1}$ be the Segre map, and let $X \subset \mathbb{P}^r \times \mathbb{P}^s$ be a subvariety of codimension k. If the class $[X] \in A^k(\mathbb{P}^r \times \mathbb{P}^s)$ is given by

$$[X] = c_0\alpha^k + c_1\alpha^{k-1}\beta + \cdots + c_k\beta^k$$

(where α and $\beta \in A^1(\mathbb{P}^r \times \mathbb{P}^s)$ are the pullbacks of the hyperplane classes, and we take $c_i = 0$ if $i > s$ or $k - i > r$),

(a) Show that all $c_i \geq 0$.

(b) Calculate the degree of the image $\sigma(X) \subset \mathbb{P}^{(r+1)(s+1)-1}$; and, using this and the first part,

(c) Show that any linear space $\Lambda \subset \Sigma_{r,s} \subset \mathbb{P}^{(r+1)(s+1)-1}$ contained in the Segre variety lies in a fiber of a projection map.

Exercise 1.54. We saw in Theorem 1.26 that an n-dimensional subvariety $Z \subset \mathbb{P}^r \times \mathbb{P}^s$ is a rationally equivalent to a linear combination of products of subspaces:

$$[Z] = \sum_{k=0}^{n} a_k[\mathbb{P}^k \times \mathbb{P}^{n-k}].$$

Show that all the coefficients a_k in this expression are nonnegative.

Exercise 1.55. Let $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ be the rational map given by

$$\varphi : (x_0, x_1, x_2) \mapsto \left(\frac{1}{x_0}, \frac{1}{x_1}, \frac{1}{x_2}\right),$$

or, equivalently,

$$\varphi : (x_0, x_1, x_2) \mapsto (x_1x_2, x_0x_2, x_0x_1)$$

and let $\Gamma_\varphi \subset \mathbb{P}^2 \times \mathbb{P}^2$ be the graph of φ. Find the class

$$[\Gamma_\varphi] \in A^2(\mathbb{P}^2 \times \mathbb{P}^2).$$

Exercise 1.56. Let $X_1, \dots, X_n \subset \mathbb{P}^n$ be hypersurfaces of degrees $d_1, \dots, d_n$. Let $p \in \mathbb{P}^n$ be a point, and suppose that the hypersurface X_i has multiplicity m_i at p; suppose moreover that the intersection of the projective tangent cones $\mathbb{P}TC_pX_i$ to X_i at p is empty. Use the description of the Chow ring of the blow-up of $\mathbb{P}^n$ at p to show that the number of points of intersection of the X_i away from p is

$$\#\left(\bigcap (X_i \setminus \{p\})\right) = \prod d_i - \prod m_i.$$

(In other words, the multiplicity $\mathrm{mult}_p(Z_1 \cdot \dots \cdot Z_k)$ is $\prod m_i$.)

Exercise 1.57. Let $X \subset \mathbb{P}^n$ be a hypersurface of degree d. Suppose that X has an ordinary double point (that is, a point $p \in X$ such that the projective tangent cone $\mathbb{P}TC_pX$ is a smooth quadric) and is otherwise smooth. What is the degree of the dual hypersurface $X^* \subset \mathbb{P}^{n*}$?

Exercise 1.58. Let $X \subset \mathbb{P}^n$ be a variety of degree d and dimension k; suppose that $p \in X$ is a point of multiplicity m. Let $B = Bl_p(\mathbb{P}^n)$ be the blow-up of $\mathbb{P}^n$ at the point p, and $\tilde{X} \subset B$ the proper transform of X in B. Find the class $[\tilde{X}] \in A^{n-k}(B)$.

Exercise 1.59. Let $p \in X \subset \mathbb{P}^n$ be as in the preceding exercise, and suppose that the projection map $\pi_p : X \to \mathbb{P}^{n-1}$ is birational onto its image. What is the degree of $\pi_p(X)$?

Exercise 1.60. Show that if X is an irreducible plane cubic with a node, then $c_1 : \mathrm{Pic}(X) \to A_1(X)$ is not a monomorphism as follows. Show that there is no biregular map from X to $\mathbb{P}^1$. Use this to show that if $p \neq q \in X$ are smooth points then the line bundles $\mathcal{O}_X(p)$ and $\mathcal{O}_X(q)$ are non-isomorphic. Show, however, that the zero loci of their unique sections, the points p and q, are rationally equivalent.

Exercise 1.61. Let X be a quadric cone in $\mathbb{P}^3$ and let B be the blowup of X at the origin. Show that B is a smooth surface. Show that if $X \subset \mathbb{P}^3$ a quadric cone with vertex p then $A_1(X) = \mathbb{Z}$, generated by the class of a line, and the image of $c_1 : Pic(X) \to A_1(X)$ is $2\mathbb{Z}$ by showing that the image consists of the subgroup of classes of curves lying on X that have even degree as curves in $\mathbb{P}^3$. In particular, the class of a line on X is not in the image. Do this by showing that any curve C passing through the vertex of X no curve $C \subset X$ of odd degree can be a Cartier divisor on X: if such a curve meets the general line of the ruling of X at δ points away from p and has multiplicity m at p, then intersecting C with a general plane through

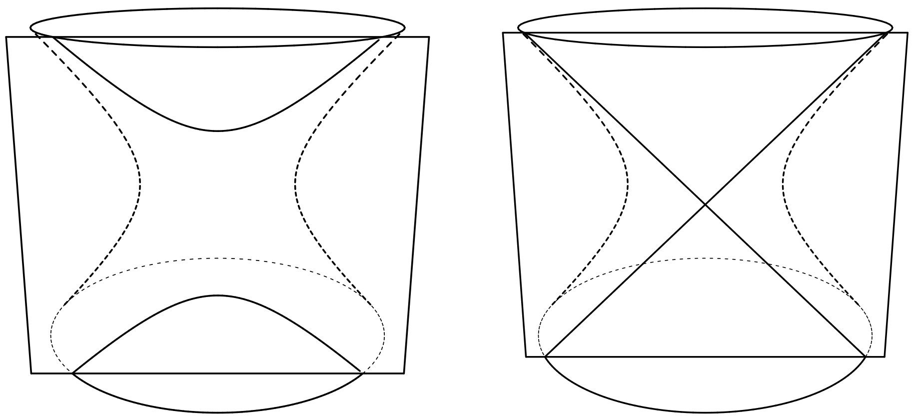


FIGURE 1.16. The diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$ is equivalent to a sum of fibers.

p we see that $\deg(C) = 2\delta + m$; it follows that m is odd, and hence that C cannot be Cartier at p. Thus, the class $[M]$ of a line of the ruling cannot be $c_1(L)$ for any line bundle L.

Exercise 1.62. Show that the Chow ring of a product of projective spaces $\mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_k}$ is

$$\begin{aligned} A(\mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_k}) &= \bigotimes A(\mathbb{P}^{r_i}) \\ &= \mathbb{Z}[\alpha_1, \dots, \alpha_k]/(\alpha_1^{r_1+1}, \dots, \alpha_k^{r_k+1}), \end{aligned}$$

where $\alpha_1, \dots, \alpha_k$ are the pullbacks of the hyperplane classes from the factors. Use this to calculate the degree of the image of the Segre embedding

$$\sigma : \mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_k} \hookrightarrow \mathbb{P}^{(r_1+1)\cdots(r_k+1)-1}$$

corresponding to the multilinear map $V_1 \times \cdots \times V_k \to V_1 \otimes \cdots \otimes V_k$.

Exercise 1.63. For $t \neq 0$, let $A_t : \mathbb{P}^r \to \mathbb{P}^r$ be the automorphism

$$[X_0, X_1, X_2, \dots, X_r] \mapsto [X_0, tX_1, t^2X_2, \dots, t^rX_r].$$

Let $\Phi \subset \mathbb{A}^1 \times \mathbb{P}^r \times \mathbb{P}^r$ be the closure of the locus

$$\Phi^\circ = \{(t, p, q) \mid t \neq 0 \text{ and } q = A_t(p)\}.$$

Describe the fiber of Φ over the point $t = 0$, and deduce once again the formula of Section 1.2.3 for the class of the diagonal in $\mathbb{P}^r \times \mathbb{P}^r$.

In the simplest case, this construction is a rational equivalence between a smooth plane section of a quadric $Q \cong \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ (the diagonal, in terms of suitable identifications of the factors with $\mathbb{P}^1$), and a singular one (the sum of a line from each ruling), as in Figure 1.16.

Exercise 1.64. Let

$$\Psi = \{(p, q, r) \in \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n \mid p, q \text{ and } r \text{ are collinear in } \mathbb{P}^n\}.$$

(Note that this includes all diagonals.) Show that this is a closed subvariety of codimension $n-1$ in $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$.

Exercise 1.65. With $\Psi \subset \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$ as in the preceding exercise, use the method of undetermined coefficients to find the class

$$\psi = [\Psi] \in A^{n-1}\left(\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n\right).$$

Assume characteristic 0, and use Kleiman's theorem as necessary.

Exercise 1.66. To extend the result of the preceding exercise to characteristic p, show in arbitrary characteristic that for a general product of subspaces $\mathbb{P}^i \times \mathbb{P}^j \times \mathbb{P}^k \subset \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$, the intersection $\Psi \cap (\mathbb{P}^i \times \mathbb{P}^j \times \mathbb{P}^k)$ is transverse.

(We will see a way to calculate the class ψ using Porteous' formula in Exercise 14.12)

Exercise 1.67. Suppose that $(F_0, \ldots, F_r)$ and $(G_0, \ldots, G_r)$ are general $(r+1)$-tuples of homogeneous polynomials in $r+1$ variables, of degrees d and e respectively, so that in particular the maps $f : \mathbb{P}^r \to \mathbb{P}^r$ and $g : \mathbb{P}^r \to \mathbb{P}^r$ sending x to $(F_0(x), \ldots, F_r(x))$ and $(G_0(x), \ldots, G_r(x))$ are regular. For how many points $x = (x_0, \ldots, x_r) \in \mathbb{P}^r$ do we have $f(x) = g(x)$?

Exercise 1.68. For $i = 0, \ldots, r+s+1$, let $F_i([X_0, \ldots, X_r], [Y_0, \ldots, Y_s])$ be a general bihomogeneous polynomial of bidegree (d, e). For how many $X = [X_0, \ldots, X_r]$ and $Y = [Y_0, \ldots, Y_s]$ do we have

$$[F_0(X, Y), \ldots, F_r(X, Y)] \sim [X_0, \ldots, X_r]$$

and

$$[F_{r+1}(X, Y), \ldots, F_{r+s+1}(X, Y)] \sim [Y_0, \ldots, Y_s]?$$

The next two exercises set up the following one, which considers when a point $p \in \mathbb{P}^2$ is collinear with its images under several maps:

Exercise 1.69. Consider the locus $\Phi \subset (\mathbb{P}^2)^4$ of fourtuples of collinear points. Find the class $\varphi = [\Phi] \in A^2((\mathbb{P}^2)^4)$ of Φ by the method of undetermined coefficients; that is, by intersecting with cycles of complementary dimension.

Exercise 1.70. With $\Phi \subset (\mathbb{P}^2)^4$ as in the preceding problem, calculate the class $\varphi = [\Phi]$ by using the result of Exercise 1.65 on the locus $\Psi \subset (\mathbb{P}^2)^3$ of triples of collinear points, and considering the intersection of the loci $\Psi_{1,2,3}$ and $\Psi_{1,2,4}$ of fourtuples (p_1, p_2, p_3, p_4) with p_1, p_2, p_3 collinear and with p_1, p_2, p_4 collinear.

Exercise 1.71. Let A, B and $C : \mathbb{P}^2 \to \mathbb{P}^2$ be three general automorphisms. For how many points $p \in \mathbb{P}^2$ are the points $p, A(p), B(p)$ and $C(p)$ collinear? Assume characteristic 0, and use Kleiman's theorem as necessary.

Exercise 1.72. Consider the map $f : \mathbb{P}^1 \to \mathbb{P}^1$ given by

$$f : [X_0, X_1] \mapsto \left[X_0^p, X_1^p\right].$$

Assuming that the ground field has characteristic p, show that the intersection of Γ_f with $\mathbb{P}^1 \times \{x\}$ is non-reduced for every x, but that the class $\gamma_f = [\Gamma_f]$ is $p\alpha + \beta$, just as it would be in characteristic 0.

Exercise 1.73. Let B be the blowup of $\mathbb{P}^n$ at a point p, with exceptional divisor E as in Section 1.2.4. With notation as in that section, show that there is an affine stratification with closed strata Γ_k for $k = 1, \dots n$ and, $E_k := \Gamma_k \cap E$ for $k = 0, \dots, n-1$. Let e_k be the class of E_k. Show that $e_{n-1} = \lambda_{n-1} - \gamma_{n-1}$ to describe the classes γ_k in terms of λ_k and e_k and vice versa. Conclude that the classes $\gamma_k = [\Gamma_k]$ and e_k form a basis for the Chow group $A(B)$.

The next few exercises deal with the blow-up of $\mathbb{P}^3$ along a line. To fix notation, let $\pi : X \to \mathbb{P}^3$ be the blow-up of $\mathbb{P}^3$ along a line $L \subset \mathbb{P}^3$; that is, the graph $X \subset \mathbb{P}^3 \times \mathbb{P}^1$ of the rational map $\pi_L : \mathbb{P}^3 \to \mathbb{P}^1$ given by projection from L; let $\alpha : X \to \mathbb{P}^1$ be projection on the second factor.

Exercise 1.74. Let $H \subset \mathbb{P}^3$ be a plane containing L, and $\tilde{H} \subset X$ its proper transform. Let $J \subset \mathbb{P}^3$ be a plane transverse to L, $\tilde{J} \subset X$ its proper transform (which equals its preimage in X), and let $M \subset J$ be a line not meeting L. Show that the subvarieties

$$X, \quad \tilde{H}, \quad \tilde{J}, \quad \tilde{J} \cap \tilde{H}, \quad M, \quad M \cap \tilde{H}$$

are the closed strata of an affine stratification of X, with open strata isomorphic to affine spaces. In particular, since only one ($M \cap \tilde{H}$) is a point, deduce that $A^3(X) \cong \mathbb{Z}$.

Exercise 1.75. Let $h = [\tilde{H}]$, $j = [\tilde{J}] \in A^1(X)$ and $m = [M] \in A^2(X)$ be the classes of the corresponding strata. Show that

$$h^2 = 0, \quad j^2 = m, \quad \text{and} \quad \deg(jm) = \deg(hm) = 1.$$

Conclude that

$$A(X) = \mathbb{Z}[h, j]/(h^2, j^3 - hj^2).$$

Exercise 1.76. Now let $E \subset X$ be the exceptional divisor of the blow-up, and $e = [E] \in A^1(X)$ its class. What is the class e^2?

Exercise 1.77. Let $\mathbb{P}^5$ be the space of conic curves in $\mathbb{P}^2$.

(a) Find the dimension and degree of the locus of double lines (in characteristic $\neq 2$).

(b) Find the dimension and degree of the locus $\Delta \subset \mathbb{P}^5$ of singular conics (that is, line pairs and double lines).

The following exercises deal with some of the loci in the space $\mathbb{P}^9$ of plane cubics described in Section 1.2.5.

Exercise 1.78. Let $\mathbb{P}^9$ be the space of plane cubics, and $\Gamma \subset \mathbb{P}^9$ the locus of reducible cubics. Let L and $C \subset \mathbb{P}^2$ be a line and a smooth conic intersecting transversely at two points $p, q \in \mathbb{P}^2$; let $L+C$ be the corresponding point of Γ. Show that Γ is smooth at $L+C$, with tangent space

$$\mathbb{T}_{L+C}\Gamma = \mathbb{P}\{\text{homogeneous cubic polynomials } F : F(p) = F(q) = 0\}.$$

Exercise 1.79. Using the preceding exercise, show that if $p_1, \ldots, p_7 \in \mathbb{P}^2$ are general points, and $H_i \subset \mathbb{P}^9$ is the hyperplane of cubics containing p_i, then the hyperplanes $H_1, \ldots, H_7$ intersect Γ transversely—that is, the degree of Γ is the number of reducible cubics through $p_1, \ldots, p_7$.

Exercise 1.80. Calculate the number of reducible plane cubics passing through 7 general points $p_1, \ldots, p_7 \in \mathbb{P}^2$, and hence, by the preceding exercise, the degree of Γ.

Exercise 1.81. We can also calculate the degree of the locus $\Sigma \subset \mathbb{P}^9$ of triangles (that is, totally reducible cubics) directly, as in the series of exercises starting with (1.78). To start, show that if $C = L_1 + L_2 + L_3$ is a triangle with the three vertices—that is, points $p_{i,j} = L_i \cap L_j$ of pairwise intersection—distinct, then Σ is smooth at C with tangent space

$$\mathbb{T}_{L+C}\Sigma = \mathbb{P}\{\text{homogeneous cubic polynomials } F : F(p_{i,j}) = 0 \; \forall\, i, j\}.$$

Exercise 1.82. Using the preceding exercise,

(a) Show that if $p_1, \ldots, p_6 \in \mathbb{P}^2$ are general points, then the degree of Σ is the number of triangles containing $p_1, \ldots, p_6$; and

(b) Calculate this number directly.

Exercise 1.83. Consider a general asterisk—that is, the sum $C = L_1 + L_2 + L_3$ of three distinct lines all passing through a point p. Show that the variety $\Sigma \subset \mathbb{P}^9$ of triangles is smooth at C, with tangent space the space of cubics double at p. Deduce that the space $A \subset \mathbb{P}^9$ of asterisks is also smooth at C.

Exercise 1.84. Let $p_1, \ldots, p_5 \in \mathbb{P}^2$ be general points. Show that any asterisk containing $\{p_1, \ldots, p_5\}$ consists, after possibly relabelling the points, of the sum of the line $L_1 = \overline{p_1\, p_2}$, the line $L_2 = \overline{p_3\, p_4}$ and the line $L_3 = \overline{p_5\, (L_1 \cap L_2)}$.

Exercise 1.85. Using the preceding two exercises, show that for general points $p_1, \ldots, p_5 \in \mathbb{P}^2$, the hyperplanes H_{p_i} intersect the locus $A \subset \mathbb{P}^9$ of asterisks transversely, and calculate the degree of A accordingly.

Exercise 1.86. Show that (in characteristic $\neq 3$) the locus $Z \subset \mathbb{P}^9$ of triple lines is a cubic Veronese surface, and deduce that its degree is 9.

Exercise 1.87. Let $X \subset \mathbb{P}^9$ be the locus of cubics of the form $2L + M$ for L and M lines in $\mathbb{P}^2$.

(a) Show that X is the image of $\mathbb{P}^2 \times \mathbb{P}^2$ under a regular map such that the pullback of a general hyperplane in $\mathbb{P}^9$ is a hypersurface of bidegree $(2, 1)$.

(b) Use this to find the degree of X.

Exercise 1.88. If you try to find the degree of the locus X of the preceding problem by intersecting X with hyperplanes $H_{p_1}, \dots, H_{p_4}$, where

$$H_p = \{C \in \mathbb{P}^9 \mid p \in C\},$$

you get the wrong answer (according to the preceding problem). Why? Can you account for the discrepancy?

Exercise 1.89. Let $\mathbb{P}^2$ denote the space of lines in the plane, and $\mathbb{P}^5$ the space of plane conics. Let $\Phi \subset \mathbb{P}^2 \times \mathbb{P}^5$ be the closure of the locus of pairs

$$\{(L, C) : C \text{ is smooth, and } L \text{ is tangent to } C\}.$$

Show that Φ is a hypersurface; and, assuming characteristic 0, find its class $[\Phi] \in A^1(\mathbb{P}^2 \times \mathbb{P}^5)$.

Exercise 1.90. Now let $\mathbb{P}^9$ be the space of plane cubic curves as before, and let $Y \subset \mathbb{P}^9$ be the closure of the locus of reducible cubics consisting of a smooth conic and a tangent line. Use the result of the first part to determine the degree of B.

Exercise 1.91. Let $\mathbb{P}^{14}$ be the space of quartic curves in $\mathbb{P}^2$, and let $\Sigma \subset \mathbb{P}^{14}$ be the closure of the space of reducible quartics. What are the irreducible components of Σ, and what are their dimensions and degrees?

Exercise 1.92. Find the dimension and degree of the locus $\Omega \subset \mathbb{P}^{14}$ of totally reducible quartics (that is, quartic polynomials that factor as a product of four linear forms).

Exercise 1.93. Again let $\mathbb{P}^{14}$ be the space of plane quartic curves, and let $\Theta \subset \mathbb{P}^{14}$ be the locus of sums of four concurrent lines. Using the result of Exercise 1.69, find the degree of Θ.

Exercise 1.94. Find the degree of the locus $A \subset \mathbb{P}^{14}$ of the preceding problem, this time by calculating the number of sums of four concurrent lines containing six general points $p_1, \dots, p_6 \in \mathbb{P}^2$, assuming transversality.

A natural generalization of the locus of asterisks, or of sums of four concurrent lines, would be the locus, in the space $\mathbb{P}^N$ of hypersurfaces of degree d in $\mathbb{P}^n$, of *cones*. We will indeed be able to calculate the degree of this locus in general, but it will require more advanced techniques than we have at our disposal here; see Section 9.10 for the answer.

Exercise 1.95. More generally, let $\mathbb{P}^N$ be the space of hypersurfaces of degree d in $\mathbb{P}^n$.

(a) Find the dimension and degree of the locus $\Gamma \subset \mathbb{P}^N$ of reducible and/or nonreduced hypersurfaces. (Note that this locus is in general reducible; by its degree we mean the degree of the union of its components of maximal dimension.)

(b) Find the dimension and degree of the locus $\Sigma \subset \mathbb{P}^N$ of totally reducible hypersurfaces.

Exercise 1.96. Let $S \subset \mathbb{P}^3$ be a smooth surface of degree d and $L \subset S$ a line. Calculate the degree of the self-intersection of the class $\lambda = [L] \in A^1(S)$ by considering the intersection of S with a general plane $H \subset \mathbb{P}^3$ containing L.

Exercise 1.97. Let S be a smooth surface. Show that if $C \subset S$ is any curve, and the corresponding point in the Hilbert scheme $\mathcal{H}$ of curves on S lies on a positive-dimension irreducible component of $\mathcal{H}$, then the degree $\deg(\gamma^2)$ of the self-intersection of the class $\gamma = [C] \in A^1(S)$ is nonnegative. Using this and the preceding exercise, prove the statement made in Section 1.4.2 that *a smooth surface $S \subset \mathbb{P}^3$ of degree 3 or more can contain only finitely many lines.*

Exercise 1.98. Let $C \subset \mathbb{P}^3$ be a smooth quintic curve. Show that

(a) if C has genus 2, it must lie on a quadric surface;

(b) if C has genus 1, it cannot lie on a quadric surface; and

(c) if C has genus 0, it may or may not lie on a quadric surface (that is, some rational quintic curves do lie on quadrics and some don't).

Exercise 1.99. Let $C \subset \mathbb{P}^3$ be a smooth quintic curve. Show that C lies on a quadric surface Q and a cubic surface S with intersection $Q \cap S$ consisting of the union of C and a line.

Exercise 1.100. Use the result of Exercise 1.99—showing that a smooth quintic curve of genus 2 is linked to a line in the complete intersection of a quadric and a cubic—to find the dimension of the subset of the Hilbert scheme corresponding to smooth curves of degree 5 and genus 2.

Exercise 1.101. Let $C \subset \mathbb{P}^3$ be a smooth quintic curve of genus 2. Show that C lies on a quadric surface Q and a cubic surface S with intersection $Q \cap S$ consisting of the union of C and a line; and use this to calculate the dimension of the open subset of the Hilbert scheme parametrizing such curves.

Exercise 1.102. Use the Poincaré-Hopf Theorem to compute the topological Euler characteristic of a smooth variety $Y = X_1 \cap X_2 \subset \mathbb{P}^n$ where X_i is a hypersurface of degree d_i.

Exercise 1.103. Let $X \subset \mathbb{A}^4$ be the affine cone defined by $xy - uv = 0$. Show that the conclusion of Theorem 0.2 fails for the subvarieties A, B given by the ideals (x, u) and (y, v).

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2

Introduction to Grassmannians and Lines in $\mathbb{P}^3$

Keynote Questions

(a) Given four general lines $L_1, \dots, L_4 \subset \mathbb{P}^3$, how many lines will meet all four? (Answer on page 103)

(b) Given four curves $C_1, \dots, C_4 \subset \mathbb{P}^3$, of degrees $d_1, \dots, d_4$, how many lines will meet general translates of all four? (Answer on page 105)

(c) If C and $C' \subset \mathbb{P}^3$ are two general twisted cubic curves, how many lines will meet each twice? (Answer on page 108)

(d) If $Q_1, \dots, Q_4 \subset \mathbb{P}^3$ are four general quadric surfaces, how many lines are tangent to all four? (Answer on page 116)

2.1 Enumerative Formulas

The subject matter of this chapter is, as its title implies, Grassmannian varieties. But it's organized around a series of enumerative problems, of which the Keynote Questions above are examples; and so it seems worthwhile to take a moment here to discuss enumerative problems and their relation to the intersection theory described in the preceding chapter.

2.1.1 What are enumerative problems, and how do we solve them?

Enumerative problems in algebraic geometry ask us to describe the set Φ of objects of a certain type satisfying a number of conditions—for example, the set of lines in $\mathbb{P}^3$ meeting each of four given lines, as in Keynote Question (a), or meeting each of four given curves $C_i \subset \mathbb{P}^3$, as in Keynote Question (b). In the most common situation, we expect Φ to be finite, and we ask for its cardinality, whence the name enumerative geometry. Enumerative problems are interesting in their own right, but—as Van der Waerden is quoted as saying in the Introduction—they are also a wonderful way to learn some of the more advanced ideas and techniques of algebraic geometry, which is why they play such a central role in this text.

There are a number of steps common to most enumerative problems, all of which will be illustrated in the examples of this chapter. If we are asked to describe the set Φ of objects of a certain type that satisfy a number of conditions, we typically carry out the following five steps:

• *Find or construct a suitable parameter space $\mathcal{H}$ for the objects we seek.* Suitable, for us, will mean that $\mathcal{H}$ should be projective and smooth, so that we can carry out calculations in the Chow ring $A(\mathcal{H})$. Most importantly, though, the locus $Z_i \subset \mathcal{H}$ of objects satisfying each of the conditions imposed should be a closed subscheme (which means in turn that the set $\Phi = \cap Z_i$ of solutions to our geometric problem will likewise have the structure of a subscheme of $\mathcal{H}$).

In our examples, the natural choice of parameter space is the Grassmannian $G = \mathbb{G}(1,3)$ parametrizing lines in $\mathbb{P}^3$, which we'll construct and describe in Sections 9.13 and 2.2.2 below; as we'll see, it is indeed smooth and projective, of dimension 4. As we'll see in Sections 2.3.1 and 2.4.2, moreover, the locus $\Sigma_C \subset G$ of lines $\Lambda \subset \mathbb{P}^3$ meeting a given curve $C \subset \mathbb{P}^3$ will indeed be a closed subscheme of codimension 1.

• *Describe the Chow ring $A(\mathcal{H})$ of $\mathcal{H}$.* This is what we'll undertake in Section 2.3 below; in the case of the Grassmannian $\mathbb{G}(1,3)$, we'll be able to give a complete description of its Chow ring. In some circumstances, we may have to work with the cohomology ring rather than the Chow ring, as in Chapter 17, or with a subring of $A(\mathcal{H})$ including the classes of the subschemes Z_i.

• *Find the classes $[Z_i] \in A(\mathcal{H})$ of the loci of objects satisfying the conditions imposed.* Thus, in the case of Question (b), we have to determine the class of the locus $Z_i \subset G$ of lines meeting the curve C_i; the answer is given in Section 2.4.2.

• *Calculate the product of the classes found in the preceding step.* If we've done everything correctly up to this point, this should be a straightforward combination of the two preceding steps.

At this point, we have what is known as an *enumerative formula*: it describes the class, in $A(\mathcal{H})$, of the scheme $\Phi \subset \mathcal{H}$ of solutions to our geometric problem, *under the assumption that this locus has the expected dimension and is generically reduced*—that is, the cycles $Z_i \subset \mathcal{H}$ intersect generically transversely. (If the cycles Z_i are all locally Cohen-Macaulay, then by Theorem 5.10, it describes the class of the subscheme $\Phi \subset \mathcal{H}$ under the weaker hypothesis that Φ has the expected dimension; that is, the cycles Z_i are dimensionally transverse.)

• *Verify that the set of solutions, viewed as a subscheme of $\mathcal{H}$, indeed has the expected dimension and investigate its geometry.* We'll discuss, in the following section, what exactly we've proven if we simply stop at the conclusion of the last step. But ideally, we'd like to complete the analysis and say when the cycles $Z_i \subset \mathcal{H}$ do in fact meet generically transversely or dimensionally transversely. In particular, if the geometric problem posed depends on choices—the number of lines meeting each of four curves C_i, for example, depends on the C_i—we'd like to be able to say that for general choices the corresponding scheme Φ is indeed generically reduced.

Thus, for example, in the case of Keynote Question (b) the analysis described above and carried out in Section 2.4.2 will tell us that if the subscheme $\Phi \subset G$ of lines meeting each of four curves $C_i \subset \mathbb{P}^3$ is zero-dimensional, it has degree $2 \cdot \prod \deg(C_i)$. But it does *not* tell us that the actual number of lines meeting each of the four curves is in fact $2 \cdot \prod \deg(C_i)$ for general C_i, or for that matter for any. That is addressed in Section 2.4.2 in characteristic 0, and again in Exercises 2.34-2.37 in positive characteristic.

One reason this last step is sometimes shortchanged is that it's often the hardest. For example, it typically involves knowledge of the local geometry of the subschemes $Z_i \subset \mathcal{H}$—their smoothness or singularity, and their tangent spaces or tangent cones accordingly—and this is usually finer information than their dimensions and classes. But it's necessary, if the result of the first four steps is to give a description of the actual set of solutions; and it's also a great occasion to learn some of the relevant geometry.

2.1.2 The content of an enumerative formula

Because the last step in the process described above is sometimes beyond our reach, it is worth saying exactly what has been proved when we carry out the first four steps in the process of the last section.

In general, the computation of the product $\alpha = \prod[Z_i] \in A(\mathcal{H})$ of the classes of some effective cycles Z_i in a space $\mathcal{H}$ tells us:

(a) If $\alpha \neq 0$ (for example, if $\alpha \in A_0(\mathcal{H})$ and $\deg(\alpha) \neq 0$), *we can conclude that* $\cap_i Z_i$ *is nonempty.* This is the source of many applications of enumerative geometry; for example, it's the basis of the Kempf/Kleiman-Laksov proof of the existence half of Brill-Noether, described in Chapter 17.

(b) If the cycles Z_i intersect in the expected dimension, then the class α is a positive linear combination of the classes of the components of the intersection $\cap Z_i$. In particular, if $\alpha \in A_0(\mathcal{H})$ has dimension 0, then *the number of points of* $\cap Z_i$ *is at most* $\deg(\alpha)$. This in turn implies:

   (i) If $\alpha \in A_0(\mathcal{H})$ and $\deg(\alpha) < 0$, we may conclude that the intersection $\cap_i Z_i$ is infinite rather than finite. More generally, if α is not the class of an effective cycle *we can conclude that* $\cap_i Z_i$ *has dimension greater than the expected dimension.*

   (ii) If $\alpha \in A_0(\mathcal{H})$ and $\deg(\alpha) = 0$, then the intersection $\cap_i Z_i$ must either be empty or infinite. (In general, if $\alpha = 0$ we can conclude that either $\cap_i Z_i = \varnothing$, or $\cap_i Z_i$ has dimension greater than the expected dimension.)

So, suppose we have carried out the first four steps in the process of the preceding section in the case of Keynote Question (a): we've described the Grassmannian $G = \mathbb{G}(1,3)$ and its Chow ring, found the class $\sigma_1 = [Z]$ of the cycle Z of lines meeting a given line $L \subset \mathbb{P}^3$, and calculated that $\deg(\sigma_1^4) = 2$. What does this tell us?

Without a verification of transversality, the formula $\deg \sigma_1^4 = 2$ really only tells us that the number of intersections is either infinite or 1 or 2. Beyond this, it says that *if* the number of "solutions to the problem"—in this case lines in $\mathbb{P}^3$ that meet the four given lines—is finite, then there are two *counted with multiplicity.* In order to say more, we need to be able to say when the intersection $\cap Z_i$ has the expected dimension; we need to be able to detect transversality and, ideally, to calculate the multiplicity of a given solution. (The third of these is often the hardest. For example, in the calculation of the number of lines meeting four given curves $C_i \subset \mathbb{P}^3$, we see in Exercises 2.34-2.37 how to check the condition of transversality; but there is no simple formula for the multiplicity in case the intersection is not transverse.)

A common aspect of enumerative problems is that they themselves may vary with parameters: if we ask how many lines meet each of four curves C_i, the problem varies with the choice of curves C_i. In these situations, a good benchmark of our understanding is whether we can count the actual number of solutions for a *general* such problem: for example, whether we can prove that if $C_1, \dots, C_4$ are general conics, then there will be exactly 32 lines meeting all four. Thus, in most of the examples of enumerative geometry we'll encounter in this book, there are two aspects to the problem. The first is to find the "expected" number of solutions by carrying out the first four steps of the preceding section to arrive at an enumerative formula. The second is to verify transversality—in other words, that the actual cardinality of the set of solutions is indeed this expected number—when the problem is suitably general.

2.2 Introduction to Grassmann varieties

A *Grassmann variety*, or *Grassmannian*, is a projective variety whose closed points correspond to the vector subspaces of a certain dimension in a given vector space. Projective spaces, which parametrize one dimensional (or one-codimensional) subspaces, are the most familiar examples. In this Chapter we will begin the study of Grassmannians in general, and then focus on the geometry and Chow ring of the Grassmannian of lines in $\mathbb{P}^3$ over the complex numbers, the first and most intuitive example beyond projective spaces.

Our goal in doing this is to introduce the reader to some ideas that will be developed in much greater generality (and complexity) in later chapters: the Grassmannian (as an example of parameter spaces), the methods of undetermined coefficients and specialization for computing intersection products more complicated than those mentioned in Chapter 1, and questions of transversality, treated via the tangent spaces to parameter spaces. For more information about Grassmannians the reader may consult the books of Harris [1992] (for basic geometry of the Grassmannian); Griffiths and Harris [1978] for the basics of the Schubert calculus and Fulton [1997] for combinatorial formulas, as well as the classic treatment in the second volume of Hodge and Pedoe [1952].

As a set, we take the Grassmann variety $G = G(k, V)$ to be the set of k-dimensional vector subspaces of the vector space V. We give this set the structure of a projective variety by giving an inclusion in a projective space, called the *Plücker embedding*, and showing that the image is the zero locus of a certain collection of homogeneous polynomials.

We will write $G(k, V)$ for the Grassmann variety of k-dimensional subspaces of V. A k-dimensional sub vector space of an n-dimensional vector space V is the same as a $(k-1)$-dimensional linear subspace of $\mathbb{P}V \cong \mathbb{P}^{n-1}$, so the Grassmannian $G(k, V)$ could also be thought of as parameterizing $(k-1)$-dimensional subspaces of $\mathbb{P}V$. We will write the Grassmannian $G(k, V)$ as $\mathbb{G}(k-1, \mathbb{P}V)$ when we wish to think of it this way. When there's no need to specify the vector space V but only its dimension, say n, we'll write simply $G(k,n)$ or $\mathbb{G}(k-1,n-1)$. Note also that there is a natural identification

$$G(k,V) = G(n-k,V^*)$$

sending a k-dimensional subspace $\Lambda \subset V$ to its annihilator $\Lambda^\perp \subset V^*$.

There are two points of potential confusion in the notation. First, if $\Lambda \subset V$ is a k-dimensional vector subspace of an n-dimensional vector space V, we will often use the same symbol Λ to denote the corresponding point in $G = G(k,V)$. When we need to make the distinction explicit, we'll write $[\Lambda] \in G$ for the point corresponding to the plane $\Lambda \subset V$. Second, when we consider the Grassmannian $G = \mathbb{G}(k, \mathbb{P}V)$ we will sometimes need to work with the corresponding vector subspaces of V. In these circumstances, if $\Lambda \subset \mathbb{P}V$ is a k-plane, we'll write $\tilde{\Lambda}$ for the corresponding $(k+1)$-dimensional vector subspace of V.

2.2.1 The Plücker embedding

To do this, we associate to a k-dimensional subspace $\Lambda \subset V$ the one-dimensional subspace

$$\wedge^k \Lambda \subset \wedge^k V;$$

that is, if Λ has basis $v_1, \dots, v_k$, we associate to it the point of $\mathbb{P}(\wedge^k V)$ corresponding to the line spanned by $v_1 \wedge \cdots \wedge v_k$ Concretely, if we take $V = \mathbb{C}^n$, we may represent Λ as the row space of a $k \times n$ matrix

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{k,1} & a_{k,2} & \dots & \dots & a_{k,n} \end{pmatrix}.$$

If we choose a basis $\{e_1, \dots, e_n\}$ for V, then a basis for $\wedge^k V$ is given by the set of products

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}_{1 \le i_1 < \cdots < i_k \le n},$$

and if $v_1, \dots, v_k$ is a basis for Λ then we may write a nonzero element of $\wedge^k \Lambda$ in the form

$$v_1 \wedge \cdots \wedge v_k = \sum_{1 \le i_1 < \cdots < i_k \le n} p_{i_1,\dots,i_k} e_{i_1} \wedge \cdots \wedge e_{i_k};$$

Here the scalar $p_{i_1,\dots,i_k}$ is the determinant of the submatrix (that is, *minor*) of A made from columns $i_1, \dots, i_k$. These $p_{i_1,\dots,i_k}$ are called the *Plücker coordinates* of Λ.

The matrix A is not unique, since we can multiply on the left by any invertible $k \times k$ matrix Ω without changing the row space, but the collection of $k \times k$ minors of A, viewed as a vector in $\mathbb{C}^{\binom{n}{k}}$, is well-defined up to scalars: multiplying by Ω multiplies each such minor by $\det(\Omega)$. Conversely, if another matrix A' has the same row space, then we can write $A' = \Omega A$ for some invertible $k \times k$ matrix Ω. This shows that the "Plücker embedding" is a well-defined map of sets from $G(k, n)$ to $\mathbb{P}^{\binom{n}{k}-1}$. On the other hand, it is not hard to show that if $v_1, \ldots, v_k$ are independent vectors in V, then a vector v annihilates $v_1 \wedge \cdots \wedge v_k$ in the exterior algebra if and only if v is in the span of $v_1, \ldots, v_k$. This proves that the "Plücker embedding" is one-to-one as a map of sets from $G(k, n)$ to $\mathbb{P}^{\binom{n}{k}-1}$.

A quick-and-dirty way to see that the image $G \hookrightarrow \mathbb{P}(\wedge^k V)$ of the Plücker embedding—the locus of vectors $\eta \in \wedge^k V$ that are expressible as a wedge product $v_1 \wedge \cdots \wedge v_k$ of k vectors $v_i \in V$—is a closed algebraic set is to use the ring structure of the exterior algebra $\wedge V$. Writing out an element $\eta \in \wedge^k V$ in coordinates as above, we see that $e_i \wedge \eta = 0$ if and only if η can be written as $e_i \wedge \eta'$ for some $\eta' \in \wedge^{k-1} V$. Since there is nothing special about the vector $e_i \in V$ we could replace it by any nonzero element $v \in V$. Repeating this idea, we see that an element $0 \neq \eta \in \wedge^k V$ can be written in the form $v_1 \wedge \cdots \wedge v_k$ for some (necessarily independent) $v_1, \ldots, v_k \in V$ if and only if the kernel of the multiplication map

$$V \xrightarrow{\wedge \eta} \wedge^{k+1} V$$

has dimension at least k. That is, the image of the Plücker embedding is

$$G = \{\eta \in \wedge^k V \mid \operatorname{rank}(V \xrightarrow{\wedge \eta} \wedge^{k+1} V) \leq n - k\},$$

and this is the zero locus of the homogeneous polynomials of degree $n-k+1$ on $\wedge^k V$ that are the $n-k+1$-order minors of the map $V \xrightarrow{\wedge \eta} \wedge^{k+1} V$, written out as a matrix.

Once we know that G is an algebraic set, it follows that G is a variety: Its ideal is the kernel of the map of polynomial rings

$$K[p_{i_1,\ldots,i_k}]_{1 \leq i_1 < \cdots < i_k \leq n} \to K[x_{i,j}]_{\substack{1 \leq i \leq k \\ 1 \leq j \leq n}}$$

sending $p_{i_1,\ldots,i_k}$ to the corresponding Plücker coordinate of the generic matrix $(x_{i,j})$, and is thus prime.

Though the equations for G just given have degree $n-k$ the generators of the ideal of homogeneous forms vanishing on $G \subset \mathbb{P}(\wedge^k V)$ are quadratic actually polynomials, known as the *Plücker relations*. We will be able to describe these quadratic polynomials explicitly following Proposition 2.2; for fuller accounts (including a proof that they do indeed generate the homogeneous ideal of $G \subset \mathbb{P}(\wedge^k V)$) and some of their beautiful combinatorial structure, we refer the reader to de Concini et al. [1980] Section 2 or Fulton [1997] Section 9.1.

From here on, we will view $G(k, V)$ as endowed with the structure of a projective variety via the Plücker embedding. As will follow from the description of its covering by affine spaces in the following subsection, it's a smooth variety of dimension $k(n-k)$. The smoothness statement follows in any case from the fact that $GL(V)$ acts transitively on it by linear transformations of the projective space $\mathbb{P}(\wedge^k V)$.

Example 2.1. The first example of a Grassmannian other than projective space is the Grassmannian $G(2,4) = \mathbb{G}(1,\mathbb{P}^3)$. Let V be a 4-dimensional vector space, and consider the Plücker embedding of $G(2,V) = G(2,4) = \mathbb{G}(1,3)$ in $\mathbb{P}\wedge^2 V \cong \mathbb{P}^5$. Since $\dim G(2,4) = 4$, this will be a hypersurface. From the discussion above we know that the equation of $G(2,4)$ in this embedding is a polynomial relation among the minors $p_{i,j}$ of a generic 2×4 matrix

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \end{pmatrix}.$$

One way to obtain this relation is to note that the determinant of the 4×4 matrix with repeated rows

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \end{pmatrix}$$

must be 0. Expanding this determinant as a sum of products of minors of the first two rows and of the last two rows, all of which are Plücker coordinates, we obtain

$$p_{1,2}p_{3,4} - p_{1,3}p_{2,4} + p_{1,4}p_{2,3} = 0. \tag{2.1}$$

As this is an irreducible polynomial (and $\dim G(2,4) = 4$), it generates the homogeneous ideal of $G(2,4) \subset \mathbb{P}^5$, which is thus a smooth quadric.

In fact, for any n, the ideal of the Grassmannian of 2-planes $G(2,n)$ is cut out by quadratic polynomials in the Plücker coordinates similar to the polynomial (2.1) above. More precisely, if $e_1, \dots, e_n$ is a basis of V and $\eta = \sum_{1 \le a < b \le n+1} p_{a,b} e_a \wedge e_b \in \wedge^2 V$, then the polynomials

$$\{g_{a,b,c,d} := p_{a,b}p_{c,d} - p_{a,c}p_{b,d} + p_{a,d}p_{b,c} = 0 \mid 1 \le a < b < c < d \le n+1\}$$

minimally generate the ideal of the Grassmannian. These are the *Plücker relations* in the special case of the Grassmannian $\mathbb{G}(1,n)$ of lines in $\mathbb{P}^n$. We'll describe the Plücker relations in general following Proposition 2.2.

Another way to characterize the collection of polynomials $\{g_{a,b,c,d}\}$ defining $\mathbb{G}(1,n)$, in characteristic not equal to 2, is that they are tantamount to the coefficients of the element $\eta^2 = 0 \in \wedge^4 V$. These coefficients may be characterized (up to a factor of 2) as the *Pfaffians* of a skew-symmetric matrix. See Exercise 2.15 for more information, and for the characteristic 2 case.

Exercises 2.16-2.23 describe a number of aspects of the projective geometry of the Grassmannian in the Plücker embedding.

2.2.2 Covering by affine spaces; local coordinates

Like a projective space, a Grassmann variety $G = G(k, V)$ can be covered by Zariski open subsets isomorphic to affine space. To see this, fix an $(n-k)$-dimensional subspace $\Gamma \subset V$, and let U_Γ be the subset of k-planes that don't meet Γ:

$$U_\Gamma = \{\Lambda \in G \mid \Lambda \cap \Gamma = 0\}.$$

This is a Zariski open subset of G: in fact, if we take $w_1, \dots, w_{n-k}$ to be any basis for Γ and set $\eta = w_1 \wedge \cdots \wedge w_{n-k}$, then we have

$$U_\Gamma = \{[\omega] \in G \subset \mathbb{P} \wedge^k V \mid \omega \wedge \eta \neq 0\}$$

from which we see that U_Γ is the complement of the hyperplane section of G corresponding to the vanishing of a Plücker coordinate. (Caution: not all hyperplane sections of G in the Plücker embedding have this form; see Exercise 2.24.)

We claim now that the open set $U_\Gamma \subset G(k, n)$ is isomorphic to affine space $\mathbb{A}^{k(n-k)}$. To see this, we first choose an arbitrary point $[\Omega] \in U_\Gamma$ that will play the role of the origin; that is, fix a k-plane $\Omega \subset V$ complementary to Γ, so that we have a direct-sum decomposition $V = \Omega \oplus \Gamma$. Any k-dimensional subspace $\Lambda \subset V$ complementary to Γ projects to Γ modulo Ω—call this map π_Γ—and projects isomorphically to Ω mod Γ—call this map π_Ω. Thus Λ is the graph of the linear map

$$\varphi : \Omega \xrightarrow{\pi_\Omega^{-1}} \Lambda \subset V = \Omega \oplus \Gamma \xrightarrow{\pi_\Gamma} \Gamma$$

Conversely, the graph of any map $\varphi : \Omega \to \Gamma$ is a subspace $\Lambda \subset \Omega \oplus \Gamma = V$ complementary to Γ. These two correspondences establish a bijection

$$U_\Gamma \cong \mathrm{Hom}(\Omega, \Gamma) \cong \mathbb{A}^{k(n-k)}.$$

To make this explicit, suppose we choose a basis for V consisting of a basis $e_1, \dots, e_k$ for Ω followed by a basis $e_{k+1}, \dots, e_n$ for Γ. If $\Lambda \in U_\Gamma$ is a k-plane then the preimages $\pi_\Omega^{-1} e_1, \dots, \pi_\Omega^{-1} e_k \in \Lambda$, form a basis for Λ. Thus Λ is the row space of the matrix

$$B = \begin{pmatrix} 1 & 0 & \dots & 0 & a_{1,1} & a_{1,2} & \dots & a_{1,n-k} \\ 0 & 1 & \dots & 0 & a_{2,1} & a_{2,2} & \dots & a_{2,n-k} \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & a_{k,1} & a_{k,2} & \dots & a_{k,n-k} \end{pmatrix}.$$

Where $A = (a_{i,j})$ is the matrix representing the linear transformation $\varphi : \Omega \to \Gamma$ in the given bases. Since there is a unique vector in Λ projecting (mod Γ) to each $e_i \in \Omega$, this matrix representation is unique. The bijection defined above sends $\Lambda \in U_\Gamma$ to the linear transformation $\Omega \to \Gamma$ given by the transpose of the matrix $A = (a_{i,j})$.

If we start with any representation of Λ as the span of the rows of a $k \times n$ matrix B' with respect to the given basis of V, then the Plücker coordinate $p_{1,2,\dots,k}$, which is the determinant of the submatrix consisting of the first k columns of B' will be nonzero. Multiplying B' on the left by the inverse of this submatrix gives us back the matrix B as above, and thus the $k \times k$ minors of B are the $k \times k$ minors of B' multiplied by the inverse of the determinant of the first $k \times k$ minor of B'.

On the other hand, we can realize the entry $a_{i,j}$ of A, up to sign, as a $k \times k$ minor of B: it is (up to sign) the determinant of the $k \times k$ submatrix in which we take all the first k columns except for the i-th, and put in instead the $k + j$-th column. Thus we may write

$$\pm a_{i,j} = \frac{p_{1,\dots,i-1,\hat{i},i+1,\dots,k,k+j}(\Lambda)}{p_{1,\dots,k}(\Lambda)},$$

and this expression shows that $a_{i,j}$ is a regular function on U_Γ. Thus the bijection $U_\Gamma \cong \mathbb{A}^{k(n-k)}$ is a biregular isomorphism.

More generally, it turns out that the ratios $p_{a_1,\dots,a_k}(\Lambda)/p_{1,2,\dots,k}(\Lambda)$ of Plücker coordinates are, up to sign, precisely the determinants of submatrices (of all sizes) of A. To express the result, suppose that I and J are sets of indices. Write A_I^J for the minor of the matrix A involving rows with indices in I and columns with indices in J. Write I' for the complement (in the set of row indices $\{1, \dots, k\}$) of I, and, if $J = \{j_1, \dots, j_t\}$, write $J + k$ for the "translated" set of indices $\{j_1 + k, \dots, j_t + k\}$. With this notation, the $t \times t$ minor A_I^J of A is equal, up to sign, to the $k \times k$ minor of B involving the columns $I' \cup J$. To see this as a regular function on U_Γ we need only divide by the minor involving columns $1, \dots, k$:

Proposition 2.2. *With notation as above, suppose that $I = \{i_1, \dots, i_{k-t}\}$ are row indices and $J = \{j_1, \dots, j_t\}$ are column indices with each $j_i > k$. We have*

$$\pm \det A_I^J = \frac{p_{I' \cup (J+k)}(\Lambda)}{p_{1,\dots,k}(\Lambda)}.$$

****make picture: The matrix B above with a square boxes around the upper left 2x2 submatrix and the lower right $(k-2) \times (k-2)$ submatrix, and caption: the determinant of the $k \times k$ submatrix of B involving columns $1, 2, n-k+3, n-k+4, \dots n$ is $\pm$ the minor of A involving rows and columns $3, \dots, n$.****

Proof. The expression in Plücker coordinates on the right is independent of the matrix representation chosen for Λ, so we may compute the two Plücker coordinates in terms of the matrix B in the form given above, so $p_{1,\dots,k}(\Lambda) = 1$, and $p_{I' \cup (J+k)}(\Lambda)$ is the minor of B involving the columns $I' \cup (J+k)$. Expanding this minor in terms of the $(k-t) \times (k-t)$ minors involving the rows of I', we see that all but the term $\pm 1 \cdot A_I^J$ is zero. □

Having established Proposition 2.2, it is easy to describe the *Plücker relations*, the quadratic polynomials in the Plücker coordinates that cut out $G(k,V) \subset \mathbb{P}(\wedge^k V)$. With notation as above, consider the expansion of any $t \times t$ minor of A along one of its rows or columns. Replacing each factor of each term that appears by the ratio of two Plücker coordinates, with denominator $p_{1,\dots,k}(\Lambda)$, and multiplying through by$p_{1,\dots,k}(\Lambda)^2$, we get a homogeneous quadratic polynomial in the p_I satisfied identically in U_Γ and hence in all of $G(k,V)$. For more information we refer the reader to de Concini et al. [1980] Section 2 or Fulton [1997] Section 9.1.

2.2.3 Universal sub and quotient bundles

In this section and the following we'll introduce the *universal bundles* on the Grassmannian $G(k,n)$ and show how to describe the tangent bundle to $G(k,n)$ in terms of them. These constructions are of fundamental importance in understanding the geometry of Grassmannians.

Let V be an n-dimensional vector space, $G = G(k,V)$ the Grassmannian of k-planes in V, and let $\mathcal{V} := V \otimes \mathcal{O}_G$ be the trivial vector bundle of rank n on G whose fiber at every point is the vector space V. We write $\mathcal{S}$ for the rank k subbundle of $\mathcal{V}$ whose fiber at a point $[\Lambda] \in G$ is the subspace Λ itself; that is,

$$\mathcal{S}_{[\Lambda]} = \Lambda \subset V = \mathcal{V}_{[\Lambda]}.$$

$\mathcal{S}$ is called the *universal subbundle* on G; the quotient $\mathcal{Q} = \mathcal{V}/\mathcal{S}$ is called the *universal quotient bundle*. In case $k = 1$—that is, $G = \mathbb{P}V \cong \mathbb{P}^{n-1}$—the universal subbundle $\mathcal{S}$ is the line bundle $\mathcal{O}_{\mathbb{P}V}(-1)$; similarly, in case $k = n-1$, so that $G = \mathbb{P}V^*$, the universal quotient bundle $\mathcal{Q}$ is the line bundle $\mathcal{O}_{\mathbb{P}V^*}(1)$.

We have said "the rank k subbundle of $\mathcal{V}$ whose fiber at a point $[\Lambda] \in G$ is the subspace Λ itself", and this certainly describes *at most* one bundle, since we have unambiguously defined a subset of $\mathcal{V}$. Who would doubt that is an algebraic vector bundle? To prove this, something more is necessary. Most primitively, we must check that it is trivial on an affine open cover, and that the transition functions are regular on the overlap of any two of the open sets of the cover. Alternately, and equivalently, we may show that the subset $\mathcal{S}$ is an algebraic subset, and that over an open cover it is isomorphic, as algebraic variety, to a trivial bundle. Here is a proof.

Proposition 2.3. *The subset $\mathcal{S}$ of $\mathcal{V}$ whose fiber over a point $[\Lambda] \in G = G(k, V)$ is the subspace $\Lambda \subset V$ is a vector bundle over G.*

Of course it follows that $\mathcal{Q} = \mathcal{V}/\mathcal{S}$ is also a vector bundle.

Proof. Let S be the incidence correspondence

$$S = \{(\Lambda, v) \in G \times V \mid v \in \Lambda\}.$$

The set S is an algebraic subset of $G \times V$, since if we represent Λ by a vector $\eta \in \wedge^k \Lambda \subset \wedge^k V$, it is given by the equation $\eta \wedge v = 0 \in \wedge^{k+1} V$. Explicitly, if Λ is the row space of the matrix A as in Section 2.2.2 then the condition $v \in \Lambda$ is equivalent to the vanishing of the $(k+1)$-st-order minors of the matrix obtained from A' by adjoining v as the $(k+1)^{\text{st}}$ row. These minors can be expressed (by expanding along the new row of A') as bilinear functions in the coordinates of v and the Plücker coordinates, proving that S is an algebraic subset.

Now pick a subspace $\Gamma \subset V$ of dimension $n-k$ and consider the preimage of $U_\Gamma \subset G$. Choosing a complement Ω to Γ as before, we can identify U_Γ with $\mathrm{Hom}(\Omega, \Gamma)$. Moreover if $\Lambda \in U_\Gamma$ then the projection $\beta_{\Omega,\Gamma} : V \to \Omega$ with kernel Γ takes $S_{[\Lambda]} = \Lambda \subset V$ isomorphically to Ω. In other words, this projection gives an isomorphism S_{U_Γ} to the trivial bundle $\Omega \times U_\Gamma$. This proves that S is actually a vector bundle, which we identify as $\mathcal{S}$. □

The following result is the reason that we refer to $\mathcal{S}$ as the "universal" subbundle. A proof may be found in Eisenbud and Harris [2000].

Theorem 2.4. *If X is any scheme then the morphisms $\varphi : X \to G$ are in one-to-one correspondence with rank k subbundles $\mathcal{F} \subset V \otimes \mathcal{O}_X$ in such a way that φ corresponds to the bundle $\mathcal{F} = \varphi^* \mathcal{S}$.* □

There is also a projective analog of the vector bundle $\mathcal{S}$. Thinking of G as $\mathbb{G}(k-1, \mathbb{P}V)$—that is, as parameterizing $(k-1)$-planes in $\mathbb{P}V$—we set

$$\Phi = \{(\Lambda, p) \in G \times \mathbb{P}V \mid p \in \Lambda\}.$$

The space Φ can also be realized as the projectivization of the universal subbundle $\mathcal{S}$, where by the *projectivization* of a vector bundle $\mathcal{E}$ on a scheme X we mean $\mathbb{P}\mathcal{E} := \mathrm{Proj}(\mathrm{Sym}\, \mathcal{E}^*)$—a locally trivial fiber bundle over X whose fiber over a point $p \in X$ is $\mathbb{P}(\mathcal{E}_p)$. (We will deal with projective bundles in general in Chapter 11.) $\Phi = \mathbb{P}\mathcal{S}$ is called the *universal* $(k-1)$-*plane* over G.

Theorem 2.4 may be interpreted by saying that the Grassmannian *represents the functor of families of k-dimensional subspaces of V*, in the sense that the contravariant functor, from schemes to sets, given on objects by $X \mapsto Morph(X, G(k, V))$ is naturally isomorphic to the functor given $X \mapsto \{\text{rank } k \text{ subbundles of } V \times X\}$. Again, with language that we will develop in 8.3, this says that *the Grassmannian* $\mathbb{G}(k-1, \mathbb{P}V)$ *is the Hilbert scheme of* $(k-1)$*-planes in* $\mathbb{P}V$. See Chapter 6 of Eisenbud and Harris [2000] for an introduction to these ideas, and a proof of this statement.

The universal bundles on the Grassmannian G are of fundamental importance in studying the geometry of the Grassmannian, and indeed in the study of vector bundles in general: in the topological or $\mathcal{C}^\infty$ categories it's the case that *any vector bundle* $\mathcal{E}$ *on a space* X *is the pullback of a universal bundle via some map* $\varphi : X \to G$ *to a Grassmannian*; in the algebraic geometry setting, this is true for all bundles generated by global sections.

In Chapter 8 we will show that the family $\Phi \to G$ has a universal property that justifies its name: if $\mathcal{X} \subset B \times \mathbb{P}V$ is a subscheme, flat over B, whose fibers are $(k-1)$-planes in $\mathbb{P}V$, then there is a unique map $\varphi : B \to G$ such that $\mathcal{X} = B \times_G \Phi$ is the pullback of Φ via φ. We will generalize all this to describe the family of $(k-1)$-planes lying on any projective variety X, called a *Fano scheme*, and, still more generally, to the universal family of subschemes of X with given Hilbert polynomial, called a *Hilbert scheme*.

2.2.4 The tangent bundle of the Grassmann variety

Knowledge of the tangent bundle of the Grassmannian is the key to its geometry. It turns out that the tangent bundle can be expressed in terms of the tautological bundles $\mathcal{S}$ and $\mathcal{Q}$:

Theorem 2.5. *The tangent bundle* T_G *to the Grassmannian* $G = G(k, V)$ *is isomorphic to* $\mathrm{Hom}_G(\mathcal{S}, \mathcal{Q})$*, where* $\mathcal{S}$ *and* $\mathcal{Q}$ *are the universal sub and quotient bundles.*

Proof. Consider the open affine set

$$U_\Gamma = \{\Lambda \in G \mid \Lambda \cap \Gamma = 0\},$$

described in Section 2.2.2, where Γ is a subspace of V of dimension $n-k$. Fixing a point $[\Omega] \in U_\Gamma$ and decomposing $V = \Omega \oplus \Gamma$ we get an identification of U_Γ with the vector space $\mathrm{Hom}(\Omega, \Gamma)$ under which the point $[\Omega]$ goes to the linear transformation 0. In particular, the tangent bundle T_G restricted to U_Γ is the trivial bundle, and the fiber over $[\Omega]$ is $\mathrm{Hom}(\Omega, \Gamma)$.

The bundle $\mathcal{S}|_{U_\Gamma}$ is isomorphic to the trivial bundle $\Omega \times U_\Gamma$ by the composite map

$$\mathcal{S}|_{U_\Gamma} \to V \times U_\Gamma \to V/\Gamma \times U_\Gamma = \Omega \times U_\Gamma,$$

and the bundle $\mathcal{Q}|_{U_\Gamma}$ is isomorphic to the trivial bundle $\Gamma \times U_\Gamma$ via the tautological projection $V \otimes \mathcal{O}_G \to \mathcal{Q}$. This gives an identification of fibers, depending on Γ:

$$(T_G)_\Omega = \mathrm{Hom}(\Omega, \Gamma) = \mathrm{Hom}(\mathcal{S}_\Omega, \mathcal{Q}_\Omega).$$

To prove that these identifications extend to an isomorphism $T_G \cong \mathrm{Hom}_G(\mathcal{S}, \mathcal{Q})$ we must check that the gluing map for T_G and that for $\mathrm{Hom}_G(\mathcal{S}, \mathcal{Q})$ on an intersection $U = U_\Gamma \cap U_{\Gamma'}$ containing the point $[\Omega]$ agree on the fiber over $[\Omega]$ (and thus agree as maps of bundles). We may regard $U \subset U_\Gamma = \mathrm{Hom}(\Omega, \Gamma)$ as the set of linear transformations whose graphs do not meet Γ', and this representation is related to the representation of $U \subset U_{\Gamma'}$ by the isomorphisms $\Gamma \xrightarrow{\alpha} V/\Omega \xleftarrow{\beta} \Gamma'$. The gluing

$$d\varphi : (T_G|_{U_\Gamma})|_{U_{\Gamma'}} \xrightarrow{\cong} (T_G|_{U_{\Gamma'}})|_{U_\Gamma}$$

along this set is by the differential of the composite linear transformation

$$\varphi : \mathrm{Hom}(\Omega, \Gamma) \xrightarrow{\alpha} \mathrm{Hom}(\Omega, V/\Omega) \xrightarrow{\beta^{-1}} \mathrm{Hom}(\Omega, \Gamma')$$

induced by these isomorphisms. Of course the differential of a linear transformation is the same linear transformation. The same isomorphisms give the gluing of the bundle $\mathrm{Hom}_G(\mathcal{S}, \mathcal{Q})$. □

From the identification of tangent vectors to $G = G(k, V)$ at Λ with the space $\mathrm{Hom}(\Lambda, V/\Lambda)$ we can see that not all tangent vectors at a given point are alike: we can associate to any tangent vector its *rank*, and this will be preserved under automorphisms of G (see Exercises 2.27 and, for a nice application, 2.26). In particular, this means that when $1 < k < \dim V - 1$, the automorphism group of $G(k, V)$ does not act transitively on nonzero tangent vectors, and hence Kleiman's Theorem 1.9 does not apply in positive characteristic. Nevertheless, the conclusions it gives for intersections of Schubert cycles are correct in all characteristics (and will be proven by a different method).

The Euler sequence on $\mathbb{P}^n$. The isomorphism of Theorem 2.5 is already useful in the case of projective space $\mathbb{P}^n = \mathbb{G}(0, n)$. In this setting Theorem 2.5 gives rise to the *Euler sequence*.

Let V be an $(n+1)$-dimensional vector space, and let $\mathbb{P}^n = \mathbb{P}V$ its projectivization. We consider the quotient map

$$q : U = V \setminus 0 \to \mathbb{P}^n$$

sending a nonzero vector $v \in V$ to the corresponding point $p = [v] \in \mathbb{P}^n$. The tangent space to U at v is the same as the tangent space to V at v, which is to say the vector space V itself, and the kernel of the differential

$$dq_v : T_v U \to T_p \mathbb{P}^n$$

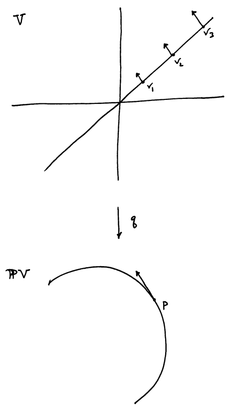


FIGURE 2.05. The differential $dq_v : T_vV = V \to T_p\mathbb{P}V$ varies linearly with $v \in q^{-1}(p)$.

is the one-dimensional subspace $\tilde{p} = \langle v \rangle \subset V$ spanned by v. Thus dq_v induces an isomorphism

$$V/\tilde{p} \xrightarrow{\cong} T_p\mathbb{P}^n,$$

as illustrated in Figure 2.05

This isomorphism does not, however, give a natural identification of the vector spaces $V/\tilde{p}$ and $T_p\mathbb{P}^n$. Even though both these vector spaces depend only on the point $p \in \mathbb{P}^n$, the isomorphism dq_v between them depends on the choice of the vector v. Indeed, if λ is any nonzero scalar, the differential $dq_{\lambda v}$ is equal to dq_v divided by λ. But if $l : \langle v \rangle \to K$ is any linear functional, then the map $l(v) \cdot dq_v$ *is* independent of the choice of v, and so we have a natural identification

$$\langle v \rangle^* \otimes V/\langle v \rangle \xrightarrow{\cong} T_{[v]}\mathbb{P}^n.$$

This is the identification

$$T_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(1) \otimes Q = Hom(\mathcal{O}_{\mathbb{P}^n}(-1), Q) = Hom(\mathcal{S}, \mathcal{Q})$$

asserted (more generally) in Theorem 2.5.

To put it another way, in terms of coordinates $x_0, \ldots, x_n$ on V, a constant vector field $\partial/\partial x_i$ on V does not give rise to a vector field on $\mathbb{P}V$, but the vector field

$$w(x) = x_j \frac{\partial}{\partial x_i}$$

on V does. This gives us a map

$$\mathcal{O}_{\mathbb{P}^n}(1) \otimes V \to T_{\mathbb{P}V},$$

whose kernel is the Euler vector field

$$e(x) = \sum x_i \frac{\partial}{\partial x_i}.$$

The resulting exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}V} \xrightarrow{1 \mapsto e} \mathcal{O}_{\mathbb{P}^n}(1) \otimes V \to T_{\mathbb{P}V} \to 0,$$

is called the *Euler sequence*. To relate this to the identification of the tangent bundle above, start with the universal sequence on $\mathbb{P}V$:

$$0 \to \mathcal{S} \to \mathcal{O}_{\mathbb{P}V} \otimes V \to \mathcal{Q} \to 0.$$

Now tensor with the line bundle $\mathcal{S}^* = \mathcal{O}_{\mathbb{P}V}(1)$; since $\mathcal{S} \otimes \mathcal{S}^* \cong \mathcal{O}_{\mathbb{P}V}$, we arrive at the sequence

$$0 \to \mathcal{O}_{\mathbb{P}V} \to \mathcal{O}_{\mathbb{P}^n}(1) \otimes V \to \mathcal{S}^* \otimes \mathcal{Q} \to 0.$$

By Theorem 2.5 the term on the right is $T_{\mathbb{P}^n}$, and we obtain the Euler sequence again.

2.2.5 Tangent spaces via the universal property

There is another way to approach the tangent space that depends on a pretty and well-known bit of algebra: Let $\mathcal{O}$ be a local ring with maximal ideal $\mathfrak{m}$, and suppose for simplicity that $\mathcal{O}$ contains a copy of its residue field K.

Proposition-Definition 2.6. There are natural one-to-one correspondences between the following sets:

(a) $Hom_K(\mathfrak{m}/\mathfrak{m}^2, K)$ (homomorphisms of K-vector spaces).

(b) $Der_K(\mathcal{O}, K)$ (K-linear derivations; that is, K-vector space homomorphisms $d : \mathcal{O} \to K$ that satisfy Leibniz' rule $d(fg) = fd(g) + gd(f)$.)

(c) $Hom_{K-\text{Algebras}}(\mathcal{O}, K[\epsilon]/(\epsilon^2))$ (homomorphisms of K-algebras).

(d) $Morph(\mathcal{T}, \operatorname{Spec} \mathcal{O})$, where $\mathcal{T} = \operatorname{Spec} K[\epsilon]/(\epsilon^2)$ (morphisms of schemes.)

The first two of these are naturally K-vector spaces, and the correspondence preserves this structure; we regard the other two as equipped with this structure as well. Any of these spaces, with its vector space structure, is called the *Zariski tangent space* of $\mathcal{O}$ (or of $\operatorname{Spec} \mathcal{O}$).

When $\mathcal{O}$ is the local ring $\mathcal{O}_{X,x}$ of a variety (or scheme) at a closed point x, we think of the Zariski tangent space of $\mathcal{O}$ as the Zariski tangent space of X at x; and of course item (d) is the same as the set of morphisms of K-schemes carrying the (unique) point of $\mathcal{T}$—which we will call 0—to x.

Proof. The sets in (c) and (d) are the same by definition. If φ as in c, then $\varphi|_{\mathfrak{m}}$ annihilates $\mathfrak{m}^2$ and induces a vector space homomorphism $\overline{\varphi} : \mathfrak{m}/\mathfrak{m}^2 \to (\epsilon)/(\epsilon^2) \cong K$ as in (a). Similarly, a Derivation d as in (b) induces a K-linear map $\overline{d|_{\mathfrak{m}}} : \mathfrak{m}/\mathfrak{m}^2 \to K$. We leave to the reader the construction of the inverse correspondences. $\square$

When X is a smooth variety and $\mathcal{O} = \mathcal{O}_{X,x}$ is it's local ring at a point, there is a fifth identification of great importance: the set of derivations (b) may be identified with the set of tangent vectors $T_{X,x}$ (send a tangent vector v to the derivative in the direction d.) Thus the Zariski tangent space recovers the ordinary tangent space in this case.

Consider the case $X = G(k, V)$. The tangent space at $x = [\Lambda]$ is, by the argument above, the collection of maps $\mathcal{T} \to G(k, V)$ sending 0 to $[\Lambda]$. By Theorem 2.4, giving a map $\mathcal{T} \to G(k, V)$ is the same as the giving a rank k sub bundles $\mathcal{W}$ of $V \times \mathcal{T}$; the map takes $0 \in \mathcal{T}$ to $[\Lambda] \in \mathbb{G}(k, V)$ if and only if the fiber $\mathcal{W}_0$ is equal to Λ.

We can understand the identification of the tangent space $T_\Lambda G(k, V)$ to the Grassmannian with the space $\mathrm{Hom}(\Lambda, V/\Lambda)$ using this description together with the universal property of the Grassmannian described in Theorem 2.4.

Since $\mathcal{T}$ is affine, a vector bundle over $\mathcal{T}$ is the same as a locally free module over $K[\epsilon]/(\epsilon^2)$. Since this ring is local, Nakayama's Lemma shows that such a module is free (see for example Eisenbud [1995] Exercise 4.11). Thus only the inclusion $\Lambda \times \mathcal{T} \to V \times \mathcal{T}$ varies.

Putting this together, we get a new way to look at the identification of the tangent spaces to the Grassmannian.

Proposition 2.7. *Let $\Lambda \subset V$ be a k-dimensional subspace, and let $\varphi : \Lambda \to V/\Lambda$ be a homomorphism. As an element of the tangent space to the Grassmannian $G(k, V)$ at the point $[\Lambda]$, φ corresponds to the sub free module*

$$\Lambda \otimes F[\epsilon]/(\epsilon^2) \to V \otimes F[\epsilon]/(\epsilon^2) \quad v \otimes 1 \mapsto v \otimes 1 + \varphi'(v) \otimes \epsilon.$$

where $\varphi' : \Lambda \to V$ is any map that, composed with the projection $V \to V/\Lambda$, gives φ.

Proof. Any map $\Lambda \times \mathcal{T} \to V \times \mathcal{T}$ that reduces to the inclusion modulo ϵ has the form $v \otimes 1 \mapsto v \otimes 1 + \varphi'(v) \otimes \epsilon$ for some φ'. If we work in the affine coordinates corresponding to a subspace Γ complementary to Λ and use the splitting $V = \Lambda \oplus \Omega$, then the point $\Lambda \subset V$ corresponds to the matrix

$$B = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \end{pmatrix}.$$

In this matrix representation, φ is represented by the last $n-k$ columns of φ', and taking a different lifting of φ corresponds to making a different choice of the first $k \times k$ block of φ'.

We can do row operations to clear all the ϵ terms from the first $k \times k$ block, adding a multiple of ϵ times certain rows to other rows. This corresponds to composing with to an automorphism of $\Lambda \times \mathcal{T}$, and thus does not change the image of $\Lambda \times \mathcal{T} \to V \times \mathcal{T}$. Since we add after multiplying by ϵ, this does not change the block representing φ. Thus we may assume that the first $k \times k$ block of φ' is 0; equivalently, the first $k \times k$ block corresponding to the map $\Lambda \times \mathcal{T} \to V \times \mathcal{T}$ is the identity. □

2.3 The Chow ring of $\mathbb{G}(1,3)$

Before launching into the geometry of general Grassmannians in the next chapter, we will we will study the geometry of $\mathbb{G}(1,3) = \mathbb{G}(1, \mathbb{P}^3_{\mathbb{C}})$, the Grassmannian of projective lines in $\mathbb{P}^3_{\mathbb{C}}$, the projective three-space over the complex numbers. This is the simplest case beyond the projective spaces. The general results are in many ways similar, but more combinatorics is involved; in the case of lines in $\mathbb{P}^3$ it is possible to visualize more of what is going on. Once the reader has absorbed the case of $\mathbb{G}(1,3)$ the general results will seem more natural.

The reason for the restriction to the complex numbers is to simplify the discussion by using Kleiman's Theorem 5.15 in its characteristic 0 form Theorem 1.9. For convenience, we write out the implication we need:

Corollary 2.8 (Kleiman's theorem for $\mathbb{G}(k, \mathbb{P}^n_{\mathbb{C}})$)**.** *The algebraic group* $\mathrm{GL}(n+1, \mathbb{C})$ *acts transitively on the Grassmannian* $\mathbb{G}(k, \mathbb{P}^n_{\mathbb{C}})$*, and given any subvarieties* $A, B \subset \mathbb{G}(k, \mathbb{P}^n_{\mathbb{C}})$ *there is an open subset* $U \subset \mathrm{GL}(n+1, \mathbb{C})$ *such that for* $g \in U$ *the subvariety* $gA \subset \mathbb{G}(k, \mathbb{P}^n_{\mathbb{C}})$ *is rationally equivalent to* A *and generically transverse to* B*.*

As we have seen, tangent vectors to the Grassmannian correspond to homomorphisms, and thus can have different ranks; it follows that the action of $\mathrm{GL}(n+1)$ on the tangent spaces is *not* transitive, and the conclusion of the Corollary actually fails in characteristic $p > 0$; see the discussion around Theorem 5.15.

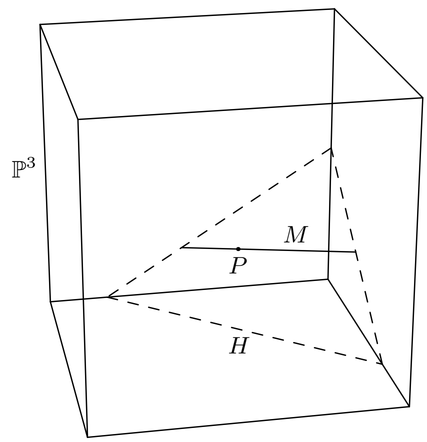


FIGURE 2.2. A complete flag $p \subset M \subset H$ in $\mathbb{P}^3$

Proof. The first statement is clear from the fact that $\mathrm{GL}(n+1,\mathbb{C})$ acts on $\mathbb{C}^{n+1}$, permuting the $(k+1)$-dimensional subspaces transitively. The second statement follows from the first by Theorem 1.9. $\square$

As we observed above, $\mathbb{G}(1,\mathbb{P}^3)$ is realized via the Plücker embedding as the nonsingular quadric in $\mathbb{P}(\wedge^2 V) \cong \mathbb{P}^5$ given by the equation (2.1).

2.3.1 Schubert cycles in $\mathbb{G}(1,3)$

****Notation: we keep Λ for the variable line in $\mathbb{P}^3$; use $p \in L \subset H \subset \mathbb{P}^3$ for the flag. We will have to ask Silvio to change the figures accordingly.****

To start, we fix a *complete flag* $\mathcal{V}$ on $\mathbb{P}^3$; that is, a choice of a point $p \in \mathbb{P}^3$, a line $L \subset \mathbb{P}^3$ containing p, and a plane $H \subset \mathbb{P}^3$ containing L (Figure 2.2).

We can give a decomposition of

$$\mathbb{G}(1,\mathbb{P}^3)$$

as a disjoint union of locally closed subsets by considering the loci of lines $\Lambda \in \mathbb{G}(1,\mathbb{P}^3)$ having specified dimension of intersection with each of the subspaces p, L and H. These are called *Schubert cells* and their closures are called *Schubert cycles*; the classes of these cycles are called, not surprisingly, *Schubert classes*. As we shall see, the Schubert cells form an affine stratification of

$$\mathbb{G}(1,\mathbb{P}^3)$$

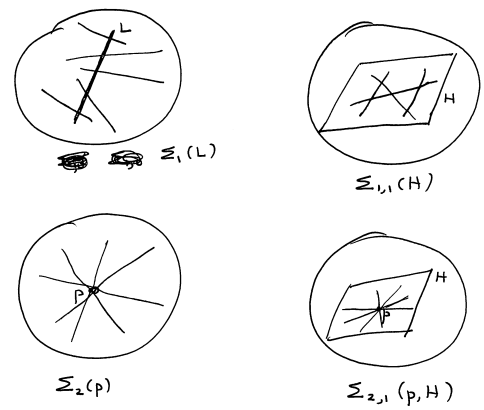


FIGURE 2.3. Schubert cycles in $\mathbb{G}(1,3)$. ****Silvio, please put these in boxes as in fig 2.2****

, and it follows from Theorem **??** that the Schubert classes generate the Chow group $A(\mathbb{G}(1,\mathbb{P}^3))$. Using intersection theory we will be able to show that in fact $A(\mathbb{G}(1,\mathbb{P}^3))$ is a free $\mathbb{Z}$-module having the Schubert classes as free generators. In the following chapter we will see that the same situation is repeated for all the Grassmann varieties.

We begin by naming and describing the Schubert cycles; the four non-trivial ones are illustrated in Figure 2.3:

$$\begin{aligned}
\Sigma_{0,0} &= \mathbb{G}(1,\mathbb{P}^3)\\
\Sigma_{1,0} &= \{\Lambda \mid \Lambda \cap L \neq \varnothing\};\\
\Sigma_{2,0} &= \{\Lambda \mid p \in \Lambda\};\\
\Sigma_{1,1} &= \{\Lambda \mid \Lambda \subset H\};\\
\Sigma_{2,1} &= \{\Lambda \mid p \in \Lambda \subset H\};\ \text{and}\\
\Sigma_{2,2} &= \{\Lambda \mid \Lambda = L\}.
\end{aligned}$$

Thus $\Sigma_{a,b}$ denotes the set of lines meeting the $(2-a)$-dimensional plane of $\mathcal{V}$ in a point, and the $(3-b)$-dimensional plane of $\mathcal{V}$ in a line. This system of indexing may seem peculiar at first, but it has good properties. For example, the codimension of $\Sigma_{a,b}$ is $a+b$, as we will soon see. Also, it behaves well with respect to the pullback maps associated to the inclusions of sub-Grassmannians $G(k,n) \hookrightarrow G(k,n+1)$ and $G(k,n) \hookrightarrow G(k+1,n+1)$, as we'll see in the next chapter when we introduce Schubert cycles and classes on general Grassmannians,.

We often drop the second index when it is 0, writing for example Σ_1 instead of $\Sigma_{1,0}$. When the choice of flag is relevant, we'll sometimes indicate the dependence by writing $\Sigma(\mathcal{V})$, or simply note the dependence on the relevant flag elements by writing, for example, $\Sigma_1(L)$ for the cycle of lines Λ meeting L.

It is easy to see that there are inclusions:

$$\begin{tikzcd}
 & & \Sigma_2 \arrow[dr, hook] & & \\
\{L\} = \Sigma_{2,2} \arrow[r, hook] & \Sigma_{2,1} \arrow[ur, hook] \arrow[dr, hook] & & \Sigma_1 \arrow[r, hook] & \mathbb{G}(1,\mathbb{P}^3). \\
 & & \Sigma_{1,1} \arrow[ur, hook] & &
\end{tikzcd}$$

For each index (a,b) we define the Schubert cell $\tilde{\Sigma}_{a,b}$ to be the complement in $\Sigma_{a,b}$ of the union of all the other Schubert cycles properly contained in $\Sigma_{a,b}$. To show that the $\Sigma_{a,b}$ form an affine stratification, it suffices to show that each $\tilde{\Sigma}_{a,b}$ is isomorphic to an affine space. We will do the most complicated case, leaving the others for the reader (the general case of a Schubert cycle in $\mathbb{G}(k,n)$ is done in Proposition 3.1).

Example 2.9. We will show that the set

$$\begin{aligned}
\tilde{\Sigma}_1 &= \Sigma_1 \setminus (\Sigma_2 \cup \Sigma_{1,1}) \\
&= \{\Lambda \mid \Lambda \cap L \neq \varnothing;\ \text{but } p \notin \Lambda \text{ and } \Lambda \not\subset H\}
\end{aligned}$$

is isomorphic to $\mathbb{A}^3$. Let H' be a general plane containing the point p but not containing the line L. Any line meeting L but not passing through p, and not contained in H, meets H' in a unique point contained in $H' \setminus (H' \cap H)$ (Figure 2.4). Thus we have maps

$$\tilde{\Sigma}_1 \to (L \setminus \{p\}) \cong \mathbb{A}^1 \quad \text{and} \quad \tilde{\Sigma}_1 \to (H' \setminus (H \cap H')) \cong \mathbb{A}^2$$

sending Λ to $\Lambda \cap L$ and $\Lambda \cap H'$ respectively. The product of these maps gives us an isomorphism

$$\tilde{\Sigma}_1 \cong \mathbb{A}^1 \times \mathbb{A}^2 = \mathbb{A}^3.$$

In Proposition 5.17 we will show that the class $[\Sigma_{a,b}] \in A^{a+b}(\mathbb{G}(1,\mathbb{P}^3))$ doesn't depend on the choice of flag. This is because any two flags differ by a transformation in GL_4; we'll denote the class of $\Sigma_{a,b}$ by

$$\sigma_{a,b} = [\Sigma_{a,b}] \in A^{a+b}(\mathbb{G}(1,\mathbb{P}^3)).$$

By Proposition 1.6, the group $A^0(\mathbb{G}(1,\mathbb{P}^3))$ is isomorphic to $\mathbb{Z}$, generated by the fundamental class $\sigma_{0,0} = [\mathbb{G}(1,\mathbb{P}^3)]$; and by Proposition 1.9, the group $A^4(\mathbb{G}(1,\mathbb{P}^3))$ is also isomorphic to $\mathbb{Z}$, generated by the class $\sigma_{2,2}$ of a point in $\mathbb{G}(1,\mathbb{P}^3)$. (In particular any two points in $\mathbb{G}(1,\mathbb{P}^3)$ are linearly equivalent.)

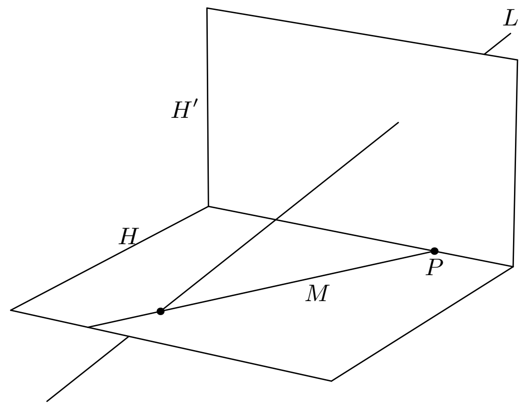


FIGURE 2.4. $L \mapsto (L \cap M, L \cap H')$ defines an isomorphism $\tilde{\Sigma}_1 \to \mathbb{A}^1 \times \mathbb{A}^2 = \mathbb{A}^3$.****In figure, replace L by Λ and M by L. After change of notation in the figure, new caption should be: $\Lambda \mapsto (\Lambda \cap L, \Lambda \cap H')$ defines an isomorphism $\tilde{\Sigma}_1 \to (L \setminus \{p\}) \times (H \setminus H \cap H') \cong \mathbb{A}^1 \times \mathbb{A}^2 \cong \mathbb{A}^3$ ****

2.3.2 Ring structure

We can now determine the structure of the Chow ring of $\mathbb{G}(1,3)$ completely.

Theorem 2.10. *The six Schubert cycles $\sigma_{a,b} \in A^{a+b}(\mathbb{G}(1,\mathbb{P}^3))$ with $0 \leq b \leq a \leq 2$ freely generate $A(\mathbb{G}(1,\mathbb{P}^3))$ as a graded abelian group, and satisfy the multiplicative relations:*

$$\begin{aligned}
\sigma_1^2 &= \sigma_{1,1} + \sigma_2; \quad (A^1 \times A^1 \to A^2)\\
\sigma_1\sigma_{1,1} &= \sigma_1\sigma_2 = \sigma_{2,1}; \quad (A^1 \times A^2 \to A^3)\\
\sigma_1\sigma_{2,1} &= \sigma_{2,2}; \quad (A^1 \times A^3 \to A^4)\\
\sigma_{1,1}^2 &= \sigma_2^2 = \sigma_{2,2};\ \sigma_{1,1}\sigma_2 = 0 \quad (A^2 \times A^2 \to A^4)
\end{aligned}$$

From these formulas we deduce $\sigma_1^3 = 2\sigma_{2,1}$, $s_1^4 = 2\sigma_{2,2}$, and $\sigma_1^2\sigma_{1,1} = \sigma_1^2\sigma_2 = \sigma_{2,2}$. Since $\dim(\mathbb{G}(1,\mathbb{P}^3)) = 4$ any product that would have degree > 4, such as $\sigma_2\sigma_{2,1}$, is 0.

Proof of Theorem 2.10. Because the degree map $\deg : A^4 \to \mathbb{Z}$ sends the class $\sigma_{2,2}$ of a point to 1, it follows that $A^4(\mathbb{G}(1,3)) = \mathbb{Z}$. The free generation of the other groups follows from the formulas for the intersections of cycles of complementary dimension above: first, from $\sigma_1\sigma_{2,1} = \sigma_{2,2}$ we see that σ_1 and $\sigma_{2,1}$ are free generators of $A^1(\mathbb{G}(1,3))$. Similarly, the formulas show that the matrix of the intersection pairing on $\sigma_{1,2}$ and s_2 is nonsingular, so $A^2(\mathbb{G}(1,3))$ is freely generated by these two classes.

It remains to prove the formulas. We will consider the intersections of pairs of cycles, taking these with respect to generically situated flags $\mathcal{V}, \mathcal{V}'$. To simplify the notation we will henceforth write $\Sigma_{a,b}$ and $\Sigma'_{a,b}$ for $\Sigma_{a,b}(\mathcal{V})$ and $\Sigma_{a,b}(\mathcal{V}')$, respectively. By Corollary 2.8, $\Sigma_{a,b}$ and $\Sigma'_{a',b'}$ are generically transverse.

We begin with the case of cycles of complementary dimension, starting with the intersection number of σ_2 with itself. By generic transversality we have

$$\sigma_2^2 = \#\big(\Sigma_2 \cap \Sigma_2'\big) \cdot \sigma_{2,2};$$

and since the intersection

$$\Sigma_2 \cap \Sigma_2' = \{\Lambda \mid p \in \Lambda \text{ and } p' \in \Lambda\};$$

consists of one point (corresponding to the unique line $\Lambda = \overline{p\,p'}$ through p and p'), we conclude that

$$\sigma_2^2 = \sigma_{2,2}.$$

Similarly,

$$\sigma_{1,1}^2 = \#\big(\Sigma_{1,1} \cap \Sigma_{1,1}'\big) \cdot \sigma_{2,2};$$

since

$$\Sigma_{1,1} \cap \Sigma_{1,1}' = \{\Lambda \mid \Lambda \subset H \text{ and } \Lambda \subset H'\}$$

consists of the unique line $\Lambda = H \cap H'$, we conclude that

$$\sigma_{1,1}^2 = \sigma_{2,2}$$

as well. On the other hand, $\Sigma_2 = \{\Lambda \mid p \in \Lambda\}$ and $\Sigma_{1,1}' = \{\Lambda \mid \Lambda \subset H'\}$ are disjoint, since $p \notin H'$, so that

$$\sigma_2 \sigma_{1,1} = 0.$$

Finally

$$\Sigma_1 \cap \Sigma_{2,1}' = \{\Lambda \mid \Lambda \cap L \neq \varnothing \text{ and } p' \in \Lambda \subset H'\}.$$

Since L will intersect H' in one point q, and any line Λ satisfying all the above conditions can only be the line $\overline{p'\,q}$ (Figure 2.5), this intersection is again a single point. Thus

$$\sigma_1 \sigma_{2,1} = \sigma_{2,2}$$

as well.

We now turn to the intersections of cycles whose codimensions sum to less than 4. The intersection $\Sigma_1 \cap \Sigma_2'$ is the locus of lines Λ meeting L and containing the point p', which is to say the Schubert cycle $\Sigma_{2,1}$ with respect to a flag containing the point p' and the plane $\overline{p'\,L}$, so we have $\sigma_1\sigma_2 = \sigma_{2,1}$. In similar fashion, the intersection of Σ_1 and $\Sigma_{1,1}'$ is a cycle of the form $\Sigma_{2,1}$, with respect to a certain flag; specifically, it is the locus of lines containing the point $L \cap H'$ and lying in H', so that $\sigma_1\sigma_{1,1} = \sigma_{2,1}$.

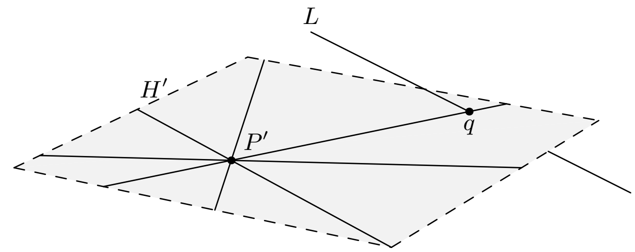


FIGURE 2.5. $\Sigma_1(L) \cap \Sigma'_{2,1}(p', H') = \{\overline{p'\,q}\}$

The last and most interesting computation to be made is the product σ_1^2. The locus $\Sigma_1 \cap \Sigma'_1$ of lines meeting each of the two general lines L and L' is not a Schubert cycle, but since $A^2(\mathbb{G}(1, \mathbb{P}^3))$ is generated by $\sigma_{1,1}$ and s_2, we may write

$$\sigma_1^2 = \alpha\sigma_2 + \beta\sigma_{1,1}, \tag{2.2}$$

Because, having proven the formulas for the intersections of pairs of cycles of complementary dimensions, we know that $A^2(\mathbb{G}(1, \mathbb{P}^3))$ actually free on the classes $\sigma_{1,1}$ and s_2, the coefficients are unique, and we can determine them by intersecting a cycle representing σ_1^2 with convenient cycles. This is called the *method of undetermined coefficients.*

We can do this by invoking the associativity of $A(\mathbb{G}(1, \mathbb{P}^3))$ and the previous calculations: we have

$$(\alpha\sigma_2 + \beta\sigma_{1,1})\sigma_2 = \sigma_1^2 \cdot \sigma_2 = \sigma_1(\sigma_1\sigma_2) = \sigma_1\sigma_{2,1} = \sigma_{2,2}$$

and since $\sigma_2^2 = 0$ and $\sigma_{1,1}\sigma_2 = \sigma_{2,2}$ we get $\beta = 1$. Similarly, from

$$(\alpha\sigma_2 + \beta\sigma_{1,1})\sigma_{1,1} = \sigma_1^2 \cdot \sigma_{1,1} = \sigma_1(\sigma_1\sigma_{1,1}) = \sigma_1\sigma_{2,1} = \sigma_{2,2}$$

and $\sigma_{1,1}^2 = 0$ we see that $a = 1$. In sum, we have

$$\sigma_1^2 = \sigma_2 + \sigma_{1,1},$$

and this completes our description of the Chow ring $A(\mathbb{G}(1, \mathbb{P}^3))$. $\square$

It is instructive to make the computations of $\sigma_1^2\sigma_2$ and $\sigma_1^2\sigma_{1,1}$ at the end of the proof above geometrically, without invoking associativity. To determine α we used

$$\sigma_1^2 \cdot \sigma_2 = (\alpha\sigma_2 + \beta\sigma_{1,1}) \cdot \sigma_2 = \alpha\sigma_{2,2}.$$

By generic transversality we have

$$\alpha = \#\left\{\Lambda \mid \begin{array}{c} \Lambda \cap L \neq \varnothing; \\ \Lambda \cap L' \neq \varnothing;\ \text{and} \\ p'' \in \Lambda \end{array}\right\}$$

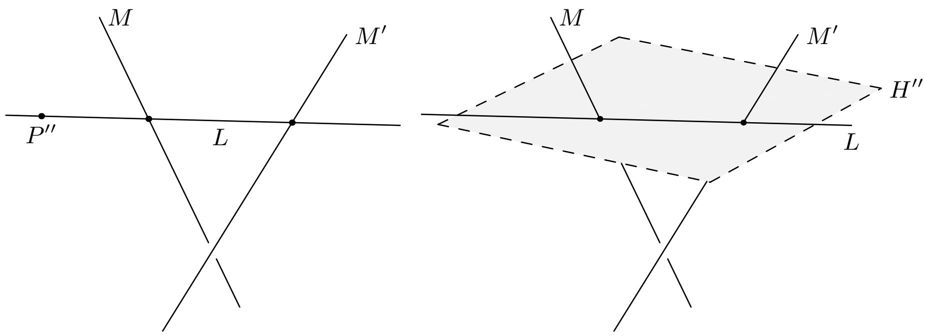


FIGURE 2.6. $\Sigma_2(p'')\cap\Sigma_1(M)\cap\Sigma_1(M') = \{L\}$; $\Sigma_{1,1}(H'')\cap\Sigma_1(M)\cap\Sigma_1(M') = \{L\}$

for L and L' general lines and p'' a general point in $\mathbb{P}^3$. Any line Λ satisfying the three conditions must lie in each of the planes $\overline{p''\,L}$ and $\overline{p''\,L'}$, and so must be their intersection; thus $\alpha = 1$.

Similarly, to determine β we used:

$$\sigma_1^2 \cdot \sigma_{1,1} = (\alpha\sigma_2 + \beta\sigma_{1,1}) \cdot \sigma_{1,1} = \beta\sigma_{2,2}.$$

Again by generic transversality we get:

$$\beta = \#\left\{ \Lambda \mid \begin{array}{c} \Lambda \cap L \neq \varnothing; \\ \Lambda \cap L' \neq \varnothing; \text{ and} \\ \Lambda \subset H'' \end{array} \right\}$$

for L and L' general lines and H'' a general plane in $\mathbb{P}^3$. The only line Λ satisfying these conditions is the line joining the points $L\cap H''$ and $L'\cap H''$, so again $\beta = 1$ (Figure 2.6).

Tangent spaces to Schubert cycles. The generic transversality of the cycles $\Sigma_{a,b}$ and $\Sigma_{a',b'}$, guaranteed by Corollary 2.8, played an essential role in the computation above. By describing the tangent spaces to the Schubert cycles, we can prove this transversality directly. This tangent space computation is in any case necessary to prove the results of this chapter in characteristic p, where we cannot use Kleiman's Theorem.

We'll carry this out here for the intersection $\Sigma_2 \cap \Sigma_2'$. Tangent spaces to other Schubert cycles in $\mathbb{G}(1,3)$ are described in Exercises 2.29 and 2.30; they will be treated in general in Theorem 3.1. The key identification is given in the following:

Proposition 2.11. *Let $\Sigma = \Sigma_2(p)$ be the Schubert cycle of lines in $\mathbb{P}^3 = \mathbb{P}(V)$ that contain p, and suppose that $L \in \Sigma_2(p)$. Writing $\tilde{L} \subset V$ for the two-dimensional subspace corresponding to L, and identifying $T_L\mathbb{G}(1,\mathbb{P}^3)$*

with $\mathrm{Hom}(\tilde{L}, V/\tilde{L})$, *we have*

$$T_L\Sigma = \{\varphi \mid \varphi(\tilde{p}) = 0\}.$$

Given Proposition 2.11, it follows immediately that for general $p, p' \in \mathbb{P}^3$ the cycles $\Sigma_2(p)$ and $\Sigma_2(p')$ meet transversely: if $p \neq p'$, then at the unique point $L = \overline{p\,p'}$ of intersection of the Schubert cycles $\Sigma = \Sigma_2(p)$ and $\Sigma' = \Sigma_2(p')$ we have

$$T_[L]\Sigma \cap T_[L]\Sigma' = \{\varphi \mid \varphi(\tilde{p}) = \varphi(\tilde{p}') = 0\} = 0$$

since $\tilde{p}$ and $\tilde{p}'$ span $\tilde{L}$.

Proof of Proposition 2.11. We choose a subspace $\Gamma \subset V$ complementary to $\tilde{L}$ and identify the open subset $U_\Gamma = \{\Lambda \in \mathbb{G}(1, \mathbb{P}^3) \mid \Lambda \cap \Gamma = \{0\}\}$ with the vector space $\mathrm{Hom}(\tilde{L}, \Gamma)$ by thinking of a 2-plane $\Lambda \in U_\Gamma$ as the graph of a linear map from $\tilde{L}$ to Γ, just as in the beginning of the proof of Theorem 2.5. It is immediate from the identification that $U_\Gamma \cap \Sigma_2$ is the linear space in $\mathrm{Hom}(\tilde{L}, \Gamma)$ consisting of maps φ such that $\tilde{p} \subset \mathrm{Ker}(\varphi)$. Thus its tangent space at a point $[L]$ has this description too. □

As a consequence of Theorem 2.10, we have the following description of the Chow ring of $\mathbb{G}(1, 3)$:

Corollary 2.12.

$$A(\mathbb{G}(1, \mathbb{P}^3)) = \frac{\mathbb{Z}[\sigma_1, \sigma_2]}{(\sigma_1^3 - 2\sigma_1\sigma_2,\ \sigma_1^2\sigma_2 - \sigma_2^2)}.$$

We will generalize this to the Chow ring of any Grassmannian, and prove it by applying the theory of Chern classes, in Chapter 7. A point to note is that the given presentation of the Chow ring has the same number of generators as relations—that is, given that the Chow ring $A(\mathbb{G}(1, 3))$ has Krull dimension 0, it is a *complete intersection*. The analogous statement is true for all of the Grassmann varieties.

2.4 Lines and curves in $\mathbb{P}^3$

In this section and the next we present several applications of the computations above.

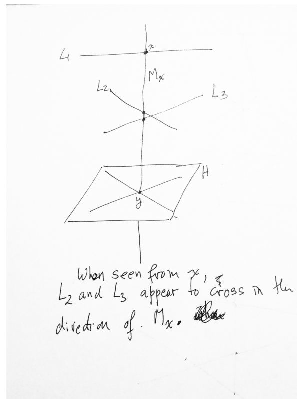


FIGURE 2.7. When looking from x the lines L_2 and L_3 appear to cross at a point in the direction M_x .

2.4.1 How many lines meet four general lines?

This is Keynote Question (a). Since σ_1 is the class of the lines meeting 1 line, and generic translates of $\Sigma_1(L)$ are generically transverse, the number is

$$\deg \sigma_1^4 = 2.$$

We can see the geometry behind this computation—and answer more refined questions about the situation—as follows. Suppose that the lines are $L_1, \dots, L_4$, and consider first the lines that meet just the first 3. If we project L_2 and L_3 from a point $x \in L_1$, to a plane $H \subset \mathbb{P}^3$ we get two general lines in H; and these lines meet in a unique point y. The line $M_x := \overline{x\,y}$ is the unique line in $\mathbb{P}^3$ containing x and meeting L_2 and L_3 as well as L_1. (Informally: if we look at L_2 and L_3, sighting from the point x, we see an "apparent crossing" in the direction of the line M_x—see Figure 2.7.) Moreover, if $x \neq x'$ then the lines $M_x, M_{x'}$ are disjoint: if they had a common point, they would lie in a plane, and all three of L_1, L_2, L_3 would be coplanar, contradicting our hypothesis of generality.

The union of the lines M_x is a surface that we can easily identify: There is a 3-dimensional family of quadratic forms on each $L_i \cong \mathbb{P}^1$. Each restriction map $H^0\mathcal{O}_{\mathbb{P}^3}(2) \to H^0\mathcal{O}_{L_i}$ is linear, so its kernel has codimension ≤ 3. Since there is a $10 = (3 \times 3 + 1)$-dimensional vector space of quadratic forms on $\mathbb{P}^3$, there is at least one quadric surface Q containing L_1, L_2 and L_3. By Bézout's Theorem, any line meeting each of L_1, L_2, L_3, and thus meeting Q at least 3 times, must be contained in Q. If Q were reducible it would be the union of two planes, and two out of every three lines on Q would

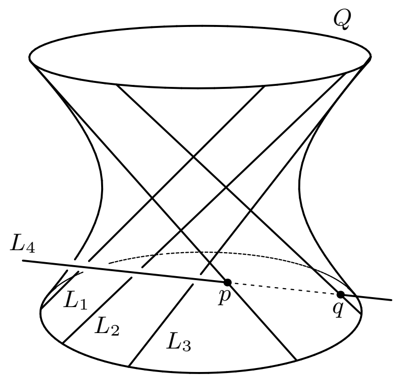


FIGURE 2.8. Two lines that meet each of $L_1, \ldots, L_4$

meet, so Q is irreducible, and it follows that Q is the disjoint union

$$\coprod_{x \in L_1} M_x = Q.$$

Since the degree of Q is 2, and L_4 is general, L_4 meets Q in two distinct points p, q; and the two lines M_x passing through p and q are the unique lines meeting all of $L_1, \ldots, L_4$ (Figure 2.8). We see again that the answer to our question is: 2.

For which sets of 4 lines are there more or fewer than 2 distinct lines meeting all 4? (It follows from the general theory that if the number is finite it must be 1 (and the line counts with multiplicity 2 in the product σ_1^4 or 2.) The geometric construction above will enable the reader to answer this question; see Exercises 2.32 and 2.33.

2.4.2 Lines meeting a curve of degree d

We do not know a geometric argument such as the one for four lines above that would enable us to answer the corresponding question for four curves, Keynote Question b; In this case intersection theory is essential. The basic computation is the following.

Proposition 2.13. *If $C \subset \mathbb{P}^3$ is a curve of degree d, then the class of the locus Γ_C of lines meeting C,*

$$\Gamma_C := \{[L] \in \mathbb{G}(1, \mathbb{P}^3) \mid L \cap C \neq \varnothing\}$$

is $d\sigma_1$.

Proof. To see that Γ_C is a divisor, consider the incidence correspondence $\Sigma \subset \mathbb{G}(1, \mathbb{P}^3) \times C$ whose points are $\{(p, L) \mid p \in C, L \text{a line containing } p\}$.

The fibers of Σ under the projection to $\mathbb{C}$ are all projective planes, so Σ is irreducible of dimension 3. On the other hand, the projection to Γ_C is generically one-to-one, so Γ_C is also irreducible of dimension 3.(See Exercise 2.21 for a generalization.) Let γ_C be the class of Γ_C in $A^1(\mathbb{G}(1,\mathbb{P}^3))$, and write$\gamma_C = \alpha \cdot \sigma_1$ for some $\alpha \in \mathbb{Z}$. To determine α, we intersect both sides with the class $\sigma_{2,1}$ and get

$$\deg \gamma_C \cdot \sigma_{2,1} = \alpha \deg(\sigma_1 \cdot \sigma_{2,1}) = \alpha.$$

If (p, H) is a general pair consisting of a point $p \in \mathbb{P}^3$ and a plane $H \subset \mathbb{P}^3$ containing it, the Schubert cycle

$$\Sigma_{2,1}(p, H) = \{L \mid p \in L \subset H\}$$

will intersect the cycle Γ_C transversely by Corollary 2.8. (This can be proven in all characteristics by using the description of the tangent spaces to $\Sigma_1(p, H)$ in Exercise 2.30 and of the tangent spaces to Γ_C in Exercise 2.34.) Therefore

$$\begin{aligned} \alpha &= \#\big(\Gamma_C \cap \Sigma_{2,1}(p, H)\big) \\ &= \#\{L \mid p \in L \subset H \text{ and } L \cap C \neq \varnothing\}. \end{aligned}$$

To evaluate this number, note that H (being general) will intersect C transversely in d points $\{q_1, \ldots, q_d\}$; and, $p \in H$ being general, no two of the points q_i will be collinear with p. Thus the intersection $\Gamma_C \cap \Sigma_{2,1}(p, H)$ will consist of the d lines $\overline{p\, q_i}$, as in Figure 4.7. It follows that $\alpha = d$, so

$$\gamma_C = d \cdot \sigma_1.$$

□

Proposition 2.13 makes it easy to answer Keynote Question (b). For if $C_1, \ldots, C_4 \subset \mathbb{P}^3$ are general translates of curves of degrees $d_1, \ldots, d_4$ then the cycles Γ_{C_i} are generically transverse by Corollary 2.8, so the number of lines meeting all four is

$$\deg \prod_{i=1}^{4} [\Gamma_{C_i}] = \deg \prod_{i=1}^{4} (d_i \sigma_1) = 2 \prod_{i=1}^{4} d_i.$$

One can verify the necessary transversality by using our description of the tangent spaces, too; as a bonus, we can see exactly when transversality fails. This is the content of Exercises 2.34-2.37.

2.4.3 Chords to a space curve

Consider now a smooth, nondegenerate space curve $C \subset \mathbb{P}^n$, of degree d and genus g. We can define the locus $S_1(C) \subset \mathbb{G}(1, n)$ of *chords*, or *secant*

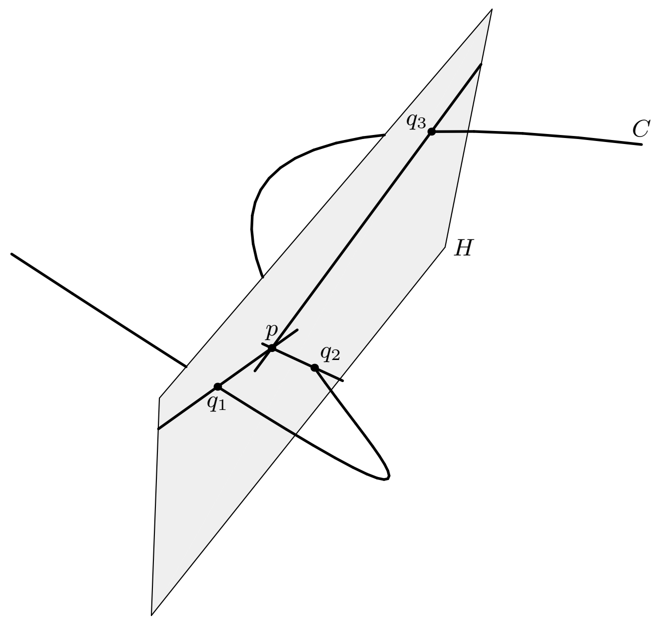


FIGURE 4.7. The intersection of Γ_C with $\Sigma_{2,1}(p, H)$

lines to C in either of two ways. We can consider the rational map $\varphi : C^{(2)} \to \mathbb{G}(1,n)$ from the symmetric square $C^{(2)}$ of C to the Grassmannian $\mathbb{G}(1,n)$ sending a pair of distinct points p, q to the line $\overline{p\,q}$, and take the variety $S_1(C) \subset \mathbb{G}(1,n)$ to be the (closed) image. Alternatively, we can define $S_1(C)$ to be the locus of lines $L \subset \mathbb{P}^n$ such that the scheme-theoretic intersection $L \cap C$ has degree at least 2. As we'll see in Exercise 2.43, these definitions (or their analogs) differ when we consider singular curves, or (as we'll see in Exercise 2.44) higher-dimensional secant planes to curves; but for smooth curves in $\mathbb{P}^n$ we'll show in Exercise 2.42 they agree, and we can adopt either one. (For much more about secant planes to curves in general, see the discussion in Section 12.3.)

Let's now restrict ourselves to the case $n = 3$ of smooth, nondegenerate space curves $C \subset \mathbb{P}^3$, and ask: what's the class, in $A^2(\mathbb{G}(1,\mathbb{P}^3))$, of the locus $S_1(C)$ of secant lines to C? We can answer this question by intersecting with Schubert cycles of complementary codimension (in this case, codimension 2). We know that

$$[S_1(C)] = \alpha\sigma_2 + \beta\sigma_{1,1}$$

for some integers α and β. To find the coefficient β we take a general plane $H \subset \mathbb{P}^3$, and consider the Schubert cycle

$$\Sigma_{1,1}(H) = \{L \in \mathbb{G}(1,\mathbb{P}^3) \mid L \subset H\}.$$

Assuming transversality (or, in characteristic 0, using Kleiman's Theorem),

$$\begin{aligned} \#\left(\Sigma_{1,1}(H) \cap S_1(C)\right) &= \deg\bigl(\sigma_{1,1} \cdot [S_1(C)]\bigr) \\ &= \deg\bigl(\sigma_{1,1} \cdot (\alpha\sigma_2 + \beta\sigma_{1,1})\bigr) \\ &= \beta. \end{aligned}$$

The cardinality of this intersection is easy to determine: the plane H will intersect C in d points $p_1, \dots, p_d$, no three of which will be collinear (Arbarello et al. [1985], Section 3.1), so that there will be exactly $\binom{d}{2}$ lines $\overline{p_i\,p_j}$ joining these points pairwise; thus

$$\beta = \binom{d}{2}$$

(Figure 2.10).

Similarly, to find α we let $p \in \mathbb{P}^3$ be a general point and

$$\Sigma_2(p) = \{L \in \mathbb{G}(1,\mathbb{P}^3) \mid p \in L\};$$

we have as before

$$\begin{aligned} \#\left(\Sigma_2(p) \cap S_1(C)\right) &= \deg\left(\sigma_2 \cdot [S_1(C)]\right) \\ &= \deg\bigl(\sigma_2 \cdot (\alpha\sigma_2 + \beta\sigma_{1,1})\bigr) \\ &= \alpha. \end{aligned}$$

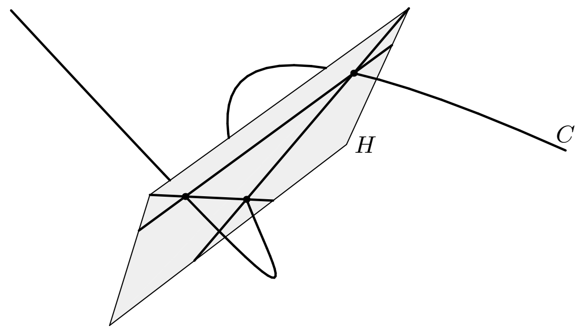


FIGURE 2.10. $\Sigma_{1,1}(H) \cap S_1(C)$ consists of $\binom{\deg C}{2}$ lines.

To count this intersection—that is, the number of chords to C through the point p—consider the projection $\pi_p : C \to \mathbb{P}^2$. This map is birational onto its image $\overline{C} \subset \mathbb{P}^2$, which will be a curve having only nodes as singularities (see Exercise 2.38), and those nodes correspond exactly to the chords to C through p. (These chords were classically called the *apparent nodes* of C (Figure 2.11): if you were looking at C with your eye at the point p, and had no depth perception, they're the nodes you would see.) By the genus formula for singular curves (Section 1.4.4), this number is

$$\alpha = \binom{d-1}{2} - g.$$

Thus we have proven:

Proposition 2.14. *If $C \subset \mathbb{P}^3_{\mathbb{C}}$ is a smooth nondegenerate curve of degree d and genus g, then the class of the locus of chords to C is*

$$[S_1(C)] = \left(\binom{d-1}{2} - g\right)\sigma_2 + \binom{d}{2}\sigma_{1,1} \in A^2(\mathbb{G}(1, \mathbb{P}^3_{\mathbb{C}}).$$

We can use this to answer the third of the keynote questions of this chapter: if C and C' are general twisted cubic curves, Then by Corollary 2.8 the cycles $S = S_1(C)$ and $S' = S_1(C')$ will intersect transversely. since the class of each is $\sigma_2 + 3\sigma_{1,1}$, we have

$$\begin{aligned} \#(S \cap S') &= \deg(\sigma_2 + 3\sigma_{1,1})^2 \\ &= 10. \end{aligned}$$

Exercises 2.45 and 2.46 explain how to use the tangent space to the Grassmannian to prove generic transversality, and thus verify this result, in all characteristics.

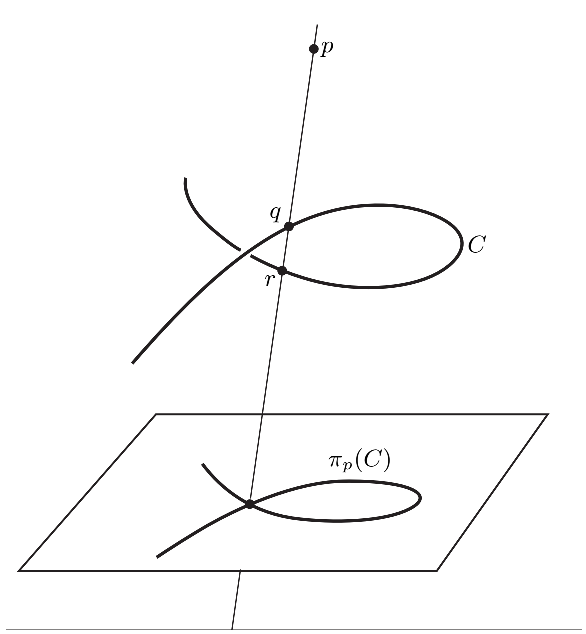


FIGURE 2.11. An “apparent node”.

2.4.4 Specialization

There is a another powerful approach to evaluating the intersection products of interesting subvarieties: *specialization.* In this section we will discuss some of its variations.

Schubert Calculus by Static Specialization. As a first illustration we show how to compute the class $\sigma_1^2 \in A(\mathbb{G}(1, \mathbb{P}^3))$ by specialization. The reader will find a far-reaching generalization to the Chow rings of Grassmannians and even to more general flag varieties in the algorithms of Vakil [2006a] and Coskun [2009].

The idea is that instead of intersecting two general cycles $\Sigma_1(L)$ and $\Sigma_1(L')$ representing σ_1, we choose a *special* pair of lines L, L'. The goal is to choose L and L' special enough that the class of the intersection $\Sigma_1(L) \cap \Sigma_1(L')$ is readily identifiable, but at the same time not so special that the intersection fails to be generically transverse.

We do this by choosing L and L' to be distinct, but incident. The intersection $\Sigma_1(L) \cap \Sigma_1(L')$ is easy to describe: if we $p = L \cap L'$ is the point of intersection of the lines and $H = \overline{L\,L'}$ the plane they span, then a line Λ meeting L and L' either passes through p or lies in H (since it then meets

L and L' in distinct points). Thus, as sets, we have

$$\begin{aligned}\Sigma_1(L) \cap \Sigma_1(L') &= \{\Lambda \mid \Lambda \cap L \neq \varnothing \text{ and } \Lambda \cap L' \neq \varnothing\} \\ &= \{\Lambda \mid p \in \Lambda \text{ or } \Lambda \subset H\} \\ &= \Sigma_2(p) \cup \Sigma_{1,1}(H).\end{aligned}$$

If we now show that the intersection is generically transverse, we get the desired formula $\sigma_1^2 = \sigma_2 + \sigma_{1,1}$. To check this transversality, we can use the description of the tangent spaces to $\Sigma_1(L)$ and $\Sigma_1(L')$ given in Exercise 2.29. First, suppose Λ is a general point of the component $\Sigma_2(p)$ of $\Sigma_1(L) \cap \Sigma_1(L')$, that is, a general line through p; we'll let $K = \overline{\Lambda L}$ and $K' = \overline{\Lambda L'}$ be the planes spanned by Λ together with L and L'. Viewing the tangent space $T_\Lambda(\mathbb{G}(1, \mathbb{P}^3))$ as the vector space of linear maps $\varphi : \tilde{\Lambda} \to V/\tilde{\Lambda}$, we have

$$T_\Lambda(\Sigma_1(L)) = \{\varphi \mid \varphi(\tilde{p}) \subset \tilde{K}/\tilde{\Lambda}\}$$

and

$$T_\Lambda(\Sigma_1(L')) = \{\varphi \mid \varphi(\tilde{p}) \subset \tilde{K}'/\tilde{\Lambda}\}.$$

Since K and K' are distinct, they intersect in Λ, so that the intersection is

$$T_\Lambda(\Sigma_1(L)) \cap T_\Lambda(\Sigma_1(L')) = \{\varphi \mid \varphi(\tilde{p}) = 0\}$$

Since this is 2-dimensional, the intersection $\Sigma_1(L) \cap \Sigma_1(L')$ is transverse at $[\Lambda]$.

Similarly, if Λ is a general point of the component $\Sigma_{1,1}(H)$ of $\Sigma_1(L) \cap \Sigma_1(L')$, so that Λ meets L and L' in distinct points q and q', we have

$$T_\Lambda(\Sigma_1(L)) = \{\varphi \mid \varphi(\tilde{q}) \subset \tilde{H}/\tilde{\Lambda}\}$$

and

$$T_\Lambda(\Sigma_1(L')) = \{\varphi \mid \varphi(\tilde{q'}) \subset \tilde{H}/\tilde{\Lambda}\},$$

so

$$T_\Lambda(\Sigma_1(L)) \cap T_\Lambda(\Sigma_1(L')) = \{\varphi \mid \varphi(\tilde{\Lambda}) \subset \tilde{H}\}.$$

Again this is 2-dimensional and we conclude that $\Sigma_1(L) \cap \Sigma_1(L')$ is transverse at $[\Lambda]$.

Lines meeting a curve by dynamic specialization. The computation of σ_1^2 above was an example of the simplest kind of specialization argument, what we may call *static specialization*: we are able to find cycles representing the two given classes that are special enough that the class of the intersection is readily identifiable, but general enough that they still intersect properly.

In general, we may not be able to find such cycles. Such situations call for a more powerful and broadly applicable technique, called *dynamic specialization*, where we consider a one-parameter family of pairs of cycles

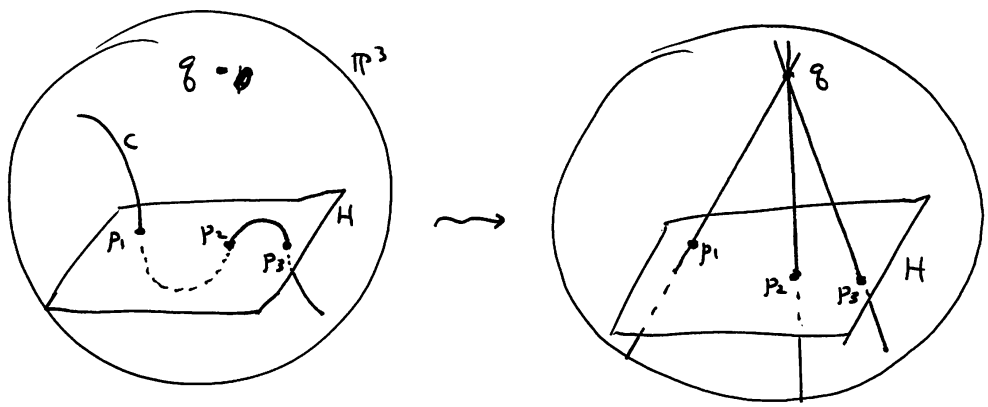


FIGURE 2.7.5. A space curve C specializes to a union of lines.

specializing from a "general" pair to a special one, and ask not for the intersection of the limiting cycles but for the limit of their intersections.

As an example, we revisit the computation of the class γ_C of the locus $\Gamma_C \subset \mathbb{G}(1,3)$ of lines meeting a curve $C \subset \mathbb{P}^3$ from Proposition 2.13, We'll use a technique called *dynamic projection*, described in much more generality in Section 4.1.1.

Let $C \subset \mathbb{P}^3$ be a curve of degree d. Choose a plane $H \subset \mathbb{P}^3$ intersecting C transversely in points $p_1, \ldots, p_d$ and $q \in \mathbb{P}^3$ any point not lying on H. Consider the one-parameter group $\{A_t\} \subset PGL_4$ with repellor plane H and attractor q; that is, choose coordinates $[Z_0, \ldots, Z_3]$ on $\mathbb{P}^3$ such that $q = [1,0,0,0]$ and H is given by $Z_0 = 0$, and consider for $t \neq 0$ the automorphisms of $\mathbb{P}^3$ given by

$$A_t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \end{pmatrix}.$$

Let $C_t = A_t(C)$, and let $\Phi \subset \mathbb{A}^1 \times \mathbb{P}^3$ be the closure of the locus

$$\{(t,p) \mid t \neq 0 \text{ and } p \in C_t\}$$

As we'll see in Section 4.1.1 (specifically, Proposition 4.5), the limit of the curves C_t as $t \to 0$—that is, the fiber of Φ over $t = 0$—is supported on the union of the d lines $\overline{p_i\, q}$, and has multiplicity 1 at a general point of each, as shown in Figure 2.7.5.

We can use this construction to give a rational equivalence between the cycle Γ_C and the sum of the Schubert cycles $\Sigma_1(\overline{p_i\, q})$ in $\mathbb{G}(1, \mathbb{P}^3)$. Explicitly, take $\Psi \subset \mathbb{A}^1 \times \mathbb{G}(1,3)$ to be the closure of the locus

$$\{(t,\Lambda) \mid t \neq 0 \text{ and } \Lambda \cap C_t \neq \varnothing\}.$$

As we'll verify in Exercises 2.39 and 2.40, the fiber Ψ_0 of Ψ over $t = 0$ is supported on the union of the Schubert cycles $\Sigma_1(\overline{p_i\, q})$ and has multiplicity one along each, establishing the rational equivalence $\gamma_C = d \cdot \sigma_1$.

The fiber of Φ over $t = 0$—that is, the flat limit $\lim_{t\to 0} C_t$ of the curves C_t—is *not* necessarily equal to the union of the d lines $\overline{p_i\, q}$: it may have an embedded point at the point q. Similarly, the flat limit Ψ_0 of the family of subvarieties $\Psi_t = \Gamma_{C_t} \subset \mathbb{G}(1,\mathbb{P}^3)$ may not be the union of the subvarieties $\Sigma_1(\overline{p_i\, q}) \subset \mathbb{G}(1,\mathbb{P}^3)$: it may have an embedded component supported on the locus of lines passing through q. But this occurs in strictly smaller dimension, and so doesn't affect the rational equivalence $\Gamma_C \sim \sum_{i=1}^{d} \Sigma_1(\overline{p_i\, q})$. If you want to see this yourself, try Exercise 2.41.

For another example of dynamic specialization, see Section 3.3.1 of the following chapter, where we consider, in the Grassmannian $\mathbb{G}(1,4)$ of lines in $\mathbb{P}^4$, the self-intersection of the cycle of lines meeting a given line in $\mathbb{P}^4$.

Chords via specialization: multiplicity problems. One of the main difficulties in using specialization is the appearance of multiplicities. We will now illustrate this problem by trying to compute, via specialization, the class of the chords a smooth curve in $\mathbb{P}^3$.

Consider again the family of curves $C_t := A_t(C) = C_t$ described in the previous section. What is the limit as $t \to 0$ of the cycles $S_1(C_t)$ of chords to C_t? To interpret this question, let $\Pi \subset \mathbb{A}^1 \times \mathbb{G}(1,\mathbb{P}^3)$ be the closure of the locus

$$\Pi^\circ = \{(t,\Lambda) \mid t \neq 0 \text{ and } \Lambda \in S_1(C_t)\}.$$

What is the fiber Π_0 of this family?

The support of Π_0 is easy to identify. It is contained in the locus of lines having intersection of degree at least 2 with the flat limit $C_0 = \lim_{t\to 0} C_t$, which is to say the union of the Schubert cycles $\Sigma_{1,1}(\overline{p_i\, p_j\, q})$ of lines lying in a plane spanned by a pair of the lines $\overline{p_i\, q}$, and the Schubert cycle $\Sigma_2(q)$ of lines containing the point q. Moreover one can show that the Schubert cycles $\Sigma_{1,1}(\overline{p_i\, p_j\, q})$ all appear with multiplicity 1 in the limiting cycle Π_0, from which we can deduce that the coefficient of $\sigma_{1,1}$ in the class of $S_1(C)$ is $\binom{d}{2}$.

The hard part is determining the multiplicity with which the cycle $\Sigma_2(q)$ appears in Π_0: this will depend in part on the multiplicity of the embedded point of C_0 at q, which will in turn depend on the genus g of C (see for example Exercises 2.50 and 2.51). Note the contrast with the calculation above of the class of the locus Γ_C of incident lines via specialization: there the embedded component of the limit scheme $\lim_{t\to 0} \Gamma_{C_t}$ also depended on the genus of C, but since it occurred in lower dimension it didn't affect the limiting cycle.

An alternative approach to this problem would be to use a different specialization to capture the coefficient of σ_2: specifically, we could take the one-parameter subgroup with repellor a general point q and attractor a general plane $H \subset \mathbb{P}^3$. The limiting scheme $C_0 = \lim_{t\to 0} C_t$ will be a plane curve of degree d with $\delta = \binom{d-1}{2} - g$ nodes $r_1, \dots, r_\delta$, with a spatial embedded point of multiplicity 1 at each node. The limit of the corresponding cycles $S_1(C_t) \subset \mathbb{G}(1,3)$ will correspondingly be supported on the union of the Schubert cycle $\Sigma_{1,1}(H)$ and the δ Schubert cycles $\Sigma_2(r_i)$. In this case the coefficient of the Schubert cycle $\Sigma_{1,1}(H)$ is the mysterious one (though calculable: given that a general line $\Lambda \subset H$ meets C_0 in d points, we can show that it's the limit of $\binom{d}{2}$ chords to C_t as $t \to 0$). On the other hand, one can show that the Schubert cycles $\Sigma_2(r_i)$ all appear with multiplicity 1 in the limit of the cycles $S_1(C_t)$, from which we can read off the coefficient δ of σ_2 in the class of $S_1(C)$.

We will fill in some of the details in this calculation in Exercise 2.52.

Common chords to twisted cubics via specialization. To illustrate the artfulness possible in specialization arguments, we give a different specialization approach to counting the chords to two twisted cubics: we won't degenerate the twisted cubics; we'll just specialize them to a general pair of twisted cubic curve C, C' lying on the same smooth quadric surface Q, of types $(1,2)$ and $(2,1)$ respectively.

The point is, no line of either ruling of Q will be a chord of both C and C' (the lines of the first ruling are chords of C but not of C' and vice versa). But since $C \cup C' \subset Q$, any line meeting $C \cup C'$ in three or more distinct points must lie in Q. It follows that *the only common chords to C and C' will be the lines joining the points of intersection $C \cap C'$ pairwise*; since the number of such points is $\#(C \cap C') = \deg([C][C']) = 5$, the number of common chords will be $\binom{5}{2} = 10$. Of course, to deduce the general formula from this analysis, we have to check that the intersection $S_1(C) \cap S_1(C')$ is transverse; we'll leave this as Exercise 2.53.

What would happen if we specialized C and C' to twisted cubics lying on Q, both having type $(1,2)$? Now there would only be 4 points of $C \cap C'$, giving rise to $\binom{4}{2} = 6$ common chords. But now the lines of the first ruling of Q *would* all be common chords to both. Thus $S_1(C) \cap S_1(C')$ would have a positive-dimensional component: explicitly, $S_1(C) \cap S_1(C')$ would consist of 6 isolated points and one copy of $\mathbb{P}^1$. It might seem that in these circumstances we couldn't deduce anything about the intersection number $\deg([S_1(C)] \cdot [S_1(C')])$ from the actual intersection, but we'll see in Chapter 15 how to apply the excess intersection formula to determine $\deg([S_1(C)] \cdot [S_1(C')])$.

2.5 Lines and surfaces in $\mathbb{P}^3$

2.5.1 Lines lying on a quadric

The family of k-planes lying on a given projective variety X has a natural structure as a subscheme of the Grassmannian, and is called the Fano scheme, denoted $F_k(X)$. (The natural scheme structure comes from a description of the Fano scheme as a *Hilbert scheme*; see Section **??**.) The Fano scheme plays an important role in geometry—for example, Fano schemes are central in the famous proof of the irrationality of the cubic threefold in $\mathbb{P}^4$ given by Clemens and Griffiths [1972]. Fano schemes will be treated in much more generality in Chapter 8.

In this section we will determine the class in $\mathbb{G}(1, \mathbb{P}^3_{\mathbb{C}})$ of simplest nontrivial Fano scheme, the family $F_1(Q)$ of lines on a smooth quadric Q in $\mathbb{P}^3$. We will treat this family set-theoretically; we will see later that the natural scheme structure is reduced (even smooth) in this case, so nothing is lost. We temporarily set $F := F_1(Q)_{\text{red}} \subset G(1, \mathbb{P}^3_{\mathbb{C}})$, the set of lines on Q. Since the family of quadrics on a line is 3-dimensional, it is 3 conditions for a line to lie in Q; thus $\dim F = 1$. (One can also see this from the fact that there are two lines in Q through each point of Q.)

Since $A^3(\mathbb{G}(1, \mathbb{P}^3_{\mathbb{C}}$ is generated by $\sigma_{2,1}$, we must have

$$[F_1(Q)] = \alpha \cdot \sigma_{2,1}$$

for some integer α. If $L \subset \mathbb{P}^3$ is a general line, and $\Sigma_1(L) \subset \mathbb{G}(1,3)$ the Schubert cycle of lines meeting L, then by Corollary 2.8 we have

$$\begin{aligned}\alpha &= \deg([F_1(Q)] \cdot \sigma_1)\\ &= \#\left(\Sigma_1(L) \cap F_1(Q)\right)\\ &= \#\{M \in \mathbb{G}(1,3) \mid M \subset Q \;\text{ and }\; M \cap L \neq \varnothing\}.\end{aligned}$$

Now L, being general, will intersect Q in two points, and through each of these points there will be two lines contained in Q; thus we have $\alpha = 4$ and

$$[F_1(Q)] = 4\sigma_{2,1}.$$

We will see how to calculate the class of the locus of linear spaces on a quadric hypersurface more generally in Section 3.5.

The variety $F_1(Q)$ is actually the union $C_1 \cup C_2$ of two disjoint curves in the Grassmannian, corresponding to the two rulings of Q; each of these curves has class $2\sigma_{2,1}$, and thus has degree 2 as a curve in the Plücker embedding in $\mathbb{P}^5$ (see Figure 2.8.5). For details see Eisenbud and Harris [2000].

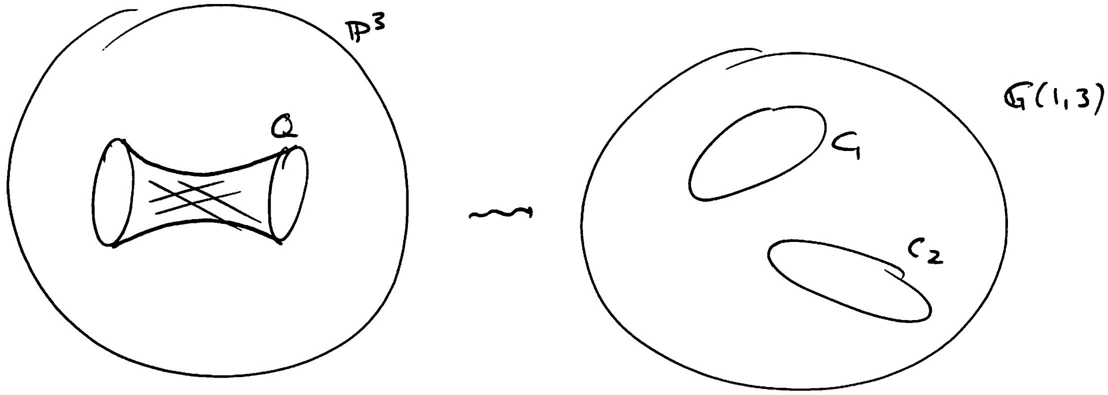


FIGURE 2.8.5. The rulings of a quadric surface $Q \subset \mathbb{P}^3$ correspond to conic curves $C_i \subset \mathbb{G}(1,3) \subset \mathbb{P}^5$; thus $[F_1(Q)] = 4\sigma_{2,1}$.

2.5.2 Tangent lines to a surface

Next, let $S \subset \mathbb{P}^3$ be any smooth surface of degree d, and consider the locus $T_1(S) \subset \mathbb{G}(1,3)$ of tangent lines to S. Let Φ be the incidence correspondence

$$\Phi = \{(p, L) \in S \times \mathbb{G}(1,3) \mid p \in L \subset \mathbb{T}_q S\},$$

where $\mathbb{T}_q S$ denotes the projective plane tangent to S at q. The projection $\Phi \to S$ on the first factor expresses Φ as a $\mathbb{P}^1$-bundle over S, from which we deduce that Φ, and hence its image $T_1(S)$ in $\mathbb{G}(1,3)$, is irreducible of dimension 3.

To find the class of $T_1(S)$, we write

$$[T_1(S)] = \alpha \cdot \sigma_1,$$

and choose a general plane $H \subset \mathbb{P}^3$ and a general point $p \in H$. Using Kleiman's Theorem 5.15 we may write

$$\begin{aligned}\alpha &= [T_1(S)] \cdot \sigma_{2,1}\\ &= \deg\left(\Sigma_{2,1}(p, H) \cap T_1(S)\right)\\ &= \#\left\{M \in \mathbb{G}(1,3) \mid M \subset \mathbb{T}_p S \text{ for some } q \in S \text{ and } p \in M \subset H\right\}.\end{aligned}$$

Now H, being general, will intersect S in a smooth plane curve $C \subset H \cong \mathbb{P}^2$ of degree d; and p being general, the line $p^* \subset \mathbb{P}^{2^*}$ dual to p will intersect the dual curve $C^* \subset \mathbb{P}^{2^*}$ transversely in $\deg(C^*)$ points. By Proposition 1.25, we have

$$\deg(C^*) = d(d-1)$$

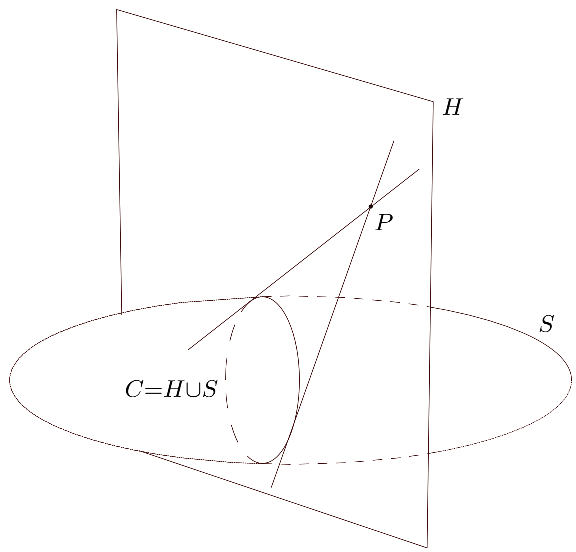


FIGURE 2.14. $\deg\big(\Sigma_{2,1}(p,H)\cap T_1(S)\big)=2.$

and hence

$$[T_1(S)] = d(d-1)\sigma_1$$

(Figure 2.14).

This gives the answer to the last keynote question of this chapter: how many lines are tangent to each of four general quadric surfaces Q_i? he cycles $T_1(Q_i)$ intersect transversely Corollary 2.8, so the answer is

$$\deg \prod [T_1(Q_i)] = \deg(2\sigma_1)^4 = 32.$$

2.6 Exercises

Exercise 2.15. Let $e_1, \ldots, e_4$ be a basis for a vector space V, and let

$$\eta = \sum_{1\le a<b\le 4} p_{a,b} e_a \wedge e_b \in \wedge^2 V.$$

Show that the polynomial in the $p_{a,b}$ corresponding to the equation

$$\eta \wedge \eta = 0 \in \wedge^4 V$$

is $2g$, where g is as in (2.1). Thus, in characteristic not 2, the equation $\eta^2 = 0$ characterizes decomposable vectors in the exterior square $\wedge^2 V$, and more generally defines the Grassmannian $\mathbb{G}(1, \mathbb{P}V) \subset \mathbb{P}(\wedge^2 V)$ for a vector space V of any dimension.

In fact, even over the integers, there is an element $\eta^{(2)} \in \wedge^4 V$ called the *divided square* of η, such that $\eta \wedge \eta = 2\eta^{(2)}$, and in every characteristic η is decomposable if and only if $\eta^{(2)} = 0$. The coefficients of $\eta^{(2)}$, when written out in terms of the basis of $\wedge^4 V$, are called the 4×4 Pfaffians of η; they are the polynomials $g_{a,b,c,d}$ of Exercise **??**. The square of the ideal of 4×4 Pfaffians of η is the ideal of 4×4 minors of the generic alternating matrix $(p_{i,j})$. See Eisenbud [1995] Appendix A2.4.

Exercise 2.16. Let Λ and $\Gamma \in G$ be two points in the Grassmannian $G = G(k, V)$. Show that the line $\overline{\Lambda\,\Gamma} \subset \mathbb{P}(\wedge^k V)$ is contained in G if and only if the intersection $\Lambda \cap \Gamma \subset V$ of the corresponding subspaces of V has dimension $k - 1$.

Exercise 2.17. Suppose that $L_1, \ldots, L_m \subset V$ are k-planes contained in a vector space, and that for every $i \neq j$ we have $\dim L_i \cap L_j = k - 1$ (equivalently, $\dim L_i + L_j = k + 1$.) Show that either all the L_i contain a subspace M of dimension $k - 1$ or all L_i are contained in a subspace N of dimension $k + 1$.

Exercise 2.18. Using Exercises 2.16 and 2.17, show that the linear subspaces of $\mathbb{P}(\wedge^k V)$ that are maximal among linear spaces contained in $G(k, V)$ are precisely those defined in one of two ways:

(a) $\Gamma_M = \{L \mid M \subset L\}$ for some subspace $M \subset V$ of dimension $k - 1$, so that $\Gamma_M \cong \mathbb{P}(V/M)$; or

(b) $\Sigma_N = \{L \mid L \subset N\}$ for some subspace $N \subset V$ of dimension $k + 1$, so that $\Sigma_N \cong \mathbb{P}(N)$.

Notice that unless $\dim V = 2k$, these two kinds of maximal linear subspaces have different dimensions. Indeed, Exercise (2.18) is key in establishing a basic fact: that for $\dim V \neq 2k$, the automorphism group of the Grassmannian $G = G(k, V)$ is the same as the automorphism group $PGL(V)$ of $\mathbb{P}V$: assuming $2k < \dim V$, we argue that an automorphism of $G(k, V)$ must be projective, that is, induced by an automorphism of the vector space $\wedge^k V$; hence must preserve linear spaces of $\mathbb{P}(\wedge^k V)$ contained in $G(k, V)$; and so induces an automorphism of the Grassmannian $G(k - 1, V)$. Continuing in this way, we arrive at an automorphism of $\mathbb{P}V$ that induces the original automorphism of $G(k, V)$. The situation when $\dim V = 2k$ is similar, except that there do exist automorphisms of $G(k, V)$ exchanging the two families of maximal linear subspaces (for example, we can choose a nondegenerate bilinear form on V and send each k-plane Λ to $\Lambda^\perp$); we arrive at an exact sequence

$$1 \to PGL(V) \to Aut(G) \to \mathbb{Z}/2 \to 1.$$

Exercise 2.19. Using the fact that the Grassmannian

$$G = G(k, V) \subset \mathbb{P}(\wedge^k V)$$

is cut out by quadratic equations, show that if $[\Lambda] \in G$ is the point corresponding to a k-plane Λ then the tangent plane $\mathbb{T}_{[\Lambda]}G \subset \mathbb{P}(\wedge^k V)$ intersects G in the locus

$$G \cap \mathbb{T}_{[\Lambda]}G = \{[\Gamma] : \dim(\Gamma \cap \Lambda) \geq k-1\};$$

that is, the locus of k-planes meeting Λ in codimension 1.

Exercise 2.20. Consider the universal k-plane over $G = \mathbb{G}(k, \mathbb{P}V)$:

$$\Phi = \{(\Lambda, p) \in G \times \mathbb{P}V \mid p \in \Lambda\},$$

whose fiber over a point $[\Lambda] \in G$ is the k-plane $\Lambda \subset \mathbb{P}V$. Show that this is a closed subvariety of $G \times \mathbb{P}V$ of dimension $k + (k+1)(n-k)$, and that it's cut out on $G \times \mathbb{P}V$ by bilinear forms on $\mathbb{P}(\wedge^k V) \times \mathbb{P}V$.

Exercise 2.21. Use the preceding exercise to show that if $X \subset \mathbb{P}^n$ is any subvariety of dimension $l < n-k$, then the locus

$$\Gamma_X = \{\Lambda \in \mathbb{G}(k, n) \mid X \cap \Lambda \neq \varnothing\}$$

of k-planes meeting X is a closed subvariety of $\mathbb{G}(k, n)$ of codimension $n-k-l$.

Exercise 2.22. Let $l < k < n$, and consider the locus of nested pairs of linear subspaces of $\mathbb{P}^n$ of dimensions l and k:

$$\mathbb{F}(l, k; n) = \{(\Gamma, \Lambda) \in \mathbb{G}(l, n) \times \mathbb{G}(k, n) \mid \Gamma \subset \Lambda\}.$$

Show that this is a closed subvariety of $\mathbb{G}(l, n) \times \mathbb{G}(k, n)$, and calculate its dimension. (These are examples of a further generalization of Grassmannians called *flag manifolds*.)

Exercise 2.23. Again let $l < k < n$, and for any $m \leq l$ consider the locus of pairs of linear subspaces of $\mathbb{P}^n$ of dimensions l and k intersecting in dimension at least m:

$$\mathbb{F}(l, k; n) = \{(\Gamma, \Lambda) \in \mathbb{G}(l, n) \times \mathbb{G}(k, n) \mid \dim(\Gamma \cap \Lambda) \geq m\}.$$

Show that this is a closed subvariety of $\mathbb{G}(l, n) \times \mathbb{G}(k, n)$, and calculate its dimension.

Exercise 2.24. Different hyperplane sections of $G(k, n)$ can have different properties; not all are given by the vanishing of a Plücker coordinate.

(a) Use (for example) Exercise 2.19 to show that a hyperplane section of $G(2, 4) = \mathbb{G}(1, 3)$ defined by a Plücker coordinate is the cone over a nonsingular quadric in $\mathbb{P}^3$. What other varieties appear as a hyperplane section of $\mathbb{G}(1, 3)$?

(b) In general, how many isomorphism classes of hyperplane sections of $\mathbb{G}(k,n)$ are there?

Exercise 2.25. We gave a description of the Plücker relations in the special case $\mathbb{G}(1,n)$ following Example 2.1, and in Exercise 2.15 above. Show that the polynomials above, obtained by expanding the determinants of submatrices of A, are the same as the polynomials described there.

Exercise 2.26. Assume that the characteristic of our ground field is 0. Let $B \subset \mathbb{G}(1,n)$ be a curve in the Grassmannian of lines in $\mathbb{P}^n$, with the property that all nonzero tangent vectors to B have rank 1. Show that the lines in $\mathbb{P}^n$ parametrized by B either

(a) all lie in a fixed 2-plane;

(b) all pass through a fixed point; or

(c) are all tangent to a curve $C \subset \mathbb{P}^n$.

(Note that the last possibility actually subsumes the first.)

Exercise 2.27. Show that an automorphism of $G(k,n)$ carries tangent vectors to tangent vectors of the same rank (in the sense of Section 2.2.4), and hence that in case $1 < k < n$ the group of automorphisms of $G(k,n)$ cannot act transitively on nonzero tangent vectors. Show, on the other hand, that the group of automorphisms of $G(k,n)$ *does* act transitively on tangent vectors of a given rank.

Exercise 2.28. In Example 2.9, we showed that the open Schubert cell $\tilde{\Sigma}_1 = \Sigma_1 \setminus (\Sigma_2 \cup \Sigma_{1,1})$ is isomorphic to affine space $\mathbb{A}^3$. For each of the remaining Schubert indices a, b, show that the Schubert cell $\tilde{\Sigma}_{a,b} \subset \mathbb{G}(1,3)$ is isomorphic to affine space of dimension $4 - a - b$.

Exercise 2.29. Consider the Schubert cycle

$$\Sigma_1 = \{\Lambda \in \mathbb{G}(1,3) \mid \Lambda \cap L \neq \varnothing\}.$$

Suppose $\Lambda \in \Sigma_1$ and that $\Lambda \neq L$, so that $\Lambda \cap L$ is a point q and the span $\overline{\Lambda\, L}$ a plane K. Show that Λ is a smooth point of Σ_1, and that its tangent space is

$$T_\Lambda(\Sigma_1) = \{\varphi \in \mathrm{Hom}(\tilde{\Lambda}, V/\tilde{\Lambda}) \mid \varphi(\tilde{q}) \subset \tilde{K}/\tilde{\Lambda}\}.$$

Exercise 2.30. Consider the Schubert cycle

$$\Sigma_{2,1} = \Sigma_{2,1}(p,H) = \{\Lambda \in \mathbb{G}(1,3) \mid p \in \Lambda \subset H\}.$$

Show that $\Sigma_{2,1}$ is smooth, and that its tangent space at a point Λ is

$$T_\Lambda(\Sigma_{2.1}) = \{\varphi \in \mathrm{Hom}(\tilde{\Lambda}, V/\tilde{\Lambda}) \mid \varphi(\tilde{p}) = 0 \text{ and } \mathrm{Im}(\varphi) \subset \tilde{H}/\tilde{\Lambda}\}.$$

Exercise 2.31. Use the preceding two exercises to show in arbitrary characteristic that general Schubert cycles Σ_1 and $\Sigma_{2,1} \subset \mathbb{G}(1,3)$ intersect transversely, and deduce the equality $\deg(\sigma_1 \cdot \sigma_{2,1}) = 1$.

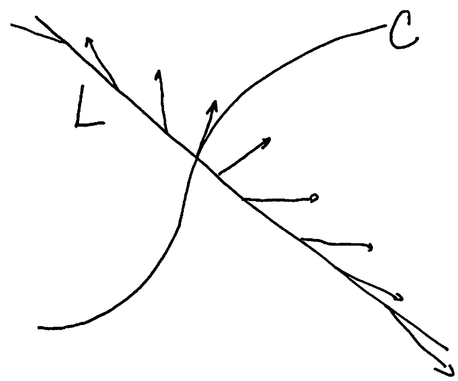


FIGURE 2.15. Deformation of a line L preserving contact with a curve C.

Exercise 2.32. Let $L_1, \dots, L_4 \subset \mathbb{P}^3$ be four arbitrary lines. Give necessary and sufficient conditions on the intersections of the L_i that there be finitely many lines meeting all 4.

Exercise 2.33. Let $L_1, \dots, L_4 \subset \mathbb{P}^3$ be four pairwise skew lines, and $\Lambda \subset \mathbb{P}^3$ a line meeting all four; set

$$p_i = \Lambda \cap L_i \quad \text{and} \quad H_i = \overline{\Lambda\, L_i}.$$

Show that $[\Lambda] \in G$ fails to be a transverse point of intersection of the Schubert cycles $\Sigma_1(L_i)$ exactly when the cross-ratio of the four points $p_1, \dots, p_4 \in \Lambda$ equals the cross-ratio of the four planes $H_1, \dots, H_4$ in the pencil of planes containing Λ.

The following series of Exercises deals with a question raised in Section 2.4.2: if $C_1, \dots, C_4 \subset \mathbb{P}^3$ are general translates of four curves in $\mathbb{P}^3$, do the corresponding cycles $\Gamma_{C_i} \subset \mathbb{G}(1,3)$ of lines meeting the C_i intersect transversely?

To start with, we have to identify the smooth locus of the cycle $\Gamma_C \subset \mathbb{G}(1,3)$ of lines meeting a given curve C, and its tangent spaces at these points; this is the content of the first exercise, which is a direct generalization of Exercise 2.29 above.

Exercise 2.34. Let $C \subset \mathbb{P}^3$ be any curve, and $L \subset \mathbb{P}^3$ a line meeting C at one smooth point p of C and not tangent to C. Show that the cycle $\Gamma_C \subset \mathbb{G}(1,3)$ of lines meeting C is smooth at the point $[L]$, and that its tangent space at $[L]$ is the space of linear maps $\tilde{L} \to \mathbb{C}^4/\tilde{L}$ carrying the one-dimensional subspace $\tilde{p} \subset \tilde{L}$ to the one-dimensional subspace $(\tilde{\mathbb{T}}_p C + \tilde{L})/\tilde{L}$ of $\mathbb{C}^4/\tilde{L}$, where we have written $\mathbb{T}_p C$ for the projective line in $\mathbb{P}^3$ that is tangent to C at p. See Figure 2.15

Next, we have to verify that, for general translates C_i of any four curves, the corresponding cycles Γ_{C_i} are smooth at each of the points of their intersection. A key fact will be the irreducibility of the relevant incidence correspondence:

Exercise 2.35. Let $B_1, \ldots, B_4 \subset \mathbb{P}^3$ be four irreducible curves, and let $\varphi_1, \ldots, \varphi_4 \in PGL_4$ be four general automorphisms of $\mathbb{P}^3$; let $C_i = \varphi_i(B_i)$. Show that the incidence correspondence

$$\Phi = \left\{(\varphi_1, \ldots, \varphi_4, L) \in (PGL_4)^4 \times \mathbb{G}(1,3) \;\middle|\; L \cap \varphi_i(B_i) \neq \varnothing \; \forall i\right\}$$

is irreducible.

Using this, we can prove the following exercise—asserting that for general translates C_i of four given curves and any line L meeting all four, the cycles Γ_{C_i} are smooth at $[L]$—simply by exhibiting a single collection $(\varphi_1, \ldots, \varphi_4, L)$ satisfying the conditions in question:

Exercise 2.36. Let $B_1, \ldots, B_4 \subset \mathbb{P}^3$ be four curves, and $\varphi_1, \ldots, \varphi_4 \subset PGL_4$ four general automorphisms of $\mathbb{P}^3$; let $C_i = \varphi_i(B_i)$. Show that the set of lines $L \subset \mathbb{P}^3$ meeting C_1, C_2, C_3 and C_4 is finite; and that for any such L

(a) L meets each C_i at only one point p_i;

(b) p_i is a smooth point of C_i; and

(c) L is not tangent to C_i for any i

Exercise 2.37. Let $C_1, \ldots, C_4 \subset \mathbb{P}^3$ be any four curves, and $L \subset \mathbb{P}^3$ a line meeting all four and satisfying the conclusions of Exercise **??**. Use the result of Exercise 2.34 to give a necessary and sufficient condition that the four cycles $\Gamma_{C_i} \subset \mathbb{G}(1,3)$ intersect transversely at $[L]$, and show directly that this condition is satisfied when the C_i are general translates of given curves.

Exercise 2.38. Let $C \subset \mathbb{P}^3$ be a smooth curve, and $p \in \mathbb{P}^3$ a general point. Show that

(a) p does not lie on any tangent line to C;

(b) p does not lie on any trisecant line to C; and

(c) p does not lie on any *stationary secant* to C; that is, a secant line $\overline{q\,r}$ to C such that the tangent lines $\mathbb{T}_q C \cap \mathbb{T}_r C \neq \varnothing$.

Deduce from these facts that the projection $\pi_p : C \to \mathbb{P}^2$ is birational onto a plane curve $C_0 \subset \mathbb{P}^2$ having only nodes as singularities.

The following three exercises deal with the approach, described in Section 2.4.4, to calculating the class of the variety $\Sigma_C \subset \mathbb{G}(1,3)$ of lines incident to a space curve $C \subset \mathbb{P}^3$ by specialization. Recall from that section that we choose a general plane $H \subset \mathbb{P}^3$ meeting C at d points p_i and a general point $q \in \mathbb{P}^3$, and let $\{A_t\}$ be the one-parameter subgroup of PGL_4 with attractor q and repellor H; we let $C_t = A_t(C)$ and take $\Psi \subset \mathbb{A}^1 \times \mathbb{G}(1,3)$ to be the closure of the locus

$$\Psi^\circ = \{(t, \Lambda) \mid t \neq 0 \ \text{ and } \Lambda \cap C_t \neq \varnothing\}.$$

Exercise 2.39. Show that the support of the fiber Ψ_0 is exactly the union of the Schubert cycles $\Sigma_1(\overline{p_i\, q})$

Exercise 2.40. Show that Ψ_0 has multiplicity 1 at a general point of each Schubert cycle $\Sigma_1(\overline{p_i\, q})$

Exercise 2.41. Suppose now that $C \subset \mathbb{P}^3$ is a general rational quartic curve. Describe the flat limit of the family of cycles $\Gamma_{C_t} \subset \mathbb{G}(1,3)$, and in particular its embedded components.

Exercise 2.42. Let $C \subset \mathbb{P}^r$ be a smooth curve. Show that the rational map $\varphi : C^{(2)} \to \mathbb{G}(1,r)$ sending a pair of distinct points $p, q \in C$ to the line $\overline{p, q}$ actually extends to a regular map on all of $C^{(2)}$ by sending the pair $2p$ to the projective tangent line $\mathbb{T}_p C$. Use this to show that the image of φ coincides with the locus of lines $L \subset \mathbb{P}^r$ such that the scheme-theoretic intersection $L \cap C$ has degree at least 2.

Exercise 2.43. Show by example that the conclusion of the preceding exercise is false in general if we do not assume $C \subset \mathbb{P}^r$ smooth. Is it still true if we allow C to have mild singularities, such as nodes?

Exercise 2.44. Similarly, show by example that the conclusion of Exercise 2.42 is false if we consider higher-dimensional secant planes: for example, the image of the map

$$\begin{aligned} \varphi : C^{(3)} &\to \mathbb{G}(2,r) \\ p+q+r &\mapsto \overline{p\,q\,r} \end{aligned}$$

need not coincide with the locus of 2-planes $\Lambda \subset \mathbb{P}^r$ whose scheme-theoretic intersection with C has degree at least 3.

Exercise 2.45. Show that the smooth locus of $S = S_1(C)$ contains the locus of lines $L \subset \mathbb{P}^3$ such that the scheme-theoretic intersection $L \cap C$ consists of two reduced points, and for such a line L identify the tangent plane $T_L S$ as a subspace of $T_L \mathbb{G}$. (When is a tangent line to C a smooth point of $S_1(C)$?)

Exercise 2.46. Use the result of the preceding Exercise to show that if C and $C' \subset \mathbb{P}^3$ are two general twisted cubic curves, then the varieties $S_1(C)$ and $S_1(C') \subset \mathbb{G}(1,3)$ of chords to C and C' intersect transversely.

Exercise 2.47. Let $C \subset \mathbb{P}^3$ be a smooth, nondegenerate curve, and let L and $M \subset \mathbb{P}^3$ be general lines.

(a) Find the number of chords to C meeting both L and M by applying the result above; and

(b) Verify this count by considering the product morphism

$$\pi_L \times \pi_M : C \to \mathbb{P}^1 \times \mathbb{P}^1$$

(where $\pi_L, \pi_M : C \to \mathbb{P}^1$ are the projections from L and M) and comparing the arithmetic and geometric genera of the image curve.

Exercise 2.48. Let $C \subset \mathbb{P}^3$ be a twisted cubic curve. Show that the locus $S_1(C) \subset \mathbb{G}(1,3) \subset \mathbb{P}^5$ of chords to C is the Veronese surface.

Exercise 2.49. Show that the twisted cubic and the elliptic quartic curves are the only smooth, irreducible, nondegenerate curves $C \subset \mathbb{P}^3$ such that $S_1(C)$ is smooth. (Hint: if C has no trisecant lines, the projection $\pi_p : C \to \mathbb{P}^2$ of C to a plane from a point $p \in C$ gives an isomorphism of C with a smooth plane curve of degree $d-1$; now use adjunction in the plane to argue that if this is the case for all $p \in C$ then either C is rational or $d = 4$.)

Exercise 2.50. Let $C \subset \mathbb{P}^3$ be a smooth, irreducible nondegenerate curve of degree d, and let $\Phi \subset \mathbb{A}^1 \times \mathbb{P}^3$ be the family of curves specializing C to a scheme supported on the union of lines joining a point $p \in \mathbb{P}^3$ to the points of a plane section of C, as constructed in Section 2.4.4. Show that C_0 may have an embedded point at p, and that the multiplicity of this embedded point may depend on the genus of the curve C, by considering the examples of curves of degrees 4 and 5.

Exercise 2.51. In the situation of the preceding problem, let $S_1(C_t) \subset G$ be the locus of chords to C_t for $t \neq 0$. Suppose that the degree of C is 4. Show that a general line L through the point p will lie in the limit $\lim_{t\to 0} S_1(C_t)$ if C is rational, but not if C is elliptic.

Exercise 2.52. Again, suppose $C \subset \mathbb{P}^3$ is any curve of degree d; choose a general plane $H \subset \mathbb{P}^3$ and point $p \in \mathbb{P}^3$ and consider the one-parameter group $\{A_t\} \subset PGL_4$ with repellor point p and attractor plane H—that is, choose coordinates $[Z_0, \dots, Z_3]$ on $\mathbb{P}^3$ such that $p = [0,0,0,1]$ and H is given by $Z_3 = 0$, and consider for $t \neq 0$ the automorphisms of $\mathbb{P}^3$ given by

$$A_t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & t \end{pmatrix}.$$

Let $C_t = A_t(C)$, and for $t \neq 0$ let $S_1(C_t) \subset G$ be the locus of chords to C_t. Show that the Schubert cycle $\Sigma_{1,1}(H)$ appears as a component of multiplicity $\binom{d}{2}$ in the limiting scheme $\lim_{t\to 0} S_1(C_t)$. (Hint: let $\Psi \subset \mathbb{A}^1 \times G$ be the closure of the family

$$\Psi^\circ = \{(t, L) \mid t \neq 0 \text{ and } L \in S_1(C_t)\},$$

and show that if $L \subset H$ is a general line, then in a neighborhood of the point $(0, L) \in \mathbb{A}^1 \times G$, the family Ψ consists of the union of $\binom{d}{2}$ smooth sheets, each intersecting the fiber $\{0\} \times G$ transversely in the Schubert cycle $\Sigma_{1,1}(H)$.)

Exercise 2.53. Let C and $C' \subset Q \subset \mathbb{P}^3$ be general twisted cubic curves lying on a smooth quadric surface Q, of types $(1,2)$ and $(2,1)$ respectively. Show that the intersection $S_1(C) \cap S_1(C')$ of the corresponding cycles of chords is transverse.

Exercise 2.54. Let $C \subset \mathbb{P}^3$ be a smooth nondegenerate curve of degree d and genus g, and let $T(C) \subset \mathbb{G}(1,3)$ be the locus of its tangent lines. Find the class $[T(C)] \in A^3(\mathbb{G}(1,3))$ of $T(C)$ in the Grassmannian $G(1,3)$.

Exercise 2.55. Let $C \subset \mathbb{P}^3$ be a smooth nondegenerate curve of degree d and genus g, and let $S \subset \mathbb{P}^3$ be a general surface of degree e. How many tangent lines to C are tangent to S?

3

Grassmannians in General

Keynote Questions

(a) If $V_1, \dots, V_4 \cong \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ are four general n-planes, how many lines $L \subset \mathbb{P}^{2n+1}$ meet all four? (Answer on page 173)

(b) Let $C \subset \mathbb{G}(1,3) \subset \mathbb{P}^5$ be a twisted cubic curve in the Grassmannian of lines in $\mathbb{P}^3$, and let

$$S = \bigcup_{[\Lambda] \in C} \Lambda \ \subset \ \mathbb{P}^3$$

be the surface swept out by the lines corresponding to points of C. What is the degree of S? How can we describe the geometry of S? (Answer on page 171)

(c) If Q_1 and $Q_2 \subset \mathbb{P}^4$ are general quadric hypersurfaces, and $S = Q_1 \cap Q_2$ their surface of intersection, how many lines does S contain? More generally, if Q_1 and Q_2 are general quadric hypersurfaces in $\mathbb{P}^{2n}$ and $X = Q_1 \cap Q_2$, how many $(n-1)$-planes does X contain? (Answer on page 149)

(d) If $G \subset \mathbb{P}^N$ is the Grassmannian $G(1,n)$ of lines in $\mathbb{P}^n$, embedded in projective space via the Plücker embedding, what is the degree of G? (Answer on page 142)

We will extend the ideas developed in Chapter 2 by introducing Schubert cycles and classes on $G(k,n)$, the Grassmannian of k-dimensional subspaces in an n-dimensional vector space V (we'll refer to these as "k-planes" in V), and analyzing their intersections, a subject that goes by the name of the *Schubert calculus*. Of course we may also consider $G(k,n)$ in its projective guise as $\mathbb{G}(k-1,n-1)$, the Grassmannian of projective $(k-1)$-planes in $\mathbb{P}^{n-1}$, and in places where projective geometry is more natural (such as Sections 3.2.2 and 3.3.1) we'll switch to the projective notation.

3.1 Schubert cells and Schubert cycles

Let $G = G(k,V)$ be the Grassmannian of k-dimensional subspaces of an n-dimensional vector space V. Generalizing the example of $\mathbb{G}(1,3) = G(2,4)$, the center of our study will be a collection of subvarieties of $G(k,n)$ called Schubert varieties or Schubert cycles, defined in terms of a chosen complete *flag* $\mathcal{V}$ in V: that is, a nested sequence of subspaces

$$0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V$$

with $\dim V_i = i$.

For any sequence $a = (a_1, \ldots, a_k)$ of integers with

$$n-k \geq a_1 \geq a_2 \geq \cdots \geq a_k \geq 0$$

we define the *Schubert cycle* $\Sigma_a(\mathcal{V}) \subset G$ to be the closed subset

$$\Sigma_a(\mathcal{V}) = \{\Lambda \in G \mid \dim(V_{n-k+i-a_i} \cap \Lambda) \geq i \ \forall i\}.$$

Proposition 5.17 shows that the class $[\Sigma_a(\mathcal{V})] \in A(G)$ doesn't depend on the choice of flag, since any two flags differ by an element of GL_n. In general, when dealing with a property independent of the choice of $\mathcal{V}$, we'll shorten the name to Σ_a and we define

$$\sigma_a := [\Sigma_a] \in A(G).$$

We shall see (Corollary 3.5) that the classes σ_a form a free basis of $A(G)$.

To simplify notation, we generally suppress trailing zeros in the indices, and write $\Sigma_{a_1,\ldots,a_s}$ in place of $\Sigma_{a_1,\ldots,a_s,0,\ldots,0}$. Also, we use the shorthand Σ_{p^r} to denote $\Sigma_{p,\ldots,p}$ with r indices equal to p.

To elucidate the rather awkward-looking definition of $\Sigma_a(\mathcal{V})$, suppose that $\Lambda \subset V$ is a k-plane. If Λ is general, then $V_i \cap \Lambda = 0$ for $i \leq n-k$, while $\dim V_{n-k+i} \cap \Lambda = i$ for $i > n-k$. Thus we may describe Σ_a as the set of Λ such that $\dim V_j \cap \Lambda \geq i$ occurs for a value of j that is a_i steps sooner than expected.

Equivalently, we may consider the sequence of subspaces of Λ:

$$0 \subset (V_1 \cap \Lambda) \subset (V_2 \cap \Lambda) \subset \cdots \subset (V_{n-1} \cap \Lambda) \subset (V_n \cap \Lambda) = \Lambda \tag{3.1}$$

Each subspace in this sequence is either equal to the one before it, or of dimension one greater, and the latter phenomenon occurs exactly k times.

The Schubert cycle $\Sigma_a(\mathcal{V})$ is the locus of planes Λ for which "the i^{th} jump in the sequence (3.1) occurs at least a_i steps early."

Here are two common special cases to bear in mind:

- The cycle of k-subspaces Λ meeting a given space of dimension l nontrivially is the Schubert cycle

$$\Sigma_{n-k+1-l}(\mathcal{V}) = \{\Lambda \mid \Lambda \cap V_l \neq 0\}.$$

  In particular, the Schubert cycle of k-dimensional subspaces meeting a given $(n-k)$-dimensional subspace nontrivially is

$$\Sigma_1(\mathcal{V}) = \{\Lambda \mid \Lambda \cap V_{n-k} \neq 0\}.$$

  This is a hyperplane section of G in the Plücker embedding (but not every hyperplane section of G is of this form; see Exercise 2.24).

- The sub-Grassmannian of k-subspaces contained in a given l-subspace is the Schubert cycle

$$\Sigma_{(n-l)^k}(\mathcal{V}) = \{\Lambda \mid \Lambda \subset V_l\}.$$

  Similarly, the sub-Grassmannian of planes containing a given r-plane is the Schubert cycle

$$\Sigma_{(n-k)^r}(\mathcal{V}) = \{\Lambda \mid V_r \subset \Lambda\}.$$

The cycles Σ_i, defined for $0 \leq i \leq n-k$, and the cycles Σ_{1^i}, defined for $0 \leq i \leq k$, are called *special* Schubert cycles. As we shall see in Section 7.4, these classes are intimately connected with the tautological sub and quotient bundles on G, and each of these sequences of cycles forms a minimal generating set for the algebra $A(G)$.

Our indexing of the Schubert cycles is by no means the only one in use, but it has several good properties:

- It reflects the partial order of the Schubert cycles by inclusion: if we order the indices termwise, that is, $(a_1, \ldots, a_k) \leq (a'_1, \ldots, a'_k)$ if and only if $a_i \leq a'_i$ for $1 \leq i \leq k$, then

$$\Sigma_a \subset \Sigma_b \iff a \geq b.$$

  This follows immediately from the definition.

- It makes the codimension of a Schubert cycle easy to compute: By Theorem 3.1 below

$$\operatorname{codim}(\Sigma_a \subset G) = \sum a_i,$$

  so that $|a| := \sum a_i$ is the degree of $\sigma_a := [\Sigma_a]$ in $A(G)$.

- It is preserved under pullback via the obvious inclusions $i : G(k,n) \hookrightarrow G(k,n+1)$ and $j : G(k,n) \hookrightarrow G(k,n+1)$, that is,
$$i^*(\sigma_a) = \sigma_a \text{ and } j^*(\sigma_a) = \sigma_a.$$
Here we adopt the convention that when $a_1 > n-k$, or when $a_{k+1} > 0$, we take $\sigma_a = 0$ as a class in $A(G(k,n))$. (This convention is consistent with the restriction to sub-Grassmannians; for example, $\Sigma_{n-k+1} \subset G(k,n+1)$ is the subset of the k-planes containing a fixed general point, and thus the intersection of Σ_{n-k+1} with the $G(k,n)$ of subspaces in a fixed general n-plane is empty, so that $j^*\sigma_{n-k+1} = 0 \in A(G(k,n))$.) It follows that if we establish a formula
$$\sigma_a\sigma_b = \sum \gamma_{a,b;c}\,\sigma_c$$
in the Chow ring of $G(k,n)$, the same formula holds true in all $G(k',n')$ with $k' \leq k$ and $n'-k' \leq n-k$. Whenever it happens that i^* or j^* is an isomorphism on $A^{|a|+|b|}$, the formula will also hold in $A(G(k,n+1))$ or $A(G(k+1,n+1))$, respectively. Conditions for this are given in Exercise 3.32.

There is a natural isomorphism $G(k,V) \cong G(n-k,V^*)$ obtained by associating to a k-dimensional subspace $\Lambda \subset V$ the $(n-k)$-dimensional subspace $\Lambda^\perp \subset V^*$ consisting of all those linear functionals on V that annihilate Λ. To see that this is an algebraic isomorphism most directly, use the result from linear algebra given in Exercise 3.13. (One can also argue that these two spaces represent the same functor; see for example the discussion in Section 8.3.1.) This duality carries each Schubert cycle to another Schubert cycle. For example, one checks immediately that $\Sigma_i(W)$, which is the set of k-planes Λ meeting a fixed $(n-k+1-i)$-plane W nontrivially, is carried into the Schubert cycle Σ_{1^i} of $(n-k)$-planes Λ' such that $\dim(\Lambda' \cap W^\perp) \geq i$; that is, such that $\Lambda' + W^\perp \subsetneq V$.

As in the case of $\mathbb{G}(1,3) = G(2,4)$, the Grassmannian $G(k,n)$ has an affine stratification. To see this, set
$$\Sigma_a^\circ = \Sigma_a \setminus \Big(\bigcup_{\substack{b \geq a \\ b \neq a}} \Sigma_b\Big).$$
The Σ_a° are called *Schubert cells.*

Theorem 3.1. *The locally closed subset $\Sigma_a^\circ \subset G$ is isomorphic to the affine space $\mathbb{A}^{k(n-k)-|a|}$; in particular it is smooth and irreducible and of codimension $|a|$ in $G(k,n)$. The tangent space to Σ_a° at a point $[\Lambda]$ is the subspace of $T_{[\Lambda]}G = \operatorname{Hom}(\Lambda, V/\Lambda)$ consisting of those elements φ that send*
$$V_{n-k+i-a_i} \cap \Lambda \ \subset\ \Lambda$$
into
$$\frac{V_{n-k+i-a_i} + \Lambda}{\Lambda} \ \subset\ V/\Lambda$$

for $i = 1, \dots, k$.

Proof. Choose a basis $(e_1, \dots, e_n)$ for V so that

$$V_i = \langle e_1, \dots, e_i \rangle.$$

Now, suppose $[\Lambda] \in \Sigma_a$, and consider the sequence (3.1) of subspaces of Λ; by definition, the first nonzero subspace in the sequence will be $V_{n-k+1-a_1} \cap \Lambda$, the first of dimension 2 will be $V_{n-k+2-a_2} \cap \Lambda$, and so on. We're going to choose a basis $(v_1, \dots, v_k)$ for Λ, accordingly, with $v_1 \in V_{n-k+1-a_1}$, $v_2 \in V_{n-k+2-a_2}$, and so on. In terms of this basis, and the basis $(e_1, \dots, e_n)$ for V, the matrix representative of Λ will look like

$$\begin{pmatrix} * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & 0 \end{pmatrix}$$

(This particular matrix corresponds to the case $k = 4$, $n = 9$ and $a = (3, 2, 2, 1)$.) Note that if Λ were general in G, and we chose a basis for Λ in this way, the corresponding matrix would look like

$$\begin{pmatrix} * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * & * & 0 \\ * & * & * & * & * & * & * & * & * \end{pmatrix}.$$

Thus the Schubert index a_i is the number of "extra zeros" in row i.

Now suppose that $\Lambda \in \Sigma_a^\circ$; that is, $\Lambda \in \Sigma_a$ but not in any of the smaller varieties $\Sigma_{a'}$ for $a' \neq a$ and $a' \geq a$ termwise. In this case $v_i \notin V_{n-k+i-a_i-1}$, so, for each i, the coefficient of $e_{n-k+i-a_i}$ in the expression of v_i as a linear combination of the e_α is nonzero, and this condition characterizes elements of Σ_a° among elements of Σ_a. It follows that Σ_a is the closure of Σ_a°. Given that the coefficient of $e_{n-k+i-a_i}$ in v_i is nonzero, we can multiply v_i by a scalar to make the coefficient 1, obtaining a basis for Λ represented by the rows of a matrix of the form

$$\begin{pmatrix} * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & 1 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & 1 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & 1 & 0 \end{pmatrix}$$

with the 1s appearing in the $(n - k + i - a_i)^{\text{th}}$ column, $i = 1, \dots, k$.

Finally, we can subtract a linear combination of $v_1, \dots, v_{i-1}$ from v_i to kill the coefficients of $e_{n-k+j-a_j}$ in the expression of v_i as a linear combination of the e_α for $j < i$, to arrive at a basis of Λ given by the row vectors of the matrix

$$A = \begin{pmatrix} * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 1 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 0 & 1 & 0 & 0 & 0 \\ * & * & 0 & * & 0 & 0 & * & 1 & 0 \end{pmatrix}$$

Setting $b = \{n-k+1-a_1, \dots, n-a_k\}$, we may describe this by saying that the submatrix of A involving columns b is the identity matrix. We claim that Λ has a unique basis of this form. Indeed, any other basis of Λ has a matrix obtained from this one by left multiplication by a unique invertible $k \times k$ matrix g, and thus has the matrix g in the columns b.

It follows that Σ_a° is contained in the the open subset $U \subset G$ consisting of planes Λ complementary to the span of the $n-k$ basis vectors whose indices are not in b. By the same argument, any element of $U = \Sigma_0^\circ$ has a unique basis given by the rows of a matrix with the identity matrix in positions b; that is, of the form

$$A = \begin{pmatrix} * & * & 1 & * & 0 & 0 & * & 0 & * \\ * & * & 0 & * & 1 & 0 & * & 0 & * \\ * & * & 0 & * & 0 & 1 & * & 0 & * \\ * & * & 0 & * & 0 & 0 & * & 1 & * \end{pmatrix}$$

Thus Σ_a° is a coordinate subspace of $U \cong \mathbb{A}^{k(n-k)}$ defined by the vanishing of $|a|$ coordinates, and it follows that Σ_a° is smooth and irreducible and of codimension $|a|$, as claimed. Since Σ_a is the closure of Σ_a°, it is also irreducible and of codimension $|a|$ in $G(k,n)$.

The statement about tangent spaces follows from the explicit coordinate description of Σ_a above. We identify the open set $U \cong \mathbb{A}^{k(n-k)}$ with the set of $k \times n$ matrices having an identity matrix in positions b. Since the tangent space to an affine space may be identified with the corresponding vector space, the tangent space $\mathrm{Hom}(\Lambda, V/\Lambda)$ to $G(k,n)$ at Λ is given by the set of matrices in the positions b' complementary to b, or more properly by the transposes of these matrices. Given such a tangent vector, we may complete it uniquely to a $k \times n$ matrix with identity block in positions b, and this (or rather its transpose) corresponds to the lifting $\Lambda \to V = V/\Lambda \oplus \Lambda$ inducing the identity map $\Lambda \to \Lambda$. Thus the set of tangent directions at Λ to the affine subspace Σ_a° is identified with the set of matrices in that subspace; and this corresponds precisely to the set of maps in $\mathrm{Hom}(\Lambda, V/\Lambda)$ whose lifting as above sends $V_{n-k+i-a_i}$ into $V_{n-k+i-a_i} + \Lambda$ as claimed. □

We may apply the description of the tangent space to a Schubert cell to deduce transversality of certain intersections:

Corollary 3.2. *If $G = G(k,n)$, then*

$$(\sigma_{n-k})^k = (\sigma_{1^k})^{n-k} = \sigma_{(n-k)^k} \in A^{k(n-k)}(G);$$

that is, $(\sigma_{n-k})^k$ *and* $(\sigma_{1^k})^{n-k}$ *are both equal to the class of a point in the Chow ring of* G.

Proof. From the affine stratification we know that $A_0(G) = \mathbb{Z}$, so it suffices to show that the degree of each of $(\sigma_{n-k})^k$ and $(\sigma_{1^k})^{(n-k)}$ is 1. We can prove this by proving the transversality of Schubert cycles associated to general flags. In characteristic 0 we could invoke Kleiman's theorem 5.15 to prove this, but the description of the tangent spaces will prove it directly and in all characteristics.

We regard G as the variety of k-dimensional subspaces Λ of the n-dimensional vector space V. If $H \subset V$ is a codimension 1 subspace, then

$$\Sigma_{1^k}(H) = \{\Lambda \subset V \ \mid \ \Lambda \subset H\}$$

and the tangent space to $\Sigma_{1^k}(H)$ at the point corresponding to Λ is

$$T_{\Sigma_{1^k}(H),[\Lambda]} = \{\varphi \in \mathrm{Hom}(\Lambda, V/\Lambda)\} \ \mid \ \varphi(\Lambda) \subset H\}.$$

If $H_1, \ldots, H_{n-k}$ are general codimension 1 subspaces, then there is a unique k-plane Λ in $\cap_{i=1}^k \Sigma_{1^k}(H_i)$, namely the intersection $\Lambda = \cap_{i=1}^k H_i$. Further, the tangent spaces intersect only in the zero homomorphism, so the intersection is transverse. This proves that $(\sigma_{1^k})^{(n-k)}$ is the class of a point.

To prove the corresponding statement for $(\sigma_{n-k})^k$ we can make an analogous argument (as suggested in Exercise 3.14), or we can simply use duality: the isomorphism $G(k,n) \cong G(n-k,n)$ introduced above carries σ_{1^k} to σ_k, as we have already remarked, and preserves the degree of 0-cycles. ☐

By Theorem **??**, the classes σ_a for $a = (a_1, \ldots, a_k)$ with $n-k \geq a_1 \geq \cdots \geq a_k \geq 0$ and $|a| = m$ generate $A^m(G)$ additively. In particular, $A^{k(n-k)}(G)$ is generated by the class of a point, $[\Sigma_{(n-k)^k}] = \sigma_{(n-k)^k}$, and thus any two points of G are linearly equivalent (as we could see directly from Proposition 5.17). The existence of the degree homomorphism $\deg : A^{k(n-k)} \to \mathbb{Z}$ that counts points shows that $A^{k(n-k)}(G)$ is actually free on the generator $\sigma_{(n-k)^k}$.

Similarly, $A^0 = \mathbb{Z} \cdot [G] = \mathbb{Z} \cdot \sigma_0$ is free on σ_0. We shall soon see that, more generally, the σ_a with $a = a_1, \ldots, a_k$ satisfying $(n-k) \geq a_1 \geq \cdots \geq a_k \geq 0$ form a free basis of $A(G)$.

3.1.1 Equations of the Schubert cycles

It is a remarkable fact that, under the Plücker embedding $G = G(k,n) \hookrightarrow \mathbb{P}^N$, every Schubert cycle $\Sigma_a \subset G$ is the intersection of G with a "coordinate subspace" of $\mathbb{P}^N$; that is, a subspace defined by the vanishing of an easily described subset of the Plücker coordinates. This is true even at the level of homogeneous ideals:

Theorem 3.3. *Let $\Sigma_a \subset G(k,n) \subset \mathbb{P}^N$ be a Schubert cycle, and let b be the strictly increasing k-tuple $b = (n-k+1-a_1, \dots, n-k+2-a_2, \dots)$. The homogeneous ideal of the Σ_a in $\mathbb{P}^N$ is generated by the homogeneous ideal of the Grassmannian (the "Plücker relations") together with those Plücker coordinates $p_{b'}$ such that $b' \not\leq b$ in the termwise partial order.*

The equations of the Σ_a were studied in Hodge [1943], and this work led to the notion of a straightening law (Doubilet et al. [1974]) and Hodge algebra (De Concini et al. [1982]). A proof of the Theorem in terms of Hodge algebras may be found in the latter publication, along with a proof that the homogeneous coordinate ring of Σ_a is Cohen-Macaulay. The ideas have also been extended to homogeneous varieties for other reductive groups by Lakshmibai, Musili, Seshadri and their co-workers (see for example Seshadri [2007]).

We will prove Theorem 3.3 in the easy case of $G(2,4) = \mathbb{G}(1,3) \subset \mathbb{P}^5$. In this variety the Schubert cycle $\Sigma_{a,b}$ consists of those 2-dimensional subspaces that meet V_{3-a} nontrivially and are contained in V_{4-b}. We must show that the homogeneous ideal of

$$\Sigma_{a,b} \subset G(2,4) \subset \mathbb{P}^5$$

is generated by the Plücker relation $g := p_{1,2}p_{3,4} - p_{1,3}p_{2,4} + p_{14}p_{2,3}$ together with the Plücker coordinates

$$\{p_{i,j} \mid (i,j) \not\leq (3-a, 4-b)\}$$

(note that the condition $(i,j) \not\leq (3-a, 4-b)$ means $i > 3-a$ or $j > 4-b$. Specifically

- Σ_1 is the hyperplane section $p_{3,4} = 0 \subset G$; that is, it's the cone over the nonsingular quadric $\overline{g} = -p_{1,3}p_{2,4} + p_{14}p_{2,3}$ in $\mathbb{P}^3$;
- Σ_2 is is the plane $p_{2,3} = p_{2,4} = p_{3,4} = 0$.
- $\Sigma_{1,1}$ is the plane $p_{1,4} = p_{2,3} = p_{2,4} = 0$.
- $\Sigma_{2,1}$ is the line $p_{1,4} = p_{2,3} = p_{2,4} = p_{3,4} = 0$.
- $\Sigma_{2,2}$ is the point $p_{1,3} = p_{1,4} = p_{2,3} = p_{2,4} = p_{3,4} = 0$.

Proof of Theorem 3.3 for $G(2,4)$. A subspace $L \in \Sigma_{a,b}$ has a basis whose first vector is in V_{3-a}, and therefore has its last $a+1$ coordinates 0; and whose second vector is in V_{4-b} and thus has its last b coordinates equal to 0. If B is the matrix whose rows are the coordinates of these two vectors, then $p_{i,j}$ is (up to sign) the determinant of the submatrix of B involving columns i, j. It follows that if $i > 3-a$ or $j > 4-b$ then $p_{i,j}(L) = 0$, so the given subsets of Plücker coordinates do vanish on the Schubert cycles as claimed.

To show that the ideals of the Schubert cycles are generated by the relation g and the given subsets, observe that each of the subsets is the ideal of the irreducible subvariety described in the Proposition, and these have the same dimensions as the Schubert cycles. For example, we know that $\dim \Sigma_{1,1} = \dim G(2,4) - (1+1) = 2$; and the ideal

$$(g, p_{1,4}, p_{2,3}, p_{2,4}) = (p_{1,4}, p_{2,3}, p_{2,4}) \subset K[p_{1,2}, \dots, p_{3,4}]$$

is the entire ideal of a plane. □

We invite the reader to give the (relatively easy) proof of the set-theoretic version of Theorem 3.3—the statement that the zero locus of the given set of Plücker coordinates on $G(k,n)$ is set-theoretically equal to Σ_a—in Exercise 3.15.

3.2 Intersection products

3.2.1 Intersections in complementary dimension

As in the case of the Grassmannian $\mathbb{G}(1,3)$, we start our description of the Chow ring of G by evaluating intersections of Schubert cycles in complementary codimension. We can do this in characteristic 0 simply by taking Schubert cycles defined relative to two general flags and counting the number of points of intersection; by Kleiman's Theorem 5.15, this will equal the degree of the intersection. In characteristic $p > 0$, we have to verify directly that the intersections are transverse. In general, we'll omit this verification; the concerned reader can supply it using the description in Theorem 3.1 of the tangent spaces to Schubert cycles.

As before, let $G = G(k,V)$ be the Grassmannian of k-dimensional linear subspaces of an n-dimensional vector space V, and choose two general flags $\mathcal{V}$ and $\mathcal{W}$ on V, consisting of subspaces

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V$$

and

$$0 = W_0 \subset W_1 \subset \cdots \subset W_{n-1} \subset W_n = V.$$

To be concrete, note that V_i will intersect W_{n-i+1} in a one-dimensional subspace; if we let e_i be any nonzero vector in this intersection then $e_1, \dots, e_n$ will be a basis for for V in terms of which we can write

$$V_i = \langle e_1, \dots, e_i \rangle \quad \text{and} \quad W_j = \langle e_{n+1-j}, \dots, e_n \rangle.$$

Let $\Sigma_a(\mathcal{V})$ and $\Sigma_b(\mathcal{W})$ be Schubert cycles of complementary codimension, that is, with

$$|a| + |b| = \dim G = k(n-k).$$

Proposition 3.4. *If $|a|+|b| = k(n-k)$ then $\Sigma_a(\mathcal{V})$ and $\Sigma_b(\mathcal{W})$ intersect transversely in a unique point if $a_i + b_{k+1-i} = n-k$ for each $i = 1, \dots, k$, and are disjoint otherwise. Thus*

$$\deg \sigma_a \sigma_b = \begin{cases} 1 & \text{if } a_i + b_{k-i+1} = n-k \text{ for all } i, \\ 0 & \text{otherwise} \end{cases}$$

Proof. Since the two flags $\mathcal{V}$ and $\mathcal{W}$ are general, the Schubert cycles will meet generically transversely. In characteristic 0, this is guaranteed by Kleiman's Theorem 5.15; in arbitrary characteristic we can deduce it from a computation of tangent spaces. Thus

$$\begin{aligned} \deg \sigma_a \sigma_b &= \#\big(\Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{W})\big) \\ &= \#\left\{ \Lambda \;\middle|\; \begin{array}{c} \dim(V_{n-k+i-a_i} \cap \Lambda) \geq i; \text{ and} \\ \dim(W_{n-k+i-b_i} \cap \Lambda) \geq i \ \forall i. \end{array} \right\} \end{aligned}$$

To evaluate the cardinality of this set, consider the conditions in pairs: that is, for each i, consider the i^{th} condition associated to the Schubert cycle $\Sigma_a(\mathcal{V})$:

$$\dim(V_{n-k+i-a_i} \cap \Lambda) \geq i$$

in combination with the $(k-i+1)^{\text{st}}$ condition associated to $\Sigma_b(\mathcal{W})$:

$$\dim(W_{n-i+1-b_{k-i+1}} \cap \Lambda) \geq k-i+1.$$

If these conditions are both satisfied, then the subspaces $V_{n-k+i-a_i} \cap \Lambda$ and $W_{n-i+1-b_{k-i+1}} \cap \Lambda$, having greater than complementary dimension in Λ, must have nonzero intersection; in particular, we must have

$$V_{n-k+i-a_i} \cap W_{n-i+1-b_{k-i+1}} \neq 0,$$

and since the flags $\mathcal{V}$ and $\mathcal{W}$ are general, this in turn says we must have

$$n-k+i-a_i+n-i+1-b_{k-i+1} \geq n+1$$

or in other words

$$a_i + b_{k-i+1} \leq n-k.$$

If we have equality in this last inequality, then the subspaces $V_{n-k+i-a_i}$ and $W_{n-i+1-b_{k-i+1}}$ will meet in a one-dimensional vector space Γ_i, which will necessarily be contained in Λ.

We have thus seen that $\Sigma_a(\mathcal{V})$ and $\Sigma_b(\mathcal{W})$ will be disjoint unless $a_i + b_{k-i+1} \leq n-k$ for all i. But from the equality

$$|a|+|b| = \sum_{i=1}^{k} (a_i + b_{k-i+1}) = k(n-k)$$

we see that if $a_i + b_{k-i+1} \leq n-k$ for all i then we must have $a_i + b_{k-i+1} = n-k$ for all i. Moreover, in this case any Λ in the intersection $\Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{W})$

must contain each of the k subspaces Γ_i, so there is a unique such Λ, equal to the span of these one-dimensional spaces, as required. $\square$

Corollary 3.5. *The classes of the Schubert cycles form a free basis for $A(G)$. In fact, the intersection forms $A^m(G) \times A^{\dim G - m}(G) \to \mathbb{Z}$ have the Schubert classes as dual bases.* $\square$

In view of the explicit duality between $A^m(G)$ and $A^{k(n-k)-m}(G)$ given by Proposition 3.4, it makes sense to introduce one more bit of notation: for any Schubert index $a = (a_1, \ldots, a_k)$, we'll define the *dual index* to be the Schubert index $a^* = (n-k-a_k, \ldots, n-k-a_1)$. In these terms, Proposition 3.4 says that $\deg(\sigma_a \sigma_b) = 1$ if $b = a^*$ and 0 otherwise.

Corollary 3.5 suggests a general approach to determining the coefficients in the expression of the class of a cycle as a linear combination of Schubert classes: if $\Gamma \subset G$ is any cycle of pure codimension m, we can write

$$[\Gamma] = \sum_{|a|=m} \gamma_a \sigma_a.$$

To find the coefficient γ_a, we intersect both sides with the Schubert cycle $\Sigma_{a^*}(\mathcal{V}) = \Sigma_{n-k-a_k,\ldots,n-k-a_1}(\mathcal{V})$ for a general flag $\mathcal{V}$; we have

$$\begin{aligned}\gamma_a &= \deg([\Gamma] \cdot \sigma_{a^*}) \\ &= \#\big(\, \Gamma \cap \Sigma_{a^*}(\mathcal{V}) \,\big).\end{aligned}$$

In fact, we have used exactly this approach—called the method of *undetermined coefficients*—in calculating classes of various cycles in $\mathbb{G}(1,3)$ in the preceding chapter; Corollary 3.5 says that it is more generally applicable in any Grassmannian. In particular, have

Corollary 3.6. *If σ_a and $\sigma_b \in A(G)$ are any Schubert classes on $G = G(k,n)$, then the product*

$$\sigma_a \sigma_b = \sum_{|c|=|a|+|b|} \gamma_{a,b;c} \sigma_c$$

where

$$\gamma_{a,b;c} = \deg(\sigma_a \sigma_b \sigma_{c^*}).$$

Since for general flags $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ the Schubert cycles $\Sigma_a(\mathcal{U}), \Sigma_b(\mathcal{V})$ and $\Sigma_{c^*}(\mathcal{W})$ are dimensionally transverse by Kleiman's theorem, the coefficients $\gamma_{a,b;c} = \deg(\sigma_a \sigma_b \sigma_{c^*})$ are nonnegative integers. They are called *Littlewood-Richardson coefficients*, and they appear in many combinatorial contexts. Note that if we adopt the convention that $\sigma_a = 0 \in A^{|a|}(G(k,n))$ if a fails to satisfy the conditions $n-k \geq a_1 \geq \cdots \geq a_k \geq 0$ and $a_l = 0\ \forall l > k$, then the Littlewood-Richardson coefficients $\gamma_{a,b;c}$ depend only on the indices a, b and c, and not on k and n.

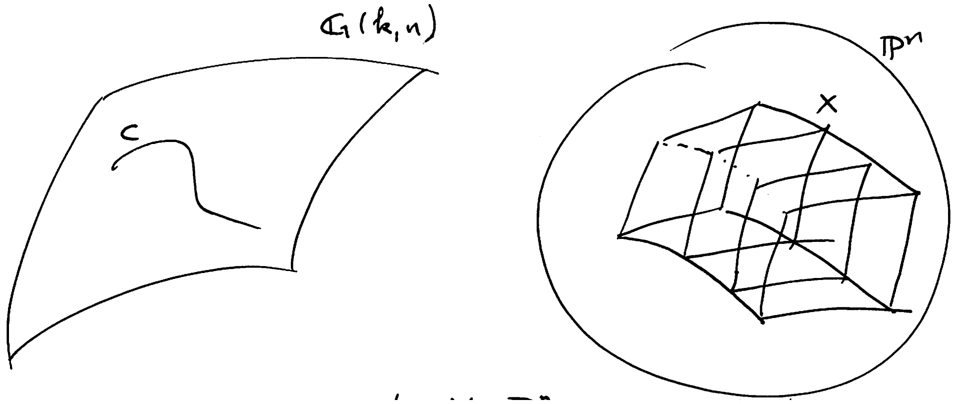


FIGURE 3.1. The variety $X \subset \mathbb{P}^n$ swept out by a 1-parameter family $C \subset \mathbb{G}(k, n)$ of k-planes

Using this approach, we'll be able to evaluate a number of products of Schubert classes in Section 3.2.3.

3.2.2 Varieties swept out by linear spaces

We'll say more about intersections of Schubert classes in Section 3.2.3. For now, we'll pause to give an application of what we know so far.

Let $C \subset \mathbb{G}(k, n)$ be a curve, and consider the variety $X \subset \mathbb{P}^n$ swept out by the linear spaces corresponding to points of C: that is,

$$X = \bigcup_{[\Lambda] \in C} \Lambda \ \subset \ \mathbb{P}^n.$$

(See Figure 3.1). We'd like to relate the geometry of X to that of C; in particular, Keynote Question (b) asks us to find the degree of X in case $C \subset \mathbb{G}(1, 3) \subset \mathbb{P}^5$ is a twisted cubic curve.

Observe to begin with that X is indeed a closed subvariety of $\mathbb{P}^n$: if

$$\Phi = \{(\Lambda, p) \in \mathbb{G}(k, n) \times \mathbb{P}^n \ \mid \ p \in \Lambda\}$$

is the universal k-plane over $\mathbb{G}(k, n)$, as described in Section **??**, and $\alpha : \Phi \to \mathbb{G}(k, n)$ and $\beta : \Phi \to \mathbb{P}^n$ are the projections, then we can write

$$X = \beta\big(\alpha^{-1}(C)\big).$$

Now, suppose that a general point $x \in X$ lies on a unique k-plane $\Lambda \in C$—that is, the map $\beta : \alpha^{-1}(C) \to X \subset \mathbb{P}^n$ is birational, so that in particular $\dim(X) = k + 1$. The degree of X is the number of points of intersection of X with a general $(n - k - 1)$-plane $V \subset \mathbb{P}^n$; since these points are general points of X, the number is the number of k-planes Λ that meet V. In other words, we have

$$\begin{aligned}\deg(X) &= \#(X \cap V)\\ &= \#(C \cap \Sigma_1(V))\\ &= \deg([C]\cdot\sigma_1) \quad \text{(by Kleiman)}\\ &= \deg(C)\end{aligned}$$

where by the degree of C we mean the degree under the Plücker embedding of $\mathbb{G}(k,n)$. Thus, for example, the answer to keynote question (b) is that the surface $X \subset \mathbb{P}^3$ swept out by the lines corresponding to a twisted cubic $C \subset \mathbb{G}(1,3) \subset \mathbb{P}^5$ is cubic. We will see a more detailed (and beautiful) picture of the geometry of X in Exercises 3.21-3.24.

If $Z \subset \mathbb{G}(k,n)$ is a variety of any dimension m, we can similarly form the variety $X \subset \mathbb{P}^n$ swept out by the planes of Z. Its degree—assuming it has the expected dimension $k + \dim Z$, and that a general point of X lies on only one plane $\Lambda \in Z$—is expressible in terms of the Schubert coefficients of the class $[Z] \in A_m(\mathbb{G}(k,n))$, though it's not in general equal to the degree of Z; this is the content of Exercise 3.20.

3.2.3 Pieri's formula

Corollary 3.6 says that to determine the products of all Schubert classes, we simply have to be able to evaluate the degrees of triple intersections of Schubert cycles in complementary dimension. Unfortunately, this is not always easy to do, and many questions remain in general. There exist beautiful algorithms for calculating the Littlewood-Richardson coefficients $\sigma_{a,b;c}$ (see Coskun [2009] and Vakil [2006a]), but even simple questions such as, "when is $\sigma_{a,b;c} \neq 0$?" and "when is $\sigma_{a,b;c} > 1$?" do not admit simple answers in general. (In case you're curious, the smallest example of a product of two Schubert classes where another Schubert class appears with multiplicity > 1 is the square $\sigma_{2,1}^2$ in $G(3,6)$, as you'll be asked to work out in Exercise 3.36.)

There is one situation, however, in which we do have a simple formula for the product of Schubert classes. This is when one of the classes has the special form $\sigma_b = \sigma_{b,0,\ldots,0}$; these are called *special Schubert classes* and the formula in this case is called *Pieri's formula*:

Proposition 3.7 (Pieri's formula)**.** *For any Schubert class $\sigma_a \in A(G)$ and any integer b, we have*

$$(\sigma_b \cdot \sigma_a) = \sum_{\substack{|c|=|a|+b\\ a_i \le c_i \le a_{i-1} \forall i}} \sigma_c$$

Proof. [1] By Corollary 3.6, Pieri's formula is tantamount to the assertion that for any Schubert index c with $|c| = |a| + b$,

$$\deg(\sigma_a \sigma_b \sigma_{c^*}) = \begin{cases} 1, & \text{if } a_i \leq c_i \leq a_{i-1} \forall i; \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

To prove this, we'll look at the corresponding Schubert cycles $\Sigma_a(\mathcal{V}), \Sigma_b(\mathcal{U})$ and $\Sigma_{c^*}(\mathcal{W})$, defined with respect to general flags $\mathcal{V}$, $\mathcal{U}$ and $\mathcal{W}$; we'll show that their intersection is empty in case c_i violates the condition $a_i \leq c_i \leq a_{i-1}$ for any i, and consists of a single point if these inequalities are all satisfied. This will establish the formula in characteristic 0, where Kleiman's theorem assures us that $\Sigma_a(\mathcal{V}), \Sigma_b(\mathcal{U})$ and $\Sigma_{c^*}(\mathcal{W})$ intersect transversely; in characteristic p we require in addition a transversality argument based on the identification of tangent spaces to Schubert cycles given in Theorem 3.1, which we leave to the reader.

Our starting point is the same as in the case of complementary dimension. We start by writing out the conditions associated to the Schubert cycles $\Sigma_a(\mathcal{V})$ and $\Sigma_{c^*}(\mathcal{W})$:

$$\Sigma_a(\mathcal{V}) = \{\Lambda \mid \dim(\Lambda \cap V_{n-k+i-a_i} \geq i \ \forall i\}$$

and

$$\Sigma_{c^*}(\mathcal{W}) = \{\Lambda \mid \dim(\Lambda \cap W_{i+c_{k+1-i}} \geq i \ \forall i\}.$$

Combining the i^{th} condition in the first and the $(k+1-i)^{\text{th}}$ condition in the second, we see that for any $\Lambda \in \Sigma_a(\mathcal{V}) \cap \Sigma_{c^*}(\mathcal{W})$ we have

$$\Lambda \cap V_{n-k+i-a_i} \cap W_{k+1-i+c_i} \neq 0$$

Of course, this necessitates that $V_{n-k+i-a_i} \cap W_{k+1-i+c_i} \neq 0$, so the first thing we see is that *if $c_i < a_i$ for any i, then $\Sigma_a(\mathcal{V}) \cap \Sigma_{c^*}(\mathcal{W}) = \varnothing$.* For the remainder of this argument, then, we'll assume that $c_i \geq a_i \ \forall i$.

Next, we introduce a sequence of k subspaces of V: for each i, we let

$$A_i = V_{n-k+i-a_i} \cap W_{k+1-i+c_i}.$$

These are nonzero vector spaces of dimensions $\dim A_i = c_i - a_i + 1$, and by what we've said, any plane $\Lambda \in \Sigma_a(\mathcal{V}) \cap \Sigma_{c^*}(\mathcal{W})$ will be spanned by its intersections with the spaces A_i. Beyond that, the key fact about them is that *they are linearly independent if and only if $c_i \leq a_{i-1}$ for all i.* (This is probably best seen in terms of the basis for V introduced in Section 3.2.1: if $V_i = \langle e_1, \ldots, e_i \rangle$ and $W_j = \langle e_{n-j+1}, \ldots, e_n \rangle$, then $A_i = \langle e_{n-k+i-c_i}, \ldots, e_{n-k+i-a_i} \rangle$; and the condition $c_i \leq a_{i-1}$ amounts to the condition that the two successive ranges of indices $n - k + i - 1 -$

[1] This proof was shown to us by Izzet Coskun

$c_{i-1}, \dots n-k+i-1-a_{i-1}$ and $n-k+i-c_i, \dots n-k+i-a_i$ don't overlap.) In other words, if we let

$$A = \langle A_1, \dots, A_k \rangle$$

be the span of the spaces A_i, then we have

$$\dim A \le \sum c_i - a_i + 1 = k + b.$$

with equality holding if and only if $c_i \le a_{i-1}$ for all i.

Now we introduce the final condition: the Schubert cycle $\Sigma_b(\mathcal{U})$ consists of k-planes that have nonzero intersection with a general linear subspace $U = U_{n-k+1-b} \subset V$ of dimension $n-k+1-b$. For there to be any $\Lambda \in \Sigma_a(\mathcal{V}) \cap \Sigma_{c^*}(\mathcal{W})$ satisfying this condition requires that $A \cap U \neq 0$, and hence, since U is general, that $\dim A \ge k + b$ Thus, if $c_i > a_{i-1}$ for any i, then we'll have $\Sigma_a(\mathcal{V}) \cap \Sigma_{c^*}(\mathcal{W}) \cap \Sigma_b(\mathcal{U}) = \varnothing$. Thus, we can assume $c_i \le a_{i-1}$ $\forall i$ and hence $\dim A = k + b$.

Finally, since $U \subset V$ is a general subspace of codimension $k+b-1$, it will meet A in a one-dimensional subspace. Choose v any nonzero vector in this intersection. Since $A = \oplus A_i$, we can write v uniquely as a sum

$$v = v_1 + \cdots + v_k \quad \text{with} \quad v_i \in A_i.$$

Suppose now that $\Lambda \in \Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{U}) \cap \Sigma_{c^*}(\mathcal{W})$ satisfies all the Schubert conditions above. Since $\Lambda \subset A$ and $\Lambda \cap U \neq 0$, Λ must contain the vector v; and since Λ is spanned by its intersections with the A_i it follows that Λ must contain the vectors v_i as well. Thus, we see that *the intersection* $\Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{U}) \cap \Sigma_{c^*}(\mathcal{W})$ *will consist of the single point corresponding to the plane* $\Lambda = \langle v_1, \dots, v_k \rangle$ *spanned by the* v_i, and we're done. □

3.3 Grassmannians of lines

Let $G = G(2, V)$ be the Grassmannian of 2-dimensional subspaces of an n-dimensional vector space V, or, equivalently, lines in the projective space $\mathbb{P}V \cong \mathbb{P}^{n-1}$. The Schubert cycles on G with respect to a flag $\mathcal{V}$ are of the form

$$\Sigma_{a_1,a_2}(\mathcal{V}) = \{\Lambda \mid \Lambda \cap V_{n-1-a_1} \neq 0 \text{ and } \Lambda \subset V_{n-a_2}\}.$$

In this case, Pieri's formula (Proposition 3.7) allows us to give a closed-form expression for the product of any two Schubert classes:

Proposition 3.8.

$$\sigma_{a_1,a_2}\sigma_{b_1,b_2} = \sum_{\substack{|c|=|a|+|b| \\ a_1+b_1 \ge c_1 \ge a_1, b_1}} \sigma_{c_1,c_2}$$

Proof. Let's start with the simplest cases, where the intersection of general Schubert cycles is again a Schubert cycle: if $b_1 = b_2 = b$, then the Schubert cycle

$$\Sigma_{b,b}(\mathcal{W}) = \{\Lambda \mid \Lambda \subset W_{n-b}\}$$

so that for any a_1, a_2 we have

$$\begin{aligned}
\Sigma_{a_1,a_2}(\mathcal{V}) \cap \Sigma_{b,b}(\mathcal{W}) &= \left\{ \Lambda \ \middle|\ \begin{array}{l} \Lambda \cap V_{n-1-a_1} \neq 0; \\ \Lambda \subset V_{n-a_2};\ \text{and} \\ \Lambda \subset W_{n-b} \end{array} \right\} \\
&= \left\{ \Lambda \ \middle|\ \begin{array}{l} \Lambda \cap (V_{n-1-a_1} \cap W_{n-b}) \neq 0 \text{ and} \\ \Lambda \subset (V_{n-a_2} \cap W_{n-b}) \end{array} \right\} \\
&= \Sigma_{a_1+b,a_2+b}\big(V_{n-1-a_1} \cap W_{n-b}, V_{n-a_2} \cap W_{n-b}\big).
\end{aligned}$$

Thus, we have

$$\sigma_{a_1,a_2}\sigma_{b,b} = \sigma_{a_1+b,a_2+b}. \tag{3.2}$$

Now, suppose we want to intersect an arbitrary pair of Schubert classes σ_{a_1,a_2} and σ_{b_1,b_2}. We can write

$$\begin{aligned}
\sigma_{a_1,a_2}\sigma_{b_1,b_2} &= (\sigma_{a_1-a_2,0}\sigma_{a_2,a_2})(\sigma_{b_1-b_2,0}\sigma_{b_2,b_2}) \\
&= \sigma_{a_1-a_2,0}\,\sigma_{b_1-b_2,0}\,\sigma_{a_2+b_2,a_2+b_2}
\end{aligned}$$

and if we can evaluate the product of the first two terms in the last expression, we can use (3.2) to finish the calculation.

But this is exactly what Pieri gives us: if $a \geq b$, Pieri says that

$$\sigma_{a,0}\sigma_{b,0} = \sigma_{a+b,0} + \sigma_{a+b-1,1} + \cdots + \sigma_{a,b}$$

and the general statement follows. $\square$

We can use this description of the Chow ring of $\mathbb{G}(1,n)$ (and a little combinatorics) to answer Keynote Question (d): what is the degree of the Grassmannian $\mathbb{G}(1,n)$ under the Plücker embedding? We observe first that, since the hyperplane class on $\mathbb{P}(\wedge^2 V)$ pulls back to the class $\sigma_1 \in A^1(\mathbb{G}(1,n))$, we have

$$\deg(\mathbb{G}(1,n)) = \deg\left(\sigma_1^{2n-2}\right).$$

To evaluate this product, we make a diagram of the Schubert classes σ_a in $\mathbb{G}(1,n)$, with the inclusions among the corresponding Schubert cycles $\Sigma_a(\mathcal{V})$ indicated by vertical or diagonal arrows (the diagram shown is the case $n = 5$):

$$\begin{tikzcd}
\sigma_0 & & & & \\
\sigma_1 \arrow[u] & & & & \\
\sigma_2 \arrow[u] & \sigma_{1,1} \arrow[ul] & & & \\
\sigma_3 \arrow[u] & \sigma_{2,1} \arrow[ul] \arrow[u] & & & \\
\sigma_4 \arrow[u] & \sigma_{3,1} \arrow[ul] \arrow[u] & \sigma_{2,2} \arrow[ul] & & \\
 & \sigma_{4,1} \arrow[ul] \arrow[u] & \sigma_{3,2} \arrow[ul] \arrow[u] & & \\
 & & \sigma_{4,2} \arrow[ul] \arrow[u] & \sigma_{3,3} \arrow[ul] & \\
 & & & \sigma_{4,3} \arrow[ul] \arrow[u] & \\
 & & & & \sigma_{4,4} \arrow[ul]
\end{tikzcd}$$

In terms of this diagram, the rule expressed in Proposition 3.8 for multiplication by σ_1 is simple: the product of any Schubert class $\sigma_{a,b}$ with σ_1 is just the sum of the Schubert classes below it—that is, connected by an inclusion—in the next row. In particular, the degree $\deg(\sigma_1^{2n-2})$ of the Grassmannian is simply the number of paths through this diagram, starting with σ_0 and ending with $\sigma_{n-1,n-1}$, of minimal length $2n-2$ (that is, always going downward). If we designate such a path by a sequence of $n-1$ "v"s and $n-1$ "d"s—corresponding to vertical and diagonal inclusions respectively—these are all the sequences of $n-1$ "v"s and $n-1$ "d"s satisfying the condition that, reading from left to right, there are never more "d"s than "v"s. Equivalently, if we associate to a "v" a left parenthesis and to a "d" a right parenthesis, this is the number of ways in which $n-1$ pairs of parentheses can appear in a grammatically correct sentence. This is called the $(n-1)^{\text{st}}$ *Catalan number*; a standard combinatorial argument (see for example Stanley [1999]) gives

$$c_{n-1} = \frac{(2n-2)!}{n!(n-1)!}.$$

In sum, we have

Proposition 3.9. *The degree of the Grassmannian* $G(2,n) \subset \mathbb{P}(\wedge^2 V)$ *is*

$$\deg G(2,n) = \frac{(2n-2)!}{n!(n-1)!}.$$

Note that this number also represents the answer to the enumerative problem: how many lines in $\mathbb{P}^n$ meet each of $2n-2$ general $(n-2)$-planes $V_1, \dots, V_{2n-2} \subset \mathbb{P}^n$?

In fact, Pieri's formula (Proposition 3.7) gives us the means to answer the generalization of Keynote Question (d) to all Grassmannians: since σ_1 is the class of the hyperplane section of the Grassmannian in its Plücker embedding, the degree of the Grassmannian in that embedding is the degree of $\sigma_1^{k(n-k)}$. This will be worked out (with the aid the of Hook formula from combinatorics) in Exercise 3.38; the answer is that

$$\deg(G(k,n)) = \big(k(n-k)\big)! \prod_{i=0}^{k-1} \frac{i!}{(n-k+i)!}.$$

We can also use the description of $A(\mathbb{G}(1,n))$ given in Proposition 3.8 to answer Keynote Question (a): if $V_1, \dots, V_4 \cong \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ are four general n-planes, how many lines $L \subset \mathbb{P}^{2n+1}$ meet all four? The answer is the cardinality of the intersection $\bigcap \Sigma_n(V_i) \subset \mathbb{G}(1, 2n+1)$; given transversality—a consequence of Kleiman's theorem 5.15 in characteristic 0, and checkable directly in arbitrary characteristic via the description of tangent spaces to Schubert cycles in Proposition 3.1—this is just the degree of the product $\sigma_n^4 \in A(\mathbb{G}(1, 2n+1))$. Applying Proposition 3.8, we have

$$\sigma_n^2 = \sigma_{2n} + \sigma_{2n-1,1} + \cdots + \sigma_{n+1,n-1} + \sigma_{n,n};$$

since each term squares to the class of a point and all pairwise products are zero, we have

$$\deg(\sigma_n^4) = n+1,$$

and this is the answer to our question.

We will see in Exercise 3.26 another way to arrive at this number, in a manner analogous to the alternative solution to the four-line problem given in Exercise **??**; and in Exercise 3.27 a nice geometric consequence.

3.3.1 Dynamic specialization

We now want to revisit the discussion, initiated in Section 2.4.4, of the method of *specialization*, in particular as it may be applied to determine the products of Schubert classes. This time we'll encounter situations where we need a stronger and more broadly applicable version of this technique, called *dynamic specialization*.

To recall: in Section 2.4.4 we described an alternative approach to establishing the relation $\sigma_1^2 = \sigma_{11} + \sigma_2$ in the Chow ring of the Grassmannian $\mathbb{G}(1,3)$. Instead of taking two general translates of the Schubert cycle $\Sigma_1(L) \subset \mathbb{G}(1,3)$—whose intersection was necessarily generically transverse, but the class of whose intersection required additional work to calculate—we considered the intersection $\Sigma_1(L) \cap \Sigma_1(L')$ where L and $L' \subset \mathbb{P}^3$ were not general, but incident lines. The tradeoff here is that now the intersection is visibly a union of Schubert cycles—specifically, if $p = L \cap L'$ is their point of intersection and $H = \overline{L\,L'}$ their span, we have

$$\Sigma_1(L) \cap \Sigma_1(L') = \Sigma_2(p) \cup \Sigma_{1,1}(H)—$$

but we have to work a little to see that the intersection is indeed generically transverse.

Suppose now we're working with the Grassmannian $G = \mathbb{G}(1,4)$ of lines in $\mathbb{P}^4$ and we try to use an analogous method to determine the product $\sigma_2^2 \in A^4(G)$—that is, the class of the locus of lines meeting each of two given lines in $\mathbb{P}^4$. Specifically, we'd like to find a pair of lines $L, M \subset \mathbb{P}^4$ such that the two cycles

$$\Sigma_2(L) = \{\Lambda \mid \Lambda \cap L \neq \varnothing\} \quad \text{and} \quad \Sigma_2(M) = \{\Lambda \mid \Lambda \cap M \neq \varnothing\}$$

representing the class σ_2 are special enough that the class of the intersection is clear, but still sufficiently general that they intersect generically transversely.

Unfortunately, we can't. If the lines L and M are disjoint, they are effectively a general pair, and the intersection is not a union of Schubert cycles. But if L meets M, say at a point p, then the locus of lines through p forms a 3-dimensional component of the intersection $\Sigma_2(L) \cap \Sigma_2(M)$, so the intersection is not even dimensionally transverse.

How do we deal with this? By considering a *family* of lines M_t in $\mathbb{P}^4$, parametrized by $t \in \mathbb{A}^1$, with M_t disjoint from L for $t \neq 0$, and with M_0 meeting L at a point p. We arrive then at a family of intersection cycles $\Sigma_2(L) \cap \Sigma_2(M_t)$: specifically, we consider the subvariety

$$\Phi^\circ = \{(t,\Lambda) \in \mathbb{A}^1 \times G \;\mid\; t \neq 0 \text{ and } \Lambda \in \Sigma_2(L) \cap \Sigma_2(M_t)\}$$

and its closure $\Phi \subset \mathbb{A}^1 \times G$. Since M_t is disjoint from L for $t \neq 0$, the fiber $\Phi_t = \Sigma_2(L) \cap \Sigma_2(M_t)$ of Φ over $t \neq 0$ represents the class σ_2^2, and it follows that Φ_0 does as well. The point is, when we look at the fiber Φ_0 we're looking *not at the intersection* $\Sigma_2(L) \cap \Sigma_2(M_0)$ *of the limiting cycles, but rather at the limit of the intersection cycles*, which is necessarily of the expected dimension.

That said, how do we characterize the fiber Φ_0? Clearly it's contained in the intersection $\Sigma_2(L) \cap \Sigma_2(M_0)$, but as we said it must be properly contained in it. In other words, a line Λ arising as the limit of lines Λ_t meeting both L and M_t must satisfy some additional condition beyond meeting both L and M_0, and we need to say what that condition is.

Fortunately, the answer to that question is reasonably straightforward. For $t \neq 0$, the lines L and M_t together span a hyperplane $H_t = \overline{L\, M_t} \cong \mathbb{P}^3 \subset \mathbb{P}^4$, and by the valuative criterion these hyperplanes H_t must have a limit H_0 as $t \to 0$. Moreover, if $\{\Lambda_t\}$ is a family of lines with Λ_t meeting both L and M_t for $t \neq 0$, we see that Λ_t must lie in H_t and hence that *the limiting line* Λ_0 *must be contained in* H_0. Now, if Λ_0 does not pass through the point $p = L \cap M_0$, then it must be contained in the 2-plane $P = \overline{L\, M_0}$ (so that the condition $\Lambda_0 \subset H_0$ is redundant); in sum, we conclude that the support of Φ_0 must be contained in the union of the two 2-dimensional Schubert cycles

$$\begin{aligned} \Phi_0 &\subset \{\Lambda \mid \Lambda \subset P\} \cup \{\Lambda \mid p_0 \in \Lambda \subset H_0\} \\ &= \Sigma_{2,2}(P) \cup \Sigma_{3,1}(p_0, H_0). \end{aligned}$$

In fact, we will see in Exercise 3.28 that the support of Φ_0 is all of $\Sigma_{2,2}(P) \cup \Sigma_{3,1}(p_0, H_0)$, and in Exercise 3.29 that Φ_0 is generically reduced. Thus, the cycle associated to the scheme Φ_0 is exactly the sum $\Sigma_{2,2}(P) + \Sigma_{3,1}(p_0, H_0)$, and we can deduce the formula

$$\sigma_2^2 = \sigma_{3,1} + \sigma_{2,2} \in A^4(\mathbb{G}(1,4)).$$

This is a good example of the method of dynamic specialization, in which we consider not a special pair of cycles representing given Chow classes and intersecting generically transversely, but a family of representative pairs specializing from pairs that do intersect transversely to a pair that may not. The goal is then to describe, as we did above, *not the intersection of the limits, but the limit of the intersections.* As we indicated, it's a very powerful and broadly applicable technique, one that is ubiquitous in the subject. For example, it can be used to give a highly effective algorithm for the product of two arbitrary Schubert classes in $G(k,n)$ (or even their generalizations to flag manifolds); the calculation we've just sketched is a starting point for the general algorithms of Coskun [2009] and Vakil [2006a]. For another example of its application, see Griffiths and Harris [1980].

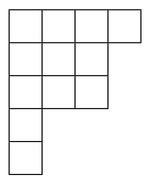
One caveat: a further wrinkle in the technique of dynamic specialization (present in all the examples cited above) is that to carry out the calculation of an intersection of Schubert cycles we may have to specialize in stages. For a relatively elementary example of this, see Exercise 3.30.

3.4 How to count Schubert cycles with Young diagrams

We can count the Schubert cycles as follows:

Corollary 3.10. *$A(G(k,n)) \cong \mathbb{Z}^{\binom{n}{k}}$ as abelian groups.*

For the proof, and for many other purposes, it is convenient to represent the Schubert class $\sigma_{a_1,\dots,a_k}$ by a *Young diagram*; that is, as a collection of left-justified rows of boxes, the i-th row having length a_i. For example, $\sigma_{4,3,3,1,1}$ would be represented by

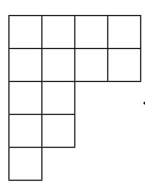
$\sigma_{4,3,3,1,1} \quad \longleftrightarrow$ 

(Warning: there are many different conventions in use for interpreting the correspondence between Schubert cycles and Young diagrams!) The condition that $n-k \geq a_1 \geq a_k \geq 0$ means that the Young diagram fits into a box with k rows and $n-k$ columns, and the rows of the diagram are weakly decreasing in length from top to bottom. As another example, the relation between a Schubert cycle and the unique Schubert cycle τ such that $\sigma\tau$ is the class of a point, described in Proposition 3.4 could be described by saying that the Young diagrams of σ and τ, after rotating the latter 180°, are complementary in the $k \times (n-k)$ box; if $\sigma = \sigma_{4,3,3,1,1} \in A(G(5,10))$, for example, then τ is shown in the picture

σ	σ	σ	σ	τ
σ	σ	σ	τ	τ
σ	σ	σ	τ	τ
σ	τ	τ	τ	τ
σ	τ	τ	τ	τ

;

that is,

$\tau \quad \longleftrightarrow$  .

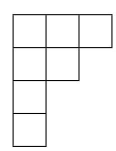
With this notation, the proof of the Corollary is easy (though clever!):

Proof of Corollary 3.10. The number of Schubert cycles is the same as the number of Young diagrams that fit into a $k \times (n-k)$ box of squares B. To count these, we associate to each Young diagram Y in B its "right boundary" L: this is the path, consisting of horizontal and vertical segments of unit length, which starts from the upper right corner of the $k \times (n-k)$ box and ends at the lower left corner of the box, such that the squares in in Y are those to the left of L. (For example, in the case of the Young diagram associated to $\sigma_{4,3,3,1,1} \subset G(5,10)$, illustrated above, we may describe L by the sequence $h, v, h, v, v, h, h, v, v$ where h and v denote horizontal and vertical segments, respectively, and we start from the upper right corner.)

Of course the number of h terms in any such boundary must be $n-k$, the width of the box, and the number of v terms must be k, the height of the box. Thus the length of the boundary is n, and giving the boundary is equivalent to specifying which k steps will be vertical; that is, the number of Young diagrams in B is $\binom{n}{k}$ as required. $\square$

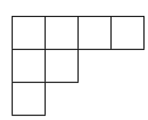
The correspondence between Schubert classes and Young diagrams behaves well with respect to many basic operations on Grassmannians. For example, under the duality $G(k,n) \cong G(n-k,n)$ the Schubert cycle corresponding to the Young diagram Y is taken to the Schubert cycle corresponding to the Young diagram Z that is the *transpose* of Y, that is, the diagram obtained by flipping Y around a 45° line running northwest-southeast. For example if

$$\sigma_{3,2,1,1} \in A(G(4,7)) \quad \longleftrightarrow$$



then the corresponding Schubert cycle in $G(3,7)$ is

$$\sigma_{4,2,1} \in A(G(3,7)) \quad \longleftrightarrow$$

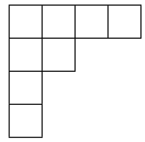


This is reasonably straightforward to verify, and is the subject of Exercise 3.31.

Note that Pieri's formula can also be described in terms of the Young diagrams. For example, in the special case of intersection with σ_1 it says that the Schubert cycles appearing in the product $\sigma_1\sigma_a$ (all with coefficient 1) correspond to Young diagrams obtained from the Young diagram of σ_a *by adding one box at the end of any row*, as long as the result is still a

Young diagram: for example, if

$$\sigma_{4,2,1,1} \in A(G(4,8)) \qquad \longleftrightarrow$$



we can add a box in either the first, second, third or fifth row, to obtain the expression

$$\sigma_1\sigma_{4,2,1,1} = \sigma_{5,2,1,1} + \sigma_{3,3,1,1} + \sigma_{3,2,2,1} + \sigma_{3,2,1,1,1}.$$

The combinatorics of Young diagrams is an extremely rich subject with many applications. See for example Fulton [1997] for an introduction.

3.5 Linear spaces on quadrics

Using these tools we can generalize the calculation in Section 2.5 of the class of the locus of lines on a quadric surface to a description of the class of the locus of planes of any dimension on a smooth quadric hypersurface of any dimension. We will assume that the characteristic of the ground field is 0, so that we can use Kleiman's Theorem 5.15 and so that a nonsingular form $Q(x)$ of degree 2 on $\mathbb{P}V$ can be written in the form $Q(x) = q(x,x)$, where $q(x,y)$ is a nonsingular bilinear form $V \times V \to K$. A small calculation, using the fact that $2 \neq 0 \in K$, shows that a linear subspace $\mathbb{P}W \subset \mathbb{P}V$ lies on the quadric $Q(x) = 0$ if and only if W is *isotropic* for q; that is, $q(W,W) = 0$. Thus we want to find the class of the locus $\Phi \subset G = G(k,V)$ of isotropic k-planes for q.

To start, we want to find the dimension of Φ. There are a number of ways to do this; probably the most elementary is to count bases for isotropic subspaces. To find a basis for an isotropic subspace, we can start with any vector v_1 with $q(v_1,v_1) = 0$; then choose $v_2 \in \langle v_1\rangle^\perp \setminus \langle v_1\rangle$ with $q(v_2,v_2) = 0$, $v_3 \in \langle v_1,v_2\rangle^\perp \setminus \langle v_1,v_2\rangle$ with $q(v_3,v_3) = 0$, and so on. Since $\langle v_1,\dots,v_i\rangle \subset \langle v_1,\dots,v_i\rangle^\perp$, this necessarily terminates when $i \geq n/2$; in other words, *a nondegenerate quadratic form will have no isotropic subspaces of dimension strictly greater than half the dimension of the ambient space.* (We could also see this by observing that q defines an isomorphism of V with its dual V^* that carries any isotropic subspace $\Lambda \subset V$ into its annihilator $\Lambda^\perp \subset V^*$.)

In this process the allowable choices for v_1 correspond to points on the quadric $Q(x) = 0$; those for v_2 correspond to the points on the quadric $Q|_{v_1^\perp}$, and so forth. In general the v_i form a locally closed subset of V of dimension $n - i$. Thus the space of all bases for isotropic k-planes has dimension

$$(n-1)+\cdots+(n-k)=k(n-k)+\binom{k}{2}.$$

Since there is a k^2-dimensional family of bases for a given isotropic k-plane, the space of such planes has dimension

$$k(n-k)+\binom{k}{2}-k^2=k(n-k)-\binom{k+1}{2},$$

or in other words the cycle Φ has codimension $\binom{k+1}{2}$ in $G(k,V)$ when $k \le n/2$, and is empty otherwise.

Having determined the dimension of Φ, we ask now for its class in $A(G(k,V))$. Following our general method for finding the decomposition of a cycle into Schubert cycles, we write

$$[\Phi]=\sum_{|a|=\binom{k+1}{2}} \gamma_a \sigma_a$$

with

$$\begin{aligned}\gamma_a &= \#\big(\ \Phi \cap \Sigma_{n-k-a_k,\dots,n-k-a_1}(\mathcal{V})\ \big)\\ &= \#\big\{\Lambda \ \mid\ q|_\Lambda \equiv 0 \ \text{ and }\ \dim(\Lambda \cap V_{i+a_i}) \ge i\ \forall i\ \big\}.\end{aligned}$$

To evaluate γ_a, suppose that $\Lambda \subset V$ is a k-plane in this intersection. The subspace $V_{a_i+i} \subset V$ being general, the restriction $q|_{V_{a_i+i}}$ of q to it will again be nondegenerate. Since $q|_{V_{a_i+i}}$ has an isotropic i-plane, we must have $a_i+i \ge 2i$, or in other words

$$a_i \ge i \quad \forall i.$$

But by hypothesis, $\sum a_i = \binom{k+1}{2}$; so in fact we must have equality in each of these inequalities. In other words, $\gamma_a = 0$ *for all* a *except the index* $a=(k,k-1,\dots,2,1)$.

It remains to evaluate the coefficient

$$\gamma_{k,k-1,\dots,2,1} = \#\{\Lambda \ \mid\ q(\Lambda,\Lambda)=0 \ \text{ and }\ \dim(\Lambda \cap V_{2i}) \ge i\ \forall i\ \}; \tag{3.3}$$

we claim that this number is 2^k.

We prove this inductively. To start, note that the restriction $q|_{V_2}$ of q to the 2-dimensional space V_2 has two one-dimensional isotropic spaces, and Λ will necessarily contain exactly one of them: it can't contain both since Φ is disjoint from any Schubert cycle $\Sigma_b(\mathcal{V})$ with $|b| > k(n-k)-\binom{k+1}{2}$.

We may thus suppose that Λ contains the isotropic subspace $W \subset V_2$, so that Λ is contained in $W^\perp$. Now, since $q(W,W)\equiv 0$, q induces a nondegenerate quadratic form q' on the $(n-2)$-dimensional quotient $W' = W^\perp/W$; and the quotient space

$$\Lambda' = \Lambda/W \subset W^\perp/W$$

is a $(k-1)$-dimensional isotropic subspace for q'. Moreover, since the spaces V_{2i} are general subspaces of V containing V_2, the subspaces

$$V'_{2i-2} = (V_{2i} \cap W^{\perp})/W \subset W^{\perp}/W$$

form a general flag in $W' = W^{\perp}/W$; and we have

$$\dim(\Lambda' \cap V'_{2i-2}) \geq i - 1 \quad \forall i.$$

Inductively, there are 2^{k-1} isotropic $(k-1)$-planes $\Lambda' \subset W'$ satisfying these conditions; and so there are 2^k planes $\Lambda \subset W$ satisfying the conditions of (3.3). We have proven:

Proposition 3.11. *Let q be a nondegenerate quadratic form on the n-dimensional vector space V, and $\Phi \subset G(k, V)$ the variety of isotropic k-planes for q. Assuming $k \leq n/2$, the class of the cycle Φ is*

$$[\Phi] = 2^k \sigma_{k,k-1,\ldots,2,1}.$$

As an immediate application of this result, we can answer Keynote Question (c). To begin with, we asked how many lines lie on the intersection of two quadrics in $\mathbb{P}^4$. To answer this, let $Q, Q' \subset \mathbb{P}^4$ be two general quadric hypersurfaces, and $X = Q_1 \cap Q_2$. The set of lines on X is just the intersection $\Phi \cap \Phi'$ of the cycles of lines lying on Q and Q'; by Kleiman's theorem 5.15 these are transverse, and so we have

$$\#(\Phi \cap \Phi') = \deg\bigl(4\sigma_{2,1}\bigr)^2 = 16.$$

More generally, if Q and $Q' \subset \mathbb{P}^{2n}$ are general quadrics, we ask how many $(n-1)$-planes are contained in their intersection; again, this is the intersection number

$$\#(\Phi \cap \Phi') = \deg\bigl(2^n \sigma_{n,n-1,\ldots,1}\bigr)^2 = 4^n.$$

3.6 Giambelli's formula

Pieri's formula tells us how to intersect an arbitrary Schubert cycle with one of the special Schubert cycles $\sigma_b = \sigma_{b,0,\ldots,0}$. Giambelli's formula is complementary, in that it tells us how to express an arbitrary Schubert cycle in terms of special ones; the two together give us (in principle) a way of calculating the product of two arbitrary Schubert cycles.

We will state Giambelli's formula without proof; see Chapter 14 for some special cases and Fulton [1997] for a proof in general.

Proposition 3.12 (Giambelli's formula).

$$\sigma_{a_1,a_2,\dots,a_k} = \begin{vmatrix} \sigma_{a_1} & \sigma_{a_1+1} & \sigma_{a_1+2} & \cdots & \sigma_{a_1+k-1} \\ \sigma_{a_2-1} & \sigma_{a_2} & \sigma_{a_2+1} & \cdots & \sigma_{a_2+k-2} \\ \sigma_{a_3-2} & \sigma_{a_3-1} & \sigma_{a_3} & \cdots & \sigma_{a_3+k-3} \\ \vdots & \vdots & & \ddots & \vdots \\ \sigma_{a_k-k+1} & \sigma_{a_k-k+2} & \sigma_{a_k-k+3} & \cdots & \sigma_{a_k} \end{vmatrix}$$

Thus, for example, we have

$$\sigma_{2,1} = \begin{vmatrix} \sigma_2 & \sigma_3 \\ \sigma_0 & \sigma_1 \end{vmatrix} = \sigma_2\sigma_1 - \sigma_3,$$

which we can then use in conjunction with Pieri to evaluate, for example, $\sigma_{2,1}^2$. Note that Giambelli's formula also reproduces some formulas we've derived already by other means: for example, in case $a_1 = a_2 = 1$ it gives

$$\sigma_{1,1} = \begin{vmatrix} \sigma_1 & \sigma_2 \\ \sigma_0 & \sigma_1 \end{vmatrix} = \sigma_1^2 - \sigma_2,$$

or in other words, $\sigma_1^2 = \sigma_2 + \sigma_{1,1}$.

In fact, as the last two examples suggest, we can deduce Giambelli's formula from Pieri's. For example, in the 2×2 case, if we know Pieri's formula then for any $a \geq b$ we can expand the determinant and apply Pieri to arrive at

$$\begin{aligned} \begin{vmatrix} \sigma_a & \sigma_{a+1} \\ \sigma_{b-1} & \sigma_b \end{vmatrix} &= \sigma_a\sigma_b - \sigma_{a+1}\sigma_{b-1} \\ &= (\sigma_{a,b} + \sigma_{a+1,b-1} + \cdots + \sigma_{a+b}) - (\sigma_{a+1,b-1} + \cdots + \sigma_{a+b}) \\ &= \sigma_{a,b}. \end{aligned}$$

In general, given Pieri, we can prove Giambelli inductively by expanding the determinant in Proposition 3.12 by cofactors along the right-hand column; Exercise 3.42 asks the reader to do this in the 3×3 case.

One word of warning. In theory, Giambelli and Pieri in tandem give us an algorithm for calculating the product of any two Schubert cycles: use Giambelli to express either as a polynomial in the special Schubert cycles, and then use Pieri to evaluate the product of this polynomial with the other. In practice, though, except in low-dimensional examples *this is an absolutely terrible idea*: as you can see from the determinantal form of Giambelli's formula, the number of calculations involved increases extremely rapidly with k and n. Nor can this method be used to prove qualitative results about products of Schubert cycles: for example, it's not even clear from this approach that such a product is necessarily a nonnegative linear combination of Schubert cycles. The algorithms of Coskun and Vakil referred to earlier are far, far better.

A final note: Giambelli's formula implies that the Chow ring $A(G)$ of a Grassmannian is generated, as a ring, by the special Schubert classes. It's natural to ask, then, what the relations are among these generators. There's a very nice (and succinct!) answer to this, which we'll give in Section 7.4, based on the fact that the special Schubert cycles are exactly the Chern classes of the universal bundles on the Grassmannian.

3.7 Exercises

Exercise 3.13. Suppose that $\varphi : \Lambda \to V$ is an inclusion of a k-dimensional vector space Λ into an n-dimensional vector space V and let $\psi : V \to \Lambda' = V/\varphi(\Lambda)$ be the quotient map, so that

$$0 \longrightarrow \Lambda \xrightarrow{\varphi} V \xrightarrow{\psi} \Lambda' \longrightarrow 0$$

is exact. Choose bases, so that φ and ψ are represented by matrices. Show that there is a nonzero scalar λ such that the $k \times k$ minor of φ involving rows with indices $i_1 < \cdots < i_k$ is equal to λ times the $(n-k)\times(n-k)$ minor of ψ involving the complementary set of column indices $j_1 < \cdots < j_{n-k}$. Conclude that the set-theoretic isomorphism $G(k, V) \cong G(n-k, V^*)$ taking a subspace of V to its annihilator in V^* is an algebraic isomorphism as well.

Exercise 3.14. Complete the proof of Corollary 3.2 without invoking duality—that is, prove directly that σ^k_{n-k} is the class of a point in $G(k, n)$.

Exercise 3.15. Use the description of the points of the Schubert cells given in Theorem 3.1 to show that Theorem 3.3 holds at least set-theoretically.

Exercise 3.16. Use the description of the Chow ring $A(\mathbb{G}(1, 4))$ given in Section 3.3 to answer the question: given 6 general 2-planes in $\mathbb{P}^4$, how many lines meet all 6? Note that this number is the same as the degree of the Grassmannian $\mathbb{G}(1, 4) = G(2, 5)$ in the Plücker embedding.

Exercise 3.17. Let $S \subset \mathbb{P}^4$ be a surface of degree d, and $\Gamma_S \subset \mathbb{G}(1, 4)$ the variety of lines meeting S.

(a) Find the class $\gamma_S = [\Gamma_S] \in A^1(\mathbb{G}(1, 4))$.

(b) Use this to answer the question: if $S_1, \dots, S_6 \subset \mathbb{P}^4$ are surfaces of degrees $d_1, \dots, d_6$, how many lines in $\mathbb{P}^4$ will meet all six?

Exercise 3.18. Let $C \subset \mathbb{P}^4$ be a curve of degree d, and $\Gamma_C \subset \mathbb{G}(1, 4)$ the variety of lines meeting C.

(a) Find the class $\gamma_C = [\Gamma_C] \in A^2(\mathbb{G}(1, 4))$.

(b) Use this to answer the question: if C_1, C_2 and $C_3 \subset \mathbb{P}^4$ are curves of degrees d_1, d_2 and d_3, how many lines in $\mathbb{P}^4$ will meet all three?

Exercise 3.19. Let $S \subset \mathbb{P}^n$ be a smooth surface of degree d whose general hyperplane section is a curve of genus g, and $T_1(S) \subset \mathbb{G}(1,n)$ the variety of lines tangent to S. Find the class of the cycle $T_1(S)$.

Exercise 3.20. Let $Z \subset \mathbb{G}(k,n)$ be a variety of dimension m, and consider the variety $X \subset \mathbb{P}^n$ swept out by the linear spaces corresponding to points of Z: that is,

$$X = \bigcup_{[\Lambda]\in Z} \Lambda \ \subset \ \mathbb{P}^n.$$

For simplicity, assume that a general point $x \in X$ lies on a unique k-plane $\Lambda \in Z$.

(a) show that Z has dimension $k+m$ and degree the intersection number $\deg(\sigma_m \cdot [Z])$.

(b) Show that this is not in general the degree of X.

Exercises 3.21-3.24 deal with the geometry of the surface described in Keynote Question (b), whose degree we worked out in Section 3.2.2: the surface $X \subset \mathbb{P}^3$ swept out by the lines corresponding to a general twisted cubic $C \subset \mathbb{G}(1,3)$.

Exercise 3.21. To start, use the fact that the dual of $\mathbb{G}(1,3) \subset \mathbb{P}^5$ has degree 2 to show that a general twisted cubic $C \subset \mathbb{G}(1,3)$ lies on two Schubert cycles $\Sigma_1(L)$ and $\Sigma_1(M)$ for some pair of skew lines $L, M \subset \mathbb{P}^3$

Exercise 3.22. Show that for skew lines L and $M \subset \mathbb{P}^3$, the intersection $\Sigma_1(L) \cap \Sigma_1(M)$ is isomorphic to $L \times M$ via the map sending a point $[\Lambda] \in \Sigma_1(L) \cap \Sigma_1(M)$ to the pair $(\Lambda \cap L, \Lambda \cap M) \in L \times M$, and that it is the intersection of $\mathbb{G}(1,3)$ with the intersection of the hyperplanes spanned by $\Sigma_1(L)$ and $\Sigma_1(M)$.

Exercise 3.23. Finally, suppose that $C \subset \Sigma_1(L) \cap \Sigma_1(M)$ is a twisted cubic curve. Using the fact that its bidegree in $\Sigma_1(L) \cap \Sigma_1(M) \cong L \times M \cong \mathbb{P}^1 \times \mathbb{P}^1$ (possibly after switching factors) is $(2,1)$, show that for some degree 2 map $\varphi : L \to M$, the family of lines corresponding to C may be realized as the locus

$$C = \{\overline{p\,\varphi(p)} \mid p \in L\}.$$

Show correspondingly that the surface

$$S = \bigcup_{[L]\in C} L \ \subset \ \mathbb{P}^3$$

swept out by the lines of C is a cubic surface double along a line, and that it's the projection of a rational normal surface scroll $X_{1,2} \subset \mathbb{P}^4$.

Exercise 3.24. For what special twisted cubic curves $C \subset \mathbb{G}(1,3)$ will the conclusion of the preceding exercise be false?

Exercise 3.25. Now let $G = \mathbb{G}(1,3) \subset \mathbb{P}^5$ be as in the preceding Exercise, and let $C \subset G$ be a general rational normal quartic curve. Can you describe the surface S swept out by the lines of C? In particular, what is the singular locus of S?

In Section 3.3 we calculated the number of lines meeting four general n-planes in $\mathbb{P}^{2n+1}$. In the following two exercises, we'll see another way to do this (analogous to the alternative count of lines meeting four lines in $\mathbb{P}^3$ given in Exercise **??**), and a nice geometric sidelight.

Exercise 3.26. Let $\Lambda_1, \dots, \Lambda_4 \cong \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ be four general n-planes. Calculate the number of lines meeting all four by showing that the union of the lines meeting Λ_1, Λ_2 and Λ_3 is a Segre variety $S_{1,n} = \mathbb{P}^1 \times \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ and using the calculation of Section 1.2.3 for the degree of $S_{1,n}$.

Exercise 3.27. By the preceding exercise, we can associate to a general configuration $\Lambda_1, \dots, \Lambda_4$ of k-planes in $\mathbb{P}^{2k+1}$ an unordered set of $k+1$ cross-ratios. Show that two such configurations $\{\Lambda_i\}$ and $\{\Lambda_i'\}$ are projectively equivalent if and only if the corresponding sets of cross-ratios coincide.

The next two exercises deal with the example of dynamic specialization given in Section 3.3.1, and specifically with the family Φ of cycles described there.

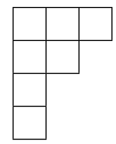
Exercise 3.28. Show that the support of Φ_0 is all of $\Sigma_{2,2}(P) \cup \Sigma_{3,1}(p_0, H_0)$.

Exercise 3.29. Verify the last assertion made in the calculation of σ_2^2; that is, show that Φ_0 has multiplicity 1 along each component. [Hint: argue that by applying a family of automorphisms of $\mathbb{P}^4$ we can assume that the plane H_t is constant and use the calculation of the preceding chapter.]

Exercise 3.30. A further wrinkle in the technique of dynamic specialization is that to carry out the calculation of an intersection of Schubert cycles we may have to specialize in stages. To see an example of this, use dynamic specialization to calculate the intersection σ_2^2 in the Grassmannian $\mathbb{G}(1,5)$. [Hint: you have to let the two 2-planes specialize first the a pair intersecting in a point, then to a pair intersecting in a line.]

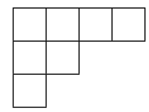
Exercise 3.31. Suppose that the Schubert class $\sigma_a =\in A(G(k,n))$ corresponds to the Young diagram Y in a $k \times (n-k)$ box B. Show that under the duality $G(k,n) \cong G(n-k,n)$, the class σ_a is taken to the Schubert class σ_b corresponding to the Young diagram Z that is the *transpose* of Y, that is, the diagram obtained by flipping Y around a 45° line running northwest-southeast. For example if

$$\sigma_{3,2,1,1} \in A(G(4,7)) \quad \longleftrightarrow$$



then the corresponding Schubert cycle in $G(3,7)$ is

$$\sigma_{4,2,1} \in A(G(3,7)) \quad \longleftrightarrow$$



Exercise 3.32. Show that the map $i^* : A^d(G,(k,n{+}1)) \to A^d(G(k,n))$ is a monomorphism if and only if $n-k \geq d$, and that $i^* : A^d(G,(k{+}1,n{+}1)) \to A^d(G(k,n))$ is a monomorphism if and only if $k \geq d$. (Thus, for example, the formula

$$\sigma_1^2 = \sigma_2 + \sigma_{11},$$

which we established in $A(\mathbb{G}(1,3))$, *holds true in every Grassmannian.*)

Exercise 3.33. Let $C \subset \mathbb{P}^r$ be a smooth, irreducible, nondegenerate curve of degree d and genus g, and let $S_1(C) \subset \mathbb{G}(1,r)$ be the variety of chords to C, as defined in Section 2.4.3 above. Find the class $[S_1(C)] \in A_2(\mathbb{G}(1,r))$.

Exercise 3.34. Let $Q \subset \mathbb{P}^n$ be a smooth quadric hypersurface, and let $T_k(Q) \subset \mathbb{G}(k,n)$ be the locus of planes $\Lambda \subset \mathbb{P}^n$ such that $\Lambda \cap Q$ is singular. Show that

$$[T_k(Q)] = 2\sigma_1.$$

Exercise 3.35. Let Q, Q' and Q'' be three general quadrics in $\mathbb{P}^8$. How many 2-planes lie on all three? (Try first to do this without the tools introduced in Section **??**.)

Exercise 3.36. Find the expression of $\sigma_{2,1}^2$ as a linear combination of Schubert classes in $A(G(3,6))$. This is the smallest example of a product of two Schubert classes where another Schubert class appears with multiplicity > 1.

Exercise 3.37. Give an alternative proof of Pieri's formula for intersections with σ_1 using the method of specialization.

Exercise 3.38. More generally, use Pieri to identify the degree of $\sigma_1^{k(n-k)}$ with the number of standard tableaux: that is, ways of filling in $k \times (n-k)$ matrix with the integers $1, \ldots, k(n-k)$ in such a way that every row and column is strictly increasing. Then use the "hook formula" (see for example, Fulton [1997]) to show that this number is

$$\big(k(n-k)\big)! \prod_{i=0}^{k-1} \frac{i!}{(n-k+i)!},$$

Exercise 3.39. Using Pieri's formula, determine all products of Schubert classes in the Chow ring of the Grassmannian $\mathbb{G}(1,4)$, and compare this with the result of Proposition 3.8.

Exercise 3.40. Using Pieri's formula, determine all products of Schubert classes in the Chow ring of the Grassmannian $\mathbb{G}(2,5)$. [Note: an extra step will be required, specifically to find the square $(\sigma_{2,1})^2$.]

Exercise 3.41. Give a statement of Pieri's formula in terms of Young diagrams.

Exercise 3.42. Deduce Giambelli's formula in the 3×3 case—that is, the relation

$$\begin{vmatrix} \sigma_a & \sigma_{a+1} & \sigma_{a+2} \\ \sigma_{b-1} & \sigma_b & \sigma_{b+1} \\ \sigma_{c-2} & \sigma_{c-1} & \sigma_c \end{vmatrix} = \sigma_{a,b,c}$$

for any $a \geq b \geq c$—by assuming Giambelli in the 2×2 case, expanding the determinant by cofactors along the last column and applying Pieri.

4
Chow Groups

Keynote Questions

(a) Let $S \subset \mathbb{P}^n$ be a surface, and $\pi : S \to \mathbb{P}^1$ a morphism. Is it possible that one fiber of π is a line in $\mathbb{P}^n$ while another fiber is a conic? (Answer on page 173)

(b) Let $L, Q \subset \mathbb{P}^3$ be a line and a nonsingular conic in $\mathbb{P}^3$. Are the schemes $(\mathbb{P}^3 \setminus L)$ and $(\mathbb{P}^3 \setminus Q)$ isomorphic? (Answer on page 171)

In this chapter and the next, we'll give a reasonably self-contained account of the main results of classical intersection theory, with Chow groups in this chapter and intersections and projective pullbacks in the next. We will refer to Chapter **??** for a few definitions and examples, but by and large the development given in this chapter and the following one is independent of what has come before.

The most important and useful results of this Chapter, Theorems 4.15 and 4.17, say that the Chow groups are functorial for projective morphisms, and allow us to compute the maps on Chow groups in the finite case. These depend in turn on a beautiful and elementary result in commutative algebra, the *Determinant Lemma*, Theorem 4.19.

We also show that the Chow groups are contravariant functors with respect to flat maps. The important result that they are also contravariant functors for projective morphism among smooth quasiprojective varieties will be treated in Chapter 5.

4.1 Chow groups and basic operations

One of the areas where algebraic geometry is most successful is the study of subvarieties of codimension 1—the theory of divisors. (The fact that there are *only* subvarieties of codimension 1 on an algebraic curve is one of the things that makes the theory of curves so tractable.) The idea of cycles, and the rational equivalence of cycles is the very useful extension of the notion of divisors that underlies all of intersection theory.

Let X be any algebraic variety or scheme. The *group of cycles* on X, denoted $Z(X)$, is the free abelian group generated by the symbols $\langle V \rangle$ where V is a subvariety (reduced irreducible subscheme) of X (most of the time we will write V in place of $\langle V \rangle$, since there will be no danger of confusion.) The group $Z(X)$ is graded by dimension: we write $Z_d(X)$ for the group of cycles that are formal linear combinations of subvarieties of dimension d. The elements of $Z_d(X)$ will be called *d-cycles* of X.

A cycle $Z = \sum_i n_i Y_i$ is called *effective* if the coefficients n_i are all non-negative. A *divisor* is a cycle whose components all have codimension 1. Note that $Z(X) = Z(X_{\mathrm{red}})$; that is, $Z(X)$ is insensitive to whatever non-reduced structure X may have.

We define an effective cycle $\langle Y \rangle \in Z(X)$ associated with any closed subscheme $Y \subset X$ by summing the irreducible components of Y, each with a multiplicity defined using the following algebraic idea:

If A is any commutative ring then we say that an A-module M has finite length if it has a finite *composition series*; that is, a sequence of submodules $M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_l = 0$ such that each factor module M_i/M_{i+1} is a simple A-module. The Jordan-Hölder Theorem asserts that the number of times a given simple A-module appears as a factor M_i/M_{i+1} does not depend on the composition series chosen; in particular, the length l of a composition series for M, written $\mathrm{length}_A(M)$, depends only on A and M.

Consider again a subscheme $Y \subset X$. Since our schemes are assumed to be of finite type over a field they are Noetherian. In particular, Y has finitely many (reduced) irreducible components $Y_1, \dots, Y_s$, and each local ring $\mathcal{O}_{Y,Y_i}$ has finite length, say l_i. We define the cycle $\langle Y \rangle$ associated to Y to be the formal combination $\sum_i l_i Y_i$.

We will next define the relation of *rational equivalence* between cycles. Our goal is to make two subvarieties rationally equivalent if there is rationally parametrized "family" of subvarieties of which they are both members. In Exercise 4.34 the reader will see that the relation we define restricts to the classical notion of linear equivalence in the case of divisors on a smooth variety.

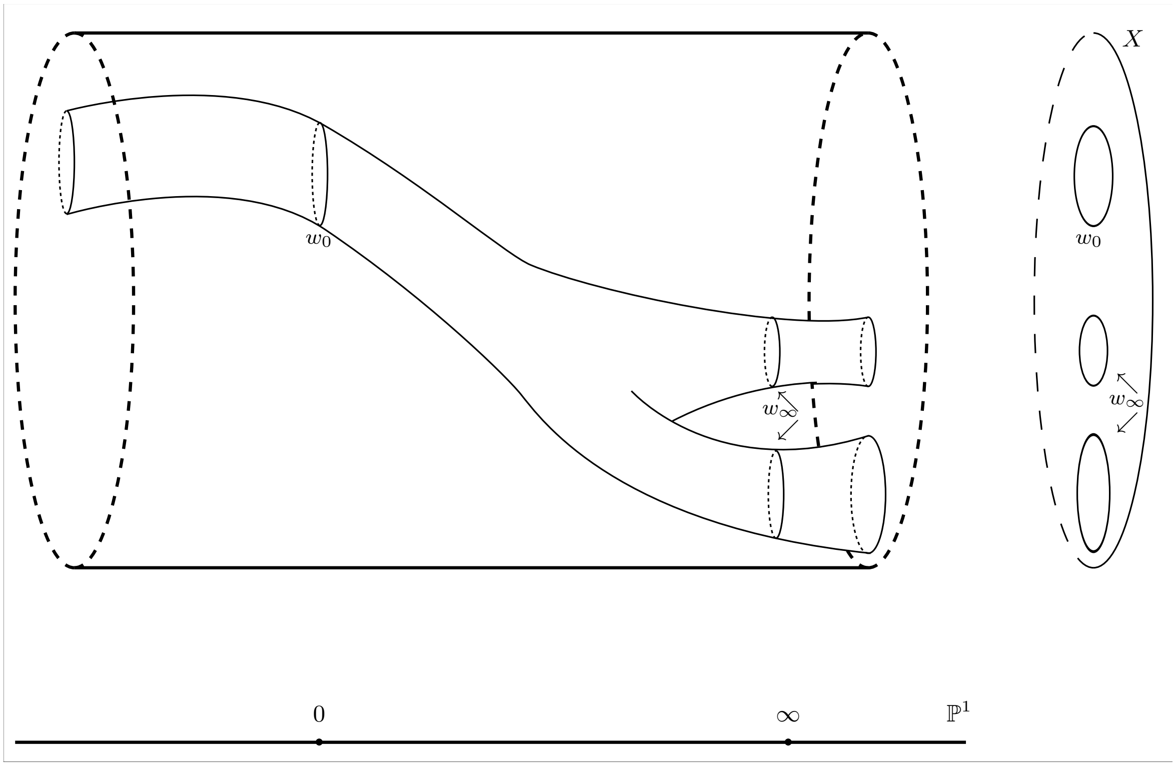


FIGURE 4.6. Rational equivalence between two cycles ω_0 and ω_∞ on X

To emphasize the analogy with ordinary cohomology theory we introduce a "boundary" map $\partial_X : Z(X \times \mathbb{P}^1) \to Z(X)$ defined on free generators as follows: Let W be a subvariety of $X \times \mathbb{P}^1$. If the projection $\pi : W \to \mathbb{P}^1$ on the second factor is not dominant—that is, if $W \subset X \times \{t\}$ for some $t \in \mathbb{P}^1$—we set $\partial_X(W) = 0$. On the other hand, if $W \to \mathbb{P}^1$ is dominant then we set $W_0 = \pi^{-1}(0) \subset X \times \{0\} = X$ and $W_\infty = \pi^{-1}(\infty) \subset X \times \{\infty\} = X$, and define $\partial_X(W) = \langle W_0 \rangle - \langle W_\infty \rangle$. We write $Rat(X) \subset Z(X)$ for the image $\partial(Z(X \times \mathbb{P}^1))$, the subgroup generated by all the cycles of the form $\langle W_0 \rangle - \langle W_\infty \rangle$.

Definition 4.1. Two cycles A and B are rationally equivalent, written $A \sim B$, if their difference lies in $Rat(X)$.

A *rational equivalence* from A to B is an element $C \in Z(\mathbb{P}^1 \times X)$ such that $\partial C = A - B$.

The *Chow group* $A(X)$ is the group of rational equivalence classes,

$$\begin{aligned} A(X) &:= Z(X)/Rat(X) \\ &= \operatorname{coker}\big(\partial_X : Z(X \times \mathbb{P}^1) \to Z(X)\big). \end{aligned}$$

With this terminology, $Rat_d(X)$ is the group of d-cycles rationally equivalent to 0.

Example 4.2 (Lines)**.** To start with the simplest of all such equivalences, let $L, L' \subset \mathbb{P}^2$ be two lines, intersecting at a point p. The set of lines in $\mathbb{P}^2$ passing through p is itself a line in the dual plane: explicitly, if we take $p = [0, 0, 1]$, $L = V(X_0)$ and $L' = V(X_1)$, we can write the family of lines L_t in $\mathbb{P}^2$ through p as

$$\{L_t = V(t_0X_0 - t_1X_1)\}_{t=[t_0,t_1]\in\mathbb{P}^1}.$$

If we then set

$$W = \{(X, t) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid t_0X_0 - t_1X_1\},$$

the subvariety $W \subset \mathbb{P}^2 \times \mathbb{P}^1$ is a rational equivalence between $W_0 = L$ and $W_\infty = L'$. Note that by taking chains of such equivalences, we see more generally any two lines in $\mathbb{P}^n$ are rationally equivalent.

Similarly, any two hypersurfaces $F = 0$ and $G = 0$ of the same degree in $\mathbb{P}^n$ are rationally equivalent via the scheme $\mathcal{W} \subset \mathbb{P}^n \times \mathbb{P}^1$ defined by the bihomogeneous equation $t_0F(x) - t_1G(x) = 0$.

Note that if $W \subset \mathbb{P}^1 \times X$ is a subvariety of dimension $d+1$ dominating $\mathbb{P}^1$ then by Krull's Principal Ideal Theorem 0.1 each W_i is a subscheme of dimension d. Thus $Rat(X)$ is generated by homogeneous elements in the grading by dimension, and it follows that the Chow group $A(X)$ is also graded by dimension, that is, $A(X) = \oplus_d A_d(X)$, with

$$A_d(X) = Z_d(X)/\partial_X(Z_{d+1}(X \times \mathbb{P}^1)).$$

We will use the notation $[Y]$ for the class modulo rational equivalence of the cycle Y; for simplicity, if $Y \subset X$ is any subscheme we will write $[Y]$ (instead of $[\langle Y\rangle]$) for the class of the cycle associated to the subscheme Y. Also, we will write ∂ in place of ∂_X when X is clear from the context.

It is often convenient to look for rational equivalences using a subvariety W' of $U \times X$ where U is $\mathbb{A}^1$ or some other open subset of $\mathbb{P}^1$. In this setting, it's still true that any two fibers $W'_p = W' \cap \{p\} \times X$ and $W_q = W' \cap \{q\} \times X$ are rationally equivalent: in order to apply the definition above, we could replace W' by its closure in $\mathbb{P}^1 \times X$, and move the points p, q to $0, \infty$ by an automorphism of $\mathbb{P}^1$.

A hidden complexity in the definition of a rational equivalence Z from A to B is that, writing $Z = \sum m_iZ_i$ with irreducible Z_i, we may have cancellation among the components of the various $Z_i \cap \{0\} \times X$ or $Z_i \cap \{\infty\} \times X$. Thus it is hard to conclude anything about the varieties Z_i from a knowledge of A and B. However, if Z is itself an irreducible variety then a lot is preserved. Here is the case of rational equivalence on projective space:

Proposition 4.3. *If $Z \subset \mathbb{P}^1 \times X$ is a closed subvariety dominating $\mathbb{P}^1$, then the fibers of Z at the points of $\mathbb{P}^1$ all have the same Hilbert polynomial; that is, $\chi(\mathcal{O}_{Z_p}(l)) = \chi(\mathcal{O}_{Z_q}(l))$ for any points $p, q \in \mathbb{P}^1$ and any integer l.*

Proof. Let (t_0, t_1) be coordinates on $\mathbb{P}^1$. We may assume that $p = 0$ and $q = \infty$. Consider the sequences

$$0 \longrightarrow \mathcal{O}_W(0,-1) \xrightarrow{t_0} \mathcal{O}_W \longrightarrow \mathcal{O}_{Z_\infty} \longrightarrow 0$$
$$0 \longrightarrow \mathcal{O}_W(0,-1) \xrightarrow{t_1} \mathcal{O}_W \longrightarrow \mathcal{O}_{Z_0} \longrightarrow 0,$$

where $\mathcal{O}_W(0,-1)$ denotes the pullback to $W \subset \mathbb{P}^1 \times \mathbb{P}^n$ of $\mathcal{O}_{\mathbb{P}^1}(-1)$. The sequences are exact because W is irreducible.

Twisting by the pullback to $\mathbb{P}^1 \times \mathbb{P}^n$ of any line bundle $\mathcal{O}_{\mathbb{P}^n}(l)$ on $\mathbb{P}^n$, and using the additivity of the Euler characteristic, we see that

$$\chi(\mathcal{O}_{Z_\infty}(l)) = \chi(\mathcal{O}_{Z_0}(l)) \text{ for every integer } l,$$

and in particular the Hilbert polynomials of Z_0 and Z_∞ are the same. □

Thus, for example, if $Z \subset \mathbb{P}^1 \times \mathbb{P}^3$ is an irreducible surface dominating $\mathbb{P}^1$, then the fibers Z_0 and Z_∞ will be one-dimensional subschemes of $\mathbb{P}^3$ having the same degree and arithmetic genus. Nonetheless, curves C and $C' \subset \mathbb{P}^3$ of the same degree d but different genera *will* be rationally equivalent, as we'll see in Theorem 4.4 below; an explanation of this phenomenon will be given following the proof of Theorem 4.4 and illustrated in Example 4.8 and Exercises **??**-**??**.

For any subscheme $Y \subset \mathbb{P}^n$ we define the *degree of* Y to be the leading coefficient of the Hilbert polynomial $l \mapsto \chi(\mathcal{O}_Y(l))$, multiplied by $(\dim Y)!$. This is of course related to the degree map deg $A_0(\mathbb{P}^n) \to \mathbb{Z}$ by the formula degree $Y = \deg[L \cap Y]$ where L is a general linear space of complementary dimension $n - \dim Y$ (or, as we'll see in **??**, if Y is Cohen-Macaulay, any linear space L of complementary dimension that meets Y in a zero-scheme). We can extend this definition to any cycle $Z = \sum_i m_i \langle X_i \rangle$ of pure dimension m by setting

$$\text{degree } Z = \sum_i m_i \text{ degree } X_i$$

If Y is any scheme, then from the definition and the additivity of the Euler characteristic we see that degree Y = degree $\langle Y \rangle$. We can now prove Theorem **??**:

Theorem 4.4. *The m-th Chow group of $\mathbb{P}^n$ is*

$$A_m(\mathbb{P}^n) \cong \mathbb{Z},$$

generated by the class of a plane L of dimension m. The class of any cycle of pure dimension m and degree d is $d[L]$.

Proof. We already know from Proposition 1.19 that the class $[L]$ of a plane L of dimension m generates $A_m(\mathbb{P}^n)$ (see the first part of the proof of Theorem 1.20). If X, X' are cycles that are rationally equivalent by a rational equivalence $Z = \sum m_i \langle Z_i \rangle$, where the $Z_i \subset \mathbb{P}^1 \times \mathbb{P}^n$ are irreducible subvarieties dominating $\mathbb{P}^1$, then for each i the fibers $(Z_i)_0$ and $(Z_i)_\infty$ have the same Hilbert polynomial, and thus the same degree. Since degree is additive, and defined on the associated cycle, we get degree $X =$ degree X'. We must have $[X] = d[L]$ and $[X'] = e[L]$ for some d, e and this shows that $d = e$. In particular, if $d[L] = 0$, then $d = 0$. $\square$

Given the constancy of Hilbert polynomials in Proposition 4.3, it may seem odd that the *only* properties of a variety in $\mathbb{P}^n$ preserved by rational equivalence are its dimension and degree. For example, as we observed above if $Z \subset \mathbb{P}^1 \times \mathbb{P}^3$ is an irreducible surface dominating $\mathbb{P}^1$ then the fibers Z_0 and Z_∞ will be one-dimensional subschemes $C, C' \subset \mathbb{P}^3$ having not only the same degree, but also the same arithmetic genus—but Theorem 4.4 asserts that two curves C and C' of the same degree but different genera will be rationally equivalent. The reason is that both can be deformed, in family parametrized by $\mathbb{P}^1$, to schemes C_0, C_0' supported on a line $L \subset \mathbb{P}^3$ and having multiplicity d, so that as cycles $\langle C \rangle \sim \langle C_0 \rangle = d\langle L \rangle$ and likewise for C'. The difference in the genera of C and C' will be reflected in two things: the scheme structure along the line in the flat limits C_0 and C_0', and the presence and multiplicity of embedded points in these limits. For an example of the first, note that the schemes $C_0 = V((x,y)^2)$ and $C_0' = V(x, y^3)$ are both supported on the line $L = V(x,y)$, and both have multiplicity 3; but the arithmetic genus of C_0 is 0, while that of C_0' is 1 (after all, it's a plane cubic!). But the mechanism by which we associate a cycle to a scheme doesn't see the difference in the scheme structure; we have $\langle C_0 \rangle = \langle C_0' \rangle = 3\langle L \rangle$. Similarly, a twisted cubic curve $C \subset \mathbb{P}^3$ can be deformed to a scheme generically isomorphic to either C_0 or C_0'; the difference in the arithmetic genus is accounted for by the fact that in the latter case the limiting scheme will necessarily have an embedded point. But again, rational equivalence doesn't "see" the embedded point; we have $\langle C \rangle = 3\langle L \rangle$ regardless.

We will see an example of this worked out in detail in Example 4.8, and more examples in Exercises 4.38-4.41.

4.1.1 Dynamic projection

As we have seen, rational equivalence in projective space is rather trivial: two equidimensional cycles are rationally equivalent if and only if they have the same dimension and degree. More concretely put: any variety of dimension m and degree d in $\mathbb{P}^n$ is rationally equivalent to the union of d distinct m-planes. In this section we will give an explicit construction of such rational equivalences by a technique called *dynamic projection* that will be central to our proof of the Moving Lemma (and thus to our construction of intersection products) in the next chapter.

The general idea is that if X is a variety and $\{\psi_t\}_{t\in\mathbb{A}^1\setminus\{0\}}$ is any arc inside the automorphism group of $\mathbb{P}^n$ with ψ_1 the identity, then the closure Z of $\cup_t\psi_t X$ will be a rational equivalence betweet $\psi_1(X) = X$ and the variety obtained as the flat limit of the $\psi_t(X)$ as $t \to 0$, as constructed below. There are many fascinating questions about the flat limits of such families, as described in Exercises 4.39-4.42.

We will use a special case of this limit construction, called *dynamic projection.* Consider a pair of disjoint spaces $A = V(x_0,\dots,x_r)$ (the *attractor*) and $R = V(y_0,\dots,y_a)$ (the *repellor*) in $\mathbb{P}^n$ that together span $\mathbb{P}^n$, so that, $(x_0,\dots,x_a,y_0,\dots,y_r)$, which we write as (x,y), is a system of homogeneous coordinates on $\mathbb{P}^n$. We may think of the projection map

$$\begin{aligned}\pi_R : \mathbb{P}^n \setminus R &\to A\\ : (x,y) &\mapsto (0,y)\end{aligned}$$

as the "limit", as t goes to 0, of the family of automorphisms

$$\psi_t : (x,y) \mapsto (tx,y); \quad t\in K^*,$$

of $\mathbb{P}^n$. (The ψ_t form a 1-parameter subgroup of $Aut(\mathbb{P}^n)$, but we will not need this.)

It is clear that the points of $A\cup R$ remain fixed under each ψ_t. On the other hand we can say intuitively that a point not in A or R will "flow toward A" as t approaches zero. More precisely, note that any point $p\notin A\cup R$ lies on a unique line that meets both A and R. (The span $\overline{p\,A}$, being an $(a+1)$-plane, must meet R. Since A and R are disjoint, $\overline{p\,A}$ can meet R only in a single point $q\in R$. The line $\overline{p\,q}$ is the unique line containing p and meeting A and R.) This line is the closure of the set $\{\psi_t(p)\}$ In particular, any point in $\mathbb{P}^n\setminus R$ has a well-defined limit in A as t approaches zero.

If X is any scheme in $\mathbb{P}^n$ then we can also define the limit of the $\psi_t(X)$ as a scheme: set

$$Z^\circ = \{(t,p)\in K^*\times\mathbb{P}^n \ \mid\ p\in\psi_t(X)\}.$$

Let $Z\subset\mathbb{A}^1\times\mathbb{P}^n$ be the closure of Z°, and define the *limit of the schemes* $\psi_t(X)$ to be the fiber X_0 of Z over $0\in\mathbb{A}^1$—see Figures 4.2 and 4.3. The

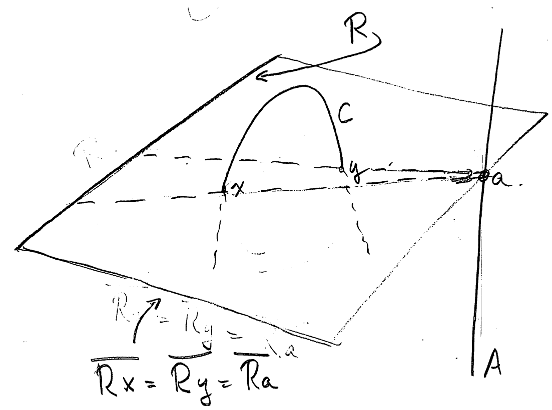


FIGURE 4.2. Dynamic projection of a conic in $\mathbb{P}^3$ from a line to a line

limit terminology is justified by the fact that Z is flat over $\mathbb{A}^1$, and there is a unique flat limit of any one-parameter flat family of subschemes; see for example Eisenbud and Harris [2000] Chapter ****.

We call X_0 the *dynamic projection of X from R.* The construction is illustrated in Figures 4.2 and 4.3. As with ordinary projection, the choice of A does not affect the isomorphism class of X_0, but in our application of this construction in the next chapter the position of X_0 in $\mathbb{P}^n$ relative to other schemes, will be important, and this does depend on A.

The following properties of the limit X_0 make it easy to analyze some interesting cases:

Proposition 4.5. *Let A, R be disjoint planes that span $\mathbb{P}^n$ defined by the vanishing of the sets of coordinates $x = (x_0, \ldots, x_r)$ and $y = (y_0, \ldots, y_a)$ respectively, and for $t \in K^*$ let ψ_t be the automorphism $\psi_t(x, y) = (tx, y)$ of $\mathbb{P}^n$. Let $X \subset \mathbb{P}^n$ be a subscheme, and let X_0 be the flat limit of the schemes $\psi_t(X)$ as t goes to 0.*

(a) *The Hilbert polynomial of X_0 is the same as the Hilbert polynomial of X; in particular,* $\dim X_0 = \dim X$ *and* $\deg X_0 = \deg X$.

(b) *If the irreducible components of X all have the same dimension, then the same is true of X_0.*

(c) $\psi_t(X_0) = X_0$ *for all $t \in K^*$.*

(d) $X_0 \cap R = X \cap R$.

(e) $(X_0)_{\mathrm{red}}$ *is contained in the cone over $X_0 \cap R$ with base A (in case $X_0 \cap R = \varnothing$, we take this to mean $(X_0)_{\mathrm{red}} \subset A$.)*

Statement d is perhaps the least obvious, but it is intuitively reasonable, at least on a set-theoretic level: we're saying that since points in $\mathbb{P}^n \setminus R$ flow away from R as $t \to 0$, the only way a point $p \in R$ can be a limit of points $\psi_t(p_t)$ is if it's there all along, that is, $p \in X \cap R$.

Proof. (a): This is a special case of Proposition 4.3.

Now let Z be the closure in $\mathbb{A}^1 \times \mathbb{P}^n$ of the set $\{(t,p) \mid p \in \psi_t(X)\}$, so that X_0 is the fiber of Z over $0 \in \mathbb{A}^1$.

(b): X_0 is a Cartier divisor Z. If the irreducible components of X have the same dimension, then this is also true of Z, and thus of X_0 as well.

(c): Consider the automorphisms of $\mathbb{A}^1 \times \mathbb{P}^n$ given by

$$\varphi_t : (s,p) \to (ts, \psi_t(p)); \quad t \in K^*.$$

Each φ_t carries Z to itself and the fiber $\{0\} \times \mathbb{P}^n$ to itself, so it carries X_0 to itself. But it acts on the fiber $\{0\} \times \mathbb{P}^n$ via the action Ψ above; thus X_0 is invariant under Ψ.

(d): One inclusion is immediate: since R is fixed pointwise by the automorphisms ψ_t, we have

$$\mathbb{P}^1 \times (X \cap R) \subset Z$$

and hence $X \cap R \subset X_0 \cap R$. To prove the other inclusion, we must show that the ideal $I_{(X\cap R)/\mathbb{P}^n} \subset I_{(X_0\cap R)/\mathbb{P}^n}$. Since $I_{X\cap R/\mathbb{P}^n}$ is generated by the y_i together with functions that are independent of the y_i, it suffices to prove that if $f \in I_{(X\cap R/R)}$ then $f \in I_{(X_0\cap R)/R}$. We may write

$$f(x) = g(x,0) \quad \text{for some } g(x,y) \in I_{X/\mathbb{P}^n}.$$

Observe that

$$I_{Z/\mathbb{P}^n} \supset \{f(x,ty) : f \in I_{X/\mathbb{P}^n}\}$$

so $h(t,x,y) = g(x,ty) \in I_{Z/\mathbb{P}^n}$. Setting $t=0$, we see that $g(x,0) \in I_{X_0/\mathbb{P}^n}$, so $f \in I_{(X_0\cap R)/R)}$ as required.

(e): note that the G_m orbit of any point not contained in $R \cup A$ is a straight line joining a point of R to a point of A. Since X_0 is stable under G_m, it is the union of such lines, together with any points of $A \cup R$ it contains. □

Here is a special case that we shall use in the next Chapter:

Corollary 4.6. *Let $X \subset \mathbb{P}^n$ be a subvariety of dimension m and degree d. Let R be a general plane of dimension $n-m$, and let A be any $(m-1)$-plane disjoint from R. The limit X_0 of the dynamic projection of X with repellor plane R and attractor plane A consists of the reduced union of d disjoint m-planes containing A, plus embedded components.*

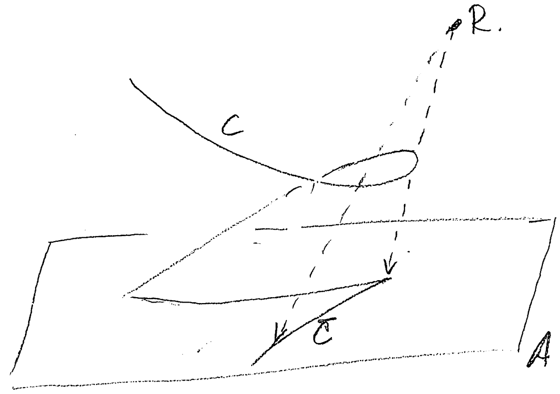


FIGURE 4.3. Dynamic projection of a twisted cubic in $\mathbb{P}^3$ from a point to a plane. The limit is a cuspidal plane cubic with an embedded point at the cusp.

Proof. By part (b) of Proposition 4.5 the irreducible components of the schemes $(X_0)_{\mathrm{red}} \subset X_0$ all have dimension m. By part (e), $(X_0)_{\mathrm{red}}$ is contained in the union of the d distinct m-planes that are the cones with base A over the $d =$ degree X points of $X \cap R = X_0 \cap R$. By part (d), X_0 contains these d points, so X_0 is the union of these d planes. It follows that X_0 has the same degree as X, so they are generically equal. $\square$

Example 4.7. We can also make an equivalence from a variety X of dimension m and degree d to a d-fold multiple plane in this way. let $A \subset \mathbb{P}^n$ be any m-dimensional subspace, and choose an $n-m-1$-plane $R \subset \mathbb{P}^n$ disjoint from D and from A, to which to apply Proposition 4.5. Since $X \cap R = \varnothing$ we see that $X_0 \cap R = \varnothing$ as well, and it follows that $(X_0)_{\mathrm{red}} \subset A$. Since $\dim X_0 = \dim X = \dim A$, we see that the support of X_0 is exactly A, and computing the degree we have $\langle X_0 \rangle = d\langle A \rangle$, as claimed.

Example 4.8 (Twisted cubics)**.** When we carry out dynamic projection, the limiting scheme X_0 may have embedded components. To see how these arise, we'll work out an example explicitly; other examples can be found in Exercises 4.39-4.41, and in Exercises 2.50-2.52; also see Chapter 2 of Eisenbud and Harris [2000].

Let $X \subset \mathbb{P}^3$ be a twisted cubic curve. Let A be any 2-plane in $\mathbb{P}^3$, and let $R \notin A \cup X$ be a point on some tangent line to X. We will show that applying the construction of Proposition 4.5, the scheme X_0 will be generically reduced, with support equal to the cubic curve with a cusp that is the projection of X from R to $\mathbb{P}^2$. Since this cuspidal curve has genus 1, while X has genus 0, the scheme X_0 cannot be reduced. In fact, it has an embedded point not contained in the plane, located at the cusp of $(X_0)_{\mathrm{red}}$. See Figure 4.3

In this case, we can compute everything:

We may take the equations of X to be the three 2×2 minors of the matrix

$$M := \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

We choose the repelling plane R to be the point $x_0 = x_1 = x_3 = 0$ and the attracting plane A to be the 2-plane $x_2 = 0$, so that the equations of tX, for $t \in G_m$, are the minors of

$$M_t := \begin{pmatrix} tx_0 & tx_1 & x_2 \\ tx_1 & x_2 & tx_3 \end{pmatrix}.$$

As usual we write $Z \subset \mathbb{A}^1 \times \mathbb{P}^3$ for the corresponding rational equivalence.

Dividing each of the three 2×2 minors of M_t by the biggest possible power of t, we see that the ideal of W contains the polynomials

$$\begin{aligned} f_1 =& x_0x_2 - tx_1^2, \\ f_2 =& tx_0x_3 - x_1x_2, \\ f_3 =& x_2^2 - t^2x_1x_3. \end{aligned}$$

Since $I(Z)$ contains the polynomial $x_1f_1 + x_2f_2 = t(x_0^2x_3 - x_1^3)$, it also contains the polynomial

$$g = x_0^2x_3 - x_1^3.$$

(A Gröbner basis computation shows that the polynomials f_i and g generate the ideal of W—see Eisenbud [1995], Chapter 15—but we will not need this.)

By setting $t = 0$ we see that the fiber X_0 of Z over $0 \in \mathbb{A}^1$ is contained in the subscheme $X_0' \subset \mathbb{P}^3$ defined by the ideal

$$(x_0x_2,\ x_1x_2,\ x_2^2,\ g).$$

The form g alone defines the cone in $\mathbb{P}^3$ over a reduced irreducible cubic curve $D = \{g = x_2 = 0\}$, lying in the plane $x_2 = 0$ and having a cusp at the point $x_0 = x_1 = x_2 = 0$. The rest of the equations

$$(x_0x_2, x_1x_2, x_2^2) = x_2(x_0, x_1, x_2) = (x_2) \cap (x_0, x_1, x_2^2)$$

define the union of the plane $x_2 = 0$ and an embedded point at $x_0 = x_1 = x_2 = 0$. Thus X_0' is the non-reduced scheme consisting of the reduced, irreducible cuspidal plane cubic curve together with an embedded point at the cusp, not contained in the plane.

Now the fiber X_0 of Z must be one-dimensional, and it follows that it must at least contain the cuspidal cubic D and be contained in X_0'. Since the arithmetic genus of X_0 must agree with that of X, which is 0, while the arithmetic genus of D is 1, we must have $D \subsetneq X_0 \subset X_0'$. But since the multiplicity of the embedded point of X_0' is only 1, it follows that $X_0 = X_0'$.

On the other hand, since the cycle associated to X_0 is insensitive to the embedded points, we have

$$[X] \sim [X_0] = [D],$$

as claimed.

There are many other interesting examples of flows ψ_t for which we could make similar constructions, such as the one-parameter subgroups $G_m \subset Aut(\mathbb{P}^n)$. Since every representation of the multiplicative group G_m can be diagonalized over an algebraically closed field, one can write such an action, in suitable coordinates, in the form

$$G_m \ni t : (x_0, \ldots, x_n) \mapsto (t^{w_0}x_0, \ldots, t^{w_n}x_n)$$

for some integral weights w_n. An example is given in Exercise 4.42.

4.2 Rational equivalence through divisors

Another important way to construct rational equivalences is by using rational functions on subvarieties of X (in fact, Corollary 4.18 shows that this method can be used to construct all rational equivalences, and thus gives an alternate presentation of $A(X)$.) ****insert picture**** To set the stage, we review the definition of the divisor associated to a nonzero rational function on an arbitrary variety.

Consider first the affine case, and the divisor associated to a nonzero principal ideal in a domain. Let U be an affine variety with coordinate ring A, and let $f \in A$ be a nonzero function. Krull's Principle Ideal Theorem 0.1 asserts that each irreducible component Y of the subscheme $V(f)$ defined by f is of codimension 1 in X. Thus the local ring $R = \mathcal{O}_{U,Y}$ of the generic point of Y is 1-dimensional, so, as above, the factor ring $R/(f)$ is of finite length. We define the *order* of f at Y to be this length, for which we will write $\mathrm{ord}_R(f)$ or $\mathrm{ord}_{Y/U}(f)$, so that

$$\mathrm{ord}_R(f) = \mathrm{ord}_{Y/U}(f) = \text{length } \mathcal{O}_{U,Y}/f\mathcal{O}_{U,Y}.$$

We define the divisor of f to be the divisor of the subscheme defined by f,

$$Div(f) := \sum_{\substack{\{Y \mid Y \text{ is a} \\ \text{component of } V(f)\}}} \mathrm{ord}_Y(f) \cdot Y.$$

Note that, by Krull's Principal Ideal Theorem all the components of $V(f)$ have codimension exactly 1 in U.

What makes the case of a principal ideal special is that the association $f \mapsto Div(f)$ turns multiplication into addition: that is,

Proposition 4.9. *If f, g are nonzerdivisors in a local Noetherian ring R of dimension 1, then*

$$Div_X(fg) = Div(f) + Div(g).$$

Proof. Since g is a nonzerodivisor, multiplication by g defines an isomorphism $R/(f) \to (g)/(fg)$. The resulting exact sequence

$$0 \longrightarrow R/(f) \xrightarrow{g} R/(fg) \longrightarrow R/(g) \longrightarrow 0,$$

gives $\mathrm{ord}_R(fg) = \mathrm{ord}_R(f) + \mathrm{ord}_R(g)$. □

This additivity has a crucially important consequence: we can define the divisor not just of a regular function, but also of a rational function. Suppose that $\alpha \in K(X)$ is a rational function on a (not necessarily affine) variety X, and Y is any codimension 1 subvariety in X, then we can choose nonzero $f, g \in R := \mathcal{O}_{X,Y}$ such that $\alpha = f/g$. The number $\mathrm{ord}_R(f) - \mathrm{ord}_R(g)$ does not depend on the choice of f and g, since if $f/g = f'/g'$ we will have $fg' = f'g$, whence by additivity $ord_R(f) + ord_R(g') = ord_R(f') + ord_R(g)$, so that $\mathrm{ord}_R(f) - \mathrm{ord}_R(g) = \mathrm{ord}_R(f') - \mathrm{ord}_R(g')$. Thus we can unambiguously define $\mathrm{ord}_Y(\alpha) = \mathrm{ord}_R(f) - \mathrm{ord}_R(g)$.

We claim that, for any nonzero rational function α, there are only finitely many codimension 1 subvarieties $Y \subset X$ such that $\mathrm{ord}_Y(\alpha) \neq 0$. To see this, take a finite covering by affine open subsets U_i. For each i, write $\alpha|_{U_i} = f_i/g_i$ with $f_i, g_i \in \mathcal{O}_X(U_i)$. If Y appears with a nonzero coefficient in $Div(\alpha)$, and U_i is one of the open sets such that $Y \cap U_i$ is dense, then $Y \cap U_i$ must be among the finitely many irreducible components of $V(f_i)$ or $V(g_i)$. This leaves only finitely many possibilities for Y.

Proposition 4.10. *Let X be a variety of dimension n. There is a homomorphism of groups*

$$Div : K(X)^* \to Rat_{n-1}(X) \subset Z_{n-1}(X)$$

taking $\alpha \in K(X)^*$ to

$$Div(\alpha) = \sum_{\substack{\{Y \mid Y \text{ is a} \\ \text{codimension 1} \\ \text{subvariety of } X.\}}} \mathrm{ord}_Y(\alpha) Y. \quad \square$$

Proof. The argument above shows that the map $Div : K(X)^* \to Z_{n-1}(X)$ is well-defined. To see that its image is in $Rat_{n-1}(X)$, let α be a rational function on X. The element α defines a rational map $\varphi : X \to \mathbb{P}^1$ and the graph of φ is a subset of $X \times \mathbb{P}^1$ birational to X. What we have called $Div(\alpha)$ is, under this isomorphism, precisely $\varphi^{-1}(0) - \varphi^{-1}(\infty)$. □

Example 4.11. Let $P \subset \mathbb{P}^3$ be a plane. Any two curves $D, D' \subset P$, both of degree d, are rationally equivalent in P and thus in $\mathbb{P}^3$: for if $g = 0$ and $g' = 0$ are the equations of D and D' respectively, then the ratio $\alpha = g/g'$ is a rational function on P because g and g' are homogeneous of the same degree. Further, $Div(\alpha) = [D] - [D']$.

Here are some observations that will help us with further examples:

Proposition 4.12. *Let X be a scheme.*

(a) $A(X) = A(X_{\text{red}})$.

(b) *If X is a variety of dimension d, then $A_d(X) \cong \mathbb{Z}$, with generator $[X]$, called the* fundamental class *of X. More generally, if X is reduced, then $A_d(X)$ is the free abelian group on the classes $[X_i]$ of the irreducible components of X that have dimension d.*

(c) *If $Y \subset X$ is a closed subscheme and $U = X \setminus Y$ is the complementary open subscheme, then there is a right-exact sequence*

$$A(Y) \xrightarrow{\beta} A(X) \xrightarrow{\alpha} A(U) \longrightarrow 0.$$

(d) *If X_1, X_2 are closed subschemes of X, then there is a right exact sequence*

$$A(X_1 \cap X_2) \longrightarrow A(X_1) \oplus A(X_2) \longrightarrow A(X_1 \cup X_2) \longrightarrow 0.$$

Proof of Proposition 4.12. (a) By definition the generators of $Z(X)$ are varieties in X, and since they are reduced they are contained in X_{red}. The same argument shows that $Z(X \times \mathbb{P}^1) = Z(X_{\text{red}} \times \mathbb{P}^1)$, so the cokernel of ∂ is unchanged if we replace X by X_{red}.

(b) Since $X \times \mathbb{P}^1$ is the only subvariety of $X \times \mathbb{P}^1$ of dimension $d+1$, the map $\partial_X : Z_{d+1}(X \times \mathbb{P}^1) \to Z_d(X)$ is zero, so $A_d(X) = Z_d(X) = \mathbb{Z} \cdot [X]$.

For the second statement, let $X_1, \ldots, X_m$ be the distinct irreducible components of X of dimension d. By definition, $A_d(X)$ is generated by $[X_1], \ldots, [X_m]$. Any relation among the $[X_i]$ comes from a variety $W \subset X \times \mathbb{P}^1$ dominating $\mathbb{P}^1$ and containing $X_i \times \{0\}$.

Since $W = \bigcup_i (W \cap (X_i \times \mathbb{P}^1))$ and W is irreducible, we must have $W \subset X_i \times \mathbb{P}^1$. Since W dominates $\mathbb{P}^1$, it is strictly larger than $X_i \times 0$, so $\dim W \geq 1 + \dim X_i = \dim(X_i \times \mathbb{P}^1)$. Thus $W = X_i \times \mathbb{P}^1$. Thus the relation on $A_d(X)$ provided by W is of the form $[X_i] = [X_i]$—the trivial relation.

(c) There is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Z(Y\times\mathbb{P}^1) \arrow[r] \arrow[d, "\partial_Y"'] & Z(X\times\mathbb{P}^1) \arrow[r] \arrow[d, "\partial_X"'] & Z(U\times\mathbb{P}^1) \arrow[r] \arrow[d, "\partial_U"'] & 0 \\
0 \arrow[r] & Z(Y) \arrow[r] \arrow[d] & Z(X) \arrow[r] \arrow[d] & Z(U) \arrow[r] \arrow[d] & 0 \\
& A(Y) \arrow[r] \arrow[d] & A(X) \arrow[r] \arrow[d] & A(U) \arrow[d] & \\
& 0 & 0 & 0 &
\end{tikzcd}$$

where the map $Z(Y) \to Z(X)$ takes the class $[A] \in Z(Y)$, where A is a subvariety of Y, to $[A]$ itself, considered as a class in X, and similarly for $Z(Y\times\mathbb{P}^1) \to Z(X\times\mathbb{P}^1)$. The map $Z(X) \to Z(U)$ takes each free generator $[A]$ to the generator $[A \cap U]$, and similarly for $Z(X \times \mathbb{P}^1) \to Z(U \times \mathbb{P}^1)$. The two middle rows and all three columns are evidently exact. A diagram chase shows that the map $A(X) \to A(U)$ is surjective, and the bottom row of the diagram above is right exact, yielding part (c).

(d) Let $Y = X_1 \cap X_2$. We may assume $X = X_1 \cup X_2$. We may argue exactly as before from the diagram

$$\begin{tikzcd}
0 \arrow[r] & Z(Y\times\mathbb{P}^1) \arrow[r] \arrow[d, "\partial"'] & Z(X_1\times\mathbb{P}^1)\oplus Z(X_2\times\mathbb{P}^1) \arrow[r] \arrow[d, "\partial\oplus\partial"'] & Z(X\times\mathbb{P}^1) \arrow[r] \arrow[d, "\partial"'] & 0 \\
0 \arrow[r] & Z(Y) \arrow[r] \arrow[d] & Z(X_1)\oplus Z(X_2) \arrow[r] \arrow[d] & Z(X) \arrow[r] \arrow[d] & 0 \\
& A(Y) \arrow[r] & A(X_1)\oplus A(X_2) \arrow[r] & A(X) \arrow[r] & 0
\end{tikzcd}$$

where, for example, the map $Z(Y) \longrightarrow Z(X_1) \oplus Z(X_2)$ takes a generator $[A] \in Z(Y)$ to $([A], -[A]) \in Z(X_1) \oplus Z(X_2)$ and the map $Z(X_1) \oplus Z(X_2) \longrightarrow Z(X)$ is addition. $\square$

Proposition 4.12 allows us to answer Keynote Question (b) in the negative. Indeed, we can generalize as follows.

Corollary 4.13. *If $X \subset \mathbb{P}^n$ is a variety of dimension m and degree d, then $A_m(\mathbb{P}^n \setminus X) \cong \mathbb{Z}/(d)$, while if $m < m' \leq n$ then $A_{m'}(\mathbb{P}^n \setminus X) = \mathbb{Z}$. In particular, m and d are determined by the isomorphism class of $\mathbb{P}^n \setminus X$.* $\square$

Proof. Part (c) of Proposition 4.12 shows that there are exact sequences $A_i(X) \to A_i(\mathbb{P}^n \to A_i(\mathbb{P}^n \setminus X \to 0$. Furthermore $A_m(X) = \mathbb{Z}$ by Part **??** of Proposition 4.12, while $A_{m'}(X) = 0$ for $m < m' \leq n$. By Theorem 1.20 we have $A_i(\mathbb{P}^n) = \mathbb{Z}$ for $0 \le i \le n$, and the image of the generator of $A_m(X)$ in $A_m(\mathbb{P}^n)$ is d times the generator of $A_i(\mathbb{P}^n)$. The results of the Corollary follow. $\square$

4.3 Proper push-forward

To get much further we need stronger tools. One of them is very compactly expressed below by saying that $A(X)$ behaves like a covariant functor with respect to proper maps, and this preserves the grading by dimension. This can be used to prove, for example, that the class of a given cycle is nonzero—we just exhibit a proper map that takes it to a cycle whose class we already know to be nonzero. We will see examples of this below.

If $f : X \to X'$ is a proper map of schemes, then the image of a subvariety $Y \subset X$ is a subvariety $f(Y) \subset X'$. This would make it possible to define a push-forward map f_* on cycles by taking any subvariety to its image. However, this map gives no information about the Chow groups because it does not preserve rational equivalence (see Exercise **??**). But if we introduce certain multiplicities, rational equivalence will be preserved, and we get a map on Chow groups. Here is how this is done:

Definition 4.14. Let $f : X \to X'$ be a proper map of schemes, and let $Y \subset X$ be a subvariety.

(a) If $f(Y)$ has lower dimension than Y, then we define $f_*(\langle Y\rangle) = 0$.

(b) If $\dim f(Y) = \dim Y$, then the map $f|_Y : Y \to f(Y)$ is generically finite. If $n := [K(Y) : K(f(Y))]$ is the degree of the extension of fields of rational functions, we say that f is *generically a cover of degree* n, and we define $f_*\langle Y\rangle = n\langle f(Y)\rangle$.

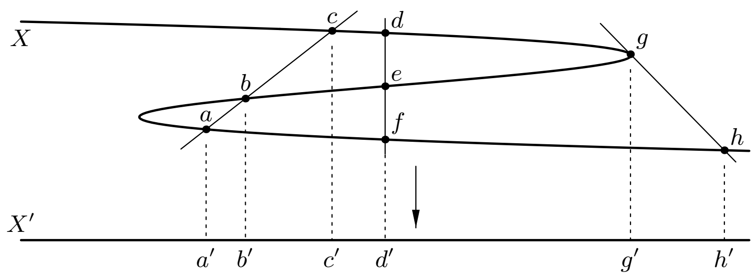
(c) If $z = \sum n_i\langle Y_i\rangle \in Z(X)$ is a formal $\mathbb{Z}$-linear combination of subvarieties, then we define $f_*(z) = \sum n_i f_*\langle Y_i\rangle$.

Theorem 4.15. *If $f : X \to X'$ is a proper map of schemes, then the map $f_* : Z(X) \to Z(X')$ defined above preserves rational equivalence, and thus induces a map $f_* : A(X) \to A(X')$.*

We will prove Theorem 4.15 and the closely related Theorem 4.17 below after developing an important algebraic tool in Lemma 4.19.

Before starting the proof of Theorem 4.15 we discuss some geometric consequences. A first hint of the importance of the apparently bland statement of the theorem is that it makes possible the existence of a degree function $A_0(X) \to \mathbb{Z}$ when X is projective over a field. In particular, it allows us to prove that $A_0(\mathbb{P}^n_K) = \mathbb{Z} \cdot [p]$ for any closed K-rational point $p \in \mathbb{P}^n$, something we could not prove in the last section.

$$a + b + c \sim d + e + f \sim 2g + h$$



$$a' + b' + c' \sim 3d' \sim 2g' + h'$$

FIGURE 4.4. Pushforwards of equivalent cycles are equivalent

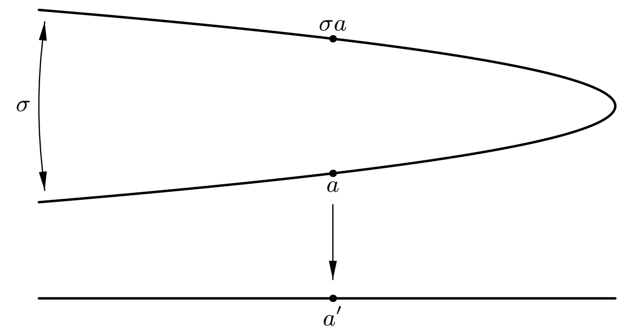
Corollary 4.16. *Let K be a field, and let X be a scheme that is proper over* $\operatorname{Spec} K$. *There is a map $A_0(X) \to \mathbb{Z}$ that sends the class $[Y]$ of a zero-dimensional subscheme $Y \subset X$ to $\deg Y := \dim_K \mathcal{O}_Y$. In particular $A_0(X) \neq 0$.*

Proof. First, the definition of $\deg : Z_0(X) \to \mathbb{Z}$ makes sense because any zero-dimensional scheme, proper over $\operatorname{Spec} K$, is finite over $\operatorname{Spec} K$. Let $f : X \to \operatorname{Spec} K$ be the constant map. If p is a closed point of X then, by definition, $f_*(\langle p \rangle) = \dim_K \kappa(p) \cdot \langle \operatorname{Spec} K \rangle$, so $\deg(\langle p \rangle) = \deg f_*(\langle p \rangle) = \dim_k \kappa(p)$. By Theorem 4.15, the map $\deg \circ f_*$ is well-defined on $A_0(X)$. $\square$

The idea of this proof also shows that we cannot drop the assumption of properness in Theorem 4.15: for example, if we could define f_* on $A_0(\mathbb{A}^n_K)$ when $f : \mathbb{A}^n_K \to \operatorname{Spec} K$, we would deduce, as above, that $A_0(\mathbb{A}^n_k) \neq 0$ and we have already seen that $A_0(\mathbb{A}^n_K) = 0$. See Exercises 4.29 and 4.30 for further examples.

The statement of Theorem 4.15 is obvious when f is a closed immersion since in this case any rational equivalence in $Z(X)$ is also a rational equivalence in $Z(X')$. But already in this case, we can use the result to answer Keynote Question (a) in the negative: If $X \subset \mathbb{P}^3$ is a surface with a map $\pi : X \to \mathbb{P}^1$ then the fibers of π are rationally equivalent in X. By Theorem 4.15 they are also rationally equivalent in $\mathbb{P}^3$. By Theorem 4.4 this implies that they have the same degree as projective curves.

We can sharpen Theorem 4.15, in the generically finite case, by giving an explicit formula for the push-forward of the divisor of a rational function.



$$\langle a' \rangle = \pi_* \langle a \rangle$$

$$\langle a \rangle + \langle \sigma a \rangle = \pi^* N \langle a \rangle$$

FIGURE 4.5. The pushforward of the divisor of a function is the divisor of its norm

Theorem 4.17. *Let $f : X \to Y$ be a proper, dominant, generically finite morphism of varieties, and let α be a rational function on X. If $N : K(X) \to K(Y)$ denotes the norm function,*

$$N(\alpha) := \det_{K(Y)}\big[K(X) \xrightarrow{\alpha} K(X)\big]$$

then $f_*(Div(\alpha)) = Div(N(\alpha)) \in Z(Y)$. In particular, the push-forward of the divisor of a rational function under a generically finite morphism is again the divisor of a rational function.

We saw (just before Proposition 4.12) that if W is a subvariety of X and α is a rational function on W, then $Div(\alpha)$, regarded as a cycle on X, is rationally equivalent to zero. As a consequence of Theorem 4.17 we can show that the divisors of rational functions suffice to generate the relation of rational equivalence:

Corollary 4.18. *For any scheme X, the kernel of the map $Z(X) \to A(X)$ is generated by the divisors of the form $Div(\alpha)$ where α ranges over rational functions on subvarieties of X.*

Proof of Corollary 4.18. We already know that such $Div(\alpha)$ are in the kernel. On the other hand, the kernel is generated by elements of the form $\langle V_0 \rangle - \langle V_\infty \rangle$ where the V_i are subschemes of X such that there exists a variety $V \subset X \times \mathbb{P}^1$ generically finite over its image in X, and dominating $\mathbb{P}^1$, with $V \cap X \times \{0\} = V_0$ and $V \cap X \times \{\infty\} = V_\infty$. As a cycle on V we have $\langle V_0 \times \{0\} \rangle - \langle V_\infty \times \{\infty\} \rangle = Div(\alpha)$, where α is the rational function corresponding to the projection map $V \to \mathbb{P}^1$.

Let V' be the image of V in X, and note that V' is a subvariety. As cycles on X we have $V_0 = f_*(\langle V_0 \times \{0\}\rangle)$, and similarly for V_∞. Thus $\langle V_0\rangle - \langle V_\infty\rangle = f_*(Div(\alpha)) = Div(N(\alpha))$, the divisor of a rational function on V', completing the argument. $\square$

4.3.1 The determinant of a homomorphism

The proof of Theorems 4.15 and 4.17 rely on a fundamental result in commutative algebra relating a homomorphism to its determinant.

Suppose that R is a domain with quotient field K. If M is a finitely-generated torsion-free R-module and $\varphi : M \to M$ is an endomorphism, then $\varphi \otimes_R K$ is an endomorphism of the finite dimensional vector space $M \otimes_R K$. We define the determinant $\det(\varphi)$ of φ to be the usual determinant of $\varphi \otimes_R K$, an element of K.

For example, if $r \in R$ and φ is multiplication by r, then $\det \varphi = r^{\operatorname{rank} M}$; this is because the endomorphism $\varphi \otimes_R K$ of $M \otimes_R K = K^{\operatorname{rank} M}$ is also multiplication by r, represented by a diagonal matrix

$$\begin{pmatrix} r & 0 & \dots & 0 \\ 0 & r & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & r \end{pmatrix}$$

Lemma 4.19 (Determinant Lemma)**.** *Suppose that R is a 1-dimensional local Noetherian domain, and let $\varphi : M \to M$ be an endomorphism of a finitely generated torsion-free R-module M. If $\det \varphi$ is nonzero, then* coker φ *is a module of finite length, and its length is equal to* $\operatorname{ord}_R(\det \varphi)$*. In particular, if $r \in R$ and M has rank n, then*

$$\operatorname{length}(M/rM) = n \cdot \operatorname{length}(R/rR).$$

The Lemma implies, in particular, that $\operatorname{ord}_R \det \varphi \geq 0$, even when $\det \varphi$ is not in R. This comes about because $\det \varphi$ is always in the integral closure of R (use, for example, Eisenbud [1995], Corollary 4.6).

When M is free and the endomorphism is defined by a diagonal matrix, the result says that

$$\text{length } R/(\prod_i x_i) = \sum_i \text{length } R/(x_i),$$

an easy special case of Proposition 4.9. The idea of the proof we will give is to reduce to this situation by moving to the integral closure of R and then localizing so that R becomes a discrete valuation ring. We can then use the structure of matrices over such rings.

Proof of Lemma 4.19. The second statement follows from the first because, taking φ to be multiplication by r on M, the remark just before the lemma shows that $\det \varphi = r^n$.

In the remainder of the proof we will assume that the domain R is a localization of an algebra finitely generated over a field, so that the integral closure R' of R is a finite R-module. This suffices for the applications we will make. See the remarks after the proof for various ways of removing this restriction and generalizing the lemma further.

Let $\varphi : M \to M$ be an endomorphism with nonzero determinant D. Since R has dimension 1, the cokernel of φ is a module of dimension zero, and thus finite length. Let R' be the integral closure of R in its quotient field K, and let $M' = M \otimes_R R'/(\text{torsion})$, so that $M \subset M' \subset K \otimes_R M$ (in fact, M' is the image of $M \otimes_R R'$ in $M \otimes_R K$).

We will use three properties of this construction. First, the endomorphism φ extends to $\varphi \otimes 1 : M \otimes_R R' \to M \otimes_R R'$, and preserves the torsion submodule, so it induces a map $\varphi' : M' \to M'$ extending the map φ on M. Second, since $M' \subset M \otimes_R K$, the quotient M'/M is torsion, and thus of finite length. Of course φ' induces an endomorphism of M'/M, which we will call $\overline{\varphi'}$. Finally, since $M \otimes_R K = M' \otimes_R K = M \otimes_{R'} K$, the maps φ and φ' have the same determinant $D = \det \varphi = \det \varphi'$. We will reduce to the case where $R = R'$, and then use the structure theorem for modules over a principal ideal domain.

With these points in mind, consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \operatorname{Ker} \overline{\varphi'} \arrow[d] & \\
0 \arrow[r] & M \arrow[r] \arrow[d, "\varphi"] & M' \arrow[r] \arrow[d, "\varphi'"] & M'/M \arrow[r] \arrow[d, "\overline{\varphi'}"] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & M' \arrow[r] \arrow[d] & M'/M \arrow[r] \arrow[d] & 0 \\
 & \operatorname{coker} \varphi \arrow[r] & \operatorname{coker} \varphi' \arrow[r] & \operatorname{coker} \overline{\varphi'} \arrow[r] & 0.
\end{tikzcd}$$

A diagram chase (the "Snake Lemma") yields an exact sequence

$$0 \to \operatorname{Ker} \overline{\varphi'} \to \operatorname{coker} \varphi \to \operatorname{coker} \varphi' \to \operatorname{coker} \overline{\varphi'} \to 0,$$

from which we deduce that

$$\begin{aligned} &\text{length } \operatorname{coker} \varphi' - \text{length } \operatorname{coker} \varphi \\ &\qquad = \text{length } \operatorname{coker} \overline{\varphi'} - \text{length } \operatorname{Ker} \overline{\varphi'}. \end{aligned}$$

But from the exact sequence

$$0 \to \mathrm{Ker}\ \overline{\varphi'} \to M'/M \xrightarrow{\overline{\varphi'}} M'/M \to \mathrm{coker}\ \overline{\varphi'} \to 0,$$

we similarly deduce that

$$\begin{aligned} 0 = \text{length } M/M' - &\text{length } M/M' \\ &= \text{length coker } \overline{\varphi'} - \text{length Ker } \overline{\varphi'}, \end{aligned}$$

proving that length coker $\varphi =$ length coker φ'.

As a special case, we may take $M = R$, and we see that length $R/D =$ length R'/D. Thus it suffices to prove the lemma in the case $R = R'$; that is, we may assume that the 1-dimensional ring R is integrally closed. By the Chinese Remainder Theorem, any R-module of finite length is the direct sum of its localizations at various maximal ideals, so we may further assume that R is a discrete valuation ring, with parameter π, say.

Any finitely-generated torsion-free module over such a ring R is free, so the endomorphism φ is defined by a square matrix of elements of R. After row and column operations, we can reduce this matrix to a diagonal matrix with powers of π on the diagonal,

$$\varphi = \begin{pmatrix} \pi^{a_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \pi^{a_m} \end{pmatrix}.$$

Since $\text{length}(R/\pi^a) = a$, we see that

$$\text{length coker } \varphi = \text{length } \bigoplus_{i=1}^{m} (R/\pi^{a_i}) = \sum_{i=1}^{m} a_i$$

while

$$\text{length } R/(\det \varphi) = \text{length } R/(\prod_{i=1}^{m} \pi^{a_i}) = \sum_{i=1}^{m} a_i$$

as required. $\square$

As we remarked in the beginning of the proof, the restriction to algebras "essentially of finite type over a field," made in order to guarantee the finiteness of the integral closure, is unnecessary. One way to avoid it is to pass to the completion, and use the result that the integral closure of any complete local Noetherian ring is finite. This is the path taken by Grothendieck [1967] 21.10.17.2. There is an attractive direct proof along very different lines in Fulton [1984], Appendix A1–A3. But the result given here and in these references is actually a special case of a far more general story involving multiplicity: the essential point is that two modules with the same multiplicity will have the same length after factoring out a regular sequence modulo which they have finite length. This idea can be used to show, for example that if $\varphi : R^p \to R^q$ is a map over a Cohen-Macaulay ring R of dimension $p-q+1$ and the cokernel of φ has finite length, then its length is the same as that of R/I, where I is the ideal generated by all the $q \times q$ minors of φ; our Lemma 4.19 contains the case $p = q$. This result is attributed in Bruns and Vetter [1986] to Angeniol and Giusti, though the paper by Angeniol and Giusti seems never to have appeared. In any case the result is generalized in the cited paper. The proof by multiplicities and specialization was given by Hochster and Huneke (also unpublished).

Example 4.20. The second statement of Lemma 4.19 is already interesting when the rank n is 1; a good example is the case where $M = K[t]$, $R = K[t^2, t^5]$ and $r = t^5$. In this case the length is simply the dimension as a K-vector space. It is easy to see that the length of

$$M/rM = K[t]/ < t^5, t^6, t^7, \cdots >$$

is 5. The reader may check that the quotient

$$R/rR = K[t^2, t^4, t^5, t^6, \dots]/ < t^5, t^7, t^9, t^{10}, t^{11}, \cdots > .$$

is generated as a vector space by $1, t^2, t^4, t^6$, and t^8, so the length is 5 in this case too, though the two vector spaces are not related in a very obvious way.

Proof of Theorem 4.15. It suffices to show that the diagram

$$\begin{array}{ccc} Z(X \times \mathbb{P}^1) & \xrightarrow{(f \times 1)_*} & Z(X' \times \mathbb{P}^1) \\ \Big\downarrow{\scriptstyle \partial_X} & & \Big\downarrow{\scriptstyle \partial_{X'}} \\ Z(X) & \xrightarrow{f_*} & Z(X') \end{array}$$

commutes. To this end, suppose $W \subset X \times \mathbb{P}^1$ is a subvariety dominating $\mathbb{P}^1$, and let $W' = (f \times 1)(W)$. Since f is proper, W' is a subvariety of $X' \times \mathbb{P}^1$. Of course the projection $W \to \mathbb{P}^1$ factors through W', so W' dominates $\mathbb{P}^1$ if and only if W does. Write W_0 and W_0' for the preimages of $0 \in \mathbb{P}^1$ in the varieties W and W'. We may think of W_0 as a subvariety of $X = (X \times \mathbb{P}^1)_0$ and similarly for W_0' and X', and from this point of view, $W_0 = f^{-1}(W_0')$ as schemes.

****insert picture****

Suppose first that W' has strictly smaller dimension than W, so that $(f \times 1)_*\langle W\rangle = 0$. By the Principal Ideal Theorem, every component of W_0 or of W_∞ has dimension one less than that of W, and similarly the components of W_0' and W_∞' have dimension one less than that of W', so $f_*(\langle W_0\rangle - \langle W_\infty\rangle) = 0$ as well.

Thus we may assume that $\dim W' = \dim W$, so that $W \to W'$ is generically finite. Write n for its degree, so that $(f \times 1)_*\langle W\rangle = n\langle W'\rangle$.

Let V' be an irreducible component of W_0', and suppose that $\langle V' \rangle$ appears in the cycle associated to W_0' with multiplicity p. We will show that $\langle V' \rangle$ appears in the cycle $f_*(\langle W_0 \rangle)$ with multiplicity np. Summing over the contributions of all the components of W_0' we see that $f_*(\langle W_0 \rangle) = n\langle W_0' \rangle$. The same argument will of course apply to W_∞, proving the Theorem.

Since f induces a surjective map $W \to W'$, and $V' \subset W'$ has codimension 1, the map $W \to W'$ must be finite over the generic point of V' (otherwise the preimage of V' would have dimension at least that of W, so W would be reducible). Because the map f is proper, the induced injection

$$f^* : R' := \mathcal{O}_{W',V'} \to f_*\mathcal{O}_W \otimes_{\mathcal{O}_{W'}} \mathcal{O}_{W',V'} = \mathcal{O}_{W,f^{-1}V'} =: R'$$

is finite.

Both R' and R have dimension 1. Saying that V' occurs with multiplicity p in W_0' means that $p = \text{length}_{R'}(R'/xR')$, where x is a coordinate on $\mathbb{P}^1$ that vanishes at 0. Since $f \times 1 : W \to W'$ has degree n, the ring R has rank n as an R'-module. The Determinant Lemma tells us that

$$\text{length}_{R'}(R/xR) = n \cdot \text{length}_{R'}(R'/xR') = np.$$

The maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_k$ of R correspond precisely to the subvarieties $V_1, \ldots, V_s \subset W_0$. If the multiplicity of x at $\mathfrak{m}_i$ is e_i, then the Chinese Remainder Theorem shows that $\text{length}_{R'} R/xR = \sum_i e_i n_i$, where n_i is the degree of the field extension $K(V_i) : K(V')$. From the definition of f_* we see that $f_*(\langle V_i \rangle) = n_i \langle V' \rangle$. Putting this together we compute that the multiplicity of $\langle V' \rangle$ in $f_*(\langle W_0 \rangle)$ is

$$f_*(\sum_i e_i \langle V_i \rangle) = \sum_i e_i n_i \langle V' \rangle = np\langle V' \rangle$$

as required. $\square$

Proof of Theorem 4.17. Write $Div(\alpha) = \sum a_i \langle V_i \rangle$, where the V_i are subvarieties. The cycle $f_*(Div(\alpha))$ is the sum of the terms $a_i f_*(\langle V_i \rangle)$. By definition, such a term is zero unless $\dim f(V_i) = \dim(V_i)$, in which case the restriction $f|_{V_i}$ is generically finite. Since the dimension of X is the same as that of Y, the subset of X over which f has finite fibers must contain an open set of any codimension 1 component of Y. Thus we may restrict to this set and assume, without loss of generality, that f is finite.

The components of $f_*(Div(\alpha))$ and $Div(N(\alpha))$ are all of codimension 1, so it suffices to prove the equality of the theorem at the generic point of a codimension 1 subvariety $V' \subset Y$. That is, we may suppose that Y is the spectrum of a one-dimensional local ring $R = \mathcal{O}_{Y,V'}$ with quotient field $K(Y)$. Since X is finite over $\text{Spec}(R)$ it is also affine, say $X = \text{Spec}(S)$, where $S = \mathcal{O}_{X,f^{-1}(V')}$ is a one-dimensional ring with quotient field $K(X)$, finite over R.

We may write $\alpha = r/s$ for some elements $r, s \in S$. Since the norm is multiplicative, $N(\alpha) = N(r)/N(s)$. Further, since Div takes multiplication to addition, it suffices to show that $f_*(Div(x)) = Div(N(x))$ for any element $x \in S$.

The components V_i of $Div(x)$ that map onto V' correspond to the maximal ideals P_i of S that contain x, and $f_*\langle V_i\rangle = [K(V_i) : K(V')]\langle V'\rangle$. By the Chinese Remainder Theorem, $S/xS = \oplus_i S_{P_i}/xS_{P_i}$, so $\mathrm{length}_R(S/xS) = \sum \mathrm{ord}_{V_i}(x)'[K(V_i) : K(V')]$. Thus $f_*Div(x) = \mathrm{length}_R(S/xS)\langle V'\rangle$.

We can now apply Lemma 4.19 to the endomorphism of S given by multiplication by x. The determinant of this map is $N(x)$, and we see that $\mathrm{length}_R(S/xS) = \mathrm{length}_R(R/N(x))$, which is the coefficient of V' in $\langle Div(N(x))\rangle$, as required. □

4.4 Flat pullback

In this section we will study and apply a different sort of functoriality of the Chow groups:

Theorem 4.21 (Flat Pullback)**.** *If $f : Y \to X$ be a flat map of equidimensional schemes, then there is a unique group homomorphism $f^* : A(X) \to A(Y)$ satisfying*

$$f^*([A]) = [f^{-1}(A)]$$

for any variety $A \subset X$. This homomorphism preserves codimension.

In the next chapter we will define an intersection product on the Chow ring of a smooth quasiprojective variety, and it will follow from the methods there that flat pullback between such varieties preserves products (Corollary 5.5).

We will use Theorem 4.21 in the next subsections to analyze affine and projective bundles. It will also be useful as a step in the proof of existence of pullbacks along arbitrary proper morphisms of smooth quasiprojective varieties, a topic of Chapter 5.

Proof of Theorem 4.21. We first prove that the flat pullback preserves codimension. Suppose that $W \subset X$ is a subvariety, and that V is an irreducible component of $f^{-1}(W)$.

Suppose that $y \in V$ is a closed point not belonging to any other component of $f^{-1}(W)$, and $x = f(y)$, which is again a closed point by the Nullstellensatz. Because $f|_V : V \to W$ is flat in a neighborhood of y, the dimension of V at y is equal to the dimension of the image of V plus the dimension at y of the fiber of f through y; that is

$$\dim V = \dim W + \dim \mathcal{O}_{f^{-1}(x),y}$$

(see for example Eisenbud [1995] Theorem 10.10). Since the same is true of the map $f : Y \to X$ itself, we have,

$$\dim V - \dim W = \dim \mathcal{O}_{f^{-1}(x),y} = \dim Y - \dim X,$$

as required.

We next show that f^{-1} preserves rational equivalence. By Corollary 4.18 the relation of rational equivalence is generated by the divisors of rational functions on irreducible subvarieties. Thus it suffices to show that if $X' \subset X$ is a subvariety and α is a rational function on X' then $f^*(Div(\alpha))$ is the divisor of the rational function $\alpha \circ f$ on $f^{-1}(X')$. The flatness of f implies the flatness of $f|_{f^{-1}(X')} : f^{-1}(X') \to X'$, so we may begin by replacing X with the variety X' and Y with $f^{-1}(X')$, and assume that X is irreducible and α is a rational function on X. Of course we can no longer assume that Y is irreducible, or even reduced, but because f is flat, the function $\alpha \circ f$ can be written, locally near any point $y \in Y$, in the form a/b where a and b are nonzerodivisors on $\mathcal{O}_{Y,y}$, so $[Div(\alpha \circ f)] \in A(Y)$. Thus proving that

$$f^*(Div(\alpha)) = Div(\alpha \circ f)$$

will show that f^* preserves rational equivalence.

Let $V \subset Y$ be any codimension 1 subvariety, and let $W \subset X$ be the closure of $f(V)$. We must show that if W has codimension 1 in X, then the coefficient of $[V]$ in $Div(\alpha \circ f)$ is equal to the coefficient of $[W]$ in in $Div(\alpha)$ times the coefficient of $[V]$ in $[f^{-1}(W)]$. This can be checked after localizing at V and W; that is, we may replace the original f by its restriction $f : \mathrm{Spec}(\mathcal{O}_{Y,V}) \to \mathrm{Spec}(\mathcal{O}_{X,W})$ corresponding to the flat, local map of 1-dimensional local rings

$$S := \mathcal{O}_{X,W} \xrightarrow{-\circ f} \mathcal{O}_{Y,V} := R.$$

To simplify the notation we set $S := \mathcal{O}_{X,W}$ and $R := \mathcal{O}_{Y,V}$. Let $\mathfrak{m}$ be the maximal ideal of S, which corresponds to the subvariety W.

Since X is a variety, S is a domain. Since R is flat over S the map $S \to R$ is an inclusion, and we will identify S with its image in R. Writing $\alpha = a/b$ for $a, b \in S$, we have $Div(\alpha) = Div(a) - Div(b)$, so it suffices to treat the case $\alpha = a \in S$.

We now write:

- p for the coefficient of $[W]$ in $Div(a)$, so $p := \mathrm{length}(S/aS)$;
- q for the coefficient of $[V]$ in $f^*[W] = [f^{-1}(W)]$, so $q := \mathrm{length}(R/\mathfrak{m}R)$;
- r for the coefficient of $[V]$ in $Div(a \circ f)$, so $r := \mathrm{length}(R/aR)$.

We must show that $pq = r$. Let

$$aS \subset M_1 \subset \cdots \subset M_p = S$$

be a chain of inclusions, where each M_{i+1}/M_i a simple module, witnessing the fact that $p := \text{length}(S/aS)$. Since S is local, it has only one simple module, namely $M_{i+1}/M_i \cong S/\mathfrak{m}$. Tensoring this module with R we get the R-module $R/\mathfrak{m}R$, whose length is q. Since R is flat over S, the chain of inclusions above gives rise to a chain of inclusions

$$aR \subset R \otimes_S M_1 \subset \cdots \subset R \otimes_S M_p = R.$$

Since each quotient $R \otimes M_{i+1}/R \otimes M_i$ has length q, we deduce that $r = pq$ as required. □

Remark. The proof of Theorem 4.21 extends easily to a slightly more general class of flat maps: we can allow X to be any scheme, but assume that there is a fixed integer n such that for each irreducible component $X' \subset X$ the preimage $f^{-1}(X')$ is equidimensional of dimension $n + \dim X'$. When this condition is satisfied, we will say that f has constant relative dimension n. Each part of the proof of Theorem 4.21 reduced to the consideration of what happened over some subvariety of X, and thus it extends at once to the more general case.

4.4.1 Affine space bundles

We have seen in Proposition 1.17 that the Chow groups of an affine space are trivial: $A(\mathbb{A}^n_K) = \mathbb{Z} \cdot [\mathbb{A}^n]$. One way to interpret this formula is to say that the flat pullback map $f^* : A^{\operatorname{Spec} K} \to A(\mathbb{A}^n_K)$ is surjective. We can extend this result to any *affine space bundle*; that is, to a map such that X is covered by open affine subsets U_i such that $f^{-1}(U_i) \cong \mathbb{A}^n \times U_i$, and $f|_{f^{-1}(U_i)}$ corresponds under this isomorphism to the projection onto the second factor. Since flatness is a local property, any affine space bundle is flat, so we can use Theorem 4.21. (By the remark following the proof of that theorem, this would work over any base scheme.)

Theorem 4.22. *If X is a variety and $f : Y \to X$ is an affine space bundle, then the map $f^* : A(X) \to A(Y)$ is surjective. In particular, if the relative dimension of f is d, then $A_m(Y) = 0$ for $m < d$.*

Proof. We may begin by replacing X with the subvariety that is the closure of $f(V)$, and Y with $f^{-1}f(V)$, and thus assume that X is a variety and f is dominant. We do induction on the dimension of X; the case $\dim X = 0$ is precisely Proposition 1.17.

Now suppose that $\dim X > 0$ and let U be an open affine subset of X such that $f^{-1}(U) \cong \mathbb{A}^n \times U$. We have a commutative diagram with exact rows given by Proposition 4.12:

$$\begin{array}{ccccccc} A(Y \setminus f^{-1}(U)) & \longrightarrow & A(Y) & \longrightarrow & A(f^{-1}U) & \longrightarrow & 0 \\ \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f^*} & & \\ A(X \setminus U) & \longrightarrow & A(X) & \longrightarrow & A(U) & \longrightarrow & 0. \end{array}$$

By induction the right hand vertical map is onto, so by a diagram chase (the "5-lemma") it suffices to show that the left-hand vertical map is onto. Thus we are reduced to the case where X is an affine variety, $Y \cong \mathbb{A}^n \times X$, and f is the projection. Since the projection $f : \mathbb{A}^n \times X \to X$ factors through projections

$$\mathbb{A}^n \times X \to \mathbb{A}^{n-1} \times X \to \cdots \to X,$$

we can also assume that $n = 1$.

Our situation is now that $X = \operatorname{Spec} S$ for some affine domain S and $Y = \operatorname{Spec} S[t]$. The variety $V \subset Y$ corresponds to a prime ideal $P \subset S[t]$. Because $f|_V$ is dominant we have $P \cap S = 0$. Let K be the field of fractions of S. Since $K[t]$ is a principal ideal domain we may write $PK[t] = (a)$ for some polynomial a, and clearing denominators we may assume $a \in S[t]$.

If $a = 0$ then $V = Y$, so $[V] = f^*[X]$ as required. Else we may consider the divisor of a, and we have $Div(a) = [V] + [V']$, where V' is not dominant. Since the closure of the image of V' has lower dimension than X, we see by induction that $[V']$ is in the image of f^*, and we are done. (The reader may also see directly that V' has the form $\mathbb{A}^1 \times V''$, so that $[V'] = f^*([V''])$.) $\square$

As a taste of what is to come, we can strengthen Theorem 4.22 in the case when Y is the total space of a vector bundle over X, or more generally when f admits a section, simply by using the existence of an intersection product (Theorem 5.3).

Corollary 4.23. *Suppose that X is a smooth quasiprojective variety. If $f : Y \to X$ is an affine space bundle, and f admits a section $\sigma : X \to Y$, then the map $f^* : A(X) \to A(Y)$ is an isomorphism.*

The result holds even without the hypothesis on X; this follows from the existence of a pullback along the "regular embedding" σ. See Fulton [1984] Chapter 6.

Proof. Suppose that $V \subset X$ is a subvariety, so that $f^*[V] = [f^{-1}(V)]$. By Theorem 5.3 there is an intersection product that preserves rational equivalence such that $[f^{-1}(V)][\sigma(X)] = [f^{-1}(V) \cap \sigma(X)]$. But the map f restricted to $\sigma(X)$ is an isomorphism, so $f_*([f^{-1}(V) \cap \sigma(X)]) = [V]$. Thus the composite

$$A(X) \xrightarrow{f^*} A(Y) \xrightarrow{f_*([\sigma(X)]\cdot -)} A(X)$$

is the identity. By Theorem 4.22, the left hand map is a surjection, so both maps are isomorphisms. □

4.5 Exercises

Exercise 4.24. Let $\mathbb{P}^m \subset \mathbb{P}^n$ be a linear subspace. Find the Chow groups of the complement $\mathbb{P}^n \setminus \mathbb{P}^m$.

Exercise 4.25. Let $C \subset \mathbb{P}^n$ be a rational normal curve. Find the Chow groups of the complement $\mathbb{P}^n \setminus C$.

Exercise 4.26. Let $S \subset \mathbb{P}^5$ be the quadratic Veronese surface. Find the Chow groups of the complement $\mathbb{P}^5 \setminus S$. Would the answer be different if we took S to be a quartic surface scroll?

Exercise 4.27. Let X be a variety, and let $f : Y \to X$ be a finite flat map, so that both f^* and f_* are well defined. Let $d := (K(Y) : K(X))$ be the degree of f. Show that the composition $f_* \circ f^* : A(X) \to A(X)$ is multiplication by d.

Exercise 4.28. Show that rational equivalence is the finest equivalence relation that is defined by graded subgroups $R_*(Y) \subset Z(Y)$ with the following properties:

(a) $R_*(Y) \subset Z(Y)$ is defined for all schemes algebraic over a field K.

(b) $R_*(Y)$ is preserved by proper push-forward and flat pull-back.

(c) $R_*(\mathbb{P}^1)$ contains $\langle 0\rangle - \langle\infty\rangle$.

Exercise 4.29. The following example shows that the assertion of Theorem 4.15 can fail if f is an open immersion: Let C be a conic in $\mathbb{P}^2$, and let U be the open set $\mathbb{P}^2 \setminus C$ in $\mathbb{P}^2$. Show that U is an affine variety, with Chow groups $A_0(U) = 0,\ A_1(U) = \mathbb{Z}/(2),\ A_2(U) = \mathbb{Z}$. There is a map on cycles $\alpha : Z(U) \to Z(\mathbb{P}^2)$ taking the class of a curve in U to the class of its closure in $\mathbb{P}^2$. If this induced a map $\alpha : A_1(U) \to A_1(\mathbb{P}^2)$, it would split the surjection $A(\mathbb{P}^2) \to A(U)$ defined in Proposition 4.12(c). Give an example of two rationally equivalent cycles Z_1, Z_2 on U such that $\alpha(Z_1)$ is not rationally equivalent to $\alpha(Z_2)$.

Exercise 4.30. Show by example that Theorem 4.15 is false without the hypothesis "proper" even when f is a birational map between smooth curves.

Exercise 4.31. Show that if $\mathcal{L}$ is a line bundle on a quasi-projective scheme X, then $\mathcal{L}$ may be embedded as a subsheaf of the total quotient sheaf $\kappa(X)$ of X as a subsheaf locally generated by one non-zerodivisor. (The result is true much more generally, but there are subtleties in the most general setting—see Kleiman [1979].)

Exercise 4.32. Suppose that X is quasi-projective and reduced. Show that if $\mathcal{L}$ and $\mathcal{M}$ are two line bundles on X embedded in $\kappa(X)$ as above, and $\varphi : \mathcal{L} \to \mathcal{M}$ is an isomorphism, then φ extends to an isomorphism $\kappa(X) \to \kappa(X)$, and that such an isomorphism is given by multiplication with a rational function. Conclude that $\operatorname{Pic}(X)$ is the same as the group of Cartier divisors modulo "principal" Cartier divisors—those defined by invertible rational functions on X.

Exercise 4.33. Suppose that X is quasi-projective and reduced. Krull's Principal Ideal Theorem 0.1 implies that in a Noetherian ring every prime ideal minimal over the ideal generated by a non-zerodivisor has codimension 1. Extend this idea to define a map from the multiplicative group of Cartier divisors on X to the group of codimension 1 cycles $Z^1(X)$, to make a commutative diagram

$$\begin{tikzcd}
\mathcal{O}_X^* \arrow[r] & \kappa(X)^* \arrow[r] \arrow[d] & \{\text{Cartier Divisors on } X\} \arrow[r] \arrow[d] & \operatorname{Pic}(X) \arrow[r] & 0 \\
 & \kappa(X)^* \arrow[r] & Z_{n-1}(X) \arrow[r] & A_{n-1}(X) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Conclude that there is a natural map $\operatorname{Pic}(X) \to A_{n-1}(X)$.

Exercise 4.34. Suppose that X is quasi-projective and normal. Using the result from commutative algebra that a normal domain is the intersection of discrete valuation rings (see for example Eisenbud [1995] §11.2), show that the map from the group of Cartier divisors to the additive group of one-cycles on X is a monomorphism. Conclude that the map $\operatorname{Pic}(X) \to A_{n-1}(X)$ is a monomorphism in this case.

Exercise 4.35. Suppose that X is quasi-projective and regular—that is, that every local ring of X is a regular local ring (it would be enough that every local ring of X is factorial.) Show that the map $\operatorname{Pic}(X) \to A_{n-1}(X)$ is an isomorphism.

Exercise 4.36. Let X consist of three concurrent, but not coplanar, lines in $\mathbb{P}^3$. Show that $A_1(X) = \mathbb{Z}$ but $\operatorname{Pic}(X) = \mathbb{Z}^3$. (Hint: Use the isomorphism $\operatorname{Pic} X = H^1(\mathcal{O}_X^*)$, where $\mathcal{O}_X^*$ is the sheaf of invertible functions on X.)

Exercise 4.37. Let X be the cone over a smooth plane conic C. Prove that the natural comparison map $\operatorname{Pic}(X) = \mathbb{Z} \to \mathbb{Z} = A_1(X)$ is multiplication by 2.

Exercises 4.38-4.42 deal with some properties and examples of, and variations on, the construction of dynamic projection introduced in Section 4.1.1

Exercise 4.38. Show that the dynamic projection of X from R to A is independent of the choice of A, in the sense that if A' is any other r-plane disjoint from R and we carry out the same process to arrive at a limit X_0', there will be an isomorphism $A \cong A'$ carrying X_0 to X_0'.)

Exercise 4.39. Let $C \subset \mathbb{P}^3$ be a twisted cubic curve, and L and $M \subset \mathbb{P}^3$ two general lines. Choose coordinates (x_0, x_1, y_0, y_1) on $\mathbb{P}^3$ with $L = V(x_0, x_1)$ and $M = V(y_0, y_1)$, and let $\psi_t : \mathbb{P}^3 \to \mathbb{P}^3$ be the automorphism given by $\psi_t : (x_0, x_1, y_0, y_1) \mapsto (tx_0, tx_1, y_0, y_1)$. Show that the flat limit of the curves $C_t = \psi_t(C) \subset \mathbb{P}^3$ is the scheme $V(((x_0, x_1)^2)$ defined by the square of the ideal of the line L, without embedded points.

Exercise 4.40. In the situation of Exercise 4.39, suppose we let $\psi_t : \mathbb{P}^3 \to \mathbb{P}^3$ be the automorphism given by $\psi_t : (x_0, x_1, y_0, y_1) \mapsto (t^2x_0, tx_1, y_0, y_1)$. Show that the flat limit of the curves $C_t = \psi_t(C) \subset \mathbb{P}^3$ is now the union of a planar triple line (the zero locus $V(x_1, x_23))$ with an embedded point.

Exercise 4.41. Let $C \subset \mathbb{P}^3$ be an irreducible, nondegenerate quartic curve, and let $H \subset \mathbb{P}^3$ and $p \in \mathbb{P}^3$ be a general hyperplane and point. Let C_0 be the dynamic projection of C from H to p. Show that if C has genus 1, then C_0 is reduced; but if C has genus 0 then C_0 will have an embedded point at p.

Exercise 4.42. Consider the G_m action on $\mathbb{P}^n$ given by

$$\varphi_t : (x_0, \dots, x_n) \mapsto (t^{w_1}x_1, \dots, t^{w_n}x_n)$$

for $t \in G_m$, with weights $w_0 > w_1 > \cdots > w_n \in \mathbb{Z}$. Let $X \subset \mathbb{P}^n$ be a subvariety. Prove analogs for each part of Proposition 4.5 for each pair of spaces $R = \{x_0 = \cdots = x_i = 0\}$ and $A = \{x_{i+1} = \cdots = x_n = 0\}$. If X has dimension m, and the coordinate system is chosen general with respect to X, then show that the limit of tX as t approaches 0 is supported on the m-plane $x_{m+1} = \cdots x_n = 0$.

Hint for Exercise 4.42: Show as in Proposition 4.5 that for any point not in R or A has G_m orbit closure meeting both of these.

Exercise 4.43. There are two isomorphism classes of subschemes $X \subset \mathbb{P}^3$ with Hilbert polynomial $p_X(m) = 3m + 1$ consisting of a planar triple line and an embedded point p: one where the embedded point is spatial (that is, $\dim T_pX = 3$), such as

$$X = V(x_0^2, x_0x_1, x_0x_2, x_1^3);$$

and one where the embedded point is planar (that is, $\dim T_pX = 2$), such as

$$X = V(x_0, x_1^3, x_1^4).$$

Show that the former is a flat limit of twisted cubics, and that the latter is not.

5

Intersection Products and Pullbacks

5.1 The Chow ring

In this chapter we will work with quasi-projective varieties over an algebraically closed field. This setting makes possible a direct and intuitive approach to intersection theory.[1] The reader who needs a more general case will find pointers below and a full treatment in Fulton [1984].

We wish to define a ring structure on the Chow group of a variety X. To give back geometric information about transverse intersections, such a product should satisfy the requirement that the intersection product be "geometric" in the simplest case:

(*) If subvarieties A, B of X intersect transversely at a general point of each component of $A \cap B$, then the product of their classes should be the class of their intersection,

$$[A][B] = [A \cap B].$$

This is not so simple: in fact, as we'll see in Section 5.5, no ring structure with property (*) can exist on the Chow groups of an arbitrary projective variety X! (This problem remains even if we replace the hypothesis (*)

[1] One word of warning: when we say "direct and intuitive," we are referring to the statements of the main results, principally Theorems 5.3, 5.4, 5.8 and 5.9, all of which are in Section 5.1. The actual proofs are another matter entirely.

with the weaker requirement that $[A][B] = [A \cap B]$ whenever A and B meet everywhere transversely, as the reader will see in Example 5.19). The goal of this chapter is to show the existence and uniqueness of the ring structure when X is smooth, and to prove certain functoriality properties that make it even more useful.

It is natural to ask whether the equality in (*) holds under the weaker hypothesis that $\operatorname{codim}(A \cap B) = \operatorname{codim} A + \operatorname{codim} B$. The answer is no in general, but there are special cases where it does hold and ways to correct the formula in general, as we shall see in Section 5.2.

The situation is analogous to that in algebraic topology. The Chow groups are like the homology groups of a space. For singular spaces the cohomology groups form a ring and homology groups do not. However, for smooth orientable manifolds, Poincaré duality allows one to identify homology and cohomology (interchanging dimension and codimension) and thus transport the ring structure to homology. Even for singular varieties one can define a ring (from Fulton-Macpherson's "bivariant theory" Fulton [1984] Chapter 17; see also Totaro [to appear]) that acts on the Chow groups much as the cohomology ring of a topological space acts on the homology via cap product, but we will not pursue this construction here.

5.1.1 Products via the Moving Lemma

To describe the path we will take, we use the following terminology:

Definition 5.1. (a) Irreducible subschemes A and B of a variety X are *dimensionally transverse* if for every component C of $A \cap B$ we have $\operatorname{codim} C = \operatorname{codim} A + \operatorname{codim} B$.

(b) Subvarieties A and B of a variety X are *transverse* at a point $p \in X$ if X, A and B are smooth at p and their tangent spaces at p satisfy $T_pA + T_pB = T_pX$.

(c) Subvarieties A and B of a variety X are *generically transverse* if every irreducible component of $A \cap B$ contains a point p at which A and B are transverse.

(d) Two cycles $\sum m_iA_i$ and $\sum n_jBj$ are dimensionally (respectively generically) transverse if A_i, B_j are dimensionally (respectively generically) transverse for every i, j.

(e) If $A = \sum m_iA_i$ and $B = \sum n_jB_j \in Z(X)$ are dimensionally transverse cycles on X, we define the intersection cycle $A \cap B$ to be the cycle

$$A \cap B = \sum m_in_j\langle A_i \cap B_j\rangle.$$

The meaning of generic transversality may be clarified by the following easy result:

Proposition 5.2. *Subvarieties A and B of a variety X are generically transverse if and only if they are dimensionally transverse and each irreducible component of $A \cap B$ is reduced and contains a smooth point of X.*

Proof. First suppose that A and B are generically transverse, so that for each irreducible component C of $A \cap B$ there is a smooth point $p \in X$ such that A and B are smooth and transverse at p. It follows that C is smooth at p, and thus is generically reduced.

To prove the converse, let C be an irreducible component of $A \cap B$. Since C is reduced there is a smooth point $p \in C$. The Zariski tangent space to C at p is the intersection of T_pA and T_pB in T_pX. By hypothesis, $\dim C = \dim A + \dim B - n$, and since $\dim T_pC = \dim C$, it follows that $\dim T_pA = \dim A$ and $\dim T_pB = \dim B$, proving that A and B are smooth at p as well. $\square$

It is important to note that for subvarieties of a smooth variety, the codimension of a component of an intersection can only be too small, not too large; this is the Generalized Principle Ideal Theorem 0.2.

The first main result of this chapter asserts the existence of an intersection product satisfying (*):

Theorem 5.3. *If X is a smooth quasiprojective variety then there is a unique bilinear product structure on $A(X)$ that satisfies*

$$[A][B] = [A \cap B]$$

whenever A and B are subvarieties that are generically transverse. This structure makes $A(X)$ into an associative, commutative ring graded by codimension.

We will call $A(X)$ with its ring structure the *Chow ring* of X.

To show that the Chow groups form a ring we must understand how to define the product of two cycles that intersect in an arbitrary way. Since the structure is to be bilinear, the problem reduces immediately to constructing a cycle representing the class of a product $[A][B]$ where A and B are subvarieties of X, of dimensions a and b respectively. Since the Chow ring is to be graded by codimension, this should be a cycle of codimension

$$\operatorname{codim} A + \operatorname{codim} B$$

We cannot define $[A][B]$ to be $[A \cap B]$ always: this may not have the right codimension, and even when it has the right codimension, $[A \cap B]$ is not always determined by the rational equivalence classes $[A]$ and $[B]$ (see Example 5.13).

Instead we will define $[A][B]$ via a strong *Moving Lemma*:

Theorem 5.4 (Moving Lemma)**.** *Let X be a smooth quasiprojective variety.*

(a) *If $\alpha \in A(X)$ is any Chow class and $B \in Z(X)$ any cycle, then there exists a cycle $A \in Z(X)$ such that $[A] = \alpha$ and A intersects B generically transversely.*

(b) *If A and $B \in Z(X)$ are cycles that intersect generically transversely, the class $[A \cap B]$ depends only on the classes $[A]$ and $[B]$.*

Note that as a consequence, if $\alpha \in A(X)$ is any class and $B_1, \dots, B_k \subset X$ any finite collection of subvarieties, there exists a cycle $A \in Z(X)$ such that $[A] = \alpha$ and A is generically transverse to each of the B_i. For other treatments of various versions of the Moving Lemma, see Roberts [1972a] and Hoyt [1971].

Given the Moving Lemma, we can prove Theorem 5.3:

Proof of Theorem 5.3. For any $\alpha, \beta \in A(X)$ we define the product $\alpha\beta$ to be the class $[A \cap B] \in A(X)$, where A and B are any cycles representing the classes α and β and intersecting generically transversely. The first part of the Moving Lemma 5.4 asserts the existence of such cycles, while the second part of the Moving Lemma assures us that this is well-defined.

The product is obviously commutative. To prove that the product is associative, let $\alpha, \beta, \gamma \in A(X)$. We claim that there are cycles A, B and C representing α, β and γ, and such that

$$(\alpha\beta)\gamma = [A \cap B \cap C] = \alpha(\beta\gamma).$$

First, we may choose cycles $B = \sum n_j B_j$ and $C = \sum o_l C_l$ representing β and γ and such that the B_j are generically transverse to the C_l; and then choose $A = \sum m_i A_i$ representing α so that the A_i are generically transverse to the $B_j \cap C_l$ *and* to all the B_j. Since A_i is generically transverse to $B_j \cap C_l$, it follows that $A_i \cap B_j \cap C_l$ is generically reduced and of codimension equal to the sum of the codimensions of A_i, B_j and C_l, so $A_i \cap B_j$ is generically transverse to C_l as well. Associativity of the intersection product now follows from that of ordinary intersections. Thus the product we have defined makes

$$A(X) = \bigoplus_k A^k(X)$$

into a commutative graded ring. $\square$

Note that even if α and β are the classes of effective cycles, we can't necessarily choose A and B effective in the Moving Lemma. For instance, in Example **??** the class e represents an effective divisor, but $e^2 = -\eta$ is the negative of an effective class, and by degree considerations cannot be represented in the form $\sum m_i n_j [A_i \cap B_j]$ with the m_i and n_j non-negative.

In the case of a flat morphism f we have already defined a group homomorphism $f^* : A(X) \to A(Y)$ by $f^*([A]) = [f^{-1}(A)]$ (Theorem 4.21). This definition clearly satisfies the condition for uniqueness in Theorem 5.8 so Theorem 4.21 implies that f^* is a ring homomorphism in this case. Using the Moving Lemma (and, again under the unnecessary hypothesis of generical separability) we will show directly that, if f is flat, then the map f^* is a ring homomorphism in the quasiprojective case:

Proposition 5.5. *If $f : Y \to X$ is a generically separable flat morphism of smooth quasi-projective varieties, then the pullback $f^* : A(X) \to A(Y)$ is a ring homomorphism.*

The assertions above will be proven in Section 5.7.

Proposition 5.5 implies that the inclusion of a closed subvariety of a projective variety defines an ideal in the Chow ring:

Corollary 5.6. *Let $\iota : X' \subset X$ be the inclusion of a closed subset. If X is a smooth projective variety, then the subgroup $\iota_*(A(X')) \subset A(X)$ is an ideal in the Chow ring.*

Proof. Set $U = X \setminus X'$. By Proposition 4.12, Part c there is a right exact sequence of Chow groups

$$A(X') \xrightarrow{\iota_*} A(X) \xrightarrow{k_U} A(U) \longrightarrow 0.$$

The scheme U is a smooth quasiprojective variety, and the restriction map $k_U : A(X) \to A(U)$ is the same as flat pullback; thus it is a ring homomorphism. This implies that the image of $(\iota_X)_*$ is an ideal. $\square$

5.1.2 From intersections to pullbacks

Theorem 5.3 can be made much more powerful by recasting it as a functoriality statement. The idea is that intersections are the simplest sorts of pullbacks. Thus we are led to try to generalize the inclusion $B \subset X$ to be a morphism $f : Y \to X$, and to generalize the intersection product $[A][B]$ to be a class, which we will call $f^*([A])$, in $A(Y)$.

Even in the case where f is the inclusion map of $Y = B$ in X, this gives something new, since the class $[A][B]$ was defined above in $A(X)$, not in $A(B)$. The two classes will be related by the formula

$$f_*(f^*([A]) = [A][B] \in A(X).$$

(Fulton's theory of refined intersection products (Fulton [1984], Chapter ****) actually gives more: it yields a well defined class in $A(A \cap B)$, which is not accessible via our Moving Lemma-based approach.)

A naive attempt at a definition of the pullback would be to take $f^*([A])$ to be the class $[f^{-1}(A)]$ of the preimage of A in Y. As in the simpler case of intersections, this can't work all the time: given a subvariety $A \subset X$, the preimage $f^{-1}(A)$ need not have the expected codimension. But by analogy with the case of intersections, we can make it a requirement that the pullback be geometric in the simplest case. We start with a definition that generalizes the definition of generic transversality for a subscheme $A \subset X$ and an inclusion $Y \subset X$.

Definition 5.7. Let $f : Y \to X$ be a projective map of quasiprojective varieties. A subvariety A of X of codimension c is said to be *generically transverse* to $f : Y \to X$ if:

(a) $f^{-1}(A)$ is generically reduced of codimension c.

(b) Y is smooth at a general point q of each irreducible component of $f^{-1}(A)$ and X and A are smooth at $f(q)$.

A cycle $A = \sum m_i A_i$ on X is said to be generically transverse to f if every component A_i is; in this case, we define the *pullback cycle* to be $f^*A = \sum m_i \langle f^{-1}(A_i)\rangle$

In these terms, we can state the second main theorem of this chapter, analogous to Theorem 5.3:

Theorem 5.8. *Let $f : Y \to X$ be a generically separable map of smooth quasiprojective varieties, and assume that X is projective.*

(a) *There exists a unique ring homomorphism $f^* : A(X) \to A(Y)$ with the property that if A is a subvariety of X generically transverse to f then $f^*([A]) = [f^{-1}(A)]$.*

(b) *(Push-Pull Formula) If we make $A(Y)$ into an $A(X)$ module via the ring homomorphism f^*, then the map $f_* : A(Y) \to A(X)$ is a map of modules; as it is often written,*

$$f_*\big(f^*(\alpha)\beta\big) = \alpha f_*(\beta).$$

(Recall that a morphism $f : Y \to X$ is *generically separable* if the field extension $K(Y)/K(X)$ is separable; in other words, there is a transcendence base $\{x_i\}_{i\in I}$ for $K(Y)$ over $K(X)$ such that $K(Y)$ is finite and separable over $K(X)(\{x_i\}_{i\in I})$.)

By introducing multiplicities, one can drop the assumption that f is generically separable, and f^* can be defined as a homomorphism of Chow groups even when Y is singular and X is only quasiprojective (but still smooth). For these refinements see Fulton [1984].

We will prove Theorem 5.8 by an analog of the Moving Lemma:

Theorem 5.9 (Moving Lemma for morphisms)**.** *Let $f : Y \to X$ be a generically separable morphism of smooth quasiprojective varieties.*

(a) *If $\alpha \in A(X)$ is any Chow class, then there exists a cycle $A \in Z(X)$ such that $[A] = \alpha$ and A is generically transverse to f.*

(b) *If $A \in Z(X)$ is a cycle that is generically transverse to f, the class $[f^{-1}A]$ depends only on the class $[A]$.*

As in the case of intersections, the first part of Theorem 5.8 follows immediately from the Moving Lemma for morphisms 5.9; the second part follows by using the Moving Lemma for morphisms to reduce the problem to the set-theoretic equality $f\big(f^{-1}(A) \cap B\big) = A \cap f(B)$ for $A \subset X$ and $B \subset Y$.

5.2 Intersection multiplicities

5.2.1 Dimensionally transverse intersections and pullbacks

We have defined intersection products by moving the intersecting cycles until they are generically transverse. But this is not always necessary. If A and B are subvarieties of a smooth quasiprojective variety X that are dimensionally transverse in the sense of Definition 5.1 above then it is possible to assign to each irreducible component Z of $A \cap B$ an *intersection multiplicity* $m_Z(A, B)$ in such a way that

$$[A][B] = \sum_Z m_Z(A, B)[Z].$$

For example, when A and B are generically transverse, Theorem 5.3 asserts that we can take all $m_Z(A, B) = 1$. More generally, if A, B are smooth, or locally complete intersections in X, or even just locally Cohen-Macaulay at a general point of each component Z, we can take $m_Z(A, B)$ to be the multiplicity of the subscheme $A \cap B$ along Z. This can be expressed in the following attractive form:

Theorem 5.10. *Let X be a smooth quasiprojective variety. If A, B are subvarieties on X that meet dimensionally transversely, and are locally Cohen-Macaulay at the generic point of every component of $A \cap B$, then*

$$[A][B] = [A \cap B].$$

For example, since plane curves are complete intersections, we have:

Corollary 5.11 (Strong Bézout)**.** *If C and $D \subset \mathbb{P}^2$ are any two plane curves of degrees d and e with no common components, then*

$$\sum_{p \in C \cap D} m_p(C, D) = de;$$

that is, the degree of the scheme $C \cap D$ is de. □

There is a natural analogue of Theorem 5.10 in the more general setting of pullbacks:

Theorem 5.12. *Let $f : X \to Y$ be a morphism of smooth projective varieties, and $A \subset Y$ a subscheme such that* $\operatorname{codim}(f^{-1}(A)) = \operatorname{codim}(A)$. *If A is locally Cohen-Macaulay, then*

$$f^*[A] = [f^{-1}(A)].$$

We will defer the proofs of Theorems 5.10 and 5.12 until Section 5.8.

5.2.2 Intersection multiplicities

In Theorem 5.10, if we do not assume that A and B are Cohen-Macaulay, the conclusion of the theorem may not hold; that is, the multiplicity $m_Z(A, B)$ may not be equal to the multiplicity of the scheme $A \cap B$ along Z, as in the following example.

Example 5.13. Let $X = \mathbb{P}^4$, and let V_1, V_2 be general planes and let $A = V_1 \cup V_2$.

Consider the intersection of A with some 2-planes. Let $p = V_1 \cap V_2$ be the point where the two planes meet and let B_1 be a 2-plane that does not pass through p, and meets each of V_1, V_2 in a single (necessarily reduced) point. Let B_2 be a 2-plane that passes through p and does not meet A anywhere else. The cycles $[B_1]$ and $[B_2]$ are rationally equivalent in $\mathbb{P}^4$. The cycle $[B_1 \cap A]$ consists of two reduced points, the intersections of B_1 with V_1 and with V_2, so $\deg[B_1 \cap A] = 2$ (see Figure 5.10).

By contrast, it's not hard to see that $\deg[B_2 \cap A] \geq 3$: since the Zariski tangent space to the scheme $A = V_1 \cup V_2$ at the point p is all of $T_p(\mathbb{P}^4)$, the tangent space to the intersection $B_2 \cap A$ at p must be all of $T_p(B_2)$. In other words, $B_2 \cap A$ must contain the "fat point" at p in the plane B_2 (that is, the scheme defined by the square of the ideal of p in B_2), and so must have degree at least 3.

Since the degree function is well-defined on rational equivalence classes of zero-cycles, this implies that $[B_1 \cap A]$ is not rationally equivalent to $[B_2 \cap A]$.

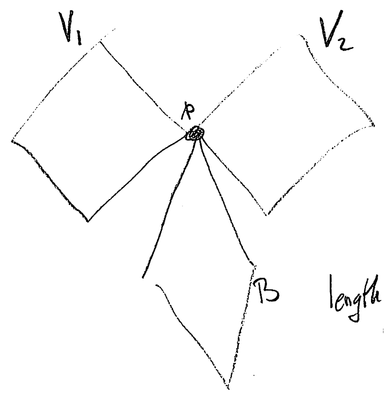


FIGURE 5.10. The degree of the product $[B_2][V_1 + V_2]$ in $A(\mathbb{P}^4)$ is 2, but the length of the local ring of $B_2 \cap (V_1 + V_2)$ is 3 ****B in the figure should be B_2****

In fact, we can see that the degree of $B_2 \cap A$ is exactly 3, by a local calculation. Since B_2 meets A only at the point p, we have to show that the length of the artinian ring $\mathcal{O}_{\mathbb{P}^4,p}/(\mathcal{I}(B_2) + \mathcal{I}(A))\mathcal{O}_{\mathbb{P}^4,p}$ is 3. Let $S = k[x_0, \ldots, x_4]$ be the homogeneous coordinate ring of $\mathbb{P}^4$. We may choose V_1, V_2 and B_2 to have homogeneous ideals:

$$\begin{aligned} I(A) &= (x_0, x_1) \cap (x_2, x_3) = (x_0x_2, x_0x_3, x_1x_2, x_1x_3), \\ I(B_2) &= (x_0 - x_2, x_1 - x_3). \end{aligned}$$

Modulo $I(B_2)$ we can eliminate the variables x_2, x_3 and the ideal $I(A)$ becomes (x_0^2, x_0x_1, x_1^2). Passing to the affine open subset where $x_4 \neq 0$, this is the square of the maximal ideal corresponding to the origin in the plane B_2. Thus $\mathcal{O}_{\mathbb{P}^4,p}/(\mathcal{I}(B_2)+\mathcal{I}(A))\mathcal{O}_{\mathbb{P}^4,p}$ has basis $\{1, x_0/x_4, x_1/x_4\}$, and thus its length is 3, as claimed.

Given that we sometimes have $[A \cap B] \neq [A][B]$, it is natural to look for a correction term. In the example above the set-theoretic intersection is a point, so this comes down to looking for a formula that will predict the difference in multiplicities $3 - 2 = 1$. Of course the correction term should reflect non-transversality, and one measure of non-transversality is the quotient $(I(A) \cap I(B_2))/(I(A) \cdot I(B_2))$. In the case above one can compute this and one finds that the quotient is a finite dimensional vector space of length 1—just the correction term we need. Now for any pair of ideals I, J in any ring R the quotient $(I \cap J)/(I \cdot J) = \mathrm{Tor}_1^R(R/I, R/J)$ (see Exercise 5.32). With this information (and knowing a special case proven earlier by Auslander and Buchsbaum), if you happened to be Jean-Pierre Serre, you might come up with the following result:

Serre [2000], originally published in 1957:

Theorem 5.14 (Serre's Formula)**.** *Suppose that $A, B \subset X$ are dimensionally transverse subschemes of a smooth scheme X, and Z is an irreducible component of $A \cap B$. The intersection multiplicity of A and B along Z is*

$$m_Z(A, B) = \sum_{i=0}^{\dim X} (-1)^i \mathrm{length}_{\mathcal{O}_{A\cap B,Z}} (\mathrm{Tor}_i^{\mathcal{O}_{X,Z}}(\mathcal{O}_{A,Z}, \mathcal{O}_{B,Z})) \quad \square$$

The first term of the alternating sum in Serre's Formula is

$$\mathrm{length}_{\mathcal{O}_{A\cap B,Z}} \mathrm{Tor}_0^{\mathcal{O}_{X,Z}}(\mathcal{O}_{A,Z}, \mathcal{O}_{B,Z}) = \mathrm{length}_{\mathcal{O}_{A\cap B,Z}} \mathcal{O}_{X,Z}/(\mathcal{I}_A + \mathcal{I}_B)$$

which is precisely the multiplicity of Z in the subscheme $A \cap B$. The rest of the terms are zero in the Cohen-Macaulay case, explaining the form of Theorem 5.10.

A different description of intersection multiplicities is given in Fulton [1984], Chapter 7.

5.3 Kleiman's Transversality Theorem

There is an easy proof of the Moving Lemma—in an even stronger form—when there is an affine algebraic group G acting transitively on X.

The basic statement is this: if A and $B \subset X$ are two subvarieties of X, then the intersection of the general translate gA of A with B is dimensionally transverse, and, if the characteristic is 0, then this intersection is generically transverse as well; moreover, the cycle $[A]$ is rationally equivalent to $[gA]$. It turns out that the same proof establishes a statement that is stronger in a very useful way:

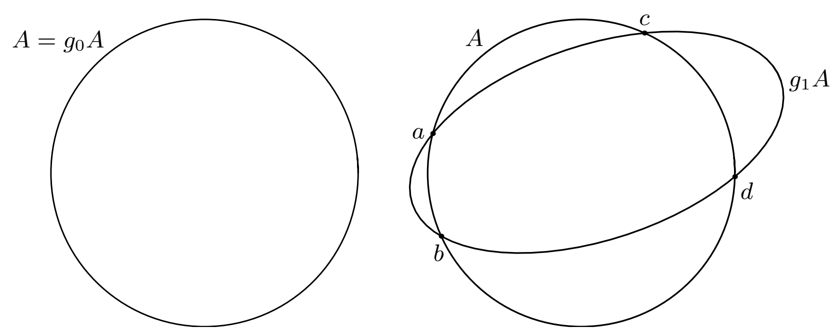
Theorem 5.15 (Kleiman [1974])**.** *Let $\varphi : Y \to X$ be a morphism of quasiprojective varieties over an algebraically closed field, and let G be an algebraic group acting transitively on X. If $A \subset X$ is a subvariety, then for general $g \in G$ the preimage $\varphi^{-1}(gA)$ had the same codimension as A. If the characteristic is 0, then $\varphi^{-1}(gA)$ is generically smooth as well.*

Proof. Let the dimensions of X, A, Y and G be n, a, b and m respectively. Set

$$\Gamma = \{(x, y, g) \in A \times Y \times G \mid gx = \varphi(y)\}.$$

Because the group action is transitive, the projection $\pi : \Gamma \to A \times Y$ is surjective. Its fibers are the cosets of stabilizers of points in X, and hence have dimension $m - n$. It follows that Γ has dimension

$$\dim \Gamma = a + b + m - n.$$



$$[A]^2 = [A]\,[g_1 A] = [a + b + c + d]$$

FIGURE 5.7. The cycle A meets a general translate of itself generically transversely. ****To be consistent with the text, in Figure 5.7 delete "$= g_0 A$" in upper left, and replace g_1 by g below figure.****

On the other hand, the fiber over g of the projection $\Gamma \to G$ is isomorphic to $\varphi^{-1}(gA)$. Thus either this intersection is empty for general g, or else it has dimension $a + b - n$, as required.

Note that, since G acts transitively, the variety X must be smooth. Now suppose that the ground field has characteristic 0. Since any algebraic group in characteristic 0 is smooth, the fibers of the projection to $A \times Y$ are also smooth, so Γ itself is smooth over $A_{\mathrm{sm}} \times Y_{\mathrm{sm}}$. Sard's Theorem then tells us that the general fiber of the projection of Γ to G is smooth outside the singular locus Γ_{sing} of Γ. If the projection of Γ_{sing} to G is not dominant, then $\varphi^{-1}(gA)$ is smooth for general g.

To complete the proof of generic transversality, we may assume that the projection $\Gamma_{\mathrm{sing}} \to G$ is dominant Again invoking the smoothness of algebraic groups in characteristic 0, each point of G is locally a complete intersection. By the Principal Ideal Theorem, every component of every fiber of $\Gamma \to G$ has codimension $\leq \dim G$, and thus every component of the general fiber has codimension exactly $\dim G$ in Γ. Since $\Gamma_{\mathrm{sing}} \to G$ is dominant, its general fiber has dimension $\dim \Gamma_{\mathrm{sing}} - \dim G < \dim \Gamma - \dim G$, so no component of a general fiber can be contained in Γ_{sing}. Thus $\varphi^{-1}(gA)$ is generically smooth for general $g \in G$. $\square$

Theorem 5.15 is particularly useful to us in the case when X is $\mathbb{P}^n$ or $\mathbb{G}(k, n)$ and the group G is $GL(n)$ acting in the standard way because replacing A with a general translate does not change the class of the cycle:

Theorem 5.16. *Suppose that $A \subset X$ is a subvariety of a quasiprojective variety X. If $G = GL(n)$ acts on X then $A \sim gA$ for any $g \in G$.*

Proof. We may regard G as an open set in the vector space of $n \times n$ matrices. The union of hA, for all h in the intersection of this open set with the line joining g to the identity gives a rational equivalence. □

Theorem 5.16 actually holds for any *affine* group G, because any point of G can be connected to the origin by a rational curve C (see Borel [1991], Theorem 18.2). It is not true, however, for more general groups; for example, if G is the group of points on an elliptic curve X, then G acts transitively on X but it is not true that any two points of X are rationally equivalent.

In Kleiman [1974] the main point is a result extending this Theorem to positive characteristic, under the stronger hypothesis that the group action is transitive on both points and nonzero tangent vectors. This stronger hypothesis is necessary, even in the case of the Grassmannian (where the automorphism group acts transitively on points, but preserves the *rank* of a tangent vector; see Exercise 2.27); examples can be found in Kleiman [1974] and Roberts [1972b]. Kleiman's theorem may seem to be applicable only in relatively special circumstance. But in fact we use it quite often.

Theorem 5.15 invokes the general fact that a connected, affine algebraic group is rationally connected. This is not an easy statement to prove in general; however, the following special case will suffice for all the applications we will make in this book.

Proposition 5.17. *Suppose that a group of the form $G = \mathrm{GL}_{n_1} \times \cdots \mathrm{GL}_{n_p}$ acts algebraically on a scheme X, and let $g \subset G$. If $V \subset X$ is a subvariety then gV is rationally equivalent to V.*

Proof. We consider G as an open subset of the affine space of p-tuples of matrices. As such we may connect $g \in G$ to the identity $e \in G$ by a family of tuples of matrices $g_t = (1-t)e + tg$ with t in an open subset U of the affine line $\mathbb{A}^1$ containing 0 and 1. Let $\Gamma \subset U \times X$ be the closure of the variety consisting of pairs $(t, x) \mid x \in g_t(V)$. Evidently, Γ dominates U and $\Gamma_0 = V$ while $\Gamma_1 = gV$. □

5.4 Linkage

There is a different construction, called *linkage,* that is sometimes useful for "moving" a cycle $A \subset X$ in the special case that it is *regularly embedded*, which means that each component of A is locally a complete intersection in X. Suppose that $A \subset X$ is a regularly embedded subvariety, and let B be any cycle in X. Choose $n-a$ general hypersurfaces $Z_1, \dots, Z_{n-a} \subset \mathbb{P}^N$ containing A, and write

$$Z_1 \cap \cdots \cap Z_{n-a} \cap X = A \cup A'.$$

By Lemma 5.21, the intersection $Z_1 \cap \cdots \cap Z_{n-a} \cap X$ is rationally equivalent to a multiple of a general linear space section E of X, so we can write

$$A \sim d \cdot E - A'.$$

The reader should be warned that, in general, $A' \subset X$ may not be a regular embedding, so continuing the linkage process becomes complicated.

While the Moving Lemma provides us with assurance that intersection products are well defined in $A(X)$, it is rarely used in practice to calculate products in the Chow ring. Exercises 5.29–**??** give examples of this: in each, we have to find cycles representing a given class and intersecting another cycle transversely; but in each the cone construction is not the easiest way to do it.

5.5 Intersections on singular varieties

In this section we discuss the problems of defining intersection products on singular varieties.

To begin with, the Moving Lemma (5.4) may fail if X is even mildly singular:

Example 5.18. (Figure 5.8)

Let $C \subset \mathbb{P}^2 \subset \mathbb{P}^3$ be a smooth conic and let $X = \overline{pC} \subset \mathbb{P}^3$ be the cone with vertex $p \notin \mathbb{P}^2$. Let $L \subset X$ be a line (which necessarily contains p). We claim that every cycle on X that is rationally equivalent to L has support containing p, and thus the conclusion of the first part of the Moving Lemma does not hold for X.

To see this, note first that the degree function is well defined on rational equivalence classes of curves in X: for if $i : X \to \mathbb{P}^3$ is the inclusion, then for any curve D on X we have

$$\deg D = \deg(\zeta \cdot i_*([D]))$$

where ζ is the class of a hyperplane in $\mathbb{P}^3$. It follows that we have a function $\deg : A_1(X) \to \mathbb{Z}$ that takes the class of each irreducible curve to its degree.

If $D \subset X$ is a curve that does not pass through p, then the projection from p is a finite map from D to C, and it follows that the degree of D is even. Thus any cycle of dimension 1 on X whose support does not contain p has even degree, and cannot be rationally equivalent to L.

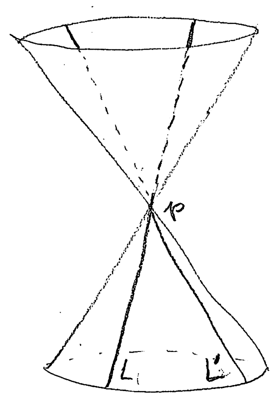


FIGURE 5.8. The degree of intersection of two lines on a quadric cone is 1/2

Continuing with the notation of Example 5.18, one might hope to define an intersection product on $A(X)$ anyway. It seems natural to think that since two distinct lines L, L' through p meet in the reduced point p, we would have $[L][L'] = [p]$. However, if ζ is the class of a general hyperplane section $H \cap X$ of X through p, then (since such a hyperplane meets each L' transversely in one point) we might also expect $\zeta[L'] = [p]$. We can't have both these things, for, if we did then

$$[p] = \zeta[L'] = 2[L][L'] = 2[p]$$

since ζ is rationally equivalent to the union of two lines through p. This contradicts the fact that the degree map is well defined on $A_0(X)$.

In some sense the problem here is that the intersection of the lines L and L', though reduced, takes place at a singular point of X. Note that in our requirement (*) of Section 5.1 we only insisted that $[A][B] = [A \cap B]$ when the generic points of the intersection were smooth points of X. And in fact we can define at least a numerical intersection product on $A(X)$ for any normal surface, satisfying (*), as long as we allow intersection numbers to be rational rather than integral. In the example above, for instance, we could have defined the intersection number of two lines through the p to be 1/2 instead of 1, and everything would have worked. See (Example 8.3.11 of Fulton [1984]) for a general statement.

In higher dimensions, however, the situation becomes truly intractable, as the following example shows.

Example 5.19. Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface and let $X = \overline{p\,Q}$ be the cone in $\mathbb{P}^4$ with vertex $p \notin \mathbb{P}^3$. The quadric Q contains two families of lines $\{M_t\}$ and $\{N_t\}$, and the cone X is correspondingly swept out by the two families of 2-planes $\{\Lambda_t = \overline{p\,M_t}\}$ and $\{\Gamma_t = \overline{p\,N_t}\}$; see Figure 5.9.

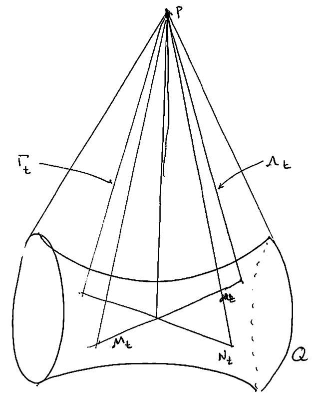


FIGURE 5.9. Cycles whose intersection product cannot be defined

Now, any line $L \subset X$ not passing through the vertex p maps, under projection from p, to a line of Q; that is, it must lie either in a plane Λ_t or in a plane Γ_t; lines on X that do pass through p lie on one plane of each type. Note that since lines M_t and $M_{t'} \subset Q$ of the same ruling are disjoint for $t \neq t'$, while lines M_t and $N_{t'}$ of opposite rulings meet in a point, a general line $M \subset X$ lying in a plane Λ_t is disjoint from $\Lambda_{t'}$ for $t \neq t'$ and meets each plane Γ_s transversely in a point, so that if there were any intersection product on $A(X)$ satisfying the fundamental condition (*) of Section 5.1, we would have

$$[M][\Lambda_t] = 0 \quad \text{and} \quad [M][\Gamma_t] = [q]$$

for some point $q \in X$. Likewise, for a general line $N \subset X$ lying in a plane Γ_t, the opposite would be true; that is, we would have

$$[N][\Lambda_t] = [r] \quad \text{and} \quad [N][\Gamma_t] = 0.$$

But the lines M and N—indeed, any two lines on X—are rationally equivalent! Specifically, the line M can be rotated in the plane Λ_t until it passes through the point p, at which point it also lies in a plane Γ_s; it can then be rotated to an arbitrary line in Γ_s. Since by Corollary 4.16 a point cannot be rationally equivalent to zero on X, we have a contradiction.

How much can one recover from these problems? The current state of the art is to define an intersection product $[A][B]$ for subvarieties of a singular variety only when one of A, B is "regularly embedded," or in other words "locally a complete intersection" in X. This leads to a notion of *Chow cohomology groups* $A^*(X)$, which play a role relative to the Chow groups analogous to that of cohomology relative to homology in the topological context: we have intersection products

$$A^c(X) \otimes A^d(X) \to A^{c+d}(X)$$

and

$$A^c(X) \otimes A_k(X) \to A_{k-c}(X)$$

analogous to cup and cap products in topology. In the present volume we will avoid all this by sticking for the most part to the case of intersections on smooth varieties, where we can simply equate $A^c(X) = A_{\dim X - c}(X)$; for the full treatment, see Fulton [1984] Chapters 6, 8 and 17, and, for a visionary account of what might be possible, Srinivas [to appear].

5.6 Proof of the Moving Lemma

5.6.1 Preliminaries

Ambient cycles. In the proof of the Moving Lemma, we will make use of a class of cycles on X that are easy to move:

Definition 5.20. Let $X \subset \mathbb{P}^N$ be a smooth quasiprojective variety. An *ambient cycle* on X is a cycle expressible as a linear combination $D = \sum m_i \langle \tilde{D}_i \cap X \rangle$, where $\tilde{D}_i$ is a subvariety of $\mathbb{P}^N$ that meets X generically transversely.

Since the subvarieties $\tilde{D}_i \subset \mathbb{P}^N$ can be moved around by applying automorphisms of $\mathbb{P}^N$, we can find lots of representative cycles for the class $[D]$ of an ambient cycle. In particular, we can apply the construction of dynamic projection (Corollary 4.6) to obtain the following:

Lemma 5.21. *Let $X \subset \mathbb{P}^N$ be a quasiprojective variety of dimension n. If $\tilde{D} \subset \mathbb{P}^N$ is a subvariety of dimension m and degree d that is generically transverse to X, then the ambient cycle $D := \langle \tilde{D} \cap X \rangle$ is rationally equivalent in X to a cycle of the form $\sum_{i=1}^d \langle L_i \rangle$ where the L_i are the intersections of X with distinct m-planes $\tilde{L}_i$, and $\cup_{i=1}^d \tilde{L}_i$ is generically transverse to X.*

Proof. If $n + m < N$ then, since $\tilde{D}$ is generically transverse to X we will have $\tilde{D} \cap X = \varnothing$. In this case the result is trivial, so we will assume that $n + m \geq N$.

Let $A, R \subset \mathbb{P}^N$ be general planes of dimension $m - 1$ and $N - m$ respectively. Dynamic projection (Corollary 4.6) produces an irreducible family $Z \subset \mathbb{P}^1 \times \mathbb{P}^N$ that is a rational equivalence between $\tilde{D}$ and $\tilde{L} := \cup_{i=1}^d \tilde{L}_i$, where the $\tilde{L}_i$ are the spans of A with the points of the intersection $R \cap \tilde{D}$.

We will show that $\tilde{L}$ is generically transverse to X. Given this, Lemma 5.27 implies that D is rationally equivalent to the cycle $\langle \tilde{L} \cap X \rangle = \sum_{i=1}^{d} \langle L_i \rangle$.

It remains to show that $\tilde{L}$ and X are generically transverse when $A \in \mathbb{G}(m-1, \mathbb{P}^N)$ and R are chosen generally. To this end let $\tilde{D}^\circ = \tilde{D} \setminus (\tilde{D} \cap X)$ and consider the incidence correspondence Λ defined by

$$\Lambda = \{(A, p, x) \in \mathbb{G}(m-1, \mathbb{P}^N) \times \tilde{D}^\circ \times X \mid q \in \overline{A\,p}\}$$

The fiber of Λ over a point $(p, x) \in \tilde{D}^\circ \times X$ is the set of $(m-1)$-planes meeting the line $\overline{p\,x}$, an irreducible subvariety of the Grassmannian, so Λ is irreducible.

Since $\tilde{D}$ is generically transverse to X, a general point of $\tilde{D}$ (and thus also of $\tilde{D}^\circ$) is not contained in the tangent plane to a general point x of X. Thus for such a point p, and a general choice of $(m-1)$-plane A meeting the line $\overline{p\,x}$, the span $\overline{A\,p}$ is transverse to X at q. It follows that the subset

$$\Lambda' = \{(A, p, x) \in \Lambda \mid \overline{A\,p} \text{ is not generically transverse to } X \text{ at } x\}$$

has smaller dimension than Λ. Let V_A be the set of points $p \in \tilde{D}^\circ$ such that the fiber of Λ over (A, p) is contained in Λ'. The inequality of dimensions shows that for general A the set V_A is contained in a proper closed subset of $\tilde{D}^\circ$. If we choose the repelling plane R generally then it will meet $\tilde{D}$ in only finitely many points, and will miss both $\tilde{D} \cap X$ and V_A. Thus for every point $p_i \in \tilde{D} \cap R$ the plane $\tilde{L}_i = \overline{A\,p_i}$ is generically transverse to X, as required. $\square$

Re-embedding X. As suggested in the discussion of ambient cycles, we'll be considering intersections of $X \subset \mathbb{P}^N$ with other subvarieties of $\mathbb{P}^N$. It will be convenient to begin by making sure that X is embedded in a good way in projective space, using the next Lemma. See Exercise 5.31 for a natural generalization.

Lemma 5.22. *Let $\nu_3 : \mathbb{P}^N \to \mathbb{P}^M$ be the third Veronese map. If $X \subset \mathbb{P}^N$ is a smooth variety, then no three points of $\nu_3(X)$ are collinear, and the tangent spaces to X at distinct points of $\nu_3(X)$ are disjoint.*

Proof. We may as well assume that $X = \mathbb{P}^n$. We claim first that any finite subscheme Y of degree $m \leq 4$ in $\mathbb{P}^N$ imposes independent conditions on cubics. In terms of the homogeneous coordinate ring S_Y of Y, this means that the component $(S_Y)_3$ of degree 3 has dimension m. To see this, note that if l is a linear form on $\mathbb{P}^N$ not vanishing on Y, then $R := S_Y/(l)$ is a homogeneous algebra of vector space dimension

$$\sum_{d=0}^{\infty} \dim_K R_d = m \leq 4.$$

If $R_d = 0$ for some particular $d \geq 1$, then $R_e = 0$ for all $e \geq d$, because R is generated as an algebra in degree 1. Thus the infinite sum above can be truncated after the 4-th term,

$$\sum_{d=0}^{3} \dim_K R_d = m.$$

From the exact sequence

$$0 \to S_Y(-1) \to S_Y \to R \to 0$$

we see by an induction starting from $e = 0$ that

$$\dim_K (S_Y)_e = \sum_{d \leq e} \dim_K R_d.$$

In particular, $\dim_K (S_Y)_3 = m$ as required.

Since linear forms on $\mathbb{P}^M$ pull back to cubics on $\mathbb{P}^N$, it follows that subschemes of length $m \leq 4$ of $\nu_3(\mathbb{P}^N)$ impose independent conditions on linear forms. In particular, no subscheme of $\nu_3(\mathbb{P}^N)$ of length ≥ 3 can be contained in a line, and no subscheme of length ≥ 4 can be contained in a 2-plane.

If the tangent spaces to $\nu_3(\mathbb{P}^N)$ at points p, q met in some point r, then the lines $L_1 = \overline{p\,r}$ and $L_2 = \overline{q\,r}$ would be contained in a 2-plane, and this plane would contain the subscheme $(L_1 \cap Y) \cup (L_2 \cap Y)$. This scheme has length at least 4, a contradiction. $\square$

Standing Hypothesis: *Henceforth in this section we will assume that $X \subset \mathbb{P}^N$ is a smooth, quasiprojective variety of dimension n, that no three points of X are collinear in $\mathbb{P}^N$, and that any two tangent planes to X at distinct points, regarded as n-dimensional projective subspaces of $\mathbb{P}^N$, are disjoint.*

5.6.2 Part (a) of the Moving Lemma

To prove the first part of the Moving Lemma, it will turn out to suffice to start with an arbitrary pair of subvarieties $A, B \subset X$ and construct a cycle A' that intersects B generically transversely and is rationally equivalent to A. (As we'll see in the course of the proof, if we can construct an A' generically transverse to one subvariety B, we can find one generically transverse to any given finite collection of subvarieties.)

In the course of the proof of Part (a) of the Moving Lemma we will actually prove a slightly stronger statement. We record it here:

Theorem 5.23 (Sharp Moving Lemma, Part (a))**.** *If A, B are subvarieties of a smooth quasiprojective variety X, with A of dimension a, then $\langle A \rangle$ can* *be written as* $\langle D\rangle - \langle A'\rangle$ *where* $\langle D\rangle$ *is an ambient* k*-cycle and where* A' *is a scheme that is generically transverse to* B *such that* $A' \cap B$ *does not contain any component of* $A \cap B$*.*

The essential ingredient in our proof of the Moving Lemma is a process called the *cone construction*, due originally to Severi and used in every proof (and attempted proof) of the Moving Lemma we know of. The cone construction expresses the cycle $\langle A\rangle$ on X as a difference $\langle D\rangle - \langle A'\rangle$ of an ambient cycle $\langle D\rangle$ and a "residual" cycle $\langle A'\rangle$ that is better situated with regard to B; iterating this process, we'll arrive at the statement of the Sharp Moving Lemma.

The cone construction itself is straightforward: we take $\Gamma \cong \mathbb{P}^{N-n-1} \subset \mathbb{P}^N$ a general $(N-n-1)$-plane, and let $\tilde{D}_\Gamma = \overline{\Gamma\, A}$ be the cone over A with vertex Γ. As we'll see in Lemma 5.24 , the intersection

$$D_\Gamma = \tilde{D}_\Gamma \cap X$$

will be generically reduced of pure dimension a, so that $\langle D_\Gamma\rangle$ will be an ambient cycle. (We can also express D_Γ in terms of the projection from Γ:

$$D_\Gamma = \pi_\Gamma^{-1}(\pi_\Gamma(A)),$$

where π_Γ denotes the projection with center Γ.) Since D_Γ has pure dimension a, A will be an irreducible component of D_Γ and we can write

$$\langle D_\Gamma\rangle = \langle A\rangle + \langle A'_\Gamma\rangle$$

with $A'_\Gamma \subset X$ the union of the remaining irreducible components of D_Γ.

The key here is to show that if A is not already generically transverse to a given cycle B on X, the residual scheme A'_Γ will be "more transverse than A" to B; this is the content of Lemma 5.25 below, the heart of the Moving Lemma. Further, the ambient cycle D_Γ can be moved so as to become transverse to B, using Lemma 5.21.

Most of the salient points in the arguments below are already present in the very simplest case, with $n = 2$ and $N = 3$ (that is, $X \subset \mathbb{P}^3$ a smooth surface), and $A = B \subset X$ a (possibly) singular curve. The reader may want to picture this case in going through the argument—see Figures 5.4 and 5.5.

We fix the following notation: Let $A \subset X \subset \mathbb{P}^N$ be a proper subvariety of dimension a. Let $\mathbb{G} = \mathbb{G}(N-n-1, \mathbb{P}^N)$ be the Grassmannian of $N-n-1$-planes in $\mathbb{P}^N$, and let $\mathbb{G}^* \subset \mathbb{G}$ be the set of $(N-n-1)$-planes in $\mathbb{P}^N$ that do not meet X. For $\Gamma \in \mathbb{G}^*$ we let $\pi_\Gamma : X \to \mathbb{P}^n$ be the linear projection with projection center Γ. Set $D_\Gamma := \pi_\Gamma^{-1}(\pi_\Gamma(A) = X \cap \overline{\Gamma\, A}$. The next Lemma shows that,when Γ is general, D_Γ is generically reduced, and thus defines an ambient cycle on X.

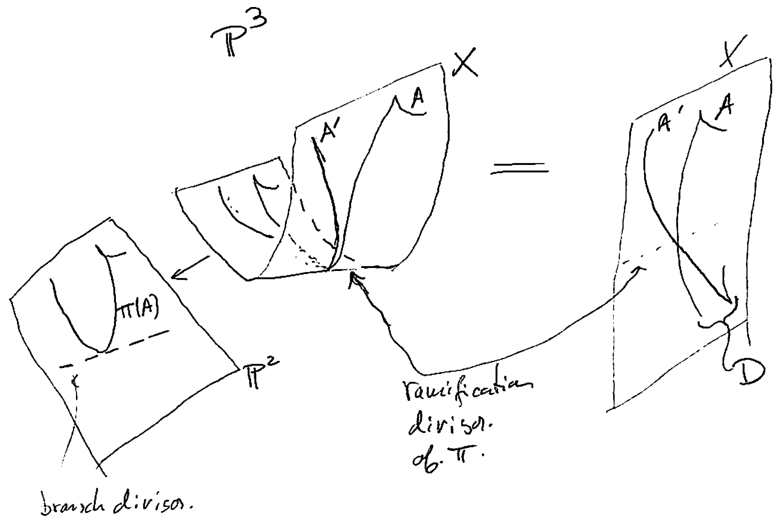
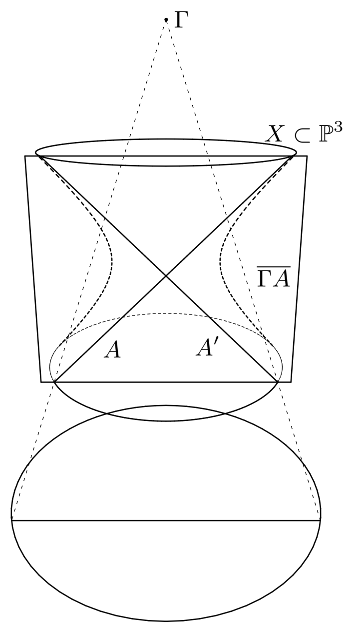
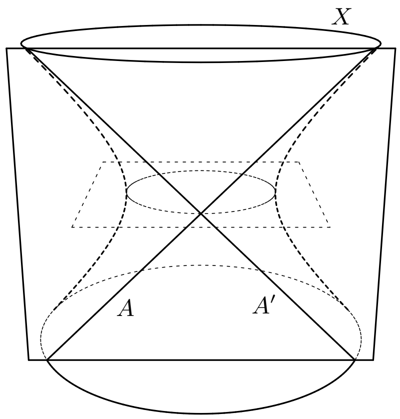


FIGURE 5.4. Cone Construction (not showing the cone)



$\overline{\Gamma A} \cap X = A + A'$

FIGURE 5.5. The cone construction (in this case the cone is a plane.) ****Silvio: Equation should be $D = \overline{\Gamma A} \cap X = A + A'$****



$A + A' \sim E$

FIGURE 5.6. Illustration of the Moving Lemma ****the dotted conic in the middle should be labeled E AND this should be amalgamated with Fig 2.2****

Lemma 5.24. *There is a dense open set of* $\Gamma \in \mathbb{G}^*$ *such that* $D_\Gamma := \pi_\Gamma^{-1}(\pi_\Gamma(A))$ *is generically reduced of pure dimension* a.

Proof. By Theorem 0.2, the components of D_Γ have codimension at most the codimension of $\pi_\Gamma(A)$. Since π_Γ is quasifinite, every irreducible component of D_Γ is of pure dimension $a = \dim A$.

It suffices to show that there exists a plane Γ disjoint from X such that D_Γ is generically reduced. To find such a Γ, we first choose a general point p of A; then we choose a general plane Σ of dimension $N - n$ containing p; and finally we take Γ to be a general hyperplane of Σ, so that $\Sigma = \overline{\Gamma\, p}$.

Being a general $(N-n)$-plane through p, Σ meets X transversely at p; and as it's the intersection of n general hyperplanes through p, Bertini's Theorem 0.5 implies that Σ meets X transversely away from p. Thus Σ meets X transversely and $\pi_\Gamma^{-1}(\pi_\Gamma(p)) = \Sigma \cap X$ is a finite collection of reduced points.

Since $\dim A < \operatorname{codim} \Sigma$ and Σ is a general plane through p we see that $\Sigma \cap A = \{p\}$, and $T_pA \cap T_p\Sigma = 0$. It follows that the map $\pi_\Gamma|_A : A \to \mathbb{P}^n$ is one-to-one over $\pi_\Gamma(p)$ and that $\pi_\Gamma|_A$ has injective differential at p. Thus $\pi_\Gamma(p)$ is a smooth point of $\pi_\Gamma(A)$, and the tangent space to $\pi_\Gamma(A)$ at $\pi_\Gamma(p)$ is the image of T_pA under π_Γ.

Since $\pi_\Gamma(A)$ is smooth at $\pi_\Gamma(p)$ and the fiber $\pi_\Gamma^{-1}(\pi_\Gamma(p)) = \Sigma \cap X$ is reduced, $D_\Gamma = \pi_\Gamma^{-1}(\pi_\Gamma(A))$ is smooth at each point q of $\pi_\Gamma^{-1}(\pi_\Gamma(p))$. We claim that there is such a point on each component of D_Γ. Since π_Γ is quasifinite and every component A_i of D_Γ has the same dimension as A, every component of D_Γ dominates $\pi_\Gamma(A)$. Since $\pi_\Gamma(p)$ is a general point of $\pi_\Gamma(A)$, every component of D_Γ must contain a point of $\pi_\Gamma^{-1}(\pi_\Gamma(p))$ as claimed. Thus $\pi_\Gamma^{-1}(\pi_\Gamma(A))$ is generically reduced. $\square$

Lemma 5.24 implies that

$$\langle A\rangle = \langle D_\Gamma\rangle - \langle A'_\Gamma\rangle$$

as cycles, where A'_Γ is a generically reduced scheme of pure dimension a that does not contain A, and D_Γ is an ambient cycle.

Since the cycle D_Γ produced in the cone construction is ambient, it is rationally equivalent to a cycle of the form $d[L]$, where L is a general plane of appropriate dimension. By Bertini's Theorem, L can be chosen transverse to any finite collection of subvarieties of X.

Now let $B \subset X$ be a fixed subvariety, and again suppose that A'_Γ, D_Γ are produced from $A \subset X \subset \mathbb{P}^N$ by the cone construction with respect to a plane $\Gamma \in \mathbb{G}^*$. The central argument of the proof of the Moving Lemma is the next Lemma, which shows that, for general Γ, the scheme A'_Γ meets B "more transversely" than does A: although $A'_\Gamma \cap B$ may still have components where the intersection is non-transverse or of the wrong dimension, they will at least be smaller than those of $A \cap B$.

Lemma 5.25 (Heart of the Moving Lemma)**.** *With notation and hypotheses as above there is a dense open set of* $\Gamma \in \mathbb{G}^*$ *such that:*

(a) *The irreducible components of $A'_\Gamma \cap B$ that are contained in A have dimension $< \dim(A \cap B)$. In particular, if A is already dimensionally transverse to B then no component of $A'_\Gamma \cap B$ is contained in A.*

(b) *The irreducible components of $A'_\Gamma \cap B$ that are not contained in A are generically transverse intersections of A'_Γ and B.*

Proof. **(a):** By choosing Γ not to meet the tangent planes of X at one chosen point of each component of $A \cap B$, we can ensure that the map $\pi_\Gamma : X \to \mathbb{P}^n$ is nonsingular at those points, and thus those points do not lie in $A \cap A'_\Gamma$. It follows that $A'_\Gamma \cap A \cap B$ has codimension strictly greater than $A \cap B$, proving the first statement of part (a).

If A and B are dimensionally transverse, then $\operatorname{codim}(A \cap B) = \operatorname{codim} A + \operatorname{codim} B$. Since X is smooth, every component of $A'_\Gamma \cap B$ has codimension $\leq \operatorname{codim} A + \operatorname{codim} B$, proving the second statement of part (a).

(b): By Lemma 5.24, there is a dense open subset $\mathbb{G}' \subset \mathbb{G}^*$ such that the scheme A'_Γ is generically reduced whenever $\Gamma \in \mathbb{G}'$.

We will prove that for general Γ the scheme D_Γ, or equivalently the scheme A'_Γ, is generically transverse to the scheme $B^* := B \setminus A \cap B$.

We can study the intersection $A'_\Gamma \cap B^*$ through the incidence correspondence that measures the set of pairs mapping to the same point under projection from Γ:

$$\begin{aligned}\Xi :&= \left\{(\Gamma, p, q) \in \mathbb{G}' \times A \times B^* \mid \pi_\Gamma(p) = \pi_\Gamma(q)\right\}\\ &= \left\{(\Gamma, p, q) \in \mathbb{G}' \times A \times B^* \mid \Gamma \cap \overline{p\,q} \neq \varnothing\right\}.\end{aligned}$$

The fiber of Ξ over any point $\Gamma \in \mathbb{G}'$ is

$$F_\Gamma := \{(p, q) \in A \times B^* \mid \pi_\Gamma(p) = \pi_\Gamma(q)\}.$$

Since π_Γ is a finite map, the projection $(p, q) \mapsto q$ maps F_Γ finite-to-one onto $A'_\Gamma \cap B^*$. Since every component of A'_Γ has dimension equal to the dimension of A, and X is smooth, Theorem **??** shows that every component of $A'_\Gamma \cap B^*$ has dimension $\geq \dim A + \dim B - n$, so the same goes for the components of F_Γ.

Of course if the image of Ξ in $\mathbb{G}'$ were not dense, then the generic fiber would be empty, so $A'_\Gamma \cap B^*$ would be empty for generic Γ, and there would be nothing further to prove; thus we may assume the projection $\Xi \to \mathbb{G}'$ is dominant. To understand this case, we will compute the dimension of Ξ and of its fibers over $\mathbb{G}'$.

To do this, observe that the variety Ξ surjects to $A \times B^*$ with fiber over p, q equal to the Schubert cycle $\Sigma_n(\overline{p\,q})$. By Theorem 3.1, $\Sigma_n(\overline{p\,q})$ has codimension n in the Grassmannian, so

$$\dim \Xi = \dim A + \dim B + \dim G - n.$$

Since we're assuming the projection $\Xi \to \mathbb{G}'$ has dense image, it follows that

$$\dim F_\Gamma \leq \dim A + \dim B - n.$$

On the other hand, F_Γ maps finite-to-one onto $A'_\Gamma \cap B^*$, which has dimension at least $\dim A + \dim B - n$ everywhere. Thus, after replacing G' by a dense open subset, we may assume that *for* $\Gamma \in \mathbb{G}'$ *every component of* $D_\Gamma \cap B^* = A'_\Gamma \cap B^*$ *has dimension exactly* $\dim A + \dim B - n$; that is, the intersections are dimensionally transverse.

The next step is to characterize when a point $q \in D_\Gamma \cap B^*$ is a non-transverse point of intersection; this is done in the following lemma.

Lemma 5.26. *With* $\pi_\Gamma : X \to \mathbb{P}^n$ *and* $A, B, D_\Gamma \subset X$ *as above, suppose that* $p \in A$ *and* $q \in \pi^{-1}\pi(p) \subset D_\Gamma$, *with* $q \notin A$. *If:*

1) B is smooth at q;

2) $\pi(A)$ is smooth at $\pi(p)$; and

3) The three linear subspaces Γ, $\mathbb{T}_p A$ and $\mathbb{T}_q B$ together span $\mathbb{P}^N$,

then the subvarieties D_Γ and B meet transversely at q.

Proof. Let E_Γ be the cone over A with vertex Γ. Because $q \in \pi^{-1}\pi(p)$, hypothesis 2 implies that E_Γ is smooth of dimension $N-n+\dim A$ at q, with tangent space the span of Γ and $\mathbb{T}_pA$. By hypothesis 1, the tangent space $\mathbb{T}_qB$ has dimension $\dim B$. By hypothesis 3, the span of $\mathbb{T}_qE_\Gamma$ and $\mathbb{T}_qB$ is all of $\mathbb{P}^N$, and so the intersection $\mathbb{T}_qE_\Gamma \cap \mathbb{T}_qB$ has dimension $\dim E_\Gamma + \dim B - N = \dim A + \dim B - n$. On the other hand, $D_\Gamma \cap B = E_\Gamma \cap B$ has dimension $\dim A + \dim B - n$ at q; it follows that $D_\Gamma \cap B$ is smooth of dimension $\dim D_\Gamma + \dim B - n = \dim A + \dim B - n$ at q. Thus D_Γ and B meet transversely at q. □

To prove that for generic Γ the intersections are in fact generically transverse, we introduce the incidence correspondence of triples consisting of a "bad" subspace Γ together with points $p \in A$ and $q \in B^*$ that constitute a "reason" why Γ is "bad." That is, we set

$$\Psi := \{(\Gamma, p, q) \in \mathbb{G}' \times A \times B^* \mid \pi_\Gamma(p) = \pi_\Gamma(q) \text{ and } D_\Gamma \text{ is not transverse to } B \text{ at } q\}.$$

We must show that the image of Ψ in $\mathbb{G}'$ is not dense, which we will do by proving that $\dim \Psi < \dim \mathbb{G}'$. Note that $\Psi \subset \Xi$, and that if Γ is in the image of Ψ, then the fiber of Ψ over Γ is equal to the corresponding fiber F_Γ of Ξ. It thus suffices to show that $\dim \Psi < \dim \Xi$.

To this end, we let $\Psi_i \subset \mathbb{G}'$, for $i = 1, 2, 3$, be the set of planes *not* satisfying condition i of Lemma 5.26, so that $\Psi \subset \Psi_1 \cup \Psi_2 \cup \Psi_3$. Thus, to start with,

$$\Psi_1 = \{(\Gamma, p, q) \in \Xi \mid q \in B_{\text{sing}}\}.$$

As for Ψ_2, the image $\pi(A)$ can be singular at $\pi(p)$ only if at least one of the following three things occurs:

(2a) $q \in \overline{\Gamma p}$ with p a singular point of A;

(2b) $q \in \overline{\Gamma p}$ for two or more distinct points $p \in A$; or

(2c) $q \in \overline{\Gamma p}$, $p \in A_{sm}$ and $\Gamma \cap \mathbb{T}_pA \neq \varnothing$.

Accordingly we set

$$\Psi_{2a} = \{(\Gamma, p, q) \in \Xi \mid p \in A_{\text{sing}}\}.$$

$$\Psi_{2b} = \{(\Gamma, p, q) \in \Xi \mid \exists p' \neq p \in A \text{ with } \Gamma \cap \overline{p'\,q} \neq \varnothing\}.$$

and

$$\Psi_{2c} = \{(\Gamma, p, q) \in \Xi \mid p \in A_{\text{sm}} \text{ and } \Gamma \cap \mathbb{T}_pA \neq \varnothing\}.$$

Finally, even if both D_Γ and B are smooth at q, the intersection may still fail to be transverse. When D_Γ is smooth at q the tangent plane to the cone $\overline{A\Gamma}$ at q is the span of Γ and the tangent space $\mathbb{T}_pA$. This span fails to intersect B transversely at q, only if the three linear spaces Γ, $\mathbb{T}_pA$ and $\mathbb{T}_qB$ fail to span all of $\mathbb{P}^N$. From our hypothesis on the embedding of X it follows that $\mathbb{T}_pA$ and $\mathbb{T}_qB$ are disjoint. Thus a necessary condition for non-transversality at q in the case where D_Γ and B are both smooth at q is that Γ is not transverse to $\overline{\mathbb{T}_pA\,\mathbb{T}_qB}$. Since $\dim \overline{\mathbb{T}_pA\,\mathbb{T}_qB} = a+b+1$ and $\dim\Gamma = N-n$, the relevant set is

$$\Psi_3 = \left\{ (\Gamma, p, q) \in \Xi \ \middle|\ \begin{array}{c} p \in A_{\mathrm{sm}}, \\ q \in B_{\mathrm{sm}};\ \text{and} \\ \dim(\Gamma \cap \overline{\mathbb{T}_pA\,\mathbb{T}_qB}) > a+b-n \geq 0 \end{array} \right\}$$

We can compute the dimension of Ψ_1 and Ψ_{2a} just as we computed the dimension of Ξ itself. Since A and B are reduced, the sets A_{sing} and B^*_{sing} have strictly smaller dimension than A and B^*. Since the fiber over (p,q) of the projection $\Xi \to A \times B^*$ is an open subset of a Schubert cycle $\Sigma_n(\overline{p\,q})$, it has codimension n in $\mathbb{G}$ so Ψ_1 and Ψ_{2a} have strictly smaller dimension than Ξ.

The sets Ψ_{2c} and Ψ_5 dominate $A_{\mathrm{sm}} \times B^*$, but with strictly smaller fibers than Ξ: By our hypothesis on the embedding of X in $\mathbb{P}^N$ we have $q \notin \mathbb{T}_pA$. Also $p \notin \Gamma$. If $\Gamma \cap \mathbb{T}_pA \neq \varnothing$ then, in addition to meeting the line $\overline{p\,q}$ in at least a point, Γ must intersect the $(a+1)$-plane $\overline{\mathbb{T}_pA\,q}$ in at least a line. Similarly, a plane Γ in the fiber of Ψ_3 over a point $(p,q) \in A_{\mathrm{sm}} \times B^*_{\mathrm{sm}}$ not only belongs to $\Sigma_n(\overline{p\,q})$ but also meets $\overline{\mathbb{T}_pA\,\mathbb{T}_qB}$ in at least a line. By Theorem 3.1, this implies in either case that that Γ belongs to a proper closed subvariety of $\Sigma_n(\overline{p\,q})$, as required.

To handle Ψ_{2b} we introduce one more incidence correspondence. Set

$$\tilde{\Psi}_{2b} = \left\{ (\Gamma, p, p', q) \in \mathbb{G}' \times A \times A \times B^* \ \middle|\ \begin{array}{c} p \neq p', \\ \Gamma \cap \overline{p\,q} \neq \varnothing,\ \ \text{and} \\ \Gamma \cap \overline{p'\,q} \neq \varnothing \end{array} \right\}$$

so that Ψ_{2b} is the image of $\tilde{\Psi}_{2b}$ under projection to $\mathbb{G}' \times A \times B^*$. It suffices to show that $\dim \tilde{\Psi}_{2b} < \dim \Xi$.

To estimate the dimension of $\tilde{\Psi}_{2b}$, consider the projection to $A \times A \times B^*$. By our hypothesis on the embedding of X in $\mathbb{P}^N$, three distinct points $p, p', q \in X$ span a 2-plane containing $\overline{pq}$. The intersection of Γ with $\overline{p\,q}$ cannot occur at q, so Γ must meet this 2-plane in at least a line. Thus the fiber over (p,p',q) consists of planes Γ contained in $\Sigma_{n,n}(\overline{p\,p'\,q})$, which has codimension n in $\Sigma_n(\overline{p\,q})$. Comparing this situation with that in the computation of the dimension of Ξ, we see that $\dim \Psi_3 \leq \dim \Xi + a - n < \dim \Xi$, completing the proof. $\square$

This completes the proof of Lemma 5.25; from here on in the argument is reasonably straightforward. To begin with, we can now deduce the Sharp Moving Lemma, and hence the first part of Theorem 5.4.

Proof of Theorem 5.23. We may immediately reduce to the case where A is reduced and irreducible. By Theorem 0.2 each $A \cap B_i$ has dimension at least $\dim A + \dim B - n$.

We do induction on the maximum of the "excess" dimensions,

$$m := \max_i \{\dim(A \cap B_i) - (\dim A + \dim B_i - n)\} \geq 0.$$

The cone construction, with general Γ, yields an expression

$$A = D_\Gamma - A'_\Gamma,$$

where D_Γ is an ambient cycle. By Lemma 5.21, $D_\Gamma \sim d[L \cap X]$, where L is a generic plane; in particular $L \cap X$ is generically transverse to B.

Lemma 5.25 tells us that if $m = 0$ then A'_Γ is generically transverse to each B_i. Further, if $m > 0$, then the maximum of the excess dimensions for the intersections of A'_Γ with the B_i,

$$m' := \max_i \{\dim(A'_\Gamma \cap B_i) - (\dim A'_\Gamma + \dim B - n)\} < m,$$

so we are done. □

5.6.3 Part (b) of the Moving Lemma

The second part of the statement of the Moving Lemma 5.4 may seem at first to be the more mysterious, but in fact it will follow from the first part. The key is to give a sufficient condition for a rational equivalence between cycles A and A' on X to "restrict" to a rational equivalence between the intersection cycles $A \cap B$ and $A' \cap B$ on a subvariety $B \subset X$, which is the content of the following Lemma—see Figure 5.1.

Lemma 5.27. *Let X be a quasiprojective variety, and let A and A' be pure dimensional cycles on X. Suppose that $Z = \sum m_i Z_i \in Z(\mathbb{P}^1 \times X)$ is a linear combination of subvarieties Z_i giving a rational equivalence between A and A'; that is, such that $Z \cap (\{0\} \times X) = A$ and $Z \cap (\{\infty\} \times X) = A'$. If B is a subvariety of X that is generically transverse to $Z_i \cap (\{0\} \times X)$ and $Z_i \cap (\{\infty\} \times X)$ for all i, then $A \cap B$ is rationally equivalent to $A' \cap B$ as cycles on B (and hence also as cycles on X).*

Proof of Lemma 5.27. We may harmlessly discard all the Z_i except those of dimension $\dim Z_i = 1 + \dim A$. The schemes $Z_i \cap (\{0\} \times B)$ and $Z_i \cap (\{\infty\} \times B)$ are Cartier divisors on $Z_i \cap (\mathbb{P}^1 \times B)$ and are reduced. It follows that there is an open set $U \subset \mathbb{P}^1$ containing $0, \infty$ such that $Z_i \cap (U \times B)$ is generically

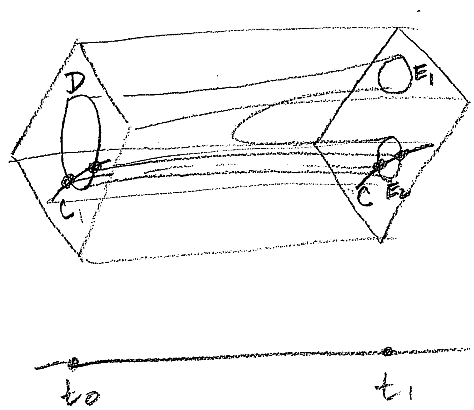


FIGURE 5.1. Restriction to a smooth subvariety preserves rational equivalence because the restriction of a transverse rational equivalence gives a rational equivalence

reduced and of dimension $1 + \dim(A \cap B)$. Thus $\sum m_i(Z_i \cap (U \times B))$ is a rational equivalence between $A \cap B$ and $A' \cap B$ as cycles on B. $\square$

Note that even if B is dimensionally transverse to the cycle $Z \cap (\{0\} \times X)$, it need not be dimensionally transverse to the individual intersections $Z_i \cap (\{0\} \times X)$: there may be cancellation among the components of the cycles $Z_i \cap (\{0\} \times X)$, as illustrated in Figure **??**. This is why we have to require the stronger condition that $Z_i \cap (\{0\} \times X)$ and $Z_i \cap (\{\infty\} \times X)$ intersect B generically transversely for all i.

Proof of Theorem 5.4 Part (b). We must show that if A and A' are rationally equivalent cycles on X and B is a cycle generically transverse to both, then $A \cap B$ is rationally equivalent to $A' \cap B$. By definition, there exists a rational equivalence $Z = \sum m_i Z_i \in Z(\mathbb{P}^1 \times X)$, a linear combination of subvarieties Z_i dominating $\mathbb{P}^1$, such that $Z_i \cap (\{0\} \times X) = A$ and $Z_i \cap (\{\infty\} \times X) = A'$. By Lemma 5.25, we may apply the cone construction to B to get an ambient cycle D and a cycle B' with $[B] = [D] - [B']$ and such that B' is generically transverse to $Z_i \cap (\{0\} \times X)$ and $Z_i \cap (\{\infty\} \times X)$ for all i. By Lemma 5.21, D is equivalent to a cycle $d\langle L \rangle$, where L is a general linear section of X and is thus generically transverse to $Z_i \cap (\{0\} \times X)$ and $Z_i \cap (\{\infty\} \times X)$ for all i. Applying Lemma 5.27 twice, we get

$$\begin{aligned}
[A \cap B] &= [A \cap D] - [A \cap B'] \\
&= d[A' \cap L] - [A' \cap B'] \\
&= [A' \cap B].
\end{aligned}$$

This completes the proof of the Moving Lemma. $\square$

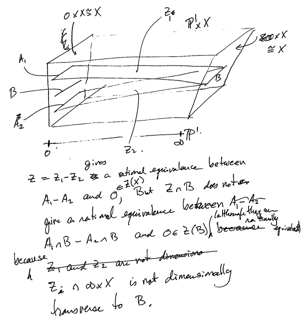


FIGURE 5.2. $S \cup T$ defines a rational equivalence between the cycles A and $A' + B - C$, and $Z \cup B$ is transverse to A, but $(S \cup T) \cap (Z \cup B)$ does not restrict to an equivalence between $A \cap (Z \cup B)$ and $A' \cap (Z \cup B)$.

5.7 Transversality for morphisms

The following result, in conjunction with the Moving Lemma, will show that, in the presence of separability, we can "move" cycles to be generically transverse to maps (see Figure 5.3).

Lemma 5.28. *Let $f : Y \to X$ be a generically separable morphism of quasiprojective varieties. There is a finite collection of subvarieties B_i of X such that if a cycle A on X is generically transverse to each B_i, then A is generically transverse to f.*

Proof. We begin by making the subvarieties B_i explicit: We will show that A is generically transverse to f as long as it is generically transverse to $f(Y)$ and dimensionally transverse to the loci where the fibers of f or the fibers of the restriction of f to the singular set of Y are "too large."

More precisely, suppose that the fiber of f over a general point $x \in f(Y)$ has dimension k, so that $\dim Y = k + \dim f(Y)$. For each $l > k$, let $Y_l \subset Y$ the the proper closed set

$$Y_l := \{y \in Y \mid \dim f^{-1}(f(y)) \geq l\},$$

which is closed in Y by the semicontinuity of fiber dimension, and let $\Phi_l \subset f(Y)$ be the closure in X of $f(Y_l)$.

Let $U \subset X$ be the open set $X \setminus \Phi_{k+1}$, and let $Y_U = f^{-1}(U)$, so that every irreducible component of every fiber of $f|_{Y_U}$ has dimension exactly k. Further, set

$$\operatorname{sing}(f) := \{y \in Y_U \mid y \text{ is a singular point of } f^{-1}(f(y))\},$$

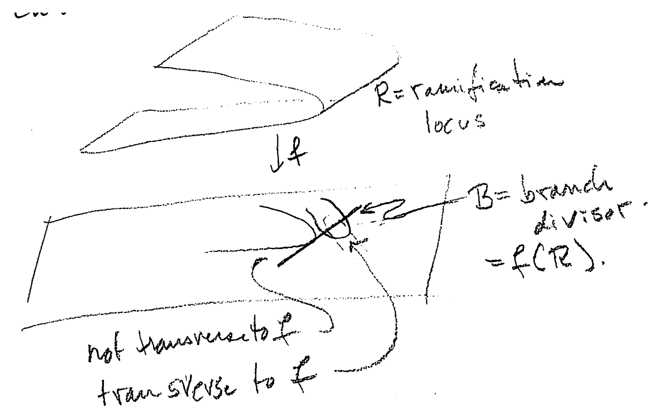


FIGURE 5.3. The Moving Lemma allows us to make a subvariety transvserse to a map

which is a closed subset of Y_U because it is the locus where the map of vector bundles $T_{Y_U} \to (f^*T_X)|_{Y_U}$ drops rank. Finally, consider the closed subset

$$(\mathrm{sing}(f))_k = \{y \in Y \mid \dim(\mathrm{sing}(f)) \cap f^{-1}(f(y)) \geq k\}$$

and let Ψ be the closure in U of $f((\mathrm{sing}(f))_k)$.

Suppose that A is generically transverse to $f(Y)$ and dimensionally transverse to each Φ_l and Ψ. With this hypotheis, we claim that A is generically transverse to f. Since each of $f(Y)$, $f(Y_l)$ and Ψ are constructible in X (Eisenbud [1995] Corollary 14.5) each contains an open set of its closure, so the statement that A meets one of the closures dimensionally transversely or generically transversely is easy to interpret.

We may assume that A is the cycle associated to a reduced closed subscheme of X. To simplify the notation, we may further replace X by $f(Y)$, and (by generic transversality) assume again that A is the cycle associated to a reduced subscheme.

To prove the claim, we first show that Φ_l has codimension $\geq l - k + 1$. This is because, for $l > k$,

$$Y_l = \{y \in Y \mid \dim f^{-1}(x) \geq l\}$$

is a proper closed subset, so $\dim \Phi_l + l < \dim Y = \dim X + k$,

Next we note that if the fiber over a point $x \in U$ has a nonreduced component, then the fiber is singular all along this component, which has dimension k; thus $x \in \Psi$. Further, $\mathrm{sing}(f)$ is a proper closed subset of Y_U because f is generically separable. Since Y_U is irreducible, $\dim \mathrm{sing}(f) <$

$\dim Y_U$. Thus the generic fiber of $f|_{\mathrm{sing}(f)}$ has dimension $< k$, so Ψ is a proper subset of U.

Now suppose that A is dimensionally transverse to each of the subvarieties Φ_l and Ψ, and let A' be a component of A. We must show that $f^{-1}(A')$ has dimension $k + \dim A'$ and is generically reduced. For the dimension statement we use the estimate on the codimension of $A' \cap \Phi_l$. For $l > k$ this yields

$$\dim(A' \cap \Phi_l) \leq \dim A' - l + k - 1,$$

from which it follows that

$$\dim\bigl(f^{-1}(A' \cap \Phi_l)\bigr) \leq \dim A' + k - 1.$$

Since every component of $f^{-1}(A')$ will have dimension at least $\dim A' + k$, it follows that no component of $f^{-1}(A')$ is contained in $f^{-1}(\Phi_l)$ for $l > k$, and hence

$$\dim f^{-1}(A') = \dim A' + k,$$

as required.

Let $B \subset f^{-1}(A')$ be an irreducible component, and let f_B be the restriction of f to B. We must show that B is generically reduced. Since $B \not\subset f^{-1}(\Phi_l)$ for $l > k$, the general fiber of f_B has dimension k, so the image $f(B)$ has the same dimension as A', and thus $f(B)$ contains a dense subset of A'.

Let $b \in B$ be a general point; we will show that B is smooth at b. We see from the previous paragraph that $f(b)$ is a general point of A'; in particular, $f(b)$ is a smooth point of A'. Further, by dimensional transversality, $f(b)$ is not contained in Ψ. Thus b is a smooth point of $f^{-1}f(b)$, and the kernel of the differential

$$T_{B,b} \to T_{A',f(b)}$$

has dimension k. Since $\dim T_{A'),f(b)} = \dim A'$ we see that $\dim T_{B,b} \leq k + \dim A' = \dim B$, so $b \in B$ is a smooth point. $\square$

Proof of Theorem 5.8 Part (a). By the Moving Lemma and Lemma 5.28, the classes of subvarieties A that are generically transverse to f generate $A(X)$ as an abelian group, so there is at most one group homomorphism $f^* : A(X) \to A(Y)$ such that $f^*[A] = [f^{-1}(A)]$ for such subvarieties.

Let $\Gamma_f \subset Y \times X$ be the graph of f and consider the projections

$$\begin{tikzcd} & \Gamma_f \subset Y \times X \arrow[dl, "\pi_Y"'] \arrow[dr, "\pi_X"] & \\ Y & & X. \end{tikzcd}$$

Since π_X is flat and π_Y is proper, there are group homomorphisms $(\pi_Y)_*$ is as in Theorem 4.15 and $(\pi_X)^*$ is as in Theorem 4.21. We define f^* by pulling back a class via π_X, multiplying with the class of Γ_f and then pushing forward via π_Y, that is,

$$f^*[A] := (\pi_Y)_*((\pi_X^*[A])\,[\Gamma_f]).$$

We begin by checking that f^* has the desired form $f^*[A] = [f^{-1}(A)]$ when A is generically transverse to f. In general we have

$$f^{-1}(A) = \pi_Y(\pi_X^{-1}(A) \cap \Gamma_f),$$

and since the restriction of π_Y to Γ_f is an isomorphism, this yields

$$[f^{-1}(A)] = \pi_{Y*}[\pi_X^{-1}(A) \cap \Gamma_f].$$

If A is generically transverse to f then $\pi^{-1}(A)$ is generically transverse to Γ_f, so

$$[\pi_X^{-1}(A) \cap \Gamma_f] = [\pi_X^{-1}(A)]\,[\Gamma_f],$$

and thus

$$[f^{-1}(A)] = (\pi_Y)_*([\pi^{-1}(A)]\,[\Gamma_f]) = f^*[A],$$

as required.

To show that f^* is a ring homomorphism, let $\alpha, \beta \in A(X)$. We claim that α and β can be represented by cycles A and B that are generically transverse to one another such that, in addition, A, B and $A \cap B$ are generically transverse to f. Indeed, let $\{B_i\}$ be, as in Lemma 5.28, a finite collection of cycles such that a cycle is generically transverse to f if it is generically transverse to each B_i. By the Moving Lemma we can choose a cycle A representing α and generically transverse to each B_i. Again by the Moving Lemma we can then choose B generically transverse to A, to all the B_i and to all the cycles $A \cap B_i$. It follows that $A \cap B$ is generically transverse to each B_i as claimed.

With these choices $f^{-1}(A)$ and $f^{-1}(B)$ intersect generically transversely and hence

$$\begin{aligned}(f^*\alpha)(f^*\beta) &= [f^{-1}(A)][f^{-1}(B)]\\ &= [f^{-1}(A) \cap f^{-1}(B)]\\ &= [f^{-1}(A \cap B)]\\ &= f^*(\alpha\beta).\end{aligned}$$

This completes the proof of Part (a) of Theorem 5.8

Part (b) The definition of the pullback map allows us to reduce statements about intersection products to statements about the intersections of subsets. For example, the push-pull formula

$$f_*\big(f^*(\alpha)\beta\big) = \alpha f_*(\beta)$$

corresponds to the set-theoretic equality

$$f(f^{-1}(A) \cap B) = A \cap f(B),$$

for subsets $A \subset X$ and $B \subset Y$. Of course to prove the equality of cycles that corresponds to the push-pull formula, we need to have enough transversality (which is possible through the Moving Lemma) and to take care of degrees.

To carry this out, note first that the formula is linear in β, so we can immediately reduce to the case where β is the class of an irreducible subvariety B. We may assume by the Moving Lemma that B is generically transverse to the generic fiber of f and thus is generically separable over its image in X (of course this is automatic in characteristic 0). Using the Moving Lemma and linearity, together with Lemma 5.28, we may assume further that $\alpha = [A]$ where $A \subset X$ is a variety generically transverse to f to $f|_B$, and to $f(B)$. Because A is generically transverse to $f|_B$ we see that $f^{-1}(A)$ is generically transverse to B.

Suppose now that B is generically finite of degree d over $f(B)$. Because A is transverse to $f|_B$ the cycle $f^{-1}(A) \cap B$ is also generically finite, and of the same degree d, over its image $A \cap f(B)$. In this case, the proof is concluded by the series of equalities below, whose justifications are given on the right.

$$\begin{aligned}
f_*(f^*(\alpha)\beta) &= f_*([f^{-1}(A)]\beta) \qquad A \text{ generically transverse to } f\\
&= f_*([f^{-1}(A) \cap B]) \qquad f^{-1}(A) \text{ generically transverse to } B\\
&= d[f(f^{-1}(A) \cap B)] \qquad \text{definition of } f_*\\
&= d[A \cap f(B)] \qquad \text{set-theoretic equality}\\
&= [A \cap f_*(B)] \qquad \text{definition of } f_*\\
&= \alpha f_*(\beta) \qquad A \text{ generically transverse to } f(B).
\end{aligned}$$

If on the other hand B is not generically finite over $f(B)$, then by the semi-continuity of fiber dimension, $f^{-1}(A) \cap B$ is not generically finite over $A \cap f(B)$. Thus the chain of equalities above is valid with $d = 0$. □

Proof of Proposition 5.5. As in the last part of the proof of part a) of Theorem 5.8, we see from Lemma 5.28 that any classes $\alpha, \beta \in A(X)$ may be represented by cycles A, B such that A, B are generically transverse and A, B and $A \cap B$ are generically transverse to f. It follows that $f^{-1}(A)$ is generically transverse to $f^{-1}(B)$, so while $\alpha\beta$ is represented by $A \cap B$ $f^*(\alpha)f^*(\beta)$ is represented by $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B)$. □

5.8 Multiplicity in the Cohen-Macaulay case

The notion of ambient cycle, introduced in the proof of the Moving Lemma, is a generally useful one; as an application, we can give proofs of Theorems 5.10 and 5.12 on intersections and pullbacks with multiplicities.

Proof of Theorem 5.10. By Theorem 5.23 there is an equality of cycles $\langle A\rangle = \langle D\rangle - \langle A'\rangle$ where D is an ambient cycle and A' is a scheme that is generically transverse to B and does not contain any component of $A \cap B$. Since A' is generically transverse to B, the intersection $A' \cap B$ is generically Cohen-Macaulay. Because of this we do not change the associated cycle $D \cap B$ by removing from X the closed subset of $D \cap B$ where it is not Cohen-Macaulay. Thus, we may assume from the outset that $D \cap B$ is Cohen-Macaulay.

Since D is ambient there is a subscheme $\mathcal{D} \subset \mathbb{P}^1 \times X$ whose fiber $\mathcal{D}_0$ over $0 \in \mathbb{P}^1$ is D and whose fiber over a general $t \in \mathbb{P}^1$ is generically transverse to B. Consider the subscheme $\mathcal{B} := \mathcal{D} \cap (\mathbb{P}^1 \times B)$. The fiber B_0 of $\mathcal{B}$ over $0 \in \mathbb{P}^1$ is $D \cap B$. Since B_0 is a Cartier divisor in $\mathcal{B}$, it follows that $\mathcal{B}$ is Cohen-Macaulay in a neighborhood of B_0, and thus flat over a neighborhood of $0 \in \mathbb{P}^1$. Thus $\mathcal{B}$ is a rational equivalence between $[B_0]$ and $[B_t]$ for generic $t \in \mathbb{P}^1$.

It follows that

$$[A \cap B] + [A' \cap B] = [\mathcal{B}_0] = [\mathcal{B}_t] = [D_t \cap B].$$

By generic transversality,

$$[D_t \cap B] = [D_t][B] = [D_0][B] = [A][B] + [A'][B].$$

Again by generic transversality $[A' \cap B] = [A'][B]$, concluding the proof. $\square$

Proof of Theorem 5.12. Essentially the same construction as in the proof of Theorem 5.10 applies here as well: there is an equality of cycles $\langle A\rangle = \langle D\rangle - \langle A'\rangle$ on Y where D is an ambient cycle and $A' \subset Y$ is a scheme that is generically transverse to the map f. This yields as before a family of cycles $\mathcal{A} \subset \mathbb{P}^1 \times Y$ with $A_0 = A$ and A_t generically transverse to f. Given that A is Cohen-Macaulay and the condition $\operatorname{codim}(f^{-1}(A)) = \operatorname{codim}(A)$ is satisfied, it follows that $f^{-1}(A) \subset X$ is also Cohen-Macaulay, and hence that the pre-image $\mathcal{B} := f^{-1}(\mathcal{A}) \subset \mathbb{P}^1 \times X$ is Cohen-Macaulay in a neighborhood of its fiber B_0. The family $\mathcal{B}$ thus gives a rational equivalence between $[B_t] = [f^{-1}(A_t)] = f^*[A]$ and $[f^{-1}(A)]$, as desired. $\square$

5.9 Exercises

Exercise 5.29. Let $S \subset \mathbb{P}^3$ be a smooth surface of degree d, and $L \subset S$ a line. Find the degree of the self-intersection $[L]^2 \in A^2(S)$ by applying the cone construction (in other words, considering the intersection of S with a general plane containing L).

Exercise 5.30. Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface, and C a twisted cubic curve lying on Q; let $c \in A^1(Q)$ be its class. Use the moving lemma construction to find the self-intersection c^2 in the Chow ring of Q.

Exercise 5.31. Show that if $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$ is the d-th Veronese mapping and $L \subset \mathbb{P}^N$ is a linear subspace of dimension $e < d$, then the length of the scheme $L \cap X$ is at most $e + 1$.

Exercise 5.32. Let R be a commutative ring, and let P, Q be ideals of R. Show that $\mathrm{Tor}_1^R(R/P, R/Q) = (P \cap Q)/(P \cdot Q)$.

Hint: Consider the long exact sequence obtained by applying the functors $\mathrm{Tor}_*^R(R/P, -)$ to the short exact sequence

$$0 \to Q \to R \to R/Q \to 0.$$

Exercise 5.33. In Example 5.13, calculate the length of the Tor modules in Serre's formula, and verify that the correct multiplicity $m_p(A, B_2)$ is 2. (Hint: this can be calculated directly, using the fact that all the ideals involved are monomial; alternatively, we can take the Koszul resolution of B_2 and tensor with $\mathcal{O}_{A,p}$.)

Exercise 5.34. Let $f : Y \to X$ be a generically finite surjective morphism of smooth projective varieties. Show that the composition

$$f_* f^* : A_k(X) \to A_k(X)$$

is multiplication by the degree of f; deduce in particular that the map $f^* : \mathbb{Q} \otimes A_k(X) \to \mathbb{Q} \otimes A_k(Y)$ is injective. Show by example that $f^* f_* \alpha$ may not be a multiple of α, so no analogous statement holds.

6

Interlude: Vector Bundles and Direct Images

We will now introduce some ideas that are of the utmost usefulness in intersection theory, and in algebraic geometry generally: vector bundles and direct images. We will use these ideas to solve enumerative problems by *linearizing* them: replacing a polynomial equation with a family of systems of linear equations, that is, a family of linear maps between families of vector spaces parametrized by the points of a variety B. It is not an exaggeration to say that in the rest of this book the different chapters are organized around the treatment of problems using different families of vector bundles.

After introducing the main idea with an example, we review the ideas of vector bundles and pullbacks. Then we present the notion of the direct image in a more systematic and leisurely way.

6.1 How vector bundles arise in intersection theory

****Make sure that normal bundles are mentioned. Say that they play a role in the description of excess intersections and are basic to blowups, both done in Ch 15 – excess intersections and blowups.****

For the purpose of this section we will assume that the reader is familiar with the language of vector bundles; we will review the necessary material later in this Chapter.

As an example, we will consider the problem of finding the number of lines on a smooth cubic surface in $\mathbb{P}^3$. We will pursue results of this type further in Chapter 8. To solve the problem by linearizing, we consider the Grassmannian $B = \mathbb{G}(1,3)$ of lines L in $\mathbb{P}^3$, and for each such L we let E_L be the space of homogeneous cubic polynomials on L. A cubic form f on $\mathbb{P}^3$ gives, by restriction, a cubic form f_L on each L, that is, a family of elements of the E_L. The element f_L is zero if and only if the form f vanishes on the line L; that is, if and only if L is contained in the cubic surface $f = 0$ in $\mathbb{P}^3$. Thus the number of lines on the cubic surface can be computed as the number of points $L \in B$ such that the element f_L vanishes.

So far we have really only renamed the problem. The next step is the crucial one: it is to "put the E_L together" into a vector bundle E on B whose fiber over the point corresponding to L is E_L, in such a way that the elements f_L become an algebraic section. As we shall see in Chapter 7, the theory of Chern classes will then make it easy to analyze the number of zeros of a section of a vector bundle. With it we can immediately compute the number 27.

To construct the vector bundle E we first note that the spaces E_L are themselves related to vector bundles: they are the spaces of global sections of the line bundles $\mathcal{O}_L(3)$ of cubic forms on the lines L, that is $E_L = H^0(L, \mathcal{O}_L(3))$. Moreover, $\mathcal{O}_L(3)$ is itself the restriction of the bundle $\mathcal{O}_{\mathbb{P}^3}(3)$ from $\mathbb{P}^3$ to L. We can make this more useful by considering the incidence correspondence

$$X = \{(L,p) \in B \times \mathbb{P}^3 \mid p \in L\},$$

and the projections $\pi_B : X \to B$ and $\pi_{\mathbb{P}^3}$ onto the two factors. The bundle $\mathcal{O}_L(3)$ appears here as the restriction of the vector bundle $\pi_{\mathbb{P}^3}^*(\mathcal{O}_{\mathbb{P}^3}(3))$ to the fiber $L = \pi_B^{-1}([L])$ of the map π_B.

Stated in this way, the problem fits into a general framework: given a map $\pi : X \to B$ of varieties and a vector bundle (or, more generally, a coherent sheaf) $\mathcal{F}$ on X, we would like to construct a vector bundle or sheaf $\mathcal{G}$ on B whose fiber $\mathcal{G}_b := \kappa(b) \otimes_{\mathcal{O}_B} \mathcal{G}$ at a point $b \in B$ is $H^0(\mathcal{F}|_{\pi^{-1}}(b))$? Does such a coherent sheaf exist? The answer is "no" in general, but "yes" in a range of useful cases, including the one above.

In general, there is a bundle $\pi_*\mathcal{F}$ on B that "best approximates" $\mathcal{F}$ (in the sense that there is a map $\pi^*\pi_*\mathcal{F} \to \mathcal{F}$ with a universal property); it is called the *direct image* of $\mathcal{F}$. For example, if B is a point, then $\pi_*\mathcal{F}$ is the space of global sections $H^0(\mathcal{F})$ (and this is true more generally when B is an affine scheme.

We will give the definition in Section 6.5, after we review the notion of vector bundles and the simpler idea of pulling back a bundle. There we will also state a condition under which $\pi_*\mathcal{F}$ is a vector bundle with fiber $(\pi_*\mathcal{F})_b$ equal to the space of of global sections $H^0(\mathcal{F}|_{\pi^{-1}(b)})$. This condition depends crucially on the notion of flatness, so we review the geometric content of that notion in Section 6.4.

6.2 Vector bundles and locally free sheaves

We will review some basic definitions and their consequences. For a more complete treatment, see for example Hartshorne [1977].

- A *vector bundle* of rank n on a scheme X is a scheme E and a morphism $\pi : E \to X$ with a section $\sigma : X \to E$ called the zero section, such that, for some open affine cover $\{U_i\}$ of X there are isomorphisms $\alpha_i : E_{U_i} := \pi^{-1}(U_i) \to \mathbb{A}^n_{U_i} = V \times U_i$, where V_i is a vector space, and such that for each $x \in U_i \cap U_j$ the induced map $\alpha_i(x)\alpha_j^{-1}(x) : \mathbb{A}^n \to \mathbb{A}^n$ is a linear isomorphism. A *section* of E (really of the pair (E, π)) over $U \subset X$ is a map $\sigma : U \to \pi^{-1}(U)$ such that $\pi\sigma = 1_U$.

The definition looks a little simpler in the case of a scheme X of finite type over an algebraically closed field K, the case treated in this book: in this case we may identify the closed points of the affine space $\mathbb{A}^n_{U_i}$ with the points of $V \times U_i$, where V is an n-dimensional vector space over K.

Sometimes—for example in the definition of the tangent bundle of a smooth variety—we don't give E and π globally, but only give the $E_{U_i} = V_i \times U_i$, with $\pi_i : E_{U_i} \to U_i$ defined as the projection. In this case we need to give isomorphisms

$$V_j \times U_j \xrightarrow{\beta_{i,j}} V_i \times U_i$$

(which, in our previous construction were $\beta_{i,j} = \alpha_i\alpha_j^{-1}$) such that for each $x \in U_i \cap U_j$ the induced map $\beta_{i,j}(x) : V_j \to V_i$ is linear; put informally, the $\beta_{i,j}(x)$ are regular functions from $U_i \cap U_j$ to the vector space $Hom(V_j, V_i)$ with values in the set of isomorphisms. These maps must satisfy a compatibility condition on triple overlaps:

$$\beta_{i,j}\beta_{j,k} = \beta_{i,k} \quad \text{ on } U_i \cap U_j \cap U_k.$$

Given this compatibility, we could construct the scheme E as the quotient of the disjoint union of the $V_i \times U_i$ by the equivalence relation generated by the $\beta_{i,j}$.

- A *locally free sheaf* of rank n on a scheme X is a coherent sheaf $\mathcal{E}$ on X such that for some open affine cover $\{U_i\}$ of X there are isomorphisms $\mathcal{E}|_{U_i} \cong \mathcal{O}_X(U_i)^n$. A *section* of $\mathcal{E}$ over U is an element of $\mathcal{E}(U)$.

Note that we are only dealing with vector bundles of finite rank, and locally free sheaves that are coherent. Nothing prevents the extension of these notions to the infinite rank/quasicoherent case, but we will never need this, and we will assume finite rank or coherence throughout.

A locally free sheaf $\mathcal{E}$ can be made from a vector bundle E by taking the sheaf of sections: $\mathcal{E}(U) := H^0(E|_U)$. A vector bundle E can be made from a locally free sheaf $\mathcal{E}$ by setting $E = \operatorname{Spec}\operatorname{Sym}(\mathcal{E}^*)$, the global spectrum of the symmetric algebra of the dual of $\mathcal{E}$.

This construction becomes clearer if we work locally: on an affine open set U of X where $\mathcal{E}$ is free we have $\mathcal{E}|_U = \mathcal{O}_X(U)^n$. Thus $(\mathcal{E}_|U)^* = \mathcal{E}^*|_U$ may be thought of as the linear functions on the fibers of the trivial vector bundle over U, $U \times K^n$, with coefficients that are regular functions on U. Thus $\operatorname{Sym}(\mathcal{E}^*)|_U = \operatorname{Sym}((\mathcal{E}|_U)^*)$ is the ring of polynomial functions on the fibers with coefficients that are regular functions on U, as required.

These identifications, which are inverse to one another, show that the notions of "vector bundle" and "locally free coherent sheaf" are equivalent. *From now on we will use the terms and the notation interchangeably.*

The category of vector bundles on a scheme X is closed under the functors Hom and $\otimes_X$, interpreted fiberwise (in the first description) or, equivalently, as operations on coherent sheaves. However, the category of vector bundles is not generally an abelian category: the quotient $\mathcal{E}/\mathcal{F}$ of locally free sheaves may not be locally free. From the vector bundle point of view, a map $\varphi : E \to F$ of vector bundles over X (that is, a morphism of schemes that commutes with the projection to X and is linear on each fiber), may have the same rank (as a linear transformation) on all the fibers; in this case we say that φ *has constant rank*, and its fiberwise kernel and cokernel are again vector bundles. But it may also have ranks that vary with the fibers; then its kernel and cokernel can be regarded as coherent sheaves, but not as vector bundles. We say that E is a subbundle of F if $E \to F$ has constant rank equal to the rank of E, and similarly for a quotient bundle.

Vector bundles are often described informally through what one might call the "Principal of Naturality". One starts with a space X and a "natural" way of describing a vector space of dimension r for each point of the space. If the description is natural enough, these spaces turn out to be identified with the fibers of a vector bundle over X; one feels that it simply could not be otherwise. This observation is experiential rather than mathematical, but the reader will see many examples below. The rigorous treatments usually involve direct images and the Theorem on Cohomology and Base Change, treated in Section 6.5.

6.3 Pullbacks

For a detailed treatment of the ideas in this section see Hartshorne [1977] Section II.5.

Given a morphism of varieties $\pi : X \to B$ and a vector bundle or coherent sheaf $\mathcal{G}$ on B, we wish to define a natural vector bundle $\mathcal{F} = \pi^*\mathcal{G}$ on X, called the pullback of $\mathcal{G}$ under the map π, whose fiber at a closed point $p \in X$ is the same as the fiber of $\mathcal{G}$ at $\pi(p)$.

In fact, given a (quasi)coherent sheaf $\mathcal{G}$ on B, there is a (quasi)coherent sheaf $\pi^*\mathcal{G}$ on X characterized by the property that for each pair of open affine sets $U \subset X$ and $V \subset B$ such that $\pi(U) \subset V$ we have $(\pi^*\mathcal{G})(U) = \mathcal{O}_X(U) \otimes_{\mathcal{O}_B(V)} \mathcal{G}(V)$ as $\mathcal{O}_X(U)$ modules. Existence is proven by observing that the module $(\pi^*\mathcal{G})(U)$ defined above is independent of the choice of V, and glues appropriately along open coverings. For example, it follows at once from the definitions that if $\pi : X \to B$ is a morphism of schemes, then $\pi^*\mathcal{O}_B = \mathcal{O}_X$.

To check that this construction implies the desired property for vector bundles, note that the fiber of $\pi^*\mathcal{G}$ at a closed point $x \in U$ is by definition $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x} \otimes_{\mathcal{O}_{X,x}} \pi^*\mathcal{G}(U)$, and

$$\begin{aligned}
&\mathcal{O}_{X,x}/\mathfrak{m}_{X,x} \otimes_{\mathcal{O}_{X,x}} \pi^*\mathcal{G}(U) \\
&\qquad = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x} \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_X(U) \otimes_{\mathcal{O}_B(V)} \mathcal{G}(V) \\
&\qquad = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x} \otimes_{\mathcal{O}_B(V)} \mathcal{G}(V).
\end{aligned}$$

Since X and B are varieties, and we are always assuming that the ground field K is algebraically closed, the Nullstellensatz shows that

$$\mathcal{O}_{B,\pi(x)}/\mathfrak{m}_{B,\pi(x)} = K = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$$

and since the map induced by π is K-linear,

$$\mathcal{O}_{X,x}/\mathfrak{m}_{X,x} \otimes_{\mathcal{O}_B(V)} \mathcal{G}(V) = \mathcal{O}_{B,\pi(x)}/\mathfrak{m}_{B,\pi(x)} \otimes_{\mathcal{O}_B(V)} \mathcal{G}(V);$$

that is, the fiber at x of $\pi^*\mathbb{G}$ is the same as the fiber of $\mathbb{G}$ at $\pi(x)$.

6.4 Flatness

We define a *family* of schemes is simply a morphism $\pi : X \to B$. But for the fibers to have much "family resemblance" some condition is necessary. We could guarantee a very strong resemblance by assuming that π is locally a product over B in some sense[1] but for many purposes in algebraic geometry

[1] Perhaps in the analytic topology, if we were working over $\mathbb{C}$, or in the étale topology generally.

this notion is too restrictive; for example it eliminates families in which smooth varieties degenerate to singular varieties, such as the family of plane curves of a given degree. Serre introduced a very algebraic but much more flexible notion that has proven itself in countless applications: *flatness.*

We say that X is flat over B or, more precisely, that π is a *flat* morphism, if for every point $x \in X$ the local ring $\mathcal{O}_{X,x}$ is a flat module over the local ring $\mathcal{O}_{B,\pi(x)}$ via the map $\mathcal{O}_{B,\pi(x)} \to \mathcal{O}_{X,x}$ taking a germ h of a function on B to the germ $h \circ \pi$ on X. This means that for every monomorphism $M' \to M$ of $\mathcal{O}_{B,\pi(x)}$-modules, the induced map $\mathcal{O}_{X,x} \otimes M' \to \mathcal{O}_{X,x} \otimes M$ is again a monomorphism.

To give some insight into this opaque definition, we will briefly review some basic results about flatness. For a more extended discussion with proofs of most of the assertions below, see for example Eisenbud and Harris [2000].)

If $X \to B$ is any morphism, with B smooth and one-dimensional, then X is flat over b_0 as long as no component of X (either irreducible or embedded) is supported on the fiber X_{b_0}.

Suppose that $\pi : X \to B$ is an arbitrary morphism from a variety X to a 1-dimensional smooth variety B, and $b \in B$ is a point. We define the *limit* $\lim_{b \to b_0} X_b$ of the family

$$X \setminus \pi^{-1}(b) \to B \setminus \{b\}$$

to be the fiber over b in the closure $Y = \overline{X} \setminus \pi^{-1}(b)$. In these terms, the statement above says that a morphism $X \to B$ to any smooth, one-dimensional target B is flat if and only if every fiber X_{b_0} is equal to the limit $\lim_{b \to b_0} X_b$.

It is crucial for the definition of limit that the base B be a smooth curve; otherwise the limit may fail to exist. For example, the normalization map $X \to B$ of a reduced curve B that is singular at a point $b \in b$ is never flat at b, and there is no way to define the limit$\lim_{b \to b_0} X_b$ as a subvariety of X. Again, if $X \to \mathbb{P}^2$ is the blowup of a point p, then the limit $\lim_{b \to p} X_b$ will not exist because if we approach p along different smooth curves in $\mathbb{P}^2$ the (well-defined) limit along the curve will depend on the direction of approach.

It is useful to observe that flatness is preserved by *base change*. That is, if

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ B' \arrow[r] & B \end{tikzcd}$$

is a pull-back diagram of schemes, and π is flat, then so is π'. This follows at once because if M is a flat module over a ring R and R' is an R algebra, then $R' \otimes M$ is a flat R' module.

Using base change we can extend the characterization above to higher-dimensional targets B: one can show that a morphism $X \to B$ to any reduced scheme Y is flat if and only if for any morphism $\iota : B' \to B$ with B' smooth and one-dimensional, the pullback $X \times_B B' \to B'$ is flat—that is, $\lim_{b \to b_0} X_{\iota(b)} = X_{\iota(b_0)}$; see Eisenbud and Harris [2000], Lemma II-30, for a proof. The algebraic definition of flatness given above extends this very geometric notion to the general setting of morphisms to possibly nonreduced schemes B.

This characterization makes clear that flatness is not a special condition, but rather one we would expect to hold in general. This is borne out by the *generic flatness theorem* of Grothendieck, which says that if one has any reasonable family of schemes $f : X \to B$ over a reduced base, then there is an open dense subset $U \subset B$ such that the restricted family $f : f^{-1}(U) \to U$ is flat (here "reasonable" includes, for example, any family of subschemes of a fixed affine or projective space). See for example Eisenbud [1995] Section 14.2.

The condition of flatness has nontrivial implications, the most striking of which is the *preservation of the Hilbert polynomial*: if $f : X \subset B \times \mathbb{P}^n$ is a flat family of projective schemes over B, then the Hilbert polynomial of the fibers X_b is locally constant. One consequence of this is the observation that if $f : X \to B$ is flat then the fiber dimension must be locally constant. (This can also be seen directly in general: if the fiber X_{b_0} over $b_0 \in B$ has dimension strictly greater than $\dim X_b$ for general b, then for a general arc $\Delta \to B$ through b_0, the fiber X_{b_0} will be an irreducible component of the pullback $X \times_B \Delta$.)

Under mild hypotheses, the preservation of the Hilbert polynomial is actually equivalent to flatness:

Proposition 6.1. *If B is reduced, and $X \subset B \times \mathbb{P}^n$ any closed subscheme, then the projection $\pi : X \to B$ is flat if and only if the Hilbert polynomial of the fiber X_b is locally constant.*

See Eisenbud and Harris [2000], Theorem III-56 for the most important special case, or Hartshorne [1977], Theorem 9.9, for the general case.

Flatness is closely connected to the Cohen-Macaulay condition as well: if B is smooth and $f : X \to B$ any morphism with constant fiber dimension, then f is flat if and only if X is Cohen-Macaulay (Hartshorne [1966]).

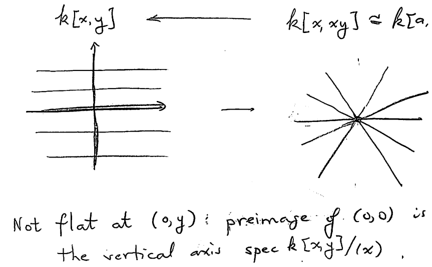


FIGURE 6.1. Not flat at $(0, y)$: the preimage of the origin $(0, 0)$ is the whole y-axis $\operatorname{Spec} k[x, y]/(x)$

6.4.1 Examples of nonflat morphisms

The most visible way in which a morphism may fail to be flat is if the dimensions of the fibers are not locally constant. The simplest example of this is the blow-up of the plane $\mathbb{A}^2 = \operatorname{Spec} k[a, b]$ at a point; here the general fiber is just one point, while the fiber over the blown up point is a line.

Morphisms with constant fiber dimension can still fail to be flat if the Hilbert polynomial of the fiber is nonconstant. For example, a finite map $f : X \to Y$ will fail to be flat if the degrees of the fibers $f^{-1}(q) \subset X$ is nonconstant. Perhaps the simplest example of this is the normalization of a singular curve: in the case of the nodal curve $C = V(b^2 - a^2(a + 1) \subset \mathbb{A}^2 = \operatorname{Spec} k[a, b]$, for example, this would be the map $\mathbb{A}^1 \to C$ sending x to $(x^2 - 1, x(x^2 - 1))$, as shown in Figure 6.2. Here the preimage of a general point in C is just one reduced point, but the fiber over the origin $p = (0, 0) \in C$ consists of two points; the map correspondingly cannot be flat over the origin. Another way to see this is to observe that if we pull the family back via the map f itself—that is, we consider the morphism $\mathbb{A}^1 \times_C \mathbb{A}^1 \to \mathbb{A}^1$—we see that the fiber product consists of the diagonal in $\mathbb{A}^1 \times \mathbb{A}^1$ plus the two isolated points $(1, -1)$ and $(-1, 1)$, and so can't be flat over $\mathbb{A}^1$.

The normalization $\mathbb{A}^1 \to C$ of a cuspidal plane cubic is another non flat example; even though the map is set-theoretically 1-1, the fiber over the cusp is a nonreduced point, so the fiber degree is higher there. Again we can see nonflatness in another way: the pullback $\mathbb{A}^1 \times_C \mathbb{A}^1 \to \mathbb{A}^1$ can't be flat because the fiber product $\mathbb{A}^1 \times_C \mathbb{A}^1$ has an embedded point at the point lying over the cusp. In fact no desingularization $\nu : \tilde{X} \to X$ of a singular variety X is flat, since the map is generically an isomorphism but there are fibers that have larger degree or dimension.

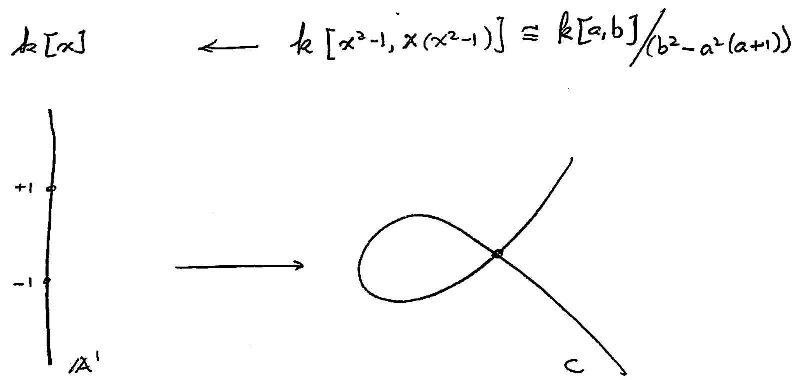


FIGURE 6.2. Not flat at $(0, \pm 1)$: the preimage of the origin $(0, 0)$ is the pair of points $\operatorname{Spec} k[x]/(x^2 - 1)$

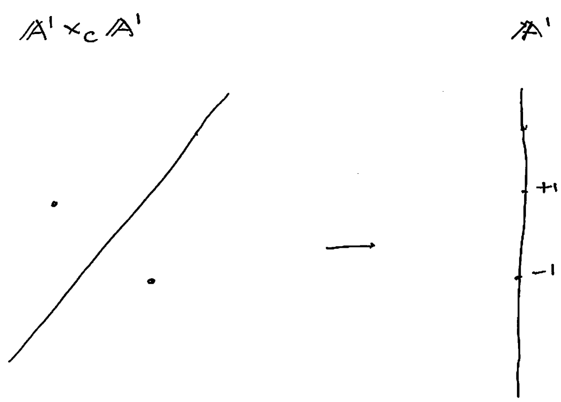


FIGURE 6.3. Not flat at $(0, \pm 1)$: the fiber product $\mathbb{A}^1 \times_C \mathbb{A}^1$ has isolated points

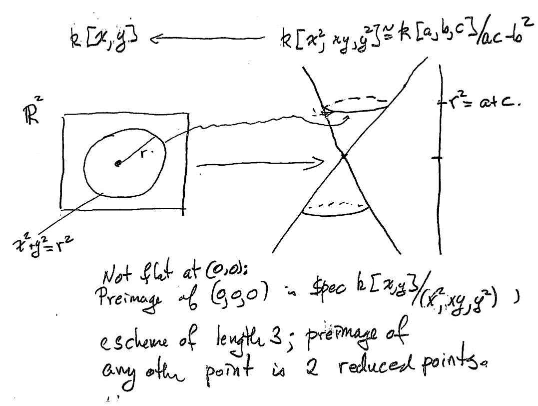


FIGURE 6.4. Not flat at $(0,0)$: the preimage of $(0,0,0)$ is the fat point $\operatorname{Spec} k[x,y]/(x^2,xy,y^2)$; the preimage of any other point is two reduced points

Another example is the map f from the plane $\mathbb{A}^2 = \operatorname{Spec} k[x,y]$ to its quotient by the involution $\tau : (x,y) \mapsto (-x,-y)$. The quotient is the quadric cone $Q = \operatorname{Spec} k[x^2,xy,y^2] \cong \operatorname{Spec} k[a,b,c]/(ac-b^2)$; the map is depicted in Figure 6.4. Here the preimage of any point in Q other than the origin is two reduced points, while the preimage of the point $(0,0,0)$ is the fat point $V(x^2,xy,y^2) \subset \mathbb{A}^2$, which has degree 3; thus the map can't be flat.

Another way to interpret the non-flatness of f in this case is to think of the quotient Q as the space parametrizing subschemes of $\mathbb{A}^2$ consisting of a pair of antipodal points $\{(x,y),(-x,-y)\}$, and $\mathbb{A}^2$ as the universal family over this parameter space. The point is, the flat limit of the scheme $\{(x,y),(-x,-y)\}$ as $(x,y) \to (0,0)$ could be any scheme of degree 2 supported at the origin, depending on the direction of approach; the actual fiber must therefore contain the union of all such subschemes, which is to say the fat point.

A final example of this phenomenon is the family of degree 2 subschemes of the plane $\mathbb{A}^2 = \operatorname{Spec} k[x,y]$, consisting of the union of the origin $(0,0)$ and a second point (a,b). This is a family parametrized by the plane $\mathbb{A}^2 = \operatorname{Spec} k[a,b]$, with equation

$$\Sigma = V\big((x,y)(x-a,y-b)\big) \subset \mathbb{A}^2 \times \mathbb{A}^2;$$

the map is depicted (crudely) in Figure 6.5.

As intended, the fiber of the projection $\Sigma \to \mathbb{A}^2_{a,b}$ over a point $(a,b) \neq (0,0)$ is just the reduced subscheme $\{(0,0),(a,b)\} \subset \mathbb{A}^2_{x,y}$. But the fiber over the origin is the fat point $V(x^2,xy,y^2) \subset \mathbb{A}^2_{x,y}$, which has degree 3; thus the map is not flat at that point. This reflects the fact that there is no

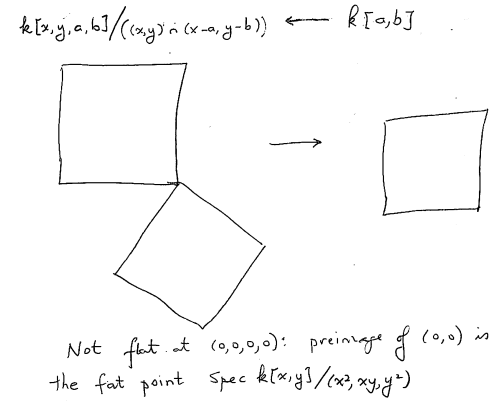


FIGURE 6.5. Not flat at $(0,0,0,0)$: the preimage of $(0,0)$ is the fat point $\operatorname{Spec} k[x,y]/(x^2, xy, y^2)$

well-defined limit of the schemes $\{(0,0),(a,b)\} \subset \mathbb{A}^2_{x,y}$ as $(a,b) \to (0,0)$. If (a,b) approaches $(0,0)$ along a line $b = \lambda a$, the limiting scheme is the degree 2 subscheme $V(x^2, xy, y^2, y - \lambda x)$. But that depends on λ, so that the actual fiber of Σ over $(0,0)$ has to contain the union of all these schemes—that is, the fat point.

Note that in this last case, the source is singular, while the target is smooth; in the preceding example, the reverse was true. But the principle in both cases is the same: since the limit of the fibers depends on the direction of approach, the map can't be flat.

6.4.2 Flatness for sheaves

It is useful to extend the definition of flatness to sheaves on the total space of a family. If $\pi : X \to B$ is any morphism of schemes and $\mathcal{F}$ is a coherent sheaf on X, then we say that $\mathcal{F}$ is flat over B if for every point $x \in X$ the stalk $\mathcal{F}_x$ is a flat module over the local ring $\mathcal{O}_{B,\pi(x)}$ via π.

One of the nice consequences of flatness, which we will prove in Section 6.7, is that the restrictions of a flat sheaf to different fibers have the same Euler characteristic:

Theorem 6.2. *Let $\pi : X \to B$ be a projective morphism of schemes, with B connected, and let $\mathcal{F}$ be a coherent sheaf on X that is flat over B. The Euler characteristic*

$$\chi(\mathcal{F}|_{\pi^{-1}(b)}) = \sum_i (-1)^i \dim H^i(\mathcal{F}|_{\pi^{-1}(b)})$$

is independent of the closed point $b \in B$.

If B is reduced and π is a projective morphism, that is, the projection of $X \subset \mathbb{P}^n_B$ to B, then the constancy of the Euler characteristic for all $\big(\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n_B}(d)\big)|_{\pi^{-1}(b)}$ is actually necessary and suffient for flatness; see for example Eisenbud and Harris [2000] III-56, where the result is proved in the special case $\mathcal{F} = \mathcal{O}_X$; the proof in general follows exactly the same lines. We will not need this fact.

6.5 Direct images

Much of the material in this section is derived from Mumford [2008].

Suppose that we are given a family of varieties and a family of sheaves on them—in other words, a morphism $X \to B$ and a sheaf $\mathcal{F}$ on X (we remind the reader of our global convention that "sheaf" means coherent sheaf unless otherwise stated). For many purposes in this book we would *like* to define a sheaf $\mathcal{G}$ on B whose fiber at each point $b \in B$ is the space $H^0(\mathcal{F}|_{X_b})$ of global section of $\mathcal{F}$ on the fiber X_b of X over b, and such that algebraic families of elements of $H^0(\mathcal{F}|X_v)$ give rise to sections of $\mathcal{G}$ in a way that is compatible with the identification of $\mathcal{G}_b$ with $H^0(\mathcal{F}|_{X_b})$. If all the $H^0(\mathcal{F}|_{X_b})$ have the same dimension, and B is a variety, then such a sheaf would be a vector bundle, and this will be an important source of vector bundles.

Does such a sheaf exist?

Often the answer is "yes". It is reasonable to interpret the phrase "algebraic family of elements" in the description above to mean "section defined over an open set of B", and this idea leads naturally to a construction of the sheaf, called the *direct image* of $\mathcal{F}$, that is the best hope:

Definition 6.3. Given a morphism of schemes $\pi : X \to B$ and a sheaf $\mathcal{F}$ on X, we define the *direct image* $\pi_*\mathcal{F}$ of $\mathcal{F}$ to be the quasicoherent sheaf on B that assigns to each open subset $U \subset B$ the space of sections of $\mathcal{F}$ on the open set $X_U = \pi^{-1}(U)$.

A priori the definition defines a presheaf, but it is easy to see that $\pi_*\mathcal{F}$ is actually a sheaf, and also that, for an open set $U \subset X$, we have

$$H^0(\pi_*(\mathcal{F})) = \pi_*(\mathcal{F}(B)) = \mathcal{F}(\pi^{-1}(B)) = H^0(\mathcal{F}).$$

We also see that there is a natural map $\pi^*\pi_*\mathcal{F} \to \mathcal{F}$ which, restricted to the preimage of an open set $U \subset B$, is the tautogical map $\mathcal{O}_{X_U} \otimes H^0(\mathcal{F}|_{X_U}) \to \mathcal{F}$.

In this section we will explain conditions under which $\pi_*\mathcal{F}$ has the desired fibers. These are encapsulated in various forms of the "theorem on cohomology and base change", and we pause to explain this name.

We may think about the fiber X_b of π as coming from the pull-back diagram

$$\begin{tikzcd} X_b \arrow[r, "\rho'"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ \{b\} \arrow[r, "\rho"] & B \end{tikzcd}$$

and from the definition of π'_* we see that $\pi'_*(\mathcal{F}|_{X_b}) = H^0(\mathcal{F}|_{X_b})$. Thus we will have succeeded in finding a sheaf with the desired fibers if we can produce a family of isomorphisms

$$\varphi_b : (\pi_*\mathcal{F})_b \to \pi'_*(\mathcal{F}|_{X_b}) = H^0(\mathcal{F}|_{X_b}).$$

We shall see that there are natural maps φ_b, but that they are not always isomorphisms—nor are $\pi'_*(\mathcal{F})|_b$ and $\pi'_*(\mathcal{F}|_{X_b})$ always isomorphic.

More generally, for any map $\rho : B' \to B$, we can consider the pullback

$$\begin{tikzcd} X' \arrow[r, "\rho'"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ B' \arrow[r, "\rho"] & B \end{tikzcd}$$

of the family $X \to B$ to B'. This gives us a morphism

$$\pi' : X' = X \times_B B' \ \to \ B'$$

and a pullback sheaf

$$\mathcal{F}' = \rho'^*\mathcal{F}$$

on X'. In these circumstances we say that the new map $\pi' : X' \to B'$ and sheaf $\mathcal{F}'$ are obtained from the map $\pi : X \to B$ and sheaf $\mathcal{F}$ by *base change*. As in the special case we shall see that there are natural maps

$$\varphi_{B'} : \rho^*\pi_*\mathcal{F} \to \pi'_*\rho'^*(\mathcal{F})$$

and we ask under what conditions they are isomorphisms. In other words, does the formation of the direct image of $\mathcal{F}$ commute with base change?

There is an easy result of this sort: If ρ is a flat map, then the maps $\varphi_{B'}$ are always isomorphisms. Unfortunately, the inclusion of a point $\{b\} \subset B$ is never a flat map in cases of interest, so this result is not very useful. (The stated result is an easy consequence of Theorem 6.10 that we will not have occasion to apply. See Hartshorne [1977] Proposition III.9.3 for a simple proof.) The more interesting cases—when ρ is not flat—are addressed by versions of the "theorem on cohomology and base change" such as Theorem 6.14.

If $X = \operatorname{Spec} T$ and $B = \operatorname{Spec} S$ are affine, we can represent the sheaf $\mathcal{F}$ as the sheafification of the T-module $H^0(\mathcal{F})$, and it follows that $\pi_*\mathcal{F}$

is the same module—but now regarded as an S-module via the map $\pi^* : S \to T$. In particular, if T is not finite over S, then even $\pi_*\mathcal{O}_X$, which is the quasicoherent sheaf on B associated to the S-module T will not be coherent.

By contrast, a fundamental result of Serre, proved below (Theorem 6.9) shows that when $\pi : X \to B$ is a projective morphism (that is, factors as the inclusion of X as a closed subset of some $\mathbb{P}^n \times B$ and the projection to B) and $\mathcal{F}$ is a coherent sheaf, then $\pi_*\mathcal{F}$ is coherent. In particular this is true whenever both X and B are projective varieties. Thus our applications will be to projective morphisms $X \to B$. With a little more effort, the results can all be extended to proper morphisms; see Grothendieck [1963] Theorem 3.2.1.

By construction, a section of $\mathcal{F}$ over an open set $U \subset B$ gives rise to a section of $\pi_*\mathcal{F}$ over U, fulfilling one of our desiderata, and doing so in the most efficient way. In general the fiber $\pi_*\mathcal{F}$ at a point $b \in B$ will *not* be $H^0(\mathcal{F}|_{X_b})$, but at least we get a natural map: for any point $b \in B$ and any neighborhood U of b in B there is a restriction map

$$\pi_*\mathcal{F}(U) = \mathcal{F}(X_U) \ \to \ H^0(\mathcal{F}|_{X_b}).$$

The map $\pi_*\mathcal{F}(U) = \mathcal{F}(X_U) \ \to \ H^0(\mathcal{F}|_{X_b})$ induces a map from the fiber of $\pi_*\mathcal{F}$ at b to $H^0(\mathcal{F}|_{X_b})$, and thus a map

$$\varphi_b : (\pi_*\mathcal{F})|_b \ \to \ H^0(\mathcal{F}|_{X_b}).$$

It follows from the definition that the image of φ_b consists of those sections of $\mathcal{F}|_{X_b}$ that extend to sections of $\mathcal{F}|_{X_U}$ for some neighborhood U of $b \in B$, while the kernel consists of sections of defined in some small neighborhod U of $\pi^{-1}(b)$ that vanish on $\pi^{-1}(b)$, but cannot be expressed in terms of functions pulled back from any small neighborhood of b and vanishing at b.

Example 6.4. Let $L \subset \mathbb{P}^2$ be a line, and let $q, r \in \mathbb{P}^2$ be two points not lying on L. Let $X = L \times \mathbb{P}^2$, and write $\pi : X \to L$ and $\alpha : X \to \mathbb{P}^2$ for the two projections. Let $\Gamma = \Gamma_1 \cup \Gamma_2 \subset X$ where

$$\Gamma_1 = L \times \{q, r\} \quad \text{and} \quad \Gamma_2 = \{(p,p) : p \in L\}.$$

Let $\mathcal{F} = \alpha^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{I}_\Gamma$, and let p_0 be the unique point in L that lies on the line in $\mathbb{P}^2$ spanned by q, r (see Figure 6.6.)

The sheaf $\mathcal{F}|_{\pi^{-1}(p)}$ has a global section if and only if $p = p_0$, and $h^0(\mathcal{F}|_{\pi^{-1}(p)}) = 1$. Since this global section does not extend to a neighborhood of the fiber over p_0, we have $\pi_*\mathcal{F} = 0$, and in particular the map φ_{p_0} is not an isomorphism.

We will be particularly interested in cases where we can show that $\pi_*(\mathcal{F})$ is a vector bundle. To get an idea what to expect, consider the simplest

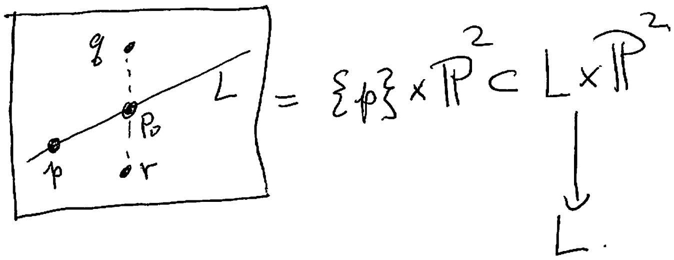


FIGURE 6.6. The fiber of Γ over $p \in L$ is the set $\{p, q, r\} \subset \{p\} \times \mathbb{P}^2$

case, where the map π is finite (and thus projective). In this simple situation it will always be the case that φ_b is an isomorphism for every b, and we can say just when $\pi_*(\mathcal{F})$ is a vector bundle—that is, M is a locally free S-module:

Proposition 6.5. *If $\pi : X \to B$ is a finite morphism of affine varieties, and $\mathcal{F}$ is a coherent sheaf on X, then the maps $\varphi_b : (\pi_*\mathcal{F})|_b \to H^0(\mathcal{F}|_{X_b})$ are isomorphisms for all closed points $b \in B$. Moreover, the following are equivalent:*

(a) *$\pi_*\mathcal{F}$ is a vector bundle on B.*

(b) *$\mathcal{F}$ is flat over B.*

(c) *The dimension of $H^0(\mathcal{F}|_{X_b})$ as a vector space over the residue class field $\kappa(b)$ is independent of the closed point $b \in B$.*

For an application of direct images for finite maps see Section 6.8.

Proof. Let $X = \operatorname{Spec} R$ and $B = \operatorname{Spec} S$, and let $M = H^0(\mathcal{F})$ be the R-module corresponding to $\mathcal{F}$. Because the varieties are affine, $\pi_*\mathcal{F}$ is represented by M regarded as a module over S via the map $\pi^* : S \to R$. The ring R is, by hypothesis, a finitely generated S-module, so M is finitely generated S-module as well. The maps φ_b are isomorphisms because, writing $\mathfrak{m}_b \subset S$ for the maximal ideal corresponding to b, both $(\pi_*\mathcal{F})|_b$ and $H^0(\mathcal{F}|_{X_b})$ may be identified canonically with $M/\mathfrak{m}_b M$. The equivalence of parts a) and c) is proven in Proposition 6.15 below. The equivalence of a) and b) is Eisenbud [1995] Exercise 6.2. $\square$

In case we suppose only that π is a projective morphism, not necessarily finite, neither condition b) nor c) of Proposition6.5 alone will imply either that $\pi_*\mathcal{F}$ is a vector bundle nor that the maps φ_b are isomorphisms. But conditions b) and c) together do imply both of these conclusions. This result is the most useful special case of the "Theorem on Cohomology and Base Change", Theorem 6.14.

Theorem 6.6 (Cohomology and Base Change—version 1)**.** *Let $\pi : X \to B$ be a projective morphism of varieties, and let $\mathcal{F}$ be a coherent sheaf on X that is flat over B. If the dimension of $H^0(\mathcal{F}|_{\pi^{-1}(b)})$ is independent of the closed point $b \in B$, then $\pi_*(\mathcal{F})$ is a vector bundle of rank equal to $h^0(\mathcal{F}|_{\pi^{-1}(b)})$, and the comparison map*

$$\varphi_b : \pi_*(\mathcal{F})|_b \to H^0(\mathcal{F}|_{\pi^{-1}(b)})$$

is an isomorphism for every closed point $b \in B$. More generally, if $\rho : B' \to B$ is any morphism of schemes, and

$$\begin{tikzcd} X' \arrow[r, "\rho'"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ B' \arrow[r, "\rho"] & B \end{tikzcd}$$

is a pullback diagram, then the natural map

$$\varphi_{B'} : \pi'_* \rho'^* \mathcal{F} \to \rho^* \pi_* \mathcal{F}.$$

is an isomorphism. □

We note that although we allow B' to be an arbitrary scheme, it is necessary to assume that B is a variety (being reduced would be enough).

Using Theorem 6.6 we can derive some results about line bundles on families of varieties that are particularly useful in the case of projective bundles (Chapter 11). We will make use of the following natural "adjunction" maps, defined for any morphism $\pi : X \to B$ of schemes:

(a) If $\mathcal{F}$ is a quasicoherent sheaf on X there is a natural map

$$\epsilon : \pi^* \pi_* \mathcal{F} \to \mathcal{F},$$

which, on the preimage of an affine set $U \subset B$ agrees with the map

$$\mathcal{O}_X(\pi^{-1}(U)) \otimes_{\mathcal{O}_Y(U)} H^0(\mathcal{F}|_U) \longrightarrow \mathcal{F}(\pi^{-1}(U))$$

sending each section to itself. This is well-defined because $\mathcal{F}(\pi^{-1}(U))$ is an $\mathcal{O}_X(\pi^{-1}(U))$-module.

(b) Given a quasicoherent sheaf $\mathcal{G}$ on B, there is a natural map

$$\eta : \mathcal{G} \to \pi_* \pi^* \mathcal{G},$$

which, on any open set $U \subset B$, agrees with the map

$$\begin{aligned} \mathcal{G}(U) = \mathcal{O}_B(U) \otimes_{\mathcal{G}(U)} &\xrightarrow{\pi^* \otimes 1} \mathcal{O}_X(\pi^{-1}(U) \otimes_{\mathcal{O}_B(U)} \mathcal{G}(U) \\ &= \pi^*(\mathcal{G})(\pi^{-1}(U) = (\pi_* \pi^* \mathcal{G})(U) \end{aligned}$$

Corollary 6.7. *Suppose $\pi : X \to B$ is a flat, projective morphism, and that all the fibers of π are reduced and connected.*

(a) $\pi_*\mathcal{O}_X = \mathcal{O}_B$.

(b) *If $\mathcal{L}, \mathcal{L}'$ are line bundles on X, then $\mathcal{L}|_{\pi^{-1}(b)} \cong \mathcal{L}'|_{\pi^{-1}(b)}$ for all $b \in B$ if and only if $\mathcal{L}$ and $\mathcal{L}'$ differ by tensoring with a line bundle pulled back from B.*

Remark. The result fails without flatness, for example in the case when π is the embedding of a proper closed subscheme of B.

Proof. To say that X is flat over B means that $\mathcal{O}_X$ is flat over B. Since flatness is a local property, this implies that any line bundle on X is flat over B, so we may apply Theorem 6.14 to line bundles on X.

(a): We will show that the natural map $\pi^* : \mathcal{O}_B \to \pi_*\mathcal{O}_X = \mathcal{O}_B$ is an isomorphism. Because X is flat, $1 \in \pi_*\mathcal{O}_X$ is not annihilated by any (local) section of $\mathcal{O}_B$. The map π^* takes $1 \in H^0(\mathcal{O}_B)$ to $1 \in H^0(\pi_*\mathcal{O}_X)$ so the map $\pi^* : \mathcal{O}_B \to \pi_*\mathcal{O}_X$ is injective.

To show surjectivity we use Theorem 6.6. Since $H^0\big((\mathcal{O}_X)|_{\pi^{-1}(b)}\big) = H^0(\mathcal{O}_{\pi^{-1}(b)})$ is 1-dimensional for every $b \in B$, the sheaf $\pi_*\mathcal{O}_X$ is a line bundle with fiber $H^0(\mathcal{O}_{\pi^{-1}(b)})$. Thus π^* is onto fiber by fiber. By Nakayama's Lemma, $\pi^* : \mathcal{O}_B \to \pi_*\mathcal{O}_X$ is surjective as desired.

(b): If $\mathcal{M}$ is a line bundle on B then $\big(\pi^*(\mathcal{M}) \otimes \mathcal{L}\big)|_{\pi^{-1}(b)} = \mathcal{M}_b \otimes \big((\mathcal{O}_X)|_{\pi^{-1}(b)}\big) \cong \big((\mathcal{O}_X)|_{\pi^{-1}(b)}\big)$.

For the converse we may replace $\mathcal{L}$ by $\mathcal{L}^{-1} \otimes \mathcal{L}'$, and it suffices to consider the case where $\mathcal{L}|_{\pi^{-1}(b)}$ is trivial for each $b \in B$. Our hypothesis implies that $H^0(\mathcal{L}|_{\pi^{-1}(b)})$ is 1-dimensional for every $b \in B$, so by Theorem 6.14, $\pi_*\mathcal{L}$ is a line bundle.

We will complete the proof by showing that the natural map $\epsilon : \pi^*\pi_*\mathcal{L} \to \mathcal{L}$ is an isomorphism. Since both sheaves are line bundles, it suffices to show that ϵ is surjective, and for this we may, by Nakayama's Lemma, restrict to a fiber. By Theorem 6.6, the fiber of $\pi_*\mathcal{L}$ at a point b is $H^0(\mathcal{L}|_{\pi^{-1}(b)})$, so $(\pi^*\pi_*\mathcal{L})|_{\pi^{-1}(b)}$ is the trivial line bundle of rank 1, generated by any nonzero global section of $\mathcal{L}|_{\pi^{-1}(b)}$. Since the latter is also a trivial line bundle of rank 1, we are done. □

For examples of higher direct images in situations where the dimension of $H^0(\mathcal{F}|_{\pi^{-1}(b)})$ jumps, see Exercises 6.18 and 6.19

6.6 Higher direct images

The direct image functor is a generalization of the functor sending a sheaf to its vector space of global sections. The *higher direct images* have the same relation to the higher cohomology. Just as the higher cohomology of a sheaf is usually used to derive information about global sections, we will use the higher direct image sheaves to shed light on the direct image itself. The higher direct image functors are also useful in their own right. In Theorem 6.14 we shall give a sufficient condition for one of them to be a vector bundle, and we shall give an application to "jumping lines" in Chapter 16.

We will deal both with the cohomology of sheaves and with the homology of complexes. To keep from confusion, we will use H^i to denote the i-th Zariski cohomology functor, while H^i denotes the i-th homology of a complex.

Suppose that $\pi : X \to B$ is a projective morphism if $\mathcal{F}$ is a sheaf on X, then since (by definition) π factors through an inclusion $X \subset \mathbb{P}^n \times B$ as a closed subvariety, we may regard $\mathcal{F}$ as a sheaf on $\mathbb{P} := \mathbb{P}^n \times B$. We write $\mathbb{P} = Proj(S)$, where S is the sheaf of graded algebras $\mathcal{O}_B[x_0, \dots, x_n]$.

Let U_i be the open subscheme $x_i \neq 0$, and let

$$\mathcal{C}^\bullet : \quad \bigoplus_i \mathcal{O}_{\mathbb{P}}|_{U_i} \longrightarrow \bigoplus_{i,j} \mathcal{O}_{\mathbb{P}}|_{U_i \cap U_j} \longrightarrow \cdots$$

be the Čech complex on $\mathbb{P}$. If $\mathcal{F}$ is a quasicoherent sheaf on X we define $R^i\pi_*\mathcal{F}$ to be the degree zero part of the homology of the complex $\mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{C}^\bullet$, that is,

$$R^i\pi_*\mathcal{F} := \big(\mathrm{H}^i(\mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{C}^\bullet)\big)_0.$$

What makes this somewhat technical definition useful is that

$$(\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}}|_{U_i})_0$$

is the sheaf of modules over

$$\mathcal{O}_B[x_0, \dots, x_{i-1}, \hat{x}_i, x_{i+1}, \dots, x_n]$$

corresponding to the restriction of the sheaf $\mathcal{F}$ to the open set $U_i = \mathbb{A}^n_B$. This and the assumption that $\mathcal{F}$ is a sheaf shows that $R^0\pi_*\mathcal{F} = \pi_*\mathcal{F}$. Also, when $U \subset B$ is affine, so that each $U_i \cap \pi^{-1}(U)$ is affine, we get

$$R^i\pi_*\mathcal{F}|_{\pi^{-1}(U)} = H^i(\mathcal{F}|_{\pi^{-1}(U)}).$$

Since any sheaf is determined by its restriction to affine open subsets, this property characterizes $R^i\pi_*\mathcal{F}$ and shows that the definition is independent of the embedding $X \subset \mathbb{P}^n_B$ that we chose.

If $b \in B$ is a point (closed or not) then

$$H^i(\mathcal{F}|_b) = \mathrm{H}^i\big(\kappa(b) \otimes \mathcal{F} \otimes \mathcal{C}^\bullet\big),$$

but this is generally not equal to the fiber

$$(R^i\pi_*\mathcal{F})_b = \kappa(b) \otimes \mathrm{H}^i(\mathcal{F} \otimes \mathcal{C}^\bullet)$$

of the higher direct image. However, if z is a cycle or boundary in $\mathcal{F} \otimes \mathcal{C}^\bullet$ then $1 \otimes z$ is a cycle or boundary in $\kappa(b) \otimes \mathcal{F} \otimes \mathcal{C}^\bullet$ so we get maps $(R^i\pi_*\mathcal{F}) \to H^i(\mathcal{F}|_{X_b})$ that in turn induce comparison maps

$$\varphi_b^i : (R^i\pi_*\mathcal{F})|_b \to H^i(\mathcal{F}|_{X_b}).$$

We ask under what circumstances these are isomorphisms. We will give a partial answer to this question in the next section, but first we prove some basic properties of the $R^i\pi_*$.

Proposition 6.8. *Let $\pi : X \to B$ be a projective morphism.*

(a) **Restriction to open sets** *Let $U \subset B$ be an open subset, and let $\pi' : \pi^{-1}(U) \to U$ be the restriction of π. If $\mathcal{F}$ is any quasicoherent sheaf on X, then $(R^i\pi_*\mathcal{F})|_U = R^i\pi'_*(\mathcal{F}|_{\pi^{-1}(U)})$.*

(b) **Long Exact Sequence:** *The functor π_* is left exact, and the functors $\mathbb{R}^i\pi_*$ are the right derived functors of π_*. In particular, if*

$$\epsilon : 0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \longrightarrow 0$$

is a short exact sequence of quasicoherent sheaves on X, then there are natural "connecting homomorphisms" η_i making the sequence

$$\cdots \longrightarrow R^i\pi_*\mathcal{F} \xrightarrow{R^i\pi_*\alpha} R^i\pi_*\mathcal{G} \xrightarrow{R^i\pi_*\beta} R^i\pi_*\mathcal{H} \xrightarrow{\eta_i} R^{i+1}\pi_*\mathcal{F} \longrightarrow \cdots$$

exact.

(c) **Push-Pull Formula:** *If $\mathcal{E}$ is a vector bundle on B and $\mathcal{F}$ is a quasicoherent sheaf on X, then*

$$R^i\pi_*(\pi^*\mathcal{E} \otimes \mathcal{F}) \cong \mathcal{E} \otimes R^i\pi_*\mathcal{F}.$$

Proof. (a): The since $\mathcal{O}_U$ is flat over $\mathcal{O}_B$, the restriction to U commutes with taking homology.

(b): The terms of the complex $\mathcal{C}^\bullet$ are flat, so when we tensor $\mathcal{C}^\bullet$ with the short exact sequence ϵ we get a short exact sequence of complexes, and thus a long exact sequence of homology sheaves. Taking the degree zero part preserves exactness. Since the long exact sequence begins with

$$0 \to \pi_*\mathcal{F} \to \pi_*\mathcal{G} \to \cdots$$

we see that π_* is left exact.

To show that $R^i\pi_*$ is the i-th right derived functor of π_* it now suffices to show that $R^i\pi_*\mathcal{F} = 0$ when $\mathcal{F}$ is injective (Eisenbud [1995] Appedix A.3.9), or more generally flasque. It suffices to prove this after restricting to an affine subset $U \subset B$. Since the restriction of a flasque sheaf to an open subset is flasque, the result follows from the corresponding result for cohomology.

(c): The sheaf $\pi^*\mathcal{E}$ is also a vector bundle, and thus flat, so tensoring with $\pi^*\mathcal{E}$ commutes with taking homology,

$$\mathrm{H}^i(\pi^*\mathcal{E} \otimes \mathcal{F} \otimes \mathcal{C}^\bullet) = \pi^*\mathcal{E} \otimes \mathrm{H}^i(\mathcal{F} \otimes \mathcal{C}^\bullet).$$

Taking the degree zero part yields the desired formula. □

The other foundational result we will use is Serre's Coherence Theorem:

Theorem 6.9. *If $\pi : X \to B$ is a projective morphism, and $\mathcal{F}$ is a coherent sheaf on X, then $R^i\pi_*\mathcal{F}$ is coherent for each i.*

The proof involves some useful ideas from homological commutative algebra.

Proof. Since the formation of $R^i\pi_*\mathcal{F}$ commutes with the restriction to an open set in the base, it suffices to treat the case where $B = \operatorname{Spec} A$ is affine. The Čech complex $\mathbb{C}$ is the direct limit of the duals of the Koszul complexes that are S-free resolutions of the of the ideals $(x_0^m, \dots x_n^m) \subset S$, and direct limits commute with taking homology. Thus if M is any finitely generated graded $S = A[x_0, \dots, x_n]$-module representing the sheaf $\mathcal{F}$, the homology of $\mathcal{F} \otimes \mathcal{C}^\bullet$ is

$$R^i\pi_*\mathcal{F} = \lim_m Ext_S^i((x_0^m, \dots, x_n^m), M)_0.$$

Write $\mathfrak{m}$ for the "irrelevant" ideal $(x_0, \dots, x_n) \subset S$. For each m there is an integer $N(m)$ such that $\mathfrak{m}^{N(m)} \subset (x_0^m, \dots, x_n^m) \subset \mathfrak{m}^m$. It follows that

$$\lim_m Ext_S^i((x_0^m, \dots, x_n^m), M) = \lim_m Ext_S^i(\mathfrak{m}^m, M).$$

Each term $Ext_S^i(\mathfrak{m}^m, M)$ of this limit is a finitely generated S-module, so its degree 0 part is a finitely generated A-module, and it suffices to show that the natural map

$$Ext_S^i(\mathfrak{m}^m, M)_0 \to Ext_S^i(\mathfrak{m}^{m+1}, M)_0$$

is an isomorphism for large m. From the long exact sequence in Ext_S, and the fact that $\mathfrak{m}^m/\mathfrak{m}^{m+1}$ is a direct sum of copies of $A(-m)$ (the free A-module of rank 1 with generator in degree m) we see that it is enough to prove that $Ext_S^i(A(-m), M)_0 = 0$ when m is large. Disentangling the degree shifts, we see that

$$Ext_S^i(A(-m), M)_0 = Ext_S^i(A, M)(m)_0 = Ext_S^i(A, M)_m.$$

However, A is annihilated (as an S-module) by $\mathfrak{m}$, so $Ext_S^i(A, M)$ is annihilated by $\mathfrak{m}$. Since it is a finitely generated S-module, it can only be nonzero in finitely many degrees, whence, indeed, $Ext_S^i(A, M)_m = 0$ when m is large. □

We remark that, using the notion of Castelnuovo-Mumford regularity it is possible to bound the degree m for which $Ext^i_S(A, M)_m = 0$ in terms of the data in a free resolution of M (similar to Smith [2000]), so the proof just given allows effective computation of the functors $R^i\pi_*(\mathcal{F})$. For a different proof see Hartshorne [1977] Theorem III.8.8.

6.7 Cohomology and base change

In this section we will study how the i-th cohomology of a sheaf can vary as the sheaf varies in a family. In particular, we will give a basic criterion for the family of i-th cohomology spaces to "fit together" to form a vector bundle, or more generally to be the fibers of a coherent sheaf.

For example, suppose that $\pi : X \to B$ is a projective morphism, and that $\mathcal{F}$ is a sheaf on X, flat over B, which we view as a family of sheaves $\mathcal{F}|_b$ on the fibers $X_b := \pi^{-1}(b)$. Suppose for simplicity that B is connected. An obvious necessary condition for the cohomology groups $H^i(\mathcal{F}|_b)$ to be the fibers of a vector bundle $\mathcal{E}$ on B is that they all have the same dimension. We will show, under some commonly satisfied hypotheses, that this is enough, and that $\mathcal{E} = R^i\pi_*(\mathcal{F})$.

Here is the result that is the key to such questions. To simplify the notation, we will identify quasicoherent sheaves over an affine scheme $B = \operatorname{Spec} A$ with their modules of global sections.

Theorem 6.10. *Let $\pi : X \to B$ be a projective morphism to an affine scheme $B = \operatorname{Spec} A$, and let $\mathcal{F}$ be a sheaf on X that is flat over B. Suppose that the maximum dimension of a fiber of π is n. There is a complex*

$$\mathcal{P}^\bullet : \quad \cdots \longrightarrow P^0 \longrightarrow \cdots \longrightarrow P^n \longrightarrow 0$$

of finitely generated projective A-modules such that

(a) $R^i\pi_*(\mathcal{F}) \cong \mathrm{H}^i(\mathcal{P}^\bullet)$ *for all i.*

(b) *For every map $b \in B$ there is an isomorphism*

$$H^i(\mathcal{F}|_{\pi^{-1}(b)}) \cong \mathrm{H}^i(\kappa(b) \otimes_A \mathcal{P}^\bullet),$$

and more generally for every pull-back diagram

$$\begin{tikzcd} X' \arrow[r, "\rho'"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ B' \arrow[r, "\rho"] & B \end{tikzcd}$$

with $B' = \operatorname{Spec} A'$

$$R^i\pi'_*\rho'^*(\mathcal{P}^\bullet) \cong H^i(A' \otimes_A \mathcal{P}^\bullet).$$

Moreover, if B is reduced, then we may choose $\mathcal{P}^\bullet$ so that $P^i = 0$ for $i < 0$.

See Example 6.13 for a simple case.

The complex $\mathcal{P}^\bullet$ in Theorem 6.10 is not unique up to isomorphism, but it is unique up to the equivalence relation, called *quasi-isomorphism*, that is generated by maps of complexes that induce isomorphisms on homology (when B is affine, this comes down in our case to homotopy equivalence). Its class, modulo quasi-isomorphism, is called the total pushforward, $R\pi_*\mathcal{F}$. It is usually treated as an element of the derived category of (left-bounded) complexes of coherent sheaves on B. We say that $\mathcal{P}^\bullet$ *represents* $R\pi_*\mathcal{F}$. The second statement of (b) could be improved to say that $\rho^* R\pi_*\mathcal{F}$ is quasi-isomorphic (or, equivalently, isomorphic in the derived category) to $R\pi_*\rho'^*\mathcal{F}$. Abstract as this result may seem, the construction of $R\pi_*\mathcal{F}$ can be performed explicitly in examples of modest size, for instance by the computer algebra package Macaulay2.

We postpone the proof and state two Corollaries through which Theorem 6.10 is used.

Corollary 6.11. *Let $\pi : X \to B$ be a projective morphism of schemes and let $\mathcal{F}$ be a sheaf on X that is flat over B. For each i the dimension function*

$$B \ni b \mapsto \dim_{\kappa(b)} H^i(\mathcal{F}|_b)$$

is a semicontinuous function (that is, takes its smallest value on an open set.) Moreover the Euler characteristic

$$\chi(\mathcal{F}|_{\pi^{-1}(b)}) := \sum_i (-1)^i \dim_{\kappa(b)} H^i(\mathcal{F}|_b)$$

is constant on each connected component of B.

Proof. It suffices to prove the results in the case where B is affine and connected, say $B = \operatorname{Spec} A$. Let $\mathcal{P}^\bullet$ be a complex of finitely generated projective A-modules with the properties given in Theorem 6.10. Restricting to some possibly smaller open set of B, we may assume that $\mathcal{P}^\bullet$ is a complex of finitely generated free modules. For each $b \in B$ we get a complex of vector spaces by taking the fiber of $\mathcal{P}^\bullet$ at b. The maps $\varphi^i : P^i \to P^{i-1}$ in $\mathcal{P}^\bullet$ are given by matrices with entries in A, and thus the rank of $\varphi^i_b := \kappa(b) \otimes_A \varphi^i$ is a semicontinuous function of b. It follows that

$$\dim_{\kappa(b)} \mathrm{H}^i(\mathcal{P}^\bullet|_b) = \dim_{\kappa(b)} P^i|_b - \operatorname{rank} \varphi_b^{i+1} - \operatorname{rank} \varphi_b^i$$

is a semicontinuous function of b. Further, the Euler characteristic

$$\begin{aligned}\chi(\mathcal{F}|_b) &= \sum (-1)^i \dim_{\kappa(b)} \mathrm{H}^i(\mathcal{P}^\bullet|_b)\\ &= \sum (-1)^i \dim_{\kappa(b)} (P^i|_b)\\ &= \sum (-1)^i \operatorname{rank} P^i\end{aligned}$$

is a constant function of b. $\square$

Here is an example showing that the hypothesis of flatness is essential:

Example 6.12. Let $B = \mathbb{A}^2$, let $\pi : X \to B$ be the blowup of B at the origin. Let $\mathbb{P}^1 \cong E \subset X$ be the exceptional divisor, and let $\mathcal{F}$ be the line bundle $\mathcal{O}(E)$. Note that $\mathcal{F}$ is not flat over B. We have $\chi(\mathcal{F}|_b) = 1$ for $b \neq 0$, but $\mathcal{F}|_0 = \mathcal{O}_{\mathbb{P}^1}(-1)$, so the dimension of $H^0(\mathcal{F}|_b)$ is not semicontinuous.and the Euler characteristic $\chi(\mathcal{F}|_b)$ is not constant.

We can sometimes deduce the structure of the $R^i\pi_*$, and even of the complex $\mathcal{P}^\bullet$ of Theorem 6.10, from the behavior of the cohomology, as in the following example.

Example 6.13. Let B be a curve of genus 1 over k, and let $p \in B$ be a point. Let $X = B \times B$, and let $\pi : X \to B$ the the projection onto the first factor. Let Δ be the diagonal in X, and consider the line bundle $\mathcal{F} = \mathcal{O}_X(\Delta - B \times p)$, so that $\mathcal{F}|_{B\times\{b\}} = \mathcal{O}_B(b-p)$. Note that $\mathcal{F}$ is flat over B. At any point $b \neq p$ we have $H^0(\mathcal{F}|_{B\times\{b\}}) = 0$, so $\pi_*\mathcal{F} = 0$, a vector bundle. However, the natural map $H^0(\mathcal{F}|_p)$ is 1-dimensional, so the natural map $H^0(\mathcal{F}|_p) \to \kappa(p) \otimes R^0\pi_*(\mathcal{F})$ is not an isomorphism.

In this case, $R^1\pi_*(\mathcal{F}) = H^0(\mathcal{F}|_{B\times\{b\}})$ is a torsion sheaf, supported at p. Let

$$\mathcal{P}^\bullet : 0 \to P^0 \to P^1 \to 0$$

be the complex of Theorem 6.10. We claim that any complex of the form

$$0 \to \mathcal{L}(-p) \to \mathcal{L} \to 0$$

represents $R\pi_*\mathcal{F}$.

With just a little more work Theorem 6.10 yields much more refined statements:

Corollary 6.14 (Cohomology and Base Change)**.** *Let $\pi : X \to B$ be a projective morphism of schemes, with B connected, and let $\mathcal{F}$ be a coherent sheaf on X.*

(a) *If B is reduced then there is a dense open set U of points $b \in B$ such that $R^i\pi_*\mathcal{F}|_U$ is a vector bundle, and such that for $b \in U$ the fiber $(R^i\pi_*\mathcal{F}|_U)_b$ is equal to $H^i(\mathcal{F}|_{\pi^{-1}(b)})$.*
(b) *Suppose that $\mathcal{F}$ is flat over B. If, for some i, $H^j(\mathcal{F}|_{\pi^{-1}(b)}) = 0$ for all $j > i$ and all closed points $b \in B$, then for every closed point $b \in B$ the comparison map*

$$\varphi^i_b : R^i\pi_*(\mathcal{F})|_b \to H^i(\mathcal{F}|_{\pi^{-1}(b)})$$

is an isomorphism.

(c) *Suppose that $\mathcal{F}$ is flat over B and B is reduced. If, for some i, the function $b \mapsto \dim_{\kappa(b)} H^i(\mathcal{F}|_{\pi^{-1}(b)})$ is constant, then $R^i\pi_*(\mathcal{F})$ is a vector bundle of rank equal to $\dim_{\kappa(b)} H^i(\mathcal{F}|_{\pi^{-1}(b)})$, and for every closed point $b \in B$ the comparison maps*

$$\varphi_b^i : R^i\pi_*(\mathcal{F})|_b \to H^i(\mathcal{F}|_{\pi^{-1}(b)})$$
$$\varphi_b^{i-1} : R^{i-1}\pi_*(\mathcal{F})|_b \to H^{i-1}(\mathcal{F}|_{\pi^{-1}(b)})$$

are isomorphisms.

We postpone this proof as well to develop some basic techniques.

6.7.1 Tools from commutative algebra

The proofs of the results in the previous section require two tools from commutative algebra. First, a fundamental method for proving that a sheaf is a vector bundle (that is, is locally free):

Proposition 6.15. *A coherent sheaf $\mathcal{G}$ on a reduced scheme B is a vector bundle if and only the dimension of the $\kappa(b)$-vector space $\mathcal{G}|_b$ is the same for all points $b \in B$; if B is quasi-projective, then it even suffices to check this for closed points. These conditions are satisfied, in particular, if $\mathcal{G}$ has a resolution*

$$\mathcal{P}^\bullet : \quad \cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow \mathcal{G} \longrightarrow 0$$

by vector bundles of finite rank that remains a resolution when tensored with $\kappa(b)$ for every $b \in B$.

Proof. If $\mathcal{G}$ is a vector bundle, then it has constant rank because B is connected, and this rank is the common dimension of $\mathcal{G}|_b$ over $\kappa(b)$.

To prove the converse, and also the last statement of the Proposition, we note that the problem is local, so we may assume that $B = \operatorname{Spec} A$, where A is a local ring with maximal ideal $\mathfrak{m}$ corresponding to the closed point $b_0 \in B$, and that $\mathcal{G} = \widetilde{G}$, where G is a finitely generated A-module. By Nakayama's Lemma, a minimal set of generators of G corresponds to a map $f : F \to G$ from a free A-module F such that the induced map $(A/\mathfrak{m}) \otimes F \to (A/\mathfrak{m}) \otimes G$ is an isomorphism. In particular, the rank of the free module F is $\dim_{A/\mathfrak{m}}(A/(\mathfrak{m}) \otimes G = \dim_{\kappa(b)} \mathcal{G}|_b$.

Let $K = \operatorname{Ker} \varphi$, and let P be a minimal prime of A. Since A is reduced, A_P is a field. Localizing at P, we get an exact sequence of finite dimensional vector spaces

$$0 \to K_P \to F_P \to G_P \to 0,$$

whence $\operatorname{rank} F = \dim_{A_P} G_P + \dim_{A_P} K_P$. By hypothesis, $\dim_{A_P} G_P = \dim_{A/\mathfrak{m}} P/\mathfrak{m}P = \operatorname{rank} F$, so $K_P = 0$. Since A is reduced, the only associated primes of F are the minimal primes of A, and thus $K \subset F$ itself must be zero.

For the last statement, suppose that $\mathcal{G}$ has a resolution $\mathcal{P}^\bullet$ with the given property. Since B is the spectrum of the local ring A, we may identify $\mathcal{P}^\bullet$ with a free resolution of $\mathcal{G}$. Since a minimal free resolution is a summand of any resolution (Eisenbud [1995] Theorem 20.2) the minimal free resolution $\mathcal{P}'^\bullet$ of $\mathcal{G}$ has the same property. But after tensoring with the residue class field $\kappa(b_0)$ the differentials in $\mathcal{P}'^\bullet$ become zero. Since by hypothesis the resolution remains acyclic, we must have $P'^0 = \mathcal{G}$, so $\mathcal{G}$ is free. $\square$

The proof of Theorem 6.10 requires one more tool, a way of approximating a complex by a complex of free modules with good properties.

Proposition 6.16. *Let A be a noetherian ring, and let*

$$\mathcal{K}^\bullet : \cdots \xrightarrow{d} K^i \xrightarrow{d} K^{i+1} \xrightarrow{d} \cdots$$

be a complex of (not necessarily finitely generated) flat A-modules whose homology modules are finitely generated and such that $K^m = 0$ for $m \gg 0$. There is a complex of finitely generated free A-modules $\mathcal{P}^\bullet$ with $P^m = 0$ for $m \gg 0$, and a map of complexes $r : \mathcal{P}^\bullet \to \mathcal{K}^\bullet$ such that, for every A-module M, the map

$$r \otimes_A M : \mathcal{P}^\bullet \otimes_A M \longrightarrow \mathcal{K}^\bullet \otimes_A M$$

induces an isomorphism on homology.

Proof. We will construct a complex of finitely generated free modules $\mathcal{P}^\bullet$ with a map r to $\mathcal{K}^\bullet$ inducing an isomorphism on homology without the assumption of flatness; and then we will use the flatness hypothesis to show that any such $r : \mathcal{P}^\bullet \to \mathcal{K}^\bullet$ induces an isomorphism on homology after tensoring with the arbitrary module M.

We will construct the complex $\mathcal{P}^\bullet$ by a downward induction. Suppose that for some i a map of complexes

$$\begin{tikzcd} K^{i+1} \arrow[r, "d^{i+1}"] & K^{i+2} \arrow[r, "d^{i+2}"] & \cdots \\ P^{i+1} \arrow[u, "r_{i+1}"] \arrow[r, "e^{i+1}"] & P^{i+2} \arrow[u, "r_{i+2}"] \arrow[r, "e^{i+2}"] & \cdots \end{tikzcd}$$

inducing isomorphisms $\mathrm{H}^j(\mathcal{P}^\bullet) \to \mathrm{H}^j(\mathcal{K}^\bullet)$ for all $j \geq i+2$ has been constructed, with the additional property that the composite map $\operatorname{Ker} e^{i+1} \to \operatorname{Ker} d^{i+1} \to \mathrm{H}^{i+1}(\mathcal{K}^\bullet)$ is surjective.

If i is sufficiently large that that $K^m = 0$ for all $m \geq i+1$, then the choice $P^m = 0$ and $r_m = 0$ for $m \geq i+1$ satisfies these conditions, giving a base for the induction.

To make the inductive step from $i+1$ to i, we choose P^i to be the direct sum of two projective modules, $P^i = P_1^i \oplus P_2^i$ where P_1^i is chosen to map onto the kernel of the composite $\operatorname{Ker} e^{i+1} \to \operatorname{Ker} d^{i+1} \to \mathrm{H}^{i+1}(\mathcal{K}^\bullet)$, and P_2^i is chosen to map onto $\mathrm{H}^i(\mathcal{K}^\bullet)$. We define the differential e^i to be the given map on P_1^i and zero on P_2^i. Also, we define r_i on P_2^i by lifting the chosen map $P_2^i \to \mathrm{H}^i(\mathcal{K}^\bullet)$ to a map $P_2^i \to \operatorname{Ker} d^i$ and composing with the inclusion $\operatorname{Ker} d^i \subset K^i$. On the other hand, since r_{i+1} carries the image of P_1^i to the kernel of the map $\operatorname{Ker} d^{i+1} \to \mathrm{H}^{i+1}(\mathcal{K}^\bullet)$, which is by definition $\operatorname{Im} d^i$, we may define r_i on P_1^i to be the lifting of this map $P_1^i \to \operatorname{Im} d^i$ to a map $P_1^i \to K^i$.

This gives a map of complexes

$$\begin{tikzcd}
K^i \arrow[r, "d^i"] & K^{i+1} \arrow[r, "d^{i+1}"] & K^{i+2} \arrow[r, "d^{i+2}"] & \cdots \\
P^i = P_1^i \oplus P_2^i \arrow[u, "r_i"] \arrow[r, "e^i"] & P^{i+1} \arrow[u, "r_{i+1}"] \arrow[r, "e^{i+1}"] & P^{i+2} \arrow[u, "r_{i+2}"] \arrow[r, "e^{i+2}"] & \cdots
\end{tikzcd}$$

It is clear from the construction that the r_i induce isomorphisms $\mathrm{H}^j(\mathcal{P}^\bullet) \to \mathrm{H}^j(\mathcal{K}^\bullet)$ for all $j \geq i+1$ and that the composite map $\operatorname{Ker} e^{i+1} \to \operatorname{Ker} d^{i+1} \to \mathrm{H}^{i+1}(\mathcal{K}^\bullet)$ is surjective, so the induction is complete.

We now use the hypothesis that the K^i are flat, and suppose that we have proven that r_j induces an isomorphism $\mathrm{H}^j(\mathcal{P}^\bullet \otimes M) \to \mathrm{H}^j(\mathcal{K}^\bullet \otimes M)$ for every $j > i$ and for every module M. This is trivial in the range where K^j and P^j are both zero, so again we can do a downward induction.

Choose a surjection $F \to M$ from a free A-module, and let L be the kernel, so that

$$0 \to L \to F \to M \to 0$$

is a short exact sequence. Since all the K^i and the P^i are flat, we get short exact sequences of complexes by tensoring with $\mathcal{P}^\bullet$ and $\mathcal{K}^\bullet$, from which we get two long exact sequences by applying the higher direct image functors. and the comparison map $r : \mathcal{P}^\bullet \to \mathcal{K}^\bullet$ induces a commutative diagram

$$\begin{tikzcd}
H^i(\mathcal{K}^\bullet \otimes L) \arrow[r] & H^i(\mathcal{K}^\bullet \otimes F) \arrow[r] & H^i(\mathcal{K}^\bullet \otimes M) \arrow[r] & H^{i+1}(\mathcal{K}^\bullet \otimes L) \arrow[r] & H^{i+1}(\mathcal{K}^\bullet \otimes F) \\
H^i(\mathcal{P}^\bullet \otimes L) \arrow[r] \arrow[u, "r_i \otimes L"] & H^i(\mathcal{P}^\bullet \otimes F) \arrow[r] \arrow[u, "r_i \otimes F", "\cong"'] & H^i(\mathcal{P}^\bullet \otimes M) \arrow[r] \arrow[u, "r_i \otimes M"] & H^{i+1}(\mathcal{P}^\bullet \otimes L) \arrow[r] \arrow[u, "r_{i+1} \otimes L", "\cong"'] & H^{i+1}(\mathcal{P}^\bullet \otimes F) \arrow[u, "r_{i+1} \otimes F", "\cong"']
\end{tikzcd}$$

where, for any module N, we have written $r_i \otimes N$ to denote the map $H^i(\mathcal{P}^\bullet \otimes N) \to H^i(\mathcal{K}^\bullet \otimes N)$ induced by r_i. The maps marked "$\cong$" are isomorphisms: $r_i \otimes F$ and $r_{i+1} \otimes F$ are isomorphisms because F is free, while $r_{i+1} \otimes L$ is an isomorphism by induction. A diagram chase (sometimes called the "five-lemma") immediately shows that the map $r_i \otimes M$ is a surjection. Since the module M was arbitrary, $r_i \otimes L$ is a surjection as well. Using this information, a second diagram chase shows that $r_i \otimes M$ is injective, completing the induction. □

Proof of Theorem 6.10. Since π is projective we may write $X \subset \mathbb{P} := \mathbb{P}^n_A$ for some n, and we let $U_i,\ i = 0, \ldots, n$ be the standard open covering of $\mathbb{P}$ as in Section6.6. Let $\mathcal{C}^\bullet$ be the Čech complex defined there. Since $\mathcal{F}$ is flat and $(\mathcal{F} \otimes \mathcal{O}_{U_i \cap U_j \cap \cdots})_0$ is the module corresponding to the restriction of $\mathcal{F}$ to the affine open set $U_i \cap U_j \cap \cdots$, the modules of the complex $(\mathcal{F} \otimes \mathcal{C}^\bullet)_0$ are flat. By Theorem6.9 the homology of this complex is finitely generated, so we may apply Proposition6.16 and obtain a complex $\mathcal{P}^\bullet$ whose i-th homology is $R^i\pi_*\mathcal{F}$. Taking $M = \kappa(b)$ in the Proposition, we see that $\mathcal{P}^\bullet$ has the second required property as well.

Finally, to show that we may choose $\mathcal{P}^\bullet$ with $P^i = 0$ for $i < 0$, note that for any choice of $\mathcal{P}^\bullet$ satisfying the Proposition the homology $\mathrm{H}^i(\mathcal{P}^\bullet) = 0$ for $i < 0$. The last statement of Proposition 6.15 shows that $P'^0 := \operatorname{coker}(P^{-1} \to P^0)$ is projective. The map r_0 induces a map $P'^0 \to \operatorname{coker}(K^{-1} \to K^0)$, and since P'^0 is projective we may lift this to a new map $r'_0 : P'^0 \to K^0$. It follows from the construction that

$$\begin{tikzcd}
\cdots \arrow[r] & K^{-1} \arrow[r] & K^{-1} \arrow[r] & K^0 \arrow[r] & K^1 \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[u, "0"] & 0 \arrow[r] \arrow[u, "0"] & P'^0 \arrow[r] \arrow[u, "r'_0"] & P^1 \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

again induces an isomorphism on homology. □

6.7.2 Proof of the theorem on cohomology and base change

Proof of Corollary 6.14 . **(a):** Removing the intersections of the components of X, and then passing to a connected component, we may harmlessly assume that X is integral. By the Generic Flatness Theorem (see Eisenbud [1995] Section 14.2) there is an open set $U_1 \subset B$ over which $\mathcal{F}$ is flat. Replacing X by this set, we may assume that $\mathcal{F}$ is flat and within that set the set U of points $b \in U_1$ where $\dim_{\kappa(b)} H^i(\mathcal{F}|_b)$ takes on its smallest value is open by part (a) of the Theorem. The rest follows from part (b) of the Theorem.

(b): Let $\mathcal{P}^\bullet$ be the complex guaranteed by Theorem 6.10. The hypothesis implies that $H^i(\mathcal{P}^\bullet|_b) = 0$ for all $j > i$. If m is the greatest integer for which $\mathcal{P}^m \neq 0$, and $m > i$, then Nakayama's Lemma implies that the map $\mathcal{P}^{m-1} \to \mathcal{P}^m$ is surjective, and thus split. Consequenly we can build a quasi-isomorphic complex $\mathcal{P}'^\bullet$ by replacing $\mathcal{P}^{m-1}$ by $\mathcal{P}'^{m-1} := \operatorname{Ker}(\mathcal{P}^{m-1} \to \mathcal{P}^m$

and truncating the complex there:

$$\begin{array}{cccccccccc}
\mathcal{P}'^{\bullet}: & \cdots & \longrightarrow & \mathcal{P}^{m-2} & \longrightarrow & \mathcal{P}'^{m-1} & \longrightarrow & 0 & & \\
& \cdots & & \downarrow & & \downarrow & & & & \\
\mathcal{P}^{\bullet}: & \cdots & \longrightarrow & \mathcal{P}^{m-2} & \longrightarrow & \mathcal{P}^{m-1} & \longrightarrow & \mathcal{P}^{m} & \longrightarrow & 0
\end{array}$$

Continuing in this way, we may assume that $\mathcal{P}^j = 0$ for all $j > i$. Thus

$$\begin{aligned}
H^i(\mathcal{F}|_{\pi^{-1}b}) &= \operatorname{coker}(\mathcal{P}_b^{i-1} \to \mathcal{P}_b^i) \\
&= (\operatorname{coker}(\mathcal{P}^{i-1} \to \mathcal{P}^i)_b \\
&= (R^i\pi_*\mathcal{F})_b.
\end{aligned}$$

(c): Suppose that the rank of the differential d_b^i is constant. Since taking fibers commutes with taking images, the fibers of the sheaf $\operatorname{Im} d^i$ have constant rank. By Proposition 6.15, the module $\operatorname{Im} d^i$ is projective, and the short exact sequence

$$0 \to \operatorname{Ker} d^i \to P^i \to \operatorname{Im} d^i \to 0$$

splits. We deduce that $(\operatorname{Ker} d^i)_b = \operatorname{Ker}(d_b^i)$, and it follows that the map

$$R^i\pi_*(\mathcal{F})_b = \mathrm{H}^i(\mathcal{P}^{\bullet})_b \xrightarrow{\varphi_b^i} \mathrm{H}^i(\mathcal{P}^{\bullet}{}_b) = H^i(\mathcal{F}|_{\pi^{-1}(b)})$$

is an isomorphism.

Now suppose that $\dim_{\kappa(b)} H^i(\mathcal{F}_{\pi^{-1}(b)})$ is constant. Since the ranks of the differentials d_b^{i-1} and d_b^i are lower semicontinuous, they must both constant as well, and we see that both displayed maps in the statement of Part (b) of the Theorem are isomorphisms. Of course then $\dim_{\kappa(b)} R^i\pi_*(\mathcal{F})_b$ is constant, and it follows from Proposition 6.15 that $R^i\pi_*(\mathcal{F})$ is a vector bundle. The Proposition also assures us that it suffices to check the constancy of the rank at closed points in the quasi-projective case. □

Remark. Suppose that $\pi : X \to B$ is a projective morphism and $\mathcal{F}$ is a coherent sheaf on X, flat over B, as in Theorem 6.14, and suppose that $b \in B$ is a point at which $\dim_{\kappa(b)} H^i(\mathcal{F}_b)$ "jumps"—that is, is larger than it is for some points in any open neighborhood of b. From the constancy of the Euler characteristic $\chi(\mathcal{F}_{\pi^{-1}(b)})$ it follows that some $\dim_{\kappa(b)} H^j(\mathcal{F}_b)$ with $j \neq i$ mod 2 must jump too. But the proof above gives a tiny bit more: since the rank of d_b^i or of d_b^{i-1} must have gone down, either $\dim_{\kappa(b)} H^{i+1}(\mathcal{F}_b)$ or $\dim_{\kappa(b)} H^{i-1}(\mathcal{F}_b)$ must jump at b.

6.8 Double covers

Not every application of direct images requires the machinery described above. Here is a case where the computation of the direct image along a finite morphism is quite illuminating.

By a *double cover* of a variety B we mean a finite flat map $\pi : X \to B$ from a reduced scheme X. We can make this description more concrete as follows: the hypotheses imply that if U is an affine open subset of X then $\mathcal{O}_X(\pi^{-1}(U)) = \pi_*(\mathcal{O}_X)(U)$ is a free module of rank 2 over $\mathcal{O}_B(U)$. Since the section 1 is nonzero locally everywhere we see that $\pi_*(\mathcal{O}_X)(U)$ is generated as a module, and thus as a ring, by 1 together with one more element z. It follows that $X|_U \subset U \times \mathbb{A}^1$, and is defined there by a monic, quadratic equation $z^2 + py + q = 0$ with $p, q \in \mathcal{O}_X(U)$. If the characteristic is not 2, then changing variable to $y = z + p/2$ we may rewrite the equation in the form $y^2 - f = 0$ with $f \in \mathcal{O}_X(U)$. Since we have assumed that X is reduced, $f \neq 0$. Conversely, any scheme that given, locally on each affine subset of B by an equation of the form $y^2 - f = 0$ with $0 \neq f \in \mathcal{O}_B(U)$

For simplicity, we assume from now on that the characteristic is not 2.

If $p \in X$, and the cover restricts on an affine neighborhood U of p to one given by an equation $y^2 - f = 0$, then the fiber $\pi^{-1}(p)$ consists of two distinct points when $f(p) \neq 0$ and one double point when $f(p) = 0$. Thus the set of points p where f vanishes is determined by the map π. But in fact, even the divisor of zeros of f is determined. To see this, we compute the pushforward to U of the relative cotangent sheaf (the sheaf of differentials) $\Omega_{\pi^{-1}(U)/U}$. To simplify the notation we set $V = \pi^{-1}(U)$.

From embedding $V \subset B \times \mathbb{A}^1$ we see that $\Omega_{V/U} = \mathcal{O}_V dy/2y$. Pushing this forward, we get

$$\pi_*\Omega_{V/U} = \pi_*\mathcal{O}_V dy/2y = \operatorname{coker}\left(\pi_*\mathcal{O}_V \xrightarrow{\pi_*(2y)} \pi_*\mathcal{O}_V\right),$$

where $\pi_*(2y)$ denotes the pushforward of the endomorphism of $\mathcal{O}_V$ that is multiplication by $2y$. Now $\mathcal{O}_V$ is a free $\mathcal{O}_U$-module on the generators $1, y$, and writing the endomorphism in this basis we get

$$\pi_*\Omega_{V/U} \cong \operatorname{coker}\left(\mathcal{O}_U \oplus \mathcal{O}_U y \xrightarrow{\begin{pmatrix} 0 & 2f \\ 2 & 0 \end{pmatrix}} \mathcal{O}_U \oplus \mathcal{O}_U y\right) \cong \mathcal{O}_U/(2f).$$

Taking into account the fact that 2 is a unit, we see that the ideal (f) is determined by the map π, as claimed. Since the construction we have just made commutes with restriction to smaller affine open sets, we get a well defined divisor on all of B, called the *branch divisor* of π. We will denote it Δ. (In general, for a finite flat map $X \to B$ of varieties one defines the branch divisor to be the 0-th Fitting ideal of $\pi_*(\Omega_{X/B}$; see Eisenbud [1995] Section ****.)

Because the element $1 \in \pi_*\mathcal{O}_X$ is nonzero locally everywhere, and $\pi_*\mathcal{O}_X$ is locally free of rank 2, the quotient

$$\mathcal{L} := \frac{\pi_*\mathcal{O}_X}{\mathcal{O}_X \cdot 1}$$

is a line bundle locally generated by the element we called y above. But in fact, $\pi_*\mathcal{O}_X$ decomposes as a direct sum. Indeed any degree 2 field extension in characteristic not 2 is Galois, and restricting to an open set V as above, we see that the Galois group is generated by the map $\sigma : y \mapsto -y$, which fixes $\mathcal{O}_X$ inside its quotient field, and has ring of invariants equal to $\mathcal{O}_U$. Thus

$$\frac{1}{2}(1+\sigma) : \pi_*\mathcal{O}_X \to \pi_*\mathcal{O}_X$$

is an $\mathcal{O}_B$-linear retraction of $\pi_*\mathcal{O}_X$ onto $\mathcal{O}_B \cdot 1$, proving that

$$\pi_*\mathcal{O}_X = \mathcal{O}_B \oplus \mathcal{L}.$$

Since $\mathcal{L}$ is locally generated by a section y whose square is a defining equation of the divisor Δ, we get

$$L^2 \;\cong\; \mathcal{O}_B(-\Delta),$$

It turns out that this data is enough to recover the double cover:

Proposition 6.17. *Let B be any variety over a field of characteristic $\neq 2$. There is a one-to-one correspondence between double covers $\pi : X \to B$ of B, and pairs $(\mathcal{L}, \Delta)$ consisting of a line bundle $\mathcal{L}$ on B and a divisor $\Delta \in |L^{-2}|$.*

Figures 6.7 and 6.8 illustrate the point that the branch divisor must be divisible by 2.

Proof. We have seen the correspondence in one direction: the double cover $\pi : X \to B$ determines both the branch divisor Δ, and $\mathcal{L} = \pi_*\mathcal{O}_X/\mathcal{O}_B \cdot 1$.

Conversely, given $(\mathcal{L}, \Delta)$ as above, the divisor Δ is the vanishing locus of a section $\alpha \in \mathcal{L}^{-2}$. The section α defines a nonzero homomorphism $\tilde{\sigma} : \mathcal{L}^2 \to \mathcal{O}_B$. Using α we make the locally free sheaf

$$\mathcal{E} \;=\; \mathcal{O}_B \oplus \mathcal{L}$$

into a sheaf of $\mathcal{O}_B$-algebras: multiplication of the summand $\mathcal{O}_B$ by itself and by $\mathcal{L}$ is the obvious one, while products of sections of the summand $\mathcal{L}$ map to $\mathcal{O}_B$ via the map $\tilde{\sigma}$. We define the corresponding double cover to be $X = \operatorname{Spec}\mathcal{E}$. To see that these descriptions are inverse, one computes what they mean in terms of the local description of the cover over an affine open set $U \subset B$. $\square$

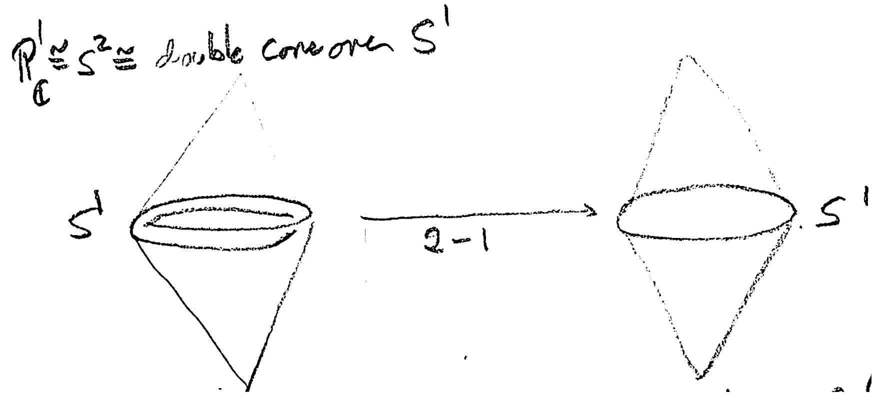


FIGURE 6.7. The unramified double cover of the circle by the circle induces a ramified double cover of the two-sphere ($\cong \mathbb{P}^1_{\mathbb{C}}$). Note that there are 2 branch points—an even number

In this situation the base change maps $\varphi_b : (\pi_* \mathcal{O}_X)|_b \to H^0(\mathcal{O}_X|_{X_b})$ are isomorphisms for every b. This is clear in case $b \notin \Delta$ is not in the branch divisor, so that the fiber X_b consists of two reduced points; and only slightly less clear when $b \in \Delta$, in which case X_b consists of a single point of degree 2.

6.9 Exercises

Exercise 6.18. Let $p \in C$ be a point on a nonsingular projective curve C of genus 1, and let $X = C \times C$. Let $\Delta \subset X$ be the diagonal, and let D be the divisor $C \times p$. Let $\mathcal{F}$ be the line bundle $\mathcal{O}_X(\Delta - D)$. Let $\pi : X \to C$ be the projection to the first factor.

Compute $\pi_*(\mathcal{F})$ and $R^1\pi_*(\mathcal{F})$. How do these sheaves reflect the values of the cohomology $H^i(\mathcal{F}_{\pi^{-1}(p')})$ as p' varies over C?

Exercise 6.19. 6.4 in the text, we showed that $\pi_*\mathcal{F} = 0$. Compute $R^1\pi_*\mathcal{F}$. How does the answer relate to the result that $\chi(\mathcal{F}|_{\pi^{-1}(p)}$ is constant?

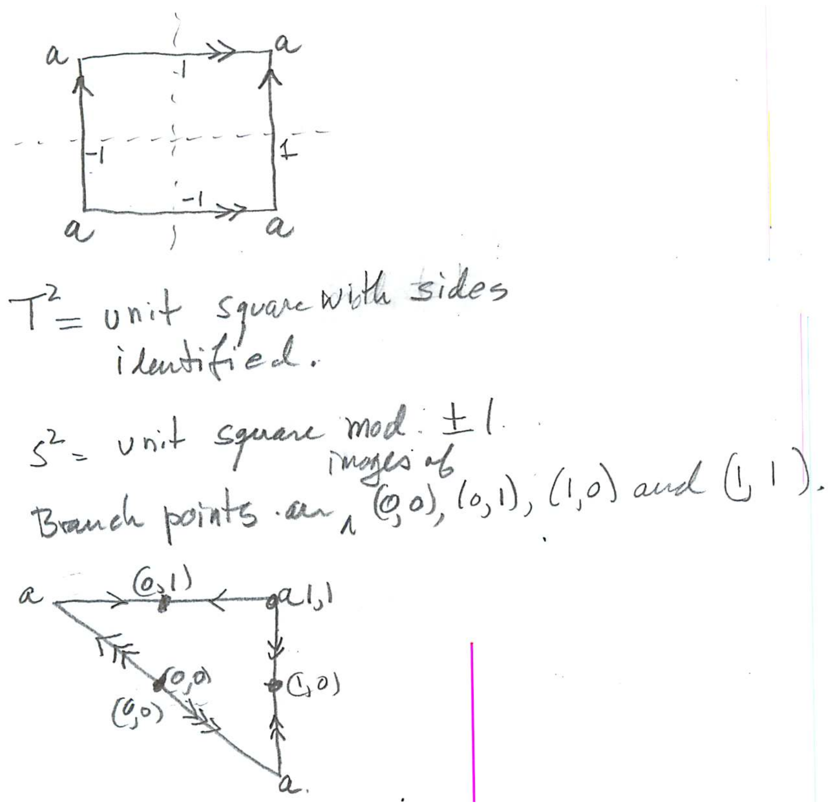


FIGURE 6.8. Double cover of a 2-sphere by a 2-torus. Note that there are 4 branch points—an even number ****the purple line should not be there!****

7

Vector Bundles and Chern Classes

Keynote Questions

(a) Let $S \subset \mathbb{P}^3$ be a smooth cubic surface. How many lines $L \subset \mathbb{P}^3$ are contained in S? Does a general quartic surface contain any lines? (Answer on page **??**)

(b) Let $\{S_t \subset \mathbb{P}^3\}_{t \in \mathbb{P}^1}$ be a general pencil of quartic surfaces—that is, let F and G be general homogeneous polynomials of degree 4 in four variables, and set $S_t = V(t_0 F + t_1 G) \subset \mathbb{P}^3$. How many members S_t of the pencil contain a line? (Answer on page **??**)

(c) Let $\{C_t \subset \mathbb{P}^2\}_{t \in \mathbb{P}^1}$ be a general pencil of plane curves of degree d—that is, let F and G be general homogeneous polynomials of degree d in three variables, and set $C_t = V(t_0 F + t_1 G) \subset \mathbb{P}^2$. How many of the curves C_t will be singular? (Answer on page **??**)

In this chapter we will introduce the machinery for answering these questions; the answers themselves will be found in Chapters 8 and 9.

7.1 Chern classes

We will begin by defining Chern classes of vector bundles generated by their global sections, because in this case the Chern classes have a simple geometrical interpretation that both motivates the definition and provides an insight into what should be true. In the following section, we'll describe the relations among Chern classes of bundles related by linear algebra constructions, and in Section 7.3 build up a vocabulary of bundles whose Chern classes we know; in combination, these will allow us to calculate the Chern classes of a wide range of bundles.

7.1.1 Chern classes as degeneracy loci

Let X be a smooth variety and $\mathcal{E}$ a vector bundle of rank r on X, which we will for now assume is generated by its global sections. The Chern classes of $\mathcal{E}$ are elements of the Chow ring $A(X)$ that measure the twisting, or non-triviality, of the bundle.

To arrive at the definition, we can reason this way: a bundle $\mathcal{E}$ of rank r is trivial if and only if it has r everywhere independent global sections. We break this down into a series of questions: does $\mathcal{E}$ have an everywhere nonzero section σ_1? Does $\mathcal{E}$ have two everywhere independent sections σ_1 and σ_2? and so on. We convert the first of these yes-or-no questions into a quantitative measure of the non-triviality of $\mathcal{E}$ by taking any suitably general section σ_1 of $\mathcal{E}$ and associating to $\mathcal{E}$ the class of its zero locus. We do the same for the second question by taking σ_1 and σ_2 suitably general sections σ_1 of $\mathcal{E}$ and associating to $\mathcal{E}$ the class of the locus where they fail to be linearly independent, and so on.

Before we can convert this idea into a definition, we have to say what the expected dimension of these loci are. Some notation: if $\sigma_1(p), \dots, \sigma_l(p)$ are global sections of $\mathcal{E}$, we define

$$V(\sigma_1 \wedge \cdots \wedge \sigma_l) = \{p \in X \mid \sigma_1(p), \dots, \sigma_l(p) \text{ are linearly dependent}\}.$$

We can give $V(\sigma_1 \wedge \cdots \wedge \sigma_l)$ the structure of a closed subscheme of X in a natural way: in any open subset $U \subset X$ over which $\mathcal{E}$ is trivial, we can represent each of the sections σ_i by an r-vector of regular functions on U; we take the defining equations of $V(\sigma_1 \wedge \cdots \wedge \sigma_l)$ in U to be the $l \times l$ minors of the $l \times r$ matrix formed by these functions.

In these terms, we have the basic

Lemma 7.1. *Let $\mathcal{E}$ be a vector bundle of rank r on the smooth variety X.*

(a) *If $\sigma_1, \dots, \sigma_l$ are any sections of $\mathcal{E}$, the locus $V(\sigma_1 \wedge \cdots \wedge \sigma_l) \subset X$ has codimension at most $r - l + 1$ everywhere.*

(b) *If $W \subset H^0(\mathcal{E})$ is a vector space of global sections that generates $\mathcal{E}$ and $\sigma_1, \dots, \sigma_r \in W$ are general elements of W, then $V(\sigma_1 \wedge \cdots \wedge \sigma_l)$ has codimension $r - l + 1$ everywhere, for all l.*

(c) *In the situation of part (b), if we assume in addition that the characteristic of our ground field K is 0, then $V(\sigma_1 \wedge \cdots \wedge \sigma_l)$ is reduced for all l.*

Proof. For part (a), we start with the case $l = 1$ of a single section σ. This is straightforward: since σ is locally defined by r equations, the Principal Ideal Theorem says that every component of $V(\sigma)$ has codimension at most r. The general case follows similarly from a version of the Principal Ideal Theorem (Macaulay's "Generalized Unmixedness Theorem", proven in general by Eagon and Northcott—see for example Eisenbud [1995] Exercise 10.9), which shows that every component of $V(\sigma_1 \wedge \cdots \wedge \sigma_l)$ has codimension at most $r - l + 1$.

For parts (b) and (c), suppose that $\dim W = m$. We have a surjection

$$W \otimes \mathcal{O}_X \cong \mathcal{O}_X^m \to \mathcal{E} \to 0,$$

and correspondingly a map

$$\varphi = \varphi_W : X \to G(m - r, W)$$

associating to each $p \in X$ the subspace $W_p \subset W$ of sections vanishing at p (that is, the kernel of the map $W \to \mathcal{E}_p$ sending $\tau \in H^0(\mathcal{E})$ to $\tau(p)$). If the sections $\sigma_1, \ldots, \sigma_l \in W$ span the l-dimensional subspace $W_l \subset W$, then the locus $V(\sigma_1 \wedge \cdots \wedge \sigma_l) \subset X$ will be simply the preimage $\varphi^{-1}(\Sigma)$ of the Schubert cycle

$$\Sigma = \Sigma_{r-l+1}(W_l) = \{\Lambda \in G(m - r, W) \mid \Lambda \cap W_l \neq 0\}$$

of $m - r$-planes in W meeting W_l nontrivially. Since $W_l \subset W$ is a general linear subspace, Σ is a general translate of a Schubert cycle, and part (b) now follows from Kleiman's Theorem 5.15. Part (c) likewise follows, albeit in two steps: Kleiman assures us that $V(\sigma_1 \wedge \cdots \wedge \sigma_l)$ is generically reduced, and Macaulay's Unmixedness Theorem implies that it has no embedded components. □

Note as a special case of this Lemma that *if $W \subset H^0(\mathcal{E})$ generates $\mathcal{E}$, and* $\operatorname{rank}(\mathcal{E}) = r > \dim X$, *then a general element $\sigma \in W \subset H^0(\mathcal{E})$ is nowhere zero.* We will see one important geometric consequence of this statement in Section 7.1.3 below.

Given Lemma 7.1, we can define the Chern classes of any globally generated vector bundle:

Proposition-Definition 7.2. Let $\mathcal{E}$ be a globally generated vector bundle of rank r on X. For $k = 1, \ldots, r$, if $\sigma_1, \ldots, \sigma_{r-k+1} \in H^0(\mathcal{E})$ are sections such that the degeneracy locus $V(\sigma_1 \wedge \cdots \wedge \sigma_{r-k+1}) \subset X$ has codimension k everywhere then the class

$$c_k(\mathcal{E}) = \big[V(\sigma_1 \wedge \cdots \wedge \sigma_{r-k+1})\big] \in A^k(X)$$

is independent of the choice of σ_i, and is called the k^{th} *Chern class* of $\mathcal{E}$.

Lemma 7.1 assures us that a globally generated bundle $\mathcal{E}$ will indeed have such collections of sections, and so this (once we have proved independence of the choice of σ_i) defines the Chern classes for any such bundle.

By convention, we take the zeroth Chern class $c_0(\mathcal{E})$ to be the fundamental class $1 \in A(X)$, and the Chern classes $c_l(\mathcal{E})$ to be zero when $l > r$. By the *total Chern class*, or simply *Chern class*, of $\mathcal{E}$ we will mean the sum

$$c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + \cdots + c_r(\mathcal{E}).$$

Proof. We start with the case $k = r$ of the top Chern class. If σ and τ are two sections of $\mathcal{E}$ whose zero loci are of codimension r, then we can interpolate between $V(\sigma)$ and $V(\tau)$ with the family

$$\Phi = \{([s,t],p) \in \mathbb{P}^1 \times X \mid s\sigma(p) + t\tau(p) = 0\}.$$

This gives a rational equivalence between $V(\sigma)$ and $V(\tau)$: since Φ has codimension at most r everywhere, components of Φ intersecting the fibers over 0 or $\infty \in \mathbb{P}^1$ must dominate $\mathbb{P}^1$, and taking the union of these components we get a rational equivalence between the class of the zero locus of σ and that of τ.

The same argument works in the general case: if $\sigma_1, \ldots, \sigma_{r-k+1}$ and $\tau_1, \ldots, \tau_{r-k+1}$ are collections of sections with degeneracy loci of the expected codimension k, we set

$$\Phi = \{([s,t],p) \in \mathbb{P}^1 \times X \mid p \in V(s\sigma_1 + t\tau_1 \wedge \cdots \wedge s\sigma_{r-k+1} + t\tau_{r-k+1})\};$$

again, the components of Φ dominating $\mathbb{P}^1$ give a rational equivalence between $V(\sigma_1 \wedge \cdots \wedge \sigma_{r-k+1})$ and $V(\tau_1 \wedge \cdots \wedge \tau_{r-k+1})$. $\square$

There is an equivalent way to characterize the Chern classes of a globally generated bundle that is often useful. If $W \subset H^0(\mathcal{E})$ is any vector space of sections generating $\mathcal{E}$, there is a morpshism

$$\varphi = \varphi_W : X \to G(m-r, W)$$

sending $p \in X$ to the kernel of the evaluation map $W \to \mathcal{E}_p$.

Proposition 7.3. *Let $\mathcal{E}$ be a vector bundle of rank r on the smooth, quasiprojective variety X and $W \subset H^0(\mathcal{E})$ an m-dimensional vector space of sections generating $\mathcal{E}$. If $\varphi : X \to G(m-r, W)$ is the associated morphism, then the k^{th} Chern class $c_k(\mathcal{E})$ is the pullback*

$$c_k(\mathcal{E}) = \varphi^*(\sigma_k)$$

of the Schubert class $\sigma_k \in A^k(G(m-r, W))$.

Proof. Suppose taht $\sigma_1, \ldots, \sigma_{r-k+1} \in W$ is a general collection of $r-k+1$ sections, and let $W_{r-k+1} \subset W$ be the subspace they span. The locus $V(\sigma_1 \wedge \cdots \wedge \sigma_{r-k+1}) \subset X$ is the preimage $\varphi^{-1}(\Sigma)$ of the Schubert cycle

$$\Sigma = \Sigma_k(W_l) = \{\Lambda \in G(m-r, W) \mid \Lambda \cap W_{r-k+1} \neq 0\}$$

of $m-r$-planes in W meeting W_{r-k+1} nontrivially. By Kleiman's Theorem, $\operatorname{codim}(\varphi^{-1}(\Sigma) \subset X) = \operatorname{codim}(\Sigma \subset G(m-r, W))$.

To conclude that $c_k(\mathcal{E}) = \varphi^*(\sigma_k)$, we use Theorem 5.12. The necessary result that the Schubert cycle $\Sigma_k(W_l)$ is Cohen-Macaulay is implied by the Cohen-Macaulay property of generic determinantal ideals, proved for example in Bruns and Herzog [1993], Theorem 7.3.1 (a). In fact all Schubert varieties are Cohen-Macaulay, and this is implied by part (b) of that Theorem, although Schubert varieties are not explicitly mentioned; see for example De Concini et al. [1982] Section 11, where the connection is made explicit. □

7.1.2 Remarks

• So far we've defined Chern classes only for globally generated bundles. Looking ahead, we'll fix this in Section 7.2.6: any bundle on a quasiprojective variety will be generated by its global sections after twisting with a sufficiently ample line bundle, and we will use this to extend our definition to arbitrary bundles. Specifically, in Section 7.2.5 we establish a relationship between the Chern classes of a bundle and the Chern classes of its tensor product with a line bundle, and in Proposition-Definition 7.11 we use this to define the Chern classes of an arbitrary bundle $\mathcal{E}$ in terms of those of $c(\mathcal{E} \otimes \mathcal{L})$ for sufficiently ample $\mathcal{L}$.

• There is another important characterization of Chern classes, given in Theorem 11.9, which is often used as a definition. The advantage of that approach is that it has nothing to do with generation by global sections, and thus avoids the passage through Proposition-Definition 7.11. We have adopted the definition here because it is geometric and intuitive, and because it is close in spirit to the applications we'll make. See Section **??** for a discussion of some other definitions of Chern classes.

• In case X is smooth and projective, the notion of Chern classes can be extended even further, to arbitrary coherent sheaves on X; this is described in Chapter **??**.

• Instead of taking $l \leq r$ sections of a globally generated bundle $\mathcal{E}$ and asking where they fail to be linearly independent, we could take $m \geq r$ sections and look at the loci where they fail to span $\mathcal{E}$. The resulting classes are called the *Segre classes* $s_k(\mathcal{E})$; the relationship between these and the Chern classes will be described in Chapter 12.

7.1.3 Application: A strong Bertini theorem

Here is an application of Lemma 7.1.

Proposition 7.4 (Strong Bertini)**.** *Let X be a smooth, n-dimensional quasiprojective variety and $\mathcal{D}$ a linear system of divisors on X whose base locus*

$$Z = \bigcap_{D \in \mathcal{D}} D$$

is a smooth k-dimensional subscheme of X. If $k < n/2$, then the general member of the linear system $\mathcal{D}$ is smooth.

Before proving this, note that the inequality $k < n/2$ is necessary, and indeed if we were to replace it by any strictly weaker inequality the statement of the Proposition would be false. The simplest nontrivial example of this would be to take $X = \mathbb{P}^4$ and $Z = \mathbb{P}^2$ a 2-plane in $\mathbb{P}^4$: if $Y = V(F) \subset \mathbb{P}^4$ is any hypersurface of degree $d > 1$ containing Z, the partial derivatives of F must have a common zero somewhere along Z and so Y must be singular. For some extensions of this example see Exercises 7.39 and 7.40.

Proof of Proposition 7.4. We start by passing to the blow-up $\pi : \tilde{X} = Bl_Z X$ of X along Z, with exceptional divisor $E = \pi^{-1}(Z)$. Let $\tilde{\mathcal{D}}$ be the proper transform in $\tilde{X}$ of the linear system $\mathcal{D}$; that is,

$$\tilde{\mathcal{D}} = \{\pi^* D - E : D \in \mathcal{D}\}.$$

(Note that a general $D \in \mathcal{D}$ will have multiplicity 1 along Z, so that the corresponding element $\tilde{D} = \pi^* D - E \in \tilde{\mathcal{D}}$ will indeed be its proper transform, but not every member of $\tilde{\mathcal{D}}$ need be the proper transform of a member of $\mathcal{D}$.)

Now, by our hypothesis that the base locus of $\mathcal{D}$ is the smooth scheme Z, the linear system $\tilde{\mathcal{D}}$ has no base locus, and so by the classical Bertini theorem the general member $\tilde{D} \in \tilde{\mathcal{D}}$ will be smooth. The question then is, does the map $\pi : \tilde{D} \to D \subset X$ introduce any singularities? Clearly it doesn't away from E.

As for what goes on along $\tilde{D} \cap E$, we observe first that *the intersection $\tilde{D} \cap \pi^{-1}(p)$ of $\tilde{D}$ with any fiber of π over a point $p \in Z$ is a reduced linear subspace of* $\pi^{-1}(p) \cong \mathbb{P}^{n-k-1}$, either a hyperplane in $\pi^{-1}(p)$ or all of $\pi^{-1}(p)$. To see this, let $D \in \mathcal{D}$ be any divisor of the original linear series, $\mathcal{L} = \mathcal{O}_X(D)$ the line bundle associated to D and $\sigma \in H^0(\mathcal{L})$ the section of $\mathcal{L}$ defining D. The restriction to Z of the section σ then gives a section τ of the tensor product $(\mathcal{I}_Z/\mathcal{I}_Z^2) \otimes \mathcal{L} = \mathcal{N}_{Z/X}^* \otimes \mathcal{L}$ of the conormal bundle of Z in X with $\mathcal{L}$; and the intersection $\tilde{D} \cap \pi^{-1}(p)$ is the zero locus, in $\pi^{-1}(p) = \mathbb{P}(\mathcal{N}_{Z/X})_p = \mathbb{P}(\mathcal{N}_{Z/X} \otimes \mathcal{L})_p$, of $\tau(p)$—either a hyperplane in $\pi^{-1}(p)$ if $\tau(p) \neq 0$, or all of $\pi^{-1}(p)$ if $\tau(p) = 0$. Moreover, we see from this that D will be smooth at p exactly when the former is the case, that is, $\tau(p) \neq 0$ and $\tilde{D} \cap \pi^{-1}(p)$ is correspondingly a proper subspace of $\pi^{-1}(p)$.

Now let $V \subset H^0(\mathcal{L})$ be the vector space of sections of $\mathcal{L}$ corresponding to the linear system $\mathcal{D}$, and let $W \subset H^0(\mathcal{N}^*_{Z/X} \otimes \mathcal{L})$ be the associated vector space of sections of $\mathcal{N}^*_{Z/X} \otimes \mathcal{L}$. We know that, as linear forms on $\mathcal{N}_{Z/X} \otimes \mathcal{L}$, the sections $\tau \in W$ have no common zeros, which is to say W generates $\mathcal{N}^*_{Z/X} \otimes \mathcal{L}$ everywhere. Given that the rank $n-k$ of $\mathcal{N}^*_{Z/X} \otimes \mathcal{L}$ is strictly greater than $\dim Z = k$, it follows from Lemma 7.1 that a general element of W is nowhere zero.

This concludes the proof of Lemma 7.4: we see that for general $D \in \mathcal{D}$, the corresponding divisor $\tilde{D} = \pi^* D - E$ is smooth and contains no fibers $\pi^{-1}(p)$ of E over Z, and so D is smooth. □

7.2 Basic properties of Chern classes

Now comes the crucial part: we're going to describe the basic properties of Chern classes, and in particular the relations among the Chern classes of bundles related by linear algebra constructions. This will serve two important purposes: first, these are the tools we will use to compute Chern classes in practice; and second, this will allow us to extend the definition of Chern classes to all vector bundles, not just those with enough sections.

7.2.1 Pullbacks

Probably the most fundamental aspect of Chern classes is how they behave with respect to pullbacks: if $f : Y \to X$ is a morphism and $\mathcal{E}$ a vector bundle of rank r of X, what is the relationship between $c(\mathcal{E})$ and $c(f^*\mathcal{E})$?

The answer is the simplest possible. Suppose that $\mathcal{E}$ is generated by global sections (so that $f^*\mathcal{E}$ will be generated by $f^*H^0(\mathcal{E})$). If $\sigma_1, \dots, \sigma_{r-k+1} \in H^0(\mathcal{E})$ are general elements we have

$$c_k(\mathcal{E}) = \big[V(\sigma_1 \wedge \cdots \wedge \sigma_{r-k+1})\big] \in A^k(X)$$

and

$$c_k(f^*\mathcal{E}) = \big[V(f^*\sigma_1 \wedge \cdots \wedge f^*\sigma_{r-k+1})\big] \in A^k(Y).$$

Note also that by definition

$$f^{-1}(V(\sigma_1 \wedge \cdots \wedge \sigma_{r-k+1})) = V(f^*\sigma_1 \wedge \cdots \wedge f^*\sigma_{r-k+1}).$$

Since

$$\operatorname{codim}\big(f^{-1}(V(\sigma_1 \wedge \cdots \wedge \sigma_{r-k+1})) \subset Y\big) = \operatorname{codim}\big(V(\sigma_1 \wedge \cdots \wedge \sigma_{r-k+1}) \subset X\big),$$

and the cycle $V(\sigma_1 \wedge \cdots \wedge \sigma_{r-k+1})$ is Cohen-Macaulay ****reference****, we can invoke Theorem 5.12 to arrive at the basic fact

$$c_k(f^*\mathcal{E}) = f^*c_k(\mathcal{E}).$$

We can also see this in terms of the alternative characterization of Chern classes given in Proposition 7.3. Let $W \subset H^0(\mathcal{E})$ be any m-dimensional space of sections generating $\mathcal{E}$, and set $V = f^*W \subset H^0(f^*\mathcal{E})$; say $\dim V = n$. Note that we have a surjection $\pi : W \to V$, whose kernel is simply the subspace $U \subset W$ of sections $\sigma \in W$ vanishing identically on the image $f(Y) \subset X$ of f.

The space $V \subset H^0(f^*\mathcal{E})$ generates the bundle $f^*\mathcal{E}$ everywhere, so that we have maps

$$\varphi_W : X \to G(m-r, W) \quad \text{and} \quad \varphi_V : Y \to G(n-r, V).$$

Unpacking the definitions, we see that

$$\varphi_V = \iota \circ \varphi_W \circ f$$

where $\iota : \Sigma \to G(n-r, V)$ is the identification of the sub-Grassmannian

$$\Sigma = \{\Lambda \in G(m-r, W) \mid U \subset \Lambda\} \cong G(n-r, V)$$

sending $\Lambda \in \Sigma$ to $\Lambda/U \in G(n-r, V)$. Since such inclusions of Grassmannians preserve Schubert classes, we have

$$c_k(f^*\mathcal{E}) = \varphi_V^*(\sigma_k) = f^*\varphi_W^*(\sigma_k) = f^*c_k(\mathcal{E})$$

as before.

7.2.2 Line bundles

Note that in case $r = 1$—that is when $\mathcal{E} = \mathcal{L}$ is a line bundle—our definition of Chern classes in general agrees with a special case of our previous definition of the first Chern class $c_1(\mathcal{L})$: when $\mathcal{L}$ is generated by global sections, $c_1(\mathcal{L})$ is the Chow class of the zero locus of any nonzero section. Combining the two, we have defined Chern classes for globally generated bundles of any rank and for arbitrary line bundles.

If $\mathcal{L}$ is an arbitrary line bundle, then we have defined its Chern class to be

$$c_1(\mathcal{L}) = [D - E]$$

where D is the divisor of zeros of a rational section σ of $\mathcal{E}$ and E its polar divisor. From this definition it follows that

$$c_1(\mathcal{L}^*) = -c_1(\mathcal{L}),$$

and also that for any two line bundles $\mathcal{L}$ and $\mathcal{M}$,

$$c_1(\mathcal{L} \otimes \mathcal{M}) = c_1(\mathcal{L}) + c_1(\mathcal{M}).$$

(The first also follows from the second by taking $\mathcal{M} = \mathcal{L}^*$.) These relations form the cornerstone of the calculus of Chern classes: we'll use them, in combination with the splitting principle and the Whitney formula below, to derive all our relations among Chern classes of bundles in general.

Note that the pullback formula $c(f^*\mathcal{L}) = f^*c(\mathcal{L})$ holds for all line bundles, not just those generated by global sections: we can always find a rational section such that the support of its divisor is generically transverse to the map f.

There is a line bundle associated to any vector bundle $\mathcal{E}$ of rank r: its top exterior power $\wedge^r\mathcal{E}$, sometimes called the *determinant bundle* of $\mathcal{E}$ and denoted $\det(\mathcal{E})$. Note that if $\mathcal{E}$ is globally generated and $\sigma_1, \dots, \sigma_r$ are general sections, the wedge product $\sigma_1 \wedge \cdots \wedge \sigma_r$ will be a global section of $\det(\mathcal{E})$, with zero locus $V(\sigma_1 \wedge \cdots \wedge \sigma_r)$; we see accordingly that

$$c_1(\det(\mathcal{E})) = c_1(\mathcal{E}).$$

We'll see in Section 7.2.8 below that this relation holds for all vector bundles.

7.2.3 The Whitney formula

Let X be as usual a smooth variety, and $\mathcal{E}$ and $\mathcal{F}$ vector bundles on X. There is a fundamental formula relating the Chern class of the direct sum $\mathcal{E} \oplus \mathcal{F}$ to the Chern classes of $\mathcal{E}$ and $\mathcal{F}$. This is the *Whitney formula*, most conveniently expressed in terms of the total Chern classes:

Theorem 7.5 (Whitney formula for direct sums)**.** *If $\mathcal{E}$ and $\mathcal{F}$ are globally generated vector bundles on a smooth, quasiprojective variety X, then*

$$c(\mathcal{E} \oplus \mathcal{F}) = c(\mathcal{E})c(\mathcal{F});$$

that is,

$$c_k(\mathcal{E} \oplus \mathcal{F}) = \sum_{0 \le l \le k} c_l(\mathcal{E})c_{k-l}(\mathcal{F}).$$

for all $k \ge 0$.

We will see below (Theorem **??**) that the formula holds more generally for extensions of $\mathcal{F}$ by $\mathcal{E}$.

Proof. Denote the ranks of $\mathcal{E}$ and $\mathcal{F}$ by e and f respectively. We start by remarking that the formula is most readily visible in the extreme cases: the first Chern class $c_1(\mathcal{E} \oplus \mathcal{F})$ and the top Chern class $c_{e+f}(\mathcal{E} \oplus \mathcal{F})$. In the first of these cases, Whitney says

$$c_1(\mathcal{E} \oplus \mathcal{F}) = c_1(\mathcal{E}) + c_1(\mathcal{F}),$$

and we can see this directly: if $\sigma_1, \ldots, \sigma_e \in H^0(\mathcal{E})$ and $\tau_1, \ldots, \tau_f \in H^0(\mathcal{F})$ are general sections, then the degeneracy locus of the $e+f$ sections

$$(\sigma_1, 0), \ldots, (\sigma_e, 0), (0, \tau_1), \ldots, (0, \tau_f) \in H^0(\mathcal{E} \oplus \mathcal{F})$$

is just the sum, as divisors, of the degeneracy loci $V(\sigma_1 \wedge \cdots \wedge \sigma_e)$ and $V(\tau_1 \wedge \cdots \wedge \tau_f)$. (Equivalently, we could just observe that

$$\wedge^{e+f}(\mathcal{E} \oplus \mathcal{F}) = \wedge^e \mathcal{E} \otimes \wedge^f \mathcal{F}$$

and invoke the equality $c_1(\det(\mathcal{E})) = c_1(\mathcal{E})$ of the preceding section.)

Similarly, at the other end Whitney says that the top Chern class of $\mathcal{E} \oplus \mathcal{F}$ is the product of the top Chern classes of $\mathcal{E}$ and $\mathcal{F}$:

$$c_{e+f}(\mathcal{E} \oplus \mathcal{F}) = c_e(\mathcal{E}) c_f(\mathcal{F}).$$

To see this, let σ and τ be general sections of $\mathcal{E}$ and $\mathcal{F}$ respectively. The zero locus $V((\sigma, \tau))$ of the section $(\sigma, \tau) \in H^0(\mathcal{E} \oplus \mathcal{F})$ is then the intersection of the zero loci $V(\sigma)$ and $V(\tau)$; by Lemma 7.1 applied to $\mathcal{F}|_{V(\sigma)}$, it will have the expected codimension $e+f$ and the equality above follows.

The general case is substantially more complicated; but it can be solved, as can so many of life's little problems, by Schubert calculus. We adopt the alternative characterization of Chern classes of Proposition 7.3: if $V \subset H^0(\mathcal{E})$ is an n-dimensional subspace generating $\mathcal{E}$, we have a map $\varphi_V : X \to G(n-e, V)$ sending p to the subspace $V_p \subset V$ of sections vanishing at p; the k^{th} Chern class of $\mathcal{E}$ is then the pullback $\varphi_V^* \sigma_k$ of the Schubert class $\sigma_k \in A^k(G(n-e, V))$.

Let $V \subset H^0(\mathcal{E})$ and $W \subset H^0(\mathcal{F})$ be generating subspaces, of dimensions n and m; let φ_V and φ_W be the corresponding maps. The subspace $V \oplus W \subset H^0(\mathcal{E} \oplus \mathcal{F})$ is again generating, and gives a map

$$\varphi_{V \oplus W} : X \to G(n+m-e-f, V \oplus W)$$

Let

$$\varphi_v \times \varphi_W : X \to G(n-e, V) \times G(m-f, W)$$

be the product map. We have

$$\varphi_{V \oplus W} = \eta \circ (\varphi_V \times \varphi_W)$$

where $\eta : G(n-e, V) \times G(m-f, W) \to G(n+m-e-f, V \oplus W)$ is the map sending a pair of subspaces of V and W to their direct sum. Given this, the Whitney formula will follow from the next Lemma:

Lemma 7.6. *Let V and W be vector spaces of dimensions n and m. For any s and t, let*

$$\eta : G(s, V) \times G(t, W) \to G(s + t, V \oplus W)$$

be the map sending a pair (Λ, Γ) to $\Lambda \oplus \Gamma$. If α and β are the projection maps on $G(s, V) \times G(t, W)$, then for any k,

$$\eta^*(\sigma_k) = \sum_{i+j=k} \alpha^*\sigma_i \cdot \beta^*\sigma_j.$$

Note that Lemma 7.6 is a direct (and substantial) generalization of the calculation, in Section 1.2.3, of the class of the diagonal $\Delta \subset \mathbb{P}^r \times \mathbb{P}^r$. Specifically, if $V = W$, $m = n = r+1$ and $s = t = 1$, then the diagonal $\Delta \subset \mathbb{P}^r \times \mathbb{P}^r$ is just the preimage, under the map $\eta : \mathbb{P}V \times \mathbb{P}V \to G(2, V \oplus V)$, of the Schubert cycle $\Sigma_n(V)$ of 2-planes intersecting the diagonal $V \subset V \oplus V$. Thus Lemma 7.6 in this case yields the formula of Section 1.2.3.

Proof. As in the earlier calculation of the class of the diagonal in $\mathbb{P}^r \times \mathbb{P}^r$, we will use the method of undetermined coefficients. Note that the product $G(s, V) \times G(t, W)$ can be stratified by products of Schubert cells, so that the products $\alpha^*\sigma_a \cdot \beta^*\sigma_b$ span $A(G(s, V) \times G(t, W))$. (In particular, we have $A_0(G(s, V) \times G(t, W)) = \mathbb{Z}$). Moreover, intersection products in complementary dimensions between classes of this type again has a simple form: we have

$$\deg\big((\alpha^*\sigma_a\beta^*\sigma_b)(\alpha^*\sigma_c\beta^*\sigma_d)\big) = \begin{cases} 1 & \begin{array}{l}\text{if } a_i + c_{s-i+1} = n - s \text{ for all } i \text{ and}\\ \quad b_j + d_{m-j+1} = m - t \text{ for all } j\end{array} \\ 0 & \text{otherwise}\end{cases}$$

From this, we see that $A(G(s, V) \times G(t, W))$ is freely generated by the classes $\alpha^*\sigma_a\beta^*\sigma_b$, and that the intersection pairing in complementary dimensions is nondegenerate. Thus, to prove the equality of Lemma 7.6 it will be enough to show that both sides have the same product with any class $\alpha^*\sigma_a \cdot \beta^*\sigma_b$. Specifically, we need to show that for products $\alpha^*\sigma_a \cdot \beta^*\sigma_b$ of dimension k—that is, with $|a| + |b| = s(n - s) + t(m - t) - k$—we have

$$\deg\big(\eta^*\sigma_k \cdot \alpha^*\sigma_a \cdot \beta^*\sigma_b\big) = \begin{cases} 1 & \begin{array}{l}\text{if } a = (n-s, \dots, n-s, n-s-i) \text{ and}\\ \quad b = (m-t, \dots, m-t, m-t-j)\\ \quad \text{for some } i + j = k; \text{ and}\end{array} \\ 0 & \text{otherwise}\end{cases}$$

We start with the "otherwise" half. Note that, by the dimension condition $|a|+|b| = s(n-s)+t(m-t)-k$, the condition $a = (n-s, \dots, n-s, n-s-i)$ and $b = (m-t, \dots, m-t, m-t-j)$ for some $i+j = k$ is equivalent to saying that the sum of the last two indices a_s and b_t is $a_s + b_t = n-s+m-t-k$; in all other cases it will be strictly greater.

Start by choosing general flags $V_1 \subset \cdots \subset V_n = V$, $W_1 \subset \cdots \subset W_m = W$ and $U_1 \subset \cdots \subset U_{n+m} = V \oplus W$. Then

$$\Sigma_a(\mathcal{V}) \subset \{\Lambda \subset V \mid \Lambda \subset V_{n-a_s}\}$$

and

$$\Sigma_b(\mathcal{W}) \subset \{\Gamma \subset W \mid \Lambda \subset W_{m-a_t}\}$$

so

$$\eta\big(\alpha^{-1}\Sigma_a \cap \beta^{-1}\Sigma_b\big) \subset \{\Omega \subset V \oplus W \mid \Omega \subset V_{n-a_s} \oplus W_{m-a_t}\}.$$

But

$$\Sigma_k(\mathcal{U}) = \{\Omega \subset V \oplus W \mid \Omega \cap U_{n-s+m-t-k+1} \neq 0\}$$

and if $a_s+b_t > n-s+m-t-k$, then $(V_{n-a_s} \oplus W_{m-a_t}) \cap U_{n-s+m-t-k+1} = 0$; thus

$$\eta^{-1}\Sigma_k \cap \alpha^{-1}\Sigma_a \cap \beta^{-1}\Sigma_b = \varnothing$$

and the product of the corresponding classes is zero.

Similarly, in case $a = (n-s, \ldots, n-s, n-s-i)$ and $b = (m-t, \ldots, m-t, m-t-j)$ for some $i+j=k$, the intersection $U = (V_{n-a_s} \oplus W_{m-a_t}) \cap U_{n-s+m-t-k+1}$ will be one-dimensional. Since

$$\Sigma_a(\mathcal{V}) = \left\{\Lambda \subset V \mid \begin{array}{l} V_{s-1} \subset \Lambda \text{ and} \\ \Lambda \subset V_{n-a_s} \end{array}\right\}$$

and

$$\Sigma_b(\mathcal{W}) = \left\{\Gamma \subset W \mid \begin{array}{l} W_{t-1} \subset \Lambda \text{ and} \\ \Lambda \subset W_{m-a_t} \end{array}\right\}$$

we see that the intersection $\eta^{-1}\Sigma_k \cap \alpha^{-1}\Sigma_a \cap \beta^{-1}\Sigma_b$ will consist of the single point (Λ, Γ), where $\Lambda \subset V$ is the span of V_{s-1} and the projection $\pi_1(U)$ and likewise $\Gamma \subset W$ is the span of W_{t-1} and the image $\pi_2(U)$. That the intersection is transverse follows from Kleiman's theorem in characteristic 0, and from direct examination of the tangent spaces in general. $\square$

Given Lemma 7.6, the Whitney formula follows: with φ_V, φ_W and $\varphi_{V\oplus W}$ as above, we have

$$\begin{aligned} c_k(\mathcal{E} \oplus \mathcal{F}) &= \varphi^*_{V\oplus W}(\sigma_k) \\ &= \sum_{i+j=k} \varphi^*_V(\sigma_i)\varphi^*_V(\sigma_j) \\ &= \sum_{i+j=k} c_i(\mathcal{E})c_j(\mathcal{F}), \end{aligned}$$

which concludes the proof of the Whitney formula for direct sums. $\square$

As indicated, the Whitney formula actually holds more generally:

Theorem 7.7 (Whitney formula for extensions)**.** *If*

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow 0$$

is an exact sequence of globally generated vector bundles, then

$$c(\mathcal{G}) = c(\mathcal{E})c(\mathcal{F}).$$

Proof. We prove this by reducing the problem to the case where $\mathcal{G} \cong \mathcal{E} \oplus \mathcal{F}$; that is, we will show that $c(\mathcal{G}) = c(\mathcal{E} \oplus \mathcal{F})$ and invoke the Whitney formula for direct sums.

Suppose that we are given the exact sequence of vector bundles on X:

$$0 \longrightarrow \mathcal{E} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{F} \longrightarrow 0.$$

We define a vector bundle $\mathcal{H}$ on $\mathbb{A}^1 \times X$ by

$$\mathcal{H} := \frac{\mathcal{E} \oplus \mathcal{G}}{(t, \alpha)(\mathcal{E})}.$$

where (t, α) denotes the map from $\mathcal{E}$ to $\mathcal{E} \oplus \mathcal{G}$ that is multiplication by t from $\mathcal{E}$ to $\mathcal{E}$ and acts by α from $\mathcal{E}$ to $\mathcal{G}$. We think of $\mathcal{H}$ as a family of vector bundles on X, parametrized by $t \in \mathbb{A}^1$; for $t \neq 0$, the bundle $\mathcal{H}_t = \mathcal{H}|_{\{t\} \times X}$ is isomorphic to $\mathcal{G}$, while for $t = 0$ it's the direct sum $\mathcal{E} \oplus \mathcal{F}$. (Those familiar with the Yoneda construction of Ext will recognize this family as the middle vector bundle in the extension that is t times the extension class of the given exact sequence.)

Now, since we assumed $\mathcal{E}$ and $\mathcal{G}$ were globally generated, $\mathcal{H}$ will be as well. We thus have a surjection

$$V \otimes \mathcal{O}_{\mathbb{A}^1 \times X} \cong \mathcal{O}^m_{\mathbb{A}^1 \times X} \to \mathcal{H} \to 0$$

for $V \subset H^0(\mathcal{H})$ a suitable m-dimensional vector space of sections of $\mathcal{H}$, and correspondingly a morphism

$$\varphi : \mathbb{A}^1 \times X \to G(m - e - f, V).$$

We think of φ as a family of morphisms $\varphi_t = \varphi|_{\{t\} \times X} : X \to G(m-e-f, V)$ parametrized by $t \in \mathbb{A}^1$, with

$$\varphi_0^*(\sigma_k) = c_k(\mathcal{E} \oplus \mathcal{F}) \quad \text{and} \quad \varphi_t^*(\sigma_k) = c_k(\mathcal{G}) \text{ for } t \neq 0.$$

We can then choose a cycle Σ in $G(m - e - f, V)$ representing the class σ_k that is generically transverse to φ_0, and hence also generically transverse to φ_t for t in a neighborhood U of 0. The preimage $\varphi^*(\Sigma) \in Z(\mathbb{A}^1 \times X)$ then gives a rational equivalence between $\varphi_0^*(\Sigma)$ and $\varphi_t^*(\Sigma)$ for $t \in U$, showing that $c_k(\mathcal{E} \oplus \mathcal{F}) = c_k(\mathcal{G})$ as desired. □

7.2.4 The splitting principle

A natural complement to the Whitney formula is the *splitting principle*, which allows us to relate an arbitrary bundle to a direct sum of line bundles. The combination of the splitting principle and the Whitney formula is a powerful tool for calculating Chern classes, as we'll see over and over in the remainder of this text (and more immediately in Exercises 7.20-7.26 below).

Theorem 7.8 (Splitting principle)**.** *Let X be any smooth variety, $\mathcal{E}$ a vector bundle of rank r on X. Then there exists a smooth variety Y and a morphism $\varphi : Y \to X$ with two properties:*

(a) *The pullback map $\varphi^* : A(X) \to A(Y)$ is injective; and*

(b) *The pullback bundle $\varphi^*\mathcal{E}$ on Y admits a filtration*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{r-1} \subset \mathcal{E}_r = \varphi^*\mathcal{E}$$

by vector subbundles $\mathcal{E}_i \subset \varphi^*\mathcal{E}$ with successive quotients $\mathcal{E}_i/\mathcal{E}_{i-1}$ locally free of rank 1.

Given this, we can use the Whitney formula and our a priori definition of the Chern class of a line bundle to describe the Chern class of the pullback:

$$c(\varphi^*\mathcal{E}) = \prod_{i=1}^{r} c(\mathcal{E}_i/\mathcal{E}_{i-1});$$

and by the first part of the Lemma this determines the Chern classes of $\mathcal{E}$.

Proof. We will exhibit such a morphism $Y \to X$ explicitly, using the notion of the *projectivization* of a vector bundle. This construction is described in detail in Chapter 11.

To start, let $\pi : \mathbb{P}\mathcal{E} \to X$ be the projectivization of the bundle $\mathcal{E}$, that is, the variety

$$\mathbb{P}\mathcal{E} = \{(x, \xi) \mid x \in X \text{ and } \xi \subset \mathcal{E}_x \text{ a 1-dimensional subspace}\}$$

with the map $\pi : (x, \xi) \mapsto x$. We observe that the pullback bundle $\pi^*\mathcal{E}$ has a tautological sub-line bundle $\mathcal{S}$, defined by specifying its fiber over a point $(x, \xi) \in \mathbb{P}\mathcal{E}$ as

$$\mathcal{S}_{(x,\xi)} = \xi \subset \mathcal{E}_x.$$

Now let $\mathcal{Q}$ be the quotient bundle $\pi^*\mathcal{E}/\mathcal{S}$ on $\mathbb{P}\mathcal{E}$ so that we have an exact sequence

$$0 \to \mathcal{S} \to \pi^*\mathcal{E} \to \mathcal{Q} \to 0.$$

We can repeat this process, taking the projectivization $\mathbb{P}\mathcal{Q}$ and introducing the tautological sub-line bundle of the pullback of $\mathcal{Q}$ to $\mathbb{P}\mathcal{Q}$. Indeed, iterating this $r-1$ times we arrive at the *flag bundle* $\varphi : \mathbb{F}\mathcal{E} \to X$, consisting of all flags in the fibers of the original bundle $\mathcal{E}$:

$$\mathbb{F}\mathcal{E} = \{(x, \xi_1, \xi_2, \dots, \xi_{r-1}) \mid x \in X \text{ and } \xi_1 \subset \xi_2 \subset \cdots \subset \xi_{r-1} \subset \mathcal{E}_x\};$$

and on $\mathbb{F}\mathcal{E}$ we have the desired filtration

$$0 \subset \mathcal{S}_1 \subset \mathcal{S}_2 \subset \cdots \subset \mathcal{S}_{r-1} \subset \varphi^*\mathcal{E}$$

where $\mathcal{S}_k$ is the rank k subbundle of $\varphi^*\mathcal{E}$ defined by

$$(\mathcal{S}_k)_{(x,\xi_1,\xi_2,\dots,\xi_{r-1})} = \xi_k.$$

It remains now to see that the pullback map $\varphi^* : A(X) \to A(\mathbb{F}\mathcal{E})$ is injective. Note that it's enough to prove the pullback map $\pi^* : A(X) \to A(\mathbb{P}\mathcal{E})$ is injective, since φ is a composition of $r-1$ such maps. This in turn follows from a direct calculation:

Lemma 7.9. *With X, $\mathcal{E}$, $\pi : \mathbb{P}\mathcal{E} \to X$ and $\mathcal{S}$ as above,*

$$\pi_*\big(c_1(\mathcal{S}^*)^{r-1}\big) = [X].$$

Proof. The class on the left has dimension equal to $\dim X$, so a priori we have

$$\pi_*\big(c_1(\mathcal{S}^*)^{r-1}\big) = c[X]$$

for some integer c; we just need to determine c. We do this as follows: let $x \in X$ be any point, and $\mathbb{P}^{r-1} = \mathbb{P}\mathcal{E}_x$ the fiber of $\mathbb{P}\mathcal{E}$ over x. By the push-pull formula and the functoriality of the first Chern class,

$$\begin{aligned} c &= \big([x] \cdot \pi_*\big(c_1(\mathcal{S}^*)^{r-1}\big)\big) \\ &= \big(\pi^*[x] \cdot c_1(\mathcal{S}^*)^{r-1}\big) \\ &= \big(c_1\left(\mathcal{S}^*|_{\mathbb{P}^{r-1}}\right)\big)^{r-1}. \end{aligned}$$

But the restriction of $\mathcal{S}^*$ to $\mathbb{P}^{r-1}$ is the bundle $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$, whose first Chern class is just the class ζ of a hyperplane in $\mathbb{P}^{r-1}$; thus

$$c = \zeta^{r-1} = 1.$$

□

Given this, Theorem 7.8 follows: by the push-pull formula in general we have for any class $\alpha \in A(X)$

$$\alpha = \pi_*\left(\pi^*\alpha \cdot c_1(\mathcal{S}^*)^{n-1}\right)$$

and so in particular $\pi^*\alpha \neq 0$ if $\alpha \neq 0$. □

7.2.5 Tensor products with line bundles

As an application of the splitting principle, we'll derive the relation between the Chern classes of a vector bundle $\mathcal{E}$ of rank r on a variety X and the Chern classes of the tensor product of $\mathcal{E}$ with a line bundle $\mathcal{L}$.

To do this, let $\varphi : Y \to X$ be a morphism such that the pullback $\varphi^* : A(X) \to A(Y)$ is injective, and such that we have a filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{r-1} \subset \mathcal{E}_r = \varphi^*\mathcal{E};$$

let $\mathcal{M}_i = \mathcal{E}_i/\mathcal{E}_{i-1},\ i = 1, \dots, r$, be the successive quotients in the filtration. Let $\alpha_i = c_1(\mathcal{M}_i) \in A^1(Y)$ be the first Chern class of $\mathcal{M}_i$, and let $\beta = c_1(\varphi^*\mathcal{L}) = \varphi^* c_1(\mathcal{L})$ be the first Chern class of the pullback of $\mathcal{L}$. By the Whitney formula, we have

$$\prod_{i=1}^{r}(1+\alpha_i) = c(\varphi^*\mathcal{E}) = \varphi^* c(\mathcal{E}),$$

so that the elementary symmetric polynomials in the α_i are the pullbacks of the Chern classes of $\mathcal{E}$:

$$\begin{aligned}
\alpha_1 + \alpha_2 + \cdots + \alpha_r &= \varphi^* c_1(\mathcal{E}) \\
\sum_{1 \le i < j \le r} \alpha_i\alpha_j &= \varphi^* c_2(\mathcal{E}) \\
&\ \vdots \\
\alpha_1\alpha_2 \dots \alpha_r &= \varphi^* c_r(\mathcal{E}).
\end{aligned}$$

On the other hand, tensoring the filtration above with $\varphi^*\mathcal{L}$ we arrive at a filtration

$$0 = \mathcal{E}_0 \otimes \varphi^*\mathcal{L} \subset \mathcal{E}_1 \otimes \varphi^*\mathcal{L} \subset \cdots \subset \mathcal{E}_{r-1} \otimes \varphi^*\mathcal{L} \subset \mathcal{E}_r \otimes \varphi^*\mathcal{L} = \varphi^*(\mathcal{E} \otimes \mathcal{L})$$

with successive quotients $\mathcal{M}_i \otimes \varphi^*\mathcal{L}$; since $c_1(\mathcal{M}_i \otimes \varphi^*\mathcal{L}) = \alpha_i + \beta$, we have

$$\varphi^* c(\mathcal{E} \otimes \mathcal{L}) = \prod_{i=1}^{r}(1 + \alpha_i + \beta). \tag{7.1}$$

Now, we can express the product on the right as a polynomial in the elementary symmetric polynomials: for example, we have

$$\varphi^* c_1(\mathcal{E} \otimes \mathcal{L}) = \sum_{i=1}^{r}(\alpha_i + \beta) = \varphi^* c_1(\mathcal{E}) + r\varphi^* c_1(\mathcal{L})$$

and likewise

$$\begin{aligned}\varphi^* c_2(\mathcal{E} \otimes \mathcal{L}) &= \sum_{1 \le i < j \le r} (\alpha_i + \beta)(\alpha_j + \beta) \\ &= \sum_{1 \le i < j \le r} \alpha_i \alpha_j + (r-1)\beta \sum_{i=1}^{r} \alpha_i + \binom{r}{2} \beta^2 \\ &= \varphi^* c_2(\mathcal{E}) + (r-1)\varphi^* c_1(\mathcal{E}) \varphi^* c_1(\mathcal{L}) + \binom{r}{2} \varphi^* c_1(\mathcal{L})^2;\end{aligned}$$

and by the injectivity of φ^* we can drop the pullbacks and deduce the corresponding relation in $A(X)$.

In general we have:

Proposition 7.10. *If $\mathcal{E}$ is a vector bundle of rank r and $\mathcal{L}$ is a line bundle, and if $\mathcal{E}$ and $\mathcal{E} \otimes \mathcal{L}$ are globally generated, then*

$$\begin{aligned}c_k(\mathcal{E} \otimes \mathcal{L}) &= \sum_{l=0}^{k} \binom{r-l}{k-l} c_1(\mathcal{L})^{k-l} c_l(\mathcal{E}) \\ &= \sum_{i=0}^{k} \binom{r-k+i}{i} c_1(\mathcal{L})^i c_{k-i}(\mathcal{E}).\end{aligned}$$

Proof. This is just a matter of collecting the terms in the expression (7.1) of degree l in the α_i and degree $k-l$ in β: we write

$$\begin{aligned}\prod_{i=1}^{r} (1 + \alpha_i + \beta) &= \sum_{1 \le i_1 \le \cdots \le i_l \le r} (1+\beta)^{r-l} \alpha_{i_1} \cdots \alpha_{i_l} \\ &= \sum_{l} c_l(\mathcal{E})(1+\beta)^{r-l}\end{aligned}$$

and the Proposition follows. □

This is our first application of the splitting principle, and it is a template for all that follow: to relate the Chern classes of tensor products of vector bundles, or symmetric or exterior powers of a given bundle, we pull them back to a suitable variety $Y \to X$ where they all split (or at any rate admit filtrations with one-dimensional quotients); we use the relations for the Chern classes of sums and products of line bundles to derive a relation there and deduce the relation on X. A colloquial formulation of the splitting principle is thus:

Any general relation among the Chern classes of bundles that can be proved for direct sums of line bundles holds true in general.

We will often apply this argument without explicitly invoking the map $Y \to X$.

7.2.6 Definition of Chern classes in general

We can now extend the definition of Chern classes to bundles that may not be globally generated. If $\mathcal{E}$ is any bundle on the quasiprojective variety X, we can find a line bundle $\mathcal{L}$ such that $\mathcal{F} := \mathcal{E} \otimes \mathcal{L}^*$ is globally generated; we can then use the formula of Proposition 7.10 to express the Chern classes of $\mathcal{E} = \mathcal{F} \otimes \mathcal{L}$ in terms of those of $\mathcal{F}$ and $\mathcal{L}$. We express this as the following:

Proposition-Definition 7.11. Suppose that $\mathcal{E}$ is a vector bundle of rank r on X, and $\mathcal{L}$ is a line bundle such that $\mathcal{F} := \mathcal{E} \otimes \mathcal{L}$ is globally generated. Note that $\mathcal{E} = \mathcal{F} \otimes \mathcal{L}^*$. The classes

$$c_k(\mathcal{E}) := \sum_{i=0}^{k} \binom{r-k+i}{i} c_1(\mathcal{L}^*)^i c_{k-i}(\mathcal{F})$$

are independent of the choice of $\mathcal{L}$, and are called the *Chern classes* of $\mathcal{E}$.

Before we prove this, we note one immediate and important consequence: the Chern classes $c_k(\mathcal{E})$ are 0 for k strictly greater than the rank r of $\mathcal{E}$.

Proof. Suppose that $\mathcal{L}$ and $\mathcal{M}$ are two line bundles such that $\mathcal{F} := \mathcal{E} \otimes \mathcal{L}$ and $\mathcal{G} := \mathcal{E} \otimes \mathcal{M}$ are globally generated; we have to show that

$$\sum_{i=0}^{k} \binom{r-k+i}{i} c_1(\mathcal{M}^*)^i c_{k-i}(\mathcal{G}) = \sum_{i=0}^{k} \binom{r-k+i}{i} c_1(\mathcal{L}^*)^i c_{k-i}(\mathcal{F}). \tag{7.2}$$

Set $\mathcal{N} = \mathcal{L}^* \otimes \mathcal{M}$, so that $\mathcal{G} = \mathcal{F} \otimes \mathcal{N}$; write $c_1(\mathcal{L}^*) = \alpha$ and $c_1(\mathcal{N}^*) = \beta$, so that $c_1(\mathcal{M}^*) = \alpha + \beta$. Since $\mathcal{F}$ and $\mathcal{G}$ are globally generated, we know by Proposition 7.10 that their Chern classes are related by

$$c_k(\mathcal{F}) = \sum_{i=0}^{k} \binom{r-k+i}{i} \beta^i c_{k-i}(\mathcal{G}).$$

Plugging this into (7.2), we have to show that

$$\begin{aligned}
&\sum_{i=0}^{k} \binom{r-k+i}{i} (\alpha+\beta)^i c_{k-i}(\mathcal{G}) \\
&\quad = \sum_{j=0}^{k} \binom{r-k+j}{j} \alpha^j \left(\sum_{l=0}^{k-j} \binom{r-k+j+l}{l} \beta^l c_{k-j-l}(\mathcal{G}) \right).
\end{aligned}$$

To do this, just compare the coefficients of $\alpha^j \beta^l c_{k-j-l}(\mathcal{H})$: setting $i = j+l$, on the left we have

$$\binom{r-k+j+l}{j+l} \binom{j+l}{j} = \frac{(r-k+j+l)!\,(j+l)!}{(j+l)!\,(r-k)!\,j!\,l!},$$

while on the right we have

$$\binom{r-k+j}{j}\binom{r-k+j+l}{l} = \frac{(r-k+j)!\,(r-k+j+l)!}{j!\,(r-k)!\,(r-k+j)!\,l!},$$

and after cancelling, these are visibly equal. □

We remark here that, by the splitting principle, the basic properties of Chern classes established so far for globally generated vector bundles hold for bundles in general. For example, we have the fundamental

Proposition 7.12. *If $f : Y \to X$ is a map of smooth projective varieties and $\mathcal{E}$ is a vector bundle on X, then*

$$c_k(f^*\mathcal{E}) = f^*c_k(\mathcal{E})$$

for any k.

Proof. We have established this for line bundles, and for globally generated vector bundles; the case of an arbitrary vector bundle $\mathcal{E}$ follows from these, using Proposition-Definition 7.11. □

Similarly, we see that the Whitney formula, established above for globally generated bundles, holds for all vector bundles by applying Proposition-Definition 7.11.

Finally, there is one last wrinkle we should mention in connection with the extension of the definition of Chern classes from globally generated bundles to arbitrary ones. In the applications of Chern classes we will generally have a vector bundle $\mathcal{E}$ on a variety X, and a collection of sections $\sigma_1, \ldots, \sigma_l$ of $\mathcal{E}$ whose degeneracy locus $V(\sigma_1 \wedge \cdots \wedge \sigma_l)$ has the right codimension, and we will want to identify the class of this locus as the Chern class $c_{r-l+1}(\mathcal{E})$. Proposition-Definition 7.2 assures us that this is the case when $\mathcal{E}$ is globally generated, but in fact it holds more generally. (In fact, in all our applications, the bundle in question will indeed be globally generated, but it's a pain to have to check this every time.) We state this as the following Proposition:

Proposition 7.13. *Suppose that $\sigma_1, \ldots \sigma_k$ are global sections of a vector bundle $\mathcal{E}$ of rank r on a smooth projective variety X. If the degeneracy locus $Y = V(\sigma_1 \wedge \cdots \wedge s_k)$ has codimension $r-k+1$ (or is empty), then $c_{r-k+1}(\mathcal{E}) = [Y]$.*

A proof for $k = 1$ is given in Corollary 11.20; the general case is treated as a special case of Porteous' Formula in Corollary 14.2.

7.2.7 Dual bundles

As an immediate application of the splitting principle, we can express the Chern classes of the dual $\mathcal{E}^*$ of a vector bundle $\mathcal{E}$ in terms of those of $\mathcal{E}$. Briefly, if $\mathcal{E}$ were the direct sum

$$\mathcal{E} = \bigoplus_{i=1}^{n} \mathcal{L}_i \quad \text{with} \quad c_1(\mathcal{L}_i) = \alpha_i,$$

we would have

$$c(\mathcal{E}) = \prod_{i=1}^{n} (1 + \alpha_i).$$

We would also have, by the same token,

$$\mathcal{E}^* = \bigoplus_{i=1}^{n} \mathcal{L}_i^*$$

and hence

$$c(\mathcal{E}^*) = \prod_{i=1}^{n} (1 - \alpha_i);$$

thus

$$c_k(\mathcal{E}^*) = (-1)^k c_k(\mathcal{E}).$$

The splitting principle then tells us that this formula holds for all vector bundles, not just split ones.

7.2.8 Determinant of a bundle

By the *determinant* $\det \mathcal{E}$ of a bundle $\mathcal{E}$ we mean the line bundle that is the highest exterior power $\det \mathcal{E} := \wedge^{\operatorname{rank} \mathcal{E}} \mathcal{E}$.

Proposition 7.14. *If $\mathcal{E}$ is a vector bundle of rank r on a smooth variety then $c_1(\mathcal{E}) = c_1(\wedge^r \mathcal{E})$.*

The formula is immediate from the definition in the case when $\mathcal{E}$ is globally generated. Here is an easy general proof illustrating the use of the splitting principle.

Proof. By the splitting principle we may assume that $\mathcal{E}$ has a filtration $0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}$ such that each $\mathcal{E}_{i+1}/\mathcal{E}_i$ is a line bundle. It follows that $\det \mathcal{E} = \prod_i \mathcal{E}_{i+1}/\mathcal{E}_i$, and thus $c_1(\mathcal{E}) = \sum c_1(\mathcal{E}_{i+1}/\mathcal{E}_i) = c_1(\prod_i \mathcal{E}_{i+1}/\mathcal{E}_i) = c_1(\det \mathcal{E})$. $\square$

7.2.9 Tensor product of two bundles

In theory, the Whitney formula and the splitting principle should yield a formula for the Chern class of the tensor product of two bundles of any rank. But, as the reader will see in Exercises 7.26-7.27, the formula in general is complicated. One case we can handle, though, is the first Chern class $c_1(\mathcal{E} \otimes \mathcal{F})$:

Proposition 7.15. *If $\mathcal{E}, \mathcal{F}$ are vector bundles of ranks e and f respectively, then*

$$c_1(\mathcal{E} \otimes \mathcal{F}) = f \cdot c_1(\mathcal{E}) + e \cdot c_1(\mathcal{F})$$

Proof. Suppose $\mathcal{E} = \bigoplus L_i$ and $\mathcal{F} = \bigoplus M_i$ are direct sums of line bundles, so that we can write

$$c(\mathcal{E}) = \prod_{i=1}^{e} (1 + \alpha_i) \quad \text{and} \quad c(\mathcal{F}) = \prod_{j=1}^{f} (1 + \beta_j)$$

with $c_1(L_i) = \alpha_i$ and $c_1(M_j) = \beta_j$; note that $c_1(\mathcal{E}) = \alpha_1 + \ldots \alpha_e$ and $c_1(\mathcal{F}) = \beta_1 + \ldots \beta_f$. We have then

$$\mathcal{E} \otimes \mathcal{F} = \bigoplus_{i,j=1,1}^{e,f} L_i \otimes M_j$$

and correspondingly

$$c(\mathcal{E} \otimes \mathcal{F}) = \prod_{i,j=1,1}^{e,f} (1 + \alpha_i + \beta_j).$$

In particular, this gives

$$\begin{aligned} c_1(\mathcal{E} \otimes \mathcal{F}) &= \sum_{i,j=1,1}^{e,f} (\alpha_i + \beta_j) \\ &= f \sum_{i=1}^{e} \alpha_i + e \sum_{j=1}^{f} \beta_j \\ &= f c_1(\mathcal{E}) + e c_1(\mathcal{F}). \end{aligned}$$

□

We will also see, in Chapter 14, a formula for the top Chern class $c_{ef}(\mathcal{E} \otimes \mathcal{F})$ of a tensor product of bundles of ranks e and f.

Is smoothness necessary?. There is one final remark to be made here. Throughout this chapter, we have considered only vector bundles $\mathcal{E}$ on *smooth* varieties X. In fact, this is needlessly restrictive: if X is any scheme and $\mathcal{E}$ a vector bundle on X, we can define the Chern classes $c_k(\mathcal{E}) \in A_{\dim X - k}(X)$. What's more, all the basic properties of Chern classes—their behavior under pullback and the Whitney formula, in particular—continue to hold.

At first glance, this may not appear to make sense: Whitney's formula, for example, involves intersection products, which are not defined on the Chow group $A(X)$ of an arbitrary X. But in fact it is possible to define products $\alpha\beta$ of Chow classes α and $\beta \in A(X)$ on an arbitrary scheme, *as long as at least one is a Chern class*: we can define the intersection of $c_k(\mathcal{E})$ with the class of an arbitrary closed subvariety $A \subset X$ simply as the k^{th} Chern class $c_k(\mathcal{E}|_A)$ of the restriction of $\mathcal{E}$ to A (or, rather, its push forward to X under the inclusion map $A \hookrightarrow X$). Indeed, this observation is a cornerstone of modern intersection theory; and of all the debatable choices we have made in writing this book, the one that will probably cause Bill Fulton to smack his forehead the hardest is this needless restriction of the definition of Chern classes to bundles on smooth varieties.

7.3 Chern classes of some interesting bundles

There are a lot of different vector bundles, but some come up again and again in geometric applications: the tautological bundles on projective spaces and Grassmannians; tangent bundles; and normal bundles of subvarieties. We'll analyze some examples of the first two types here (with generalizations in chapters to come). Normal bundles will be used, for example, in Chapter 8.

7.3.1 Universal bundles on projective space

We start with the most basic of all bundles: the bundle $\mathcal{O}_{\mathbb{P}^r}(1)$ on projective space $\mathbb{P}^r$. We have

$$c_1(\mathcal{O}_{\mathbb{P}^r}(1)) = \zeta \in A^1(\mathbb{P}^r)$$

where ζ is the hyperplane class; and similarly

$$c_1(\mathcal{O}_{\mathbb{P}^r}(n)) = n \cdot \zeta \in A^1(\mathbb{P}^r)$$

for any $n \in \mathbb{Z}$.

This in turn allows us to compute the Chern class of the universal quotient bundle $\mathcal{Q}$ on $\mathbb{P}^r$: if $\mathbb{P}^r = \mathbb{P}V$, from the exact sequence

$$0 \to \mathcal{S} = \mathcal{O}_{\mathbb{P}^r}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}^r} \to \mathcal{Q} \to 0$$

we have

$$\begin{aligned} c(\mathcal{Q}) &= \frac{1}{c(\mathcal{O}_{\mathbb{P}^r}(-1))} \\ &= \frac{1}{1-\zeta} \\ &= 1 + \zeta + \zeta^2 + \cdots + \zeta^r. \end{aligned}$$

Note that we could also arrive at this directly from the description of Chern classes as degeneracy loci of sections: an element $v \in V$ gives rise to a global section σ of the bundle $\mathcal{Q}$; given k elements $v_1, \dots, v_k \in V$ the corresponding sections $\sigma_1, \dots, \sigma_k$ of $\mathcal{Q}$ will be linearly dependent at a point $x \in \mathbb{P}^r$ exactly when x lies in the $\mathbb{P}^{k-1}$ corresponding to the subspace $W = \langle v_1, \dots, v_k \rangle \subset V$ spanned by the v_i. Thus

$$c_{r-k+1}(\mathcal{Q}) = [\mathbb{P}^{k-1}] = \zeta^{r-k+1} \in A^{r-k+1}(\mathbb{P}^r).$$

7.3.2 Chern classes of tangent bundles

Chern classes in general are invariants of a pair $(X, \mathcal{E})$ consisting of a variety X and a vector bundle $\mathcal{E}$ on X. Since the tangent and cotangent bundles of a smooth variety (or, more generally the sheaf of differentials on any scheme) are intrinsically associated to that variety, their Chern classes are intrinsic invariants of the variety alone, and they are indeed important ones. On the whole, perhaps the most important things we can know about a variety, after a description of its Chow or cohomology ring, are the Chern classes of its tangent bundle. We will make some computations of these in the next few sections.

7.3.3 Tangent bundles of projective spaces

We start by calculating the Chern classes of the tangent bundle $\mathcal{T}_{\mathbb{P}^n}$ of projective space. This is straightforward, given the Euler sequence of Section 2.2.4: we have

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \to \mathcal{T}_{\mathbb{P}^n} \to 0$$

and hence

$$c(\mathcal{T}_{\mathbb{P}^n}) = (1+\zeta)^{n+1}$$

where $\zeta \in A^1(\mathbb{P}^n)$ is the hyperplane class.

We could also derive this from the identification $\mathcal{T} = \mathrm{Hom}(\mathcal{S}, \mathcal{Q}) = \mathcal{S}^* \otimes \mathcal{Q}$, where $S = \mathcal{O}_{\mathbb{P}^n}(-1)$ and $\mathcal{Q}$ are the universal sub- and quotient bundles, by applying Proposition 7.10.

7.3.4 Tangent bundles to hypersurfaces

We can combine the formula above for the Chern classes of the tangent bundle to projective space $\mathbb{P}^r$ and the Whitney formula to calculate the Chern classes of the tangent bundle to a smooth hypersurface $X \subset \mathbb{P}^r$ of degree d.

To do this, we use the standard normal bundle sequence

$$0 \to \mathcal{T}_X \to \mathcal{T}_{\mathbb{P}^r}|_X \to \mathcal{N}_{X/\mathbb{P}^r} \to 0$$

and the identification

$$\mathcal{N}_{X/\mathbb{P}^r} = \mathcal{O}_{\mathbb{P}^r}(X)|_X = \mathcal{O}_X(d).$$

Letting ζ_X denote the restriction to X of the hyperplane class on $\mathbb{P}^r$, we can write

$$\begin{aligned} c(\mathcal{T}_X) &= \frac{c(\mathcal{T}_{\mathbb{P}^r})|X}{\mathcal{N}_{X/\mathbb{P}^r}} \\ &= \frac{(1+\zeta_X)^{r+1}}{1+d\zeta_X} \\ &= \left(1+(r+1)\zeta_X + \binom{r+1}{2}\zeta_X^2 + \dots\right)\left(1 - d\zeta_X + d^2\zeta_X^2 + \dots\right). \end{aligned}$$

There is a beautiful consequence for the topology of a smooth hypersurface over the complex numbers. In the classical, or analytic topology on X, the Hopf Index Theorem **ref** says that the Euler characteristic of X is equal to the degree of the top Chern class of its tangent bundle. Applying the formula above, then, and remembering that $\deg(\zeta_X^{r-1}) = d$, we have

$$\chi(X) = c_{r-1}(\mathcal{T}_X) = \sum_{k=0}^{r-1}(-1)^k\binom{r+1}{k+2}d^{k+1}.$$

In fact, we know more: the Lefschetz Hyperplane Theorem **ref** tells us that the integral cohomology groups of X, except for the middle one $H^{r-1}(X)$, are all equal to the corresponding cohomology groups of projective space: in particular, the Betti numbers are 1 in even dimensions and 0 in odd. (In fact, the analogous statement is true for any smooth complete intersection: all the cohomology groups except the middle are equal to those of projective space.) Given that, knowing the Euler characteristic determines the middle Betti number as well. In Table 7.1 we give the results of this calculation in a few of the cases where it is actually used.

variety	χ	middle Betti number
quadric surface	4	2
cubic surface	9	7
quartic surface	24	22
quintic surface	55	53
quadric threefold	4	0
cubic threefold	-6	10
quartic threefold	-56	60
quintic threefold	-200	204
quadric fourfold	6	2
cubic fourfold	27	23

TABLE 7.1. Euler Characteristics of Favorite Hypersurfaces

Note that some of these we already knew: the quadric surface is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, from which we can see directly both the Euler characteristic and the second Betti number; the quadric fourfold may also be viewed as the Plücker embedding of the Grassmannian $\mathbb{G}(1,3)$, whose cohomology has as basis its six Schubert cycles, and whose middle cohomology in particular has basis given by the two Schubert cycles $\sigma_{1,1}$ and σ_2.

7.3.5 Bundles on Grassmannians

Let's consider next the case of the Grassmannian $G = G(k,n)$ of k-planes in an n-dimensional vector space V, and its universal sub- and quotient bundles $\mathcal{S}$ and $\mathcal{Q}$.

We'll start with $\mathcal{Q}$, since this bundle is globally generated, so that we can determine its Chern classes directly as degeneracy loci. Specifically, elements $v \in V$ give rise to sections σ of $\mathcal{Q}$ simply by taking their images in each quotient of V: that is, for a k-plane $\Lambda \subset V$, we set

$$\sigma(\Lambda) = \overline{v} \in V/\Lambda.$$

Now, given a collection $v_1, \dots, v_m \in V$, the corresponding sections will fail to be independent at a point $\Lambda \in G$ exactly when the corresponding $\overline{v}_i \in V/\Lambda$ are dependent, which is to say when Λ intersects the span $W = \langle v_1, \dots, v_m \rangle \subset V$ in a nonzero subspace—that is, when

$$\mathbb{P}\Lambda \cap \mathbb{P}W \neq \varnothing.$$

We may recognize this locus as the Schubert cycle $\Sigma_{n-k-m+1}(W)$, from which we conclude that the Chern class of $\mathcal{Q}$ is the sum

$$c(\mathcal{Q}) = 1 + \sigma_1 + \sigma_2 + \cdots + \sigma_{n-k}.$$

Unlike $\mathcal{Q}$, the universal subbundle $\mathcal{S}$ doesn't have nonzero global sections, so we can't use the characterization of Chern classes as degeneracy loci. But the dual bundle $\mathcal{S}^*$ does: if $l \in V^*$ is a linear form, we can define a section τ of $\mathcal{S}^*$ by restricting l to each k-plane $\Lambda \subset V$ in turn; in other words, we set

$$\tau(\Lambda) = l|_\Lambda.$$

Now, if we have m independent linear forms $l_1, \ldots, l_m \in V^*$, the corresponding sections of $\mathcal{S}^*$ will fail to be independent at the point $\Lambda \in G$—that is, some linear combination of the l_i will vanish identically on Λ—exactly when Λ fails to intersect the common zero locus U of the l_i properly; that is, when

$$\dim(\mathbb{P}\Lambda \cap \mathbb{P}U) \geq k - m.$$

Again, this locus is a Schubert cycle in G, specifically the cycle $\Sigma_{1,1,\ldots,1}(U)$; and we conclude that

$$c(\mathcal{S}^*) = 1 + \sigma_1 + \sigma_{1,1} + \cdots + \sigma_{1,1,\ldots,1};$$

from this we can deduce in turn that

$$c(\mathcal{S}) = 1 - \sigma_1 + \sigma_{1,1} + \cdots + (-1)^k \sigma_{1,1,\ldots,1}.$$

In theory, we should be able to use the identification of the Chern classes $c(\mathcal{S})$ and $c(\mathcal{Q})$ to derive the Chern class of the tangent bundle $\mathcal{T}_G$, which we saw in Section **??** is isomorphic to $\mathrm{Hom}(\mathcal{S}, \mathcal{Q}) = \mathcal{S}^* \otimes \mathcal{Q}$. In general, unfortunately, this knowledge remains theoretical: as we indicated in Section 7.2.9, the formula for the Chern class of the tensor product of two bundles of higher rank is complicated. But we can at least use Proposition 7.15 to give the first Chern class $c_1(\mathcal{T}_G)$: since $c_1(\mathcal{S}^*) = c_1(\mathcal{Q}) = \sigma_1$, we have

Proposition 7.16. *The first Chern class of the tangent bundle of the Grassmannian $G = G(k, n)$ is*

$$c_1(\mathcal{T}_G) = n \cdot \sigma_1.$$

We see from this also that the canonical class K_G of G is $-n\sigma_1$. Note that this agrees with our prior calculations in the case $k = 1$ of projective space $\mathbb{P}^{n-1}$, and in case $k = 2$ and $n = 4$, where the Grassmannian $G(2, 4)$ may be realized as a quadric hypersurface in $\mathbb{P}^5$ and we can apply the results of Section 7.3.4.

7.4 Generators and relations for $A(G(k, n))$

We have seen in Corollary 3.5 that the Chow ring of the Grassmannian is a free abelian group generated by the Schubert cycles. We will now show that it is generated multiplicatively by just the the *special Schubert cycles*, which are the Chern classes of the universal subbundle. At the same time, we will see that the Whitney formula and the fact that the Chern classes of a bundle vanish above the rank of the bundle provide a complete description of the relations among the special Schubert cycles, and that these form a complete intersection.

Theorem 7.17. *The Chow ring of the Grassmannian $G(k,n)$ has the form*

$$A(G(k,n)) = \mathbb{Z}[c_1,\dots,c_k]/I$$

where $c_i \in A^i(G(k,n))$ is the i-th Chern class of the universal subbundle $\mathcal{S}$ and the ideal I is generated by the terms of total degree $n-k+1,\dots,n$ in the power series expansion

$$\frac{1}{1+c_1+\dots+c_k} = 1-(c_1+\dots+c_k)+(c_1+\dots+c_k)^2-\dots \in \mathbb{Z}[[c_1,\dots,c_k]].$$

Moreover, I is a complete intersection.

For example, the Chow ring of $G(3,7)$ is $\mathbb{Z}[c_1,c_2,c_3]/I$ where I is generated by the elements

$$c_1^5+4c_1^3c_2+3c_1c_2^2+3c_1^2c_3+2c_2c_3,$$
$$c_1^6+5c_1^4c_2+6c_1^2c_2^2+c_2^3+4c_1^3c_3+6c_1c_2c_3+c_3^2,$$
$$c_1^7+6c_1^5c_2+10c_1^3c_2^2+4c_1c_2^3+5c_1^4c_3+12c_1^2c_2c_3+3c_2^2c_3+3c_1c_3^2,$$

and these elements form a regular sequence.

The proof uses two results from commutative algebra, Proposition 7.18 and Lemma 7.19, which are variations on some frequently used results; the reader may wish to familiarize himself with them before reading the proof of Theorem 7.17. Recall that the *socle* of a finite dimensional graded algebra T is the submodule of elements annihilated by all elements of positive degree. In particular, if d is the largest degree such that $T_d \neq 0$, then the socle of T contains T_d. For a somewhat different proof, and the generalization to Flag bundles of arbitrary vector bundles, see Grayson and Stillman [$\geq$ 2013].

Proof. Set $A = A(G(k,n))$ and write t_i for the degree i part of the power series expansion of $1/(1+c_1+\dots+c_k)$, so that $t_0=1,\ t_1=-c_1,\ t_2=c_1^2-c_2,\dots$. Let $J=(t_{n-k+1},\dots,t_n)$ and let $R=\mathbb{Z}[c_1,\dots,c_k]/J$.

The Whitney formula (Theorem 7.7) applied to the tautological sequence of vector bundles

$$0\to\mathcal{S}\to\mathcal{O}^n_{G(k,n)}\to\mathcal{Q}\to 0$$

on $G(k,n)$ shows that $c(\mathcal{Q})=1/c(\mathcal{S})$. Since $\mathcal{Q}$ has rank $n-k$, the classes $c_i(\mathcal{Q})$ vanish for all $i>n-k$, and it follows that there is a ring homomorphism

$$\varphi : R\to A;\quad t_i\mapsto c_i(\mathcal{Q}).$$

Under this homomorphism the class c_i goes to the Schubert cycle $c_i(\mathcal{S}) = \sigma_{1^i}$ (where the subscript denotes a sequence of i ones.) Recall from Corollary 3.2 that $\sigma_{1^k}^{n-k}$ is the class of a point.

We will show that for any field F the sequence $t_{n-k+1}, \dots, t_n$ is a regular sequence in $R \otimes_{\mathbb{Z}} F$, and the induced map

$$R' := R \otimes_{\mathbb{Z}} F \xrightarrow{\varphi' := \varphi \otimes_{\mathbb{Z}} F} A' := A \otimes_{\mathbb{Z}} F$$

is an isomorphism. Since A is a finitely generated abelian group, the surjectivity of φ follows from this result using Nakayama's Lemma and the two cases $F = \mathbb{Z}/(p)$ and $F =$
CQ. On the other hand, by Corollary 3.5, A is a free abelian group so, as an abelian group, $\varphi(R)$ is free. Thus the kernel of φ is a summand of R, so the injectivity of φ follows from the injectivity, for every choice of F, of φ'. Using Lemma 7.19 inductively, it also follows that $t_{n-k+1}, \dots, t_n$ is a regular sequence, proving the Theorem.

To show that $t_{n-k+1}, \dots, t_n$ is a regular sequence in R' it suffices, since the t_i have positive degree, to show that

$$F[c_1, \dots, c_k]/J$$

has Krull dimension zero. Since F was arbitrary it suffices, by the Nullstellensatz, to show that if $f_i \in F$ for the c_i in such a way that $t_{n-k+1} = \cdots = t_n = 0$, then all the f_i are zero.

Indeed, after such a substitution we see that $1/(1 + f_1x + f_2x^2 \cdots + f_kx^k) = p(x) + q(x)$ where $p(x)$ is a polynomial of degree $\leq n-k$ and $q(x)$ is a rational function vanishing to order at least $n+1$ at 0. We may rewrite this as

$$\frac{1 - p(x)(1 + f_1x + f_2x^2 \cdots + f_kx^k)}{1 + f_1x + \cdots + f_kx^k} = q(x).$$

However, the denominator of the left hand side is nonzero at the origin, and the numerator has degree at most n. Since $q(x)$ vanishes to order at least $n+1$ at the origin, both sides must be identically zero; that is $p(x) = 1, q(x) = 0$, and thus all $f_i = 0$ as required. (See Exercise 7.38 for a more explicit version.)

Combining this information with Proposition **??** we get:

- The dimension of R' (as a vector space over F) is $\binom{n}{k}$;
- The highest degree d such that $R'_d \neq 0$ is $k(n-k)$.
- Since a complete intersection is Gorenstein (Eisenbud [1995] Theorem ****) every nonzero ideal of R' contains $R'_{k(n-k)}$.

.

We now return to the map φ'. By Corollary 3.10, the rank of $A(G(k,n))$ is also $\binom{n}{k}$; thus to show that φ' is an isomorphism, it suffices to show that its kernel is zero. We know that $(\sigma_{1^k})^{n-k}$ is in the image of φ', so $\operatorname{Ker}\varphi'$ does not contain $R_{k(n-k)}$. Since $R'_{k(n-k)}$ is the socle of R', the kernel of φ' must be zero. $\square$

We have used the following two results from commutative algebra:

Proposition 7.18. *Suppose that F is a field and that*

$$T = F[x_1, \ldots, x_k]/(g_1, \ldots, g_k)$$

is a zero-dimensional graded complete intersection with $degx_i = \delta_i > 0$ and $\deg g_i = \epsilon_i > 0$. *The Hilbert series of T is*

$$H_T(d) := \sum_{u=0}^{\infty} \dim_F T_u d^u = \frac{\prod_{i=1}^k (1-d^{\epsilon_i})}{\prod_{i=1}^k (1-d^{\delta_i})}$$

The degree of the socle of T is $\sum_{i=0}^k \epsilon_i - \sum_{i=0}^k \delta_i$, and the dimension of T is

$$\dim_F T = \frac{\prod_{i=1}^k (\epsilon_i - 1)}{\prod_{i=1}^k (\delta_i - 1)}.$$

Proof. We begin with the Hilbert series. The polynomial ring $F[x_1, \ldots, x_k]$ is the tensor product of the polynomial rings in 1 variable $F[x_i]$ so

$$H_{F[x_1,\ldots,x_k]}(d) := \frac{1}{\prod_{i=1}^k (1-d_i^{\delta})}.$$

We can put in the relations one by one using the exact sequences

$$\begin{aligned} 0 \longrightarrow F[x_1, \ldots, x_k]/(g_1, \ldots, g_i)(-\epsilon_i) &\xrightarrow{g_{i+1}} F[x_1, \ldots, x_k]/(g_1, \ldots, g_i) \\ &\longrightarrow F[x_1, \ldots, x_k]/(g_1, \ldots, g_{i+1}) \longrightarrow 0, \end{aligned}$$

and using induction we see that

$$H_T(d) = H_{F[x_1,\ldots,x_k]/(g_1,\ldots,g_k)}(d) = \frac{\prod_{i=1}^k (1-d^{\epsilon_i})}{\prod_{i=1}^k (1-d^{\delta_i})}.$$

A priori this is a rational function of degree $s := \sum_{i=1}^k \epsilon_i - \sum_{i=1}^k \delta_i$. Since we know from the computation above that T is a finite dimensional vector space over F, the Hilbert series must be a polynomial. Thus it is a polynomial of degree s, so the largest degree in which T is nonzero is s.

The dimension of T is the value of $H_T(d)$ at $d = 1$. The product $(1-d)^k$ obviously divides both numerator and denominator of the expression for the Hilbert series above. After dividing, we get:

$$H_T(d) = \frac{\prod_{i=1}^k \sum_{j=0}^{\epsilon_i - 1} d^j}{\prod_{i=1}^k \sum_{j=0}^{\delta_i - 1} d^j}.$$

Setting $d = 1$ in this expression gives us the desired result. $\square$

The other result from commutative algebra that we used is a version of the fact that regular sequences in a local ring can be permuted (Eisenbud [1995], Corollary 17.2). The same result holds in the local case when every element of the regular sequence has positive degree, but the case we need is slightly different, since one element of the regular sequence is an integer. The result may also be viewed as a variation on the local criterion of flatness (Eisenbud [1995]Section 6.4).

Lemma 7.19. *Suppose that R is a finitely generated graded algebra over $\mathbb{Z}$, with algebra generators in positive degrees, and that $f \in R$ is a homogeneous element. If R is free as a $\mathbb{Z}$-module and $f \otimes_{\mathbb{Z}} \mathbb{Z}/(p)$ is a monomorphism for every prime p, then f is a monomorphism and $R/(f)$ is free as a $\mathbb{Z}$-module as well.*

Proof. Since R is free, so is every submodule; in particular fR is free, and the kernel K of multiplication by f is a free summand of R. It follows that $K \otimes_{\mathbb{Z}} \mathbb{Z}/(p) \subset R \otimes_{\mathbb{Z}} \mathbb{Z}/(p)$. Since this ideal is obviously contained in the kernel of multplication by f on $R \otimes \mathbb{Z}/(p)$, we see that $K \otimes_{\mathbb{Z}} \mathbb{Z}/(p) = 0$. Since K is free, this implies that $K = 0$ as well; that is, f is a nonzerodivisor on R, and the diagram

$$\begin{tikzcd}
0 \arrow[r] & R(-1) \arrow[r, "f"] \arrow[d, "p"] & R \arrow[r] \arrow[d, "p"] & R/fR \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & R(-1) \arrow[r, "f"] & R \arrow[r] & R/fR \arrow[r] & 0
\end{tikzcd}$$

has exact rows. A diagram chase (the *Snake Lemma*) shows that p is a nonzerodivisor on R/fR. Since p was an arbitrary prime, R/fR is a torsion free abelian group. Since R is finitely generated and f is homogenous, R/fR is a direct sum of finitely generated generated abelian groups, and torsion-freeness implies freeness. $\square$

7.5 Exercises

Many of the following exercises give applications of the Whitney formula and splitting principle. We will be assuming the basic facts that if

$$\mathcal{E} = \bigoplus_{i=1}^{e} \mathcal{L}_i \quad \text{and} \quad \mathcal{F} = \bigoplus_{i=1}^{f} \mathcal{M}_i$$

are direct sums of line bundles, then

$$\operatorname{Sym}^k \mathcal{E} = \bigoplus_{1 \le 1_1 \le \cdots \le i_k \le r} \mathcal{L}_{i_1} \otimes \ldots \cdots \otimes \mathcal{L}_{i_k};$$

$$\wedge^k \mathcal{E} = \bigoplus_{1 \le 1_1 < \cdots < i_k \le r} \mathcal{L}_{i_1} \otimes \ldots \cdots \otimes \mathcal{L}_{i_k}; \text{ and}$$

$$\mathcal{E} \otimes \mathcal{F} = \bigoplus_{i,j=1,1}^{e,f} \mathcal{L}_i \otimes CM_j.$$

Exercise 7.20. Let $\mathcal{E}$ be a vector bundle of rank 3. Express the Chern classes of $\wedge^2\mathcal{E}$ in terms of those of $\mathcal{E}$ by invoking the splitting principle and the Whitney formula

Exercise 7.21. Verify your answer to the preceding exercise by observing that wedge product map

$$\mathcal{E} \otimes \wedge^2\mathcal{E} \to \wedge^3\mathcal{E} = \det(\mathcal{E})$$

yields an identification $\wedge^2\mathcal{E} = \mathcal{E}^* \otimes \det(\mathcal{E})$, and applying the formula for tensor product with a line bundle.

Exercise 7.22. Let $\mathcal{E}$ be a vector bundle of rank 4. Express the Chern classes of $\wedge^2\mathcal{E}$ in terms of those of $\mathcal{E}$.

Exercise 7.23. Let $\mathcal{E}$ be a vector bundle of rank 3. Express the Chern classes of $\mathrm{Sym}^2 \mathcal{E}$ in terms of those of $\mathcal{E}$.

Exercise 7.24. Let $\mathcal{E}$ be a vector bundle of rank 4. Express the Chern classes of $\mathrm{Sym}^2 \mathcal{E}$ in terms of those of $\mathcal{E}$.

Exercise 7.25. Let $\mathcal{E}$ be a vector bundle of rank 2. Express the Chern classes of $\mathrm{Sym}^3 \mathcal{E}$ in terms of those of $\mathcal{E}$.

Exercise 7.26. Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles of rank 2. Express the Chern classes of the tensor product $\mathcal{E} \otimes \mathcal{F}$ in terms of those of $\mathcal{E}$ and $\mathcal{F}$.

Exercise 7.27. Just to get a sense of how rapidly this gets complicated: do the preceding exercise for a pair of vector bundles $\mathcal{E}$ and $\mathcal{F}$ of ranks 2 and 3.

Exercise 7.28. Apply the preceding exercise to find all the Chern classes of the tangent bundle $\mathcal{T}_G$ of the Grassmannian $G = G(2, 4)$

Exercise 7.29. Find all the Chern classes of the tangent bundle $\mathcal{T}_Q$ of a quadric hypersurface $Q \subset \mathbb{P}^5$.

Exercise 7.30. Check that your answers to the last two exercises agree!

Exercise 7.31. Calculate the Chern classes of the tangent bundle to a product $\mathbb{P}^n \times \mathbb{P}^m$ of projective spaces

Exercise 7.32. Find the Euler characteristic of a smooth hypersurface of bidegree (a, b) in $\mathbb{P}^m \times \mathbb{P}^n$.

Exercise 7.33. Using the Whitney formula, show that for $n \geq 2$ the tangent bundle $\mathcal{T}_{\mathbb{P}^n}$ of projective space is not a direct sum of line bundles.

Exercise 7.34. An extension of the preceding exercise: show that for $n \geq 2$ the tangent bundle $\mathcal{T}_{\mathbb{P}^n}$ of projective space is not a nontrivial direct sum of vector bundles of any positive rank.

Exercise 7.35. Find the Betti numbers of the smooth intersection of a quadric and a cubic hypersurface in $\mathbb{P}^4$, and of the intersection of three quadrics in $\mathbb{P}^5$. (Both of these are examples of *K3 surfaces*, which are diffeomorphic to a smooth quartic surface in $\mathbb{P}^3$.)

Exercise 7.36. Find the Betti numbers of the smooth intersection of two quadrics in $\mathbb{P}^5$. This is the famous *quadric line complex*, about which you can read more in [GH], Chapter 6.

Exercise 7.37. Show that the cohomology groups of a smooth quadric threefold $Q \subset \mathbb{P}^4$ are isomorphic to those of $\mathbb{P}^3$ ($\mathbb{Z}$ in even dimensions, 0 in odd), but its cohomology ring is different (the square of the generator of $H^2(Q, \mathbb{Z})$ is twice the generator of $H^4(Q, \mathbb{Z})$). (This is a useful example if you're ever teaching a course in algebraic topology.)

Exercise 7.38. With notation as in Theorem 7.17 and the convention that $t_i = 0$ for $i < 0$, show that for $i \geq 1$

$$t_i = \sum_{j=1}^{k} (-1)^j c_j t_{i-j} \quad \text{when } i \geq 1.$$

Deduce that J contains all t_j with $j > n - k$ (this is a little stronger than the argument given in the text, which shows that these t_i are in the radical of J. Use this to show that if we substitute elements f_i for the c_i making $t_{n-k+1} = \cdots = t_n = 0$ then the inverse of the polynomial in one variable $p(x) := 1 + f_1 x + \cdots + f_k x^k \in F[x]$ is itself a polynomial. It follows that $p(x)$ is constant, so all f_i are zero.

Exercise 7.39. Let $S \subset \mathbb{P}^4$ be a smooth complete intersection of hypersurfaces of degrees d and e, and let $Y \subset \mathbb{P}^4$ be any hypersurface of degree f containing S. Show that if f is not equal to either d or e, then Y is necessarily singular.

(Hint: Assume Y is smooth, and apply the Whitney formula to the sequence

$$0 \to \mathcal{N}_{S/Y} \to \mathcal{N}_{S/\mathbb{P}^4} \to \mathcal{N}_{Y/\mathbb{P}^4}|_S \to 0$$

to arrive at a contradiction.)

Exercise 7.40. Let $S \subset \mathbb{P}^n$ be a smooth k-dimensional complete intersection of hypersurfaces of degrees $d_1, \dots d_{n-k}$, and let $Y \subset \mathbb{P}^n$ be any hypersurface of degree e containing S. Show that if $k > n/2$ and $e \neq d_i$ for any i, then Y is necessarily singular.

8

Lines on Hypersurfaces

Keynote Questions

(a) Let $X \subset \mathbb{P}^4$ be a general quintic hypersurface. How many lines $L \subset \mathbb{P}^4$ does X contain?

(b) Let $\{X_t \subset \mathbb{P}^3\}_{t \in \mathbb{P}^1}$ be a general pencil of quartic surfaces—that is, let f and g be general homogeneous polynomials of degree 4 in four variables, and set $X_t = V(t_0 f + t_1 g) \subset \mathbb{P}^3$. How many of the surfaces X_t contain a line?

(c) Now let $\{X_t \subset \mathbb{P}^3\}_{t \in \mathbb{P}^1}$ be a general pencil of cubic surfaces, and consider the locus of all lines contained in some member of this pencil. What is the degree of the surface swept out by these lines? What is the genus of the curve parametrizing them?

(d) Can a smooth quartic hypersurface in $\mathbb{P}^4$ contain a 2-parameter family of lines?

In this Chapter we will study the schemes parametrizing lines (and planes of higher dimension) on a hypersurface. These are called *Fano schemes*. There are two phases to the treatment. It turns out that the enumerative content of the keynote questions above, and many others, can be answered through a single type of Chern class computation. But there is another side of the story, involving beautiful and important techniques for working with the tangent spaces of Hilbert schemes, of which Fano schemes are examples. These ideas will allow us to verify that the "numbers" we compute really correspond to the geometry that they are meant to reflect. We will go even beyond these techniques and explore a little of the local structure of the Fano scheme. There are many open questions in this area, and the chapter ends with and exploration of one of them.

8.1 What to expect

For what n and d should we expect a general hypersurface $X \subset \mathbb{P}^n$ of degree d to contain lines? What dimensional family of lines we would expect it to contain? When the dimension is zero, how many lines will there be? In this chapter e will treat such questions, and explain some cases where the "expectations" are realized.

8.1.1 *Definition of the Fano scheme*

A projective space $\mathbb{P}^{n*} = \mathbb{P}V*$ may be thought of as parametrizing the hyperplanes in $\mathbb{P}^n := \mathbb{P}V$, and more generally the Grassmannian $\mathbb{G}(k,n)$ parametrizes the planes of any dimension k. Basic to this chapter is the fact that for closed scheme $X \subset \mathbb{P}^n$ there is a natural subscheme of the Grassmannian $\mathbb{G}(k,\mathbb{P}^n)$ whose points correspond to the k-planes contained in X, called the *Fano scheme* $F_k(X)$. Moreover, there is a "family of k-planes in X" defined as a subvariety $\Phi(X,k) \subset X \times \mathbb{G}(k,n)$ with nice properties such that the fiber of $\Phi(X,k)$ over the point p of the Grassmannian corresponding to a k-plane L is $L \times \{p\} \subset X \times \{p\} = X$.

The simplest example of such a family is perhaps the *universal hyperplane*

$$\Phi = \Phi(n-1,\mathbb{P}^n) \subset \mathbb{P}^n \times \mathbb{G}(n-1,n) = \mathbb{P}^n \times \mathbb{P}^{n*}.$$

If we write $\mathbb{P}^n = \mathbb{P}V$, and choose dual bases $\{a_i\}$ and $\{z_i\}$ for V and V^* respectively, then Φ is defined by the equation $\sum_i a_i x_i = 0$.

This description coincides with that of the *universal* $(n-1)$*-plane* given in Section **??**. Recall that this was defined to be $\mathbb{P}S$, where S is the universal subbundle on $\mathbb{G}(n-1,n) = \mathbb{P}^{n*}$:

$$0 \longrightarrow S \longrightarrow \mathcal{O}_{\mathbb{P}}^{n+1} \xrightarrow{(x_0,\dots x_n)} \mathcal{O}_{\mathbb{P}^{n*}}(1) \longrightarrow 0$$

Indeed, the inclusion $S \longrightarrow \mathcal{O}_{\mathbb{P}}^{n+1}$ gives an embedding

$$\mathbb{P}S \subset \mathbb{P}\mathcal{O}_{\mathbb{P}}^{n+1} = \mathbb{P}^n \times \mathbb{P}^{n*},$$

and the exact sequence says precisely that the lines in $\mathbb{P}S$ are those whose homogeneous coordinates $(a_0,\dots,a_n)$ satisfy $\sum a_i x_i = 0$. The name "universal family" will be justified in Section 8.3.2.

The first nontrivial examples of Fano schemes are the schemes of lines on smooth surfaces. The family of lines is a union of linear equivalence classes, each isomorphic to a projective space. For instance the set of lines on a nonsingular quadric in $\mathbb{P}^3$ is the disjoint union of two copies of $\mathbb{P}^1$. We know already that the lines on a quadric cone in $\mathbb{P}^3$ form a single 1-dimensional family, which can also be parametrized by $\mathbb{P}^1$, though one might suspect that, since the singular quadric is the limit of nonsingular quadrics, this family of lines should count twice, and indeed, as a scheme, we shall see that it has mulptiplicity two (Exercise 8.46).

We begin by giving a direct definition of this scheme that is local on $\mathbb{G}(k,n)$. We will return to the definition twice later in this chapter to give a global description and a universal property that justifies the idea that we are taking the "right" scheme structure.

It will suffice to give the definition of the Fano scheme $F_k(X)$ in case X is a hypersurface in $\mathbb{P}^n$; for an arbitrary scheme $Y \subset \mathbb{P}^n$ we define

$$F_k(Y) = \bigcap_{\substack{Y \subset X \subset \mathbb{P}^n \\ X \text{ is a hypersurface}}} F_k(X).$$

To define $F_k(X)$ for a hypersurface X given by an equation $G = 0$ we use the idea that a plane L lies on X if and only if the restriction of G to L is zero. If we have a parametrization $\alpha : \mathbb{P}^k \to L$ of L, then we can pull back G via α; the condition $L \subset X$ is given by the vanishing of the coefficients of $\alpha^*(G)$. However, this is not quite what we want: it gives the equations, not for $F_k(X)$, but for the variety of hypersurfaces containing L.

To turn this around, we use an incidence correspondence, described set-theoretically by the formula

$$\begin{aligned}\Phi = \Phi(n,d,k) = \{(X, L) \mid\ & X \text{ is a hypersurface of degree } d \text{ in } \mathbb{P}^n \\ & \text{and } L \subset X \text{ is a } k\text{-plane}\}.\end{aligned}$$

We will define Φ as a closed subscheme of $\mathbb{P}^N \times \mathbb{G}(k, \mathbb{P}^n)$ where $\mathbb{P}^N = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^n}(d))$ is the projective space of hypersurfaces of degree d in $\mathbb{P}^n$ (here $N = \binom{n+d}{n} - 1$). The scheme Φ itself will be quite useful in other ways; we call it the *universal Fano scheme* of hypersurfaces of degree d in $\mathbb{P}^n$.

To do this it is enough to work over the sets U of the open cover of $\mathbb{G}(k, \mathbb{P}^n)$ defined in Section 2.2.2 of Chapter 2. Such an open set U is defined as the set of all k-planes not meeting a fixed $(n-k-1)$-plane. If the latter is given by the vanishing of the first $k+1$ coordinates, then U may be identified with the affine space of $(k+1) \times (n-k)$ matrices: Any

k-plane L belonging to U is the row-space of a a matrix of the form

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 & a_{0,k+1} & \cdots & a_{0,n+1} \\ 0 & 1 & \cdots & 0 & a_{1,k+1} & \cdots & a_{1,n+1} \\ & & \cdots & & & \cdots & \\ 0 & 0 & \cdots & 1 & a_{k,k+1} & \cdots & a_{k,n+1} \end{pmatrix}$$

We can thus give a parametrization of L in the form

$$\begin{aligned} \mathbb{P}^k \ni (s_0, \ldots, s_k) \mapsto & (s_0 \ \cdots \ s_k)A = \\ & (s_0, \cdots, s_k, \sum_i a_{i,k+1}s_i, \ldots, \sum_i a_{i,n+1}s_i) \end{aligned}$$

We now regard the $a_{i,j}$ as coordinates on U. If we substitute the $n+1$ coordinates of the parametrization into a generic form of degree d in $n+1$ variables x_i with coefficients z_δ

$$g := \sum_{|\delta|=d} z_\delta x^\delta$$

then we get a form $g(s_0, \ldots, s_k)$ of degree d in the s_i, and the coefficients of this form are polynomials in the $a_{i,j}$ and z_δ, homogeneous (of degree 1) in the latter. We define $\Phi \cap \mathbb{P}^N \times U$ to be the subscheme defined by these polynomials. If we perform the same construction for a different basic open set U' of $\mathbb{G}(k, \mathbb{P}^n)$ then the two subschemes agree on the intersection $(U \cap U') \times \mathbb{G}(k, \mathbb{P}^n)$, and thus define a scheme

$$\Phi = \Phi(n, d, k) \subset \mathbb{P}^N \times \mathbb{G}(k, \mathbb{P}^n)$$

as required. It is not hard to check this agreement directly, and we soon show that Φ is globally defined as the vanishing locus of a section of a vector bundle in a way that will make the agreement on overlaps obvious.

Finally, we can give the rest of the definition: The *Fano scheme* $F_k(X) \subset \mathbb{G}(k, n)$ of a hypersurface $X \subset \mathbb{P}^n$ of degree d is the fiber of $\Phi(n, d, k)$ over the point of $\mathbb{P}^N$ corresponding to X.

8.1.2 Dimension of the universal Fano scheme

For any fixed k-plane $L \subset \mathbb{P}^n$, the fiber of Φ over L is a linear subspace of $\mathbb{P}^N$ of codimension $\binom{k+d}{k}$, since it corresponds to the linear space of those forms of degree d that restrict to zero on $L \cong \mathbb{P}^k$. This makes it easy to analyze Φ:

Proposition 8.1. *Let $N = \binom{n+d}{d} - 1$. The universal Fano scheme $\Phi = \Phi(n, d, k) \subset \mathbb{P}^N \times \mathbb{G}(k, \mathbb{P}^n)$ is a smooth irreducible variety of dimension*

$$\dim \Phi(n, d, k) = N + (k+1)(n-k) - \binom{k+d}{k}.$$

Proof. Each fiber of the projection $\Phi \to \mathbb{G}(k, \mathbb{P}^n)$ is a projective space of dimension $N - \binom{k+d}{k}$. □

Though Proposition 8.1 does not allow us to deduce the existence of k-planes on a given hypersurface, it does imply a lower bound on the dimension of the family of such planes should there be any:

Corollary 8.2. (a) *The dimension of any component of the family of k-planes on any hypersurface of degree d in $\mathbb{P}^n$ is at least*

$$\varphi(n, d, k) := (k+1)(n-k) - \binom{k+d}{k}.$$

(b) *If $\varphi(n, d, k) < 0$ then the general hypersurface of degree d in $\mathbb{P}^n$ contains no k-planes.*

(c) *If $\varphi(n, d, k) \geq 0$ and the general hypersurface of degree d contains any k-planes, then every hypersurface of degree d contains k-planes, and every component of the family of lines on a general hypersurface of degree d has dimension exactly $\varphi(n, d, k)$.*

Proof. We use the notation of Proposition 8.1. (a): Since a fiber of $\Phi(n, d, k)$ over $\mathbb{P}^N$ is cut out by N equations, the Principal Ideal Theorem 0.1 gives the desired lower bound.

(b): $\dim \Phi(n, d, k)$ is the sum of the dimension of its image in $\mathbb{P}^N$ and the dimension of a general fiber. By part (a) the image cannot be of dimension N in this case.

(c): If the general hypersurface of degree d contains a k-plane then $\Phi(n, d, k)$ dominates $\mathbb{P}^N$, and since $\Phi(n, d, k)$ is projective, the map to $\mathbb{P}^N$ is surjective. Further, by the formula used in (b), the general fiber must have dimension $\varphi(n, d, k)$. □

We shall eventually show (Corollary 8.32 and Theorem 8.28) that when $\varphi(n, d, k) \geq 0$ then, except in some cases where $k > 1$ and $d = 2$, the general hypersurface actually does contain k-planes, so the results above apply. For example, if $d > 2n - 3$, a general hypersurface of degree d in $\mathbb{P}^n$ contains no lines, but the family of lines on a general hypersurface of degree $d \leq 2n - 3$ has dimension $\varphi(n, d, 1) = 2n - 3 - d$.

An example already mentioned the case of quadrics in $\mathbb{P}^3$, where the estimate above corresponds with the fact that a the Fano scheme of a smooth surface has dimension 1 (the same is true for the cone over s smooth conic; but the dimension jumps to 2 for the union of two planes, or a double plane.)

Corollary 8.2 shows that the general surface in $\mathbb{P}^3$ of degree $d \geq 4$ contains no lines. But we can say more, using the same sort of incidence correspondence argument made above. For example, we'll see in Exercise 8.47 that a general surface $S \subset \mathbb{P}^3$ of degree $d \geq 4$ containing a line contains only one. This implies that the locus $\Sigma \subset \mathbb{P}^N$ of surfaces that do contain a line has codimension $d - 3$.

Of course special hypersurfaces may well contain families of planes of dimension $> \varphi(n, d, k)$. We can easily give an upper bound on the possible dimension:

Proposition 8.3. *If $X \subset \mathbb{P}^n$ is an m-dimensional scheme, then*

$$\dim F_k(X) \leq (k+1)(m-k) = \dim \mathbb{G}(k, \mathbb{P}^m),$$

with equality if and only if X is an m-plane.

Note that this is not strong enough to answer keynote question (d), for which smoothness is essential.

Proof. We may assume without loss of generality that X is nondegenerate. Let $U \subset X^{k+1}$ be the open set consisting of $(k+1)$-tuples of linearly independent points, and let

$$\Gamma = \{\big((p_0, \dots, p_k), L\big) \in U \times F_1(X) \mid p_i \in L \text{ for all } i\}.$$

Via the projection $\Gamma \to U$, we see that $\dim \Gamma \leq m(k+1)$. Since the fibers of the projection $\Gamma \to F_1(X)$ have dimension $(k+1)^2$ we conclude that $\dim F_1(X) \leq m(k+1) - (k+1)^2 = (k+1)(m-k)$ as required.

Equality of dimensions can hold only if the projection $\Psi \to U$ is dominant, that is, X contains the plane spanned by any $k+1$ general points of X, and this can happen only if X is a linear space. $\square$

In Section 8.8 we will discuss some open questions about these dimensions.

8.2 Fano schemes and Chern classes

To get global information about the Fano scheme of a hypersurface we will express it as the zero locus of a section of a vector bundle on the Grassmannian. To understand the idea, suppose that $X \subset \mathbb{P}^n$ is the hypersurface $g = 0$, where g is a homogeneous form of degree d. As we have seen, the condition that X contain a particular k-dimensional linear space L is that g goes to 0 under the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(\mathcal{O}_L(d)).$$

To describe the scheme $F_k(X) \subset \mathbb{G}(k,n)$ of k-planes on X, we will realize the family of vector spaces $H^0(\mathcal{O}_L(d))$, with varying k-planes L as the fibers of a vector bundle in such a way that the images of g in these vector spaces are the values of a section σ_g of the bundle:

Proposition 8.4. *Let V be an $(n+1)$-dimensional vector space, and let Let $S \subset V \otimes \mathcal{O}_{\mathbb{G}}$ be the tautological rank $k+1$ subbundle on the Grassmannian $\mathbb{G} := \mathbb{G}(k, \mathbb{P}^n)$ of k-planes in $\mathbb{P}^n$. A form g of degree d on $\mathbb{P}^n = \mathbb{P}V$ gives rise to a global section σ_g of $\operatorname{Sym}^d S^*$ whose zero locus is $F_k(X)$, where X is the hypersurface $g = 0$.*

Thus when $F_k(X)$ has "expected" codimension $\binom{k+d}{k} = \operatorname{rank} \operatorname{Sym}^d S^*$ in $\mathbb{G}$, we have

$$[F_k(X)] = c_{\binom{k+d}{k}}(\operatorname{Sym}^d(S^*)) \in A(\mathbb{G}).$$

Since an empty scheme always has the "expected dimension" in this sense, Corollary 8.32 and Theorem 8.28 could in principle be deduced from this Proposition (together with a messy computation of symmetric functions.)

Proof. The fiber of S over the point$[L] \in \mathbb{G}(k,n)$ representing the subspace $\mathbb{P}^k \cong L \subset \mathbb{P}V$ is the corresponding $(k+1)$-dimensional subspace of V. The fiber of the dual bundle S^* at $[L]$ is thus the space of linear forms on L, that is to say $H^0(\mathcal{O}_{\mathbb{P}L}(1))$; and the dual map $V^* \otimes \mathcal{O}_{\mathbb{G}} \to S^*$ evaluated at a point $[L]$, takes a linear form $\varphi \in V^*$, thought of as a constant section of the trivial bundle $V^* \otimes \mathcal{O}_{\mathbb{G}}$, to the restriction of φ to L. The vector space of forms of degree d is on $\mathbb{P}^n$ is $\operatorname{Sym}^d V^* = H^0(\mathcal{O}_{\mathbb{P}V}(d))$, and the induced map on symmetric powers

$$\operatorname{Sym}^d V^* \to \operatorname{Sym}^d S^*$$

evaluated at L takes a form g of degree d to its restriction to L, as required.

Let $\sigma_g \in H^0(\operatorname{Sym}^d S^*)$ be the global section of $\operatorname{Sym}^d S^*$ that is the image of g. We claim that $F_k(X) \subset \mathbb{G}(k,n)$ is the zero locus of this section. It is enough to check this locally on an open covering of $\mathbb{G}(k,n)$, and we use the open covering by basic affine sets described in Chapter 2. Without loss of generality we may assume that U is the open set of planes that are row-spaces of matrices A as in Section 8.1.1. On this open set the bundle S is trivial, with the inclusion

$$S|_U = \mathcal{O}_U^{k+1} \to V \otimes \mathcal{O}_U$$

given by the transpose of the matrix A. It follows that the dual map

$$A^* : H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_U = V^* \otimes \mathcal{O}_U \to S^*|_U$$

is the restriction of linear forms from $\mathbb{P}^n$, and its d-th symmetric power is the restriction of forms of degree d. Thus the value of σ_g at the point of U

corresponding to a plane L is the restriction of g to L, or in other words the result of the substitution given in Section 8.1.1, as required. $\square$

8.2.1 Counting lines on cubics

Let's see how this will work for the case of lines on a cubic surface $X \subset \mathbb{P}^3$. In the language above, we want to compute something about the Fano scheme $F_1(X)$ in the Grassmannian $\mathbb{G} = \mathbb{G}(1,3)$. We saw in Section 7.3.5 that the Chern class of S^* is

$$c(S^*) = 1 + \sigma_1 + \sigma_{1,1}.$$

Since S has rank 2, the rank of $\operatorname{Sym}^3 S^*$ is 4, so we want to compute $c_4(\operatorname{Sym}^3 S^*)$. To do this, we will apply the splitting principle (Section 7.2.4), which implies that to compute the Chern class we may pretend that S^* splits into a direct sum of two line bundles L and M. Suppose that

$$c(L) = 1 + \alpha \qquad \text{and} \qquad c(M) = 1 + \beta.$$

By the Whitney formula

$$c(S^*) = (1+\alpha)(1+\beta)$$

so that

$$\alpha + \beta = \sigma_1 \qquad \text{and} \qquad \alpha \cdot \beta = \sigma_{1,1}.$$

If S^* were split as above then the bundle $\operatorname{Sym}^3 S^*$ would split as well:

$$\operatorname{Sym}^3 S^* = L^3 \oplus (L^2 \otimes M) \oplus (L \otimes M^2) \oplus M^3$$

so that we have

$$c(\operatorname{Sym}^3 S^*) = (1+3\alpha)(1+2\alpha+\beta)(1+\alpha+2\beta)(1+3\beta).$$

In particular, the top Chern class can be written

$$\begin{aligned} c_4((\operatorname{Sym}^3 S^*) &= 3\alpha(2\alpha+\beta)(\alpha+2\beta)3\beta \\ &= 9\alpha\beta(2\alpha^2 + 5\alpha\beta + 2\beta^2) \\ &= 9\alpha\beta(2(\alpha+\beta)^2 + \alpha\beta) \end{aligned}$$

Re-expressing this in terms of the Chern classes of S^* itself, we get

$$\begin{aligned} &= 9\sigma_{1,1}(2\sigma_1^2 + \sigma_{1,1}) \\ &= 27\sigma_{2,2}, \end{aligned}$$

so

$$\deg\left(c_4((\operatorname{Sym}^3 S^*)\right) = 27.$$

The whole Chern class of $\operatorname{Sym}^3 S^*$ can also be computed by hand in this way, or with the following commands in Macaulay2:

```
loadPackage "Schubert2"
G = flagBundle({2,2}, VariableNames=>{s,q})
-- sets G to be the Grassmannian of 2-planes in 4-space,
-- and gives the names $s_i$ and $q_i$ to the Chern classes
-- of the sub and quotient bundles, respectively.
(S,Q)=G.Bundles
-- names the sub and quotient bundles on G
chern symmetricPower(3,dual S)
```

which returns the output

```
                   2                        2
o4 = 1 + 6q  + (21q  - 10q ) + 42q q  + 27q
           1       1      2       1 2      2

                 QQ[][s , s , q , q ]
                       1   2   1   2
o4 : -------------------------------------------
     (s  + q , s  + s q  + q , s q  + s q , s q )
       1    1   2    1 1    2   2 1    1 2   2 2
```

The answer, on the first output line "o4" is written in terms of the Chern classes $q_i := c_i(Q)$, which generate the (rational) Chow ring of the Grassmannian, described on the second output line "o4".

Since the class $\sigma_4 \operatorname{Sym}^3 S^*$ is nonzero, we deduce that every cubic surface must contain lines, and thus that a general cubic surface contains only finitely many. Moreover, if a particular cubic surface $X \subset \mathbb{P}^3$ contains only finitely many lines, then the number of these lines, counted with the appropriate multiplicity (that is, the degree of the corresponding component of the zero-scheme of σ_g), is 27. As we will soon see, the Fano scheme $F_1(X)$ of a smooth cubic surface X is necessarily of dimension zero and smooth, so the actual number of lines is always 27. In the next section we will develop a general technique that will allow us to prove this statement and much more.

8.3 Definition and existence of Hilbert schemes

It was Grothendieck's brilliant observation that the Grassmannian and the Fano scheme are special cases of a very general construction, the *Hilbert scheme*. Hilbert schemes are defined by a universal property that we will explain in this section, after making the property explicit for the Grassmannian and Fano schemes.

One of the useful consequences of being a Hilbert scheme is a general technique for determining tangent spaces that we will explain in the next section. For more remarks about Hilbert schemes in general, see Section 10.4.1.

8.3.1 A universal property of the Grassmannian

Recall from Theorem 2.4 that the Grassmannian $G := G(k+1, V) = \mathbb{G}(k, \mathbb{P}V)$ of $k+1$-planes in an $(n+1)$-dimensional vector space V, with its tautological subbundle $\mathcal{S} \subset V \otimes \mathcal{O}_G$, has the following universal property: Given any scheme B and any rank $k+1$ subbundle $\mathcal{F}$ of the trivial bundle $V \otimes \mathcal{O}_B$, there is a unique morphism $\varphi : B \to G$ such that $\mathcal{F} = \varphi^*\mathcal{S}$. We could, of course, just as well express this in terms of a universal property of the quotient bundle $\mathcal{Q} = V \otimes \mathcal{O}_B/\mathcal{S}$, or of the $(n-k)$-subbundle $\mathcal{Q}^* \subset V^* \otimes \mathcal{O}_B$, which is the most convenient for what we will do in this section.

Similarly, if we think of G as parametrizing k-planes in $\mathbb{P}V$, we can introduce the *universal k-plane in $\mathbb{P}V$*:

$$\Phi = \{(\Lambda, p) \in G \times \mathbb{P}V \mid p \in \Lambda\}$$

This is a universal family of k-planes in $\mathbb{P}V$ in the following sense. For any scheme B, we will say that a subscheme $\mathcal{L} \subset B \times \mathbb{P}V$ is a *flat family of k-planes in* $\mathbb{P}V$ if the restriction $\pi : \mathcal{L} \to B$ of the projection $\pi_1 : B \times \mathbb{P}V \to B$ is flat, and the fibers over closed points of B are linearly embedded k-planes in $\mathbb{P}V$. We have then

Proposition 8.5. *If $\pi : \mathcal{L} \subset B \times \mathbb{P}V \to B$ is a flat family of k-planes in $\mathbb{P}V$, then there is a unique map $\alpha : B \to G$ such that $\mathcal{L}$ is equal to the pullback of the family $\mathcal{L}$ via α:*

$$\begin{tikzcd}
\mathcal{L} \arrow[drr, "\pi"'] & = & B \times_G \Phi \arrow[r] \arrow[d] & \Phi \arrow[d, "\pi"] \\
& & B \arrow[r, "\alpha"'] & \mathbb{G}(k, n).
\end{tikzcd}$$

Proof. We will prove the proposition by showing that the desired property can be reduced to the universal property of Theorem 2.4. Though the reduction may appear technical, it is really just an application Theorem 6.6 together with the remark that the ideal of any k-plane in $\mathbb{P}V$ is generated by $n-k$ independent linear forms.

To simplify the notation we denote the ideal sheaves of $\Phi \subset G \times \mathbb{P}V$ and $\mathcal{L} \subset B \times \mathbb{P}V$ respectively by $\mathcal{I}$ and $\mathcal{J}$, and we write $\mathcal{J}(1)$ for $\mathcal{J} \otimes \pi^*\mathcal{O}_{\mathbb{P}V}(1)$ where π denotes the projection onto $\mathbb{P}V$. We define $\mathcal{I}(1)$ similarly. For any scheme B we denote the trivial bundle with fiber V^* and base B by V_B^*.

The proof consists of the following steps: we will begin by showing that $\pi_*\mathcal{J}(1)$ is a subbundle of V_B^*. Since Φ satisfies the same hypotheses as $\mathcal{L}$, the same reasoning will show that the sheaf $\pi_*\mathcal{I}(1)$ is a subbundle of V_G^*. We will see that this subbundle is equal to the subbundle $\mathcal{Q}^*$. It follows that there is a unique map $\alpha : B \to G$ such that

$$\alpha^*(\pi_*\mathcal{I}(1)) = \pi_*\mathcal{J}(1).$$

Finally, we will show that this last equation is equivalent to the equality

$$\mathcal{L} = B \times_\alpha \Phi$$

as families of k-planes in $\mathbb{P}V$, where $B \times_\alpha \mathcal{L}$ denotes the pullback $B \times_G \mathcal{L}$ defined using the map α.

The fact that $\pi_*(\mathcal{J}(1))$ is a bundle follows from Theorem 6.6 and the remark that the restriction of $\pi_*(\mathcal{J}(1))$ to a fiber b is the $n-k$-dimensional linear space of forms vanishing on the k-plane $\mathcal{L}'_b \subset \{b\} \times \mathbb{P}V \cong \mathbb{P}V$. The natural map $\pi_*\mathcal{J}(1) \to \pi_*\mathcal{O}_{B\times\mathbb{P}V}(1) = V_B^*$ is an inclusion on fibers, so $\pi_*\mathcal{J}(1)$ is a subbundle as claimed.

To identify $\pi_*\mathcal{I}(1)$ with $\mathcal{Q}^*$ we remark that both are subbundles of V_B^*, and at each point their fibers are the same subspace—namely, the space of linear forms vanishing on L. It now follows from the universal property of Theorem 2.4 that there is a unique morphism $\alpha : B \to G$ such that $\alpha^*\pi_*\mathcal{I}(1) = \pi_*\mathcal{J}(1)$ as subbundles of V_B^*.

We claim that this property of α implies the equality $\mathcal{L} = B \times_\alpha \Phi$. To prove this, it suffices to show that $\mathcal{J} = (\alpha \times 1)^*\mathcal{I}$, or equivalently $\mathcal{J}(1) = (\alpha \times 1)^*\mathcal{I}(1)$. If we restrict $\mathcal{L}$ to the fiber over $b \in B$ we get a subspace of $\mathbb{P}V$ whose ideal is generated by the linear linear forms it contains. For $b \in B$, Theorem 6.6 identifies this space of linear forms with the fiber of $\pi_*\mathcal{J}(1)$ at b. Thus there is a surjection

$$\pi^*\pi_*\mathcal{J}(1) \to \mathcal{J}(1).$$

Similar remarks hold for $\mathcal{I}$. Thus the commutative diagram

$$\begin{tikzcd}
\pi^*\pi_*\mathcal{J}(1) \arrow[dr] & =\pi^*\alpha * \pi_*\mathcal{I}(1)= & (\alpha\times 1)^*\pi^*\pi_*\mathcal{I}(1) \arrow[dl] \\
& \mathcal{O}_{B\times P} &
\end{tikzcd}$$

shows that the ideal sheaves of $\mathcal{L}$ and $B \times_\alpha \Phi$ are equal.

Finally, we prove the uniqueness of α. Suppose that $\mathcal{L} = B \times_{\alpha'} \Phi$ for some morphism α'. We will show that $\alpha' = \alpha$ by showing that $\pi_*\mathcal{J}(1) = \alpha'^*\pi_*\mathcal{I}(1)$. But the hypothesis implies that $\mathcal{J}(1) = (\alpha' \times 1)^*\mathcal{I}(1)$. From the definition of the pushforward we get a natural map

$$\alpha'^*\pi_*\mathcal{I}(1) \to \pi_*(\alpha' \times 1)^*\mathcal{I}(1) = \pi_*\mathcal{J}(1)$$

that is an isomorphism fiber by fiber, so we are done. □

8.3.2 A universal property of the Fano scheme

We realized the Fano scheme of a projective variety X as the subscheme of the Grassmannian consisting of planes lying in X, and as such it inherits a universal property :

Proposition 8.6. *If $X \subset \mathbb{P}^n$ is a subscheme, then the scheme $F_k(X)$ represents the functor of k-planes on X, in the sense that the correspondence above induces a one-to-one correspondence between morphisms of schemes $B \to F_k(X) \subset \mathbb{G}(k,n)$ and families of k-planes $\mathcal{L} \subset B \times X \subset B \times \mathbb{P}^n$ that are flat over B.*

Proof. This is a corollary of the statement for the Grassmannian: Suppose that $\mathbb{P}^n = \mathbb{P}V$ and that X is defined by some homogeneous forms $g_i \in Sym_{d_i} V^*$. Let S be the universal subbundle on$\mathbb{G}(k,n)$, so that the fiber of S^* at a point $[L] \in \mathbb{G}(k,n)$ is the space of linear forms on the corresponding k-plane $L \subset \mathbb{P}V$. Writing δ_{g_i} for the section of $\operatorname{Sym}^{d_i} S^*$ that is the image of the form g_i, we see that g_i vanishes on L if and only if the sections σ_{g_i} vanish at the point $[L]$. $\square$

8.3.3 The Hilbert scheme and its universal property

Grothendieck's idea was to ask, more generally: given any projective scheme X and a subscheme Y, "how Y can move within X?" More precisely and ambitiously: is there a flat family $X \times B \supset \mathcal{Y} \to B$ of subschemes of X, universal in some sense, which contains Y?

When B is reduced then a family $\mathcal{Y}$ as above is flat if and only if the fibers all have the same Hilbert polynomial; in particular, any family over a reduced base whose fibers are all k-planes is automatically flat. (See for example Eisenbud and Harris [2000] Proposition III-56.) Gorthendieck's idea was to define "the family of all subschemes" of X with Hilbert polynomial equal to $P_Y(d)$, the Hilbert polynomial of Y ****check notation for Hilbert poly.****.

We might worry that this goes too far to be a generalization of the Fano scheme—could there be a subschemes of X that is not a k-plane but whose Hilbert polynomial is equal to that of a k-plane? The following result shows that all is well:

Proposition 8.7. *A subscheme $Y \subset \mathbb{P}^n$ is a linearly embedded k-plane $\mathbb{P}^k$ if and only if the Hilbert polynomial of Y is*

$$P(d) = \frac{(d+k)(d+k-1)\cdots(d+1)}{k(k-1)\cdots 1},$$

Proof. Since the dimension of the d-th graded component of a polynomial ring on $k+1$ variables is $\binom{d+k}{k}$, the Hilbert polynomial of a linearly embedded k-plane is $P(d)$.

Conversely, suppose that Y has Hilbert polynomial P. From the degree and leading coefficient of P we see that Y is is a scheme of dimension k and degree 1. Thus $L := Y_{red} \subset Y$ is a linearly embedded k-plane. This inclusion induces a surjection of homogeneous coordinate rings, $S_Y \to S_L$, and the equality of Hilbert polyomials shows that it is an isomorphism in high degrees. Since the inclusions $L \subset Y \subset \mathbb{P}^n$ can be recovered as $\mathrm{Proj}(S_L) \subset \mathrm{Proj}(S_Y) \subset \mathrm{Proj}(S)$, where S is the homogeneous coordinate ring of $\mathbb{P}^n$, and since $\mathrm{Proj}(S_Y)$ depends only on the high degree part of S_Y, this shows $L \subset Y$ is actually an equality. □

Here is the general definition and existence theorem for Hilbert schemes, showing that there is a unique "most natural" scheme structure:

Proposition-Definition 8.8. Let $X \subset \mathbb{P}^n$ be a closed subscheme, and let $P(d)$ be a polynomial. There exists a unique scheme $\mathcal{H}_P(X)$, called the *Hilbert Scheme* of X for the Hilbert polynomial P, with a flat family

$$X \times \mathcal{H}_P(X) \supset \mathcal{Y} \xrightarrow{\pi} \mathcal{H}_P(X),$$

of subschemes of X, called the *universal family* of subschemes of X with Hilbert polynomial P, having the following properties:

- The fibers of π all have Hilbert polyomial equal to $P(d)$.
- For any flat family

$$X \times \mathcal{H}_P(X) \supset \mathcal{Y}' \xrightarrow{\pi} B$$

  whose fibers have Hilbert polynomial $P(d)$, there is a unique morphism $\alpha : B \to \mathcal{H}_P(X)$ such that $\mathcal{Y}'$ is isomorphic to the pullback of $\mathcal{Y}$:

$$\begin{tikzcd}
\mathcal{Y}' \arrow[dr, "\pi"'] & \cong B \times_{\mathcal{H}_P(X)} \mathcal{Y} \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d, "\pi"'] \\
 & B \arrow[r, "\alpha"'] & \mathcal{H}_P(X).
\end{tikzcd}$$

****fix the spacing of $\cong$ in the diagram****

A compact way of stating the existence of $\mathcal{Y} \to \mathcal{H}_P(X)$ and its universal property is to say that the contravariant functor from schemes to sets

$$\begin{aligned}
F_{X,P} : B \mapsto \\
&\{\text{Flat families } X \times B \supset \mathcal{Y}' \to B \text{ of subschemes of } X \subset \mathbb{P}^n \\
&\text{whose fibers over closed points all have Hilbert polynomial P}\}
\end{aligned}$$

and maps $B' \to B$ to the map of sets taking a flat family over B to its pullback to a family over B', is *representable* by the scheme $\mathcal{H}_P(X)$ in the sense that

$$F_{X,P} \cong \mathrm{Mor}(-, \mathcal{H}_P(X))$$

as functors. The universal family in $F_{X,P}(\mathcal{H}_P(X))$ then corresponds to the identity map in $\mathrm{Mor}(\mathcal{H}_P(X), \mathcal{H}_P(X))$. See for example Eisenbud and Harris [2000] Chapter VI for more about this idea.

Proof of Uniqueness in Proposition 8.8. As with any object with a universal property, the uniqueness of a map $\pi : \mathcal{Y} \to \mathcal{H}_P(X)$ with the given properties is easy: Given another, $\pi' : \mathcal{Y}' \to B$, the universal properties of the two produce maps $B \to \mathcal{H}_P(X)$ and $\mathcal{H}_P(X) \to B$ whose composition $\mathcal{H}_P(X) \to B \to \mathcal{H}_P(X)$ is the unique map guaranteed by the definition that corresponds to the family $\pi : \mathcal{Y} \to \mathcal{H}_P(X)$ itself—that is, the identity map—and similarly for the composite $B \to \mathcal{H}_P(X) \to B$. $\square$

8.3.4 Sketch of the construction of the Hilbert scheme

The construction of $\mathcal{H}_P(X)$ and the universal family is also relatively easy to describe, though the proofs of the necessary facts are deeper. There are several approaches, but the following (from the Thesis Bayer [1982] is perhaps the most explicit.

We first treat the case when $X = \mathbb{P}^n$, since (as in the case of the Fano schemes) we shall see that the general case reduces to this. Let $S = K[x_0, \ldots, x_n]$ be the homogeneous coordinate ring of $\mathbb{P}^n$. The Hilbert scheme $\mathcal{H}_P(X)$ is constructed as a subscheme of the Grassmannian of $P(d)$-dimensional subspaces of S_d, the space of homogeneous forms of degree d, for suitably large d. The possibility of doing this is provided by the following basic result from commutative algebra, which combines ideas of Macaulay and Gotzmann (see Green [1989] for a coherent account.)

Theorem 8.9. *With notation as above, there is an integer $d_0(P)$ (explicitly computable from the coeffcients of P) such that if $d \geq d_0(P)$, the saturated homogeneous ideal I of any subscheme of X with Hilbert polynomial P is generated in degrees $\leq d$, and $\dim(S_d/I_d) = P(d)$. Further, a subspace $U \subset S_d$ of dimension $P(d)$ generates an ideal with Hilbert polynomial $P(d)$ if and only if*

$$\dim(S_{d+1}/S_1U) \geq P(d+1),$$

in which case

$$\dim(S_{d+1}/S_1U) = P(d+1). \quad \square$$

Example 8.10. For example, if X is any hypersurface of degree s in $\mathbb{P}^n$, then the Hilbert function of S_X is

$$\dim(S_X)_d = \binom{n+d}{n} - \binom{n+d-s}{n},$$

which is equal to a polynomial $P(d)$ of degree $n-1$ for all d such that $d \geq s$, as one checks immediately. Conversely, given any scheme $X \subset \mathbb{P}^n$ with this Hilbert polynomial, we see that $\dim X = n-1$, so X is a hypersurface, and the leading coefficient of the Hilbert polynomial tells us that $\deg X = s$. It follows that the saturated ideal of X is generated by a single form of degree s. In this case, every subspace $U \subset S_d$ generates an ideal with this Hilbert polynomial; the growth condtion of the Theorem is automatically satisfied.

Given Theorem 8.9, we choose $d \geq d_0(P)$, and take $\mathcal{H}_P(\mathbb{P}^n)$ to be the closed subscheme of the Grassmannian $G := G(\dim S_d - P(d), S_d)$ equations saying that, with $U \in G$, the vector space $S_1 U$ has the smallest possible dimension, which is $\dim S_{d+1} - P(d+1)$. As such, it is defined by the following determinantal condition: Writing $\mathcal{S}$ for the universal subbundle of the trivial vector bundle $S_d \otimes \mathcal{O}_G$ on G, is defined by the condition that the rank of the composite map

$$S_1 \otimes \mathcal{S} \to S_1 \otimes S_d \otimes \mathcal{O}_G \to \otimes S_{d+1} \otimes \mathcal{O}_G$$

has corank $\geq P(d+1)$.

Further, we can construct the universal family $\mathcal{Y} \subset \mathbb{P}^n \times \mathcal{H}_P(\mathbb{P}^n)$ as follows. Let

$$\mathbb{P}^n \xleftarrow{pi_1} \mathbb{P}^n \times G \xrightarrow{\pi_2} G$$

be the projection maps, there is a natural map $S_d \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n \times G}$, and composing this with the inclusion we get a map of sheaves

$$\pi_2^* U \otimes \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n \times G}.$$

Let $\tilde{\mathcal{Y}}$ be the subscheme of $P^n \times G$ defined by the image of this map, and let $\mathcal{Y} \to \mathcal{H}_P(\mathbb{P}^n)$ be the restriction to $\mathcal{H}_P(X) \subset G$ of the (non)flat family given by the composite

$$\tilde{\mathcal{Y}} \subset \mathbb{P}^n \times G \xrightarrow{\pi_2} G.$$

The universal property (which we will not prove) shows that these construction are (up to canonical isomorphism) independent of the choice of $d \geq d_0$.

So far we have only defined $\mathcal{H}_P(\mathbb{P}^n)$, but we can use this to construct $\mathcal{H}_P(X)$ for any $X \subset \mathbb{P}^n$. Let $I = I(X) \subset S$ be the ideal corresponding to X, and suppose that I is generated in degrees $\leq e$. Given the Hilbert polynomial P we choose $d \geq max(d_0(P), e)$. Then to define $\mathcal{H}_P(X)$, we simply add equations to $\mathcal{H}_P(\mathbb{P}^n)$ implying that $I_d \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^n}(-d)$ is contained in $U \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^n}(-d)$. This can be translated into a rank condition on a map of vector bundles, as before.

Example 8.11 (Example , continued.)**.** The argument above shows that the Hilbert scheme of a hypersurface of degree s in $\mathbb{P}^n$ is the projective space $\mathbb{P}^N$ of all homogeneous forms of degree s, and the universal family is the universal hypersurface

$$\mathcal{X} = \{(x, X) \subset \mathbb{P}^n \times \mathbb{P}^N \mid x \in X\}$$

as one would hope.

Here is one way to understand the integer $d_0(P)$ that plays a central role in the construction. Recall that the set of monomials of given degree d can be ordered *lexicographically*, where

$$m := x_0^{e_0} \cdots x_n^{e_n} < x_0^{f_0} \cdots x_n^{f_n} =: n$$

if $e_i > f_i$ for the smallest i such that $e_i \neq f_i$—informally put, m involves more of the lowest-index variables than n. A monomial ideal $I \subset S$ is called *lexicographic* if, whenever $m < n$ are monomials of degree d and $n \in I$, then $m \in I$ too. It follows easily that the saturation of a lexicographic ideal is lexicographic.

Proposition 8.12. *Let $S = K[x_0, \ldots, x_n]$.*

(a) *If I is any homogeneous ideal of S then there is a lexicographic ideal J such that the Hilbert function of S/J is the same as that of S/I.*

(b) *If $P = P_I$ is the Hilbert polynomial S/I, then there is a unique saturated lexicographic ideal J_P with Hilbert polynomial P.*

The integer $d_0(P)$ may be taken to be the maximal degree of a generator of J_P. □

For example, if I is the principal ideal generated by a form of degree s as in the example above, the Proposition gives $d_0 = s$. See Green [1989] for further information.

8.4 Tangent spaces to Fano and Hilbert schemes

In order to use the Chern class calculation of Section 8.2.1 to count the number of distinct lines on a cubic surface, we need to know when the Fano scheme is reduced. In the zero-dimensional case, this is the same as being smooth, and can thus be approached through a computation of Zariski tangent spaces. Happily, we can give a simple description of the Zariski tangent spaces of any Hilbert scheme.

We first state the main assertions for Fano schemes. They will allow us to deduce the exact number of lines on a general hypersurface $X \subset \mathbb{P}^n$ of degree $d = 2n - 3$ and other geometric facts. We will then make the computation of the tangent spaces in the general setting of Hilbert schemes (Theorem 8.21). In section 8.7 below, we'll show how to calculate the multiplicity of $F_1(X)$ at L by writing down explicit local equations for $F_1(X) \subset \mathbb{G}(1, n)$.

8.4.1 Normal bundles and the smoothness of the Fano scheme

We will make use of the universal property of Fano schemes to give a geometric condition for the smoothness of $F_k(X)$ at a given point. Recall that if $Y \subset X$ is a smooth subvariety of the smooth variety X then the *normal bundle* $N_{Y/X}$ of Y in X is the quotient of the map of tangent bundles $N_{Y/X} = \operatorname{coker}\big(T_Y \to T_X|_Y\big)$ induced by the inclusion of $Y \subset X$. Recall also that the Zariski tangent space of a scheme F at a point p is by definition $\operatorname{Hom}_{\mathcal{O}_p}(\mathfrak{m}_p/\mathfrak{m}_p^2, \mathcal{O}_p/\mathfrak{m}_p)$, where $\mathfrak{m}_p$ is the maximal ideal of the local ring $\mathcal{O}_p$ of F at p.

The following Theorem is a special case of a general result on Hilbert schemes, Theorem 8.21, which we will prove in the next section.

Theorem 8.13. *Suppose that $L \subset X$ is a k-plane in a smooth variety $X \subset \mathbb{P}^n$, and let $[L] \in F_k(X)$ be the corresponding point. The Zariski tangent space of $F_k(X)$ at $[L]$ is isomorphic to $H^0(N_{L/X})$ as vector spaces.*

The result is intuitively plausible if we think of a section of $N_{L/X}$ as providing an infinitesimal normal vector at each point in X, with a corresponding infinitesimal motion of X.

For a case that is easy to understand, take $k = 0$. The Hilbert scheme of points on a variety X is X itself, as one checks from the definition. The tangent space at a point $x \in X$ is thus the Zariski tangent space to X at X, and this is—by definition—equal to $\operatorname{Hom}_{\mathcal{O}_X}(\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2, \mathcal{O}_x) = N_{x/X}$. Before introducing the general machinery of the proof, we explain how the result can be used.

Corollary 8.14. *Suppose that $L \subset X$ is a k-plane in a smooth variety $X \subset \mathbb{P}^n$, and let $[L] \in F_k(X)$ be the corresponding point. The dimension of $F_k(X)$ at $[L]$ is at most $\dim H^0(N_{L/X})$. Moreover, $F_k(X)$ is smooth at $[L]$ if and only if equality holds.*

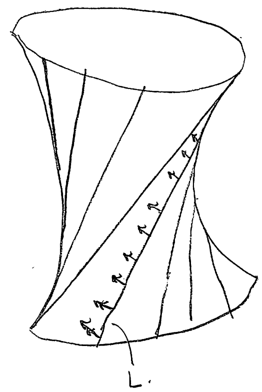


FIGURE 8.1. A tangent vector to the Fano scheme $F_1(X)$ at $[L]$ corresponds to a normal vector field along L in X.

Proof of Corollary 8.14. By the Principal Ideal Theorem the dimension of the Zariski tangent space of a local ring is always at least the dimension of the ring; and equality holds if and only if the ring is regular. See Eisenbud [1995]. $\square$

To apply Corollary 8.14 we need to be able to compute normal bundles, and this is often easy. For example, we have:

Proposition-Definition 8.15. Suppose that $Y \subset X$ are schemes.

(a) If X and Y are smooth varieties then $N_{Y/X} = Hom_{\mathcal{O}_Y}(\mathcal{I}_Y/\mathcal{I}_Y^2, \mathcal{O}_Y)$. *For arbitrary schemes $Y \subset X$ we define $N_{Y/X}$ by this formula.*

(b) If $Y \subset X \subset W$ are schemes, and X is locally a complete intersection in W, then there is a left exact sequence of normal bundles

$$0 \longrightarrow N_{Y/X} \longrightarrow N_{Y/W} \xrightarrow{\alpha} N_{X/W}|_Y.$$

If all three schemes are smooth, then α is an epimorphism.

(c) If Y is a Cartier divisor on X then $N_{Y/X} = \mathcal{O}_X(Y)$. More generally, if Y is the zero locus of a section of a bundle E of rank e on X, and Y has codimension e in X, then

$$N_{Y/X} = E|_Y.$$

Proof of Proposition 8.15. **(a):** For any inclusion of subschemes $Y \subset X$ there is a right exact sequence involving the cotangent sheaves of X and Y:

$$\mathcal{I}_{Y/X}/\mathcal{I}_{Y/X}^2 \xrightarrow{d} \Omega_X|_Y \longrightarrow \Omega_Y \longrightarrow 0,$$

where d is the map taking the class of a (locally defined) function $f \in \mathcal{I}_{Y/X}$ to its differential $df \in \Omega_X|_Y$; see for example Eisenbud [1995] Proposition 16.12. Since X and Y are smooth, Y is locally a complete intersection in X, so $\mathcal{I}_{Y/X}/\mathcal{I}_{Y/X}^2$ is a locally free sheaf on Y of rank equal to $\dim X - \dim Y = \operatorname{rank} \Omega_X|_Y - \operatorname{rank} \Omega_Y$. If the left-hand map d were not a monomorphism, then the image of d would have strictly smaller rank, so the sequence could not be exact at $\Omega_X|_Y$. Thus d is a monomorphism, and we have an exact sequence

$$0 \longrightarrow \mathcal{I}_{Y/X}/\mathcal{I}_{Y/X}^2 \xrightarrow{d} \Omega_X|_Y \longrightarrow \Omega_Y \longrightarrow 0,$$

of bundles. Since Y is smooth, Ω_Y is locally free, so dualizing preserves exactness, and we get an exact sequence

$$0 \longleftarrow Hom_{\mathcal{O}_Y}(\mathcal{I}_{Y/X}/\mathcal{I}_{Y/X}^2, \mathcal{O}_Y) \longleftarrow T_X|_Y \longleftarrow T_Y \longleftarrow 0,$$

where the right hand map is the differential of the inclusion $Y \subset X$, proving that $N_{Y/X} = Hom_{\mathcal{O}_Y}(\mathcal{I}_Y/\mathcal{I}_Y^2, \mathcal{O}_Y)$.

(b): From the inclusions $Y \subset X \subset W$ we derive an exact sequence of ideal sheaves

$$0 \to \mathcal{I}_{X/W} \to \mathcal{I}_{Y/W} \to \mathcal{I}_{Y/X} \to 0.$$

Applying the functor $Hom_{\mathcal{O}_W}(-, \mathcal{O}_Y)$ gives a left exact sequnce

$$0 \to N_{Y/X} \to N_{Y/W} \to Hom(\mathcal{I}_{Y/X}, \mathcal{O}_Y).$$

Since $Hom(\mathcal{I}_{Y/X}, \mathcal{O}_Y) \cong Hom(\mathcal{I}_{Y/X} \otimes \mathcal{O}_Y, \mathcal{O}_Y)$ and $\mathcal{I}_{Y/X} \otimes \mathcal{O}_Y \cong \mathcal{I}_{Y/X}/\mathcal{I}_{Y/X}^2$, we get the desired sequence.

In the smooth case we start with the exact sequence

$$0 \to T_X \to T_W|_X \to N_{X/W} \to 0,$$

that defines $N_{X/W}$. We restrict to Y and factor out the subbundle T_Y from both T_X and $T_W|_X$ to get the required exact sequence

$$0 \to N_{Y/X} \to N_{Y/W} \to N_{X/W}|_Y \to 0.$$

(c) The first formula follows at once from Part (a), since in that case $\mathcal{I}_{Y/X} = \mathcal{O}_X(-Y)$, and taking the dual of a bundle commutes with restriction.

For the second statement of Part (c) we first give a geometric argument that works in the smooth case, and then a proof in general. Let Z be the total space of the bundle E. The tangent bundle to Z restricted to the zero section $X \subset Z$ is $T_X \oplus E$.

Along the zero locus Y of σ, the derivative $D\sigma$ of σ is thus a map $T_X|_Y \to T_X|_Y \oplus E_Y$. Since the component of $D\sigma$ that maps $T_X|_Y$ to E_Y is zero along Y, the composite

$$T_Y \longrightarrow T_X|_Y \xrightarrow{D\sigma} T_X|_Y \oplus E_Y \longrightarrow E_Y$$

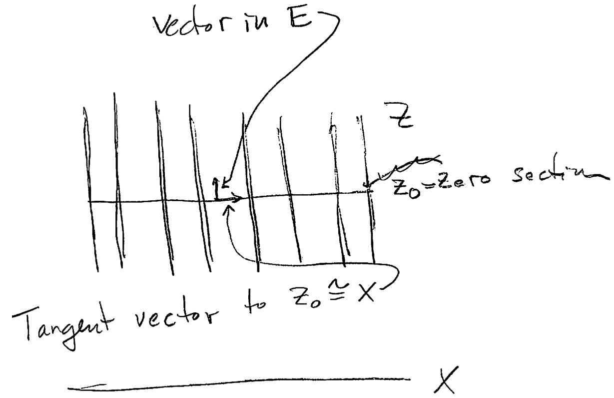


FIGURE 8.2. The tangent bundle to Z along the zero section is $T_X \oplus E$

is zero. Locally at each point $y \in Y$ the image of $(T_X)_y$ in $(T_X)_y \oplus E_y$ is the tangent space to $\sigma(X) \subset Z$. Since Y is smooth of codimension equal to the rank of E the manifold $\sigma(X)$ meets the zero locus $X \subset Z$ transversely. This means that $(T_X)_y$ projects onto E_y, and tells us that the composite map of bundles

$$T_X|_Y \xrightarrow{D\sigma} T_X|_Y \oplus E_Y \to E_Y$$

is surjective. Considering the ranks, it follows that the sequence

$$0 \longrightarrow T_Y \longrightarrow T_X|_Y \longrightarrow E_Y \longrightarrow 0$$

is exact; that is, $N_{Y/X} = E_Y$.

With a more algebraic approach we can avoid the hypothesis that X or Y is smooth. We may think of σ as defining the map $\mathcal{O}_X \to E$ that sends $1 \in \mathcal{O}_X$ to $\sigma \in E$. Dualizing, the statement that Y is the zero locus of σ means that the ideal sheaf $\mathcal{I}_{Y/X}$ is the image of the map $\sigma^* : E^* \to \mathcal{O}_X$. Since the codimension of Y is e we see that Y is locally a complete intersection. Thus the kernel of σ^* is generated by the Koszul relations; that is, the sequence

$$\cdots \wedge^2 E^* \xrightarrow{\kappa} E^* \longrightarrow \mathcal{I}_{Y/X} \longrightarrow 0$$

is exact, where $\kappa(e \wedge f) = \sigma^*(e)f - \sigma^*(f)e$. Because the coefficients in the map κ lie in $\mathcal{I}_{Y/X}$ they become zero on tensoring with $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{I}_Y$, so we we get the right exact sequence

$$\cdots \wedge^2 E^*|_Y \xrightarrow{0} E^*|_Y \longrightarrow \mathcal{I}_{Y/X}/\mathcal{I}_{Y/X}^2 \longrightarrow 0.$$

This shows that $E^*|_Y \cong \mathcal{I}_{Y/X}/\mathcal{I}_{Y/X}^2$, whence $E = E^{**} = N_{Y/X}$. $\square$

In the special case where Y is a complete intersection of X with divisors on $\mathbb{P}^n$ of degrees d_i the normal bundle is $N_{Y/X} = \oplus \mathcal{O}_X(d_i)$, so the last statement of Proposition 8.15 takes a particularly simple form. We can make it even more explicit when both X and Y are complete intersections:

Corollary 8.16. *Suppose that $Y \subset X \subset \mathbb{P}^n$ are (not necessarily smooth) complete intersections of hypersurfaces with homogeneous ideals*

$$I_X = (g_1, \dots g_s) \subset I_Y = (f_1, \dots, f_t); \qquad g_i = \sum_j a_{i,j} f_j.$$

If $\deg f_i = \varphi_i$ and $\deg g_i = \gamma_i$, then

$$N_{Y/\mathbb{P}^n} = \bigoplus_{i=1}^{t} \mathcal{O}_Y(\varphi_i); \qquad N_{X/\mathbb{P}^n} = \bigoplus_{i=1}^{s} \mathcal{O}_X(\gamma_i)$$

and $N_{Y/X}$ is the kernel of the induced map $\alpha :\ N_{Y/\mathbb{P}^n} \to N_{X/\mathbb{P}^n}|_Y$ given by the matrix $(\overline{a}_{j,i})$, where $\overline{a}_{j,i}$ denotes the restriction of $a_{j,i}$ to Y.

Proof. The complete intersection X is the zero locus of the section $(g_1, \dots, g_s)$ of the bundle $\mathcal{O}_{\mathbb{P}^n}(\gamma_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(\gamma_s)$ and similarly for Y. Using the formula of Part(c) we see that

$$N_{X/\mathbb{P}^n} = \bigoplus_{i=1}^{s} \mathcal{O}_X(\gamma_i),$$

and similarly for Y. The identification of α follows at once from Part (a). $\square$

As an immediate application we can finally show that there are exactly 27 distinct lines on every smooth cubic surface (of course pending the proof of Theorem 8.13):

Corollary 8.17. *If $X \subset \mathbb{P}^3$ is a smooth surface of degree $d \geq 3$. If $F_1(X) \neq \varnothing$, then $F_1(X)$ is smooth and zero-dimensional. In particular, X contains at most finitely many lines, and if $d = 3$ then X contains exactly 27 distinct lines.*

See Corollary 8.27 for a strengthening.

Proof. Suppose $L \subset X$ is a line. By Corollary **??**, the self-intersection number of L on X is negative, so the normal bundle $N_{L/X}$ is a line bundle of negative degree. It follows that $\dim H^0(N_{L/X}) = 0$, and Corollary 8.14 now implies that L is isolated and $F_1(X)$ is smooth at $[L]$. $\square$

In particular, in the case of the cubic surface, the fact that the class of the Fano scheme is 27 points implies, with this result, that the Fano scheme actually consists of 27 reduced points.

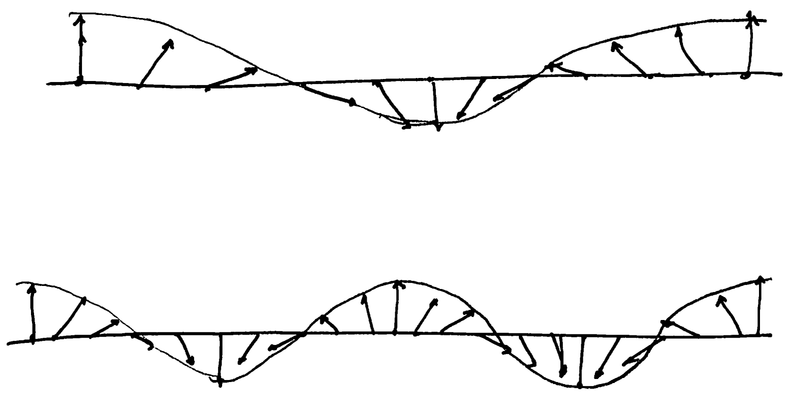


FIGURE 8.3. The tangent planes to a smooth quadric surface along a line wind once around the line; but in the case of a smooth cubic surface they wind around twice.

We can also see Corollary 8.17 geometrically, via the Gauss map $\mathcal{G}_X : X \to \mathbb{P}^{3*}$ sending $p \in X$ to the tangent plane $\mathbb{T}_p X \subset \mathbb{P}^3$. The restriction of $\mathcal{G}_X$ to a line $L \subset X \subset \mathbb{P}^3$ sends L to the dual line

$$L^\perp = \{H \in \mathbb{P}^{3*} \mid L \subset H\},$$

and this map, being given by the partial derivatives of the defining equation of X, has degree $d-1$. Thus, for example, as we travel along a line on a smooth quadric surface Q, the tangent planes to Q rotate once around the line; on a smooth cubic surface X, by contrast, they wind twice around the line (see Figure 8.3). But if $\tilde{L}$ is a first-order deformation of L in $\mathbb{P}^3$, the direction of motion of a point $p \in L$—that is, the 2-plane spanned by L and the normal vector $\sigma(p)$, where σ is the section of the normal bundle $N_{L/\mathbb{P}^3}$—is linear in p. It is thus impossible to find a first-order deformation of L on X, or on any smooth surface of higher degree.

Note that if X is singular at a point of L, the partial derivatives of the defining equation of X have a common zero along L, and so the degree of $\mathcal{G}_X : L \to L^\perp$ will be less than $d-1$. Thus, for example, the tangent planes to a quadric cone are constant along a line of its ruling; and if $L \subset X$ is a line on a cubic surface with an ordinary double point on L the Gauss map will have degree 1 on L. In this case, there *will* exist first-order deformations of L on X—as we'll see shortly in Section 8.7

8.4.2 First-order deformations as tangents to the Hilbert scheme

The proof of Theorem 8.13 and its generalization involves the idea of a *first order deformation* of a subscheme, which is the main content of this section. Suppose that Y is a closed subscheme of a scheme X, defined over the field K. By a *deformation of* $Y \subset X$ *over a scheme* T we mean a flat family $\mathcal{Y} \to T$ having a distinguished closed fiber over $\operatorname{Spec} K$ that is isomorphic to Y, and represented as a closed sub-family of a trivial family $X \times T \to T$:

$$\begin{tikzcd}
Y \arrow[r, hook] \arrow[d, "\alpha"'] & \mathcal{Y} \arrow[r, hook] \arrow[d, "\beta"'] & X \times T \arrow[dl, "\text{projection}"] \\
\operatorname{Spec} K \arrow[r, hook] & T. &
\end{tikzcd}$$

We think of the image of $\operatorname{Spec} K \hookrightarrow T$ as a distinguished point of T, and we will denote it by [Y].

A deformation is called *first-order* if its base, T, is the spectrum of a local ring of the form $R_m = K[\epsilon_1, \ldots, \epsilon_m]/(\epsilon_1, \ldots, \epsilon_m)^2$ for some m. We set $T_m := \operatorname{Spec} R_m$. Note that this is a scheme with a unique closed point, which we shall denote by 0. We think of it as a first-order neighborhood of a point on a smooth m-dimensional variety.

It follows from the universal property of the Hilbert scheme that a first order deformation of Y over T_m is the same thing as a morphism $T_m \to H$ sending 0 to $[Y]$.

In general, we will denote the set of morphisms of T_m into a K-scheme Z sending 0 to a point $z \in Z$ by $\operatorname{Mor}_z(T_m, Z)$, so we have

$$\{\text{Deformations of } Y \subset X \text{ over } T_m\} = \operatorname{Mor}_0(T_m, H).$$

For simplicity we restrict ourselves for a while to the case $m = 1$, and consider deformations over T_1.

The identification of first order deformations with morphisms from T_1 to H is the key to identifying the tangent space of H (and thus, in our case, of the Fano scheme.) Indeed, for any closed K-rational point z on any scheme Z we can identify the set $\operatorname{Mor}_z(T_1, Z)$ with the Zariski tangent space to Z at z. To describe the identification, recall that for any morphism $t : T_1 \to Z$ sending 0 to z we have a pullback map on functions denoted $t^* : \mathcal{O}_{Z,z} \to R_1$. Restricting this map to $\mathfrak{m}_{Z,z}$ we get

$$t^*|_{\mathfrak{m}_{Z,z}} : \mathfrak{m}_{Z,z} \to \mathfrak{m}_{T,0} = K\epsilon \cong K.$$

Since t^* sends $\mathfrak{m}_{Z,z}^2$ to zero, we may identify $t^*|_{\mathfrak{m}_{Z,z}}$ with the induced map

$$t^*|_{\mathfrak{m}_{Z,z}} : \mathfrak{m}_{Z,z}/\mathfrak{m}_{Z,z}^2 \to \mathfrak{m}_{T,0} = K\epsilon \cong K.$$

Lemma 8.18. *Let $z \in Z$ be a K-rational point on a K-scheme. The map*

$$\mathrm{Mor}_z(T_1, Z) \to T_{z/Z} = \mathrm{Hom}_K(\mathfrak{m}_{z/Z}/\mathfrak{m}_{z/Z}^2, K),$$

sending a morphism t to the restriction of the pullback map on functions $t^*|_{\mathfrak{m}_{Z,z}}$, is bijective.

Proof. Giving a morphism $t : T_1 \to Z$ is equivalent to giving the local map of K-algebras $t^* : \mathcal{O}_{Z,z} \to R_1$ that induces the identity map $K \cong \mathcal{O}_{Z,z}/\mathfrak{m}_{Z,z} \to R_1/(\epsilon_1) = K$. Thus t^* is determined by the induced map of vector spaces $\mathfrak{m}_{Z,z}/\mathfrak{m}_{Z,z}^2 \to (\epsilon_1) \subset R_1$.

Conversely, any map $\mathfrak{m}_{Z,z}/\mathfrak{m}_{Z,z}^2 \to (\epsilon)$ extends to a local algebra homomorphism $t^* : \mathcal{O}_{Z,z} \to K[\epsilon]/(\epsilon^2) = R_1$. □

As we have explained, the universal property of the Hilbert scheme of $Y \subset X$ also allows us to identify $\mathrm{Mor}_{[Y]}(T_1, H)$ with the set of first order deformations of $Y \subset X$ over T_1. Such deformations admit another very concrete description.

Theorem 8.19. *Suppose that $Y \subset X$ are schemes. There is a one-to-one correspondence between flat families of subschemes of X over the base T_m with central fiber Y and homomorphisms of $\mathcal{O}_Y$-modules $\mathcal{I}_Y/\mathcal{I}_Y^2 \to \mathcal{O}_Y^n$. In particular, flat families of deformations of Y in X over T_1 correspond to global sections of the normal sheaf of Y in X.*

We will use the following characterization of flatness over T_m:

Lemma 8.20. *If M is a (not necessarily finitely generated) module over the ring R_n, then M is flat if and only if the map*

$$M^n \xrightarrow{(\epsilon_1,\ldots,\epsilon_n)} M$$

induces an isomorphism $(M/(\epsilon_1, \ldots, \epsilon_n)M)^n \cong (\epsilon_1, \ldots, \epsilon_n)M$.

Proof. The general criterion of Eisenbud [1995] Proposition 6.1 says that M is flat if and only if the multiplication map $\mu_I : I \otimes_R M \to IM$ is an isomorphism for all ideals I. But every nontrivial ideal of R is a summand of $(\epsilon_1, \ldots, \epsilon_n) = (\epsilon)$; and since $(R/(\epsilon))^n \cong (\epsilon)$, the map $\mu_{(\epsilon)}$ may be identified with the given map $(M/(\epsilon)M)^n \to (\epsilon)M$. □

Proof of Theorem 8.19. The problem is local, so we may assume that X and Y are affine. Since any homomorphism $\mathcal{I}_Y \to \mathcal{O}_Y^m$ must annihilate $\mathcal{I}_Y^2$, we may identify a homomorphism

$$\varphi : \mathcal{I}_Y/\mathcal{I}_Y^2 \xrightarrow{(\varphi_1,\ldots,\varphi_m)} \mathcal{O}_Y^n$$

with the composition $\mathcal{I}_Y \to \mathcal{I}_Y/\mathcal{I}_Y^2 \to \mathcal{O}_Y^m$. Let $\mathcal{I}_\varphi \subset \mathcal{O}_X \otimes R_m$ be the ideal

$$\mathcal{I}_\varphi := (\{g + \sum_j g_j \epsilon_j \mid g \in \mathcal{I}_Y \text{ and } g_j \equiv \varphi_j(g) \bmod \mathcal{I}_Y\}),$$

and note that $\mathcal{I}_\varphi \supset \sum_j \epsilon_j \mathcal{I}_Y = (\epsilon)\mathcal{I}_Y$.

From $\mathcal{I}_\varphi$ we construct the family

$$\begin{tikzcd}
Y \arrow[r, hook] \arrow[d, "\alpha"'] & Y_{T_m} \arrow[r, hook] \arrow[d, "\beta"'] & X \times T_m \arrow[dl, "\text{projection}"] \\
\operatorname{Spec} K \arrow[r, hook] & T_m &
\end{tikzcd}$$

where Y_{T_m} is defined by $\mathcal{I}_\varphi$. If we set all $\epsilon_j = 0$, then $\mathcal{I}_\varphi$ becomes equal to $\mathcal{I}_Y$, so α is indeed the pullback of β.

We may identify $\mathcal{I}_\varphi/((\epsilon)\mathcal{I}_Y)$ with the graph of $\varphi : \mathcal{I}_Y \to \mathcal{O}_Y^n$ in

$$\begin{aligned}
\mathcal{I}_Y \oplus \mathcal{O}_Y^n &\cong \mathcal{I}_Y \oplus \bigoplus \mathcal{O}_Y \epsilon_j \\
&\subset \mathcal{O}_X \oplus \bigoplus \mathcal{O}_Y \epsilon_j \\
&= \mathcal{O}_X[\epsilon]/((\epsilon)^2 + (\epsilon)\mathcal{I}_Y).
\end{aligned}$$

Thus $\mathcal{I}_\varphi \cap (\epsilon)\mathcal{O}_X = (\epsilon)\mathcal{I}_Y$, and it follows that

$$\begin{aligned}
(\epsilon)(\mathcal{O}_X/\mathcal{I}_\varphi) &= (\epsilon)\mathcal{O}_X/\mathcal{I}_\varphi \cap (\epsilon)\mathcal{O}_X \\
&= (\epsilon)\mathcal{O}_X/(\epsilon)\mathcal{I}_Y \\
&\cong (\mathcal{O}_X/\mathcal{I}_Y)^n \cong \mathcal{O}_Y^n.
\end{aligned}$$

By Lemma 8.20, $\mathcal{O}_X/\mathcal{I}_\varphi$ is flat over R_m.

Conversely, given an R_m-algebra of the form

$$S := \mathcal{O}_X[\epsilon]/((\epsilon)^2 + \mathcal{I}),$$

the statement that $Y_{T_m} := \operatorname{Spec} S$ has Y as its pullback over $\operatorname{Spec} K \subset T_m$ means that $\mathcal{I}$ is congruent to $\mathcal{I}_Y$ modulo (ϵ). Multiplying by (ϵ) and using $(\epsilon)^2 = 0$ we see that $\mathcal{I} \supset (\epsilon)\mathcal{I}_Y$. If S is flat over R then by Lemma 8.20 we must have $\mathcal{I} \cap (\epsilon) = (\epsilon)\mathcal{I}_Y$. Putting these facts together, we see that $\mathcal{I}/(\epsilon)\mathcal{I}_Y$ is the graph of a homomorphism $\mathcal{I}_Y \to (\epsilon)\mathcal{O}_X/(\epsilon)\mathcal{I}_Y \cong \mathcal{O}_Y^n$, and this is the inverse of the construction above. $\square$

These results already identify both the Zariski tangent space $T_{[Y],H}$ of the Hilbert scheme H of $Y \subset X$ at the point corresponding to Y, and also the vector space of global sections of the normal sheaf with the *set* of first order deformations of Y in X, identified with the set $\operatorname{Mor}_{[Y]}(T_1, H)$. But our goal is to compute the dimension of one of these two vector spaces in terms of the other! So we must ensure that the identification preserves the vector space structure.

Theorem 8.21. *Suppose that $Y \subset X$ is a subscheme of a K-scheme $X \subset \mathbb{P}^n$, and let H be the Hilbert scheme of Y. If $[Y] \in H$ denotes the point corresponding to Y, then*

$$T_{[Y]/H} \cong H^0\left(\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}_{Y/X}/\mathcal{I}^2_{Y/X}, \mathcal{O}_Y)\right)$$

as vector spaces.

Theorem 8.13 is the special case where Y is a k-plane in X.

Proof of Theorem 8.21. We will show how to introduce a vector space structure directly into the set of morphisms $T_1 \to H$, and prove that this third structure is compatible with the bijections we have already given.

The definitions for addition and for scalar multiplication in the set $\mathrm{Mor}_{[Y]}(T_1, H)$ are similar, and the one for addition is more complicated, so will define addition, and check that it is compatible with the identifications of Lemma 8.18 and Theorem 8.19. We leave the analogous treatment of scalar multiplication to the reader.

As before we set $R_m = K[\epsilon_1, \dots, \epsilon_m]/(\epsilon_1, \dots, \epsilon_m)^2$ and $T_m = \operatorname{Spec} R_m$ (we will only use the cases $m = 1$ and $m = 2$). A morphism of schemes $\Psi : T_2 \to H$ sending the closed point to $[Y]$ corresponds to a homomorphism $\psi : \mathfrak{m}_{H,[Y]} \to K\epsilon_1 \oplus K\epsilon_2$ or, equivalently, a pair of homomorphisms $\psi_1, \psi_2 : \mathfrak{m}_{H,[Y]} \to K$, or a pair of morphisms $\Psi_1, \Psi_2 : T_1 \to H$ (in fancy language: T_2 is the coproduct of T_1 with itself in the category of pointed schemes). Moreover the embedding

$$T_1 \xrightarrow{(plus)} T_2$$

that corresponds to factoring out $\epsilon_1 - \epsilon_2$ from R_2, has the property that $\Psi \circ (plus) : T_1 \to H$ is the morphism corresponding to the sum $\psi_1 + \psi_2 : \mathfrak{m}_{H,[Y]} \to K$.

Let $\mathcal{Y}_{\varphi_i}$ be the family obtained by pulling back the universal family along Ψ_i, and let $\varphi_i : \mathcal{I}_Y/\mathcal{I}_Y^2 \to \mathcal{O}_Y$ be the homomorphism corresponding to this flat family. We have a pullback diagram

$$\begin{tikzcd}
\mathcal{Y}_{\varphi_1} \arrow[r] \arrow[d] & \mathcal{Y}_2 \arrow[d] & \mathcal{Y}_{\varphi_2} \arrow[l] \arrow[d] \\
T_1 \arrow[r] & T_2 & T_1 \arrow[l]
\end{tikzcd}$$

of flat families, where $\mathcal{Y}_2 \to T_2$ is the family obtained by pulling back along Ψ. It suffices to show that the pullback of $\mathcal{Y}_2$ along the map $+ : T_1 \to T_2$ is the family $\mathcal{Y}_{\varphi_1+\varphi_2}$.

Let $\varphi_i : \mathcal{I}_Y/\mathcal{I}_Y^2 \to \mathcal{O}_Y^2$ be the homomorphism corresponding to $\mathcal{Y}_2$, so that the ideal of $\mathcal{Y}_2$ is the graph of the corresponding homomorphism

$\mathcal{I}_Y \to \mathcal{O}_Y\epsilon_1 \oplus \mathcal{O}_Y\epsilon_2$. Reducing mod ϵ_2 gives us the map corresponding to φ_1, and similarly for ϵ_1 and φ_2, so φ is in fact the map

$$\mathcal{I}_Y/\mathcal{I}_Y^2 \xrightarrow{\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}} \mathcal{O}_Y^2.$$

Thus if we pull back along the map $+$, that is, factor out $\epsilon_1 - \epsilon_2$, the resulting algebra corresponds to the the map $\varphi = \varphi_1 + \varphi_2$ as required. □

Associated to any family

$$\mathcal{Y} \subset X \times B \xrightarrow{\pi} B$$

of subschemes of X is the *union of the schemes in the family*, defined to be the image of $\mathcal{Y}$ under the projection to X. In this spirit, if $Y \subset X$ are projective schemes, and B is a subscheme of the Hilbert scheme of Y in X, then the *subscheme swept out by* B to be the union $Y' = Y'_B$ of the schemes in the restriction to B of the universal family over H.

We can now give a bound on the Zariski tangent spaces to Y' in the case where Y and X are smooth. Suppose that $p \in Y$ is a point of one of the schemes Y represented by points of B. The tangent space to Y' at p obviously contains the tangent space to Y at p, so it is enough to bound the image of T_pY' in T_pX/T_pY, which is the fiber at p of the normal bundle $(N_{Y/X})_p$ of Y in X.

Intuitively, the amount the tangent space T_pY "moves" as Y moves in B is measured by the tangent space to B at $[Y]$, although some tangent vectors to B may produce trivial motions of T_pY. Of course $T_{[Y]}B \subset T_{[Y]}H$, and by Theorem 8.21 this last is $H^0(N_{Y/X})$. Let $\varphi_{p,Y}$ be the evaluation map

$$\varphi_{p,Y} : H^0(N_{Y/X}) \to (N_{Y/X})_p = T_pX/T_pY.$$

Proposition 8.22. *Let $Y \subset X$ be smooth projective schemes, and let $B \subset H$ be a closed subscheme of the Hilbert scheme of Y in X containing the point $[Y]$. If $p \in Y$ and Y' is the subscheme swept out by B, then*

$$T_pY'/T_pY \subset \varphi_{p,Y}(T_{[Y]}B).$$

Proof. ???? ****this was supposed to prove the $d = 4$ case of deBarre-deJong, in the last section; but at the moment we can only use it in char 0, since we need to know that generically every tangent vector from the union of the deformed schemes lifts to a tangent vector on the abstract deformation, and this is at least generically true in char 0.**** □

Lemma 8.23. *Let $Z \subset Y$ be closed subschemes of a scheme X, and let $Z_\sigma, Y_\tau \subset \operatorname{Spec} K[\epsilon]/(\epsilon^2) \times X$ be first-order deformations of Z and Y in X corresponding to the sections $\sigma \in H^0(N_{Z/X})$ and $\tau \in H^0(N_{Y/X})$. The*

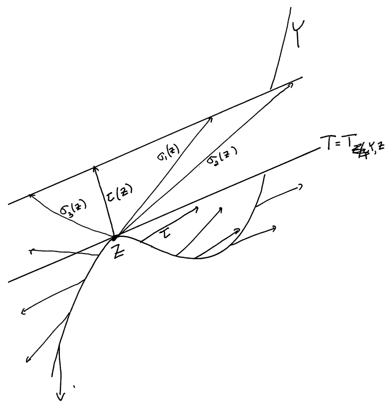


FIGURE 8.4. $\sigma_i(Z) \equiv \tau(Z)$ modulo the tangent line T to Y at Z, so each of the deformations of the point Z corresponding the the σ_i keep it inside the deformation of Y corresponding to σ.

scheme Z_σ is contained in Y_τ if and only if the images of σ and τ are equal under the maps

$$\begin{array}{c} \sigma \in H^0(N_{Z/X}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_{Z/X}, \mathcal{O}_Z) \\ \big\downarrow \\ \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_{Y/X}, \mathcal{O}_Z) \\ \big\uparrow \\ \tau \in H^0(N_{Y/X}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_{Y/X}, \mathcal{O}_Y) \end{array}$$

induced by the inclusion $\mathcal{I}_{Y/X} \subset \mathcal{I}_{Z/X}$ and the projection $\mathcal{O}_Y \to \mathcal{O}_Z$. If Y and X are smooth, or more generally $Y \subset X$ is locally a complete interesection, then $Hom_{\mathcal{O}_X}(\mathcal{I}_{Y/X}, \mathcal{O}_Z) \cong N_{Y/X}|_Z$.

See Figureeffig8-2-A.

Proof. The statement is local, so we can assume Z, Y and X are affine. We regard the global sections σ and τ as module homomorphisms $\mathcal{I}_{Z/X} \to \mathcal{O}_Z$ and $\mathcal{I}_{Y/X} \to \mathcal{O}_Y$.

The schemes $Z_\sigma, Y_\tau \subset X \times T_1$ are given by the ideals

$$\mathcal{I}_\sigma = \{f + \epsilon f' \mid f \in \mathcal{I}_{Z/X} \text{ and } f' \equiv \sigma(f) \bmod \mathcal{I}_{Z/X}\}$$

and

$$\mathcal{I}_\tau = \{g + \epsilon g' : g \in \mathcal{I}_{Y/X} \text{ and } g' \equiv \tau(g) \bmod \mathcal{I}_{Y/X}\}$$

in $\mathcal{O}_X \otimes R_1 = \mathcal{O}_X \oplus \mathcal{O}_X \epsilon$.

Accordingly, we have $Z_\sigma \subset Y_\tau$—that is, $\mathcal{I}_\tau \subset \mathcal{I}_\sigma$—if and only if

$$\sigma(f) \equiv \tau(f) \bmod \mathcal{I}_{Z/X} \quad \text{for all } f \in \mathcal{I}_{Y/X},$$

which is the statement of the Lemma. □

8.4.3 Normal bundles of k-planes on hypersurfaces

In order to apply the description of the tangent space to a Fano scheme $F_k(X)$ of k-planes on a hypersurface X, we need to know something about its normal bundle.

Suppose that $L \subset X \subset \mathbb{P}^n$ is a k-plane on a (not necessarily smooth) hypersurface X of degree d in $\mathbb{P}^n$. Choose coordinates so that the ideal of L is $I_L = (x_{k+1}, \dots, x_n)$, and let $I_X = (g) \subset I_L$. There is a unique expression

$$g = \sum_{i=k+1}^{n} x_i g_i(x_0, \dots, x_k) + h.$$

with $h \in (x_{k+1}, \dots, x_n)^2$. Differentiating, we see that g_i, as a form on L, is the restriction to L of the derivative $\partial g/\partial x_i$.

Since the ideal of $L \subset \mathbb{P}^n$ is generated by $n-k$ linear forms, the normal bundle of L in $\mathbb{P}^n$ is $\mathcal{O}_L^{n-k}(1)$, and, similarly, the normal bundle of X in $\mathbb{P}^n$ is $CO_X(d)$. Thus the restriction $N_{X/\mathbb{P}^n}|_L$ is $\mathcal{O}_L(d)$, and the left exact sequence of Part (b) of Proposition 8.15 takes the form

$$0 \longrightarrow N_{L/X} \longrightarrow \mathcal{O}_L^{n-k}(1) \xrightarrow{\alpha=(g_{k+1},\dots,g_n)} \mathcal{O}_L(d).$$

Proposition 8.24. *With notation as above, let*

$$\alpha = (g_{k+1}, \dots, g_n) : \mathcal{O}_L^{n-k}(1) \to \mathcal{O}_L(d).$$

(a) *The hypersurface X is smooth along L if and only if α is a surjection of sheaves.*

(b) *The point $[L]$ is a smooth point on $F_k(X)$ and the dimension of $F_k(X)$ at $[L]$ is equal to the "expected dimension" $(k+1)(n-k) - \binom{k+d}{k}$ if and only if α is surjective on global sections.*

(c) *The point $[L]$ is an isolated reduced (that is, smooth) point of $F_k(X)$ if and only if α is injective on global sections.*

Proof. (a): Since $L \subset X$ the derivatives of g along L are all zero, so X is smooth at a point $p \in L$ if and only if at least one of the normal derivatives $g_i = \partial g/\partial x_i$, for $i > k$, is nonzero at p. This condition is the condition that α is surjective as a map of sheaves.

(b): By Corollary 8.2 the dimension of $F_k(X)$ at any point is at least $D := (k+1)(n-k) - \binom{k+d}{k}$, so $F_k(X)$ is smooth of dimension D at $[L]$ if and only if the tangent space $T_{[L]}F_k(X) = H^0 N_{L/X}$ has dimension D. Since $\dim H^0(\mathcal{O}_L^{n-k}(1)) = (k+1)(n-k)$ and $\dim H^0(\mathcal{O}_L(d)) = \binom{d+k}{k}$, we see from the exact sequence just before the Proposition that $\dim H^0 N_{L/X} = D$ if and only if α is surjective on global sections.

(c): The condition that $[L]$ is an isolated reduced point of $F_k(X)$ is the condition that $T_{[L]}F_k(X) = H^0 N_{L/X} = 0$, and by the argument of part (b) this happens if and only if α is injective on global sections. $\square$

We can unpack the conditions of Proposition 8.24 as follows: The condition of part (a) is equivalent to saying that the components g_i of the map α don't all vanish simultaneously at a point of L. This gives another proof of Corollary 8.26.

Using the exact sequence preceding the Proposition, and assuming that X is smooth along L so that α is a surjection of sheaves, we see that the condition of part (b) that α is surjective on global sections is equivalent to the condition $H^1 N_{L/X} = 0$. On the other hand, the global sections $x_0, \dots, x_k$ of the i-th summand $\mathcal{O}_L(1) \subset \mathcal{O}_L^{n-1}(1)$ map by α to the sections $x_0 g_{k+i}, \dots, x_k g_{k+i}$, so the condition of surjectivity on sections is also equivalent to the condition that the ideal $(g_{k+1}, \dots, g_n)$ contains every form of degree d.

Similarly, it follows from the exact sequence that the condition of part c is equivalent to the condition $H^0 N_{L/X} = 0$. This means that there are no maps $\mathcal{O}_L$ to the kernel of α or, more concretely, that the g_i have no linear syzygies.

Although part (a) of Proposition 8.24 tells only about smoothness along L, we can do a little better: Bertini's Theorem 0.5 tells us that if the general member of a linear series is not multiple, then it can only be singular along the base locus of the series, and it follows that the general X with a given map α is smooth except possibly along L. Thus if $\alpha : \mathcal{O}_L^{n-k}(1) \to \mathcal{O}_L(d)$ is any surjective map of sheaves, there is smooth hypersurface X containing L such that $N_{L/X} = \operatorname{Ker} \alpha$.

Example 8.25. (Cubic Surfaces Again) The following gives another treatment of Corollary 8.17. In the case of a cubic surfaces $X \subset \mathbb{P}^3$ we have $n = d = 3$ and the expected dimension of $F_1(X)$ is $D = 0$. If we choose

$$g_2 = x_0^2,\ g_3 = x_1^2$$

then the conditions in all three parts of Proposition 8.24 apply: g_2 and g_3 obviously have no common zeros in $\mathbb{P}^1$; because g_2 and g_3 are relatively prime quadratic forms, they have no linear syzygies; and since

$$(x_0, x_1)(x_0^2, x_1^2) = (x_0^3, x_0^2 x_1, x_0 x_1^2, x_1^3) \subset K[x_0, x_1, x_2, x_3],$$

the map α is surjective on global sections. Since the numbers of global sections of the source and target of α are equal, the map α is injective on global sections as well. We can see this directly, too: because g_2, g_3 are relatively prime quadratic forms, the kernel of α is

$$\mathcal{O}_L(-1) \xrightarrow{\begin{pmatrix} -g_3 \\ g_2 \end{pmatrix}} \mathcal{O}_L(1)^2.$$

so $N_{L/X} = \mathcal{O}_L(-1)$, and we see again that $H^0(N_{L/X}) = 0$

From all this we see that L will be an isolated smooth point of $F_1(X)$, where X is the hypersurface defined by the equation $x_0^2x_2 + x_1^2x_3 = 0$. Although this hypersurface is not smooth, Bertini's theorem, as above, shows that there are smooth cubics having the same map α. Since the rank of a linear transformation is upper semicontinuous as the transformation varies, this will also be true for the general cubic surface containing a line. By Corollary 8.2, every cubic surface in $\mathbb{P}^3$ contains lines.

One special case of Proposition 8.24 shows that a smooth hypersurface cannot contain a plane of more than about half its dimension:

Corollary 8.26. *Let $X \subset \mathbb{P}^n$ be a hypersurface of degree $d > 1$. If $L \subset X$ is a k-plane on X, and X is smooth along L, then*

$$k \leq \frac{n-1}{2}.$$

For example, there are no 2-planes on a quadric surface in $\mathbb{P}^4$—even though the "expected dimension" $\varphi(4,2,2)$ is 0. This implies that all singular quadrics contain families of 2-planes of dimension $\geq 1 > 0 = \varphi(4,2,2)$—of course it is easy to see this directly.

Proof. If $k > (n-1)/2$ then $k+1 > n-k$, so $n-k$ forms on P^k of strictly positive degree must have a common zero, and we can apply Part (a) of Proposition 8.24. $\square$

Remark. Corollary 8.26 is a special case of a Corollary of the Lefschetz hyperplane theorem (see Appendix **??**): if $X \subset \mathbb{P}^n_{\mathbb{C}}$ is smooth and $Y \subset X$ is any subvariety of dimension $k > (n-1)/2$, then

$$\deg(X) | \deg(Y).$$

****we need a reference for the char p case.****

In the case of planes of the minimum dimension allowed by Corollary 8.26, Proposition 8.24 gives us particularly sharp information: Note that this applies, in particular, to lines on surfaces in $\mathbb{P}^3$, and thus generalizes Corollary 8.17.

Corollary 8.27. *Let $X \subset \mathbb{P}^n$ be a hypersurface of degree $d \geq 3$ containing a k-plane L with $k = (n-1)/2$. If X is smooth along L then $[L]$ is an isolated smooth point of the Fano scheme $F_k(X)$. If $n = d = 3$—that is, $X \subset \mathbb{P}^3$ is a cubic surface—then the converse is also true.*

If, in the setting of Proposition 8.24 we take an example where the g_i are general forms of degree $d-1$ in $k+1$ variables vanishing at some point of $\mathbb{P}^k$ with $d = 2, k > 1$ or $d > 3, k \geq 1$ then the g_i will have no linear syzygies, so the corresponding $L \subset X$ will be a smooth point on the Fano scheme, though X is singular at a point of L. Thus the "converse" part of the Corollary cannot be extended to these cases.

Proof of Corollary 8.27. If X is smooth along L, then, by Proposition 8.24, the $k+1$ forms g_i of degree $d-1$ have no common zeros. It follows that they are a regular sequence, so all the relations among them are also of degree $d-1 \geq 2$, so again by Proposition 8.24 $[L]$ is a smooth point of $F_k(X)$.

In the case of a cubic surface, g_2, g_3 are quadratic forms in 2 variables. If they have a zero in common then they have a linear common factor, so they have a linear syzygy. □

Despite the non-existence of 2-planes on quadric hypersurfaces $X \subset \mathbb{P}^4$ and other examples coming from Corollary 8.26, the situation becomes uniform for hypersurfaces of degree $d \geq 3$. The proof for the general case is quite complicated, and we only sketch it. In the next section we give a complete and independent treatment for the case of lines.

Theorem 8.28. *Set $D = (k+1)(n-k) - \binom{k+d}{k}$.*

(a) *If $k = 1$ or $d \geq 3$ and $D \geq 0$ then every hypersurface of degree d contains k-planes, and the general hypersurface X of degree d in $\mathbb{P}^n$ has $\dim F_k(X) = D$.*

(b) *If $D \leq 0$ and X is a general hypersurface containing a given k-plane L, then L is an isolated smooth point of $F_k(X)$.*

See Exercise 8.59 for an example that can be worked out directly.

Proof. (a): The first part follows from the second using Corollary 8.2. For the second part we use Proposition 8.24. We must show that, under the given hypotheses, a general $(n-k)$-dimensional vector space of forms of degree forms of degree $d-1$ generates an ideal containing all the forms of degree d.

On the other hand, for part (b) we must show that a general $(n-k)$-dimensional vector space of forms of degree forms of degree $d-1$ generates an ideal without linear syzygies.

These two statements together say that if $g_{k+1}, \dots g_{k+n}$ is a general collection of of $n-k$ forms of degree $d-1$ in $k+1$ variables, then the degree d component of the ideal $(g_{k+1}, \dots g_{k+n})$ has dimension equal to $\min((k+1)(n-k), \binom{k+d}{k})$. This is a special case of the formula for the maximal Hilbert function of a homogeneous ideal with generators in given degrees conjectured in Fröberg [1985]. This particular case of Fröberg's Conjecture that was proven in Hochster and Laksov [1987], Theorem 1. $\square$

8.4.4 The case of lines

The case $k=1$ of lines is special because, very much in contrast with the general situation, we can classify vector bundles on $\mathbb{P}^1$ completely: we have the

Theorem 8.29. *Any vector bundle E on $\mathbb{P}^1$ is a direct sum of line bundles; that is,*

$$E = \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(e_i)$$

for some integers $e_1, \dots, e_r$.

Note that the analogous statement is clearly false in general for bundles on projective space $\mathbb{P}^n$ of dimension $n \geq 2$ (see for example Exercise 8.53).

Proof. The proof uses the Riemann-Roch theorem for vector bundles on curves. Riemann-Roch theorems in general will be discussed in Chapter 16, where we will also discuss more aspects of the behavior of vector bundles on $\mathbb{P}^1$. The reader may wish to glance ahead or, since we will not make logical use of Theorem 8.29, defer reading this proof until then.

That said, we start with a basic observation: an exact sequence of vector bundles

$$0 \longrightarrow E \longrightarrow F \xrightarrow{\alpha} G \longrightarrow 0$$

on any variety X splits if and only if there exists a map $\beta : G \to F$ such that $\alpha \circ \beta = Id_G$. This will be the case whenever the map $\mathrm{Hom}(G, F) \to \mathrm{Hom}(G, G)$ given by composition with α is surjective on global sections; from the exactness of the sequence

$$0 \to \mathrm{Hom}(G, E) \to \mathrm{Hom}(G, F) \to \mathrm{Hom}(G, G) \to 0$$

this will in turn be the case whenever $H^1(\mathrm{Hom}(G, E)) = H^1(G^* \otimes E) = 0$.

Now suppose E is a vector bundle of rank 2 on $\mathbb{P}^1$, with first Chern class of degree d. By Riemann-Roch, we have

$$h^0(E) \geq d+2;$$

from this we may deduce the existence of a nonzero global section σ of E vanishing at $m \geq d/2$ points of $\mathbb{P}^1$, or equivalently of an inclusion of vector bundles $\mathcal{O}_{\mathbb{P}^1}(m) \hookrightarrow E$ with $m \geq d/2$. We thus have an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(m) \to E \to \mathcal{O}_{\mathbb{P}^1}(d-m) \to 0,$$

and since $H^1(\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(d-m), \mathcal{O}_{\mathbb{P}^1}(m))) = H^1(\mathcal{O}_{\mathbb{P}^1}(2m-d)) = 0$ in this case, we conclude that $E \cong \mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(d-m)$.

The case of a bundle E of general rank r follows by induction: if we let $L \subset E$ be a sub-line bundle of maximal degree m, we get a sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(m) \longrightarrow E \xrightarrow{\alpha} F \longrightarrow 0$$

with F by induction a direct sum of line bundles $L_i \cong \mathcal{O}_{\mathbb{P}^1}(e_i)$. Moreover, $e_i \leq m$ for all i: if $e_i > m$ for some i, then $\alpha^{-1}(L_i)$ would be a bundle of rank 2 and degree $> 2m$; by the rank 2 case, this would contradict the maximality of m. Thus this sequence splits, and we're done. $\square$

We should remark in passing that vector bundles on higher-dimensional projective spaces $\mathbb{P}^n$ remain mysterious, even for $n = 2$, and open problems regarding them abound. To mention just one, it is unknown whether there exist vector bundles of rank 2 on $\mathbb{P}^n$ (other than direct sums of line bundles) when $n \geq 6$. Interestingly, though, Theorem 8.29 provides a tool for the study of bundles on higher-dimensional projective spaces, via the notion of *jumping lines*, which we'll discuss in Section 16.5

To return to our discussion of linear spaces on hypersurfaces, suppose that $X \subset \mathbb{P}^n$ is a hypersurface of degree d and $L \subset X$ a line. We choose coordinates so that L is defined by $x_2 = \cdots x_n = 0$. As before, we write the equation of X in the form

$$\sum_{i=2}^{n} x_i g_i(x_0, x_1) + h,$$

with $h \in (x_0, \ldots, x_n)^2$, and we let α be the map $(g_2, \ldots, g_n) : \mathcal{O}_L^{n-1} \to \mathcal{O}_L(d)$. In this situation, the expected dimension of the Fano scheme $F_1(X)$ is $D := 2n - 3 - d$. We will make use of this notation throughout this subsection.

We can say exactly what normal bundles of lines in hypersurfaces are possible. Since any vector bundle on $L \cong \mathbb{P}^1$ is a direct sum of line bundles, we may write $N_{L/X} \cong \oplus_1^{n-1} \mathcal{O}_{\mathbb{P}^1}(e_i)$.

Proposition 8.30. *Suppose that $n \geq 3$ and $d \geq 1$. There exists a smooth hypersurface X in $\mathbb{P}^n$ of degree d, and a line $\mathbb{P}^1 \cong L \subset X$ such that $N_{L/X} \cong \oplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^n}(e_i)$ if and only if*

$$e_i \leq 1 \textit{ for all } i \quad \textit{and} \quad \sum_{i=1}^{n-2} e_i = n - 1 - d.$$

Proof. If the normal bundle is $N_{L/X} \cong \oplus \mathcal{O}_{\mathbb{P}^1}(e_i)$ then from the fact that there is an inclusion $N_{L/X} \to N_{L/\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^1}^{n-1}(1)$ it follows that $e_i \leq 1$ for all i. Computing Chern classes from the exact sequence of sheaves on $\mathbb{P}^1$

$$0 \to \bigoplus_{i=1}^{n-2} \mathcal{O}_{\mathbb{P}^1}(e_i) \to \mathcal{O}_{\mathbb{P}^1}^{n-1}(1) \to \mathcal{O}_{\mathbb{P}^1}(d) \to 0$$

we get $\sum_i e_i = n - 1 - d$.

Conversely, suppose the e_i satisfy the given conditions. To simplify the notation, let $F = \oplus_{i=1}^{n-2} \mathcal{O}_{\mathbb{P}^1}(e_i)$ and let $G = \mathcal{O}_{\mathbb{P}^1}^{n-1}(1)$. Let $\beta : F \to G$ be any map, and let α be the map $G \to \mathcal{O}_{\mathbb{P}^1}(d)$ given by the matrix of $(n-2) \times (n-2)$ minors of the matrix of β, with appropriate signs. [1] The composition $\alpha\beta$ is zero because the i-th entry of the composite matrix is the Cauchy expansion of the determinant of a matrix obtained from β by repeating the i-th column.

If we take β of the form

$$\beta = \begin{pmatrix} x_0^{1-e_1} & 0 & 0 & \cdots & 0 & 0 \\ x_1^{1-e_2} & x_0^{1-e_2} & 0 & \cdots & 0 & 0 \\ 0 & x_1^{1-e_3} & x_0^{1-e_3} & \cdots & 0 & 0 \\ 0 & 0 & x_0^{1-e_4} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & x_1^{1-e_{n-3}} & x_0^{1-e_{n-3}} \\ 0 & \cdots & 0 & 0 & 0 & x_1^{1-e_{n-2}} \end{pmatrix}$$

****this matrix needs cosmetic work to make the diagonals line up!**** then the top $(n-2) \times (n-2)$ minor will be x_0^{d-1} and the bottom $(n-2) \times (n-2)$ minor will be x_1^{d-1}. This shows that the map α will be an epimorphism of sheaves, so that the general hyperplane X containing L

---

[1] More formally and invariantly, α is the composite map

$$G \cong \mathcal{O}_{\mathbb{P}^1}(n-1) \otimes \wedge^{n-2} G^* \xrightarrow{\mathcal{O}_{\mathbb{P}^1}(n-1) \otimes \wedge^{n-2}\beta^*} \mathcal{O}_{\mathbb{P}^1}(n-1) \otimes \wedge^{n-2} F^* \cong \mathcal{O}_{\mathbb{P}^1}(d).$$

where we have used an identification of G with $\mathcal{O}_{\mathbb{P}^1}(n-1) \otimes \wedge^{n-2} G^*$ corresponding to a global section of $\mathcal{O}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-n+1) \otimes \wedge^{n-1} G$.

induced map $\alpha : N_L \to N_X$ will be smooth. By Eisenbud [1995] Theorem 20.9, the sequence

$$0 \longrightarrow F \xrightarrow{\beta} G \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^1}(d) \longrightarrow 0$$

is exact, so $N_{L/X} \cong F$. □

Corollary 8.31. *If $d \le 2n-3$, then there exists a pair (X, L) with $X \subset \mathbb{P}^n$ a smooth hypersurface of degree d and $L \subset X$ a line such that $F_1(X)$ is smooth of dimension $2n-3-d$ in a neighborhood of $[L]$.*

Proof. Using Proposition 8.30: we observe that, if $d \le 2n-3$, we can choose all the $e_i \ge -1$. With this choice $\dim H^0(\mathcal{O}_{\mathbb{P}^1}(e_i)) = e_i + 1$ for all i and hence $\dim H^0(N_{L/X}) = 2n-3-d$. Since $F_1(X)$ has dimension at least $2n-3-d$ everywhere, the result follows. □

Corollary 8.32. *If $d \le 2n-3$, then every hypersurface of degree d in $\mathbb{P}^n$ contains a line.*

Proof. The univeral Fano scheme $\Phi(n, d, 1)$ is irreducible, and has dimension $N - d + 2n - 3$. Moreover, Corollary 8.31 asserts that at some point $(X, L) \in \Phi$ the fiber dimension of the projection $\Phi(n, d, 1) \to \mathbb{P}^N$ is $2n-d-3$. It follows that this projection is surjective. □

We have seen above that the Fano scheme of any smooth cubic surface in $\mathbb{P}^3$ is reduced and of the correct dimension. We can now say something about the higher-dimensional case as well:

Corollary 8.33. *The Fano scheme of lines on any smooth hypersurface of degree $d \le 3$ is smooth and of dimension $2n-3-d$. But if $n \ge 4$ and $d \ge 4$ then there exist smooth hypersurfaces of degree d in $\mathbb{P}^n$ whose Fano schemes are singular or of dimension $> 2n-3-d$.*

Proof. We follow the notation of Proposition 8.30. If $d \le 3$ then for any $e_1, \dots, e_{n-2}$ allowed by the conditions of the Proposition we will have all $e_i \ge -1$, and thus $h^0(N_{L/X}) = \chi(N_{L/X}) = 2n-3$, proving that the Fano scheme is smooth and of expected dimension at L.

On the other hand, if $n \ge 4$ and $d \ge 4$ then we can take $e_1 = \cdots = e_{n-3} = 1$ and $e_{n-2} = 2 - d \le -2$. In this case $h^0(N_{L/X}) = 2n-6 > 2n-3-d$, so the Fano scheme is singular or of "too large" dimension at L. □

The first statement of Corollary 8.33 is an easy case of the conjecture of Debarre and de Jong that we will discuss further in Section 8.8.

8.5 Lines on quintic threefolds and beyond

We can now answer the first of the keynote questions of this Chapter: how many lines are contained in a general quintic threefold $X \subset \mathbb{P}^4$? More generally, we can now compute the number of distinct lines on a general hypersurface X of degree $d = 2n - 3$ in $\mathbb{P}^n$, the case in which the expected dimension of the family of lines is zero.

The set-up is the same as that for the lines on a cubic surface: the defining equation g of the hypersurface X gives a section σ_g of the bundle $\operatorname{Sym}^d S^*$ on the Grassmannian $\mathbb{G}(1, n)$; the zero locus of σ_g is then the Fano scheme $F_1(X)$ of lines on X; and (assuming $F_1(X)$ has the expected dimension 0) the degree D of this scheme is the degree of the top Chern class $c_{d+1}(\operatorname{Sym}^d S^*) \in A^{d+1}(\mathbb{G}(1, n))$. If we can show in addition that $H^0(N_{L/X}) = 0$ for each line $L \subset X$, then it follows as in the previous section that the Fano scheme is zero-dimensional and reduced, so the actual number of distinct lines on X is exactly D.

To calculate the Chern class we could use the splitting principle. The computation is reasonable for $n = 4, d = 5$, the case of the quintic threefold, but becomes successively more complicated for larger n and d. Schubert2 (in Macaulay2) instead deduces it from a Gröbner basis for the Chow ring. Here is a Schubert2 script that computes the numbers for $n = 3, \ldots, 20$, followed by its output:

```
loadPackage "Schubert2"
grassmannian = (m,n) -> flagBundle({m+1, n-m})
time for n from 3 to 10 do(
     G=grassmannian(1,n);
     (S,Q) = G.Bundles;
     d = 2*n-3;
     print integral chern symmetricPower(d, dual S))
```

```
27
2875
698005
305093061
210480374951
210776836330775
289139638632755625
520764738758073845321
1192221463356102320754899
3381929766320534635615064019
11643962664020516264785825991165
47837786502063195088311032392578125
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231191601420598135249236900564098773215
1298451577201796592589999161795264143531439
83866260295124407255717362657730471722899221 29
6173084437050848781779832818903892339718128038 4657
51368728776479020796032943406584459797840143884179 6875
479849240965383456367278060519107076039364076181726998 5515
     -- used 119.123 seconds
```

The following result gives a geometric meaning to these numbers beyond the fact that they are degrees of certain Chern classes.

Theorem 8.34. *If $X \subset \mathbb{P}^n$ is a general hypersurface of degree $d \geq 1$, then the Fano scheme $F_1(X)$ of lines on X is reduced and has the expected dimension $2n - d - 3$.*

We now have the definitive answer to the keynote question (a)

Corollary 8.35. *A general quintic threefold $X \subset \mathbb{P}^4$ contains exactly 2875 lines. More generally, the numbers in the table above are equal to the number of distinct lines on general hypersurfaces of degrees $3, 5, \dots, 37$ and dimensions $2, 3, \dots, 19$.*

We have seen that *every* smooth cubic surface has exactly 27 distinct lines. By contrast, the hypothesis "general" in the preceding Corollary is really necessary for quintic threefolds: by Corollary 8.33, the Fano scheme of a smooth quintic threefold may be singular or positive-dimensional (we'll see in Exercises 8.68 and 8.74 that both possibilities actually occur).

The 2875 lines on a quintic threefold have played a significant role in algebraic geometry, and even show up in physics. For example, the Lefschetz Hyperplane Theorem (see Appendix **??**) implies that all $2,875$ are homologous to each other, but one can show that they are linearly independent in the group of cycles mod algebraic equivalence ****reference!****. On the other hand, the number of rational curves of degree d on a general quintic threefold, of which the $2,875$ lines are the first example, is one of the first predictions of *mirror symmetry* (see for example Cox and Katz [1999].)

Proof of Theorem 8.34. We already know that for general X of degree $d > 2n - 3$ the Fano scheme $F_1(X)$ is empty, so we henceforward assume that $d \leq 2n - 3$. We have seen in Corollary 8.31 that there exists a pair (X, L) with $X \subset \mathbb{P}^n$ a smooth hypersurface of degree d and $L \subset X$ a line such that $\dim T_L F_1(X) = 2n - 3 - d$—that is, $F_1(X)$ is smooth of the expected dimension in a neighborhood of L. We now use an incidence correspondence to deduce that for general X, an open dense set of $L \in F_1(X)$ have this property (and in particular, if $d = 2n - 3$ then X will contain a finite number of lines, every one of which will be a reduced point of $F_1(X)$).

Let $\mathbb{P}^N$ be the projective space of forms of degree d in $n+1$ variables, whose points we think of as hypersurfaces in $\mathbb{P}^n$. Consider the projection maps from the universal Fano scheme $\Phi := \Phi(n,d,k)$:

$$\begin{tikzcd}[column sep=small] & \Phi = \{(X,L) \in \mathbb{P}^N \times \mathbb{G}(1,n) \mid L \subset X\} \arrow[dl, "\varphi"'] \arrow[dr, "\gamma"] & \\ \mathbb{P}^N & & \mathbb{G}(1,n), \end{tikzcd}$$

so that the fiber of φ over the point X of $\mathbb{P}^N$ is the Fano scheme of X. As we have seen in Proposition 8.1 Φ is smooth and irreducible of dimension $N+2n-3-d$. It follows that the fiber of φ through any point of Φ has dimension $\geq 2n-3-d$.

The set of points of Φ where the fiber dimension is equal to $2n-3-d$ is open; and within that the set of points U where the fiber is smooth is also open. Corollary 8.31 shows that this open set is nonempty; given this, Lemma 8.36 with $B = \Phi \setminus U$ shows that if X is a generic hypersurface of degree d, then any component of $F_1(X)$ is generically smooth of dimension D. Since $F_1(X)$ is defined by the vanishing of a section of a bundle of rank $d+1$, it is locally a complete intersection. Thus $F_1(X)$ cannot have embedded components, and the fact that it is generically smooth implies that it is reduced. $\square$

Lemma 8.36. *Suppose that $\varphi : X \to P$ is a dominant morphism of (not necessarily projective) varieties. If $B \subset X$ is a closed subset that contains a component of each fiber of φ over an open subset of P, then $B = X$. In particular, the generic fiber is equidimensional.*

Proof. We may suppose for simplicity that $\varphi(X) = P$. Let $X' \subset X$ be the subset of points x such that the dimension of $\varphi^{-1}(\varphi(x))$ is minimal. By the semicontinuity of fiber dimension X' is open, and for $x \in X'$ we have $\dim X = \dim P + \dim \varphi^{-1}(\varphi(x))$. Since B contains a component of the generic fiber, a similar computation shows that $\dim B \geq \dim X$, proving that $B = X$. $\square$

8.6 The universal Fano scheme and the geometry of families of lines

In keynote question (c) we asked: what is the degree of the surface S in $\mathbb{P}^3$ swept out by the lines on a cubic surface as the cubic surface moves in a general pencil? What is the genus of the curve $C \subset \mathbb{G}(1,3)$ consisting of the points corresponding to lines on the various elements of the pencil of cubic surfaces? We can answer such questions by giving a "global" view of the universal Fano scheme as the zero locus of a section of a vector bundle, just as we have done for Fano schemes of individual hypersurfaces.

We will compute the degree of S as the number of times S intersects a general line. The task of computing this number is made easier by the fact that a general point of the surface lies on only one of the lines in question (Reason: a general point that lay on two lines would have to lie on lines from different surfaces in the pencil, and thus would lie in the base locus of the pencil, contradicting the assumption that it was a general point). Thus the degree of the surface is the same as the degree of the curve $C \subset \mathbb{G}(1,3)$ in the Plücker embedding.

The incidence correspondence

$$\Phi = \Phi(n,d,1) = \{(X,L) \in \mathbb{P}^N \times \mathbb{G}(1,n) \mid L \subset X\}.$$

which we may call the *universal* or *relative* Fano scheme of lines on cubic surfaces, was introduced in Section 8.1. We can learn about its global geometry by realizing it as the zero locus of a section of a bundle, just as with the case of the Fano scheme of a given hypersurface.

We have seen that the maps of vector spaces

$$\left\{\begin{matrix}\text{homogeneous cubic}\\ \text{polynomials on } \mathbb{P}^3\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{homogeneous cubic}\\ \text{polynomials on } L\end{matrix}\right\}$$

for different $L \in \mathbb{G}(1,3)$ fit together to form a bundle map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \otimes \mathcal{O}_{\mathbb{G}(1,3)} \to \operatorname{Sym}^3 S^*$$

on the Grassmannian $\mathbb{G}(1,3)$. Also, if $V = H^0(\mathcal{O}_{\mathbb{P}^3}(3))$ is the 20-dimensional vector space of all cubic polynomials, then the inclusions

$$\langle F \rangle \hookrightarrow V$$

fit together to form a map of vector bundles on $\mathbb{P}V \cong \mathbb{P}^{19}$

$$T = \mathcal{O}_{\mathbb{P}^{19}}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}^{19}}$$

where T is the universal subbundle on $\mathbb{P}^{19}$.

We will put these two constructions together to understand not only $\Phi(n,d,1)$ but its restriction to any linear space of forms $M \subset \mathbb{P}^N$. We denote the restriction of the universal Fano scheme to M by $\Phi(n,d,1)|_M$.

Theorem 8.37. *The universal Fano scheme $\Phi(n,d,1)|_M$ of lines on a general m-dimensional linear family $M = \mathbb{P}^m$ of hypersurfaces of degree d in $\mathbb{P}^n$ is reduced and of codimension $d+1$ in the $(2n-2+m)$-dimensional space $\mathbb{P}^m \times \mathbb{G}(2,n)$. It is the zero locus of a section of the rank $d+1$ vector bundle $E = \pi_2^* Sym_d S^* \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^m}(1)$ on that space, so its class is $c_{d+1}(E)$.* $\square$

Proof. We will first show that the class of $\Phi(n,d,1)|_M$ is the zero locus of the given section, and for this it suffices to treat the case $M = \mathbb{P}^N$, the space of all forms of degree d.

Consider the product of $\mathbb{P}(\operatorname{Sym}^d V^*)$ and the Grassmannian $\mathbb{G}(1,n)$ and its projections,

$$\mathbb{P}(\operatorname{Sym}^d V^*) \xleftarrow{\pi_1} \mathbb{P}(\operatorname{Sym}^d V^*) \times \mathbb{G}(1,n) \xrightarrow{\pi_2} \mathbb{G}(1,n),$$

On the product, we have both a universal rank 1 subbundle $\mathcal{O}_{\mathbb{P}(\operatorname{Sym}^d V^*)}(-1)$ of the trivial bundle $\pi_1^* \operatorname{Sym}^d V^*$, and a universal k-subbundle S of the trivial bundle $\pi_2^* V$. We thus have maps

$$\pi_1^* \mathcal{O}_{\mathbb{P}(\operatorname{Sym}^d V^*)}(-1) \longrightarrow \pi_1^* \operatorname{Sym}^d V^* \cong \pi_2^* \operatorname{Sym}^d V^* \longrightarrow \pi_2^* \operatorname{Sym}^d S^*.$$

Restricted to the fiber over the point of $\mathbb{P}(\operatorname{Sym}^d V^*)$ corresponding to f the composite map takes a generator of $\pi_1^* \mathcal{O}_{\mathbb{P}(\operatorname{Sym}^d V^*)}(-1)|_{\langle f \rangle}$ to σ_f. Thus the zero locus of the composite map is the incidence correspondence $\Phi(n,d,1)$. Tensoring with $\pi_1^* \mathcal{O}_{\mathbb{P}(\operatorname{Sym}^d V^*)}(1)$ we get a map

$$\mathcal{O}_{\mathbb{P}(\operatorname{Sym}^d V^*)} \to \pi_2^* \operatorname{Sym}^d S^* \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^n}(1).$$

Let σ be the global section of the bundle

$$E := \pi_2^* \operatorname{Sym}^d S^* \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^n}(1)$$

that is the image of $1 \in \mathcal{O}_{\mathbb{P}(\operatorname{Sym}^d V^*)}$. The zero locus of the composite map is the same as the zero locus of σ. Moreover, if we restrict to an open subset of the Grassmannian over which the universal subbundle S is trivial, then the vanishing of σ is given by the local equations we originally used to define the scheme structure on Φ.

We must still show that $\Phi(n,d,1)|_M$ is reduced and of codimension $\operatorname{rank} E = d+1$ in $M \times G(1,n)$. For the case when M is a point, this is Theorem 8.34. On the other hand, the projection $\Phi(n,d,1)|_M \to M$ is surjective, so the fiber of $\Phi(n,d,1)$ over a general point of M has codimension at least $\dim M$ in $\Phi(n,d,1)|_M$. Moreover, this fiber is cut out by the pullbacks of $\dim M$ linear forms on M. The fact that $\Phi(n,d,1)|_M$ has the expected dimension now follows from the Generalized principal ideal theorem. The generic smoothness of the fiber also follows from a very general result, given in the next Proposition, applied with R equal to the local ring of $\Phi(n,d,1)_M$ at a general point. $\square$

Proposition 8.38. *Suppose that R is a local ring with maximal ideal $\mathfrak{m}$, and that $x_1, \dots, x_c \in \mathfrak{m}$. If $\overline{R} := R/(x_1, \dots x_c)$ is a regular local ring, and $\dim \overline{R} \le \dim R - c$, then in fact equality holds and R is regular.*

Proof. The equality of dimensions follows from the Principal Ideal Theorem. A ring is regular if and only if the number of generators of its maximal ideal is equal to the dimension of the ring. By the Principal Ideal Theorem, the number of generators is always at least this large. Thus the minimal number of generators of the maximal ideal of $\overline{R}$ is $\dim \overline{R}$, so the minimal number of generators of $\mathfrak{m}$ is at most $c + \dim \overline{R} = \dim R$, and we are done. $\square$

Theorem 8.37 immediately gives the answers to keynote questions (b) and (c).

Corollary 8.39. *The class of the universal Fano scheme $\Phi(3,3,1)$ of lines on cubic surfaces in $\mathbb{P}^3$ is*

$$\begin{aligned}[\Phi(3,3,1)] &= c_4\big(\pi_2^* Sym_3 S^* \otimes \pi_1^*\mathcal{O}_{\mathbb{P}^1}(1)\big)\\ &= 27\sigma_{2,2} + 42\sigma_{2,1}\zeta + (11\sigma_2 + 21\sigma_{1,1})\zeta^2 + 6\sigma_1\zeta^3 + \zeta^4.\end{aligned}$$

while the class of the universal Fano scheme $\Phi(4,3)$ of lines on quartic surfaces in $\mathbb{P}^3$ is

$$\begin{aligned}[\Phi(3,4,1)] &= c_5\big(\pi_2^* Sym_4 S^* \otimes \pi_1^*\mathcal{O}_{\mathbb{P}^1}(1)\big)\\ &= 320\sigma_{2,2}\zeta + 220\sigma_{2,1}\zeta^2 + (30\sigma_2 + 55\sigma_{1,1})\zeta^3 + 10\sigma_1\zeta^4 + \zeta^5.\end{aligned}$$

If C is the curve of lines on a general pencil of cubic surfaces then the degree of C is 42 and the genus of C is 70. The number of quartic surfaces that contain a line in a general pencil of quartic surfaces is 320.

Restricting to a point $t \in \mathbb{P}^1$, we see again that a particular cubic surface X_t will contain

$$[\Phi(3,3,1)] \cdot \zeta^{19} = 27$$

lines.

Proof of Corollary 8.39. The identifications of $[\Phi(3,3,1)]$ and $[\Phi(3,4,1)]$ with the given Chern classes is part of Theorem 8.37.

For the explicit computations of the Chern classes one can use the splitting principle or appeal to Schubert2. Here is the computation, via the splitting principle, for the case of $\Phi(3,3,1)$, the 4-th Chern class of the bundle E on $\mathbb{P}^{19} \times \mathbb{G}(1,3)$. We will use use the symbol ζ for the pullback to $\mathbb{P}^{19} \times \mathbb{G}(1,3)$ of the hyperplane class on $\mathbb{P}^{19}$; and we'll use the symbols $\sigma_{i,j}$ for the pullbacks to $\mathbb{P}^{19} \times \mathbb{G}(1,3)$ of the corresponding classes in $A(\mathbb{G}(1,3))$.

Formally factoring the Chern class of $\nu^* S^*$ as

$$c(\nu^* S^*) = 1 + \sigma_1 + \sigma_{1,1} = (1+\alpha)(1+\beta),$$

we can write

$$\begin{aligned}&c\big(\mu^*\mathcal{O}_{\mathbb{P}^{19}}(1) \otimes \nu^* \operatorname{Sym}^3 S^*\big) =\\ &\qquad (1+3\alpha+\zeta)(1+2\alpha+\beta+\zeta)(1+\alpha+2\beta+\zeta)(1+3\beta+\zeta)\end{aligned}$$

and in particular the top Chern class is given by

$$c_4(\mu^*\mathcal{O}_{\mathbb{P}^{19}}(1) \otimes \nu^* \operatorname{Sym}^3 S^*) = \\ (3\alpha+\zeta)(2\alpha+\beta+\zeta)(\alpha+2\beta+\zeta)(3\beta+\zeta) \in A^4(\mathbb{P}^1 \times \mathbb{G}(1,3)).$$

Evaluating, we first have

$$(3\alpha+\zeta)(3\beta+\zeta) = 9\sigma_{1,1} + 3\sigma_1\zeta + \zeta^2;$$

and then

$$(2\alpha+\beta+\zeta)(\alpha+2\beta+\zeta) = 2\sigma_1^2 + \sigma_{1,1} + 3\sigma_1\zeta + \zeta^2.$$

Multiplying out, we have

$$[\Phi] = 27\sigma_{2,2} + 42\sigma_{2,1}\zeta + (11\sigma_2 + 21\sigma_{1,1})\zeta^2 + 6\sigma_1\zeta^3 + \zeta^4.$$

Here is the corresponding Schubert2 code:

```
n=3
d=3
m=19

P = flagBundle({1,m}, VariableNames=>{z,q1})
(Z,Q1)=P.Bundles
V = abstractSheaf(P,Rank =>n+1)
G = flagBundle({2,n-1},V,VariableNames=>{s,q})
(S,Q) = G.Bundles
p = G.StructureMap
ZG = p^*(dual Z)
chern_4 (ZG**symmetricPower_d dual S)
```

Replacing the line "$d = 3$" with "$d = 4$" we get the corresponding result for $\Phi(3,4)$.

From the computation of $[\Phi(3,3,1)]$ we see that the number of lines on members of our pencil of cubics meeting a given line is

$$[\Phi] \cdot \sigma_1 \cdot \zeta^{18} = 42$$

from which we deduce that the degree of C, which is equal to the degree of the surface swept out by the lines on our pencil of cubics, is 42. For the genus $g(C)$ of C we use Part (c) of Proposition 8.15 to conclude that the normal bundle of C is the bundle $E|_C$, where E is the restriction to $\mathbb{P}^1 \times \mathbb{G}(1,n)$ of the bundle $\pi_2^* Sym_3 S^* \otimes \pi_1^*\mathcal{O}_{\mathbb{P}^1}(1)$, whose section defines $\Phi(3,3,1)_{\mathbb{P}^1}$, as in Corollary 8.39. From the exact sequence

$$0 \to T_C \to T_{\mathbb{P}^1 \times \mathbb{G}(1,n)}|_C \to N_{C/\mathbb{P}^1 \times \mathbb{G}(1,n)} \to 0$$

we deduce that the degree of T_C, which is $2-2g(C)$, is

$$\begin{aligned}\deg T_C = \deg c_1(T_C) &= \deg\bigl([C]c_1(T_{\mathbb{P}^1\times\mathbb{G}(1,n)})\bigr) - \deg c_1(N_{C/\mathbb{P}^1\times\mathbb{G}(1,n)})\\ &= c_4(E)(4\sigma_1+2\zeta) - c_4(E)c_1(E),\end{aligned}$$

where we have used the computation $c_1(T_{\mathbb{G}(1,n)}) = (n+1)\sigma_1$ from Corollary 7.15.****in Ch 5****. We can compute $c_1(E)$ by the splitting principle or by calling

```
chern_1 (ZG**symmetricPower_d dual S)
```

and we get $c_1(E) = 6\sigma_1 + 4\zeta$. Using the fact that ζ^2 restricts to zero on the preimage of a line in $\mathbb{P}^19$, this gives

$$\begin{aligned}2-2g(C) &= \deg(27\sigma_{2,2}+42\sigma_{2,1}\zeta)\bigl(4\sigma_1+2\zeta-6\sigma_1-4\zeta\bigr)\\ &= \deg(-138\sigma_{2,2}\zeta) = -138,\end{aligned}$$

whence $g = 70$. Another view of this computation is suggested in Exercise 8.57

Finally, the number of quartic surfaces that contain a line in a general pencil of quartic surfaces is the number of lines that lie on some quartic surface in the pencil, that is, the degree of $\Phi(3,4,1)\cap\mathbb{P}^1$. (If a line lay on more than one element of the pencil, it would be a component of the base locus—but the pencil being general, the base locus is smooth and connected. ****we need to refer to the argument that a surface in the pencil can't contain more than one line!****) Writing σ again for the section of $\pi_2^*\operatorname{Sym}^4 S^*\otimes\pi_1^*\mathcal{O}_{\mathbb{P}^1}(1)$ defined above, this is

$$\deg\zeta^{18}c_5\bigl(\pi_2^*\operatorname{Sym}^4 S^*\otimes\pi_1^*\mathcal{O}_{\mathbb{P}^1}(1)\bigr).$$

By the computation of $[\Phi(3,4,1)]$, this is 320. □

The coefficients of higher powers of ζ in the class of $\Phi(3,3,1)$ computed above have to do with the geometry of larger linear systems of cubics: for example, we'll see how to answer questions about lines on a net of cubics in Exercise 8.55.

8.6.1 Lines on the quartic surfaces in a pencil

Here is a slightly different approach to keynote question (b). Given that the set of quartic surfaces that contain some line is a hypersurface Γ in the projective space $\mathbb{P}^{34}$ of quartic surfaces, we are asking for the degree of Γ.

To find that number we look again at the bundle E on the Grassmannian $\mathbb{G}(1,3)$ whose fiber over a point $L\in\mathbb{G}(1,3)$ is the vector space

$$E_L = H^0(\mathcal{O}_L(4))$$

—that is, the fourth symmetric power $\mathrm{Sym}^4 S^*$ of the dual of the universal subbundle on $\mathbb{G}(1,3)$. As before, the polynomials f and g generating the pencil define sections σ_f and σ_g of the bundle E. The locus of lines $L \subset \mathbb{P}^3$ that lie on some element of the pencil is the locus where the values of the sections σ_f and σ_g are dependent so the degree of this locus is the fourth Chern class $c_4(E) \in A^4(\mathbb{G}(1,3)) \cong \mathbb{Z}$. As before this can be computed either with the splitting principle or with Schubert2, and one finds again the number 320.

We'll see another way of calculating the genus of the curve Φ in the following Chapter (after we've determined the number of singular cubic surfaces in a general pencil), by expressing Φ as a 27-sheeted cover of $\mathbb{P}^1$ and using Hurwitz' Theorem.

8.7 Lines on a cubic with a double point

Identifying the Fano scheme $F_1(X)$ as the Hilbert scheme of lines on X has allowed us to give a necessary and sufficient condition for its smoothness, and to show that it is indeed smooth in certain cases. But there are aspects of geometry that we can't get at in this way, such as the multiplicity of $F_1(X)$ at a point L where it is not smooth. We might want to know, for example, if we can find a smooth hypersurface $X \subset \mathbb{P}^n$ of degree $2n-3$ whose Fano scheme of lines includes a point of multiplicity exactly 2, as in Harris [1979]; or we might ask, if X has an ordinary double point, how does this affect the number of lines it will contain? To answer such questions we must go back to the local equations of $F_1(X)$ introduced (in more generality) at the beginning of this Chapter.

We will describe the lines on a cubic surface with one ordinary double point. Other examples can be found in Exercise **??**, where we'll consider the case of cubic surfaces with more than one double point; in Exercise 8.68, where we'll show that it's possible to find a smooth quintic hypersurface $X \subset \mathbb{P}^4$ whose Fano scheme contains an isolated double point, and in Exercise 8.71, where we'll look at a case of lines on higher-dimensional singular hypersurfaces.

********Need to recall a bit of the local equations from 1.1.1 and the tan space stuff from 1.3.3******** To this end we will adapt the notation of Section 8.1.1 to the case of cubic surfaces. We work in an open neighborhood $U \subset \mathbb{G}(1,3)$ of the line

$$L : x_2 = x_3 = 0,$$

where U consists of the lines not meeting the line $x_0 = x_1 = 0$. Any line in U can be written uniquely as the row space of a matrix of the form

$$A = \begin{pmatrix} 1 & 0 & a_2 & a_3 \\ 0 & 1 & b_2 & b_3. \end{pmatrix}$$

(so $U \cong \mathbb{A}^4$, with coordinates a_2, a_3, b_2, b_3). Such a line has the parametrization

$$\mathbb{P}^1 \ni (s_0, s_1) \to (s_0, s_1)A = (s_0, s_1, a_2s_0 + b_2s_1, a_3s_0 + b_3s_1) \in \mathbb{P}^3.$$

Now let $X \subset \mathbb{P}^3$ be a cubic surface containing L and suppose that the point $p = (1, 0, 0, 0) \in L$ an ordinary double point of X; that is, the tangent cone to X at p is the cone over a smooth conic curve. We assume that X has no other singularities along L.

We may suppose that the tangent cone to X at p is given by the equation $x_1x_3 + x_2^2 = 0$. With these choices, the defining equation $g(x)$ of X can be written in the form

$$g(x) = x_0x_1x_3 + x_0x_2^2 + \alpha x_1^2x_2 + \beta x_1^2x_3 + \gamma x_1x_2^2 + \delta x_1x_2x_3 + \epsilon x_1x_3^2 + k,$$

where $k \in (x_2, x_3)^3$. The condition that X be smooth along L except at p says that $\alpha \neq 0$; otherwise the coefficients $\alpha, \dots, \epsilon$ are arbitrary.

As we saw in Section 8.4.3, the normal bundle $N_{L/X}$ can be computed from the short exact sequence

$$0 \longrightarrow N_{L/X} \longrightarrow \mathcal{O}_L^2(1) \xrightarrow{(g_2\ g_3)} \mathcal{O}_L(3) \longrightarrow 0.$$

where the g_3, g_3 are the coefficients of x_2, x_3 in the part of g that is not contained in (x_2, x_3); that is, $g_2 = \alpha x_1^2$ and $g_3 = x_0x_1 + \beta x_1^2$.

Since the polynomial ring in s, t has unique factorization, the syzygies between these two forms are generated by the linear syzygy $\alpha x_1 g_3 - (x_0 + \beta x_1)g_2 = 0$, so $N_{L/X} \cong \mathcal{O}_L$, and the tangent space to the Fano scheme is given by $T_{[L]}F_1(X) = H^0N_{L/X}$, which is 1-dimensional. In particular, the Fano scheme is *not* "smooth of the expected dimension" at $[L]$.

We can now write down the local equations of $F_1(X)$ near L: if we substitute the 4 coordinates from the parametrization of a line in U into g we get

$$g(s, t, a_2s + b_2t, a_3s + b_3t) = c_0s^3 + c_1s^2t + c_2st^2 + c_3t^3,$$

where the c_i are the polynomial in the $a_{i,j}$ that define the Fano scheme withing U. Writing this out, we find that

$$\begin{aligned}
c_0 &= a_2^2 \\
c_1 &= a_3 + 2a_2b_2 + \gamma a_2^2 + \delta a_2a_3 + \epsilon a_3^2 \\
c_2 &= b_3 + \alpha a_2 + \beta a_3 + b_2^2 + 2\gamma a_2b_2 + \delta(a_2b_3 + a_3b_2) + 2\epsilon a_3b_3 \\
c_3 &= \alpha b_2 + \beta b_3 + \gamma b_2^2 + \delta b_2b_3 + \epsilon b_3^2
\end{aligned}$$

Examining these equations, we see that c_1, c_2 and c_3 have independent differentials at the origin $a_2 = a_3 = b_2 = b_3 = 0$; thus in a neighborhood of the origin the zero locus of these three is a smooth curve. Moreover, the tangent line to this curve is not contained in the plane $a_2 = 0$, so that $c_0 = a_2^2$ vanishes to order exactly 2 on this curve. Thus the component of $F_1(X)$ supported at L is 0-dimensional, and is isomorphic to Spec $K[\epsilon]/(\epsilon^2)$. In particular, it has multiplicity 2.

Having come this far, we can answer the question: if $X \subset \mathbb{P}^3$ is a cubic surface with one ordinary double point p, and X is otherwise smooth, how many lines will X contain? We have seen that the lines $L \subset X$ passing through p count with multiplicity 2, and those not passing through p with multiplicity one. Since we know that the total count, with multiplicity, is 27, the only question is: how many distinct lines on X pass through p?

To answer this, take $p = (1, 0, 0, 0)$ as above and expand the defining equation $g(x)$ of X around p. Since p is a double point of X, we can write

$$g(x_0, x_1, x_2, x_3) = x_0 A(x_1, x_2, x_3) + B(x_1, x_2, x_3)$$

where A is homogeneous of degree 2 and B homogeneous of degree 3. The lines on X through p then correspond to the common zeros of A and B. Moreover, if we write a line L through p as the span $L = \overline{p\,q}$ with $q = (0, x_1, x_2, x_3)$, then by Exercise 8.63, the condition that X be smooth along $L \setminus \{p\}$ is exactly the condition that the zero loci of A and B intersect transversely at (x_1, x_2, x_3). Thus there will be exactly 6 lines on X through p. Summarizing:

Proposition 8.40. *Let $X \subset \mathbb{P}^3$ be a cubic surface with an ordinary double point p. If X is smooth away from p, it will contain exactly 21 lines: six through p and 15 not passing through p.* □

****There's something odd about this. It seems that we proved that the equations given hold on U, and thus that within U there is only one line on X through p. It's true that we have 6 open sets to cover the Grassmannian, but it seems unlikely that each contains exactly 1 line! What's up? We should add an explanation.****

(Compare this with the discussion in Chapter 4 of Griffiths and Harris [1978]. ****Joe: just what did you have in mind? Ch 4 of GH is 200 pages long. . . ****)

In Exercises 8.64-8.67 we'll take up the case of cubics with more than one singularity, arriving ultimately at the statement that *a cubic surface $X \subset \mathbb{P}^3$ can have at most four isolated singular points.*

We have used the local equations of the Fano scheme only to describe the locus of lines on a single hypersurface. A similar approach gives some information about the lines on a linear system of hypersurfaces. As a sample application, we'll see in Exercises 8.72 and 8.73 how to describe the singular locus of and tangent spaces to the locus $\Sigma \subset \mathbb{P}^{34}$ of quartic surfaces in $\mathbb{P}^3$ that contain a line.

8.8 The Debarre–de Jong Conjecture

By Theorem 8.34, general hypersurfaces $X \subset \mathbb{P}^n$ of degree d all have Fano schemes $F_1(X)$ of the "expected" dimension $D = 2n - d - 3$. On the other hand, it is easy to find smooth hypersurfaces of degree > 3 whose Fano schemes have dimension$> D$—any smooth surface of degree > 3 in $\mathbb{P}^3$ that contains a line is such an example..

However, Corollary 8.33 shows that *every* smooth hypersurface of degree ≤ 3 has Fano scheme dimension D (that is, the open set of hypersurfaces for which $F_1(X)$ has the expected dimension contains the open set of smooth hypersurfaces when the degree is ≤ 3.) Further, it was shown by Harris et al. [1998] that when $d \ll n$, every smooth hypersurface of degree d in $\mathbb{P}^n$ has a Fano scheme of lines of the correct dimension—in fact, all the $F_k(X)$ have the expected dimension when both d and k are much smaller than n. But the lower bound on n given there is very large, and examples are few. In general, we have no idea what to conjecture for the true bound required!

There is a conjecture, however, for the Fano schemes of lines. To motivate it, note that a general hypersurface $X \subset \mathbb{P}^{2m+1}$ that contains an m-plane will be smooth (Exercise 8.75) and will contain a copy of the Grassmannian of lines in the m-plane, a variety of dimension $2m-2$. When $d > 2m+1$, this is larger than the expected dimension $D = 2d-3$ of $F_1(X)$. Another family of such examples is given in Exercise 8.74, but, just as in the examples above, that construction requires $d > n$.

Conjecture 8.41 (Debarre–de Jong)**.** *If $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree d with $d \leq n$, then the Fano scheme $F_1(X)$ of lines on X has dimension $2n - 3 - d$.*

One striking aspect of the Debarre-de Jong conjecture is that the inequality $d \leq n$ for a smooth hypersurface $X \subset \mathbb{P}^n$ is exactly equivalent to the condition that the anticanonical bundle ω_X^* is ample, though it's not clear what role this might play in a proof.

Conjecture 8.41 has been proven for $d \leq 5$ by de Jong and Debarre, and for $d = 6$ by Beheshti in Beheshti [2006]. One might worry that proving the conjecture, even for small d, would involve high-dimensional geometry, but as we will now show, it would be enough to prove the conjecture for $d = n$.

Proposition 8.42. *If* $\dim F_1(X) = d-3$ *for every smooth hypersurface of degree* d *in* $\mathbb{P}^d$, *and* $d \leq n$, *then* $\dim F_1(X) = 2n-d-3$ *for every smooth hypersurface of degree* d *in* $\mathbb{P}^n$.

Proof. We have already treated the case of quadrics (Proposition 3.11) so we may assume that $3 \leq d \leq n$. Suppose that $X \subset \mathbb{P}^n$ is a smooth hypersurface and $L \subset X$ a line. Let Λ be a general d-plane in $\mathbb{P}^n$ containing L and let $Y = \Lambda \cap X$. By Lemma 8.43, below, Y is a smooth hypersurface of degree d in $\Lambda = \mathbb{P}^d$.

The Fano scheme $F_1(Y)$ is the intersection of $F_1(X)$ with the Schubert cycle $\Sigma_{m-n,m-n}(\Lambda) \subset \mathbb{G}(1,m)$; by the Generalized Principal Ideal Theorem,

$$d-3 = \dim_L F_1(Y) \geq \dim_L F_1(X) - 2(n-d),$$

whence $\dim_L F_1(X) \leq 2n-d-3$ as required. □

We have used a special case of the following extension of Bertini's Theorem:

Lemma 8.43. *Let* $X \subset \mathbb{P}^m$ *be a smooth hypersurface and* $L \cong \mathbb{P}^k \subset X$ *a* k*-plane contained in* X. *If* $\Lambda \cong \mathbb{P}^n \subset \mathbb{P}^m$ *is a general* n*-plane containing* L, *then the intersection* $Y = X \cap \Lambda$ *is smooth if and only if* $n-1 \geq 2k$.

Proof. If $n-1 < 2k$ then Y must be singular by Corollary 8.26.

For the converse we may assume by an obvious induction that $n = m-1$. Bertini's Theorem implies that Y is smooth away from L. On the other hand, the locus of tangent hyperplanes $\mathbb{T}_pX$ to X at points $p \in L$ is a subvariety of dimension at most k in the dual projective space $\mathbb{P}^{m*}$, while the locus of hyperplanes containing L will be the $(m-k-1)$-plane $L^\perp \subset \mathbb{P}^{m*}$. Thus, if $n-1 = m-2 \geq 2k$, so that $k < m-k-1$, then not every hyperplane containing L is tangent to X at a point of L. It follows that, for general Λ, the intersection $Y = \Lambda \cap X$ is smooth. □

****the proof for $d = 4$ below is incomplete! Maybe drop this case? We need the reference to Debarre, in any case.****

We can now prove Conjecture 8.41 for $d = 4$, and thereby answer keynote question (d) in the negative:

Theorem 8.44. *If* $X \subset \mathbb{P}^n$ *is a smooth hypersurface of degree* 4, *then the Fano scheme* $F_1(X)$ *has dimension* 1.

Proof. The cases $d \leq 2$ are trivial.

Proposition 8.42 shows that it is enough to consider the case $n = 4$. Suppose $F \subset F_1(X)$ is an irreducible component with $\dim F \geq 2$, and let $L \in F$ be a general point. By Proposition 8.30, the normal bundle $N = N_{L/X}$ must be either $\mathcal{O}_L \oplus \mathcal{O}_L(-1)$ or $\mathcal{O}_L(1) \oplus \mathcal{O}_L(-2)$. Either way, all global sections of N take values in line bundle contained in N. It follows that for any point $p \in L$ the map $H^0(N_{L/X}) \to (N_{L/X})_p = T_pX/T_pL$ has rank at most 1.

(Since $\dim H^0(N) \geq \dim T_L F \geq 2$, the normal bundle must in fact be $\mathcal{O}_L(1) \oplus \mathcal{O}_L(-2)$, but we don't need this.)

Let $Y \subset X$ be the subvariety swept out by the lines of $F \subset F_1(X)$. By Proposition 8.22, Y can have dimension at most 2. But by hypothesis, Y contains a two-dimensional family of lines. From Proposition 8.3 we conclude that Y is a 2-plane. Corollary 8.26 tells us this is impossible, and we're done. $\square$

8.8.1 Further open problems

The Debarre-de Jong conjecture deals with the dimension of the family of lines on a hypersurface $X \subset \mathbb{P}^n$, but we can also ask further questions about the geometry of $F_1(X)$: for example, whether it's irreducible and/or reduced. Exercises 8.77-8.80, in which we show that the Fano scheme $F_1(X)$ of lines on the Fermat quartic hypersurface $X \subset \mathbb{P}^4$ is neither, shows that the statement of Debarre/de Jong cannot be strengthened for all $d \leq n$. But—based on our knowledge of examples—it does seem to be the case that the smaller d is relative to n, the better behaved $F_1(X)$ is for an arbitrary smooth hypersurface $X \subset \mathbb{P}^n$ of degree d. For example, the following questions are open:

(a) Is $F_1(X)$ is reduced and irreducible if $d \leq n-1$ and $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree d? (Assuming it has the expected dimension, $F_1(X)$ is Cohen-Macaulay, ****explain!**** so reduced is equivalent to generically reduced.)

(b) Can we bound the dimension of the singular locus of $F_1(X)$ in terms of d? (The arguments above show that for $d = 3$, the Fano scheme $F_1(X)$ is smooth. while for $d \geq n$ it may not be reduced. What about the range $4 \leq d \leq n-1$?)

The analogous questions for $F_k(X)$ with $k > 1$ are completely open. We can ask, for example: given d and k, what is the largest n such that there exists a smooth hypersurface $X \subset \mathbb{P}^n$ of degree d with $\dim F_k(X) > (k+1)(n-k) - \binom{k+d}{d}$? Again, Harris et al. [1998] says that such n are bounded; but the bound given there is probably far too large.

Finally, we can ask: why the Fano schemes instead of other Hilbert schemes? Why not look, for example, at rational curves of any degree e on a hypersurface? Here the field is wide open. Specifically, we have an "expected" dimension: since a rational curve $C \subset \mathbb{P}^n$ is given parametrically as the image of a map $f : \mathbb{P}^1 \to \mathbb{P}^n$, which is specified by $n+1$ homogeneous polynomials of degree e on $\mathbb{P}^1$, and two such $(n+1)$-tuples have the same image if and only if they differ by a scalar or by an automorphism of $\mathbb{P}^1$, the space $\mathcal{H}$ of such curves has dimension $(n+1)(e+1)-4$. On the other hand, it is $ed+1$ conditions on a hypersurface $X \subset \mathbb{P}^n$ of degree d to contain a general such curve C, since if $X = V(F)$, the condition is that $f^*F = 0 \in H^0(\mathcal{O}_{\mathbb{P}^1}(de))$. Thus we'd expect the fibers of the incidence correspondence

$$\Psi = \{(X, C) \in \mathbb{P}^N \times \mathcal{H} \mid C \subset X\}$$

over $\mathcal{H}$ to have dimension $N - (de+1)$, and Ψ correspondingly to have dimension

$$(n+1)(e+1) - 4 + N - (de+1) = N + (n-d)e + n + e - 4.$$

This leads us to:

Conjecture 8.45. *If $X \subset \mathbb{P}^n$ is a general hypersurface of degree d, then X contains a rational curve of degree e if and only if*

$$\lambda(n, d, e) := (n-d)e + n + e - 4 \geq 0;$$

and when this inequality is satisfied the family of such curves on X has dimension $\lambda(n, d, e)$.

We proved the conjecture in this chapter for $e = 1$, and it's also known when d is small enough relative to n and e ****reference!****.

Note that the analog of the Debarre-de Jong conjecture in this setting—that the dimension estimate of Conjecture 8.45 holds for an arbitrary smooth $X \subset \mathbb{P}^n$ of degree $d \leq n$—is false; one counterexample is given in Exercise 8.81. But it might hold when d satisfies a stronger inequality with respect to n, perhaps for $d \leq n/e$.

8.9 Exercises

****In the exercises, need to change homogeneous forms F to f and G to g; also watch out for τ_G that should be σ_G and $G(k, n)$ that should be $\mathbb{G}(k, n)$.****

Exercise 8.46. Let U be the open set of $\mathbb{G}(1, \mathbb{P}^3)$ of lines that can be written as the row-space of a matrix

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}.$$

Let Q be the smooth quadric $x_0x_1 + x_2x_3 = 0$ and let Q' be the singular quadric $x_0x_1 + x_2^2 = 0$. Show using the equations defined in Section 8.1.1 that $F_1(Q) \cap U$ is the union of two lines, while $F_1(Q') \cap U$ is a double line.

Exercise 8.47. Let $\mathbb{P}^N$ be the space of all surfaces of degree d in $\mathbb{P}^3$. Show that the set of surfaces of degree d containing two or more lines is a subvariety of codimension $2d - 6$ in $\mathbb{P}^N$. (Thus a general surface $X \subset \mathbb{P}^3$ of degree $d \geq 4$ containing a line contains only one line, and no surface in a general pencil of such surfaces will contain more than one line.)

Exercise 8.48. Show that the expected number of lines on a hypersurface of degree $2n-3$ in $\mathbb{P}^n$ is always positive, and deduce that *every hypersurface of degree* $2n - 3$ *in* $\mathbb{P}^n$ *must contain a line.* (This is just a special case of Corollary 8.32; the idea here is to do it without a tangent space calculation.)

Exercise 8.49. Let $X \subset \mathbb{P}^4$ be a general quartic threefold. By Theorem 8.44, X will contain a one-parameter family of lines. Find the class of the Fano scheme $F_1(X)$, and the degree of the surface $Y \subset \mathbb{P}^4$ swept out by these lines.

Exercise 8.50. Find the class of the scheme $F_2(Q) \subset \mathbb{G}(2, 5)$ of 2-planes on a quadric $Q \subset \mathbb{P}^5$. (Do the problem first, then ompare your answer to the result in Proposition 3.11.)

Exercise 8.51. Find the expected number of 2-planes on a general quartic hypersurface $X \subset \mathbb{P}^7$, that is, the degree of $c_{15}(\mathrm{Sym}^4 S^*) \in A(\mathbb{G}(2, 7))$.

Exercise 8.52. We can also use the calculation carried out in this Chapter to count lines on complete intersections $X = Z_1 \cap \cdots \cap Z_k \subset \mathbb{P}^n$, simply by finding the classes of the schemes $F_1(Z_i)$ of lines on the hypersurfaces Z_i and multiplying them in $A(\mathbb{G}(1, n))$. Do this to find the number of lines on the intersection $X = Y_1 \cap Y_2 \subset \mathbb{P}^4$ of two cubic hypersurfaces in $\mathbb{P}^5$

Exercise 8.53. Use Chern classes to prove that the following bundles are not direct sums:

(a) The tangent bundle to $\mathbb{P}^n$

(b) The universal subbundle on the Grassmannian $G(k, n)$

Exercise 8.54. Find the Chern class $c_3(\mathrm{Sym}^3 S^*) \in A^3(\mathbb{G}(1, 3))$. Why is this the degree of the curve of lines on the cubic surfaces in a pencil? Note that this computation does not use the incidence correspondence Φ.

Exercise 8.55. Let $\{X_t = V(t_0F + t_1G + t_2H)\}$ be a general net of cubic surfaces in $\mathbb{P}^3$.

(a) Let $p \in \mathbb{P}^3$ be a general point. How many lines lying on some member X_t of the net pass through p?

(b) Let $H \subset \mathbb{P}^3$ be a general plane. How many lines lying on some member X_t of the net lie in H?

Compare your answer to the second half of this question to the calculation in Chapter 3 of the degree of the locus of reducible plane cubics!

Exercise 8.56. Let $X \subset \mathbb{P}^3$ be a surface of degree $d \geq 3$. Show that if $F_1(X)$ is positive-dimensional, then either X is a cone or X has a positive-dimensional singular locus.

Exercise 8.57. Let $X \subset \mathbb{P}^4$ be a smooth cubic threefold, and

$$\{S_t = X \cap H_t\}_{t \in \mathbb{P}^1}$$

a general pencil of hyperplane sections of X. What is the degree of the surface swept out by the lines on the surfaces S_t, and what is the genus of the curve parametrizing them?

Exercise 8.58. Prove Theorem 8.13 using the methods of Section 8.7, that is, by writing the local equations of $F_k(X) \subset \mathbb{G}(k, n)$

Exercise 8.59. Using the methods of Section 8.7, show that there exists a pair (X, Λ) with $X \subset \mathbb{P}^7$ a quartic hypersurface and $\Lambda \subset X$ a 2-plane such that Λ is an isolated, reduced point of $F_2(X)$.

Exercise 8.60. Using the result of Exercise 8.59, show that the number of 2-planes on a general quartic hypersurface $X \subset \mathbb{P}^7$ is the number calculated in Exercise 8.51.

The following two exercises use a theorem of Hochster, which says that

****add statement of theorem and reference****

Exercise 8.61. Use Hochster's theorem above to prove the analog of Corollary 8.31 for k-planes in general—that is, that if $\binom{k+d}{d} \leq (k+1)(n-k)$, then there exists a pair (X, Λ) with $X \subset \mathbb{P}^n$ a smooth hypersurface of degree d and $\Lambda \subset X$ a k-plane such that $F_k(X)$ is smooth of dimension $(k+1)(n-k) - \binom{k+d}{d}$ in a neighborhood of Λ—using methods similar to those of Exercise 8.59.

Exercise 8.62. Use the result of the preceding exercise (and the simple estimate on the dimension of the incidence correspondence $\Phi(n, d, 1)$ introduced in Section 8.1) to deduce Theorem 8.28

Exercise 8.63. To complete the proof of Proposition 8.40, let $X \subset \mathbb{P}^3$ is a cubic surface with one ordinary double point $p = (1, 0, 0, 0)$, given as the zero locus of the cubic

$$F(Z_0, Z_1, Z_2, Z_3) = Z_0 A(Z_1, Z_2, Z_3) + B(Z_1, Z_2, Z_3)$$

where A is homogeneous of degree 2 and B homogeneous of degree 3. If we write a line $L \subset X$ through p as the span $L = \overline{p\,q}$ with $q = (0, Z_1, Z_2, Z_3)$, show that X is smooth along $L \setminus \{p\}$ if and only if the zero loci of A and B intersect transversely at (Z_1, Z_2, Z_3).

Exercise 8.64. Extending the results of Section 8.7, suppose that X is a general cubic surface having two ordinary double points $p, q \in X$. Describe the scheme structure of $F_1(X)$ at the point corresponding to the line $L = \overline{p\,q}$, and in particular determine the multiplicity of $F_1(X)$ at L.

Exercise 8.65. Now let $X \subset \mathbb{P}^3$ be a cubic surface and $P, q \in X$ isolated singular points of X; let $L = \overline{p\,q}$. Show that L is an isolated point of $F_1(X)$, and that the multiplicity

$$\operatorname{mult}_L F_1(X) \geq 4$$

Exercise 8.66. Let $X \subset \mathbb{P}^3$ be a cubic surface and $p_1, \ldots, p_\delta$ isolated singular points of X. Show that no three of the points p_i are collinear.

Exercise 8.67. Use the result of the preceding two exercises to deduce the statement that *a cubic surface $X \subset \mathbb{P}^3$ can have at most four isolated singular points.*

Exercise 8.68. Show that there exists a smooth quintic threefold $X \subset \mathbb{P}^4$ whose scheme $F_1(X)$ of lines contains an isolated point of multiplicity 2.

Exercise 8.69. Using the result of the preceding exercise, show that there exists a smooth quintic threefold whose scheme of lines consists of one double point and exactly 2873 simple points (that is, which contains exactly 2874 lines in all).

Exercise 8.70. Let Φ be the incidence correspondence of triples consisting of a hypersurface $X \subset \mathbb{P}^n$ of degree $d = 2n-3$, a line $L \subset X$ and a singular point p of X lying on L: that is,

$$\Phi = \{(X, L, p) \in \mathbb{P}^N \times \mathbb{G}(1, n) \times \mathbb{P}^n \mid p \in L \subset X \text{ and } p \in X_{\text{sing}}\}.$$

Show that Φ is irreducible.

Exercise 8.71. Let Φ be as in the preceding exercise, and suppose that $(X, L, p) \in \Phi$ is a general point. Find the multiplicity of the Fano scheme $F_1(X)$ at L.

Exercise 8.72. Suppose F and G are two quartic polynomials on $\mathbb{P}^3$ and $\{X_t = V(t_0F + t_1G)\}$ the pencil of quartics they generate; let σ_F and σ_G be the sections of $\operatorname{Sym}^4 S^*$ corresponding to F and G. Let X_{t_0} be a member of the pencil containing a line $L \subset \mathbb{P}^3$.

(a) Find the condition on F and G that L is a reduced point of $V(\sigma_F \wedge \sigma_G) \subset \mathbb{G}(1, 3)$.

(b) Show that this is equivalent to the condition that the point $(t_0, L) \in \mathbb{P}^1 \times \mathbb{G}(1, 3)$ is a simple zero of the map $\mu^*\mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \nu^* \operatorname{Sym}^d S^*$.

Exercise 8.73. Let $\Sigma \subset \mathbb{P}^{34}$ be the space of quartic surfaces in $\mathbb{P}^3$. Interpret the condition of the preceding problem in terms of the geometry of the pencil $\mathcal{D}$ around the line L, and use this to answer two questions:

(a) What is the singular locus of Σ?

(b) What is the tangent hyperplane $\mathbb{T}_X\Sigma$ at a smooth point corresponding to a smooth quartic surface X containing a single line?

The following two exercises give constructions of smooth hypersurfaces containing more than the expected families of lines.

Exercise 8.74. Let $Z \subset \mathbb{P}^{n-2}$ be any smooth hypersurface. Show that the cone $\overline{p\,Z} \subset \mathbb{P}^{n-1}$ over Z in $\mathbb{P}^{n-1}$ is the hyperplane section of a smooth hypersurface $X \subset \mathbb{P}^n$, and hence that for $d > n$ there exist smooth hypersurfaces $X \subset \mathbb{P}^n$ whose Fano scheme $F_1(X)$ of lines has dimension strictly greater than $2n - 3 - d$.

Exercise 8.75. Take $n = 2m + 1$ odd, and let $\Lambda \subset \mathbb{P}^n$ be an m-plane. Show that there exist smooth hypersurfaces $X \subset \mathbb{P}^n$ of any given degree d containing Λ, and deduce once more that for $d > n$ there exist smooth hypersurfaces $X \subset \mathbb{P}^n$ whose Fano scheme $F_1(X)$ of lines has dimension strictly greater than $2n - 3 - d$.

Note that the construction of Exercise 8.75 cannot be modified to provide counterexamples to the de Jong/Debarre conjecture, since by Corollary 8.26 there do not exist smooth hypersurfaces $X \subset \mathbb{P}^n$ containing linear spaces of dimension strictly greater than $(n-1)/2$. The following exercise shows that the construction of Exercise 8.74 is similarly extremal, but is harder: it requires use of the Second Fundamental Form of a hypersurface (see Harris [1992]).

Exercise 8.76. Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d > 2$. Show that X can have at most finitely many hyperplane sections that are cones.

To see some of the kinds of odd behavior the variety of lines on a smooth hypersurface can exhibit, short of having the wrong dimension, the following series of exercises will look at the Fermat quartic $X \subset \mathbb{P}^4$, that is, the zero locus

$$X = V(Z_0^4 + Z_1^4 + Z_2^4 + Z_3^4 + Z_4^4).$$

The conclusion is that $F_1(X)$ has 40 irreducible components, each of which is everywhere non-reduced! We start with a useful more general fact:

Exercise 8.77. Let $S = \overline{p\,C} \subset \mathbb{P}^3$ be the cone with vertex p over a plane curve C of degree $d \geq 2$, and $L \subset S$ any line. Show that the tangent space

$T_L F_1(S)$ has dimension at least two, and hence that $F_1(S)$ is everywhere nonreduced.

Exercise 8.78. Show that X has 40 conical hyperplane sections Y_i, each a cone over a quartic Fermat curve in $\mathbb{P}^2$.

Exercise 8.79. Show that the reduced locus $F_1(Y_i)_{\mathrm{red}}$ has class $4\sigma_{3,2}$.

Exercise 8.80. Using your answer to Exercise 8.49, conclude that

$$F_1(X) = \bigcup_{i=1}^{40} F_1(Y_i);$$

in other words, $F_1(X)$ is the union of 40 double curves.

Exercise 8.81. Show that

(a) There exist smooth quintic hypersurfaces $X \subset \mathbb{P}^5$ containing a 2-plane $\mathbb{P}^2 \subset \mathbb{P}^5$; and

(b) For such a hypersurface X, the family of conic curves on X has dimension strictly greater than the number $\lambda(5, 5, 2)$ of Conjecture 8.45.

9

Singular Elements of Linear Series

Keynote Questions

(a) If F and G are general forms of degree d on $\mathbb{P}^2$, how many curves in the pencil $\{C_t = V(t_0F + t_1G)\}_{t=(t_0,t_1)\in\mathbb{P}^1}$ are singular?

(b) With F and G as above, let H be a third general form of degree d, and consider $\{C_t = V(t_0F+t_1G+t_2H)\}_{t=(t_0,t_1,t_2)\in\mathbb{P}^2}$, a *general net of plane curves of degree* d. What is the degree and genus of the curve $\Gamma \subset \mathbb{P}^2$ traced out by the singular points of members of the net? What is the degree and genus of the discriminant curve $\mathcal{D} = \{t : C_t \text{ is singular}\}$?

(c) If $C \subset \mathbb{P}^2_{\mathbb{C}}$ (note that we are in characteristic 0) is a general plane curve of degree d, then C has $3(d-2)d$ flex lines (lines that have order of contact at least 3 with C at some point); these are the intersections of C with its *Hessian*, a curve of degree $3(d-2)$. What about a curve $C \subset \mathbb{P}^n$? For example, is there a formula for the number of hyperplanes that have order of contact at least $n+1$ with C at some point?

(d) If $\{C_t = V(t_0F+t_1G)\}_{t\in\mathbb{P}^1}$ is a general pencil of plane curves of degree d, how many of the curves C_t will have hyperflexes (that is, lines having contact of order 4 with C_t)?

In this chapter we introduce the *bundle of principal parts* associated with a line bundle. Its sections represent the Taylor series expansions of sections of the line bundle up to a given order. We will use the techniques we've developed to compute Chern classes of this bundle, and the computation will enable us to answer many questions about singular points and other special points of varieties in families. We'll start out by discussing hypersurfaces in projective space; but the techniques we develop are much more broadly applicable to families of hypersurfaces in any smooth projective variety X, and in Section 9.5.2 we'll see how to generalize our formulas to that case.

In the last section we introduce a different approach to such questions, the "topological Hurwitz formula".

Since quadrics play a special role in our theory (see the definition of ordinary double point, below, and its use in the proof of Proposition 9.1) *we will assume throughout this chapter that the characteristic of our ground field is not 2*; the case of characteristic 2 requires a more subtle treatment.

9.1 Singular hypersurfaces and the universal singularity

Before starting on this path, we'll take a moment out to talk about loci of singular plane curves, and more generally hypersurfaces in $\mathbb{P}^n$. Let $\mathbb{P}^N = \mathbb{P}^{\binom{d+n}{n}-1}$ be the projective space parametrizing all hypersurfaces of degree d in $\mathbb{P}^n$. Our primary object of interest is the *discriminant locus* $\mathcal{D} \subset \mathbb{P}^N$, defined as the set of singular hypersurfaces.

To establish some notation, we remind the reader that a hypersurface X, locally defined as the zero locus of a function $f(x)$, is *singular* at a point p if f and all its first derivatives vanish at p. It will be useful for us to think of this in terms of the Taylor expansion f at p. Changing coordinates so that p is the origin 0, this takes the form

$$f(x) = f_0 + f_1(x) + f_2(x) + \cdots,$$

where f_i is a form of degree i. Of course f vanishes at 0 if $f_0 = 0$, and is then singular at 0 if $f_1(x) = 0$. The *projectivized tangent cone* of X at p is the variety in $\mathbb{P}^{n-1}$ defined by the vanishing of the form f_i, where i is the smallest index for which $f_i \neq 0$.

For example, the simplest possible singularity of a hypersurface X, generalizing the case of a node of a plane curve, is called an *ordinary double point*. This is a point $p \in X$ such that the equation of X can be written in local coordinates with $p = 0$ as above with $f_0 = f_1 = 0$ and where $f_2(x)$ is a *non-degenerate quadratic form*—that is, the projectivized tangent cone to X at p is a smooth quadric. Indeed, examples are the cones over smooth quadrics—see Figure 9.0. (Here is why we will assume that the characteristic is not 2: a quadric in $\mathbb{P}^{n-1}$ is smooth if the generator f_2 of its ideal, together with the derivatives of f_2, is an irrelevant ideal; when the characteristic is not 2, Euler's formula $2f_2 = \sum x_i \partial f_2/\partial x_i$ shows that it is

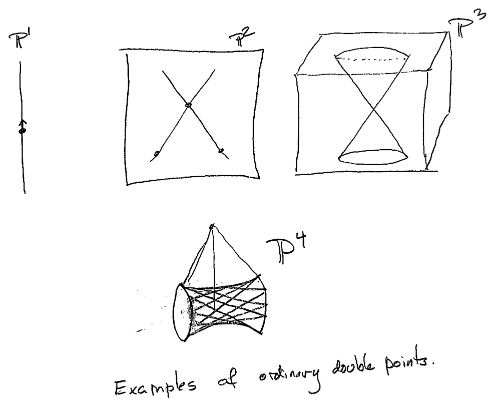

FIGURE 9.0. Ordinary Double Points of Hypersurfaces of Dimension $0, 1, 2, 3$.

equivalent to assume that the partial deriviatives of f_2 are linearly independent, and this is the property we will actually use. In characteristic 2 *no* quadratic form in an odd number of variables has this property.) As with nodes of plane curves, we shall see that the general singular hypersurface has only one singularity, which is an ordinary double point.

A central role in this chapter will be played by the *universal singular point* $\Sigma = \Sigma_{n,d}$ of hypersurfaces of degree d in $\mathbb{P}^n$, defined as follows:

$$\begin{array}{ccc} \Sigma := \{(Y,p) \in \mathbb{P}^N \times \mathbb{P}^n \mid p \in Y_{\text{sing}}\} & \xrightarrow{\pi_2} & \mathbb{P}^n \\ \Big\downarrow{\scriptstyle \pi_1} & & \\ \{\text{hypersurfaces } Y \text{ of degree } d \text{ in } \mathbb{P}^n\} = \mathbb{P}^N. & & \end{array}$$

If we write the general form of degree d on $\mathbb{P}^n$ as $F = \sum_{|I|=d} a_I x^I$ and think of it as a bihomogeneous form of bidegree $(1, d)$ in the coordinates a_I of $\mathbb{P}^N$ and the coordinates $x_0, \dots, x_n$ of $\mathbb{P}^n$, then Σ is defined by the bihomogeneous equations

$$F(x) = 0 \quad \text{ and } \quad \frac{\partial F}{\partial x_i} = 0 \text{ for } i = 0, \dots, n,$$

so Σ is an algebraic set. Note that in characteristic 0 (or more generally in characteristic prime to the degree d) the first of these equations is implied by the others; given the dimension statement of Proposition 9.1 below, this means that Σ is a complete intersection of $n+1$ hypersurfaces of bidegree $(1, d-1)$ in $\mathbb{P}^N \times \mathbb{P}^n$.

The image $\mathcal{D}$ of Σ in $\mathbb{P}^N$ is the set of singular hypersurfaces; it is called the *discriminant*. The next proposition shows that $\mathcal{D}$ is a hypersurface, and that $\Sigma \to \mathcal{D}$ is a resolution of singularities:

Proposition 9.1. *With notation as above, suppose that $d \geq 2$.*

(a) *The variety Σ is smooth and irreducible of dimension $N-1$ (that is, codimension $n+1$); in fact, it is a $\mathbb{P}^{N-n-1}$-bundle over $\mathbb{P}^n$.*

(b) *The general singular hypersurface of degree d has a unique singularity, which is an ordinary double point. In particular, Σ is birational to the discriminant $\mathcal{D}$, which is its image in $\mathbb{P}^N$.*

(c) *$\mathcal{D}$ is an irreducible hypersurface in $\mathbb{P}^N$.*

Proof. Let $p \in \mathbb{P}^n$ be a point, and let $x_0, \dots, x_n$ be homogeneous coordinates of $\mathbb{P}^n$ such that $p = (1, 0 \dots, 0)$. Let $f(x_1/x_0, \dots, x_n/x_0) = x_0^{-d} F(x_0, \dots, x_n) = 0$ be the affine equation of the hypersurface $F = 0$. For $d \geq 1$ the $n+1$ coefficients of the constant and linear terms f_0 and f_1 in the Taylor expansion of f at p are equal to certain coefficients of F, so the fiber of Σ over p is a projective subspace of $\mathbb{P}^N$ of codimension $n+1$. The first part of the Proposition follows from this, and implies that the discriminant $\mathcal{D} = \pi_1(\Sigma)$ is irreducible.

To prove the statements in the second part of the Proposition, note that the fiber of Σ over a point $p \in \mathbb{P}^n$ contains the hypersurface that is the union of $d-2$ hyperplanes not containing p with a cone over a nonsingular quadric in $\mathbb{P}^{n-1}$ with vertex p. This hypersurface has an ordinary double point at p, and is generically smooth. By the previous argument, the hypersurfaces corresponding to points of the fiber of Σ over p form a linear system of hypersurfaces, with no base points other than p. Bertini's Theorem **??** shows that a general member of this system is smooth away from p. Thus the fiber of the map $\pi_1 : \Sigma \to \mathbb{P}^N$ over a general point of $\mathbb{P}^N$ consists of just one point, showing that the map is birational onto its image, $\mathcal{D}$. Since smoothness is an open condition on a quadratic form, the the general member has only an ordinary double point at p.

The fact that Σ, which has dimension $N-1$, is birational to $\mathcal{D}$ shows that $\mathcal{D}$ also has dimension $N-1$, completing the proof. $\square$

The defining equation of $\mathcal{D} \subset \mathbb{P}^N$ is difficult to write down explicitly, though of course it can be computed in principle by elimination theory. There are determinantal formulas in a few cases: see for example Gelfand et al. [2008] and Eisenbud et al. [2003], Even in very simple cases the discriminant locus has a lot of interesting features, as even a picture of the real points of the discriminant of a quartic $f(a) = a^4 + xa^2 + ya + z$ variables suggests:

****check this. Eqn of Swallowtail $0 = 256x^3 - 27y^4 - 128x^2z^2 + 144yxy^2z + 16xz^4 - 4y^2z^3$—from Donal O'Shea's article in Contemp Math. 40: Singularities: proceedings of the IMA Participating Institutions Conference, ed R. Randell.****

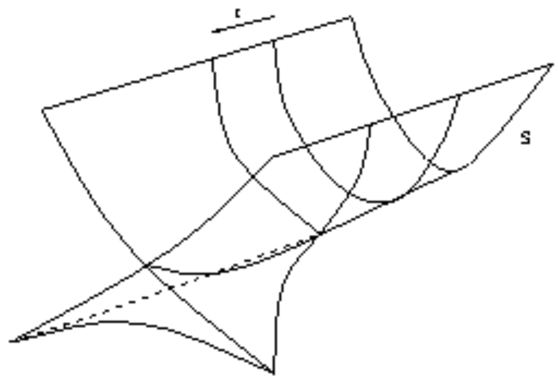

FIGURE 9.4. Discriminant of a quartic polynomial.[[Note that the dotted line is wrong]]

In view of Proposition 9.1, we can rephrase the first keynote question of this chapter as asking for the number of points of intersection of a general line $L \subset \mathbb{P}^N$ with the hypersurface $\mathcal{D}$; that is, the degree of $\mathcal{D}$. How can we determine this if we can't write down the form?

There is an interpretation of the discriminant hypersurface in $\mathbb{P}^N$ that relates $\mathcal{D}$ to an object previously encountered in Chapter 1. The d-th Veronese map ν_d embeds P^n in the dual $\mathbb{P}^{N*}$ of the projective space $\mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^n}(d))$, in such a way that the intersection of $\nu_d(\mathbb{P}^n)$ with the hyperplane corresponding to a point $F \in \mathbb{P}^N$ is isomorphic, via ν_d, to the corresponding hypersurface $F = 0$ in $\mathbb{P}^n$. Thus the discriminant is the set of hyperplanes in $\mathbb{P}^{N*}$ that have singular intersection with $\nu_d(\mathbb{P}^n)$; or, equivalently, those that contain a tangent plane to $\nu_d(\mathbb{P}^n)$. This is the definition of the dual variety to $\nu_d(\mathbb{P}^n)$, which we first encountered in Section 1.2.2. Proposition 9.1 shows that the dual of $\nu_d(\mathbb{P}^n)$ is a hypersurface, and that the general tangent hyperplane is tangent at just one point, at which the intersection has an ordinary double point.

9.2 Linear systems

The family of hypersurfaces of degree d on $\mathbb{P}^n$ is the simplest and most concrete example of a linear system of divisors, a notion that permeates algebraic geometry (see for example the book Lazarsfeld [2004]). The main construction of this chapter, the bundle of principal parts, applies to any linear system, and we will apply it in that generality, so we pause to review the idea.

Definition 9.2. By a *linear system* (also called a *linear series*) ****later we seem often to use "series". Also, though defined here, linear systems occur starting in the Overture, and in many other places. Figure out the right place for this.**** on a scheme X we mean a pair $\mathcal{W} = (\mathcal{L}, W)$, where $\mathcal{L}$ is a line bundle on X and $W \subset H^0\mathcal{L}$ is a vector space of global sections of $\mathcal{L}$. (It is sometimes convenient to consider the case where there is simply a homomorphism $W \to H^0\mathcal{L}$, but we will not need this version.)

If X is a variety, then each global section σ of a line bundle gives rise to a Cartier divisor $D = V(\sigma)$, the zero locus of σ. Moreover $\mathcal{L}$ can be recovered from D as $\mathcal{L} \cong I(D)^*$ and σ is also determined by D up to a nonzero scalar multiple. Thus we can recover $(\mathcal{L}, W)$ from the set

$$\mathcal{W} = |(\mathcal{L}, W)| := \{D = V(\sigma) \mid \sigma \in W\},$$

which is naturally identified with the projective space $\mathbb{P}W$, and by abuse of notation we often say that the linear system is this set of divisors.

We will follow classical usage, and define the dimension of a linear system $(\mathcal{L}, W)$ to be the dimension of $\mathbb{P}(W)$. A linear system of dimension 1 is called a *pencil* (presumably from the french *pinceau*, brush: the elelements of the linear system are like the hairs of a flat brush.) A linear system of dimension 2 is called a *net*, and (though we will not treat this case specifically) a linear system of dimension 3 the system is called a *web*. Pencils may be characterized as families of divisors parametrized by $\mathbb{P}^1$ such that a general point $x \in X$ lies on a unique member of the family.

The most important examples of linear systems are those that arise from embeddings $X \subset \mathbb{P}^n$. Given such an embedding, we take $|(\mathcal{L}, W)|$ to be the space of hyperplane sections of X. That is, $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(1)|_X$ and W is the space of sections of $\mathcal{O}_{\mathbb{P}^n}(1)$, restricted to X. (If X is contained in a hyperplane, we say that the embedding is *degenerate*, and then the restriction map $H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(\mathcal{L})$ is not a monomorphism). A linear system that corresponds to an embedding is said to be *very ample*.

Associated with any linear system $\mathcal{W}$ is its *base locus*, consisting of the (scheme theoretic) intersection

$$\bigcap_{D \in \mathcal{W}} D.$$

Somewhat more general than very ample systems are those that are *base-point free*—that is, where the base locus is empty. It is easy to see that $\mathcal{W}$ is basepoint free if and only if W generates $\mathcal{L}$ as a sheaf. Basepoint free systems correspond to maps from X to projective spaces (set theoretically, the map corresponding to $\mathcal{W}$ sends $x \in X$ to the point of $\mathbb{P}W^*$ corresponding to the hyperplane in W of sections that vanish at x.)

Other natural examples of linear systems are given by simply choosing some forms $g_0 \dots, g_r$: we let $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}$ and we take W to be the subspace of global sections spanned by the g_i. The elements of $|(\mathcal{L}, W)|$ are then the hypersurfaces corresponding to linear combinations of the g_i.

A basic result about linear systems is Bertini's Theorem (Hartshorne [1977] and Flenner [1977]), which says that, in characteristic 0, the general element of a linear series on a smooth variety is nonsingular outside the base locus. More precisely:

Theorem 9.3. *If $X \subset \mathbb{P}^n$ is a smooth quasiprojective variety, then a general hyperplane section of X is smooth and irreducible. More generally, if $(\mathcal{L}, W)$ is a linear system on any quasiprojective variety X, with base locus B, and the morphism $X \setminus B \to \mathbb{P}(W^*)$ defined by $(\mathcal{L}, W)$ is separable, then there is an open dense subset $U \subset |(\mathcal{L}, W)|$ such that the divisors $D \in U$ are singular at most along B and the singularities of X.* $\square$

We can imitate the construction of the universal divisor, and the universal singularity, replacing the family of hypersurfaces of degree d on $\mathbb{P}^n$ by any linear system, and we will pursue this idea in Section 9.5.2 below.

9.3 Bundles of principal parts

We can simplify the problem of describing the discriminant by linearizing it. We ask not, "Is the hypersurface $C = V(F)$ singular?" but rather ask for each point $p \in \mathbb{P}^n$ in turn the simpler question "Is C singular at p?" As in the context of the problem of counting lines on hypersurfaces, this converts a higher-degree equation into a family of systems of linear equations, which we can then express as the vanishing of a section of a vector bundle.

For each point $p \in \mathbb{P}^n$ we have an $(n+1)$-dimensional vector space

$$E_p = \frac{\{\text{germs of sections of } \mathcal{O}_{\mathbb{P}^n}(d) \text{ at } p\}}{\{\text{germs vanishing to order} \geq 2 \text{ at } p\}}.$$

This space should be thought of as the vector space of first-order Taylor expansions of forms of degree d. We will see that the spaces E_p fit together to form a vector bundle, called the bundle of *first order principal parts*, which we will write as $\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d))$. A form F of degree d will give rise to a section τ_F of this vector bundle whose value at the point p is the first-order Taylor expansion of F locally at p, and whose vanishing locus is thus the set of singular points of the hypersurface $F = 0$.

An important feature of the situation is that each vector space E_p has a naturally defined subspace, the space of germs vanishing at p. These subspaces will, as we'll see, glue together into a subbundle of $\mathcal{P}^1$. Using the Whitney formula, this will help evaluate the Chern classes of the bundle.

If we replace the forms of degree d by the space of sections of a line bundle, or a subspace of the space of sections, we could again form the bundle of Taylor series of germs of sections. But in fact we can work still more generally with sections of a sheaf. Let $\mathcal{L}$ be a quasicoherent sheaf on a K-scheme X, and write $\pi_1, \pi_2 : X \times X \to X$ for the projections onto the two factors. Let $\mathcal{I}$ be the ideal of the diagonal in $X \times X$. We set

$$\mathcal{P}^m(\mathcal{L}) = \pi_{2*}\Big(\pi_1^*\mathcal{L} \otimes \mathcal{O}_{X\times X}/\mathcal{I}^{m+1}\Big),$$

which is a quasicoherent sheaf on X. We will parse and explain this rather technical expression below; but first we list its very useful properties:

****We really should restrict to the case where $\mathcal{L}$ is a vector bundle, K is algebraically closed, and perhaps even X is smooth.****

Theorem 9.4. *The sheaves $\mathcal{P}^m(\mathcal{L})$ have the following properties:*

(a) *If $p \in X$ is a K-rational point, then there is a canonical identification of the fiber $\mathcal{P}^m(\mathcal{L}) \otimes \kappa_p$ of $\mathcal{P}^m(\mathcal{L})$ at p with the sections of the restriction of $\mathcal{L}$ to the m-th order neighborhood of p; that is,*

$$\mathcal{P}^m(\mathcal{L}) \otimes \kappa_p = H^0(\mathcal{L} \otimes \mathcal{O}_{X,p}/\mathfrak{m}_{X,p}^{m+1})$$

as vector spaces over $\kappa(p) = \mathcal{O}_{X,p}/\mathfrak{m}_{X,p} = K$. In other words,

$$\mathcal{P}^m(\mathcal{L}) \otimes \kappa_p = \frac{\{\text{germs of sections of } \mathcal{L} \text{ at } p\}}{\{\text{germs vanishing to order} \geq m+1 \text{ at } p\}}.$$

(b) *If $F \in H^0(\mathcal{L})$ is a global section, then there is a global section $\tau_F \in H^0(\mathcal{P}^m(\mathcal{L}))$ whose value at p is the class of F in $H^0(\mathcal{L} \otimes \mathcal{O}_{X,p}/\mathfrak{m}_{X,p}^{m+1})$*

(c) *$\mathcal{P}^0(\mathcal{L}) = \mathcal{L}$, and for each m there is a natural right exact sequence*

$$\mathcal{L} \otimes \operatorname{Sym}^m(\Omega_X) \to \mathcal{P}^m(\mathcal{L}) \to \mathcal{P}^{m-1}(\mathcal{L}) \to 0,$$

where $\Omega_{X/K}$ denotes the sheaf of K-linear differential forms on X.

(d) *If X is smooth and of finite type over K and $\mathcal{L}$ is a vector bundle on X, then $\mathcal{P}^m(\mathcal{L})$ is a vector bundle on X, and the right exact sequences of part (c) are left exact as well.*

Example 9.5. In the simplest and most interesting case, where $m = 1$, $X = \mathbb{P}^n$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$, this can be made very explicit. In characteristic 0,

$$\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d)) \cong \begin{cases} \Omega_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n} & \text{if } d = 0 \\ \mathcal{O}_{\mathbb{P}^n}(d-1)^{n+1} & \text{if } d \neq 0, \end{cases}$$

while in characteristic p the conditions are replaced by $d \equiv 0$ or $d \not\equiv 0$ mod p. This curious dichotomy is explained by the answer to a more refined question: By Part (d) we have a short exact sequence

$$0 \to \Omega_{\mathbb{P}^n}(d) \to \mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d)) \to \mathcal{O}_{\mathbb{P}^n}(d) \to 0.$$

and we can ask for its class in

$$\mathrm{Ext}^1_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}(d), \Omega_{\mathbb{P}^n}(d)) \cong \mathrm{Ext}^1_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}, \Omega_{\mathbb{P}^n}) = H^1(\Omega_{\mathbb{P}^n}) = K.$$

More generally, for any line bundle $\mathcal{L}$ on a smooth variety X, the short exact sequence in Part(d) gives us a class in

$$\mathrm{Ext}^1_X(\mathcal{L}, \Omega_X \otimes \mathcal{L}) = H^1(\Omega_X)$$

called the *Atiyah class* $at(\mathcal{L})$ of $\mathcal{L}$ (Atiyah [1957a] and Illusie [1972].) The formula for $\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d))$ follows at once from the more refined and more uniform result that

$$at(\mathcal{O}_{\mathbb{P}^n}(d)) = d \cdot \eta,$$

where $\eta \in \mathrm{Ext}^1_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}, \Omega_{\mathbb{P}^n})$ is the class of the tautological sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \xrightarrow{(x_0,\dots,x_n)} \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0.$$

See Perkinson [1996] 2.II, where the theorem is stated just for the characteristic 0 case, but the technique works more generally, and Re [2011] for an analysis of all the $\mathcal{P}^m$.

We will not make use of these explicit computations below.

Proof of Theorem 9.4. Since the constructions all commute with restriction to open sets, we may harmlessly suppose that $X = \operatorname{Spec} R$ is affine. Thus also $X \times X = \operatorname{Spec} S$, where $S := R \otimes_K R$. We may think of $\mathcal{L}$ as coming from an R-module L, and then $\pi_1^* L := L \otimes_K R$. Pushing a (quasi-coherent) sheaf $\mathcal{M}$ on $X \times X$ forward by π_{2*} simply means considering the corresponding S-module as an R-module via the ring map $R \to S$ sending r to $1 \otimes_K r$.

In this setting the sheaf of ideals $\mathcal{I}$ defining the diagonal embedding of X in $X \times X$ corresponds to the ideal $I \subset S$ that is the kernel of the multiplication map $S = R \otimes_K R \to R$. If R is generated as a K-algebra by elements x_i, then I is generated as an ideal of S by the elements $x_i \otimes 1 - 1 \otimes x_i$.

With this notation we see that the R-module corresponding to the sheaf $\mathcal{P}^m(\mathcal{L})$ can be written as

$$P^m(L) = (L \otimes_K R)/I^{m+1}(L \otimes_K R)$$

regarded as an R-module by the action $f \mapsto 1 \otimes r$ as above.

Part (a) now follows: If the K-rational point p corresponds to the maximal ideal

$$\mathfrak{m} = \mathrm{Ker}(R \xrightarrow{\varphi} K); \quad \varphi : x_i \mapsto a_i.$$

then in $R/\mathfrak{m} \otimes_R S \cong R$ the class of $x_i \otimes_K 1 - 1 \otimes_K x_i$ is $x_i \otimes_K 1 - 1 \otimes_K a_i = x_i - a_i$. Thus

$$R/\mathfrak{m} \otimes_R P^m(L) = L/(\{x_i \otimes_K 1 - 1 \otimes_K x_i\})^{m+1} L = L/(\{x_i - a_i\})^{m+1} L,$$

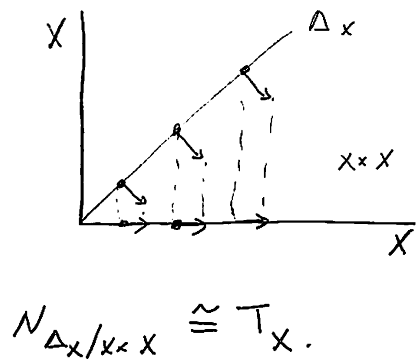

FIGURE 9.0.5. The normal bundle of the diagonal $\Delta_X \subset X \times X$ is isomorphic to the tangent bundle of X.

as required.

Part (b) is similarly obvious from this point of view: the section τ_F can be taken to be the image of the element $F \otimes_K 1$ in $(L \otimes_K R)/I^{m+1}(L \otimes_K R)$. As the construction is natural, these elements will glue to a global section when we are no longer in the affine case.

Part (c) requires another important idea: the module of K-linear differentials $\Omega_{R/K}$ is isomorphic, as an R-module, to I/I^2, which has universal derivation $\delta : R \to I/I^2$ given by $\delta(f) = f \otimes_K 1 - 1 \otimes_K f$. This is plausible, since when X is smooth one can see geometrically that the normal bundle of the diagonal, which is $\mathrm{Hom}(I/I^2, R)$, is isomorphic to the tangent bundle of X, which is $\mathrm{Hom}(\Omega_{R/K}, R)$. See Eisenbud [1995], Section 16.8 for further discussion and a general proof. Given this fact, the obvious surjection $\mathrm{Sym}^m(\Omega_{R/K}) \cong \mathrm{Sym}^m(I/I^2) \to I^m/I^{m+1}$ yields the desired right exact sequence.

Finally, it is enough to prove part (d) locally at a point $q \in X \times X$. If q is not on the diagonal then, after localizing, I is the unit ideal, and the result is trivial, so we may assume that $q = (p, p)$. Locally at p the module L is free, so it suffices to prove the result when $L = R$.

Write $d : R \to \Omega_{R/K}$ for the universal K-linear derivation of R. Since X is smooth, $\Omega_{R/K}$ is locally free at p, and is generated there by elements $d(x_1), \ldots, d(x_n)$, where $x_1, \ldots, x_n$ is a system of parameters at p, and thus $\mathrm{Sym}^m(\Omega_{R/K})$ is the free module generated by the monomials of degree m in the $d(x_i)$. Since R is a domain, I is a prime ideal.

Because $\mathrm{Sym}^m(\Omega_{R/K})$ is free, it suffices to show that the map

$$\mathrm{Sym}^m(\Omega_{R/K}) \to S/I^{m+1}$$

is a monomorphism after localizing at the prime ideal I. Since $I/I^2 \cong \Omega_{R/K}$ is free on the classes mod I^2 of the elements $x_i \otimes_K 1 - 1 \otimes_K x_i$ that correspond to the $d(x_i)$, Nakayama's Lemma shows in the local ring S_I, I_I is generated by the images of the $x_i \otimes_K 1 - 1 \otimes_K x_i$ themselves, and it follows that these are a regular sequence. Thus the associated graded ring $S_I/I_I \oplus I_I/I_I^2 \oplus \cdots$ is a polynomial ring on the classes of the elements $x_i \otimes_K 1 - 1 \otimes_K x_i$, and in particular the monomials of degree m in these elements freely generate I_I^m/I_I^{m+1}. Consequently, the map $S_I \otimes_S \mathrm{Sym}^m(\Omega_{R/K}) \to I_I^m/I_I^{m+1}$ is an isomorphism as desired. $\square$

Remark. The name "bundle of principle parts," first used by Grothendieck and Dieudonné, was presumably suggested by the (conflicting) usage that the "principal part" of a meromorphic function at a point is the sum of the terms of negative degree in the Laurent expansion of the function around the point—a finite power series, albeit in the inverse variables. It is not the only terminology in use: $\mathcal{P}^m(\mathcal{L})$ would be called the bundle of *m-jets of sections of $\mathcal{L}$* by those studying singularities of mappings (see for example Golubitsky and Guillemin [1973] II.2) and some algebraic geometers (for example Perkinson [1996].) On the other hand, the m-jet terminology is in use in another conflicting sense in algebraic geometry: the "scheme of m-jets" of a scheme X is used to denote the scheme parametrizing mappings from $\operatorname{Spec} K[x]/(x^{m+1})$ into X. So we have thought it best to stick to the Grothendieck-Dieudonné usage.

9.4 Singular elements of a pencil

9.4.1 From pencils to degeneracy loci

Using the bundle of principle parts we can tackle a slightly more general version of keynote question (a): how many linear combinations of general polynomials F and G of degree d on $\mathbb{P}^n$ have singular zero loci? By Proposition 9.1, none of the hypersurfaces $C_t = V(t_0F + t_1G)$ of the pencil will be singular at more than one point. Furthermore, no two elements of the pencil will be singular at the same point since otherwise every member of the pencil would be singular there. Thus, the keynote question is equivalent to the question: For how many points $p \in \mathbb{P}^n$ is some element C_t of the pencil singular at p? This, in turn, amounts to asking, at how many points $p \in \mathbb{P}^n$ are the values $\tau_F(p)$ and $\tau_G(p)$ in the fiber of $\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d))$ at p linearly dependent, given that they are dependent at finitely many points? We can do this with Chern classes, provided that the degeneracy locus is reduced.

To see that this locus is reduced, consider the behavior of the sections τ_F and τ_G around a point $p \in \mathbb{P}^n$ where they are dependent. At such a point, some linear combination $t_0F + t_1G$—which we might as well take to be F—vanishes to order 2. If G were also zero at p, then the scheme $V(F, G)$ would have (at least) a double point at p. But Bertini's Theorem shows that a general complete intersection such as $V(F, G)$ is reduced, so this cannot happen.

To show that $V(\tau_F \wedge \tau_G)$ is reduced at p, we restrict our attention to an affine neighborhood of p where all our bundles are trivial. By Proposition 9.1 the hypersurface $C = V(F)$ has an ordinary node at p, so if we work on an affine neighborhood where the bundle $\mathcal{O}_{\mathbb{P}^n}(d)$ is trivial, and take p to be the origin with respect to coordinates $x_1, \dots, x_n$ we may assume that the functions F and G have Taylor expansions at p of the form

$$f = f_2 + (\text{terms of order} > 2)$$
$$g = 1 + (\text{terms of order} \geq 1).$$

The sections τ_F and τ_G are then represented locally by the rows of the matrix

$$\begin{pmatrix} f & \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \\ g & \frac{\partial g}{\partial x} & \cdots & \frac{\partial g}{\partial x_n} \end{pmatrix}.$$

The vanishing locus of $\tau_F \wedge \tau_G$ near p is by definition defined by the 2×2 minors of this matrix, and to prove that it is a reduced point we need to see that it contains n functions (vanishing at p) with independent linear terms. Suppressing all the terms of the functions in the matrix that could not contribute to the linear terms of the minors, get the matrix

$$\begin{pmatrix} 0 & \partial f_2/\partial x_1 & \cdots & \partial f_2/\partial x_n \\ 1 & 0 & \cdots & 0. \end{pmatrix}.$$

Thus there are 2×2 minors whose linear terms are $\partial f_2/\partial x_1, \dots, \partial f_2/\partial x_n$, and these are linearly independent because $f_2 = 0$ is a smooth quadric and the characteristic is not 2.

As usual, if we assign multiplicities appropriately we can extend the calculations to pencils whose degeneracy locus $V(\tau_F \wedge \tau_G)$ is nonreduced. In Section 9.8.2 we'll see one way to calculate these multiplicities.

9.4.2 The Chern class of a bundle of principal parts

Once again, let F, G be general forms of degree d on $\mathbb{P}^n$. As we saw in the previous section, the linear combinations $t_0F + t_1G$ that are singular correspond exactly to points where the two sections τ_F and τ_G are dependent. The degeneracy locus of τ_F and τ_G is the n-th Chern class of the rank $n + 1$ bundle $\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d))$, so we turn to the computation of this class. For brevity, we'll shorten $\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d))$ to $\mathcal{P}^1(d)$, but the reader should keep in mind that this is *not* "a bundle $\mathcal{P}^1$ tensored with $\mathcal{O}(d)$""!

Stated explicitly, if $\zeta \in A^1(\mathbb{P}^n)$ denote the class of a hyperplane in $\mathbb{P}^n$, we want to compute the coefficient of ζ^n in

$$c\big(\mathcal{P}^1(d)\big) \in A(\mathbb{P}^n) \cong \mathbb{Z}[\zeta]/(\zeta^{n+1}).$$

Parts c) and d) of Theorem 9.4 give us a short exact sequence

$$0 \to \Omega_{\mathbb{P}^n}(d) \to \mathcal{P}^1(d) \to \mathcal{O}_{\mathbb{P}^n}(d) \to 0,$$

so $c\big(\mathcal{P}^1(d)\big) = c\big(\mathcal{O}_{\mathbb{P}^n}(d)\big) \cdot c\big(\Omega_{\mathbb{P}^n}(d)\big)$. (See Proposition 9.7 for the other $\mathcal{P}^m(d)$.) On the other hand, $\Omega_{\mathbb{P}^n}$ fits into a short exact sequence

$$0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0.$$

Tensoring with $\mathcal{O}(d)$ we get an exact sequence

$$0 \to \Omega_{\mathbb{P}^n}(d) \to \mathcal{O}_{\mathbb{P}^n}(d-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n}(d) \to 0$$

similar to the one involving $\mathcal{P}^1(d)$. This doesn't mean that $\mathcal{P}^1(d)$ and $\mathcal{O}_{\mathbb{P}^n}(d-1)^{n+1}$ are isomorphic (they are not), but by the Whitney formula (Proposition **??**) their Chern classes agree:

$$c\big(\mathcal{P}^1(d)\big) = c\big(\mathcal{O}_{\mathbb{P}^n}(d-1)^{n+1}\big) = \big(1 + (d-1)\zeta\big)^{n+1}.$$

Putting this formula together with the idea of the previous section we deduce:

Proposition 9.6. *The degree of the discriminant hypersurface in the space of forms of degree d on $\mathbb{P}^n$ is*

$$\deg c_n(\mathcal{P}^1(d)) = (n+1)(d-1)^n,$$

and this is the number of singular hypersurfaces in a general pencil of hypersurfaces of degree d in $\mathbb{P}^n$. □

In particular this answers keynote question (a): a general pencil of plane curves of degree d will have $3(d-1)^2$ singular elements.

It is pleasant to observe that the conclusion agrees with what we get from elementary geometry in the cases where it is easy to check, such as those of plane curves ($n = 2$) with $d = 1$ or $d = 2$. For $d = 1$ the statement $c_2 = 0$ simply means that there are no singular lines in a pencil of lines. The case $d = 2$ corresponds to the number of singular conics in a general pencil $\{C_t\}$ of conics. To see that this is really $3(d-1)^2 = 3$, note that the pencil $\{C_t\}$ consists of all conics passing through the four (distinct) base points, and a singular element of the pencil will thus be the union of a line joining two of the points with the line joining the other two. There are indeed three such pairs of pairs of lines (Fig. 9.1).

We could also get the number 3 by viewing the pencil of conics as given by a 3×3 symmetric matrix M of linear forms on $\mathbb{P}^2$ whose entries vary

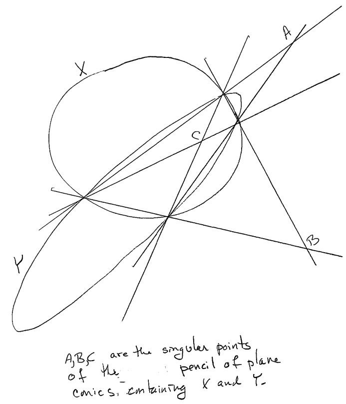

FIGURE 9.1. A, B, C are the singular points of the pencil of plane conics containing X and Y. [[SILVIO X and Y should be conics.]]

linearly with a parameter t; the determinant of M will then be a cubic polynomial in t.

There is a simpler way to arrive at the formula of Propositions9.6, at least when the characteristic of the ground field does not divide d. By the Euler relation

$$d \cdot F(Z_0, \ldots, Z_n) = \sum_{k=0}^{n} Z_k \frac{\partial F}{\partial Z_k},$$

the condition that the hypersurface $V(F)$ be singular at p is just that all the partial derivatives of F vanish at p. Given a pencil $\{X_t = V(t_0 F + t_1 G)\}$ of such hypersurfaces, the locus of pairs (t, p) with $p \in (X_t)_{\mathrm{sing}}$ is just the simultaneous zero locus of the $n+1$ polynomials

$$H_k(t, Z) = \frac{\partial}{\partial Z_k}(t_0 F + t_1 G).$$

Now, H_k is bihomogeneous of bidegree $(1, d-1)$, and so its zero locus is a hypersurface in $\mathbb{P}^1 \times \mathbb{P}^n$ with class $\zeta_{\mathbb{P}^1} + (d-1)\zeta_{\mathbb{P}^n}$. Assuming transversality, then, the number of common zeros of the $n+1$ polynomials H_k is the degree of the product

$$\big(\zeta_{\mathbb{P}^1} + (d-1)\zeta_{\mathbb{P}^n}\big)^{n+1} \in A(\mathbb{P}^1 \times \mathbb{P}^n) = \mathbb{Z}[\zeta_{\mathbb{P}^1}, \zeta_{\mathbb{P}^n}]/(\zeta_{\mathbb{P}^1}^2, \zeta_{\mathbb{P}^n}^{n+1}),$$

which is $(n+1)(d-1)^n$.

Why did we adopt the approach via principal parts, given this alternative? The answer is that, as we'll see in Section 9.5.2, the principal parts approach can be applied to linear series on arbitrary smooth varieties; the alternative we've just given applies only to projective space.

It is easy to extend the computation above to all the $\mathcal{P}^m(\mathcal{O}_{\mathbb{P}^n}(d))$, and this will be useful in the rest of this chapter:

Proposition 9.7. $c\big(\mathcal{P}^m(\mathcal{O}_{\mathbb{P}^n}(d))\big) = \big(1+(d-m)\zeta\big)^{\binom{n+m}{n}}$.

Proof. We will again use the exact sequences of Theorem 9.4. With the Whitney formula they immediately give

$$c\big(\mathcal{P}^m(\mathcal{O}_{\mathbb{P}^n}(d)\big) = \prod_{j=0}^{m} c\Big(\mathrm{Sym}^j(\Omega_{\mathbb{P}^n})(d)\Big).$$

To derive the formula we need, we apply Lemma 9.8 below to the exact sequence

$$0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0,$$

and the line bundle $\mathcal{L} := \mathcal{O}_{\mathbb{P}^n}(d)$. To simplify the notation, we set $\mathcal{U} = \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1}$. The Lemma yields

$$c\Big(\mathrm{Sym}^j(\Omega_{\mathbb{P}^n})(d)\Big) = c\Big(\mathrm{Sym}^j(\mathcal{U})(d)\Big)\cdot c\Big(\mathrm{Sym}^{j-1}(\mathcal{U})(d)\Big)^{-1},$$

for all $j \geq 1$. Combining this with the obvious $\mathrm{Sym}^0(\Omega_{\mathbb{P}^n})(d) = \mathcal{O}_{\mathbb{P}^n}(d)$, we see that the product in the formula for $c\big(\mathcal{P}^m(\mathcal{O}_{\mathbb{P}^n}(d)\big)$ is

$$c\Big(\mathcal{O}_{\mathbb{P}^n}(d)\Big)\cdot\frac{c\Big(\mathrm{Sym}^1(\mathcal{U})(d)\Big)}{c\Big(\mathcal{O}_{\mathbb{P}^n}(d)\Big)}\cdot\frac{c\Big(\mathrm{Sym}^2(\mathcal{U})(d)\Big)}{c\Big(\mathrm{Sym}^1(\mathcal{U})(d)\Big)}\cdots$$

which collapses to

$$c\Big(\mathcal{P}^m(\mathcal{O}_{\mathbb{P}^n}(d))\Big) = c\Big(\mathrm{Sym}^m(\mathcal{U})(d)\Big).$$

But

$$\begin{aligned} c\Big(\mathrm{Sym}^m(\mathcal{U})(d)\Big) &= \\ c\Big(\mathrm{Sym}^m(\mathcal{O}_{\mathbb{P}^n}(-1)^{n+1})(d)\Big) &= c\Big(\mathcal{O}_{\mathbb{P}^n}(-m)^{\binom{n+m}{n}}(d)\Big) \\ &= c\Big(\mathcal{O}_{\mathbb{P}^n}(d-m)^{\binom{n+m}{n}}\Big) \\ &= \big(1+(d-m)\zeta\big)^{\binom{n+m}{n}} \end{aligned}$$

yielding the formula of the Proposition. □

Lemma 9.8. *If*

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

is a short exact sequence of vector bundles on a projective variety X with $\operatorname{rank}\mathcal{C} = 1$*, then for any line bundle $\mathcal{L}$ on X and any $j \geq 1$,*

$$c\big(Sym_j(\mathcal{A})\otimes\mathcal{L}\big) = c\big(Sym_j(\mathcal{B})\otimes\mathcal{L}\big)\,\cdot\, c\big(Sym_{j-1}(\mathcal{B})\otimes\mathcal{C}\otimes\mathcal{L}\big)^{-1}.$$

Proof. For any right exact sequence of coherent sheaves

$$\mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

the universal property of the symmetric powers (see for example Eisenbud [1995], Proposition A.2.2.d) shows that there is for each $j \geq 1$ a right exact sequence

$$\mathcal{E} \otimes \mathrm{Sym}^{j-1}(\mathcal{F}) \to \mathrm{Sym}^j(\mathcal{F}) \to \mathrm{Sym}^j(\mathcal{G}) \to 0,$$

Since $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are vector bundles the dual of the exact sequence in the hypothesis is exact, and we may apply the result on symmetric powers with $\mathcal{G} = \mathcal{A}^*, \mathcal{F} = \mathcal{B}^*$ and $\mathcal{E} = \mathcal{C}^*$.

In this case, since $\operatorname{rank} \mathcal{E} = \operatorname{rank} \mathcal{C} = 1$, the sequence

$$0 \to \mathcal{E} \otimes \mathrm{Sym}^{j-1}(\mathcal{F}) \to \mathrm{Sym}^j(\mathcal{F}) \to \mathrm{Sym}^j(\mathcal{G}) \to 0$$

is left exact as well, as one sees immediately by comparing the ranks of the three terms (this is a special case of a longer exact sequence, independent of the rank of $\mathcal{E}$, derived from the Koszul complex.)

Since these are all bundles, dualizing preserves exactness, and we get an exact sequence

$$0 \to \mathrm{Sym}^j(\mathcal{A}^*)^* \to \mathrm{Sym}^j(\mathcal{B}^*)^* \to \mathrm{Sym}^{j-1}(\mathcal{B}^*)^* \otimes \mathcal{C} \to 0.$$

Of course the double dual of a bundle is the bundle itself, and in characteristic 0, or characteristic $p > j$, the dual of the j-th symmetric power is naturally isomorphic to the j-th symmetric power of the dual, so in such cases all the *s cancel. When this occurs, we need only tensor with $\mathcal{L}$ and we can deduce the Lemma from the Whitney formula.

However, when the characteristic of the ground is less than j the bundles $\mathrm{Sym}^j(\mathcal{A}^*)^*$ and $\mathrm{Sym}^j(\mathcal{A})$ are not in general isomorphic (thanks to Torsten Ekedahl and Tom Goodwillie for showing us examples. ********the examples aren't so accessible to this audience. A footnote?********) Nevertheless the corresponding equality of Chern classes is always true, and remains true if we tensor with a line bundle.

To prove this, suppose that $\mathcal{H}$ is any vector bundle (in any characteristic.) We wish to show that

$$c\Big((Sym_m(\mathcal{H}^*))^* \otimes \mathcal{L})\Big) = c\Big(Sym_m(\mathcal{H}) \otimes \mathcal{L})\Big).$$

we again use the splitting principle, by which it suffices to do the case where $\mathcal{H}$ has a filtration whose successive quotients are line bundles $\mathcal{L}_i$. In this case $\mathrm{Sym}^m(\mathcal{H}^*)$ has a filtration with successive quotients $\mathcal{L}_{i_1}^{-1} \otimes \cdots \otimes \mathcal{L}_{i_m}^{-1}$, so both $\mathrm{Sym}^m(\mathcal{H}^*)^* \otimes \mathcal{L}$ and $\mathrm{Sym}^m(\mathcal{H}^*) \otimes \mathcal{L}$ have filtrations with successive quotients $\mathcal{L}_{i_1} \otimes \cdots \otimes \mathcal{L}_{i_m} \otimes \mathcal{L}$. By the Whitney formula, $c\big(Sym_m(\mathcal{H}^*)^* \otimes \mathcal{L}\big)$ and $c\big(Sym_m(\mathcal{H}) \otimes \mathcal{L}\big)$ are equal. With the Whitney formula, and the fact that Chern classes are invertible elements of the Chow ring, this yields the conclusion of the Lemma. $\square$

9.4.3 Triple points of plane curves

We can adapt the preceding ideas to compute the number of points of higher order in linear families of hypersurfaces. By way of example we consider the case of triple points of plane curves.

Suppose that C is a curve of degree d defined by the equation $F = 0$. We say that F has a triple point at p if the Taylor expansion of the restriction f of F to an affine space of which p is the origin has the form

$$f_3 + f_4 + \dots$$

with f_i of degree i.

Let $\mathbb{P}^N$ be the projective space of all plane curves of degree d, and let

$$\Sigma' = \{(C, p) \mid C \text{ has (at least) a triple point at } p\}.$$

The condition that a curve have a triple point at a given point $p \in \mathbb{P}^2$ is six independent linear conditions on the coefficients of the defining equation of C, from which we see that the fibers of the projection map $\Sigma' \to \mathbb{P}^2$ on the second factor are linear spaces $\mathbb{P}^{N-6} \subset \mathbb{P}^N$, and hence that Σ' is irreducible of dimension $N-4$. It follows that the set of curves with a triple point is irreducible as well. An argument similar to that for double points also shows that a general curve $F = 0$ with a triple point has only one. In particular, the projection map $\Sigma' \to \mathbb{P}^N$ on the first factor is birational onto its image. It follows in turn that *the locus $\Phi \subset \mathbb{P}^N$ of curves possessing a point of multiplicity 3 or more is an irreducible variety of dimension $N-4$.* We also see that if C is a general curve with a triple point at p then the cubic term f_3 of the Taylor expansion defines a smooth variety (3 reduced points in $\mathbb{P}^1$)

We ask now for the degree of the variety of curves with a triple point, or, equivalently, the answer to the question: if $F_0, \dots, F_4$ are general polynomials of degree d on $\mathbb{P}^2$, for how many linear combinations $F_a = a_0F_0 + \cdots + a_4F_4$ (up to scalars) will the corresponding plane curve $C_a = V(F_a) \subset \mathbb{P}^2$ have a triple point?

If we write τ_F for the section defined by F in $\mathcal{P}^2(\mathcal{O}_{\mathbb{P}^2}(d))$ then C has a triple point at p if and only if τ_F vanishes at p. An argument analogous to the one given in Section 9.4.1, together with the smoothness of the tangent cone at a general triple point shows that the 5×5 minors of the map $\mathcal{O}_{\mathbb{P}^2}^5 \to \mathcal{P}^2(\mathcal{O}_{\mathbb{P}^2}(d))$ generate the maximal ideal locally at a generic point where a linear combination of the F_i defines a curve with a triple point.

Thus the number of triple points in the family is the degree of the second Chern class $c_2\big(\mathcal{P}^2(\mathcal{O}_{\mathbb{P}^2}(d))\big)$. By Proposition 9.7,

$$c\big(\mathcal{P}^2(\mathcal{O}_{\mathbb{P}^2}(d))\big) = (1 + (d-2)\zeta)^6 = 1 + 6(d-2)\zeta + 15(d^2 - 4d + 4)\zeta^2.$$

Proposition 9.9. *If $\Phi = \Psi_{d,n} \subset \mathbb{P}^N$ is the locus, in the space of all hypersurfaces of degree d in $\mathbb{P}^n$, of hypersurfaces having a point of multiplicity 3 or more, then for $d \geq 2$*

$$\deg(\Phi) = 15(d^2 - 4d + 4).$$

In case $d = 1$, the number 15 computed is of course meaningless, because the expected dimension $N - 4$ of Φ is negative—any five global sections τ_{F_i} of the bundle $\mathcal{P}^2(\mathcal{O}(1))$ are everywhere dependent. On the other hand, the number 0 computed in the case $d = 2$, which is 0, really does reflect the fact that no conics have a triple point. For $d = 3$ the computation above gives 15, a number we already computed as the degree of the locus of "asterisks" in Section 1.2.5.

9.4.4 Cones

As we remarked, the calculation in the preceding section is a generalization of the calculation, in Section 1.2.5, of the degree of the locus Φ parametrizing triples of concurrent lines ("asterisks") in the space $\mathbb{P}^9$ parametrizing plane cubic curves, There is another generalization of this problem: we can ask for the degree, in the space $\mathbb{P}^N$ parametrizing hypersurfaces of degree d in $\mathbb{P}^n$, of the locus Ψ of *cones*. We are now in a position to answer that more general problem, which we'll do here.

We won't go through the steps in detail, since they're exactly analogous to the last calculation; the upshot is that the degree of Ψ is the degree of the n^{th} Chern class of the bundle $\mathcal{P}^{d-1}(\mathcal{O}_{\mathbb{P}^n}(d))$. By Proposition 9.7,

$$c\big(\mathcal{P}^{d-1}(\mathcal{O}_{\mathbb{P}^n}(d))\big) = \big(1 + \zeta\big)^{\binom{n+d-1}{n}}.$$

and so we have the

Proposition 9.10. *If $\Psi = \Psi_{d,n} \subset \mathbb{P}^N$ is the locus of cones in the spave of all hypersurfaces of degree d in $\mathbb{P}^n$, then*

$$\deg(\Psi) = \binom{\binom{n+d-1}{n}}{n}.$$

Thus, for example, in case $d = 2$ we see again that the locus of singular quadrics in $\mathbb{P}^n$ is $n + 1$; and in case $d = 3$ and $n = 2$ the locus of asterisks has degree 15. Likewise, in the space $\mathbb{P}^{14}$ of quartic plane curves the locus of concurrent fourtuples of lines has degree

$$\binom{\binom{5}{2}}{2} = \binom{10}{2} = 45.$$

Compare this to the calculation in Exercises 1.93 and 1.94!

9.5 Singular elements of linear series in general

Let X be a smooth projective variety of dimension n and let $\mathcal{W} = (\mathcal{L}, W)$ be a linear system on X. We think of the elements of $\mathbb{P}W$ as divisors in X, and as in Section 9.1, we introduce the incidence correspondence

$$\Sigma_{\mathcal{W}} = \{(Y, p) \in \mathbb{P}W \times X \mid p \in Y_{\text{sing}}\}$$

with the projection maps $\pi_1 : \Sigma \to \mathbb{P}W$ and $\pi_2 : \Sigma \to X$. Also as in Section 9.1, we denote by $\mathcal{D} = \pi_1(\Sigma) \subset \mathbb{P}W$ the locus of singular elements of the linear series $\mathcal{W}$, which we again call the *discriminant*.

As mentioned in the introduction to this chapter, the techniques developed so far apply as well in this generality. What's missing is the analog of Proposition 9.1: we don't know in general that Σ is irreducible of codimension $n + 1$; we don't know that it maps birationally onto $\mathcal{D}$ (as we'll see more fully in Section 12.6, the discriminant $\mathcal{D}$ may have dimension strictly smaller than that of Σ) and we don't know that the general singular element of $\mathcal{W}$ has one ordinary double point as its singularity. Thus the formulas we derive in this generality are only *enumerative formulas*, in the sense of Section 2.1: they apply subject to the hypothesis that the loci in question do indeed have the expected dimension, and even then only if multiplicities are taken into account.

That said, we can still calculate the Chern classes of the bundle of principle parts $\mathcal{P}^1(\mathcal{L})$, and derive an enumerative formula for the number of singular elements of a pencil of divisors (that is, the degree of $\mathcal{D} \subset \mathbb{P}W$, in case $\mathcal{D}$ is indeed a hypersurface); we'll do this in Section 9.5.1 below.

We note one interpretation of $\mathcal{D}$ in case the linear series $\mathbb{P}W$ is very ample. When $X \subset \mathbb{P}^N$ is a smooth variety, and $\mathcal{W} = (\mathcal{O}_X(1), W = H^0(\mathcal{O}_{\mathbb{P}^N}(1))|_X)$ is the linear series of hyperplane sections of X then a section in W is singular if and only if the corresponding hyperplane is tangent to X. Thus the set of points in $\mathbb{P}W$ corresponding to such sections is the *dual variety* to X, and the number of singular elements in a general pencil of these sections is the degree of the dual variety. We will treat dual varieties carefully in Section 12.6.

9.5.1 *Number of singular elements of a pencil*

Let X be a smooth projective variety of dimension n, and $\mathcal{W} = (\mathcal{L}, W)$ a pencil of divisors on X (typically, a general pencil in a larger linear series). We can use the Chern class machinery to compute the expected number of singular elements of $\mathcal{W}$. To simplify the notation, we will denote the first Chern class of the line bundle $\mathcal{L}$ by $\lambda \in A^1(X)$, and the Chern classes of the cotangent bundle Ω_X of X simply by $c_1, c_2, \ldots, c_n$.

From the exact sequence

$$0 \to \Omega_X \otimes \mathcal{L} \to \mathcal{P}^1(\mathcal{L}) \to \mathcal{L} \to 0$$

and Whitney's Theorem, we see that the Chern class of $\mathcal{P}^1(\mathcal{L})$ is the Chern class of $\mathcal{L} \otimes (\mathcal{O}_X \oplus \Omega_X)$. Since $c_i(\mathcal{O}_X \oplus \Omega_X) = c_i(\Omega_X) = c_i$, we may apply the formula for the Chern class of a tensor product of a line bundle (Proposition 7.10) to arrive at

$$c_k(\mathcal{P}^1(\mathcal{L})) = \sum_{i=0}^{k} \binom{n+1-i}{k-i} \lambda^{k-i} c_i.$$

In particular,

$$\begin{aligned} c_n(\mathcal{P}^1(\mathcal{L})) &= \sum_{i=0}^{n} (n+1-i)\lambda^{k-i} c_i \\ &= (n+1)\lambda^n + n\lambda^{n-1}c_1 + \cdots + 2\lambda c_{n-1} + c_n. \end{aligned}$$

As remarked above, this represents only an enumerative formula for the number of singular elements of a pencil. But the calculations of Section 9.4.1 hold here as well: a singular element Y of a pencil corresponds to a reduced point of the relevant degeneracy locus if Y has just one ordinary double point as its singularity. Thus we have the following

Proposition 9.11. *Let X be a smooth projective variety of dimension n. If $\mathcal{W} = (\mathcal{L}, W)$ is a pencil of divisors on X such that each singular element of $\mathcal{W}$ has just one ordinary double point not contained in the base locus of the pencil, then the number of singular elements is the degree of the class*

$$\gamma(\mathcal{L}) = (n+1)\lambda^n + n\lambda^{n-1}c_1 + \cdots + 2\lambda c_{n-1} + c_n \in A^n(X),$$

where $\lambda = c_1(\mathcal{L})$ and $c_i = c_i(\Omega_X)$.

Three remarks about this statement. First, if the degree in question is negative, we may conclude that every element of the pencil is singular. Moreover, if the class $(n+1)\lambda^n + n\lambda^{n-1}c_1 + \cdots + 2\lambda c_{n-1} + c_n \neq 0$, then any such pencil must have singular elements. Thus if $\mathcal{W}$ is a general pencil of hyperplane sections of a projectively embedded $X \subset \mathbb{P}^N$, the degree of $\gamma(\mathcal{L})$ must be non-negative; and if it's nonzero, then the dual variety X^* of X is necessarily a hypersurface.

Finally, we will see in Section 9.8.2 below a way of calculating multiplicities of the relevant degeneracy locus topologically, so that even in case the singular elements of $\mathcal{W}$ don't satisfy the hypothesis of having only one ordinary double point we can say something about the number of singular elements. (The conclusions of Section 9.8.2 are stated only for pencils of curves on a surface, but analogous statements hold in higher dimension as well.)

9.5.2 Pencils of curves on a surface

By way of an example, we'll apply the results of Proposition 9.11 to pencils of curves on surfaces.

Suppose that $X \subset \mathbb{P}^3$ is a smooth surface of degree d and that W is the linear series of intersections of X with hypersurfaces of degree e, so that $\mathcal{L} = \mathcal{O}_X(e)$. Letting $\zeta \in A^1(X)$ denote the restriction of the hyperplane class, we have $c_1(\mathcal{L}) = e\zeta$; and as we've seen,

$$\begin{aligned} c(T_X) &= \frac{c(T_{\mathbb{P}^3})}{c(N_{X/\mathbb{P}^3})} \\ &= \frac{(1+\zeta)^4}{1+d\zeta} \\ &= (1+4\zeta+6\zeta^2)(1-d\zeta+d^2\zeta^2) \\ &= 1+(4-d)\zeta+(d^2-4d+6)\zeta^2. \end{aligned}$$

Thus $c_1 = (d-4)\zeta$ and $c_2 = (d^2-4d+6)\zeta^2$. From formula (**??**) above we see that

$$c_2(\mathcal{P}(\mathcal{L})) = (3e^2+2(d-4)e+d^2-4d+6)\zeta^2.$$

Finally, since $\deg(\zeta^2) = d$, the number of singular elements in a general pencil of plane sections of a smooth surface $X \subset \mathbb{P}^3$ of degree d is

$$\deg c_2(\mathcal{P}(\mathcal{L})) = d(3e^2+2(d-4)e+d^2-4d+6). \tag{9.1}$$

As explained above, this will be the degree of the dual surface of the e-th Veronese image $\nu_3(X)$ of X. For example, when $e = 1$ this reduces to

$$\deg X^* = d(d-1)^2,$$

as calculated in Section 1.2.2.

When $e = 2$ we are computing the expected number singular points in the intersection of X with a general pencil $\{Q_t \subset \mathbb{P}^3\}_{t\in\mathbb{P}^1}$ of quadric surfaces in $\mathbb{P}^3$, and we find that it is equal to

$$d^3+2d.$$

The reader should check the case $d = 1$ directly! We invite the reader to work out some more examples, and to derive analogous formulas in higher (and lower!) dimensions, in Exercises 9.22-9.29.

9.6 Inflection points of curves in $\mathbb{P}^n$

The bundles of principal parts are very useful for studying maps of curves to projective space. The connection with "singular elements of linear series" comes from the fact that a plane in projective space is tangent to a curve if and only if its intersection with the curve—an element of the linear system corresponding to the embedding—is singular. If the plane meets the curve with a higher degree of tangency—think of the tangent line at a flex point of a plane curve—then that will be reflected in a higher order singularity. Thus the technique we developed in Section **??** will allow us to solve the third of the keynote questions of this chapter: how to extend the notion of flexes to curves in $\mathbb{P}^n$, and how to count them.

Recall that if $C \subset X$ is a reduced curve on a scheme X and $D \subset X$ an effective Cartier divisor not containing any component of C, then for any closed point $p \in D \cap C$ we defined the multimplicity of intersection of C with D at p to be the length (or the dimension over the ground field K, which are the same since we are supposing that K is algebraically closed) of $\mathcal{O}_{C,p}/\mathcal{I}(D) \cdot \mathcal{O}_{C,p}$. Thus for example when $p \notin C \cap D$ the multiplicity is 0, and the multiplicity is 1 if and only if C and D are both smooth at p and meet transversely there.

For the purpose of this chapter it is convenient to expand this notion. Suppose that C is a smooth curve, $f : C \to X$ is a morphism and D is any subscheme of X such that $f^{-1}(D \cap C)$ is a finite scheme. We define the *order of contact* of D with C at $p \in C$ to be

$$\mathrm{ord}_p f^{-1}(D) := \dim_{\kappa(p)} \mathcal{O}_{C,p}/f^*(\mathcal{I}(D)).$$

Since we have assumed that C is smooth, the local ring $\mathcal{O}_{C,p}$ is a discrete valuation ring, so $\mathrm{ord}_p f^{-1}(D)$ is the minimum of the lengths of the algebras $\mathcal{O}_{C,p}/f^*(g)$, where g ranges over the local sections of $\mathcal{I}(D)$ at p, or over the generators of this ideal.

If f is the inclusion map of a smooth curve $C \subset X = \mathbb{P}^r$, and $D = \Lambda \subset \mathbb{P}^r$ is a linear subspace, then the order of contact $\mathrm{ord}_p f^{-1}(\Lambda)$ is the minimum, over the set of hyperplanes H containing Λ, of the intersection multiplicity $m_p(C, H)$. (Note that the intersection multiplicity of a curve with a subscheme of codimension at least 2 (for example the *intersection multiplicity* of a curve and a line in 3-space) is always 0; the order of contact is a very different quantity.)

For example, if $p \in C \subset \mathbb{P}^2$ is a smooth point of a plane curve and $L \supset p$ is any line through p then the order of contact of L with C at p is at least 1; L is tangent to C at p if and only if it is at least 2. The line L is called a *flex tangent* if the order is at least 3, and in this case p is called a *flex* of C. Carrying this further, we say that p is a *hyperflex* if the tangent line L at p meets C with order ≥ 4. We adopt similar definitions in the situation where $f : C \to \mathbb{P}^2$ is a nonconstant morphism from a smooth curve. For a curve in 3-space we can consider both the orders of contact with lines and the orders of contact with hyperplanes.

9.6.1 Vanishing sequences and osculating planes

We will systematize these ideas by considering a linear system $\mathcal{W} := (\mathcal{L}, W)$ on a smooth curve C with $\dim_K W = r + 1$. Given a point $p \in C$ and a section $\sigma \in W$, the order of vanishing $\mathrm{ord}_p\sigma$ of σ at p is defined to be the length of the $\mathcal{O}_{C,p}$-module $\mathcal{L}_p/(\mathcal{O}_{C,p}\sigma)$. Again, because the ground field K is algebraically closed we have $\kappa(p) = K$, so

$$\mathrm{ord}_p\sigma = \dim_K \mathcal{L}_p/(\mathcal{O}_{C,p}\sigma).$$

Given $p \in C$, consider the collection of all orders of vanishing of sections $\sigma \in W$ at p.

We define the *vanishing sequence* $a(\mathcal{W}, p)$ of the linear system $\mathcal{W}$ at p to be the sequence of integers that occur as orders of vanishing at p of sections in W, arranged in strictly increasing order:

$$a(\mathcal{W}, p) := (a_0(\mathcal{W}, p) < a_1(\mathcal{W}, p) < \cdots),$$

Since sections vanishing to distinct orders are linearly independent, $a(\mathcal{W}, p)$ has at most $\dim_K W$ elements. On the other hand, we can find a basis for W consisting of sections vanishing to distinct orders at p (start with any basis; if two sections vanish to the same order replace one with a linear combination of the two vanishing to higher order, and repeat.). It follows that the number of elements in $a(\mathcal{W}, p)$ is exactly $\dim_K W = r + 1$,

$$a(\mathcal{W}, p) = (a_0(\mathcal{W}, p) < \cdots < a_r(\mathcal{W}, p)).$$

We call the associated weakly increasing sequence $(\alpha_0(\mathcal{W}, \ldots, \alpha_r)$ with $\alpha_i = a_i - i$ the *ramification sequence* of $\mathcal{W}$ at p. When the linear system $\mathcal{W}$ or the point p we are referrring to is clear from context, we will drop it from the notation and write $a_i(p)$ or just a_i in place of $a_i(\mathcal{W}, p)$, and similarly for α_i.

For example, p is a base point of $\mathcal{W}$ if and only if $a_0(p) = \alpha_0(p) > 0$, and more generally $a_0(p)$ is the order to which p occurs in the base locus of $\mathcal{W}$. If p is a base point of $\mathcal{W}$ then, since C is a smooth curve, we may remove it; that is, W is in the image of the monomorphism $H^0(\mathcal{L}(-a_0p)) \to H^0(\mathcal{L})$, and we may thus consider W as defining a linear series $\mathcal{W}' := (\mathcal{L}(-a_0p), W)$. Thus most questions about linear systems on smooth curves can be reduced to the basepoint-free case.

When p is not a base point of $\mathcal{W}$, so that $\mathcal{W}$ defines a map $f : C \to \mathbb{P}^r$ that is a morphism in a neighborhood of p, we have $a_1(p) = 1$ if and only if f is an embedding near p. If $r = 2$ and $\mathcal{W}$ is very ample, so that f is an embedding, we thus have $\alpha_0(p) = \alpha_1(p) = 0$ for all p and $\alpha_2(p) > 0$ for some particular p if and only if there is a line meeting the embedded curve with multiplicity > 2 at p; that is, p is an inflection point of the embedded curve. The geometric meaning of the vanishing sequence is given in general by the next result.

Proposition 9.12. *Let $\mathcal{W} = (\mathcal{L}, W)$ be a linear series on a smooth curve C, and let $p \in C$. If p is not a basepoint of $\mathcal{W}$ we let $\mathcal{W}' = \mathcal{W}$; in general, let $\mathcal{W}' = (\mathcal{L}(-a_0(\mathcal{W}, p)), W)$.*

(a) $a_i(\mathcal{W}', p) = a_i(\mathcal{W}, p) - a_0(\mathcal{W}, p)$.

(b) *Choose $\sigma_0, \dots, \sigma_r \in W$ such that σ_j vanishes at p to order $a_j(\mathcal{W}', p)$, and let H_j be the hyperplane in $\mathbb{P}(W^*)$ corresponding to σ_j. The plane*

$$L_i = H_{i+1} \cap \cdots \cap H_r$$

is the unique linear subspace of dimension i with highest order of contact with C at p, and that order is $a_{i+1}(\mathcal{W}, p)$.

The planes L_i are called the *osculating planes* to $f(C)$ at p. We always have $L_0 = p$. If $f(C)$ is smooth at $f(p)$ then L_1 is the tangent line, and in general it is the tangent cone to the branch of $f(C)$ that is the image of an analytic neighborhood of $p \in C$.

Proof. Part (a): A section $(\mathcal{L}(-d)$ that vanishes to order m as a section of $(\mathcal{L}(-d)$ will vanish to order $m + d$ as a section of $\mathcal{L}$.

Part(b): Writing f for the germ at p of the morphism defined by $\mathcal{W}'$, It follows from the definitions that $\mathrm{ord}_p L_i = a_{i+1}$. If there were an i-plane L' with higher order of contact, and we write

$$L' = H'_{i+1} \cap \cdots \cap H'_r$$

for some hyperplanes H'_r, then each H'_r would have order of contact with C at p strictly greater than a_{i+1}. But these would correspond to independent sections in W, and taking linear combinations of these sections we would get $r - i$ sections with vanishing orders at p strictly greater than a_{i+1}. This contradicts the assumption that the highest $r - i$ elements of the vanishing sequence are $a_{i+1}, \dots, a_r$. □

9.6.2 Total inflection: the Plücker formula

We say that p is an *inflection point* for a linear system $(\mathcal{L}, W)$ of dimension r if the ramification sequence $(\alpha_0, \dots, \alpha_r)$ is not $(0, \dots, 0)$, or equivalently, if $\alpha_r > 0$, which is the same as $a_r > r$. In case $\mathcal{W}$ arises from a morphism $f : C \to \mathbb{P}^r$ that is an embedding near p, the number $a_i(W, p)$ can be computed from the orders of contact with which planes meet $f(C)$ at $q = f(p)$: the equations of an i-plane L pulls back to a vector space of $r - i$ independent sections of W, and the order $\mathrm{ord}_p(f^{-1}L)$ is the minimum of the orders of vanishing of these sections at p. Thus for $i \geq 1$,

$$a_i = \max\{\mathrm{ord}_p f^{-1} L \mid L \subset \mathbb{P}^r \text{ is a plane of dimension } i - 1\}.$$

In particular, p is an inflection point of $\mathcal{W}$ if and only if some hyperplane has contact $\geq r+1$ at p.

We define the *weight* of $p \in C$ with respect to $\mathcal{W}$ to be

$$w(\mathcal{W}, p) := \sum_{i=0}^{r} \alpha_i.$$

This number is a measure of what might be called the "total inflection" of $\mathcal{W}$ at p.

In characteristic 0 we can compute the sum $\sum_{p \in C} w(\mathcal{W}, p)$ as a Chern class of the bundle of principal parts of $\mathcal{L}$.

Theorem 9.13 (Plücker Formula)**.** *Let C be a smooth projective curve of genus g over an algebraically closed field of characteristic 0. If $\mathcal{W}$ is a linear system of degree d and dimension r on C then*

$$\sum_{p \in C} w(W, p) = (r+1)d + (r+1)r(g-1).$$

The result is false in positive characteristic because of the possible presence of wild ramification and inseparable maps. For a simple example, consider the curve $y - x^p = 0$ in the affine x, y-plane over a field of characteristic p. The tangent line at the point (a, a^p) is $y - a^p = 0$, and this meets the curve in a p-fold point; thus every point of the curve is a flex point, and the sum in the Plücker formula is not even finite. See for example Laksov [1984] and Osserman [2006] for more information.

Proof. The key observation is that both sides of the desired formula are equal to the degree of the first Chern class of the bundle $\mathcal{P}^r(\mathcal{L})$. We can compute the class of this bundle from Theorem 9.4 as

$$c\big(\mathcal{P}^r(\mathcal{L})\big) = \prod_{j=0}^{r} c\big(\mathrm{Sym}^j(\Omega_C) \otimes \mathcal{L}\big).$$

Since Ω_C is a line bundle we have $\mathrm{Sym}^j(\Omega_C) = \Omega_C^j$, and thus $c((\mathrm{Sym}^j(\Omega_C) \otimes \mathcal{L}) = 1 + jc_1(\Omega_C) + c_1(\mathcal{L})$. It follows that

$$c_1\big(\mathcal{P}^r(\mathcal{L})\big) = (r+1)c_1(\mathcal{L}) + \binom{r+1}{2} c_1(\Omega_C).$$

Since the degree of Ω_C is $2g-2$, the degree of this class is

$$\deg c_1\big(\mathcal{P}^r(\mathcal{L})\big) = (r+1)d + (r+1)r(g-1),$$

the right hand side of the Plücker formula.

We may define a map $\varphi : \mathcal{O}_C^{r+1} \to \mathcal{P}^r(\mathcal{L})$ by choosing any basis $\sigma_0, \cdots, \sigma_r$ of W and sending the i-th basis element of $\mathcal{O}_C^{r+1}$ to τ_{σ_i}. We will complete the proof by showing that for any point $p \in C$ the determinant of the map

φ vanishes at p to order exactly $w(W,p)$, and that there are only finitely many points $w(W,p)$ where the determinant is nonzero.

To this end fix a point $p \in C$. Since the determinant of φ depends on the choice of basis $\sigma_0, \dots, \sigma_r$ only up to scalar, we may choose the basis σ_i so that the order of vanishing $\operatorname{ord}_p(\sigma_i) = a_i$ at p is $a_i(W,p)$. Trivializing $\mathcal{L}$ in a neighborhood of p, we may think of the section σ_i locally as a function, and φ is represented by the matrix

$$\begin{pmatrix} \sigma_0 & \sigma_1 & \dots & \sigma_r \\ \sigma_0' & \sigma_1' & \dots & \sigma_r' \\ \vdots & \vdots & & \vdots \\ \sigma_0^{(r)} & \sigma_1^{(r)} & \dots & \sigma_r^{(r)} \end{pmatrix}.$$

where σ_i' denotes the derivative, and $\sigma_i^{(r)}$ the r-th derivative. Because σ_i vanishes to order $\geq i$ p, the matrix evaluated at p is lower triangular, and the entries on the diagonal are all nonzero if and only if $a_i = i$ for each i; that is, if and only if p is not an inflection point for W.

We can compute the exact order of vanishing of $\det \varphi$ at an inflection point as follows. Denote by $v(z)$ the $(r+1)$-vector $(\sigma_0(z), \dots, \sigma_r(z))$, so that the determinant of φ is the wedge product

$$\det(\varphi) = v \wedge v' \wedge \cdots \wedge v^{(r)}.$$

Applying the product rule, the n^{th} derivative of $\det(\varphi)$ is then a linear combination of terms of the form

$$v^{(\beta_0)} \wedge v^{(\beta_1+1)} \wedge \cdots \wedge v^{(\beta_r+r)}$$

with $\sum \beta_i = n$. Now, $v^{(\beta_0)}(p) = 0$ unless $\beta_0 \geq \alpha_0$; similarly, $v^{(\beta_0)}(p) \wedge v^{(\beta_1)}(p) = 0$ unless $\beta_0 + \beta_1 \geq \alpha_0 + \alpha_1$, and so on. We conclude that *any derivative of* $\det(\varphi)$ *of order less than* $w = \sum \alpha_i$ *vanishes at* p; and the expression for the w^{th} derivative of $\det(\varphi)$ has exactly one term nonzero at p, namely

$$v^{(\alpha_0)} \wedge v^{(\alpha_1+1)} \wedge \cdots \wedge v^{(\alpha_r+r)}.$$

Since this term appears with nonzero coefficient (we have used characteristic 0 here), we conclude that $\det(\varphi)$ vanishes to order exactly w at p.

It remains to show that not every point of C can be an inflection point for W—that is, that $\det \varphi$ is not identically zero. To prove this, suppose that $\det(\varphi)$ does vanish identically, that is, that

$$v \wedge v' \wedge \cdots \wedge v^{(k)} \equiv 0 \tag{9.2}$$

for some $k \leq r$. Suppose in addition that k is the smallest such integer, so that at a general point $p \in C$ we have

$$v(p) \wedge v'(p) \wedge \cdots \wedge v^{(k-1)}(p) \neq 0;$$

in other words, $v(p), \dots, v^{(k-1)}(p)$ are linearly independent, but $v^{(k)}(p)$ lies in their span Λ. Again using the product rule to differentiate the expression 9.2, we see that

$$\frac{d}{dz}\big(v \wedge v' \wedge \cdots \wedge v^{(k-1)} \wedge v^{(k)}\big) = v \wedge v' \wedge \cdots \wedge v^{(k-1)} \wedge v^{(k+1)} \equiv 0;$$

so that $v^{(k+1)}(p)$ also lies in the span of $v(p), \dots, v^{(k-1)}(p)$. Similarly, taking the second derivative of 9.2, we see that

$$\frac{d^2}{dz^2}\big(v \wedge v' \wedge \cdots \wedge v^{(k-1)} \wedge v^{(k)}\big) = v \wedge v' \wedge \cdots \wedge v^{(k-1)} \wedge v^{(k+2)} \equiv 0,$$

where are all the other terms in the derivative are zero because they are $(k+1)$-fold wedge products of vectors lying in a k-dimensional space. Continuing in this way, we see that $v^{(m)}(p) \in \Lambda$ for all m. Since we are in characteristic 0 it follows by integration that $v(z) \in \Lambda$ for all z. This implies that the linear system W has dimension $k < r+1$, contradicting our assumptions. □

Flexes of plane curves. The Plücker formula gives the answer to keynote question (c). We don't even need to assume C is smooth; if C is singular, as long as it's reduced and irreducible we view it as the image of the map $\nu : \tilde{C} \to \mathbb{P}^r$ from its normalization. If we apply the Plücker formula to this linear system corresponding to this map, we see that C has

$$(r+1)d + r(r+1)(g-1) = 3d + 6g - 6$$

flexes, where g is the genus of $\tilde{C}$, that is, geometric genus of C. If the curve C is indeed smooth, then $2g - 2 = d(d-3)$, and so this yields

$$3d + 6g - 6 = 3d + 3d(d-3) = 3d(d-2).$$

To be explicit, this formula counts points $p \in \tilde{C}$ such that for some line $L \subset \mathbb{P}^2$, the multiplicity of the pullback divisor $\nu^* L$ at p is at least 3. In particular:

(a) It does not necessarily count nodes of C, even though at a node p of C, there will be lines having intersection multiplicity 3 or more with C at p.

(b) It does count singularities where the differential $d\nu$ vanishes, for example cusps.

Some applications of the general Plücker formula will be given in Exercises 9.43-9.45

We should mention that there is an alternative notion of a flex point of a (possibly singular) curve $C \subset \mathbb{P}^2$: a point $p \in C$ such that for some line $L \subset \mathbb{P}^2$ through p, we have

$$m_p(C \cdot L) \geq 3.$$

In this sense, a node p of a plane curve C is a flex point, since the tangent lines to the branches of the curve at the node will have intersection multiplicity at least 3 with C at p. (The Plčker formula, applied to the linear system $\nu^* H^0(\mathcal{O}_{\mathbb{P}^2}(1))$ on the normalization $\nu : \tilde{C} \to C \subset \mathbb{P}^2$, will not in general count a node as a flex.) When we want to talk about flexes in this sense, we'll refer to them as *Cartesian* flexes, since they are defined in terms of the defining equation of $C \subset \mathbb{P}^2$ rather than its parametrization by a smooth curve.

There is a classical way to calculate the number of flexes of a plane curve that does count Cartesian flexes. Briefly, if C is the zero locus of a homogeneous polynomial $F(X,Y,Z)$, we define the *Hessian* of C be be the zero locus of the polynomial

$$H = \begin{vmatrix} \frac{\partial^2 F}{\partial X^2} & \frac{\partial^2 F}{\partial X \partial Y} & \frac{\partial^2 F}{\partial X \partial Z} \\ \frac{\partial^2 F}{\partial X \partial Y} & \frac{\partial^2 F}{\partial Y^2} & \frac{\partial^2 F}{\partial Y \partial Z} \\ \frac{\partial^2 F}{\partial X \partial Z} & \frac{\partial^2 F}{\partial Y \partial Z} & \frac{\partial^2 F}{\partial Z^2} \end{vmatrix}$$

For a smooth plane curve C, the flexes are exactly the points of intersection of C with its Hessian, even counting multiplicity. This (and the behavior of the Hessian when C is singular) will be explored in Exercises 9.46-9.48; we'll also see, in Exercise 9.41, what happens to the flexes on a smooth plane curve when it acquires a node.

Hyperflexes. First, the bad news: we're not going to answer keynote question (c) here. The question itself is well-posed: we know that a general plane curve $C \subset \mathbb{P}^2$ of degree $d \geq 4$ has only ordinary flexes, and it's not hard to see that the locus of those curves that do is a hypersurface in the space $\mathbb{P}^N$ of all such curves (see Exercise 9.49). Surely the techniques we've employed in this chapter will enable us to calculate the degree of that hypersurface? Unfortunately, they don't; and indeed the reason we included Question (c) is so that we could point out the problem.

Very much by analogy with the analysis of lines on surfaces and singular points on curves, we'd like to determine the class of the "universal hyperflex:" that is, in the universal curve

$$\Phi = \{(C,p) \in \mathbb{P}^N \times \mathbb{P}^2 \mid p \in C\},$$

the locus

$$\Gamma = \{(C,p) \in \Phi \mid \ p \text{ is a hyperflex of } C \ \}.$$

Moreover, it seems as if this would be amenable to a Chern class approach: we'd define a vector bundle E on Φ whose fiber at a point $(C,p) \in \Phi$ would be the vector space

$$E_{(C,p)} = \frac{\{\text{germs of sections of } \mathcal{O}_C(1) \text{ at } p\}}{\{\text{germs vanishing to order} \geq 4 \text{ at } p\}}.$$

We would then have a map of vector bundles on Φ from the trivial bundle with fiber $H^0(\mathcal{O}_{\mathbb{P}^2}(1))$ to E; and the degeneracy locus of this map would be the universal hyperflex Γ. We could conclude then that

$$[\Gamma] = c_2(E).$$

As we indicated, though, there is a problem with this approach. The description above of the fibers of E makes sense *as long as p is a smooth point of C, but not otherwise*. Reflecting this fact, if we were to try to define E by taking $\Delta \subset \Phi \times_{\mathbb{P}^N} \Phi$ the diagonal and setting

$$E = \pi_{1*}\big(\pi_2^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{O}_{\Phi \times_{\mathbb{P}^N} \Phi}/\mathcal{I}_\Delta^4\big),$$

the bundle E would have fiber as desired over the open set of (C, p) with C smooth at p, but would not even be locally free on the complement. The fact that bundles of principal parts do not behave well in families (except, of course, smooth families) is a real obstruction to carrying out this sort of calculation; and while there are often other approaches, it would be very desirable to have a theory of characteristic classes that would work in this setting.

And now, the good news: there is another way to approach keynote question (c), and we will explain it in Section **??**.

9.6.3 The situation in higher dimension

Is there an analogue of the Plücker formula for linear series on varieties of dimension greater than 1? Assuming that the linear series yields an embedding $X \subset \mathbb{P}^r$ we might ask, for a start, what sort of singularities we should expect the intersection $X \cap \Lambda$ of X with linear spaces $\Lambda \subset \mathbb{P}^r$ of a given dimension to have at a point p, and ask for the locus of points that are "exceptional" in this sense.

We do not know satisfying answers to these questions. One issue is that, while the singularities of subschemes of a smooth curve are simply classified by their multiplicity, there is already a tremendous variety of singularities of subschemes of surfaces. Another is that the analogue of the the final step in the proof of Theorem 9.13—showing that not every point on a smooth curve $C \subset \mathbb{P}^r$ can be an inflectionary point—is false in general. By way of illustrating this last point, we mention first (without proof) a positive result, analogous to the statement that a general point of a smooth curve $C \subset \mathbb{P}^2$ is not a flex:

Proposition 9.14. *If $S \subset \mathbb{P}^3$ is any smooth surface of degree $d \geq 2$, $p \in S$ a general point and $H = \mathbb{T}_p S \subset \mathbb{P}^3$ the tangent plane to S at p, then the intersection $H \cap S$ has an ordinary double point at p.*

We invite the reader to prove this for a general surface $S \subset \mathbb{P}^3$ (Exercise 9.51); to prove it for an arbitrary smooth $S \subset \mathbb{P}^3$ requires the use of the second fundamental form of S, as described in Harris [1992] and Griffiths and Harris [1979].

On the other hand, a dimension count might lead us to expect that for a general point p on a smooth, nondegenerate surface $S \subset \mathbb{P}^5$, no hyperplane $H \subset \mathbb{P}^5$ intersects S in a curve $C = H \cap S$ with a triple point at p; but there are such surfaces for which this is false (see Exercises 9.31-9.32); we do not know a classification of such surfaces.

We'll revisit this question in Chapter **??**, where we'll describe the behavior of plane sections of a general surface $S \subset \mathbb{P}^3$.

9.7 Nets of plane curves

We now want to consider larger families of curves, and in particular to answer the second keynote question of this chapter. A key step will be to compute the class of the universal singular point $\Sigma = \{(C, p) \mid p \in C_{\text{sing}}\}$ as a subvariety of $\mathbb{P}^N \times \mathbb{P}^2$ where $\mathbb{P}^N = \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^2}(d))$.

9.7.1 Class of the universal singular point

We can easily compute the class of

$$\begin{aligned}\Sigma_{n,d,m} = \{(H, p) \mid & H \subset \mathbb{P}^n \text{ a hypersurface;} \\ & p \in H \text{ a singular point of order } \geq m\}\end{aligned}$$

as a class in $\mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \times \mathbb{P}^n$. To simplify the notation, we set $W := H^0(\mathcal{O}_{\mathbb{P}^n}(d))$. Let

$$\begin{tikzcd} \mathbb{P}W \times \mathbb{P}^n \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \mathbb{P}^n \\ \mathbb{P}W & \end{tikzcd}$$

be the projection maps.

Proposition 9.15. $\Sigma_{n,d,m}$ *is the zero locus of a section of the vector bundle*

$$\mathcal{P}^m := \pi_1^*\mathcal{O}_{\mathbb{P}W}(1) \otimes \pi_2^*\mathcal{P}^m\big(\mathcal{O}_{\mathbb{P}^n}(d)\big)$$

which has Chern class

$$c(\mathcal{P}^m) = \big(1 + (d-m)\zeta_n + \zeta_W\big)^{\binom{n+m}{n}},$$

where ζ_n and ζ_W are the pullbacks of the hyperplane classes on $\mathbb{P}^n$ and $\mathbb{P}W$ respectively. Thus the class of $\Sigma_{n,d,m}$ in $A(\mathbb{P}W \times \mathbb{P}^n)$ is the sum of the terms of total degree $\binom{n+m}{n}$ in this expression. For example in the case $n = 2, m = 1$ this is

$$[\Sigma] = \zeta_W^3 + 3(d-1)\zeta_2\zeta_W^2 + 3(d-1)^2\zeta_2^2\zeta_W \in A^3(\mathbb{P}W \times \mathbb{P}^2).$$

Proof. The computation is similar to the one used in the calculation of the class of the universal line in Section 8.6. Since every polynomial $F \in W$ defines a section τ_F of $\mathcal{P}^m(\mathcal{O}_{\mathbb{P}^n}(d))$, we have a map

$$W \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{P}^m(\mathcal{O}_{\mathbb{P}^n}(d))$$

of vector bundles on $\mathbb{P}^n$. Likewise, we have the tautological inclusion

$$\mathcal{O}_{\mathbb{P}W}(-1) \to W \otimes \mathcal{O}_{\mathbb{P}W}$$

on $\mathbb{P}W$. We pull these maps back to the product $\mathbb{P}W \times \mathbb{P}^2$ and compose them to arrive at a map

$$\pi_1^*\mathcal{O}_{\mathbb{P}W}(-1) \to \pi_2^*\mathcal{P}^m(\mathcal{O}_{\mathbb{P}^n}(d)),$$

or equivalently, a section of the bundle $\mathcal{P}^m$. The zero locus of this map is $\Sigma \subset \mathbb{P}W \times \mathbb{P}^n$, so the class of $\Sigma_{n,d,m}$ in $A(\mathbb{P}W \times \mathbb{P}^n)$ is the class of a section of $\mathcal{P}^m$ as claimed.

To compute the Chern class of $\mathcal{P}^m$ we follow the argument of Proposition 9.7, pulling back the sequences

$$0 \to \mathrm{Sym}^i(\Omega_{\mathbb{P}^n})(d) \to \mathcal{P}^i(\mathcal{O}_{\mathbb{P}^n}(d)) \to \mathcal{P}^{i-1}(\mathcal{O}_{\mathbb{P}^n}(d)) \to 0$$

and tensoring with the line bundle $\pi_1^*\mathcal{O}_{\mathbb{P}W}(1)$ to get

$$c(\mathcal{P}^m) = \prod_{j=0}^{m} c\Big(\mathrm{Sym}^j(\pi_2^*\Omega_{\mathbb{P}^n}) \otimes \mathcal{O}(d\zeta_n + \zeta_W)\Big).$$

where we have written $\mathcal{O}(d\zeta_n + \zeta_W)$ for the line bundle associated to the pullback of d times the hyperplane section of $\mathbb{P}^n$ tensored with the pullback of the hyperplane section of $\mathbb{P}W$. Using the exact sequences

$$0 \to \mathrm{Sym}^i(\Omega_{\mathbb{P}^n}) \to \mathrm{Sym}^i(\mathcal{O}_{\mathbb{P}^n}(-1)) \to \mathrm{Sym}^{i-1}(\mathcal{O}_{\mathbb{P}^n}(-1)) \to 0$$

we get a collapsing product as before, yielding the desired formula for the Chern class of $\mathcal{P}^m$. To deduce the special case at the end of the Proposition it suffices to remember that since ζ_2 is the pullback from a 2-dimensional variety we have $\zeta_2^2 = 0$. $\square$

9.7.2 The discriminant of a net of plane curves

We will assume in this section that the characteristic of the ground field is zero. ****Is there a reason why the map from the universal singularity to the discriminant is separable in general? If so, we could avoid the assumption.****

We return to the case of a net of plane curves of degree d, with notation introduced at the beginning of Section 9.7. If $W = H^0(\mathcal{O}_{\mathbb{P}^2}(d))$ and $\mathcal{B} \subset \mathbb{P}W$ is a linear subspace of dimension 2, then the set of curves $\{C \mid C \in \mathcal{B}\}$, or the 2-plane $\mathcal{B}$, is called a *net* of curves.

Throughout this section we fix a general net of curves $\mathcal{B}$. Let $\mathcal{D} \subset \mathcal{B}$ be the set of singular curves, called the *discriminant curve* of the net $\mathcal{B}$. Since $\mathcal{D}$ is the intersection of $\mathcal{B}$ with the discriminant hypersurface in $\mathbb{P}W$, its degree is $\deg \mathcal{D} = 3(d-1)^2$ by Proposition 9.6. Next, let $\Gamma \subset \mathbb{P}^2$ be the plane curve traced out by the singular points of members of the net, so that if we set

$$\Sigma_{\mathcal{B}} := \Sigma \cap (\mathcal{B} \times \mathbb{P}^2),$$

then the projection maps π_i on Σ restrict to surjections

$$\begin{tikzcd} \Sigma_{\mathcal{B}} \arrow[r, "\pi_2|_{\mathcal{B}}"] \arrow[d, "\pi_1|_{\mathcal{B}}"'] & \Gamma \\ \mathcal{D}. & \end{tikzcd}$$

Since Σ is smooth of codimension 3, Bertini's Theorem shows that $\Sigma_{\mathcal{B}}$ is a smooth curve in $\mathcal{B} \times \mathbb{P}^2$. Since the generic singular plane curve is singular at only one point, the map $\Sigma_B \to \Gamma$ is birational. Since the fiber of Σ over a given point $p \in \mathbb{P}^2$ is a linear space of dimension $N-3$, the general 2-plane $\mathcal{B}$ containing a curve singular at p will contain a unique such curve. Thus the map $\Sigma_B \to \mathcal{D}$ is also birational, and $\Sigma_{\mathcal{B}}$ is the normalization (resolution of singularities) of each of Γ and $\mathcal{D}$. In particular the geometric genus of $\mathcal{D}$ and of Γ are the same as the genus of $\Sigma_{\mathcal{B}}$.

From the previous section we know that $\Sigma_{\mathcal{B}}$ is the zero locus of a section of the rank 3 bundle $\mathcal{P}^1|_{\mathcal{B}\times\mathbb{P}^2}$ on $\mathcal{B} \times \mathbb{P}^2$. This makes it easy to compute the degree and genus of $\Sigma_{\mathcal{B}}$, and we will derive the degree and genus of Γ, answering keynote question (b).

Proposition 9.16. *With notation as above, the map $\Sigma_{\mathcal{B}} \to \Gamma$ is an isomorphism, so both curves are smooth. The curve Γ has degree $3d-3$ and thus has genus $\binom{3d-4}{2}$. When $d \geq 2$ the curve $\mathcal{D}$ is singular.*

We will see how the singularities of $\mathcal{D}$ arise, what they look like and how many there are, in Chapter 13.

Proof. We begin with the degree of Γ, the number of points of intersection of Γ with a line $L \subset \mathbb{P}^2$. Since $\Sigma_{\mathcal{B}} \to \Gamma$ is birational, this is the same as the degree of the product $[\Sigma_{\mathcal{B}}]\zeta_2 \in A^4(\mathcal{B} \times \mathbb{P}^2)$. (More formally, $\pi_{2*}[\Sigma_{\mathcal{B}}] = \Gamma$, and $\pi_{2*}[\Sigma_{\mathcal{B}}][L] = [\Sigma_{\mathcal{B}}]\zeta_2$.) Write $\zeta_{\mathcal{B}}$ for the restriction of ζ_W, the pullback of the hyperplane section from $\mathbb{P}^N$, to $\mathcal{B} \times \mathbb{P}^2$. The degree of a class in $\mathcal{B} \times \mathbb{P}^2$ is the coefficient of $\zeta_2^2\zeta_{\mathcal{B}}^2$ in its expression in

$$A(\mathcal{B} \times \mathbb{P}^2) = \mathbb{Z}[\zeta_2, \zeta_{\mathcal{B}}]/(\zeta_2^3, \zeta_{\mathcal{B}}^3).$$

Since $\zeta_{\mathcal{B}}^3 = 0$ the last formula in Proposition 9.15 gives

$$\begin{aligned}\deg(\Gamma) &= \deg \zeta_2\big(3(d-1)\zeta_2\zeta_{\mathcal{B}}^2 + 3(d-1)^2\zeta_2^2\zeta_{\mathcal{B}}\big)\\ &= \deg 3(d-1)\zeta_2^2\zeta_{\mathcal{B}}^2\\ &= 3d-3.\end{aligned}$$

Since Γ is a plane curve, the arithmetic genus of the curve Γ is $\binom{3d-4}{2}$.

Next we compute the genus g_{Σ_B} of the smooth curve $\Sigma_{\mathcal{B}}$. The normal bundle of $\Sigma_{\mathcal{B}}$ in $\mathbb{P}^2 \times \mathcal{B}$ is the restriction of the rank 3 bundle $\mathcal{P}^1$, and the canonical divisor on $\mathbb{P}^2 \times \mathcal{B}$ has class $-3\zeta_2 - 3\zeta_{\mathcal{B}}$ so by the Adjunction Formula (Hartshorne [1977] Proposition 8.20) the degree of the canonical class of $\Sigma_{\mathcal{B}}$ is is the degree of the line bundle obtained by tensoring the canonical bundle of $\mathcal{B} \times \mathbb{P}^2$ with $\wedge^3\mathcal{P}^1$ and restricting the result to $\Sigma_{\mathcal{B}}$. This is the degree of the class

$$\begin{aligned}&(-3\zeta_2 - 3\zeta_{\mathcal{B}} + c_1(\mathcal{P}^1))[\Sigma_B]\\ &\qquad = (-3\zeta_2 - 3\zeta_{\mathcal{B}} + c_1(\mathcal{P}^1)) \cdot (3(d-1)\zeta_2\zeta_{\mathcal{B}}^2 + 3(d-1)^2\zeta_2^2\zeta_{\mathcal{B}}).\end{aligned}$$

Substituting the value $c_1(\mathcal{P}^1) = 3((d-1)\zeta_2 + \zeta_W)$ from Proposition 9.15 and taking account of the fact that $\zeta_W\zeta_{\mathcal{B}} = \zeta_{\mathcal{B}}^2$, this becomes

$$(3d-6)\zeta_2 \cdot (3(d-1)\zeta_2\zeta_{\mathcal{B}}^2 + 3(d-1)^2\zeta_2^2\zeta_{\mathcal{B}}) = (3d-6)(3d-3)\zeta_2^2\zeta_{\mathcal{B}}^2$$

with degree $2g_{\Sigma_B} - 2 = (3d-3)(3d-6)$, and we see that

$$g(\Sigma_{\mathcal{B}}) = \frac{(3d-4)(3d-5)}{2} = \binom{3d-4}{2}.$$

Since this coincides with the arithmetic genus of Γ computed above, we see that Γ is smooth and the map $\Sigma_{\mathcal{B}} \to \Gamma$ is an isomorphism. On the other hand the degree $3(d-1)^2$ of $\mathcal{D}$ is different from that of Γ for all $d \geq 2$, so in these cases the arithmetic and geometric genuses of $\mathcal{D}$ differ, and $\mathcal{D}$ must be singular, completing the proof. $\square$

Here is a different method for computing the degree of Γ: the net $\mathcal{B}$ of curves, having no base points, defines a regular map

$$\varphi_{\mathcal{B}} : \mathbb{P}^2 \to \Lambda$$

where $\Lambda \cong \mathbb{P}^2$ is the projective plane dual to the plane parametrizing the curves in the net $\mathcal{B}$. This map expresses $\mathbb{P}^2$ as a d^2-sheeted branched cover of Λ, and the curve $\Gamma \subset \mathbb{P}^2$ is the ramification divisor of this map.

By definition,

$$\varphi^* \mathcal{O}_\Lambda(1) = \mathcal{O}_{\mathbb{P}^2}(d);$$

so that if we denote by ζ_Λ the hyperplane class on Λ, we have $\varphi^* \zeta_\Lambda = d\zeta$.

The Hurwitz formula to the cover $\varphi : \mathbb{P}^2 \to \Lambda$ says that

$$K_{\mathbb{P}^2} = \varphi^* K_\Lambda + \Gamma,$$

and since $K_\Lambda = -3\zeta_\Lambda$, this yields

$$-3\zeta = -3d\zeta + [\Gamma]$$

or $[\Gamma] = (3d - 3)\zeta$.

These ideas work on an arbitrary smooth projective surface S, as long as we know the classes c_1, c_2 and λ and can evaluate the degrees of the relevant products. See Exercise 9.39 for an example.

9.8 The topological Hurwitz formula

Over the complex numbers we can use the topological Euler characteristic to give a different approach to the whole question of singular elements of linear series. It sheds additional light on the formula of Proposition 9.6, and is applicable in many circumstances in which Proposition 9.6 cannot be used. In addition, it will allow us to describe the local structure of the discriminant hypersurface (its tangent planes and tangent cones in particular).

Suppose that X is a smooth projective variety over $\mathbb{C}$, and $Y \subset X$ a divisor. Denoting the topological Euler characteristic (in the classical, or analytic topology) by χ_{top} we have

$$\chi_{\text{top}}(X) = \chi_{\text{top}}(Y) + \chi_{\text{top}}(X \setminus Y)$$

This relation is not hard to prove: we apply the Mayer-Vietoris sequence to the covering of X by $U = X \setminus Y$ and a small open neighborhood V of Y, and argue that, since Y is the zero locus of a section of a line bundle in X, the Euler characteristic of the intersection $U \cap V$ is 0. (In fact, the formula $\chi_{\text{top}}(X) = \chi_{\text{top}}(Y) + \chi_{\text{top}}(X \setminus Y)$ applies much more generally, to an arbitrary subvariety Y of an arbitrary X. But this requires a much harder result, that we can triangulate X in such a way that Y forms a subcomplex; even the statement that an arbitrary variety admits a triangulation is difficult. See Hironaka [1975].)

****insert something about tubular neighborhoods. Joe to ask Jun Li. This is very standard for smooth subvarieties, less so for hypersurfaces or locally complete intersections, still less for arbitrary singular subvarieties. Is there a good reference in Milnor?****

Let X be a smooth projective variety and let $f : X \to B$ be a map to a smooth curve B of genus g. This being characteristic 0, there are only a finite number of points $p_1, \dots, p_\delta$ over which the fiber X_{p_i} is singular. We can apply the relation on Euler characteristics to the divisor

$$Y = \bigcup_{i=1}^{\delta} X_{p_i} \subset X.$$

Naturally, $\chi_{\text{top}}(Y) = \sum \chi_{\text{top}}(X_{p_i})$; and on the other hand the open set $X \setminus Y$ is a fiber bundle over the complement $B \setminus \{p_1, \dots, p_\delta\}$, so that

$$\begin{aligned}\chi_{\text{top}}(X \setminus Y) &= \chi_{\text{top}}(X_\eta)\chi_{\text{top}}(B \setminus \{p_1, \dots, p_\delta\}) \\ &= (2 - 2g - \delta)\chi_{\text{top}}(X_\eta)\end{aligned}$$

where again η is a general point of B. Combining these, we have

$$\begin{aligned}\chi_{\text{top}}(X) &= (2 - 2g - \delta)\chi_{\text{top}}(X_\eta) + \sum_{i=1}^{\delta} \chi_{\text{top}}(X_{p_i}) \\ &= \chi_{\text{top}}(B)\chi_{\text{top}}(X_\eta) + \sum_{i=1}^{\delta}\big(\chi_{\text{top}}(X_{p_i}) - \chi_{\text{top}}(X_\eta)\big).\end{aligned}$$

In this form, we can extend the last summation over all points $q \in B$. We have proven:

Theorem 9.17 (Topological Hurwitz Formula)**.** *Let $f : X \to B$ be a morphism from a smooth projective variety to a smooth projective curve; let $\eta \in B$ be a general point. Then*

$$\chi_{\text{top}}(X) = \chi_{\text{top}}(B)\chi_{\text{top}}(X_\eta) + \sum_{q \in B}\big(\chi_{\text{top}}(X_q) - \chi_{\text{top}}(X_\eta)\big).$$

In English: the Euler characteristic of X is what it would be if it were a fiber bundle over B—that is, the product of the Euler characteristics of B and the general fiber X_η—with a "correction term" coming from each singular fiber, equal to the difference between its Euler characteristic and the Euler characteristic of the general fiber.

To see why Theorem 9.17 is a generalization of the classical Hurwitz formula (see for example Hartshorne [1977] Theorem *****), consider the case where X is a smooth curve of genus h, and $f : X \to C$ a branched cover of degree d. For each point $p \in C$, we write the fiber X_p as a divisor:

$$f^*(p) = \sum_{q \in f^{-1}(p)} m_q \cdot q.$$

We call the integer $m_q - 1$ the *ramification index* of f at q; we define the *ramification divisor* R of f to be the sum

$$R = \sum_{q \in X} (m_q - 1) \cdot q$$

and we define the *branch divisor* B of f to be the image of R (as a divisor, not as a scheme!)—that is,

$$B = \sum_{p \in C} b_p \cdot p \quad \text{where} \quad b_p = \sum_{q \in f^{-1}(p)} m_q - 1.$$

Now, since the degree of any fiber $X_p = f^{-1}(p)$ of f is equal to d, for each $p \in C$ the cardinality of $f^{-1}(p)$ will be $d - b_p$, so its contribution to the topological Hurwitz formula is $-b_p$. The formula then yields

$$2 - 2h = d(2 - 2g) - \deg(B),$$

the classical Hurwitz formula.

9.8.1 Pencils of curves on a surface, revisited

To apply the topological Hurwitz formula to keynote question (a), suppose that $\{C_t = V(t_0F + t_1G) \subset \mathbb{P}^2\}$ is a general pencil of plane curves of degree d. The polynomials F and G being general, the base locus $\Gamma = V(F, G)$ of the pencil will consist of d^2 reduced points, and the graph of the rational map $[F, G] : \mathbb{P}^2 \to \mathbb{P}^1$, which is

$$X = \{(p, t) : t_0F + t_1G = 0\} \subset \mathbb{P}^2 \times \mathbb{P}^1,$$

is the blow-up of $\mathbb{P}^2$ along Γ. In particular, X is be smooth, so Theorem 9.17 can be applied to the map $f : X \to \mathbb{P}^1$ that is the projection on the second factor.

Since X is the blow-up of $\mathbb{P}^2$ at d^2 points, we have

$$\chi_{\text{top}}(X) = \chi_{\text{top}}(\mathbb{P}^2) + d^2 = d^2 + 3.$$

Next, we know that a general fiber C_η of the map f is a smooth plane curve of degree d; as we've seen in Section **??** and many times since, its genus is $\binom{d-1}{2}$ and hence

$$\chi_{\text{top}}(C_\eta) = -d^2 + 3d.$$

We know from Proposition 9.1 that each singular fiber C appearing in a general pencil of plane curves has a single node as singularity. Thus its normalization $\tilde{C}$ will be a curve of genus $\binom{d-1}{2} - 1$ and hence Euler characteristic $-d^2 + 3d + 2$. Since C is obtained from $\tilde{C}$ by identifying two points, we have

$$\chi_{\text{top}}(C) = -d^2 + 3d + 1,$$

so the contribution of each singular fiber of f to the topological Hurwitz formula is exactly 1. It follows that the number of singular fibers is

$$\begin{aligned}\delta &= \chi_{\mathrm{top}}(X) - \chi_{\mathrm{top}}(\mathbb{P}^1)\chi_{\mathrm{top}}(C_\eta)\\ &= d^2 + 3 - 2(-d^2 + 3d)\\ &= 3d^2 - 6d + 3,\end{aligned}$$

as we saw before.

This same analysis can be applied to a pencil of curves on any smooth surface S. Let $\mathcal{L}$ be a line bundle on S with first Chern class $c_1(\mathcal{L}) = \lambda \in A^1(S)$, and let $W = \langle \sigma_0, \sigma_1 \rangle \subset H^0(\mathcal{L})$ be a two-dimensional vector space of sections with

$$\{C_t = V(t_0\sigma_0 + t_1\sigma_1) \subset S\}_{t \in \mathbb{P}^1}$$

the corresponding pencil of curves. We make—for the time being—two assumptions:

(a) The base locus $\Gamma = V(\{\sigma\}_{\sigma \in W})$ of the pencil is reduced, that is, consists of $\deg(\lambda^2)$ points; and

(b) Each of the finitely many singular elements of the pencil has just one node as singularity.

We also denote by $c_i = c_i(\Omega_S)$ the Chern classes of the cotangent bundle to S.

Given this, the calculation proceeds as before: we let X be the blow-up of S along Γ, and apply the topological Hurwitz formula to the natural map $f : X \to \mathbb{P}W^* \cong \mathbb{P}^1$. To start, we have

$$\chi_{\mathrm{top}}(X) = \chi_{\mathrm{top}}(S) + \#(\Gamma) = c_2 + \lambda^2.$$

(We're omitting the "deg" here for simplicity.) Next, by the adjunction formula, the Euler characteristic of a smooth member C_η of the pencil is given by

$$\chi_{\mathrm{top}}(C_\eta) = -\deg(\omega_{C_\eta}) = -(c_1 + \lambda)\cdot\lambda = -\lambda^2 - c_1\lambda.$$

and we know that the Euler characteristic of each singular element of the pencil is one greater than the Euler characteristic of the general one. In sum, then, the number of singular fibers is

$$\begin{aligned}\delta &= \chi_{\mathrm{top}}(X) - \chi_{\mathrm{top}}(\mathbb{P}^1)\chi_{\mathrm{top}}(C_\eta)\\ &= \lambda^2 + c_2 - 2(-\lambda^2 - c_1\lambda)\\ &= 3\lambda^2 + 2\lambda c_1 + c_2,\end{aligned}$$

agreeing with our previous calculation.

We'll see how this may be applied in higher dimensions in Exercises 9.56-9.57

9.8.2 Multiplicities of the discriminant hypersurface

One striking thing about this derivation of the formula for the number of singular elements in a pencil is that it gives a description of the multiplicities with which a given singular element counts that allows us to determine these multiplicities at a glance. In the derivation of the formula, we assumed that the singular elements of the pencil had only nodes as singularities. But what if an element C of the pencil has a cusp? In that case the geometric genus of the curve—the genus of its normalization $\tilde{C}$—is again 1 less than the genus of the smooth fiber, but this time instead of identifying two points we're just "crimping" the curve at one point. (In the analytic topology, C and $\tilde{C}$ are homeomorphic.) Thus

$$\chi_{\text{top}}(C) = \chi_{\text{top}}(\tilde{C}) = \chi_{\text{top}}(C_\eta) + 2,$$

and the fiber C counts with multiplicity 2. Similarly, if C has a tacnode, we have $g(\tilde{C}) = g(C_\eta) - 2$, so that $\chi_{\text{top}}(\tilde{C}) = \chi_{\text{top}}(C_\eta) + 4$; but we identify two points of $\tilde{C}$ to form C so in all

$$\chi_{\text{top}}(C) = \chi_{\text{top}}(C_\eta) + 3,$$

and the fiber C counts with multiplicity 3. Again, if C has a triple point, then $g(\tilde{C}) = g(C_\eta) - 3$, but we identify 3 points of $\tilde{C}$ to form C, so

$$\chi_{\text{top}}(C) = \chi_{\text{top}}(C_\eta) + 4,$$

and the fiber C counts with multiplicity 4. Moreover, if a fiber has more than one isolated singularity, the same analysis shows that the multiplicity with which it appears in the formula above is just the sum of the contributions coming from the individual singularities.

In addition to giving us a way of determining the contribution of a given singular fiber to the expected number, this approach tells us something about the geometry of the discriminant locus $\mathcal{D} \subset \mathbb{P}^N$. To see this, suppose that $C \subset \mathbb{P}^2$ is any plane curve of degree d with isolated singularities. Let D be a general plane curve of the same degree, and consider the pencil $\mathcal{B}$ of plane curves they span—in other words, take $\mathcal{B} \subset \mathbb{P}^N$ a general line through the point $C \in \mathbb{P}^N$. By what we've said, the number of singular elements of the pencil $\mathcal{B}$ other than C will be $3(d-1)^2 - (\chi_{\text{top}}(C) - \chi_{\text{top}}(C_\eta))$, where C_η is a smooth plane curve of degree d; it follows that the multiplicity of the intersection $\mathcal{B} \cap \mathcal{D}$ at C is $\chi_{\text{top}}(C) - \chi_{\text{top}}(C_\eta)$. We deduce the

Proposition 9.18. *Let $C \subset \mathbb{P}^2$ be any plane curve of degree d with isolated singularities. Then*

$$\text{mult}_C(\mathcal{D}) = \chi_{\text{top}}(C) - \chi_{\text{top}}(C_\eta).$$

Thus, a plane curve with a cusp (and no other singularities) corresponds to a double point of $\mathcal{D}$; a plane curve with a tacnode is a triple point, and so on. Note also that a curve C with one node and no other singularities is necessarily a smooth point of $\mathcal{D}$.

9.8.3 Tangent cones of the discriminant hypersurface

Let $\mathbb{P}^N$ be the projective space of plane curves of degree d. We can use the ideas above to describe the tangent spaces and tangent cones to the discriminant hypersurface $\mathcal{D} \subset \mathbb{P}^N$. To do this, we have to remove the first assumption in our application of the topological Hurwitz formula to pencils of curves, and deal with pencils whose base locus is not reduced.

For example, suppose that a point in a line in the plane, say $p \in L \subset \mathbb{P}^2$ are given, and that F, G are general forms of degree d such that $V(F)$ and $V(G)$ pass through p and are tangent to L at p. Let $\Gamma = V(F, G)$ be the base locus of the pencil $C_t = V(t_0 F + t_1 G)$, so that Γ consists of a scheme of length 2 whose reduced scheme is p together with $d^2 - 2$ reduced points. Since begin singular at p is one more linear condition on the elements of the pencil, exactly one member C_0 of the pencil will be singular at p, while all the others are smooth at p and have a common tangent line at p.

Let X be the minimal smooth blow-up resolving the indeterminacy of the map $\mathbb{P}^2 \to \mathbb{P}^1$ associated to the pencil—that is, X is obtained by blowing up $\mathbb{P}^2$ at Γ_{red} and then blowing up the resulting surface at the point p' on the exceptional divisor corresponding to the common tangent line L to the smooth members of the pencil at p. (This is not the blow up of S along the scheme Γ, which is singular! See for example Eisenbud and Harris [2000], IV.2.3.)

The fiber of the map $X \to \mathbb{P}^1$ corresponding to C_0 is the union of the proper transform of C_0 and the proper transform E of the first exceptional divisor, so that, for example, if C_0 had a node at p the fiber is the union of its normalization $\tilde{C}_0$ and a copy of $\mathbb{P}^1$, meeting at the two points of $\tilde{C}_0$ lying over the node p (Fig 9.2).

In sum, the Euler characteristic of the fiber is

$$\chi_{\text{top}}(\tilde{C} \cup E) = \chi_{\text{top}}(\tilde{C}) + \chi_{\text{top}}(E) - 2 = \chi_{\text{top}}(\tilde{C}) = \chi_{\text{top}}(C_\eta) + 2$$

and the fiber counts with multiplicity 2. We can use this to analyze the tangent planes to $\mathcal{D}$ at its simplest points:

Proposition 9.19. *Let C be a plane curve with a node at p and no other singularities. The tangent plane $\mathbb{T}_C\mathcal{D} \subset \mathbb{P}^N$ is the hyperplane $H_p \subset \mathcal{D}$ of curves containing the point p.*

Proof. If $C \subset \mathbb{P}^2$ is a plane curve with one node p and no other singularities, then by Proposition 9.18 C is a smooth point of $\mathcal{D}$, so it suffices to show

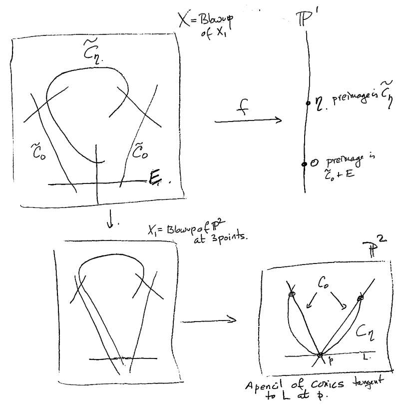

FIGURE 9.2. The morphism $f : X \to \mathbb{P}^1$ coming from the pencil of conics tangent to L at p [[SILVIO C_η should be a conic.]]

that H_p is contained in the tangent space to $\mathcal{D}$. But if $\mathcal{B} \subset \mathbb{P}^N$ is a general pencil including C and having p as a base point, $\mathcal{B}$ will meet $\mathcal{D}$ in exactly $3(d-1)^2-1$ points. By Bertini's Theorem it can't be tangent to $\mathcal{D}$ anywhere except at C, and so it must be tangent at C. $\square$

The argument given shows that, more generally, if C is a plane curve with a unique singular point p, the tangent cone to $\mathcal{D}$ at C will be a multiple of the hyperplane H_p; and more generally still, if C has isolated singularities $p_1, \dots, p_\delta$ the tangent cone $\mathbb{T}_C\mathcal{D}$ is supported on the union of the planes H_{p_i}.

There is also a sort of converse to Proposition 9.18:

Proposition 9.20. *The smooth locus of $\mathcal{D}$ consists exactly of those curves with a single node and no other singularity.*

Proof. Proposition 9.18 gives one inclusion: if C has a node and no other singularity, it is a smooth point of $\mathcal{D}$. Moreover, if C has more than one (isolated) singular point, then the projection map $\Sigma \to \mathcal{D}$ is finite but not one-to-one over C; it is intuitively clear (and follows from Zariski's Main Theorem) that $\mathcal{D}$ is analytically reducible and hence singular at C. Moreover, we observe that if $d \geq 3$ any curve with multiple components is a limit of curves with isolated singularities and at least 3 nodes—just deform each multiple component $mC_0 \subset C$ to a union of m general translates of C_0—so these must also lie in the singular locus of $\mathcal{D}$.

It remains to see that if C is a singular curve having a singularity p other than a node, then $\mathcal{D}$ is singular at C. This follows from an analysis of plane curve singularities: if C has isolated singularities including a point

p of multiplicity $k \geq 3$, then as we saw in Section ** of Chapter **??** the genus of the normalization $\tilde{C}$ is at most

$$g(\tilde{D}) \leq \binom{d-1}{2} - \frac{k(k-1)}{2}$$

and since at most k points of the normalization lying over p are identified in C,

$$\chi_{\text{top}}(C) \geq 2 - 2g(\tilde{C}) + k - 1 \geq -d(d-3) + (k-1)^2.$$

As for double points p other than a node, we've already done the case of a cusp; other double points will drop the genus of the normalization by 2 or more, and since we have at most two points of the normalization lying over p, we must have $\chi_{\text{top}}(C) \geq -d(d-3) + 3$. $\square$

Finally, note that the techniques of this section can be applied in exactly the same way in one dimension less! the result is the

Proposition 9.21. *Assume the characteristic is 0. Let $\mathbb{P}^n = \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ be the space of polynomials of degree n on $\mathbb{P}^1$, and $\mathcal{D} \subset \mathbb{P}^n$ the discriminant hypersurface, that is, the locus of polynomials with a repeated root. If $F \in \mathcal{D}$ is a point corresponding to a polynomial with exactly one double root p and $d-2$ simple roots, then $\mathcal{D}$ is smooth at F with tangent space the space of polynomials vanishing at p.*

We leave the proof via the Hurwitz formula as Exercise 9.62.

We should add that that there are many, many problems having to do with the local geometry of $\mathcal{D}$ and its stratification by singularity type, only a small fraction of which we know how to answer. The statements above barely scratch the surface; for more, see ****insert a reference—eg Brieskorn's plane alg curves and/or Teissier, Hunting INvariants in the geometry of discriminants. DE to ask Greuel.****.

9.9 Exercises

Exercise 9.22. Let $S = \mathbb{P}^1 \times \mathbb{P}^1$, and let $\{C_t \subset S\}_{t \in \mathbb{P}^1}$ be a general pencil of curves of type (a, b) on S. How many of the curves C_t are singular? (Make sure your answer agrees with (9.1) in case $(a, b) = (1, 1)$!)

Exercise 9.23. Let $S \subset \mathbb{P}^3$ be a smooth surface of degree d, and let $\{X_t \subset \mathbb{P}^3\}_{t \in \mathbb{P}^1}$ be a general pencil of surfaces of degree e. How many of the surfaces X_t are tangent to S? (Check your answer by comparing the case $d = 1$ to the relevant case of Proposition 9.6.)

Exercise 9.24. Let $S \subset \mathbb{P}^3$ be a smooth cubic surface, and $L \subset S$ a line. Let $\{C_t\}_{t \in \mathbb{P}^1}$ be the pencil of conics on S cut out by the pencil of planes $\{H_t \subset \mathbb{P}^3\}$ containing L. How many of the conics C_t are singular? Use this to answer to question: how many other lines on S meet L?

Exercise 9.25. Let $p \in \mathbb{P}^2$ be a point, and let $\{C_t \subset \mathbb{P}^2\}$ be a general pencil of plane curves singular at p—in other words, let F and G be two general polynomials vanishing to order 2 at p, and take $C_t = V(t_0F + t_1G)$. How many of the curves C_t will be singular somewhere else?

Exercise 9.26. Let $S = X_1 \cap X_2 \subset \mathbb{P}^4$ be a smooth complete intersection of hypersurfaces of degrees e and f. If $\{H_t \subset \mathbb{P}^4\}_{t \in \mathbb{P}^1}$ is a general pencil of hyperplanes in $\mathbb{P}^4$, how many of the hyperplane sections $S \cap H_t$ will be singular? (Equivalently: if $\Lambda \cong \mathbb{P}^2 \subset \mathbb{P}^4$ is a general 2-plane, how many tangent planes to S will intersect Λ in a line?)

Exercise 9.27. In case $r = 1$, show that Proposition 9.6 gives a special case of the *Hurwitz formula*: if C is a smooth projective curve of genus g and $f : C \to \mathbb{P}^1$ a branched cover of degree d, the number b of branch points of f is

$$b = 2d + 2g - 2.$$

(In characteristic $p > 0$ we must as usual assume that f is tamely ramified; that is, the characteristic does not divide any of the ramification exponents.)

Exercise 9.28. Let X be a smooth projective threefold, and $\mathcal{L}$ a line bundle on X. In analogy with the derivation of (**??**), find the Chern class $c_3(\mathcal{P}(\mathcal{L}))$.

Exercise 9.29. Let $X \subset \mathbb{P}^4$ be a smooth hypersurface of degree d. Using the result of Exercise 9.28, find the number of singular hyperplane sections of X in a pencil. Again, compare your answer to the result of Section 1.2.2.

Exercise 9.30. Let $X \cong \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ be the Segre threefold. Using the result of Exercise 9.28, find the number of singular hyperplane sections of X in a pencil. (Check: by Example **??**, your answer should be 0.)

Exercise 9.31. Let $S \subset \mathbb{P}^r$ be a smooth surface, and $W = H^0(\mathcal{O}_{\mathbb{P}^r}(1))|_S$ the linear system cut by hyperplanes. Show that if S is ruled by lines, then

$$\dim \Sigma_3 > r - 4.$$

Exercise 9.32. Find an example of a smooth surface $S \subset \mathbb{P}^r$ with $r \geq 5$ such that S is *not* ruled by lines, but such that if $W = H^0(\mathcal{O}_{\mathbb{P}^r}(1))|_S$ is the linear system cut by hyperplanes, then

$$\dim \Sigma_3 > r - 4.$$

Exercise 9.33. Let $X \subset \mathbb{P}^r$ be a smooth n-fold and $W = H^0(\mathcal{O}_{\mathbb{P}^r}(l))|_X$ the linear series cut on X by hypersurfaces of degree l in $\mathbb{P}^r$. Show that if $l \geq m$, then

$$\dim \Sigma_m = \dim \mathbb{P}W + n - \binom{m+n-1}{n}.$$

Exercise 9.34. Let $S = X_1 \cap X_2 \subset \mathbb{P}^4$ be a smooth complete intersection of hypersurfaces of degrees e and f. How many hyperplane sections of S will have a triple point? (Check this in case $e = f = 2$!)

Exercise 9.35. Let $(\mathcal{L}, W)$ be a linear series, and consider the "universal m-fold point"

$$\Sigma_m = \{(Y, p) \in \mathbb{P}W \times X \mid \operatorname{mult}_p Y \geq m\}.$$

We would expect that the fibers of $\pi_2 : \Sigma_m \to X$ would have codimension $\binom{m+n-1}{n}$ in $\mathbb{P}W \cong \mathbb{P}^N$, and that Σ_m would correspondingly have dimension $N + n - \binom{m+n-1}{n}$. Actually, this may fail even when W is very ample. The question of when the dimension estimate may fail is explored in Exercises 9.31-9.33.

Assuming now that Σ_m has the expected dimension and that the projection $\Sigma_m \to \mathbb{P}W$ is birational onto its image, show that the number of elements Y in a general sub-linear series $\mathbb{P}W \subset \mathbb{P}W$ of dimension $l = \binom{m+n-1}{n} - n$ that have an m-fold point is the degree of the Chern class $c_n(\mathcal{P}^{m-1}(\mathcal{L}))$.

Exercise 9.36. With notation as in Exercise 9.35 suppose that X is a smooth surface with Chern classes c_1, c_2, and that the Chern class of $\mathcal{L}$ is $1 + \lambda$. Show that

$$c_2\big(\mathcal{P}^2(\mathcal{L})\big) = 15\lambda^2 + 20c_1\lambda + 5c_1^2 + 5c_2,$$

by using the splitting principle to compute the class of $\operatorname{Sym}^2(\Omega_S) \otimes \mathcal{L}$, and using the Whitney formula. This is our formula for the expected degree of the locus $\pi_1(\Sigma_3) \subset \mathbb{P}W$ of curves with a triple point. For some applications, see Exercises 9.34-13.19.

Exercise 9.37. An *Eckhart point* of a smooth cubic surface $S \subset \mathbb{P}^3$ is a point $p \in S$ where three lines of the cubic surface meet; that is, such that $S \cap \mathbb{T}_p S$ consists of three concurrent lines.

(a) Show that a general cubic surface has no Eckhart points.

(b) Show that the locus $\Gamma \subset \mathbb{P}^{19}$ of cubic surfaces that do have an Eckhart point is a hypersurface.

(c) Find the degree of Γ; that is, the number of cubic surfaces in a general pencil that do have an Eckhart point.

Exercise 9.38. How much of the analysis of Section 9.7.2 extends to the case of an n-dimensional family $\mathbb{P}^n \subset \mathbb{P}(\mathcal{O}_{\mathbb{P}^n}(d))$ of hypersurfaces of degree d in $\mathbb{P}^n$?

Exercise 9.39. Let $S \subset \mathbb{P}^3$ be a general surface of degree d, and $\mathcal{B}$ a general net of plane sections of S (that is, intersections of X with planes containing a general point $p \in \mathbb{P}^3$. What are the degree and genus of the curve $\Gamma \subset S$ traced out by singular points of this net? What are the degree and genus of the discriminant curve?

Exercise 9.40. Verify that for a general curve $C \subset \mathbb{P}^2$ of degree d the number $3d(d-2)$ is the actual number of flexes of C; that is, all inflection points of C will have weight 1.

Exercise 9.41. Let $\{C_t = V(t_0F + t_1G)\}$ be a general pencil of plane curves of degree $d \geq 3$; suppose C_0 is a singular element of C (so that in particular by Proposition 9.1, C_0 will have just one node as singularity). By our formula, C_0 will have 6 fewer flexes than the general member C_t of the pencil. Where do the other 6 flexes go? If we consider the incidence correspondence

$$\Phi = \{(t,p) : C_t \text{ is smooth and } p \text{ is a flex of } C_t\} \subset \mathbb{P}^1 \times \mathbb{P}^2,$$

what is the geometry of the closure of Φ near $t = 0$?

Here are some special cases of the Plücker formula in higher-dimensional space:

Exercise 9.42. Find the points on $\mathbb{P}^1$, if any, that are ramification points for the maps

$$\mathbb{P}^1 \ni (s,t) \mapsto (s^3, s^2t, st^2, t^3 \in \mathbb{P}^3$$

and

$$\mathbb{P}^1 \ni (s,t) \mapsto (s^4, s^3t, st^3, t^4 \in \mathbb{P}^3.$$

Exercise 9.43. Show that the only smooth, irreducible and nondegenerate curve $C \subset \mathbb{P}^r$ with no inflection points is the rational normal curve.

Exercise 9.44. Observe that in case $g = 1$ and $d = r + 1$—that is, the curve is an elliptic normal curve E—the Plücker formula yields the number $(r+1)^2$ of inflection points. Show that these are exactly the translates of any one by the points of order $r + 1$ on E, each having weight 1.

Exercise 9.45. Let C be a smooth curve of genus $g \geq 2$. A point $p \in C$ is called a *Weierstrass point* if there exists a nonconstant rational function on C with a pole of order g or less at p and regular on $C \setminus \{p\}$.

(a) Show that the Weierstrass points of C are exactly the inflection points of the canonical map $\varphi : C \to \mathbb{P}^{g-1}$; and

(b) Use this to count the number of Weierstrass points on C.

For the following, let $C \subset \mathbb{P}^2$ be a plane curve and $H \subset \mathbb{P}^2$ its Hessian, as defined in Section **??**

Exercise 9.46. Let $p \in C$ be a smooth point. Show that p is a flex of C if and only if it's a point of intersection of C with D.

Exercise 9.47. With $p \in C$ as in the preceding exercise, show that the weight of p as an inflection point is equal to the intersection multiplicity $m_p(C \cdot D)$, and deduce that a smooth plane curve has exactly $3d(d-2)$ flexes, counting weight.

Exercise 9.48. Now suppose C is a general singular curve, $p \in C$ its node. What is the intersection multiplicity of C with D at p? If C is special, under what circumstances might $m_p(C \cdot D)$ be larger?

The following exercises deal with inflectionary points of weight greater than 1. As we've asked you to verify in Exercise 9.40, a general plane curve has only inflection points of weight 1. What about the curves that *do* have a flex of order ≥ 2?

Exercise 9.49. Let $\mathbb{P}^N$ be the space of all plane curves of degree $d \geq 4$, and let $H \subset \mathbb{P}^N$ be the closure of the locus of smooth curves with a hyperflex. Show that H is a hypersurface.

We'll be able to calculate the degree of the hypersurface of Exercise 9.49 once we have developed the techniques of Chapter 13.

If we try to generalize these ideas, there are many open questions. For example: *Let $\mathcal{H}$ be any component of the Hilbert scheme parametrizing curves in $\mathbb{P}^r$ whose general point corresponds to a smooth, irreducible nondegenerate curve $C \subset \mathbb{P}^r$. Do all inflection points of C have weight 1?*

Here is one case in which the answer is yes:

Exercise 9.50. Let $C \subset \mathbb{P}^r$ be a general complete intersection of hypersurfaces of degrees $d_1, \dots, d_{r-1} \geq 2$. Show that C has no inflection points of weight 2 or more.

Exercise 9.51. Let $S \subset \mathbb{P}^3$ be a general surface of degree $d \geq 2$, $p \in S$ a general point and $H = \mathbb{T}_p S \subset \mathbb{P}^3$ the tangent plane to S at p. Show that the intersection $H \cap S$ has an ordinary double point at p.

Exercise 9.52. Let $S = \mathbb{P}^1 \times \mathbb{P}^1$, and let $\{C_t \subset S\}_{t \in \mathbb{P}^1}$ be a general pencil of curves of type (a, b) on S. Use the topological Hurwitz formula to say how many of the curves C_t are singular. (Compare this with your answer to Exercise 9.22.)

Exercise 9.53. Let $p \in \mathbb{P}^2$ be a point, and let $\{C_t \subset \mathbb{P}^2\}$ be a general pencil of plane curves singular at p, as in Exercise 9.25. Use the topological Hurwitz formula to count the number of curves in the pencil singular somewhere else.

Exercise 9.54. Let $S \subset \mathbb{P}^3$ be a smooth surface of degree d, and let $\{X_t \subset \mathbb{P}^3\}_{t \in \mathbb{P}^1}$ be a general pencil of surfaces of degree e. Use the topological Hurwitz formula to count the number of surfaces X_t tangent to S. (Compare this with your answer to Exercise 9.23.)

Exercise 9.55. Suppose now that $\{C_t \subset \mathbb{P}^2\}$ is a pencil of plane curves all of which are singular at a specific point $p \in \mathbb{P}^2$—in other words, let F and G be two general polynomials vanishing to order 2 at p, and take $C_t = V(t_0 F + t_1 G)$. How many of the curves C_t will be singular somewhere else?

Exercise 9.56. Let $X \subset \mathbb{P}^r$ be a general hypersurface of degree d, and $X_0 \subset \mathbb{P}^r$ a hypersurface of the same degree having one ordinary double point as singularity. Show that

$$\chi_{\text{top}}(X_0) = \chi_{\text{top}}(X) + (-1)^{r-1}.$$

Exercise 9.57. Use the topological Hurwitz formula, and the preceding exercise, to derive the formula of Proposition 9.6.

Exercise 9.58. Prove Proposition 9.18 directly in the case of a cusp and a tacnode by analyzing the defining equations of the universal singular point $\Sigma \subset \mathbb{P}^N \times \mathbb{P}^2$ defined in Section 9.1. Can you deduce other cases of Proposition 9.18 as well in this way?

Exercise 9.59. Similarly, prove Proposition 9.19 directly by analyzing the equations of the universal singularity $\Sigma \subset \mathbb{P}^N \times \mathbb{P}^2$.

Exercise 9.60. Let $\mathbb{P}^5$ be the space of conic plane curves, and $\mathcal{D} \subset \mathbb{P}^5$ the discriminant hypersurface. Let $C \in \mathcal{D}$ be a point corresponding to a double line. What is the multiplicity of $\mathcal{D}$ at C, and what is the tangent cone?

Exercise 9.61. Let $\mathbb{P}^{14}$ now be the space of quartic plane curves, and $\mathcal{D} \subset \mathbb{P}^{14}$ the discriminant hypersurface. Let $C \in \mathcal{D}$ be a point corresponding to a double conic. What is the multiplicity of $\mathcal{D}$ at C, and what is the tangent cone?

Exercise 9.62. Used the Hurwitz formula to prove Proposition 9.21: if $\mathbb{P}^n$ is the space of polynomials of degree n on $\mathbb{P}^1$, $\mathcal{D} \subset \mathbb{P}^n$ the discriminant hypersurface and $F \in \mathcal{D}$ is a point corresponding to a polynomial with exactly one double root $p$and $d-2$ simple roots, show a general pencil $\langle F, G \rangle$ on $\mathbb{P}^1$ including F has $b = 2d + 2g - 2$ distinct branch points; and that a general pencil of the form $\langle F, G \rangle$ with $G(p) = 0$ has exactly $b - 1$.

10

Compactifying Parameter Spaces

Keynote Questions

(a) (Five Conic Problem): Given five general plane conics $C_1, \dots, C_5 \subset \mathbb{P}^2$, how many smooth conics are tangent to all 5?

(b) Given 11 general points $p_1, \dots, p_{11} \in \mathbb{P}^2$ in the plane, how many rational quartic curves $C \subset \mathbb{P}^2$ contain them all?

All the applications of intersection theory to enumerative geometry exploit the fact that interesting classes of algebraic varieties—lines, hypersurfaces and so on—are parametrized by the points of an algebraic variety themselves, the *parameter space*, and our efforts have all been toward counting intersections on these spaces. But to use intersection theory to count something, the parameter space must be projective (or at least proper) so that we have a degree map, as defined in Chapter 1. In the first case we treated in this book, that of the family of planes of a certain dimension in projective space, the natural parameter space was the Grassmannian, and the fact that it is projective is what makes the Schubert calculus so useful for enumeration. When we studied the questions about hypersurfaces containing lines, we were similarly concerned with parameter spaces that were projective—either the projective space of hypersurfaces itself, or the universal hypersurface. These spaces have an additional feature of importance: a universal family of the geometric objects we are studying, or what comes to the same thing, the property of representing a functor we understand.

This property is useful in many ways, first of all for understanding tangent spaces, and thus transversality questions.

In many interesting cases, however, the "natural" parameter space for a problem is *not* projective. To use the tools of intersection theory to count something, we must add points to the parameter space to complete it to a projective (or at least proper) variety. It is customary to call these new points the *boundary*, although this is not a topological boundary in any ordinary sense—the boundary points may look like any other point of the space—and (more reasonably) to call the enlarged space a *compactification* of the original space. If we are lucky, the boundary points of the compactification still parametrize some sort of geometric object we understand. In such cases we can use this structure to solve geometric problems. But as we shall see, the boundary can also get in the way, even when it seems quite natural. In such cases, we might look for a "better" compactification... but just how to do this is a matter of art rather than of science.

Perhaps the first problem in enumerative geometry where this tension became clear is the Five Conic Problem, which was solved in a naive way, not taking the difficulty into account (and therefore getting the wrong answer) by Steiner [1848], and again, with the necessary subtlety and correct answers, by Chasles [1864]. In this case there is a very beautiful and classical construction of a good parameter space, the space of *complete conics*. In this chapter we will explore the construction, and briefly discuss two more general constructions: Hilbert schemes and Kontsevich spaces.

10.1 Approaches to the five conic problem

Conics in characteristic 2 are interesting, but rather different than conics in characteristic not 2. For simplicity: *We will assume throughout this chapter that the characteristic of the ground field is not 2.*

To reiterate the problem: Given five general plane conics $C_1, \dots, C_5$, how many smooth conics are tangent to all five? Here is a naive approach:

(a) The set of plane conics is parametrized by $\mathbb{P}^5$. The locus of conics tangent to each given C_i is an irreducible hypersurface $Z_i \subset \mathbb{P}^5$, as one sees this by considering the incidence correspondence

$$\begin{tikzcd}[column sep=small] & \{(C,p) \in \mathbb{P}^5 \times C_i \mid C \text{ a conic tangent to } C_i \text{ at } p\} \arrow[dl] \arrow[dr, "\pi_2"] & \\ Z_i & & C_i, \end{tikzcd}$$

and noting that the fibers of π_2 are linear subspaces of $\mathbb{P}^5$ of codimension 3. (Here, "tangent to C_i at p" means $m_p(C \cdot C_i) \geq 2$, that is, the restriction to C_i of the defining equation of C vanishes to order at least 2 at p.)

(b) The degree of Z_i is 6. To see this, we intersect Z_i with a general line in $\mathbb{P}^5$—that is, we take a general pencil of conics and count how many are tangent to C_i. The conic C_i may be thought of as the embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$ by the complete linear system of degree 2. Thus a general pencil of conics cuts out a general linear series on C_i of degree 4, and the degree of the Z_i is the number of divisors in this family with less than 4 distinct points. The linear series defines a general map $C_i \to \mathbb{P}^1$ of degree 4 with distinct branch points, and by Hurwitz' theorem the number of branch points of this map is 6.

(c) Thus the number of points of intersection of $Z_1, \dots Z_5$, *assuming they intersect transversely,* will be $6^5 = 7,776$.

Alas, 7,776 is *not* the answer to the question we posed. The problem is not hard to spot: so far from being transverse, the hypersurfaces Z_i don't even meet in a finite set. To be sure, the part of the intersection within the open set $U \subset \mathbb{P}^5$ of smooth conics (which is what we wanted to count) *is* transverse, as we'll verify below. The trouble is with the compactification: we used the space of all (possibly singular) conics, and "excess" intersection of the Z_i takes place along the boundary.

In detail: the hypersurface Z_i is the *closure* in $\mathbb{P}^5$ of the locus of smooth conics C tangent to C_i. A smooth conic C is tangent to C_i exactly when the defining equation F of C, restricted to $C_i \cong \mathbb{P}^1$ and viewed as a quartic polynomial on $\mathbb{P}^1$, has a multiple root. When we extend this characterization to arbitrary conics C we see that *a double line is tangent to every conic.* Thus the five hypersurfaces $Z_1, \dots, Z_5 \subset \mathbb{P}^5$ will all contain the locus $S \subset \mathbb{P}^5$ of double lines, which is a Veronese surface in the $\mathbb{P}^5$ of conics. As we shall see, the intersection $\bigcap Z_i$ is the union of S and the finite set of smooth conics tangent to the five C_i. The presence of this extra component S means that the number we seek has little to do with the intersection product $\prod [Z_i] \in A^5(\mathbb{P}^5)$.

There are at least three successful approaches to dealing with this issue:

Blowing up the excess locus. Suppose we are interested in intersections inside some quasi-projective variety U, and we have a compactification V of U; in the example above U is the space of smooth conics, and V the space of all conics. We could blow up some locus in the boundary $V \setminus U$ to obtain a new compactification. This is the classical way of separating subvarieties of a given variety that we don't want to meet. In the Five Conic Problem we would blow up the surface S in $\mathbb{P}^5$ and consider the proper transforms $\tilde{Z}_i$ of the hypersurfaces Z_i in the blow-up $X = Bl_S\mathbb{P}^5$. If we are lucky (and in this case we are) we will have eliminated the excess intersection—that is, the $\tilde{Z}_i$ will not intersect anywhere in the exceptional divisor of the blow up ((If this were not the case we would have to blow up the common intersection inside the first blowup again.) In our case the $\tilde{Z}_i$ intersect transversely, and only inside U. To finish the argument, we could determine the Chow ring $A(X)$ of the blow-up, find the class $\zeta \in A^1(X)$ of the $\tilde{Z}_i$, and evaluate the product $\zeta^5 \in A^5(X)$.

This approach has the virtue of being universally applicable, at least in theory: any component of any intersection of cycles can be eliminated by blowing up repeatedly. It is not too hard to carry out this attack in the case of the the Five Conic Problem—see Griffiths and Harris [1979]. But often we cannot recognize the blowup as the parameter space of any nice geometric objects, and this makes the computations less intuitive and sometimes unwieldy. For example, if we look at cubics tangent to nine curves rather than conics tangent to five, the problem analogous to the Five Conic Problem, solved heuristically by Schubert in the 19th century and rigorously in Aluffi [1990], requires multiple blow-ups of the space $\mathbb{P}^9$ of cubics and complex calculations. ****The following is really a variant of the compactification argument!****

Excess intersection formulas. Excess intersection problems were already considered by Salmon in 1847, and were much generalized by Cayley around 1868. The Excess Intersection Formula of Fulton and MacPherson (Fulton [1984], Chapter 9) is a general formula that assigns to every connected component of an intersection $\bigcap Z_i \subset X$ a class in the appropriate dimension, in such a way that the sum of these classes (viewed as classes on the ambient variety X via the inclusion) equals the product of the classes of the intersecting cycles. The formula requires that all but at most one of the subvarieties Z_i are local complete intersections in X; in our case all are hypersurfaces. We will give an exposition of the formula in Chapter 15, and show in Section 15.6.1 how it may be applied to the five-conic problem, as was originally carried out in Fulton and MacPherson [1978].

As a general method, excess intersection formulas are often an improvement on blowing up. But, as with the blow-up approach, they require some knowledge of the normal bundles (or, more generally, normal cones) of the various loci involved.

Changing the parameter space. To understand what sort of compactification is "right" for a given problem is, as we have said, an art. In the case of the Five Conic Problem, we can take a hint from the fact that the problem is about tangencies. The set of lines tangent to a nonsingular conic is again a conic in the dual space (we will identify it explicitly below.) But when a conic degenerates to the union of two lines or a double line, the dual conic seems to disappear!—the dual of a line is only a point. This leads us to ask for a compactification of the space of smooth conics that preserves information about limiting positions of tangents.

It turns out that a more general compactness argument can be used as well, in cases (such as this one) where carefully chosen degenerations of the data—in our case the conics C_i don't cause the solutions to the problem (in our case the conics tangent to all the C_i) to approach the "degenerate" solutions that we are not interested in. ******For an approach related to Schubert's original ideas, which is capable of handling problems of tangency and incidence for plane curves of any degree, see Fulton [1984] Section 10.4.)

There are at least two ways to make a compactification that encodes the necessary information. One is to use the *Kontsevich space*. It parametrizes, not subschemes of $\mathbb{P}^2$, but rather maps $f : C \to \mathbb{P}^2$, with C a nodal curve of arithmetic genus 0. This is an important construction, which generalizes to a parametrization of curves of any degree and genus in any variety. We will discuss it informally in the second half of this Chapter. But proving even the existence of Kontsevich spaces requires a considerable development, and we will not take this route; the reader will find an exposition in Fulton and Pandharipande [1997].

The other way to describe a compactification of the space of smooth conics that preserves the tangency information is through the idea of *complete conics*. The space of complete conics is very well-behaved, and we will spend the first half of this Chapter on this beautiful construction. It turns out that the space we will construct is isomorphic to the Kontsevich space for conics (and, for that matter, to the blow-up $Bl_S\mathbb{P}^5$ of $\mathbb{P}^5$ along the surface of double lines), but generalizes in a different direction: there are analogues for quadric hypersurfaces of any dimension, for linear transformations ("complete colineations") and more generally for symmetric spaces (see De Concini and Procesi [1983], De Concini and Procesi [1985], De Concini et al. [1988] and Bifet et al. [1990]), but not for curves of higher degree or genus.

10.2 Complete conics

We begin with an informal discussion. Later in this section we will provide a rigorous foundation for what we describe. Recall that the *dual* of a smooth conic $C \subset \mathbb{P}^2$ is the set of lines tangent to C, regarded as a curve $C^* \subset \mathbb{P}^{2*}$. As we shall see, C^* is also a smooth conic (this would not be true in characteristic 2.)

10.2.1 Informal description

Degenerating the dual. Consider what happens to the dual conic as a smooth conic degenerates to a singular conic—either two distinct lines or a double line. That is, let $\mathcal{C} \to B$ be a one-parameter family of conics with parameter t, with C_t smooth for $t \neq 0$. Associating to each curve C_t the dual conic $C_t^* \subset \mathbb{P}^{2*}$ we get a regular map from the punctured disc $B \setminus \{0\}$ to the space $\mathbb{P}^{5*}$ of conics in $\mathbb{P}^{2*}$. (If $\mathbb{P}^2 = \mathbb{P}V$ and $\mathbb{P}^{2*} = \mathbb{P}V^*$, the space of conics on each are respectively $\mathbb{P}\operatorname{Sym}^2 V^*$ and $\mathbb{P}\operatorname{Sym}^2 V$—in particular, they're naturally dual to one another, so if we write the former as $\mathbb{P}^5$ it makes sense to write the latter as $\mathbb{P}^{5*}$.) Since the space of all conics is proper, this extends to a regular map on all of B—in other words, there is a well-defined conic $C_0^* = \lim_{t\to 0} C_t^*$. However, as we'll see this limit depends, in general, on the family $\mathcal{C}$ and not just on the curve C_0: in other words, the limit of the duals C_t^* is *not* determined by the limit of the curves C_t.

To provide a compactification of the space U of smooth conics that captures this phenomenon, we realize U as a locally closed subset of $\mathbb{P}^5 \times \mathbb{P}^{5*}$: as we'll see in the following section, the map $C \mapsto C^*$ is regular on smooth conics, so U is isomorphic to the graph of the map $U \to \mathbb{P}^{5*}$ sending a smooth conic C to its dual. In other words, we set

$$U = \{(C, C^*) \in \mathbb{P}^5 \times \mathbb{P}^{5*} \mid C \text{ a smooth conic in } \mathbb{P}^2 \text{ and } C^* \subset \mathbb{P}^{2*} \text{ its dual.}\}$$

The desired compactification, the *variety of complete conics* is the closure

$$X = \overline{U} \subset \mathbb{P}^5 \times \mathbb{P}^{5*}.$$

The dual of the dual of a smooth conic is the original conic, as we shall soon see (in fact the same statement holds for varieties much more generally, and will be proven in Section 12.6) so the set U is symmetric under exchanging $\mathbb{P}^5$ and $\mathbb{P}^{5*}$. It follows that X is symmetric too. (As one consequence of this symmetry, note that if $(C, C^*) \in X$ and either C or C' is smooth, then the other is too.) The set $U \subset \mathbb{P}^5$ of smooth conics is by definition dense in X, and it follows that X is irreducible and of dimension 5 as well.

What happens when C is singular? Let's first consider the case of a family $\{C_t\}$ of smooth conics approaching a conic C_0 of rank 2, that is, $C_0 = L \cup M$ is the union of a pair of distinct lines; for example, the family given (in affine coordinates on $\mathbb{P}^2$) as

$$\mathcal{C} = \{(t, x, y) \in B \times \mathbb{P}^2 \mid y^2 = x^2 - t\}$$

The picture makes it easy to guess what happens: any collection $\{L_t\}$ of lines with L_t tangent to C_t for $t \neq 0$ approaches a line L_0 through the point $p = L \cap M$, and conversely any line L_0 through p is a limit of lines L_t tangent to C_t. (Actually, the second half follows from the first, given that the limit $C_0' = \lim_{t\to 0} C_t^*$ is one-dimensional.) Since C_0' is by definition a conic, it

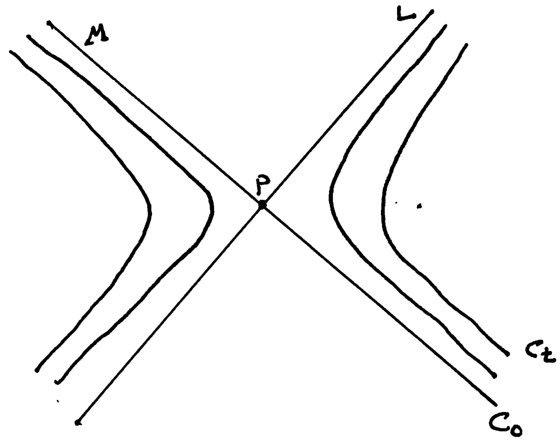

FIGURE 10.1. Conics specializing to a conic of rank 2

must be the double of the line in $\mathbb{P}^{2*}$ dual to the point p, irrespective of the family $\{C_t\}$ used to construct it, or of the positions of the lines L and M.

Things are much more interesting when we consider a family of smooth conics $\{C_t \subset \mathbb{P}^2\}$ specializing to a double line $C_0 = 2L$, and ask what the limit $\lim_{t\to 0} C_t^*$ of the dual conics $C_t^* \subset \mathbb{P}^{2*}$ may be. One way to realize such a family of conics is as the images of a family of maps $\varphi_t : \mathbb{P}^1 \to \mathbb{P}^2$. (We will see in Section **??** that any family of conics can be described in this way locally in the étale or analytic topology on the base.) Such a family of maps is given by a triple of polynomials $(f_t(x), g_t(x), h_t(x))$, homogeneous of degree 2 in $x = (x_0, x_1)$, whose coefficients are regular functions in t. In our present circumstances, our hypotheses say that f, g and h are linearly independent (and so span $H^0(\mathcal{O}_{\mathbb{P}^1}(2))$) for $t \neq 0$, but span only a two-dimensional vector space $W \subset H^0(\mathcal{O}_{\mathbb{P}^1}(2))$ for $t = 0$. For now, we'll make the additional assumption that the linear system W is base point free; the case where it's not will be dealt with below.

To see what the limit of the dual conics C_t^* will be in this situation, let $u, v \in \mathbb{P}^1$ be the ramification points of the map $\varphi_W : \mathbb{P}^1 \to \mathbb{P}^1$ associated to W (note that the map φ_0 is just the composition of this map with the inclusion of the target $\mathbb{P}^1$ as the line $L \subset \mathbb{P}^2$), and let $p = \varphi_0(u)$ and $q = \varphi_0(v) \in L$ be their images. We claim that in this case the limit $\lim_{t\to 0} C_t^*$ of the dual conics is the conic $C_0^* = p^* + q^* \subset \mathbb{P}^{2*}$ consisting of lines through p and lines through q.

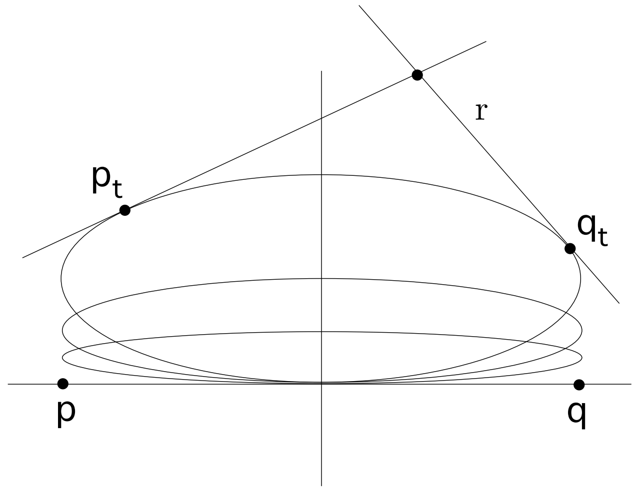

FIGURE 10.3. The family of conics $y^2 = t(x^2 - 2y)$

To prove this, let $r \in \mathbb{P}^2$ be any point not in L and not in all curves C_t, and let $\pi_r : \mathbb{P}^2 \to L$ be the projection from r to L. The composition $\pi_r \circ \varphi_t : \mathbb{P}^1 \to L$ is the degree 2 map associated to a 2-dimensional vector space $W_t \subset H^0(\mathcal{O}_{\mathbb{P}^1}(2))$; let $u_t, v_t \in \mathbb{P}^1$ be the ramification points of this map and $p_t, q_t \in L$ the corresponding branch points. As $t \to 0$, the linear system W_t approaches the linear system W; correspondingly, the divisor $u_t + v_t$ approaches $u + v$ and $p_t + q_t$ approaches $p + q$. In other words, the tangent lines to C_t passing through r—which are exactly the lines $\overline{r\,\varphi_t(u_t)} = \overline{r\,p_t}$ and $\overline{r\,\varphi_t(v_t)} = \overline{r\,q_t}$—approach the lines $\overline{r\,p}$ and $\overline{r\,q}$, *independently of* r. Thus every line through p or q is a limit of tangent lines to C_t, and conversely.

It is important to note that in this situation, unlike the case where C_0 is the union of two distinct lines, the limit of the dual conics C_t^* is not determined by the conic C_0. As we'll see in Section **??** below, the points p and q may be any pair of points of L, depending on the path along which C_t approaches C_0.

The remaining case to consider is when the branch points $p_t, q_t \in L$ of the maps $\pi_r \circ \varphi_t$ approach the same point $p \in L$. (Typically, this corresponds to the case where W has a base point: when W has a base point u, the ramification of W is concentrated at this point, which must then be the limit as $t \to 0$ of both the ramification points u_t and v_t of W_t.) In this case, the same logic shows that the limit of the dual conics C_t^* will be the double $2p^*$ of the line $p^* \subset \mathbb{P}^{2*}$ dual to the image point $p = \varphi_0(u)$.

Types of complete conics. In conclusion, there are four types of complete conics, that is, points $(C, C') \in X$:

(a) C and C' are both smooth, and $C' = C^*$. We will call these *smooth* complete conics.

(b) $C = L \cup M$ is of rank 2, and $C' = 2p^*$, where $p^* \subset \mathbb{P}^{2*}$ is the line dual to $p = L \cap M$;

(c) $C = 2L$ is of rank 1, and $C' = p^* \cup q^*$ is the union of the lines in $\mathbb{P}^{2*}$ dual to two points $p, q \in L$; or

(d) $C = 2L$ is of rank 1, and $C' = 2p^*$ is the double of the line in $\mathbb{P}^{2*}$ dual to a point $p \in L$.

Note that the description is exactly the same if we reverse the roles of C and C', except that the second and third types are exchanged. Note also that the points of each type form a locally closed subset of X, with the first open and the last closed; and all four are orbits of the action of PGL_3 on $\mathbb{P}^5 \times \mathbb{P}^{5*}$.

As we have already explained, the locus of complete conics of type (a) is isomorphic to U; in particular, it has dimension 5. It is easy to see that those of type (b) are determined by the pair of lines L, M, and thus form a set of dimension 4. By symmetry (or inspection) the same is true for type (c). Finally, those of type (d) are determined by the line L and the point $p \in L$; thus these form a set of dimension 3, which is in fact the intersection of the closures of the sets of points described in (b) and (c).

10.2.2 Rigorous description

Let's now verify all these statements, using the equations defining the locus $X \subset \mathbb{P}^5 \times \mathbb{P}^{5*}$. We could do this explicitly in coordinates, but it will save a great deal of ink if we use a little multilinear algebra. The reader to whom this is new will find more than enough background in Appendix 2 of Eisenbud [1995]. The multilinear algebra allows us to treat some basic properties in all dimensions with no extra effort, so we begin with some general results about duality for quadrics.

Duals of quadrics. Let V be a vector space. Recall that, since we are assuming the characteristic of the ground field K is not 2, the following three notions are equivalent:

- A symmetric linear map $\varphi : V \to V^*$;
- A quadratic map $q : V \to K$;
- An element $q' \in \mathrm{Sym}^2(V^*)$.

Explicitly, if we start with a symmetric map $\varphi : V \to V^*$ then we take $q(x) = \langle \varphi(x), x\rangle$, and the element $q' \in \mathrm{Sym}^2(V^*)$ comes about from the identification of $\mathrm{Sym}(V^*)$ with the ring of polynomial functions on V.

Any one of these objects, if nonzero, defines a quadric hypersurface $Q \subset \mathbb{P}V$, defined as the zero locus $Q = V(q)$ of q, or equivalently the locus

$$\{v \in \mathbb{P}(V) : \langle \varphi(v), v\rangle = 0\},$$

where we abuse notation and use the same symbol v to denote both a nonzero vector $v \in V$ and the corresponding point in $\mathbb{P}V$. The quadric $Q \subset \mathbb{P}V$ will be smooth if and only if φ is an isomorphism; more generally, the singular locus of Q will be the (projectivization of the) kernel of φ. The *rank* of Q is defined to be, equivalently, the rank of the linear map φ, or $n - \dim(Q_{\mathrm{sing}})$ (where we adopt, for the present purposes only, the convention that $\dim(\varnothing) = -1$); another way to characterize it is to say that a quadric of rank k is the cone, with vertex a linear space $Q_{\mathrm{sing}} \cong \mathbb{P}^{n-k} \subset \mathbb{P}^n$, over a smooth quadric hypersurface $\overline{Q} \subset \mathbb{P}^{k-1}$.

Now, recall from Section **??** that the dual of any variety $X \subset \mathbb{P}^n$ is defined to be the closure, in $\mathbb{P}^{n*}$, of the locus of hyperplanes tangent to X at a smooth point of X. (It is not, in particular, the set of H such that $H \cap X$ is singular). Given the description in the last paragraph of a quadric Q of rank k as a cone, we see that the dual of a quadric of rank k has dimension $k-2$. That said, we ask: what, in these terms, is the dual to Q?

To state the result, recall that if $\varphi : V \to W$ is any map of vector spaces of dimension $n+1$ then there is a *cofactor map* $\varphi^c : W \to V$ represented by a matrix whose entries are signed $n \times n$ minors of φ, satisfying $\varphi \circ \varphi^c = det(\varphi) Id_W$ and $\varphi^c \circ \varphi = det(\varphi) Id_V$. In invariant terms, φ^c is the composite

$$W \cong \wedge^n W^* \xrightarrow{\wedge^n \varphi^*} \wedge^n V^* \cong V$$

where the identifications $W \cong \wedge^n W^*$ and $\wedge^n V^* \cong V$ are defined by choices of nonzero vectors in the 1-dimensional spaces $\wedge^{n+1} W$ and $\wedge^{n+1} V^*$ respectively. Note that when the rank of φ is $< n$ the map φ^c is zero.

In the following we will abuse notation by using the same symbols for nonzero vectors in V and the points of $\mathbb{P}(V)$ that they represent.

Proposition 10.1. *Let $Q \subset \mathbb{P}(V) = \mathbb{P}^n$ be the quadric corresponding to the symmetric map $\varphi : V \to V^*$, and let $v \in V$ be a nonzero vector such that $\langle \varphi(v), v\rangle = 0$, so that $v \in Q$. The tangent hyperplane to Q at v is*

$$\mathbb{T}_v Q = \{w \in \mathbb{P}(V) \mid \langle \varphi(v), w\rangle = 0\}.$$

The dual of Q is thus

$$Q^* = \{\varphi(v) \in \mathbb{P}(V^*) \mid v \in Q \text{ and } \varphi(v) \neq 0\}.$$

In particular, if Q is nonsingular (that is, if the rank of φ is $n+1$), then Q^* is the image $\varphi(Q)$ of Q under the induced map $\varphi : \mathbb{P}V \to \mathbb{P}V^*$, and Q^* is the quadric corresponding to the cofactor map φ^c.

On the other and, if the rank of Q is n, and Q^c is the quadric corresponding to the cofactor map φ^c, then Q^c is the unique double hyperplane containing Q^*; that is, the support of Q^c is the hyperplane corresponding to the annihilator of the singular point of Q.

Proof. For any $w \in V$, the line $\overline{vw} \subset \mathbb{P}V$ spanned by v and w is tangent to Q at v if and only if

$$\langle \varphi(v + \epsilon w), v + \epsilon w\rangle = 0 \quad \mathrm{mod}(\epsilon^2)$$

Expanding this out we get

$$\langle \varphi(w), v\rangle + \langle \varphi(v), w\rangle = 0$$

and by the symmetry of φ and the assumption that we are not in characteristic 2 this is the case if and only if

$$\langle \varphi(v), w\rangle = 0,$$

proving the first statement and identifying the dual variety as $Q^* = \varphi(Q)$.

Suppose the rank of Q is n or $n+1$. Let φ^c be the matrix of cofactors of φ, so that $\varphi^c\varphi = \det\varphi \circ I$, where I is the identity map. Since $\operatorname{rank} Q = \operatorname{rank}\varphi \geq n$, the map φ^c is nonzero. The quadric Q^c is by definition the set of all $w \in V^*$ such that $\langle w, \varphi^c(w) = 0$. If $v \in Q$ then

$$\langle \varphi(v), \varphi^c\varphi(v)\rangle = (\det\varphi)\langle \varphi(v), v\rangle = 0,$$

so $\varphi(Q)$ is contained in Q^c.

If $\operatorname{rank}\varphi = n+1$, so that φ is an isomorphism, then $Q^* = \varphi(Q)$ is again a quadric hypersurface, and we must have $Q^* = Q^c$. If $\operatorname{rank}\varphi = n$ then since $\varphi^c\varphi = 0$ the rank of φ^c is 1, and the associated quadric is a double plane. On the other hand, Q is the cone over a nonsingular quadric in $\mathbb{P}^{n-1}$, and Q^* is the dual of that quadric inside a hyperplane (corresponding to the vertex of Q) in $\mathbb{P}^{n*}$. Thus Q^* spans the plane contained in Q^c. $\square$

The following easy consequence will be useful for the Five Conic Problem.

Corollary 10.2. *If Q and Q' are smooth quadrics, then Q and Q' have the same tangent hyperplane $l = 0$ at some point of intersection $v \in Q \cap Q'$ if and only if Q^* and Q'^* have the common tangent hyperplane $v = 0$ at the point of intersection $l \in Q^* \cap Q'^*$. In particular, if D is a smooth plane conic then the divisor $Z_D \subset X$, which is the closure of the set of complete conics (C, C') such that C is smooth and tangent to D is equal to the divisor defined similarly starting from the dual conic D^*.*

Proof. Suppose that Q, Q' correspond to symmetric maps φ and ψ. Since the tangent planes at v are the same, Proposition 10.1 shows that $\varphi(v) = \psi(v) \in Q^* \cap Q'^*$. Since $v = \varphi^{-1}(\varphi(v)) = \psi^{-1}(\psi(v)) \sim \psi^{-1}(\varphi(v))$, we see that Q^* and Q'^* are in fact tangent at $\varphi(v)$. (In addition to the fact that the duality interchanges points and planes, we are really proving that the dual of Q^* is Q, and similarly for Q'. Such a thing is actually true for any nondegenerate variety, as we will see in Section 12.6.)

The second statement follows at once from the first. $\square$

Equations for the variety of complete conics. We now return to the case of conics in $\mathbb{P}^2$, and suppose that V is 3-dimensional.

Proposition 10.3. *The variety*

$$X \subset \mathbb{P}(\mathrm{Sym}^2 V^*) \times \mathbb{P}(\mathrm{Sym}^2 V) = \mathbb{P}^5 \times \mathbb{P}^{5*}$$

of complete conics is smooth and irreducible. Thinking of $(\varphi, \psi) \in \mathbb{P}^5 \times \mathbb{P}^{5*}$ as coming from a pair of symmetric matrices $\varphi : V \to V^*$ and $\psi : V^* \to V$, the scheme X is defined by the ideal I generated by the 8 bilinear equations specifying that the product $\psi \circ \varphi$ has its diagonal entries equal to one another (two equations) and its off-diagonal entries equal to zero (6 equations).

(For the experts: It follows from the Proposition that the ideal I has codimension 5, and that its saturation, in the bihomogeneous sense, is prime. Computation shows that (in characteristic 0, anyhow) the polynomial ring modulo I is Cohen-Macaulay. With the Proposition, this implies that I is prime. In particular, I is preserved under the interchange of factors φ and ψ, which does not seem evident from the form given.)

Proof. Let Y be the subscheme defined by the given equations. We first show that Y agrees set-theoretically with X on at least the locus of those points (φ, ψ) where $\operatorname{rank} \varphi \geq 2$ or $\operatorname{rank} \psi \geq 2$, that is, where φ or ψ corresponds to a smooth conic or the union of two distinct lines. On the locus of smooth conics, φ has rank 3 and $(\varphi, \psi) \in Y$ if and only if $\psi = \varphi^{-1}$ up to scalars, so Proposition 10.1 shows that the dual conic is defined by ψ. Moreover, if the rank of φ is 2, and $(\varphi, \psi) \in Y$ then we see from the equations that $\psi \circ \varphi = 0$. Up to scalars, $\psi = \varphi^c$ is the unique possibility, and again Proposition 10.1 shows that the corresponding conic C' is the dual of C. To see the uniqueness (up to scalars) in terms of matrices, note that in suitable bases

$$\varphi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the symmetric matrices annihilating the image have the form

$$\psi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix} = a\varphi^c.$$

The same arguments show that when $\operatorname{rank}\psi \geq 2$ then $\varphi = \psi^c$ and again they correspond to dual conics. (Note that, since $\operatorname{rank}\psi^c = 1$ on this locus we do *not* have $\psi = \varphi^c$ there.)

Since X is defined as the closure of U, we see now in particular that $X \subset Y$. We will show next that Y is smooth of dimension 5 locally at any point $(\varphi, \psi) \in Y$ where both φ and ψ have rank 1. We will use this to show that Y is everywhere smooth of dimension 5.

To this end, suppose that $(\varphi, \psi) \in Y$ and that both φ and ψ have rank 1. The tangent space to Y at the point (φ, ψ) may be described as the locus of pairs of symmetric matrices $\alpha : V \to V^*$, $\beta : V^* \to V$ such that

$$(\psi + \epsilon\beta) \circ (\varphi + \epsilon\alpha) \mod(\epsilon^2)$$

has its diagonal entries equal and its off-diagonal entries zero. Since both φ and ψ have rank 1, the rank of $\psi \circ \alpha + \beta \circ \varphi$ is at most 2, so this is equivalent to saying that

$$\psi \circ \alpha + \beta \circ \varphi = 0.$$

We must show that this linear condition on the entries of the pair (α, β) is equivalent to 5 independent linear conditions. In suitable coordinates the maps φ, ψ will be represented by the matrices

$$\varphi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Multiplying out, we see that

$$\psi \circ \alpha = \begin{pmatrix} 0 & 0 & 0 \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} \\ 0 & 0 & 0 \end{pmatrix}, \quad \beta \circ \varphi = \begin{pmatrix} \beta_{1,1} & 0 & 0 \\ \beta_{2,1} & 0 & 0 \\ \beta_{3,1} & 0 & 0 \end{pmatrix}.$$

Thus the equation $\psi\alpha + \beta\varphi = 0$ is equivalent to the equations $\alpha_{2,1} + \beta_{2,1} = 0$ and $\alpha_{2,2} = \alpha_{2,3} = \beta_{1,1} = \beta_{3,1} = 0$; five independent linear conditions, as required.

To complete the proof of smoothness, note that Y is preserved scheme-theoretically by the action of the orthogonal group G. (Proof: if $(\varphi, \psi) \in Y$ and α is orthogonal, then $(\alpha\varphi\alpha^*, \alpha\psi\alpha^*) \in Y$ since $\alpha^*\alpha = 1$. Any closed point on Y where $\operatorname{rank}\varphi \geq 2$ degenerates under the action of G to a point where $\operatorname{rank}\varphi = 1$. (Proof: If α is orthogonal, that is, $\alpha\alpha^* = 1$, then the matrix $\varphi\psi$ is diagonal if and only if $\alpha\varphi\alpha^*\alpha\psi\alpha^* = \alpha\varphi\psi\alpha^*$ is diagonal. Thus, in a basis for which φ is diagonal, stretching one of the coordinates will make the correponding entry of φ approach zero, and $\psi = \varphi^c$ moves at the same time; a similar argument works when rank $\psi \geq 2$.) Consequently, if the singular locus of Y were not empty it would have to intersect the locus of pairs of matrices of rank 1, and we have seen that this is not the case.

To complete the proof that equal to X scheme-theoretically it is enough to show that the open of Y where each of φ and ψ have rank 3, along which Y is equal to X, is dense in Y. We use the fact that each point (φ, ψ) of Y corresponds to a unique pair of quadrics (Q, Q'). When φ has rank 2, Q corresponds to a pair of distinct lines, and Q' is uniquely determined. Thus this set is 4-dimensional. The same goes for the case when ψ has rank 2. On the other hand, when both φ and ψ have rank 1, Q is the double of a line L and Q' is the double of a line corresponding to one of the points of L; thus this set is only 3-dimensional. Since Y is everywhere smooth of dimension 5, any component of Y must intersect the set where φ and ψ have rank 3, as required. $\square$

The classification of the points of X into the four types above follows from Proposition 10.3: if $(\varphi, \psi) \in Y$, then:

(a) (smooth complete conics) If φ is of rank 3, then ψ must be its inverse;

(b) If φ is of rank 2, then (since X is symmetric) the products $\psi \circ \varphi$ and $\varphi \circ \psi$ must both be zero; it follows that ψ is the unique (up to scalars) symmetric map $V^* \to V$ whose kernel is the image of φ and whose image is the kernel of φ;

(c) If φ is of rank 1, ψ may have rank 1 or 2; if the latter, it may be any symmetric map $V^* \to V$ whose kernel is the image of φ and whose image is the kernel of φ; and finally

(d) If φ and ψ both have rank 1, they simply have to satisfy the condition that the kernel of ψ contains the image of φ and vice versa.

10.2.3 Solution to the five conic problem

Now that we've established that the space X of complete conics is smooth and projective, we'll show how to solve the Five Conics Problem. To any smooth conic $D \subset \mathbb{P}^2$ we associate a divisor $Z = Z_D \subset X$, which we define to be the closure in X of the locus of pairs $(C, C^*) \in X$ with C smooth and tangent to D, and let $\zeta = [Z_D] \in A^1(X)$ be its class. We will address each of the following issues:

Outline.

(a) We have to show that in passing from the "naive" compactification $\mathbb{P}^5$ of the space U of smooth conics to the more sensitive compactification X, we have in fact eliminated the problem of extraneous intersection: in other words, that for five general conics C_i the corresponding divisors $Z_{C_i} \subset X$ intersect only in points $(C, C') \in X$ with C and $C' = C^*$ smooth.

(b) We have to show that the five divisors Z_{C_i} are transverse at each point where they intersect.

(c) We have to determine the Chow ring of the space X, or at least the structure of a subring of $A^*(X)$ containing the class ζ of the hypersurfaces Z_{C_i} we wish to intersect;

(d) We have to identify the class ζ in this ring and find the degree of the fifth power $\zeta^5 \in A^5(X)$.

Complete conics tangent to five general conics are smooth. We begin by recalling that X is symmetric under the operation of interchanging the factors $\mathbb{P}^5$ and $\mathbb{P}^{5*}$.

Let's start by showing that no complete conic (C, C') of type (b) lies in the intersection of the divisors $Z_i = Z_{C_i}$ associated to five general conics C_i. The first thing we need to do is to describe the points (C, C') of type (b) lying in Z_D for a smooth conic D. This is straightforward: if $C = L \cup M$ is a conic of rank 2 which is a limit of smooth conics tangent to D, then C also must have a point of intersection multiplicity 2 or more with D; thus either L or M is a tangent line to D, or the point $p = L \cap M$ lies on D. (Note that by symmetry, a similar description holds for the points of type (c): the complete conic $(2L, p^* + q^*)$ will lie on Z_D only if L is tangent to D, or p or q lie on D.)

We now show that no complete conic (C, C') of type (b) lies in the intersection of the divisors $Z_i = Z_{C_i}$ associated to five general conics C_i. Write $C = L \cup M$, and set $p = L \cap M$. We note that since the C_i are general, no three are concurrent; this p can lie on at most two of the conics C_i. We'll proceed by considering the cases in turn

(a) p lies on none of the conics C_i. This is the most immediate case: since the conics C_i^* are also general, it is likewise the case that no three of them are concurrent. In other words, no line in the plane is tangent to more than two of the C_i; and correspondingly $(L \cup M, p) \in Z_{C_i}$ for at most 4 of the C_i.

(b) p lies on two of the conics C_i, say C_1 and C_2. Since C_3, C_4 and C_5 are general with respect to C_1 and C_2, none of the finitely many lines tangent to two of them passes through a point of $C_1 \cap C_2$; thus L and M can each be tangent to at most one of the conics C_3, C_4 and C_5, and again we see that $(L \cup M, p) \in Z_{C_i}$ for at most 4 of the C_i.

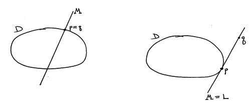

Fig 10.5 : The two types of complete conics of type (d) tangent to D.

FIGURE 10.5.

(c) p lies on exactly one of the conics C_i, say C_1. Now, since C_1 is general with respect to C_2, C_3, C_4 and C_5, it will not contain any of the finitely many points of pairwise intersection of lines tangent to two of them. Thus L and M can't each be tangent to two of the conics $C_2, \ldots, C_5$, and once more we see that $(L \cup M, p) \in Z_{C_i}$ for at most 4 of the C_i.

Thus no complete conic of type (b) can lie in the intersection of the Z_{C_i}; and by symmetry no complete conic of type (c) can either.

It remains to verify that no complete conic (C, C') of type (d) can lie in the intersection $\bigcap Z_{C_i}$, and again we have to start by characterizing the intersection of a cycle $Z = Z_D$ with the locus of complete conics of type (d).

To do this, write an arbitrary complete conic of type (d) as $(2M, 2q^*)$, with $q \in M$. If $(2M, 2q^*) \in Z_D$, then there is a one-parameter family $(C_t, C'_t) \in Z_D$ with C_t smooth and $C'_t = C^*_t$ for $t \neq 0$, and with $(C_0, C'_0) = (2M, 2q^*)$; let $p_t \in C_t \cap D$ be the point of tangency of C_t with D and set $p = \lim_{t \to 0} p_t \in M$. The tangent line $\mathbb{T}_{p_t} C_t = \mathbb{T}_{p_t} D$ to C_t at p_t will have as limit the tangent line L to D at p so $L \in q^*$. Thus both p and q are in both L and M. If $p = q$ then in particular $q \in D$. On the other hand, if $p \neq q$ then we must have $M = \overline{pq} = L$, so $M \in D^*$. We conclude, therefore, that *a complete conic* $(2M, 2q^*)$ *of type (d) can lie in* Z_D *only if either* $q \in D$ *or* $M \in D^*$.

Given this, we see that the first condition ($q \in C_i$) can be satisfied for at most two of the C_i, and the latter ($M \in C^*_i$) likewise for at most two; thus no complete conic $(2M, 2q^*)$ of type (d) can lie in Z_{C_i} for all $i = 1, \ldots, 5$.

Transversality. In order to prove that the cycles $Z_{C_i} \subset X$ intersect transversely when the conics $C_1, \dots, C_5$ are general we need a description of the tangent spaces to the Z_{C_i} at points of $\cap Z_{C_i}$. We have just shown that such points are represented by smooth conics, and the open subscheme parametrizing smooth conics is the same whether we are working in $\mathbb{P}^5$ or in the space of complete conics, so we may express the answer in terms of the geometry of $\mathbb{P}^5$.

Lemma 10.4. *Let $D \subset \mathbb{P}^2$ be a smooth conic curve, and $Z_D^\circ \subset \mathbb{P}^5$ the variety of smooth plane conics C tangent to D.*

(a) *If C has a point p of simple tangency with D and is otherwise transverse to D, then Z_D° is smooth at $[C]$; and*
(b) *In this case, the projective tangent plane $\mathbb{T}_{[C]} Z_D^\circ$ to Z_D° at $[C]$ is the hyperplane $H_p \subset \mathbb{P}^5$ of conics containing the point p.*

Proof. First, identify D with $\mathbb{P}^1$ and consider the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^2}(2)) \to H^0(\mathcal{O}_D(2)) = H^0(\mathcal{O}_{\mathbb{P}^1}(4)).$$

This map is surjective, with kernel the one-dimensional subspace spanned by the section representing D itself. In terms of projective spaces, the restriction induces the linear projection map

$$\pi_D : \mathbb{P}^5 = \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^2}(2)) \to \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^1}(4)) = \mathbb{P}^4$$

from the point $D \in \mathbb{P}^5$ to $\mathbb{P}^4$. The closure Z_D° in $\mathbb{P}^5$ is thus the cone, with vertex $D \in \mathbb{P}^5$, over the hypersurface $\mathcal{D} \subset \mathbb{P}^4$ of singular divisors in the linear system $|\mathcal{O}_{\mathbb{P}^1}(4)|$; and Lemma 10.4 will follow directly from the

Proposition 10.5. *Assume the characteristic is not 2. Let $\mathbb{P}^n = \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ be the space of polynomials of degree n on $\mathbb{P}^1$, and $\mathcal{D} \subset \mathbb{P}^n$ the discriminant hypersurface, that is, the locus of polynomials with a repeated root. If $F \in \mathcal{D}$ is a point corresponding to a polynomial with exactly one double root p and $d-2$ simple roots, then $\mathcal{D}$ is smooth at F with tangent space the space of polynomials vanishing at p.*

Proof. Note that we have already seen this statement: it's the content of Proposition 9.21 (proved in Exercise 9.62). For another proof, in local coordinates and more general characteristic, we can introduce the incidence correspondence

$$\Psi = \{(F, p) \in \mathbb{P}^n \times \mathbb{P}^1 \mid ord_p(F) \geq 2\},$$

and write down its equations in local coordinates (a, x) in $\mathbb{P}^n \times \mathbb{P}^1$: Ψ is the zero locus of the polynomials

$$R(a, t) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

and

$$S(a,t) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2 x + a_1.$$

Evaluated at a general point (a, x) where $a_1 = a_0 = x = 0$, all the partial derivatives of R and S vanish except

$$\begin{pmatrix} \frac{\partial R}{\partial a_1} & \frac{\partial R}{\partial a_0} & \frac{\partial R}{\partial x} \\ \frac{\partial S}{\partial a_1} & \frac{\partial S}{\partial a_0} & \frac{\partial S}{\partial x} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2a_2 \end{pmatrix}.$$

The fact that the first 2×2 minor is nonzero assures us that Ψ is smooth at the point, and the fact that $a_2 \neq 0$ and the characteristic is not 2 assures us that the differential $d\pi : T_{(a,0)}\Psi \to T_a\mathbb{P}^n$ of the projection $\pi : \mathcal{D} \to \mathbb{P}^n$ is injective, with image the plane $a_0 = 0$. Finally, the fact that π is one-to-one at such a point tells us the image $\mathcal{D} = \pi(\Psi)$ is smooth at the image point. $\square$

Getting back to the statement of Lemma 10.4, if $C =\subset \mathbb{P}^2$ is a conic with a point p of simple tangency with D and is otherwise transverse to D, then by Lemma 10.5 $\mathcal{D}$ is smooth at the image point in $\mathbb{P}^4$, with tangent space the space of polynomials vanishing at p. Since Z_D is the cone over $\mathcal{D}$ it follows that Z_D is smooth at C; the tangent space statement follows as well. $\square$

In order to apply Lemma 10.4, we need to establish some more facts about a conic tangent to five general conics:

Lemma 10.6. *Let $C_1, \ldots, C_5 \subset \mathbb{P}^2$ be general conics, and $C \subset \mathbb{P}^2$ any smooth conic tangent to all five. Each conic C_i is simply tangent to C at a point p_i and otherwise transverse to C; and the points $p_i \in C$ are distinct.*

Proof. Let U be the set of smooth conics and consider incidence correspondences

$$\begin{aligned}\Phi &= \{(C_1, \ldots, C_5; C) \in (U^5 \times U) \mid \text{each } C_i \text{ is tangent to } C\} \\ &\subset \Phi' = \{(C_1, \ldots, C_5; C) \in ((\mathbb{P}^5)^5 \times U) \mid \text{each } C_i \text{ is tangent to } C\}\end{aligned}$$

The set Φ is an open subset of the set Φ'. Since U is irreducible of dimension 5 and the projection map $\Phi' \to U$ on the last factor has irreducible fibers $(Z_C)^5$ of dimension 20, we see that Φ', and thus also Φ, is irreducible of dimension 25.

There are certainly points in Φ where the conditions of the Lemma are satisfied: just choose a conic C and five general conics C_i tangent to it. Thus the set of $(C_1, \ldots, C_5; C) \in \Phi$ where the conditions of the Lemma are not satisfied is a proper closed subset, and as such it can have dimension at most 24, and cannot dominate U^5 under the projection to the first factor. This proves the Lemma. $\square$

To complete the argument for transversality, let $[C] \in \cap Z_{C_i}$ be a point corresponding to the conic $C \subset \mathbb{P}^2$. By Lemma 10.6 the points p_i of tangency of C with the C_i are distinct points on C. Since C is the unique conic through these five points, the intersection of the tangent spaces to Z_{C_i} at $[C]$

$$\bigcap \mathbb{T}_{[C]} Z_{C_i} = \bigcap H_{p_i} = \{[C]\}$$

is zero-dimensional, proving transversality.

10.2.4 Chow ring of the space of complete conics

Having confirmed that the intersection $\bigcap Z_{C_i}$ indeed behaves well, let us turn now to computing the intersection number. We start by describing the relevant subgroup of the Chow group $A(X)$.

First, let α and $\beta \in A^1(X)$ be the pullbacks to $X \subset \mathbb{P}^5 \times \mathbb{P}^{5*}$ of the hyperplane classes on $\mathbb{P}^5$ and $\mathbb{P}^{5*}$. These are respectively represented by the divisors

$$A_p = \{(C, C^*) : p \in C\}$$

(for any point $p \in \mathbb{P}^2$), and

$$B_L = \{(C, C^*) : L \in C^*\}$$

(for any point $L \in \mathbb{P}^{2*}$).

Also, let $\gamma, \varphi \in A^4(X)$ be the classes of the curves Γ and Φ that are the pullbacks to X of general lines in $\mathbb{P}^5$ and $\mathbb{P}^{5*}$. These are the classes of the loci of complete conics (C, C^*) such that C contains four general points in the plane, and such that C^* contains four lines.

Lemma 10.7. *The group $A^1(X)$ of divisor classes on X has rank 2, and is generated over the rationals by α and β. The intersection number of these divisors with Γ and Φ are given by the table*

$$\begin{array}{cc} & \begin{matrix} \alpha & \beta \end{matrix} \\ \begin{matrix} \gamma \\ \varphi \end{matrix} & \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \end{array}$$

Proof. We first show that the rank of $A^1(X)$ is at most two. The open subset $U \subset X$ of pairs (C, C^*) with C and C^* smooth is isomorphic to the complement of a hypersurface in $\mathbb{P}^5$, and hence has torsion Picard group: any line bundle on U extends to a line bundle on $\mathbb{P}^5$, a power of which is represented by a divisor supported on the complement $\mathbb{P}^5 \setminus U$. Thus, if L is any line bundle on X, a power of L is trivial on U and hence represented by a divisor supported on the complement $X \setminus U$. But the complement of U in X has just two irreducible components, the closures D_2 and D_3 of the loci of complete conics of type (b) and (c). Any divisor class on X is thus a rational linear combination of the classes of D_2 and D_3, from which we see that the rank of the Picard group of X is at most 2.

Since passing through a point is one linear condition on a quadric we have $\deg(\alpha\gamma) = 1$ and dually $\deg(\beta\varphi) = 1$. Arguing as for the degree of the divisor Z_{C_i} above, we get $\deg(\alpha\varphi) = 2$ and again by duality $\deg(\beta\gamma) = 2$. Since the matrix of intersections between α, β and γ, φ is nonsingular, we conclude that α and β generate $\operatorname{Pic}(X) \otimes \mathbb{Q}$. □

The class of the divisor of complete conics tangent to C. It follows from Lemma 10.7 that we can write $\zeta = p\alpha + q\beta \in A^1(X) \otimes \mathbb{Q}$ for some $p, q \in \mathbb{Q}$. To compute p and q we use the fact that, restricted to the open set $U \subset X$, the divisor Z is a sextic hypersurface; it follows that $\deg \zeta\gamma = p + 2q = 6$, and since ζ is symmetric in α and β we get $\deg \zeta\gamma = q + 2p = 6$ as well. Thus

$$\zeta = 2\alpha + 2\beta \in A^1(X) \otimes \mathbb{Q}.$$

From this we see that $\deg(\zeta^5) = 32 \deg(\alpha+\beta)^5$, and it suffices to evaluate $\deg \alpha^{5-i}\beta^i \in A^5(X)$ for $i = 0 \dots 5$. By symmetry, it's enough to do this for $i = 0$, 1 and 2.

We will use the fact that the projection of $X \subset \mathbb{P}^5 \times \mathbb{P}^{5*}$ onto the first factor is an isomorphism on the set of pairs U_1 consisting of the pairs of quadrics (Q, Q') such that $\operatorname{rank} Q \geq 2$. Since all conics passing through three given general points have rank ≥ 2, the intersections needed will occur only in U_1. Since the degree of a zero-dimensional intersection is equal to the degree of the intersection scheme, this implies that we can make the computations on $\mathbb{P}^5$ instead of on X. For this we will use Bézout's theorem.

i=0: Passing through a point is a linear condition on quadrics. There is a unique quadric through 5 general points, and the intersection of five hyperplaces in $\mathbb{P}^5$ has degree 1, so $\deg(\alpha^5) = 1$.

i=1: The quadrics tangent to a given line form a quadric hypersurface in $\mathbb{P}^5$. Since not conics in the 1-dimensional linear space of conics through 4 general points will be tangent to a general line, $\deg(\alpha^4\beta) = 2$.

i=2: Similarly, we see that the conics passing through three given general points and tangent to a general line form a conic curve in $U_1 \subset \mathbb{P}^5$. Not all these conics are tangent to another given general line. (For example, after fixing coordinates we may think of circles as the conics passing through the points $\pm\sqrt{-1}$ on the line at ∞. Certainly there are circles through a given point and tangent to a given line that are not tangent to to another given line.) It follows that $\deg(\alpha^3\beta^2)$ is the degree of the zero-dimensional intersection of a plane with two quadrics, that is, 4.

Thus

$$\begin{aligned}\deg((\alpha+\beta)^5) &= \binom{5}{0} + 2\binom{5}{1} + 4\binom{5}{2} + 4\binom{5}{3} + 2\binom{5}{4} + \binom{5}{5}\\ &= 1 + 10 + 40 + 40 + 10 + 1\\ &= 102\end{aligned}$$

and correspondingly,

$$\zeta^5 = 2^5 \cdot 102 = 3264.$$

This proves:

Theorem 10.8. *There are 3264 plane conics tangent to five general plane conics.* □

Of course, the fact that we are imposing the condition of being tangent to a conic is arbitrary; we can use the space of complete conics to count conics tangent to five general plane curves of any degree, as Execises 10.10 and 10.11 show; and indeed we can extend this to the condition of tangency with singular curves, as Exercises 10.12-10.14 indicate.

10.3 Complete quadrics

There are beautiful generalizations of the construction of the space of complete conics to the case of quadrics in $\mathbb{P}^n$ and more general bilinear forms or homomorphisms. The paper Laksov [1987] gives an excellent account and many references. Here is a sketch of the beginning of the story. As usual we restrict ourselves the the case where the ground field has characteristic $\neq 2$.

As in the case of conics, we represent a quadric in $\mathbb{P}^n = \mathbb{P}(V)$ by a symmetric transformation $\varphi : V \to V^*$, or equivalently a symmetric bilinear form in $\mathrm{Sym}_2(V^*)$. To this transformation we associate the sequence of symmetric transformations

$$\varphi_i : \wedge^i V \to \wedge^i(V^*) = (\wedge^i V)^* \quad \text{for } i = 1, \ldots, n.$$

Here the identification $\wedge^i(V^*) = (\wedge^i V)^*$ is canonical—see for example Eisenbud [1995], Section ****.

We think of φ_i as an element of $\mathrm{Sym}_2(\wedge^i(V^*)$, and we define *the variety of complete quadrics in* $\mathbb{P}^n$, which we will denote by Φ, to be the closure in

$$\prod_{i=1}^{n} \mathbb{P}(\mathrm{Sym}_2(\wedge^i(V^*))$$

of the image of the set of smooth quadric under the map $\varphi \mapsto (\varphi_1, \ldots, \varphi_n)$.

The space $\mathbb{P}(\wedge^i(V^*))$ in which the quadric corresponding to φ_i lies is the ambient space of the Grassmannian G_i of $(i-1)$-planes in $\mathbb{P}^n$, and in fact an $(i-1)$-plane $\Lambda \subset \mathbb{P}^n$ is tangent to Q if the point $[\Lambda] \in G_i$ lies in this quadric.

From the definition we see that Φ has an open set U isomorphic to the open set corresponding to quadrics in the projective space of quadratic forms on $\mathbb{P}^n$. As with the case of complete conics, there is a beautiful description of the points that are not in U.

To start, let $\mathbb{P}^n = \mathbb{P}V$ and consider a flag $\mathcal{V}$ of subspaces of V of arbitrary length r and dimensions $k = \{k_1 < \cdots < k_r\}$:

$$0 \subset V_{k_1} \subset V_{k_2} \subset \cdots \subset V_{k_r} \subset V.$$

Now consider the variety F_k of pairs $(\mathcal{V}, Q)$ where $\mathcal{V}$ is a flag as above, and $Q = (Q_1, \ldots, Q_{r+1})$, with Q_i a smooth quadric hypersurface in the projective space $\mathbb{P}(V_{k_i}/V_{k_{i-1}})$; this is an open subset of a product of projective bundles over the variety of flags $\mathcal{V}$. We have then

Proposition 10.9. *There is a stratification of Φ whose strata are the varieties F_k, where k ranges over all strictly increasing sequences $0 < k_1 < \cdots < k_r < r$.*

One can also describe the limit of a family of smooth quadrics $q_t \in U = F_\varnothing$ when the family approaches a quadric q_0 of rank $n+1-k$, as in

$$\varphi_t := \begin{pmatrix} t \cdot I_k & 0 \\ 0 & I_{n+1-k} \end{pmatrix}.$$

The limit lies in the stratum $F_{\{k\}}$, where the flag consists of one intermediate space $0 \subset V_k \subset V$; the k-plane V_k will be the kernel of φ_0, the quadric Q_2 on $\mathbb{P}(V/V_k)$ will be the quadric induced by Q_0 on the quotient, and Q_1 the quadric on $\mathbb{P}V_k$ associated to the limit

$$\lim_{t \to 0} \frac{\varphi_t|_{V_k}}{t}.$$

In general, the stratum F_k lies in the closure of F_l exactly when $l \subset k$; the specialization relations can be defined inductively, using the above example.

10.4 Parameter spaces of curves

So far in this chapter we have been studying parameter spaces of smooth conics. The most obvious is perhaps $\mathbb{P}^5$, which we can identify as the space of all subschemes of $\mathbb{P}^2$ having pure dimension 1 and degree 2 (and thus arithmetic genus zero), and we have shown how the compactification by complete conics was more useful for dealing with tangencies. Here we have used the fact that the dual of a smooth conic is again a smooth conic. It would have been a different story if the problem had involved twisted cubics in $\mathbb{P}^3$ rather than conics in $\mathbb{P}^2$—if we had asked, for example, for the number of twisted cubic curves meeting each of 12 lines, or tangent to each of 12 planes, or, as in one classical example, the number of twisted cubic curves tangent to each of 12 quadrics. In that case it is not so clear how to make any parameter space and compactification at all![1]

In this section, we'll discuss two general approaches to constructing parameter spaces for curves in general: the Hilbert scheme of curves and the Kontsevich space of stable maps. Each of these has advantages, as we'll see.

10.4.1 Hilbert schemes

Recall from Section 8.3 that the Hilbert scheme $\mathcal{H}_{P(m)}(\mathbb{P}^n)$ is a parameter space for subschemes of $\mathbb{P}^n$ with Hilbert polynomial $P(m)$; in the case of curves (one-dimensional subschemes) this means all subschemes with fixed degree and arithmetic genus.

The Hilbert scheme of twisted cubics. We saw in Section 8.3 that the Hilbert scheme of hypersurfaces of degree s was just the projective space of all forms of degree s. The next simplest case is the Hilbert scheme containing twisted cubic curves in $\mathbb{P}^3$ (that is, parametrizing subschemes of $\mathbb{P}^3$ with Hilbert polynomial $H(m) = 3m + 1$). It has one component of dimension 12 whose general point corresponds to a twisted cubic curve, but it has a second component, whose general point corresponds to the union of a plane cubic $C \subset \mathbb{P}^2 \subset \mathbb{P}^3$ and a point $p \in \mathbb{P}^3$. Moreover, this second component has dimension 15 (the choice of plane has 3 degrees of freedom; the cubic inside the plane 9 more; and the point gives an additional 3). These two components meet along the 11-dimensional subscheme of singular plane cubics C with an embedded point at the singularity, not contained in the plane spanned by C. See Piene and Schlessinger [1985]

Report card for the Hilbert scheme. The Hilbert scheme is from some points of view the most natural parameter space that is generally available for projective schemes. Among its advantages: As shown in Section 8.3, it represents a functor that is easy to understand. There is a useful cohomological description of the tangent spaces to the Hilbert scheme, and, beyond that, a deformation theory that in some cases can describe its local structure. It was proven by Hartshorne (in characteristic 0) and (with some

[1]Though this does not seem to have slowed down Schubert, who found the correct number ** in his work

elaboration) by Bigatti, Hulett Reeves and Pardue in general that it is connected, and the graph of its connected components is small—the radius of the graph is at most the dimension of the varieties being parametrized. And, of course, associated to a point on the Hilbert scheme is all the rich structure of a homogenous ideal in the ring $K[x_0, \ldots, x_n]$ and its free resolution.

However, as a compactification of the space of smooth curves, the Hilbert scheme has drawbacks that sometimes make it difficult to use:

(a) **It has extraneous components, often of differing dimensions.** We see this phenomenon already in the case of twisted cubics, above. Of course we could take just the closure $\mathcal{H}^\circ$ in the Hilbert scheme of the locus of smooth curves, but we would lose some of the nice properties, like the description of the tangent space. Thus while it is relatively easy to describe the singular locus of $\mathcal{H}$, we don't know how to describe the singular locus of $\mathcal{H}^\circ$ along the locus where it intersects other components.
In fact, we don't know for curves of higher degree how many such extraneous components there are, or their dimensions: for $r \geq 3$ and large d the Hilbert scheme of zero-dimensional subschemes of degree d in $\mathbb{P}^r$ will have an unknown number of extraneous components of unknown dimensions, and this creates even more extraneous components in the Hilbert schemes of curves.

(b) **No one knows what's in the closure of the locus of smooth curves**. If we do choose to deal with the closure of the locus of smooth curves rather than the whole Hilbert scheme—as it seems we must—we face another problem: Except in a few special cases, we can't tell if a given point in the Hilbert scheme is in this closure. That is, we don't know how to tell whether a given singular 1-dimensional scheme $C \subset \mathbb{P}^r$ is smoothable.

(c) **It has many singularities**. The singularities of the Hilbert scheme are, in a precise sense, arbitrarily bad: Vakil [2006b] has shown that the completion of every affine local K-algebra appears (up to adding variables) as the completion of a local ring on a Hilbert scheme of curves.

10.4.2 The Kontsevich space

In the case of curves in a variety, the *Kontsevich space* is an alternative compactification. A precise treatment of this object is given in Fulton and Pandharipande [1997]; here we will treat it informally, sketch some of its properties, and indicate how it is used, with the hope that this will give the interested reader a taste of what to expect.

The Kontsevich space $\overline{M}_{g,0}(\mathbb{P}^r, d)$ parametrizes what are called *stable maps* of degree d and genus g to $\mathbb{P}^r$. These are morphisms

$$f : C \to \mathbb{P}^r$$

with C a connected curve of arithmetic genus g having only nodes as singularities, such that the image $f_*[C]$ of the fundamental class of C is equal to d times the class of a line in $A_1(\mathbb{P}^r)$, and satisfying the one additional condition that the automorphism group of the map f—that it, automorphisms φ of C such that $f \circ \varphi = f$—is finite. (This last condition is automatically satisfied if the map f is finite; it's relevant only for maps that are constant on an irreducible component of C, and amounts to saying that any smooth, rational component C_0 of C on which f is constant must intersect the rest of the curve C in at least three points.) Two such maps $f : C \to \mathbb{P}^r$ and $f' : C' \to \mathbb{P}^r$ are said to be the same if there exists an isomorphism $\alpha : C \to C'$ with $f' \circ \alpha = f$. There's an analogous notion of a family of stable maps, and the Kontsevich space $\overline{M}_{g,0}(\mathbb{P}^r, d)$ is a coarse moduli space for the functor of families of stable maps. Note that we're taking the quotient by automorphisms of the domain, but not of the image, so that $\overline{M}_{g,0}(\mathbb{P}^r, d)$ shares with the Hilbert scheme $\mathcal{H}_{dm-g+1}(\mathbb{P}^r)$ a common subset parametrizing smooth curves $C \subset \mathbb{P}^r$ of degree d and genus g.

There are naturally variants of this: the space $\overline{M}_{g,n}(\mathbb{P}^r, d)$ parametrizes maps $f : C \to \mathbb{P}^r$ with C a nodal curve having n marked distinct smooth points $p_1, \dots, p_n \in C$. (Here an automorphism of f is an automorphism of C fixing the points p_i and commuting with f; the condition of stability is thus that any smooth, rational component C_0 of C on which f is constant must have at least three distinguished points, counting both marked points and points of intersection with the rest of the curve C.) More generally, for any projective variety X and numerical equivalence class $\beta \in \mathrm{Num}_1(X)$, we have a space $\overline{M}_{g,n}(X, \beta)$ parametrizing maps $f : C \to X$ with fundamental class $f_*[C] = \beta$, again with C nodal and f having finite automorphism group.

It is a remarkable fact that the Kontsevich space is proper: in other words, if $\mathcal{C} \subset \mathcal{D} \times \mathbb{P}^r$ is a flat family of subschemes of $\mathbb{P}^r$ parametrized by a smooth, one-dimensional base $\mathcal{D}$, and the fiber C_t is a smooth curve for $t \neq 0$, then no matter what the singularities of C_0 there is a unique stable map $f : \tilde{C}_0 \to \mathbb{P}^r$ which is the limit of the inclusions $\iota_t : C_t \hookrightarrow \mathbb{P}^r$. Note that this limiting stable map $f : \tilde{C}_0 \to \mathbb{P}^r$ depends on the family, not just on the scheme C_0; the import of this in practice is that the Kontsevich space is often locally a blow-up of the Hilbert scheme along loci of curves with singularities worse than nodes. (This is not to say we have in general a regular map from the Kontsevich space to the Hilbert scheme; as we'll see in the examples below, the limiting stable map $f : \tilde{C}_0 \to \mathbb{P}^r$ doesn't determine the flat limit C_0 either.) We'll see how this plays out in four relatively simple cases:

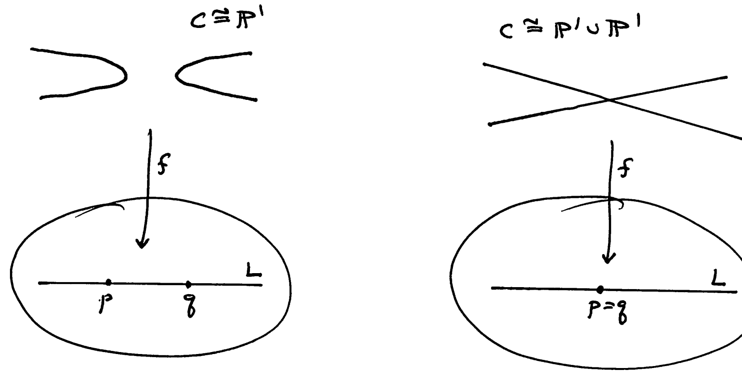

FIGURE 10.6. Stable maps of degree 2 with image a line

Plane conics. One indication of how useful the Kontsevich space can be is that, in the case of $\overline{M}_0(\mathbb{P}^2, 2)$ (that is, plane conics), the Kontsevich space is actually equal to the space of complete conics:

To begin with, if $C \subset \mathbb{P}^2$ is a conic of rank 2 or 3—that is, anything but a double line—then the inclusion map $\iota : C \hookrightarrow \mathbb{P}^2$ is a stable map; thus the open set $W \subset \mathbb{P}^5$ of such conics is likewise an open subset of the Kontsevich space $\overline{M}_0(\mathbb{P}^2, 2)$.

But when a family $\mathcal{C} \subset \mathcal{D} \times \mathbb{P}^2$ of conics specializes to a double line $C_0 = 2L$, the limiting stable map is a finite, degree 2 map $f : C \to L$, with C either isomorphic to $\mathbb{P}^1$, or two copies of $\mathbb{P}^1$ meeting at a point. Such a map is characterized, up to automorphisms of the domain curve, by its branch divisor $B \subset L$, a divisor of degree 2. (If B consists of two distinct points, $C \cong \mathbb{P}^1$, while if $B = 2p$ for some $p \in L$, the curve C will be reducible.)

Thus we have a birational morphism

$$\pi : \overline{M}_0(\mathbb{P}^2, 2) \to \mathcal{H}_{2m+1}(\mathbb{P}^2) = \mathbb{P}^5$$

from the Kontsevich space to the Hilbert scheme, with 2-dimensional fibers over the locus in $\mathbb{P}^5$ corresponding to double lines.

Conics in space. By contrast, there is not a regular map in either direction between the Hilbert scheme of conics in space and the Kontsevich space $\overline{M}_0(\mathbb{P}^3, 2)$. Of course there is a common open set: its points correspond to reduced curves of degree 2 embedded in $\mathbb{P}^3$ (such a curve is either a smooth conic in a plane or the union of two coplanar lines). To see that the identification of this open set does not extend to a regular map in either direction, note first that, as before, if $\mathcal{C} \subset \mathcal{D} \times \mathbb{P}^3$ is a family of conics specializing to a double line C_0, the limiting stable map is a finite, degree 2 cover $f : C \to L$, and this cover is not determined by the flat limit C_0 of the schemes $C_t \subset \mathbb{P}^3$. Thus there is no map from the Hilbert scheme to the Kontsevich space. On the other hand the scheme C_0 is contained in a plane—the limit of the unique planes containing the C_t. Since it has degree 2, the plane containing it is unique. But this plane is not determined by the data of the map f. Thus there is no map from the Kontsevich space to the Hilbert scheme.

The birational equivalence between the Hilbert scheme and the Kontsevich space is of a type that appears often in higher-dimensional birational geometry: the Kontsevich space is obtained from the Hilbert scheme $\mathcal{H}$ by blowing up the locus of double lines, and then blowing down the exceptional divisor along another ruling. (The blow-up of $\mathcal{H}$ along the double line locus can be described as the space of pairs $(H; (C, C^*))$, where $H \subset \mathbb{P}^3$ is a plane and (C, C^*) a complete conic in $H \cong \mathbb{P}^2$.)

Plane cubics. Here, we do have a regular map from the Kontsevich space $\overline{M}_1(\mathbb{P}^2, 3)$ to the Hilbert scheme $\mathcal{H}_{3m}(\mathbb{P}^2) \cong \mathbb{P}^9$, and it does some interesting things: it blows up the locus of triple lines, much as in the example of plane conics, and the locus of cubics consisting of a double line and a line as well. But it also blows up the locus of cubics with a cusp, and cubics consisting of a conic and a tangent line, and these are trickier: the blow-up along the locus of cuspidal cubics, for example, can be obtained either by three blow-ups with smooth centers, or one blow-up along a nonreduced scheme supported on this locus.

But what we really want to illustrate here is that the Kontsevich space $\overline{M}_{1,0}(\mathbb{P}^2, 3)$ is not irreducible—in fact, it's not even 9-dimensional! For example, maps of the form $f : C \to \mathbb{P}^2$ with C consisting of the union of an elliptic curve E and a copy of $\mathbb{P}^1$, with f mapping $\mathbb{P}^1$ to a nodal plane cubic C_0 and mapping E to a smooth point of C_0 form a 10-dimensional family of stable maps; in fact, these form an open subset of a second irreducible component of $\overline{M}_1(\mathbb{P}^2, 3)$, as illustrated in Figure 10.7.

And there's also a third component, whose general member is illustrated in Figure 10.8.

Twisted cubics. Here the shoe is on the other foot. The Hilbert scheme $\mathcal{H} = \mathcal{H}_{3m+1}$ has, as we saw, a second irreducible component besides the closure $\mathcal{H}_0$ of the locus of actual twisted cubics, and the presence of this component makes it difficult to work with. For example, it takes quite a bit of analysis to see that $\mathcal{H}_0$ is smooth, since we have no simple way of describing its tangent space; see **Piene-Schlessinger** for details. By contrast, the Kontsevich space is irreducible, and has only relatively mild (finite quotient) singularities.

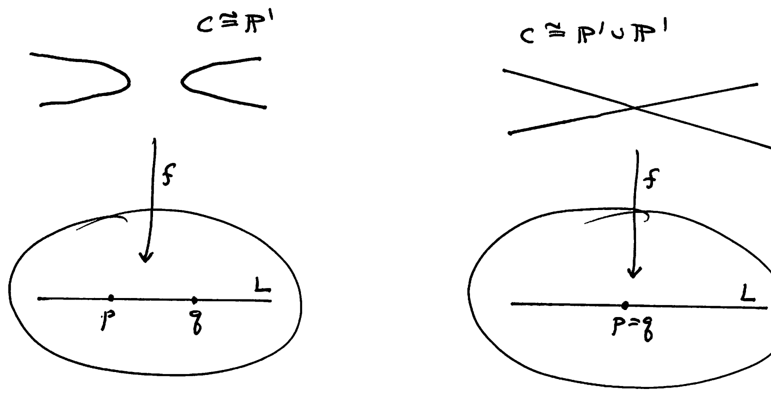

FIGURE 10.7. A typical point in the 10-dimensional component of $\overline{M}_{1,0}(\mathbb{P}^2, 3)$

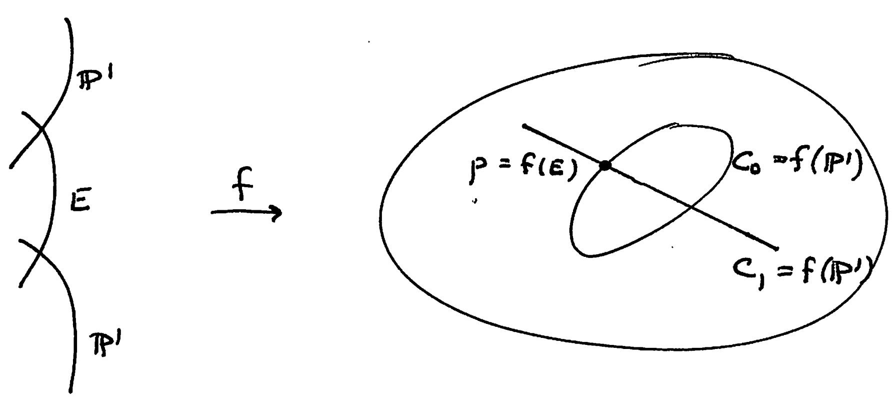

FIGURE 10.8. General member of a third component of $\overline{M}_{1,0}(\mathbb{P}^2, 3)$

Report card for the Kontsevich space. As with the Hilbert scheme, there are difficulties in using the Kontsevich space:

(a) **It has extraneous components**. These arise in a completely different way from the extraneous components of the Hilbert scheme, but they're there. A typical example of an extraneous component of the Kontsevich space $\overline{M}_g(\mathbb{P}^r, d)$ would consist of maps $f : C \to \mathbb{P}^r$ in which C was the union of a rational curve $C_0 \cong \mathbb{P}^1$, mapping to a rational curve of degree d in $\mathbb{P}^r$, and C_1 an arbitrary curve of genus g meeting C_0 in one point and on which f was constant; if the curve C_1 does not itself admit a nondegenerate map of degree d to $\mathbb{P}^r$, this map can't be smoothed.
So, using the Konsevitch space rather than the Hilbert scheme doesn't solve this problem, but it does provide a frequently useful alternative: there are situations where the Kontsevich space has extraneous components and the Hilbert scheme not—like the case of plane cubics described above—and also situations where the reverse is true, such as the case of twisted cubics.

(b) **No one knows what's in the closure of the locus of smooth curves**. This, unfortunately, remains an issue with the Kontsevich space. Even in the case of the space $\overline{M}_g(\mathbb{P}^2, d)$ parametrizing plane curves, where it might be hoped that the Kontsevich space would provide a better compactification of the Severi variety than simply taking its closure in the space $\mathbb{P}^N$ of all plane curves of degree d, the fact that we don't know which stable maps are smoothable represents a real obstacle to its use.

(c) **It has points corresponding to highly singular schemes, and these tend to be in turn highly singular points of $\overline{M}_g(\mathbb{P}^r, d)$**. Still true; but in this respect, at least, it might be said that the Kontsevich space represents an improvement over the Hilbert scheme: even when the image $f(C)$ of a stable map $f : C \to \mathbb{P}^r$ is highly singular, the fact that the domain of the map is at worst nodal makes the deformation theory of the map relatively tractable.

Finally, we should mention one other virtue of the Kontsevich space: it allows us to work with tangency conditions, without modifying the space and without excess intersection. The reason is simple: if $X \subset \mathbb{P}^r$ is a smooth hypersurface, the closure Z_X in $\overline{M}_g(\mathbb{P}^r, d)$ of the locus of embedded curves tangent to X is contained in the locus of maps $f : C \to \mathbb{P}^r$ such that the preimage $f^{-1}(X)$ is nonreduced or positive-dimensional. Thus, for example, a point in $\overline{M}_g(\mathbb{P}^2, d)$ corresponding to a multiple curve—that is, a map $f : C \to \mathbb{P}^2$ that is multiple-to-one onto its image—is not necessarily in Z_X.

10.5 How the Kontsevich space is used: rational plane curves

One case in which the Kontsevich space is truly well-behaved is the case $g = 0$. Here the space $\overline{M}_0(\mathbb{P}^r, d)$ is irreducible—it has no extraneous components—and moreover its singularities are at worst finite quotient singularities (in fact, it's the coarse moduli space of a smooth, Deligne-Mumford stack). Indeed, the use of the Kontsevich space has been phenomenally successful in answering enumerative questions about rational curves in projective space. We'll close out this chapter with an example of this; specifically, we'll answer the second keynote question, and more generally: how many rational curves $C \subset \mathbb{P}^2$ of degree d are there passing through $3d-1$ general points in the plane?

Since we have not even defined the Kontsevich space, this analysis will be far from complete. The paper Fulton and Pandharipande [1997] provides enough background to complete it.

Before starting the calculation, let's check that we do in fact expect a finite number. Maps of degree d from $\mathbb{P}^1$ to $\mathbb{P}^2$ are given by triples $[F, G, H]$ of homogeneous polynomials of degree d on $\mathbb{P}^1$ with no common zeros; since the vector space of polynomials of degree d on $\mathbb{P}^1$ has dimension $d+1$, the space U of all such triples has dimension $3d+3$. Now look at the map $U \to \mathbb{P}^N$ from U to the space $\mathbb{P}^N$ of plane curves of degree d, sending such a triple to the image (as divisor) of the corresponding map $\mathbb{P}^1 \to \mathbb{P}^2$. This has four-dimensional fibers (we can multiply F, G and H by a common scalar, or compose the map with an automorphism of $\mathbb{P}^1$), so we conclude that the image has dimension $3d-1$. In particular, we see that there are no rational curves of degree d passing through $3d$ general points of $\mathbb{P}^2$; and we expect a finite number (possibly 0) through $3d-1$. We'll denote the number by $N(d)$.

We will work on the space $M_d := \overline{M}_{0,4}(\mathbb{P}^2, d)$ of stable maps from curves with 4 marked points. This is convenient because on M_d we have a rational function φ, given by the *cross-ratio*: at a point of M_d corresponding to a map $f : (C; p_1, p_2, p_3, p_4) \to \mathbb{P}^2$ with $C \cong \mathbb{P}^1$ irreducible, it is the cross-ratio of the points $p_1, p_2, p_3, p_4 \in \mathbb{P}^1$; that is, in terms of an affine coordinate z on $\mathbb{P}^1$,

$$\varphi = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)},$$

where $z_i = z(p_i)$. The cross-ratio takes on the values $0, 1$ and ∞ only when two of the points coincide, which in our setting corresponds to when the curve C is reducible: for example, if C has two components C_1 and C_2, with $p_1, p_2 \in C_1$ and $p_3, p_4 \in C_2$, then by blowing down the curve C_1 in the total space of the domain family we can realize $(C, p_1, \ldots, p_4)$

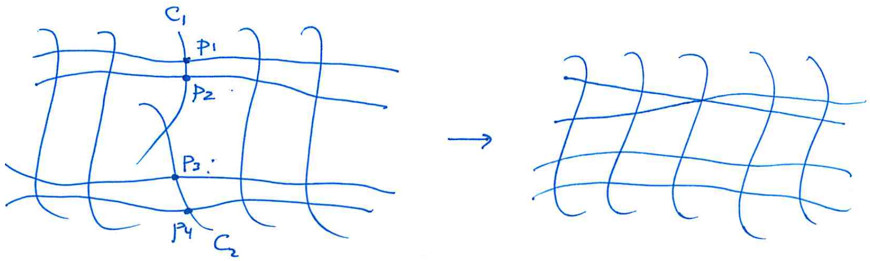

FIGURE 10.11. A family of maps that blows down C_1.

as a limit of pointed curves $(C_t, p_1(t), \ldots, p_4(t))$ with C_t irreducible and $\lim_{t\to 0} p_1(t) = \lim_{t\to 0} p_2(t)$ (see Figure 10.11). Thus φ has a zero at such a point. Similarly, if three of the points p_i, or all four, lie on one component of C, then φ will be equal to the cross-ratio of four distinct points on $\mathbb{P}^1$, and so will not be 0, 1 or ∞.

We now introduce a curve $B \subset M_d$ on which we will make the calculation. Fix a point $p \in \mathbb{P}^2$ and two lines $L, M \subset \mathbb{P}^2$ passing through p; fix two more general points $q, r \in \mathbb{P}^2$ and a collection $\Gamma \subset \mathbb{P}^2$ of $3d-4$ general points. We consider the locus

$$B = \left\{ f : (C; p_1, p_2, p_3, p_4) \to \mathbb{P}^2 \;\middle|\; \begin{array}{l} f(p_1) = q;\ f(p_2) = r; \\ f(p_3) \in L;\ f(p_4) \in M \text{ and} \\ \Gamma \subset f(C) \end{array} \right\} \subset M_d.$$

Since as we said the space of rational curves of degree d in $\mathbb{P}^2$ has dimension $3d-1$, and we're requiring the curves in our family to pass through $3d-2$ points (the points q and r, and the $3d-4$ points of Γ), our locus B will be a curve.

There may be points in B for which the domain C of the corresponding map $f : (C; p_1, p_2, p_3, p_4) \to \mathbb{P}^2$ is reducible. But in these cases C *will have no more than two components*. To see this, note that if the image of C has components $D_1, \ldots, D_k$ of degrees $d_1, \ldots, d_k$, by the above the curve D_i can contain at most $3d_i - 1$ of the $3d-2$ points $\Gamma \cup \{q, r\}$. Thus

$$3d - 2 \geq \sum_{i=1}^{k} (3d_i - 1) = 3d - k,$$

whence $k \leq 2$. And if the map f is nonconstant on at most two components, by the stability condition it can't be constant on any.

This argument also shows that there are only finitely many points in B for which the domain C is reducible: if $D = D_1 \cup D_2 \subset \mathbb{P}^2$, with D_i a rational curve of degree d_i, and $\Gamma \cup \{q, r\} \subset D$, then by the above D_i must contain exactly $3d_i - 1$ of the $3d-2$ points $\Gamma \cup \{q, r\}$. The number of such plane curves D is thus

$$\binom{3d-2}{3d_1-1} N(d_1)N(d_2).$$

Moreover, for each such plane curve D there are $d_1 d_2$ stable maps $f : C \to \mathbb{P}^2$ with image D: we can take C the normalization of D at all but any one of the points of intersection $D_1 \cap D_2$. (By Exercise 10.16, D_1 and D_2 will intersect transversely.)

On with the calculation! We equate the number of zeros and the number of poles of φ on B. To begin with, we consider points $f : (C; p_1, p_2, p_3, p_4) \to \mathbb{P}^2$ of B with C irreducible. Since $f(p_1) = q$ and $f(p_2) = r$ are fixed and lie off the lines L and M, the only way any two of the points p_i can coincide on such a curve is if

$$f(p_3) = f(p_4) = p \quad \text{where} \quad L \cap M = \{p\}.$$

Such points are zeros of φ; the number of these zeros is the number of rational plane curves of degree d through the $3d-1$ points p, q, r and Γ, that is to say, $N(d)$.

What about zeros and poles of φ coming from points

$$f : (C; p_1, p_2, p_3, p_4) \to \mathbb{P}^2$$

in B with $C = C_1 \cup C_2$ reducible? As we've observed, we get a zero of φ at such a point iff the points p_1 and p_2 lie on one component of C and p_3 and p_4 on the other. How many such points are there? Well, letting d_1 be the degree of the component C_1 of C containing p_1 and p_2, and $d_2 = d - d_1$ the degree of the other component C_2, we see that $f(C_1)$ must contain q, r and $3d_1 - 3$ of the points of Γ, while C_2 contains the remaining $3d - 4 - (3d_1 - 3) = 3d_2 - 1$ points of Γ. For any subset of $3d_1 - 3$ points of Γ, the number of such plane curves is $N(d_1)N(d_2)$, and for each such plane curve there are d_2 choices of the point $p_3 \in C_2 \cap f^{-1}(L)$ and d_2 choices of the point $p_4 \in C_2 \cap f^{-1}(M)$, as well as $d_1 d_2$ choices of the point $f(C_1 \cap C_2) \in f(C_1) \cap f(C_2)$. We thus have a total of

$$\sum_{d_1=1}^{d-1} d_1 d_2^3 \binom{3d-4}{3d_1-3} N(d_1)N(d_2)$$

zeros of φ arising in this way.

The poles of φ are counted similarly. These can occur only at points $f : (C; p_1, p_2, p_3, p_4) \to \mathbb{P}^2$ in B with $C = C_1 \cup C_2$ reducible, specifically with the points p_1 and p_3 lying on one component of C and p_2 and p_4 on the other. Again letting d_1 be the degree of the component C_1 of C containing p_1 and p_3, and $d_2 = d - d_1$ the degree of the other component C_2, we see that $f(C_1)$ must contain q and $3d_1 - 2$ points of Γ, and $f(C_2)$ the remaining

$3d-4-(3d_1-2)=3d_2-2$ points of Γ, plus r. For any subset of $3d_1-2$ points of Γ, the number of such plane curves is $N(d_1)N(d_2)$, and for each such plane curve there are d_1 choices of the point $p_3 \in C_2 \cap f^{-1}(L)$ and d_2 choices of the point $p_4 \in C_2 \cap f^{-1}(M)$, as well as d_1d_2 choices of the point $f(C_1 \cap C_2) \in f(C_1) \cap f(C_2)$. We thus have a total of

$$\sum_{d_1=1}^{d-1} d_1^2 d_2^2 \binom{3d-4}{3d_1-2} N(d_1)N(d_2)$$

poles of φ arising in this way. Now adding up the poles and zeros, we conclude that

$$N(d) = \sum_{d_1=1}^{d-1} d_1 d_2 \left[d_1 d_2 \binom{3d-4}{3d_1-2} - d_2^2 \binom{3d-4}{3d_1-3} \right] N(d_1)N(d_2),$$

a recursive formula that allows us to determine $N(d)$ if we know $N(d')$ for $d' < d$. To show that this gives the actual number of curves, we would have to show that the the intersections described above occur with multiplicity 1, and we will not do this.

For example, there is a unique line through two points, and a unique conic through 5 general points, so $N(d_1) = N(d_2) = 1$. Thus if we take $d = 3$ we have

$$N(3) = 2\left[2\binom{5}{1} - 4\binom{5}{0}\right] + 2\left[2\binom{5}{4} - \binom{5}{3}\right] = 12$$

as we've already seen.

Continuing to $d = 4$, we have

$$\begin{aligned} &N(4) \\ &\quad = 3 \cdot 12 \left[3\binom{8}{1} - 9\binom{8}{0}\right] + 4\left[4\binom{8}{4} - 4\binom{8}{3}\right] + 3 \cdot 12\left[3\binom{8}{7} - \binom{8}{6}\right] \\ &\quad = 620. \end{aligned}$$

Always ignoring the question of multiplicity, this answers Keynote Question (b): there are 620 rational quartic curves through 11 general points of $\mathbb{P}^2$.

Exercises 10.19-10.19 suggest some additional problems that can be solved using spaces of stable maps.

10.6 Exercises

Exercise 10.10. Let $D \subset \mathbb{P}^2$ be a smooth curve of degree d, and let $Z \subset X$ be the closure, in the space X of complete conics, of the locus of smooth conics tangent to D. Find the class $[Z_D] \in A^1(X)$ of the cycle Z.

Exercise 10.11. Now let $D_1, \dots, D_5 \subset \mathbb{P}^2$ be general curves of degrees $d_1, \dots, d_5$. Show that the corresponding cycles $Z_{D_i} \subset X$ intersect transversely, and using the result of the preceding exercise, find the number of conics tangent to all five .

Exercise 10.12. Now let $D \subset \mathbb{P}^2$ be a curve of degree d with a node at a point $p \in D$ (and smooth otherwise), and let $Z \subset X$ be the closure, in the space X of complete conics, of the locus of smooth conics tangent to D at a smooth point of D. Find the class $[Z_D] \in A^1(X)$ of the cycle Z.

Exercise 10.13. Next, let $\{D_t\}$ be a family of plane curves of degree d, with D_t smooth for $t \neq 0$ and D_0 having a node at a point p. What is the limit of the cycles Z_{D_t} as $t \to 0$?

Exercise 10.14. Finally, here's a very 19^{th} century way of deriving the result of Exercise 10.10 above. Let $\{D_t\}$ be a pencil of plane curves of degree d, with D_t smooth for general t and D_0 consisting of the union of d general lines in the plane. Using the description of the limit of the cycles Z_{D_t} as $t \to 0$ in the preceding exercise, find the class of the cycle Z_{D_t} for t general.

Exercise 10.15. True or False: There are only finitely many PGL_4 orbits in the Kontsevich space $\overline{M}_0(\mathbb{P}^3, 3)$.

Exercise 10.16. Let Γ_1 and Γ_2 be collections of $3d_1 - 1$ and $3d_2 - 1$ general points in $\mathbb{P}^2$, and $D_i \subset \mathbb{P}^2$ any of the finitely many rational curves of degree d_i passing through Γ_i. Show that D_1 and D_2 intersect transversely.

Exercise 10.17. Let $p_1, \dots, p_7 \in \mathbb{P}^2$ be general points and $L \subset \mathbb{P}^2$ a general line. How many rational cubics pass through $p_1, \dots, p_7$ and are tangent to L?

Exercise 10.18. (Harder): More generally, let $p_1, \dots, p_k \in \mathbb{P}^2$ be general points and $L_1, \dots, L_{8-k} \subset \mathbb{P}^2$ general lines. Given the result of Exercise 10.19 below, how many rational cubics pass through all the points and are tangent to all the lines?

Exercise 10.19. (a) Let $M = \overline{M}_0(\mathbb{P}^2, d)$ be the Kontsevich space of rational plane curves of degree d, and let $U \subset M$ be the open set of immersions $f : \mathbb{P}^1 \to \mathbb{P}^2$ that are birational onto their images. For $D \subset \mathbb{P}^2$ a smooth curve, let $Z_D^\circ \subset U$ be the locus of maps $f : \mathbb{P}^1 \to \mathbb{P}^2$ such that $f(\mathbb{P}^1)$ is tangent to D at a smooth point of $f(\mathbb{P}^1)$, and $Z_D \subset M$ its closure. Verify the statement above: that is, show that Z_D is contained in the locus of maps $f : C \to \mathbb{P}^r$ such that the preimage $f^{-1}(D)$ is nonreduced or positive-dimensional.

(b) Given this, show that for $D_1, \dots, D_{3d-1}$ general curves the intersection $\cap Z_{D_i}$ is contained in U.

11

Projective Bundles and their Chow Rings

Keynote Questions

(a) Given eight general lines $L_1, \dots, L_8 \subset \mathbb{P}^3$, how many plane conic curves in $\mathbb{P}^3$ meet all eight?

(b) Can a ruled surface (that is, a $\mathbb{P}^1$-bundle over a curve) contain more than one curve of negative self-intersection?

Many interesting varieties, such as scrolls, blowups of linear subspaces of projective spaces, and some natural parameter spaces for enumerative problems, can be described as projective bundles over simper varieties. In this chapter we'll investigate such varieties and compute their Chow rings. This will allow us to answer the question above. It will also help us to describe the Chow ring of a blowup, which we'll do in Chapter 15.

11.1 Projective bundles and the tautological divisor class

Definition. A *projective bundle* over a scheme X is a map $\pi : Y \to X$ such that there is an open covering $\{U_i\}$ of X with the property that, for all i,

$$\pi^{-1}U_i \cong U_i \times \mathbb{P}^r,$$

in such a way that π corresponds to the projection onto the first factor.

One can make projective bundles from vector bundles as follows: First, if $\mathcal{E} \cong \mathcal{O}_X^{r+1}$ is a trivial vector bundle, then

$$X \times \mathbb{P}^r = \mathrm{Proj}(\mathcal{O}_X[x_0, \ldots, x_r]) = \mathrm{Proj}(\mathrm{Sym}(\mathcal{E}^*)),$$

and the structure map $\mathcal{O}_X \to \mathrm{Sym}(\mathcal{E}^*)$ corresponds to the projection $\pi : X \times \mathbb{P}^r \to X$. By definition, any vector bundle $\mathcal{E}$ becomes trivial on an open cover of X, so $\mathbb{P}\mathcal{E} := \mathrm{Proj}\,\mathrm{Sym}(\mathcal{E}^*) \to X$ is a projective bundle, called the *projectivization of* $\mathcal{E}$. In fact, every projective bundle can be written as $\mathbb{P}\mathcal{E}$ for some vector bundle $\mathcal{E}$. Before we can prove this, we need to know a little more about the schemes $\mathbb{P}\mathcal{E}$. [1]

From the local description of $\mathbb{P}(\mathcal{E})$ as a product it follows that the points of $\mathbb{P}\mathcal{E}$ correspond to pairs (x, ξ) with $x \in X$ and ξ a one-dimensional subspace $\xi \subset \mathcal{E}_x$ of the fiber $\mathcal{E}_x$ of $\mathcal{E}$. The bundle $\pi^*\mathcal{E}$ on $\mathbb{P}\mathcal{E}$ thus comes equipped with a *tautological subbundle* of rank 1, whose fiber at a point $(x, \xi) \in \mathbb{P}\mathcal{E}$ is the the subspace $\xi \subset \mathcal{E}_x$ of the fiber $\mathcal{E}_x$ corresponding to the point $\xi \in \mathbb{P}E_x$. This subbundle is denoted $\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1) \subset \pi^*\mathcal{E}$. On an open set $U \subset X$ where $\mathcal{E}$ becomes trivial, so that $\pi^{-1}U = U \times \mathbb{P}^r$, the bundle $\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1)$ is the pullback of $\mathcal{O}_{\mathbb{P}^r}(-1)$ from the second factor. We write $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1) = Hom(\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1), \mathcal{O}_{\mathbb{P}\mathcal{E}})$ for the dual bundle. Dualizing the inclusion of the tautological bundle we get a surjection $\pi^*\mathcal{E}^* \to \mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$.

We can get an idea of the relation between $\mathbb{P}\mathcal{E}$ and $\mathcal{E}$ from the case where $\mathcal{E}$ is a line bundle. In this case $\mathbb{P}\mathcal{E}$ is locally $X \times \mathbb{P}^0$, so the projection $\pi : \mathbb{P}\mathcal{E} \to X$ is an isomorphism. Identifying $\mathbb{P}\mathcal{E}$ with X via π, we see that $\pi^*(\mathcal{E}) = \mathcal{E}$, and moreover $\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1) = \mathcal{E}$.

From this example we see that the bundles $\mathcal{E}$ and $\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1)$ are not determined by the scheme $\mathbb{P}\mathcal{E}$ or even by the map $\pi : \mathbb{P}\mathcal{E} \to \mathcal{E}$—rather, the bundle $\mathcal{E}$ is an additional piece of data that determines the bundle $\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1)$. We shall soon see that, in general, the projective bundle $\mathbb{P}\mathcal{E} \to X$ alone determines $\mathcal{E}$ up to tensor product with a line bundle (Proposition 11.3), and that the line bundle $\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1)$ on $\mathbb{P}\mathcal{E}$ determines $\mathcal{E}$ completely (Proposition 11.2.

[1] There is a conflicting definition that is also in use. Some sources, following Grothendieck, define the projectivization of $\mathcal{E}$ to be what we would call the projectivization of $\mathcal{E}^*$, that is, $\pi : \mathrm{Proj}(\mathrm{Sym}\,\mathcal{E}) \to X$. Its points correspond to 1-quotients of fibers of $\mathcal{E}$. We are following the classical tradition, which is also that adopted in Fulton [1984]. Grothendieck's convention is better adapted to the generalization from vector bundles to arbitrary coherent sheaves, which we will not use.

11.1.1 Example: rational normal scrolls

Before continuing with the general theory we pause to work out the case of projective bundles over $\mathbb{P}^1$. Vector bundles on $\mathbb{P}^1$ are particularly simple: each one is a direct sum of line bundles, $\oplus_{i=0}^r \mathcal{O}_{\mathbb{P}^1}(a_i)$.

Write $\mathbb{P}^1 = \mathbb{P}(V)$, where V is a vector space of dimension 2, with homogeneous coordinates $s, t \in V^*$. Recall that for $a \geq 1$, the *rational normal curve* of degree a is the image of the morphism

$$\varphi : \mathbb{P}^1 \to \mathbb{P}^a : \ (s,t) \mapsto (s^a, s^{a-1}t, \dots, t^a).$$

When $a = 0$ we take $\varphi : \mathbb{P}^1 \to \mathbb{P}^0$ to be the constant map. More invariantly, for $a \geq 0$, we can think of φ as the map $\mathbb{P}(V) \to \mathbb{P}^a = \mathbb{P}(W^*)$ given by the complete linear series

$$|\mathcal{O}_{\mathbb{P}^1}(a)| := \big(\mathcal{O}_{\mathbb{P}^1}(a), W_a\big),$$

where $W_a = H^0(\mathcal{O}_{\mathbb{P}^1}(a)) = \mathrm{Sym}_a(V^*)$. Identifying $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-a))$ with $\mathbb{P}^1$ via the projection π, we can rephrase what we have just reviewed by saying that for $a > 0$ the complete linear series $|\mathcal{O}_{\mathbb{P}\mathcal{O}_{\mathbb{P}^1}(a)}(1)|$ embeds $\mathbb{P}\mathcal{O}_{\mathbb{P}^1}(-a)$ as the rational normal curve of degree a.

Now fix a sequence of $r+1$ non-negative integers $a_0, \dots, a_r$. Let $\mathcal{E} = \oplus_i \mathcal{O}_{\mathbb{P}^1}(-a_i)$. We will analyze the projective bundle $\mathbb{P}\mathcal{E}$ by embedding it in a projective space $\mathbb{P}^N$ using the line bundle $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$.

Set $V_i = H^0\mathcal{O}_{\mathbb{P}^1}(a_i)$ and $V := H^0\mathcal{E}^* = \oplus_i V_i$. and write

$$N = \dim V - 1 = \sum_i (a_i + 1) - 1 = r + \sum_i a_i.$$

Inside $\mathbb{P}\mathcal{E}$ we consider the $r+1$ rational curves

$$C_i = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-a_i)) \cong \mathbb{P}^1.$$

There are natural maps

$$V = H^0\mathcal{E} \to H^0\pi^*\mathcal{E}^* \to H^0\mathcal{O}_{\mathbb{P}\mathcal{E}}(1),$$

and from the commutative diagram

$$\begin{array}{ccc}
\oplus_i V_i = V & \longrightarrow & H^0\mathcal{O}_{\mathbb{P}\mathcal{E}}(1) \\
\text{projection}\big\downarrow & & \big\downarrow\text{restriction to } C_i \\
V_i = H^0\mathcal{O}_{\mathbb{P}^1}(a_i) & \xrightarrow{\cong} & H^0\mathcal{O}_{\mathbb{P}\mathcal{O}_{\mathbb{P}^1}(-a_i)}(1),
\end{array}$$

we see that $V \to H^0\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ is a monomorphism, and that its restriction to C_i is the complete linear series $|\mathcal{O}_{\mathbb{P}^1}(a_i)|$. Let $\varphi_i : \mathbb{P}^1 \to \mathbb{P}(V_i) \subset \mathbb{P}(V)$ be the corresponding morphism, which embeds C_i as the rational normal curve of degree a_i as above.

For each $p \in \mathbb{P}^1$, the restriction of the linear series $\mathcal{W} := (\mathcal{O}_{\mathbb{P}\mathcal{E}}(1), V)$ to the fiber $\mathbb{P}^r = \pi^{-1}(p)$ is a sub-series of $|\mathcal{O}_{\mathbb{P}^r}(1)|$. Since the image contains the $r+1$ linearly independent points $\varphi_i(p)$, it is the complete linear series, and this plane is mapped isomorphically to the $\mathbb{P}^r$ that is the linear span of the points $\varphi_i(p)$. Thus the linear series $\mathcal{W}$ is base point free, and defines a morphism $\varphi : \mathbb{P}\mathcal{E} \to \mathbb{P}^N$.

We define the *rational normal scroll*

$$S(a_0, \dots, a_r) \subset \mathbb{P}(\oplus_i V_i) = \mathbb{P}^N$$

to be the image $\varphi(\mathbb{P}\mathcal{E})$. It is the union of the r-dimensional planes spanned by $\varphi_0(p), \dots, \varphi_r(p)$ as p runs over $\mathbb{P}^1$,

$$S := S(a_0, \dots, a_r) = \bigcup_{p \in \mathbb{P}^1} \overline{\varphi_0(p) \, \dots \, \varphi_r(p)}.$$

Since each $\mathbb{P}\mathcal{O}_{\mathbb{P}^1}(-a_i)$ is embedded by the restriction of $\mathcal{W}$, and the distinct $\varphi_i(p)$ are linearly independent for every $p \in \mathbb{P}^1$, it is already clear that φ is set-theoretically an injection.

In the next section we will show that $\mathcal{W}$ is the complete linear series $|\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)|$, and that when all $a_i > 0$ the map φ induces an isomorphism $\mathbb{P}\mathcal{E} \cong S$. The ideal of forms vanishing on a rational normal scroll is also easy to describe (Exercise 11.31).

We will also show that

$$\mathbb{P}(\oplus_{i=0}^r \mathcal{O}_{\mathbb{P}^1}(a_i)) \cong \mathbb{P}(\oplus_{i=0}^r \mathcal{O}_{\mathbb{P}^1}(b_i))$$

if and only if there is an integer b such that $b_i = a_i + b$ for all i; thus the description above can also be applied to describe the bundles $\mathbb{P}(\oplus_{i=0}^r \mathcal{O}_{\mathbb{P}^1}(a_i))$ even when some of the a_i are negative.

Some examples of this construction are already familiar. In the case $r = 0$ we have already noted that $S(a_0)$ is the rational normal curve of degree a_0 (or a point, if $a_0 = 0$.) From the construction of $S(1,1) \subset \mathbb{P}^3$ above as the union of lines joining corresponding points on two given disjoint lines, the images of φ_0 and φ_1, we see that $S(1,1)$ is the nonsingular quadric in $\mathbb{P}^3$: the lines in the union are the lines in one of the two rulings, while the images of φ_0 and φ_1 are two of the lines in the other ruling (see Figure 11.1).

If $a_r = 0$ then from the construction we see that $S(a_0, \dots, a_r)$ is a cone over $S(a_0, \dots, a_{r-1})$, and similarly for the other a_i. This remark allows us to reduce most questions about scrolls to the case where all $a_i > 0$ for all i. For example, the quadric in $\mathbb{P}^3$ with an isolated singularity, that is, the cone over a nonsingular conic in $\mathbb{P}^2$, can be described as $S(2,0)$ or $S(0,2)$.

To describe the first example beyond these, the scroll $S(1,2) \subset \mathbb{P}^4$, we choose an isomorphism between a line L and a nonsingular conic C lying

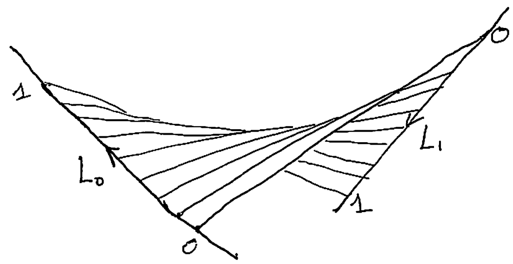

FIGURE 11.1. $S(1,1)$, the union of lines joining corresponding points on the parametrized lines L_0 and L_1, is a nonsingular quadric in $\mathbb{P}^3$.

in a plane disjoint from L. The scroll is then the union of the lines joining the points of L to the corresponding points of C.

There is much more to say about the geometry of rational normal scrolls, some of which will be deduced from the more general situation of projective bundles in the next sections. For more information see Eisenbud and Harris [1987] or Harris [1992] (look in the index under rational normal scrolls).

11.2 Maps to a projective bundle

Given a projective bundle $\pi : Y \to X$ we will choose an appropriate line bundle $\mathcal{L}$ on Y, and show that $Y \cong \mathbb{P}\mathcal{E}$, where $\mathcal{E} = \pi_*\mathcal{L}$, in such a way that $\mathcal{L} = \mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$. To construct the isomorphism $Y \to \mathbb{P}\mathcal{E}$, we will then use the following universal property, which generalizes the one for projective spaces:

Proposition 11.1 (Universal Property of Proj)**.** *Commutative diagrams of maps of schemes*

$$\begin{tikzcd} Y \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & \mathbb{P}\mathcal{E} \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

are in natural one-to-one correspondence with line subbundles $L \to p^*\mathcal{E}$.

Proof. Given φ we pull back $S \to \pi^*\mathcal{E}$ via φ and get $\varphi^*S \to \varphi^*\pi^*\mathcal{E} = p^*\mathcal{E}$. Conversely, given $L \to p^*\mathcal{E}$, we may cover X by open sets on which $\mathcal{E}$ and L are trivial, and get a unique map over each of these using the universal property of ordinary projective space. By uniqueness, these maps glue together to give a map over all of X. $\square$

To prepare for the next step we need, at the least, to know how to reconstruct $\mathcal{E}$ from a line bundle on $\mathbb{P}_X\mathcal{E}$. For future use, we will treat an easy generalization. Write $\mathcal{O}_{\mathbb{P}\mathcal{E}}(m)$ for the m-th tensor power of $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$. Thus (for any integer m) the sheaf $\mathcal{O}_{\mathbb{P}\mathcal{E}}(m)$ is the sheaf on $\mathrm{Proj}(\mathrm{Sym}(\mathcal{E}^*)$ associated to the sheaf of $\mathrm{Sym}\,\mathcal{E}^*$-modules $(\mathrm{Sym}\,\mathcal{E}^*)(m)$ on X, obtained by shifting the grading of $\mathrm{Sym}(\mathcal{E}^*)$. For any quasicoherent sheaf $\mathcal{F}$ on $\mathbb{P}\mathcal{E}$ we write $\mathcal{F}(m)$ to denote $\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}\mathcal{E}}(m)$.

The surjection $\pi^*\mathcal{E}^* \to \mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$, restricted to the fiber over a point $(x,\xi) \in \mathbb{P}\mathcal{E}$, sends a linear form on $\mathcal{E}_x$ to its restriction to the subspace $\xi \subset \mathcal{E}_x$. Thus any global section σ of $\mathcal{E}^*$ gives rise to a global section $\tilde{\sigma}$ of $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$. The following result strengthens and extends this observation.

Proposition 11.2. *If $\pi : \mathbb{P}\mathcal{E} \to X$ is a projectivized vector bundle on X, then for $m \geq 0$,*

$$\pi_*\mathcal{O}_{\mathbb{P}\mathcal{E}}(m) = \mathrm{Sym}_m\,\mathcal{E}^*,$$

and $R^i\pi_*\mathcal{O}_{\mathbb{P}\mathcal{E}}(m) = 0$ for $i > 0$.

Taking $m = 1$ we see that the map $\pi : \mathbb{P}\mathcal{E} \to X$, together with the tautological line bundle $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ (or $\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1)$), determines $\mathcal{E}$.

Proof. Suppose that $\mathcal{E}$ has rank $r+1$. Over an affine open set $U \subset X$ where $\mathcal{E}|_U \cong \mathcal{O}_U^{r+1}$, the natural maps $H^0(\pi^*\,\mathrm{Sym}_m\,\mathcal{E}|_U) \to H^0(\mathcal{O}_{\mathbb{P}\mathcal{E}}(m)|_U)$ are isomorphisms, while $H^i(\mathcal{O}_{\mathbb{P}\mathcal{E}}(m)|_U) = 0$ for $i > 0$ so the Proposition follows immediately from the definition of the direct image functors. □

Remark. Proposition 11.2 is a direct generalization of the usual computation of $H^0(\mathcal{O}_{\mathbb{P}^r}(m)$—the case when X is a point. Though we will not make use of these facts, the rest of the computation of the cohomology of line bundles on a projective space, and Serre duality, also generalizes, and one can show that

$$R^i\pi_*\mathcal{O}_{\mathbb{P}\mathcal{E}}(m)) = \begin{cases} \mathrm{Sym}_m\,\mathcal{E}^* & \text{for } i = 0 \\ 0 & \text{for } 0 < i < r-1 \\ \mathrm{Sym}_{m-r-1}\,\mathcal{E} & \text{for } i = r \end{cases}$$

As a part of our computation of the Chow ring of $\mathbb{P}\mathcal{E}$ in the next section we will see that every line bundle on $\mathbb{P}\mathcal{E}$ has the form $\pi^*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}\mathcal{E}}(m)$ for a unique line bundle $\mathcal{L}$ on X and integer m; that is, $\mathrm{Pic}(\mathbb{P}\mathcal{E}) \cong \mathrm{Pic}\,X \oplus \mathbb{Z}$. From the push-pull formula of Proposition 6.8, we get a computation of the direct images of any line bundle:

$$R^i\pi_*(\pi^*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}\mathcal{E}}(m)) = \mathcal{L} \otimes R^i\pi_*(\mathcal{O}_{\mathbb{P}\mathcal{E}}(m)).$$

SeeDieudonné [1969], p.308 for equivalent material, with references to EGA.

Serre duality also generalizes to a *relative duality.* For example, setting

$$\omega_{\mathbb{P}/B} = \wedge^r\mathcal{E}(-r-1)$$

we have $R^r\pi_*(\omega_{\mathbb{P}/B}) = \mathcal{O}_B$, and more generally

$$R^r\pi_*(\mathcal{M}) = \mathrm{Hom}(\pi_*(\omega \otimes \mathcal{M}^{-1}), \mathcal{O}_B)$$

for any line bundle $\mathcal{M}$ on $\mathbb{P}$.

SeeAltman and Kleiman [1970], in particular Theorem 3.8, for most of this.

Supposing that $\mathcal{E}^*$ has a global section σ, the proof of Proposition 11.2 shows that the corresponding section $\tilde{\sigma}$ of $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ vanishes on the locus of pairs (x, ξ) such that σ_x vanishes on ξ; thus the divisor $(\tilde{\sigma})$ meets a general fiber of $\pi : \mathbb{P}\mathcal{E} \to X$ in a hyperplane. It will not in general meet *every* fiber of $\mathbb{P}\mathcal{E} \to X$ in a hyperplane, however; the section σ of $\mathcal{E}^*$ may have zeros $x \in X$; and the divisor $(\tilde{\sigma}) \subset \mathbb{P}\mathcal{E}$ will contain the corresponding fibers $(\mathbb{P}\mathcal{E})_x = \pi^{-1}(x)$.

Using these ideas we can characterize the schemes over X that are projective bundles:

Proposition 11.3. *Let $\pi : Y \to X$ be a smooth finite type morphism of projective schemes whose closed scheme-theoretic fibers are all isomorphic to $\mathbb{P}^r$. The following are equivalent:*

(a) *$Y = \mathbb{P}\mathcal{E}$ is the projectivization of a vector bundle $\mathcal{E}$ on X.*
(b) *$\pi : Y \to X$ a projective bundle: that is, it is locally isomorphic to a product in the Zariski topology on X.*
(c) *There exists a line bundle L on Y whose restriction to each fiber $Y_x \cong \mathbb{P}^r$ of π is isomorphic to $\mathcal{O}_{\mathbb{P}^r}(1)$.*
(d) *There exists a Cartier divisor $D \subset Y$ intersecting a general fiber $Y_x \cong \mathbb{P}^r$ of π in a hyperplane.*

Proof. Condition (a) clearly implies (b) and (c): the projectivization of a vector bundle is locally trivial in the Zariski topology, since a vector bundle is, and comes with the line bundle $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$. We have already shown that (b) implies (c). Also, it is easy to see that (c) and (d) are equivalent: if D is a divisor as in (d), the line bundle $L = \mathcal{O}_Y(D)$ satisfies condition (c) and conversely if L is a line bundle as in (c), tensoring with the pullback of an ample line bundle from X we can assume the existence of a nonzero global section of L, whose zero locus will be the divisor of part (d).

To complete the argument we take L as in part (c), and we must prove that Y is as in part (a). For any $p \in X$ we have $H^1(L_p) = H^1\mathcal{O}_{\mathbb{P}^r}(1) = 0$, so Theorem **??** shows that $\mathcal{E} := \pi_*L$ is a vector bundle whose fiber at p is $H^0(\mathcal{O}_{\mathbb{P}^r}(1))$.

We claim that there is an isomorphism $\alpha : Y \to \mathbb{P}\mathcal{E}$ such that $\pi = \pi_{\mathcal{E}}\alpha$. By Proposition 11.1 we can define the morphism α by giving a line bundle that is a subbundle of $\pi^*(\mathcal{E})$, or equivalently a line bundle that is a homomorphic image of $\pi^*(\mathcal{E}^*) = \pi^*\pi_* L$.

There is a natural map $\pi^*\pi_* L \to L$ coming from the definitions of π^* and π_*. Restricted to the fiber over a point p this map becomes the surjection $\mathcal{E}_p \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}_p)} \to \mathcal{O}_{\mathbb{P}(\mathcal{E}_p)}(1)$, so $\pi^*\pi_* L \to L$ is surjective. Let $\alpha : Y \to \mathbb{P}\mathcal{E}$ be the corresponding morphism.

The map α is an isomorphism on each fiber of π because it restricts to the map $\pi^{-1}(p) \cong \mathbb{P}^r \to \mathbb{P}^r$ given by the complete linear series $|\mathcal{O}_{\mathbb{P}^r}(1)|$. This shows that α is a set-theoretic isomorphism.

To prove that α is a scheme-theoretic isomorphism we need to show that if α carries $y \in Y$ to a point $q \in \mathbb{P}\mathcal{E}$, then the map of local rings $\alpha^* : \mathcal{O}_{\mathbb{P}\mathcal{E},q} \to \mathcal{O}_{Y,y}$ is an isomorphism. Of course it is enough to prove this after completing both rings. Set $p = \pi(y)$. By smoothness, the completions of both local rings are isomorphic to $\hat{\mathcal{O}}_{X,p}[[z_0, \dots z_r]]$. Since α commutes with the projections, it induces the identity modulo the maximal ideal of $\mathcal{O}_{X,p}$, and thus induces an isomorphism. $\square$

****We should think again what the right statement is! $Y \to Z$ is a flat map of schemes over a scheme X, and $Y \to Z$ is an isomorphism on scheme-theoretic fibers, is it an isomorphism?****

We can also use Proposition 11.1 to see when two vector bundles give the same projective bundle:

Corollary 11.4. *Let X be a scheme. Two projective bundles $\pi : \mathbb{P}\mathcal{E} \to X$ and $\pi' : \mathbb{P}\mathcal{E}' \to X$ are isomorphic (as schemes over X) if and only if there is a line bundle L on X such that $L \otimes \mathcal{E}' = \mathcal{E}$. In this case the line bundle $\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1)$ corresponds under the isomorphism to $\pi^*(L) \otimes \mathcal{O}_{\mathbb{P}\mathcal{E}'}(-1)$.*

Proof. Let L be a line bundle and set $\mathcal{E}' = L \otimes \mathcal{E}$. Tensoring the tautological subbundle $\mathcal{O}_{\mathbb{P}\mathcal{E}'}(-1) \to \pi'^*\mathcal{E}' = \pi'^*L \otimes \pi'^*\mathcal{E}$ with $\pi'^*(L^{-1})$ we get a subbundle $\pi'^*(L^{-1}) \otimes \mathcal{O}_{\mathbb{P}\mathcal{E}'}(-1) \to \pi'^*\mathcal{E}$. By Proposition 11.1 this determines a unique morphism of X-schemes $\varphi : \mathbb{P}\mathcal{E}' \to \mathbb{P}\mathcal{E}$ such that

$$\varphi^*\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1) = \pi'^*(L^{-1}) \otimes \mathcal{O}_{\mathbb{P}\mathcal{E}'}(-1).$$

The inverse map is defined similarly. The proof that they are inverse to each other is that the composite $\mathbb{P}\mathcal{E} \to \mathbb{P}(L \otimes \mathcal{E}) \to \mathbb{P}\mathcal{E}$ corresponds to the original subbundle $\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1) \subset \pi^*\mathcal{E}$.

Conversely, suppose that $\mathcal{E}'$ is a vector bundle on X and let $\pi' : \mathbb{P}\mathcal{E}' \to X$ be the projection. If $\varphi : \mathbb{P}\mathcal{E}' \to \mathbb{P}\mathcal{E}$ is an isomorphism commuting with the projections to X, then since any isomorphism from $\mathbb{P}^n$ to itself preserves the bundle $\mathcal{O}_{\mathbb{P}^n}(1)$, it follows that $\varphi^*\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ restricts on each fiber $\mathbb{P}(\mathcal{E}'_x) \cong \mathbb{P}^n$ to the bundle $\mathcal{O}_{\mathbb{P}^n}(1)$. By Corollary 6.7, $\mathcal{O}_{\mathbb{P}\mathcal{E}'}(1) = \pi'^*(L) \otimes \varphi^*\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ for some line bundle L on X. Thus

$$\begin{aligned}\mathcal{E}'^* = \pi'_*(\mathcal{O}_{\mathbb{P}\mathcal{E}'}(1)) &= \pi'_*(\pi'^* L \otimes \varphi^* \mathcal{O}_{\mathbb{P}\mathcal{E}}(1)) \\ &= L \otimes \pi'_* \varphi^* \mathcal{O}_{\mathbb{P}\mathcal{E}}(1) \\ &= L \otimes \pi'_* \varphi_*^{-1} \mathcal{O}_{\mathbb{P}\mathcal{E}}(1) \\ &= L \otimes \pi_* \mathcal{O}_{\mathbb{P}\mathcal{E}}(1) \\ &= L \otimes \mathcal{E}^*.\end{aligned}$$

and also $\varphi^* \mathcal{O}_{\mathbb{P}\mathcal{E}}(-1) = \pi^*(L) \otimes \mathcal{O}_{\mathbb{P}\mathcal{E}'}(-1)$, as claimed. □

11.3 Ample vector bundles

Recall that a linear series on a scheme X is called *very ample* if it defines an embedding *very* ample if the complete $|L|$ is very ample for $n \gg 0$. A vector bundle $\mathcal{F}$ on a scheme X is called *very ample* if the line bundle $\mathcal{O}_{\mathbb{P}\mathcal{F}^*}(1)$ very ample. We will show that a bundle is very ample if it can be written as the tensor product of a bundle generated by global sections and a very ample bundle line bundle. This implies, in particular that the bundle $\oplus_i \mathcal{O}_{\mathbb{P}^1}(a_i)$ on $\mathbb{P}^1$ is very ample (or, for that matter, ample) if $a_i > 0$ for all i, since it can then be written as $\oplus_i \mathcal{O}_{\mathbb{P}^1}(a_i - 1) \otimes \mathcal{O}_{\mathbb{P}^1}(1)$. A general criterion for a bundle to be very ample is given, with an application, in Exercise 11.36.

There is also a useful relative notion: If $\pi : Z \to X$ is a morphism of schemes and $\mathcal{W}$ is a base-point free linear series on Z whose restriction to each fiber of π is very ample, then we say that $\mathcal{W}$ is *relatively very ample*.

Lemma 11.5. *Suppose that $\pi : Z \to X$ is a morphism of projective schemes. If $\mathcal{W}$ is a relatively very ample linear series on Z and $\mathcal{W}'$ is a very ample linear series on X, then the series $\mathcal{W}(\pi^*\mathcal{W}')$ is very ample on Z.*

Proof. Let $\mathcal{W} = (\mathcal{L}, W)$ and let $\mathcal{W}' = (\mathcal{L}', W')$, so that, by definition,

$$\mathcal{W}\pi^*(\mathcal{W}') = (\mathcal{L} \otimes \pi^*\mathcal{L}', W\pi^*(W')).$$

We must show that vanishing on any subscheme $Y \subset Z$ of degree 2 imposes two independent conditions on elements of $W\pi^*(W')$. Let $p \in Y$ *subsetZ* be a reduced point. We consider two cases:

First if Y is contained in a fiber of π, then, since there is a section $\sigma' \in W'$ that does not vanish on $\pi(Y)$, and a section $\sigma \in W$ that vanishes on p but not on Y; thus $\sigma\pi^*(\sigma')$ vanishes on p but not Y.

Next suppose that Y is not contained in a fiber of π. Since $\mathcal{W}$ is base point free we can find a section $\sigma \in W$ that does not vanish at p. Since Y

is not contained in a fiber, the image πY in X is again a scheme of degree 2. Because $\mathcal{W}'$ is very ample there is a section $\sigma' \in W'$ that vanishes on $\pi(p)$ but does not vanish on $\pi(Y)$ and again $s\pi^*(\sigma')$ vanishes on p but not Y. $\square$

Proposition 11.6. *If $\mathcal{F}$ is a vector bundle on a scheme X that is generated by global sections, and $\mathcal{L}$ is a very ample line bundle on X, then $\mathcal{L} \otimes \mathcal{F}$ is very ample. In particular, if $\mathcal{L}$ is an ample line bundle and $\mathcal{G}$ is any vector bundle, then $\mathcal{L}^n \otimes \mathcal{G}$ is an ample vector bundle when n is sufficiently large.*

Proof of Proposition 11.6. Since $\mathcal{F}$ is generated by global sections, so is $\pi^*\mathcal{F}$, and since this bundle surjects onto $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$, this line bundle is also generated by global sections. Moreover, by Proposition 11.2 we have $\pi_*\mathcal{O}_{\mathbb{P}\mathcal{E}}(1) = \mathcal{F}$, so $H^0(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)) = H^0\mathcal{F}$. Since $\mathcal{F}$ is generated by global sections, this surjects onto each fiber $\mathcal{F}_p = H^0\mathcal{O}_{\mathbb{P}\mathcal{E}}|_{\pi^{-1}(p)}$; thus the restriction of the complete linear series $|(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)|$ to any fiber of π is very ample. Lemma 11.5 completes the proof. $\square$

Example 11.7 (Elliptic Quintic Scrolls)**.** The condition for ampleness in Proposition 11.6 is sufficient but not necessary. A necessary and sufficient condition is given in Exercise 11.36. With the theory of vector bundles on an elliptic curve developed in Exercise 11.37, this yields the existence of a famous class of projective surfaces, the elliptic quintic scrolls, which are embeddings of varieties of the form $\mathbb{P}\mathcal{E}$ where $\mathcal{E}$ is an indecomposable vector bundle of rank 2 and degree 5 on an elliptic curve.

One of the many results suggesting that the definition of very ampleness for vector bundles given above is the "right" one is the following.

Proposition 11.8. *If $\mathcal{E}$ is a very ample vector bundle on X, then $\mathcal{E}$ is generated by global sections.*

For other evidence of this type see Hartshorne [1970].

Proof. By hypothesis, $L := \mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ is very ample, so the restriction of H^0L to each fiber $\pi^{-1}(x) \cong \mathbb{P}^r$ of $\mathbb{P}(\mathcal{E}) \to X$ is a very ample linear series of linear forms. It follows that $H^0(L)$ restricts to an $r+1$-dimensional space of sections on $\pi^{-1}(x)$. There is an isomorphism

$$H^0\mathcal{E} \to H^0\pi_*L = H^0L$$

induced by the composite isomorphism

$$\mathcal{E} \to \pi_*\pi^*\mathcal{E} \to \pi_*L.$$

It follows that, under this isomorphism, any global section of $\mathcal{E}$ that vanishes at $x \in X$ maps to a global section of L that vanishes along $\pi^{-1}(x)$, so $H^0\mathcal{E}/(H^0(\mathcal{I}_{x/X}\mathcal{E})$ has dimension at least $r+1$. As this group is contained in

$H^0(\mathcal{E}_x)$, and the latter has only dimension $r+1$, we see that they are equal; that is, the global sections of $\mathcal{E}$ generate $\mathcal{E}$ at each point, as required. $\square$

11.4 Chow ring of a projective bundle

We now turn to the central problem of this Chapter: to describe the Chow ring of a projective bundle $Y = \mathbb{P}\mathcal{E} \to X$. We will see that the Chow groups of Y depend only on the rank of $\mathcal{E}$, but the ring structure reflects the Chern classes of $\mathcal{E}$.

As we mentioned in Section **??** the Künneth Theorem holds for the Chow ring of the product of any smooth variety with a projective space. Thus if

$$Y = \mathbb{P}(\mathcal{O}_{\mathbb{P}^r}^{r+1}) = X \times \mathbb{P}^r,$$

then

$$\begin{aligned} A(Y) &\cong A(X) \otimes_{\mathbb{Z}} A(\mathbb{P}^r) \\ &\cong A(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta]/(\zeta^{r+1}) \\ &\cong A(X)[\zeta]/(\zeta^{r+1}). \end{aligned}$$

where ζ is the pullback of the hyperplane class on $\mathbb{P}^r$. In particular,

$$A(Y) = \bigoplus_{i=0}^{r} \zeta^i A(X)$$

as groups. Here and in what follows we think products of the form $\alpha\beta$ with $\alpha \in A(Y)$ and $\beta \in A(X)$ as meaning $\alpha(\pi^*\beta) \in A(Y)$.

The general case is not much more complicated:

Theorem 11.9. *Let $\mathcal{E}$ be a vector bundle of rank $r+1$ on a smooth projective scheme X, and let $\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)) \in A^1(\mathbb{P}\mathcal{E})$. Let $\pi : \mathbb{P}\mathcal{E} \to X$ be the projection. The map $\pi^* : A(X) \to A(\mathbb{P}\mathcal{E})$ is an injection of rings, and via this map*

$$A(\mathbb{P}\mathcal{E}) \cong A(X)[\zeta]/\big(\zeta^{r+1} + c_1(\mathcal{E})\zeta^r + \cdots + c_{r+1}(\mathcal{E})\big).$$

In particular, the group homomorphism $A(X)^{\oplus r+1} \to A(\mathbb{P}\mathcal{E})$ given by $(\alpha_0, \dots, \alpha_r) \mapsto \sum \zeta^i \pi^*(\alpha_i)$ is an isomorphism, so that

$$A(\mathbb{P}\mathcal{E}) \cong \bigoplus_{i=0}^{r} \zeta^i A(X)$$

as groups.

It's worth remarking that much of the statement of Theorem 11.9 remains true without the assumption that X is smooth: if E is a vector bundle of rank $r+1$ over an arbitrary scheme X and $\mathbb{P}E = \mathrm{Proj}(\mathrm{Sym}\, E^*)$

its associated projective bundle, then we have a well-defined line bundle $\mathcal{O}_{\mathbb{P}E}(1)$ on $\mathbb{P}E$ such that $\zeta = c_1(\mathcal{O}_{\mathbb{P}E}(1)$ restricts to the hyperplane class on each fiber, and we can show that

$$A(\mathbb{P}\mathcal{E}) \cong \bigoplus_{i=0}^{r} \zeta^i A(X)$$

just as in the smooth case (see Fulton [1984] Chapter 3). Indeed, the result was inverted and used by Grothendieck to *define* the Chern classes of $\mathcal{E}$ as the coefficients in the (unique) expression of ζ^{r+1} as a linear combination of the classes $1, \zeta, \ldots, \zeta^r$. As a definition, this has the advantage of not depending on global sections (as in the definition we have chosen), but it does not seem to us to shed much light on the properties and uses of the Chern classes.

We isolate part of the proof that will be useful elsewhere:

Lemma 11.10. *Let the hypotheses be as in Theorem 11.9. If $\alpha \in A(X)$ then*

$$\pi_*(\zeta^i \alpha) = \begin{cases} \alpha & \text{if } i = r \\ 0 & \text{if } i < r. \end{cases}$$

Proof. By the Push-Pull formula (Proposition 6.8), $\pi_*(\zeta^i\alpha) = \pi_*(\zeta^i)\alpha$. If $i < r$, then $\pi_*(\zeta^i)$ is zero for dimension reasons. If $i = r$ we see similarly that $\pi_*(\zeta^r)$ must be a multiple $m[X] \in A^0(X)$ of the fundamental class of X. Let η be the class of a point $x \in X$, and $f = \pi^*(\eta) = [\mathbb{P}\mathcal{E}_x]$ the class of the fiber $\mathbb{P}\mathcal{E}_x \cong \mathbb{P}^r$. Intersecting both sides of the equality $\pi_*(\zeta^r) = m[X]$ with η and taking degrees, we have

$$\begin{aligned} m &= \deg(\pi_*(\zeta^r) \cdot \eta) \\ &= \deg(\zeta^r \cdot [\mathbb{P}^r]) \\ &= 1 \end{aligned}$$

since the restriction of ζ to a fiber is the hyperplane class. $\square$

Proof of Theorem 11.9. Let $\psi : A(\mathbb{P}\mathcal{E}) \to \oplus_{i=0}^r A(X)\zeta^i$ be the map

$$\beta \mapsto \sum_i \pi_*(\zeta^{r-i}\beta)\zeta^i.$$

and let $\varphi : \oplus_{i=0}^r A(X) \to A(\mathbb{P}\mathcal{E})$ be the sum of the multiplications by powers of ζ,

$$\varphi : (\alpha_0, \ldots, \alpha_r) \mapsto \sum_i \zeta^i \alpha_i$$

By Lemma 11.10 the composite $\psi\varphi$ is upper triangular with ones on the diagonal; in particular, φ is a monomorphism.

To prove the additive part of Theorem 11.9 it now suffices to show that the subgroups $\zeta^i A(X)$ generate $A(\mathbb{P}\mathcal{E})$ additively. This is a relative version of the fact that the linear subspaces of a projective space generate its Chow ring, and the proof runs along the same lines. In the case of a single projective space, we used the technique of dynamic projection to degenerate a given subvariety $Z \subset \mathbb{P}^n$ to a multiple of a linear space; we do the same thing here, but in a family of projective spaces.

We start with a definition. If $Z \subset \mathbb{P}\mathcal{E}$ is a k-dimensional subvariety, we say that Z has *footprint* l if the image $W = \pi(Z)$ has dimension l, or equivalently, if the general fiber of the map $\pi : Z \to W$ has dimension $k - l$.

Lemma 11.11. *If $Z \subset \mathbb{P}\mathcal{E}$ is a subvariety of dimension k and footprint l then*

$$Z \sim Z' + \sum n_i B_i$$

for some subvarieties $B_i \subset \mathbb{P}\mathcal{E}$ such that:

(a) $[Z'] = \zeta^{r-k+l}\alpha$ *for a class* $\alpha \in A(X)$*; and*

(b) *each B_i has footprint strictly less than l.*

Applying the Lemma repeatedly, we can express the class of an arbitrary subvariety as a sum of classes of the form $\zeta^i\alpha$, establishing the group isomorphism $A(\mathbb{P}\mathcal{E}) \cong \oplus\zeta^i A(X)$.

Proof of Lemma 11.11. By Proposition 11.6 the bundle $\mathcal{E}^*$ will become very ample if we tensor it with a sufficiently ample line bundle $\mathcal{L}$. By Corollary 11.4, this does not change $\mathbb{P}\mathcal{E}$ but has the effect of replacing the class ζ by $\zeta + c_1(\pi^*\mathcal{L})$, so it doesn't affect the truth of our assertion. Thus we may assume from the outset that $\mathcal{E}^*$ is very ample. By Proposition 11.8, $\mathcal{E}$ is generated by its global sections.

This done, we choose a point $x \in \pi(Z) \subset X$ and a general collection $\tau_0, \dots, \tau_r$ of global sections of $\mathcal{E}^*$, making sure that the τ_i satisfy two conditions:

(a) $\tau_0(x), \dots, \tau_r(x)$ are independent, that is, span the fiber $\mathcal{E}_x$; and

(b) The zero locus $(\tau_0(x) = \cdots = \tau_{k-l}(x) = 0) \subset \mathbb{P}\mathcal{E}_x$ is disjoint from the fiber $Z_x = Z \cap \mathbb{P}\mathcal{E}_x$ of Z over x.

These are both open conditions; let $U \subset X$ be the locus of $x \in X$ where they hold. Note in particular that by the first condition, the bundle $\mathbb{P}\mathcal{E}$ is trivial over U, the sections $\tau_0, \dots, \tau_r$ giving an isomorphism $\mathbb{P}\mathcal{E}_U \cong U \times \mathbb{P}^r$.

Now consider the one-parameter group of automorphisms A_t of $\mathbb{P}\mathcal{E}_U \cong U \times \mathbb{P}^r$ given, in terms of this trivialization, by the matrix

$$\begin{pmatrix} I_{k-l+1} & 0 \\ 0 & t \cdot I_{r-k+l} \end{pmatrix}.$$

Let $\tilde{Z} = Z \cap \mathbb{P}\mathcal{E}_U$ be the preimage of U in Z; let Z_t be the closure of the image $A_t(\tilde{Z})$ and let Z_0 be the limiting cycle, as $t \to 0$, of the subvarieties Z_t. In other words, let $\tilde{\Phi} \subset \mathbb{A}^1 \times \mathbb{P}\mathcal{E}$ be the incidence correspondence

$$\tilde{\Phi} = \{(t, p) \in \mathbb{A}^1 \times \mathbb{P}\mathcal{E} \mid t \neq 0 \text{ and } p \in A_t(\tilde{Z})\};$$

let Φ be the closure of $\tilde{\Phi}$ and let Z_0 be the fiber of Φ over $t = 0$.

What does Z_0 look like? Over the open subset $U \subset X$ the original cycle Z has been flattened to a multiple of the zero locus $\tau_{k-l+1} = \cdots = \tau_r = 0$. There is thus a unique component Z' of Z_0 dominating $W = \pi(Z)$, and it is the closure of the intersection of the common zero locus $\tau_{k-l+1} = \cdots = \tau_r = 0$ with the preimage $\pi^{-1}(W \cap U)$.

Now, the zero loci $\tau_i = 0$ in $\mathbb{P}\mathcal{E}$ are zero loci of general sections of $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$, and since we have arranged that $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ is very ample, we may regard these as general hyperplane sections of $\mathbb{P}\mathcal{E}$ in a projective embedding. Thus the common zero locus $\tau_{k-l+1} = \cdots = \tau_r = 0$ of $r - k + l$ of them intersects the subvariety $\pi^{-1}(W)$ generically transversely, in an irreducible, k-dimensional subvariety of $\mathbb{P}\mathcal{E}$ with class $[W] \cdot \zeta^{r-k+l}$; in sum,

$$[Z'] = m[W] \cdot \zeta^{r-k+l}$$

for some multiplicity m.

To complete the proof we note that we don't need to know what happens over the complement of $U \cap W$ in W, because any component of Z_0 not dominating W necessarily has footprint smaller than l. □

From this description of the Chow groups we see that we can write ζ^{r+1} as a linear combination of products of (pullbacks of) classes in $A(X)$ with lower powers of ζ—that is, ζ satisfies a monic polynomial f of degree $r + 1$ over $A(X)$. Thus the ring homomorphism $A(X)[\zeta] \to A(\mathbb{P}\mathcal{E})$ factors through the quotient $A(X)[\zeta]/(f)$. Since $A(X)[\zeta]/(f) \cong \oplus \zeta^i \mathbb{A}(X)$ as groups, it follows that the map $A(X)[\zeta]/(f) \to A(\mathbb{P}\mathcal{E})$ is an isomorphism of rings.

It remains to identify the polynomial f. Let $S = \mathcal{O}_{\mathbb{P}\mathcal{E}}(-1)$, and let Q be the cokernel of the natural inclusion $S \to \pi^* E$, a bundle of rank r. We have an exact sequence

$$0 \to S \to \pi^* E \to Q \to 0.$$

Identifying $A(X)$ with $\pi^*(A(X))$ as before, we have

$$c(S) \cdot c(Q) = c(E)$$

by the Whitney formula, Proposition ??.

We defined the class ζ to be the first Chern class of the line bundle $\mathcal{O}_{\mathbb{P}E}(1)$, which is the dual of S; thus $c(S) = 1 - \zeta$ and we can write this as

$$c(Q) = c(E) \cdot c(S)^{-1} \tag{11.1}$$
$$= c(E)(1 + \zeta + \zeta^2 + \dots). \tag{11.2}$$

Since Q is a vector bundle of rank r, we conclude that

$$0 = c_{r+1}(Q) = \zeta^{r+1} + c_1(E)\zeta^r + c_2(E)\zeta^{r-1} + \cdots + c_r(E)\zeta + c_{r+1}(E),$$

so the polynomial f is given by the formula in the Theorem. $\square$

If L is a line bundle on X then Corollary 11.4 shows that $\mathbb{P}\mathcal{E} \cong \mathbb{P}(E \otimes L)$, but the class ζ is different in the two representations; the two classes differ by multiplication with the pull-back of L. The relation between the two resulting descriptions of the Chow ring is addressed in Exercises 11.32 and 11.33.

Using Theorem 11.9 we can immediately compute the degrees of rational normal scrolls, or more generally of any projectivized vector bundle $\mathbb{P}\mathcal{E}$ over a curve X, embedded by $|\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)|$:

Corollary 11.12. *If $a_0, \dots, a_r$ are positive integers then the degree of the rational normal scroll $S(a_0, \dots, a_r)$ is $\sum_i a_i$. More generally, if X is a smooth curve, and $\mathcal{E}$ is a very ample vector bundle on X, then the degree of the image of $\mathbb{P}\mathcal{E}$ under the embedding given by $|\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)|$ is $-\deg c_1\mathcal{E}$.*

Note that degree and codimension of a scroll S satisfies the equation

$$\deg S = 1 + \operatorname{codim} S.$$

This is the minimal degree for any subvariety of projective space not contained in a hyperplane. The Veronese surface in $\mathbb{P}^4$, and any cone over it, also satisfy this equation, but these are the only "varieties of minimal degree". See Harris [1992] Theorem 19.19.

Proof. if the rank of $\mathbb{P}\mathcal{E}$ is $r+1$ then the dimension of $\mathbb{P}\mathcal{E}$ is $r+1$, so the degree of the image of $\mathbb{P}\mathcal{E}$ under the embedding given by $|\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)|$ is $\deg \zeta^{r+1}$. Since X is a 1-dimensional we have $c_i\mathcal{E} = 0$ for $i > 1$, so $\zeta^{r+1} = -c_1\mathcal{E}$. If $X = \mathbb{P}^1$ and

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(-a_0) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-a_r),$$

then $c_1\mathcal{E} = -\sum_i a_i$ and $S(a_0, \dots, a_r)$ is the embedding of $\mathbb{P}(\mathcal{E})$ by $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$. $\square$

11.4.1 The universal k-plane over the Grassmannian $\mathbb{G}(k,n)$

In this section and the next, we'll use Theorem 11.9 to give a description of the Chow ring of some varieties that arise often in algebraic geometry: the universal k-plane over the Grassmannian $\mathbb{G}(k,n)$, and the blowup of $\mathbb{P}^n$ along a linear space.

For the first of these, let $G = \mathbb{G}(k, \mathbb{P}V)$ be the Grassmannian parametrizing k-planes $\Lambda \subset \mathbb{P}V$ in the projectivization of an $(n+1)$-dimensional vector space V, and let Φ be the universal plane

$$\Phi = \big\{(\Lambda, p) \in G \times \mathbb{P}V \mid p \in \Lambda\big\}$$

We can recognize Φ, via the projection $\pi : \Phi \to G$ on the first factor, as the projectivization $\mathbb{P}\mathcal{S}$ of the universal subbundle on G, and use Theorem 11.9 to describe $A(\Phi)$. We'll use the notation introduced above: we'll identify $A(G)$ with its image in $A(\Phi)$ via the pullback map π^*, and denote the first Chern class of the tautological bundle $\mathcal{O}_{\mathbb{P}\mathcal{S}}(1)$ by $\zeta \in A^1(\Phi)$.

Note that a linear form $l \in V^*$ on V gives rise to a section of $\mathcal{S}^*$ by restriction in turn to each subspace of V, hence to a section of $\pi^*\mathcal{S}$, and ultimately to a section of $\mathcal{O}_{\mathbb{P}\mathcal{S}}(1)$ via the surjection $\pi^*\mathcal{S}^* \to \mathcal{O}_{\mathbb{P}\mathcal{S}}(1)$ dual to the tautological inclusion $\mathcal{O}_{\mathbb{P}\mathcal{S}}(-1) \hookrightarrow \pi^*\mathcal{S}$. Simply put, if we think of $\Phi = \mathbb{P}\mathcal{S}$ as the variety of pairs $(\tilde{\Lambda}, \xi)$ with $\tilde{\Lambda} \subset V$ a $(k+1)$-dimensional subspace and $\xi \subset \tilde{\Lambda}$ a one-dimensional subspace, then we can define a section σ_l of $\mathcal{O}_{\mathbb{P}\mathcal{S}}(1)$ by setting

$$\sigma_l(\tilde{\Lambda}, \xi) = l|_{\xi}.$$

In particular, we see that the zero locus of the section σ_l is just the locus of $(\tilde{\Lambda}, \xi)$ such that ξ is contained in the hyperplane $\mathrm{Ker}(l) \subset V$, and hence *the tautological class $\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{S}}(1) \in A^1(\Phi)$ is just the pullback of the hyperplane class on $\mathbb{P}V$ via the projection map $\eta : \Phi \to \mathbb{P}V$ on the second factor.*

Recalling the calculation of the Chern classes of the universal bundles on $\mathbb{G}(k,n)$ from Section 7.3.5 and applying Theorem 11.9, we conclude the

Proposition 11.13. *Let $G = \mathbb{G}(k,n)$ be the Grassmannian of k-planes in $\mathbb{P}^n$ and $\Phi \subset G \times \mathbb{P}^n$ as above the universal k-plane, with $\pi : \Phi \to G$ and $\eta : \Phi \to \mathbb{P}^n$ the projection maps. We have then*

$$A(\Phi) = A(G)[\zeta]/(\zeta^{k+1} - \sigma_1\zeta^k + \sigma_{1,1}\zeta^{k-1} + \cdots + (-1)^{k+1}\sigma_{1,1,\dots,1}),$$

where $\zeta \in A^1(\Phi)$ is the tautological class, or equivalently the pullback via η of the hyperplane class in $\mathbb{P}^n$.

The two special cases occurring most commonly are the cases $k = n-1$ of the universal hyperplane, and the case $k = 1$ of the universal line. In the first case,

$$\Phi = \big\{(H, p) \in \mathbb{P}^{n*} \times \mathbb{P}^n \mid p \in H\big\}$$

and if we let ω be pullback to Φ of the hyperplane class in $\mathbb{P}^{n*}$, we have

$$A(\Phi) = \mathbb{Z}[\omega, \zeta]/(\omega^{n+1}, \zeta^{n+1}, \zeta^n - \omega\zeta^{n-1} + \cdots + (-1)^n\omega^n)$$

We have written the ideal of relations in this way to emphasize the symmetry, but it's redundant: we could drop either ω^{n+1} or ζ^{n+1}. Note that when $a + b = \dim(\Phi) = 2n - 1$ we have

$$\deg(\omega^a\zeta^b) = \begin{cases} 1, & \text{if } (a, b) = (n, n-1) \text{ or } (n-1, n) \\ 0 & \text{otherwise,} \end{cases}$$

which we could also see from the fact that $\Phi \subset \mathbb{P}^{n*} \times \mathbb{P}^n$ is a hypersurface of bidegree $(1, 1)$.

The universal line will also come up a lot in the following chapters; in this case we have

$$A(\Phi) = A(\mathbb{G}(1, n))[\zeta]/(\zeta^2 - \sigma_1\zeta + \sigma_{1,1}).$$

We'll leave it to the reader to calculate the degrees of monomials $\sigma_1^a\sigma_{1,1}^b\zeta^c$ of top degree $a + 2b + c = \dim(\Phi) = 2n - 1$ in Exercise 11.40.

11.4.2 The blowup of $\mathbb{P}^n$ along a linear space

In Chapter 1 we saw how to describe the Chow ring of the blowup of projective space at a point. We can now analyze much more generally and systematically the Chow ring of the blowup $Z = Bl_\Lambda\mathbb{P}^n$ of projective space $\mathbb{P}^n = \mathbb{P}V$ along any linear subspace $\Lambda \cong \mathbb{P}^{r-1}$. The key is to realize Z as the total space of a projective bundle.

To understand the picture, first recall that the blowup is the graph of the rational map $\pi_\Lambda : \mathbb{P}^n \to \mathbb{P}^{n-r}$ given by projection from Λ. Thus $Z \subset \mathbb{P}^n \times \mathbb{P}^{n-r}$. We will show that the projection $Z \to \mathbb{P}^{n-r}$ to the second factor makes Z into a projective bundle. Certainly, each fiber of the projection is an r-dimensional projective space (see Figure 11.2). Concretely, if we choose an $(n - r)$-plane $\Gamma \subset \mathbb{P}^n$ disjoint from Λ, we can write

$$Z = \{(p, q) \in \mathbb{P}^n \times \Gamma \mid p \in \overline{\Lambda\, q}\}.$$

The fiber over a point $q \in \Gamma$ is thus the linear subspace $\overline{\Lambda\, q} \cong \mathbb{P}^{n-r+1} \subset \mathbb{P}^n$. If we write $\mathbb{P}^n$ as $\mathbb{P}(V)$, then Λ corresponds to an r-dimensional linear subspace $V' \subset V$ and Γ corresponds to a complementary $(n - r + 1)$-dimensional subspace W. The fiber of Z over $q \in \Gamma$ corresponds to the subspace spanned by V' and the one-dimensional subspace $\tilde{q}$ corresponding to q in W. Here V' is fixed, while the one-dimensional subspace varies over all such subspaces of W. This suggests that Z is the projectivization of the bundle $\mathcal{O}_{\mathbb{P}^{n-r}}(-1) \oplus V' \otimes \mathcal{O}_{\mathbb{P}^{n-r}}$, which we will now prove.

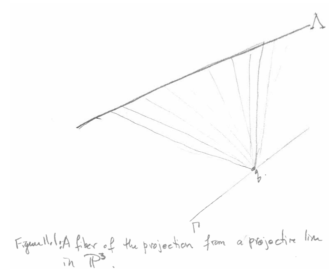

FIGURE 11.2. The fiber over a point under the projection of $\mathbb{P}^3$ from the line Λ.

Proposition 11.14. *Let $V' \subset V$ be an r-dimensional subspace of an $(n+1)$-dimensional vector space V, and let*

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^{n-r}}(-1) \oplus (V' \otimes \mathcal{O}_{\mathbb{P}^{n-r}}),$$

so that $\mathcal{E}$ is a vector bundle of rank $(r+1)$ on $\mathbb{P}^{n-r} = \mathbb{P}(V/V')$. The blowup Z of $\mathbb{P}(V)$ along the $(r-1)$-dimensional subspace $\mathbb{P}(V')$, together with its projection to $\mathbb{P}^{n-r}$, is isomorphic to the projective bundle $\pi : \mathbb{P}\mathcal{E} \to \mathbb{P}^{n-r}$. Under this isomorphism the blowup map $Z \to \mathbb{P}^n$ corresponds to the complete linear series $|\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)|$.

Proof. Choose a complement $V/V' \cong W \subset V$ to V', so that $V = W \oplus V'$. With $\mathcal{E}$ as in the Proposition, the natural inclusion $\mathcal{O}_{\mathbb{P}W}(-1) \subset (W \otimes \mathcal{O}_{\mathbb{P}W})$ induces an inclusion

$$\mathcal{E} \subset (W \otimes \mathcal{O}_{\mathbb{P}W}) \oplus (V' \otimes \mathcal{O}_{\mathbb{P}W}) = V \otimes \mathcal{O}_{\mathbb{P}W}.$$

The dual map, which is a surjection, induces an isomorphism $V^* \to H^0\mathcal{E}^* = V'^* \oplus W^*$. Thus $\mathcal{E}^*$ is generated by its global sections and the complete linear series $|\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)|$ corresponds to a map $\mathbb{P}\mathcal{E} \to \mathbb{P}V$.

The fiber of $\mathcal{E}$ over a point $q \in \mathbb{P}W$ is, as a subspace of V, equal to $\overline{V'\,\tilde{q}}$, whose projectivization is the fiber over q of the blowup Z of $\mathbb{P}V'$ in $\mathbb{P}V$. Thus, together with the projection map $\pi : \mathbb{P}\mathcal{E} \to \mathbb{P}W$ we get a closed immersion $\varphi : \mathbb{P}\mathcal{E} \to \mathbb{P}V \times \mathbb{P}W$, that maps the fiber of $\mathbb{P}\mathcal{E}$ isomorphically to Z. $\square$

Corollary 11.15. *Let $Z \subset \mathbb{P}^n \times \mathbb{P}^{n-r}$ be the blowup of an $(r-1)$-plane Λ in $\mathbb{P}^n$. Writing $\alpha, \zeta \in A^1(Z)$ for the pullbacks of the hyperplane classes on $\mathbb{P}^{n-r}$ and $\mathbb{P}^n$ respectively, we have*

$$A(Z) = \mathbb{Z}[\alpha, \zeta]/(\alpha^{n-r+1}, \zeta^{r+1} - \alpha\zeta^r).$$

With this notation the class of the exceptional divisor $E \subset Z$, the preimage of Λ in Z, is

$$[E] = \zeta - \alpha.$$

Proof. The Chern class of $\mathcal{E} = \mathcal{O}_{\mathbb{P}^{n-r}}(-1) \oplus V' \otimes \mathcal{O}^r_{\mathbb{P}^{n-r}}$, is $1 - \alpha$ so the formula for $A(Z)$ follows at once from Theorem 11.9. Since ζ is the class of the preimage of a hyperplane $H \subset \mathbb{P}^n$ (which could contain Λ), and α is represented by the proper transform of a hyperplane containing Λ, we have $[E] = \zeta - \alpha$ as claimed. □

For example, in the case of the blowup of the plane at a point we have

$$[E]^2 = (\zeta - \alpha)^2 = \zeta^2 - 2\alpha\zeta + \alpha^2 = -\zeta^2,$$

that is, minus the class of a point, as we already knew. But we can now compute $\deg([E]^n)$ in general (Exercise 11.45).

11.5 Projectivization of a subbundle

If $\mathcal{F} \subset \mathcal{E}$ are bundles on a variety X then $\mathbb{P}\mathcal{F}$ is naturally a subscheme of $\mathbb{P}\mathcal{E}$. Understanding its class is a crucial in understanding the Chow ring of a blow-up (Section **??**), and will allow us to answer Keynote Question (b).

Let $\pi : \mathbb{P}\mathcal{E} \to X$ be the projection and let $\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1) \subset \pi^*\mathcal{E}$ be the universal subbundle. A point $p \in \mathbb{P}\mathcal{E}$ over a point $x \in X$ corresponds to the one dimensional space that is the fiber of $\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1)$ at p. Thus $p \in \mathbb{P}\mathcal{F}$ if and only if this space is contained in the fiber of $\mathcal{F}$. In other words, $p \in \mathbb{P}\mathcal{F}$ if and only if the composite map

$$\varphi : \mathcal{O}_{\mathbb{P}\mathcal{E}}(-1) \to \pi^*\mathcal{E} \to \pi^*(\mathcal{E}/\mathcal{F}).$$

vanishes at p. We can view φ as a global section of the bundle

$$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1), \pi^*(\mathcal{E}/\mathcal{F})) \cong \mathcal{O}_{\mathbb{P}\mathcal{E}}(1) \otimes \pi^*(\mathcal{E}/\mathcal{F})$$

If we write everything in local coordinates then we see that $\mathbb{P}\mathcal{F}$ is scheme-theoretically defined by the vanishing of φ. Since the codimension of $\mathbb{P}\mathcal{F}$ is the same as the rank of $\mathcal{E}/\mathcal{F}$, it follows that $[\mathbb{P}\mathcal{F}] \in A(\mathbb{P}\mathcal{E})$ is given by a Chern class, which we can compute using the formula for the Chern class of the tensor product of a bundle with a line bundle (Proposition 7.10):

Proposition 11.16. *If X is a smooth projective variety and $\mathcal{F} \subset \mathcal{E}$ are vector bundles on X of ranks s and r respectively, then*

$$[\mathbb{P}\mathcal{F}] = c_{r-s}\big(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1) \otimes \pi^*(\mathcal{E}/\mathcal{F})\big) = \zeta^{r-s} + \gamma_1\zeta^{r-s-1} + \cdots + \gamma_{r-s} \in A^{r-s}(\mathbb{P}\mathcal{E})$$

where $\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1))$ and $\gamma_k = c_k(\mathcal{E}/\mathcal{F})$. □

An important reason to consider projectivized subbundles is suggested by the following characterization of sections. Giving a section—that is, a map $\alpha : X \to \mathbb{P}\mathcal{E}$ such that $\pi \circ \alpha$ is the identity—is the same as giving the image the section; and we will therefore refer to the image as a section as well.

Proposition 11.17. *If $\mathcal{L} \subset \mathcal{E}$ is a line subbundle of a vector bundle $\mathcal{E}$ on a variety X, then $\mathbb{P}\mathcal{L} \subset \mathbb{P}\mathcal{E}$ is the image of a section $X \to \mathbb{P}\mathcal{E}$ of the projection $\mathbb{P}\mathcal{E} \to X$, and every section has this form.*

Informally: giving a section is the same as specifying point of $\mathbb{P}\mathcal{E}$ over each point of X; that is, giving a 1-dimensional subspace of each fiber of $\mathcal{E}$.

Proof. By the universal property of $\pi : \mathbb{P}\mathcal{E} \to X$, giving a map $\alpha : X \to \mathbb{P}\mathcal{E}$ that "commutes with" the identity map $X \to X$ is the same as given a line subbundle of $\mathcal{E}$. $\square$

11.5.1 Ruled surfaces

Recall that a *ruled surface* is by definition the projectivization of a vector bundle of rank 2 over a smooth curve. We can now answer Keynote Question (b):

Proposition 11.18. *A ruled surface can contain at most one irreducible curve of negative self-intersection.*

Proof. Let X be a smooth curve, let $\pi : \mathbb{P}\mathcal{E} \to X$ be a ruled surface, and suppose that $C_1, C_2 \subset \mathbb{P}\mathcal{E}$ are two irreducible curves of strictly negative self-intersection. A fiber $\pi^{-1}(x)$, satisfies $[\pi^{-1}(x)]^2 = \pi^*([x]^2) = 0$, so the induced maps $\pi : C_i \to X$ are finite. Let $C_1' \to C_1$ be the normalization of C_1, and let $\alpha : C_1' \to C \subset X$ be the corresponding map. Consider the pullback diagram

$$\begin{tikzcd} \mathbb{P}\alpha^*\mathcal{E} = C_1' \times_X \mathbb{P}\mathcal{E} \arrow[r, "\beta"] \arrow[d] & \mathbb{P}\mathcal{E} \arrow[d, "\pi"'] \\ C_1' \arrow[r, "\alpha"'] & X \end{tikzcd}$$

The preimage $\beta^{-1}(C_1) = C_1' \times_X C_1$ represents a cycle $m\Sigma_1 + D_1$, where Σ_1 is a section, D_1 has no component in common with Σ_1, and $m > 0$. Hence

$$\begin{aligned} m^2 \deg[\Sigma_1]^2 &= \deg[\Sigma_1][\beta^* C_1] - \deg[\Sigma_1][D_1] \\ &\le \deg[\Sigma_1][\beta^* C_1] \\ &= \deg[\beta_*\Sigma][C_1] \\ &= \deg[C_1]^2, \end{aligned}$$

so $\deg[\Sigma_1]^2 < 0$.

Since a section pulls back to a section with the same self-intersection, we can repeat the process with a component of $\beta^{-1}C_2$ to arrive at a situation where we have two sections sections Σ_i of negative self-intersection. We can analyze this case using Proposition 11.17. Suppose that $\Sigma_i = \mathbb{P}\mathcal{L}_i \subset \mathbb{P}\mathcal{E}$.

By Theorem 11.9, we have

$$A(\mathbb{P}\mathcal{E}) = A(X)[\zeta]/(\zeta^2 + c_1(\mathcal{E})\zeta),$$

where $\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1))$. Now $\deg(c_1(\mathcal{E})\zeta) = \deg \pi_*(c_1(\mathcal{E})\zeta) = \deg c_1(\mathcal{E})$ because ζ meets each fiber of π in degree 1. It follows that $\deg \zeta^2 = -\deg c_1(\mathcal{E})$. By Proposition 11.16,

$$[\Sigma_i] = \zeta + c_1(\mathcal{E}) - c_1(\mathcal{L}_i),$$

so

$$0 > \deg[\Sigma_i]^2 = \deg \zeta^2 + 2\deg c_1(\mathcal{E}) - 2\deg \mathcal{L}_i.$$

Thus $2\deg \mathcal{L}_i > \deg c_1(\mathcal{E})$. (Exercise 11.55 strengthens this conclusion slightly.)

Supposing now that $\Sigma_1 \neq \Sigma_2$, we get an exact sequence

$$0 \to \mathcal{L}_1 \oplus \mathcal{L}_2 \to \mathcal{E} \to \mathcal{G} \to 0$$

where $\mathcal{G}$ is a sheaf with finite support, and it follows that $\deg \mathcal{E} \geq \deg \mathcal{L}_1 + \deg \mathcal{L}_2 > \deg \mathcal{E}$, a contradiction. $\square$

By contrast, it is possible for a (non-ruled) smooth projective surface to contain infinitely many irreducible curves of negative self-intersection; Exercises 11.50- 11.52 show how to construct an example. It is an open problem (in characteristic 0) whether the self intersections of irreducible curves on a surface S (in characteristic 0) are bounded below: that is, whether a surface can contain a sequence $C_1, C_2, \dots$ of irreducible curves with $(C_n \cdot C_n) \to -\infty$. (In characteristic $p > 0$, János Kollár has shown us an example, described in Exercise 11.53.)

11.5.2 Self-intersection of the zero section of a vector bundle

As we explained in Chapter 6, a vector bundle $\mathcal{E}$ on a scheme X may itself be considered as a scheme $\mathbb{A}(\mathcal{E}) := \mathrm{Spec}(\mathrm{Sym}\, \mathcal{E}^*)$ over X.

It is useful to have a compactification of $\mathbb{A}(\mathcal{E})$; that is, a variety proper over X that includes $\mathbb{A}(\mathcal{E})$ as a dense open subset. It is natural to try to compactify each fiber by putting it inside a projective space of the same dimension, and we can do this globally by taking the projectivization of the direct sum $\mathcal{E} \oplus \mathcal{O}_X$; that is, we set

$$\overline{\mathcal{E}} := \mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X).$$

Let r be the rank of $\mathcal{E}$. Since $c(\mathcal{E} \oplus \mathcal{O}_X) = c(\mathcal{E})$, we have

$$A(\overline{\mathcal{E}}) = A(X)[\zeta]/(\zeta^{r+1} + c_1(\mathcal{E})\zeta^r + \cdots + c_r(\mathcal{E})\zeta),$$

In terms of coordinates, $\mathbb{A}(E) \subset \overline{\mathcal{E}}$ is "the locus where the last coordinate is nonzero." It's complement is the divisor $\mathbb{P}\mathcal{E} \subset \mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X)$, which we therefore call the "hyperplane at infinity". Since this is the locus where the section of $\mathcal{O}_{\overline{E}}(1)$ corresponding to $1 \in \mathcal{O}_X \subset \mathcal{E} \oplus \mathcal{O}_X$ vanishes, we get

$$\zeta := c_1(\mathcal{O}_{\overline{E}}(1)) = [\mathbb{P}\mathcal{E}].$$

(One can also see this from Proposition 11.16.)

The section $\mathbb{P}\mathcal{O}_X \subset \overline{\mathcal{E}}$ is the locus where all the coordinates in $\mathcal{E}^*$ vanish; it is thus the zero-section of $\mathbb{A}(\mathcal{E})$, which we will call Σ_0. By Proposition 11.16 we have $[\Sigma_0] = \zeta^r + c_1(\mathcal{E})\zeta^{r-1} + \cdots + c_{r-1}(\mathcal{E})\zeta + c_r(\mathcal{E})$. More generally, if τ is a global section of $\mathcal{E}$, then $(\tau, 1)$ is a nowhere vanishing section of $\mathcal{E} \oplus \mathcal{O}_X$, and the line subbundle it generates corresponds to a section of $\overline{\mathcal{E}}$, which we will call Σ_τ. Using Proposition 11.16 or the family $\Sigma_{t\tau}$, which is a rational equivalence between Σ_τ and Σ_0, we see that $[\Sigma_\tau] = [\Sigma_0]$. If τ vanishes in codimension r then

$$\pi_*([\Sigma_0]^2) = \pi_*\big([[\Sigma_0][\Sigma_\tau]]\big) = [(\tau)_0] = c_r(\mathcal{E}).$$

We claim that this formula holds in general:

Proposition 11.19. *Let $\mathcal{E}$ be a vector bundle of rank r on a smooth variety X, and let $\pi : \overline{\mathcal{E}} = \mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X) \to X$ be the projection, and let $\iota : X \to \mathbb{A}(\mathcal{E}) \subset \overline{\mathcal{E}}$ be the zero section, with image $\Sigma_0 = \mathbb{P}(\mathcal{O}_X)$. We have*

$$\pi_*([\Sigma_0]^2) = c_r(\mathcal{E}),$$

and, for any class $\alpha \in A(X)$,

$$\iota^*\iota_*\alpha = \alpha c_r(\mathcal{E}).$$

Proof. By Proposition 11.16,

$$[\Sigma_0] = \zeta^r + c_1(\mathcal{E})\zeta^{r-1} + \cdots + c_{r-1}(\mathcal{E})\zeta + c_r(\mathcal{E}).$$

Since Σ_0 is disjoint from the hyperplane at infinity $\mathbb{P}\mathcal{E} \subset \mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X)$, which has class ζ, we get $[\Sigma_0]\zeta = 0$. (This also follows from the computation of $A(\overline{E})$.)

Thus

$$\begin{aligned}[\Sigma_0]^2 &= [\Sigma_0]\Big(\zeta^r + c_1(\mathcal{E})\zeta^{r-1} + \cdots + c_{r-1}(\mathcal{E})\zeta + c_r(\mathcal{E})\Big) \\ &= [\Sigma_0]c_r(\mathcal{E}) \in A(\overline{\mathcal{E}})\end{aligned}$$

From the push-pull formula we get $\pi_*([\Sigma_0]^2) = (\pi_*[\Sigma_0])c_r(\mathcal{E}) = c_r(\mathcal{E})$, proving the first assertion.

For the second assertion, we use the fact that π induces an isomorphism from Σ_0 to X, and thus $\iota^*\beta = \pi_*(\beta \cap [\Sigma_0])$ for any cycle β on $\overline{\mathcal{E}}$. Thus

$$\iota^*\iota_*\alpha = \iota^*(\iota_*\alpha[\Sigma_0]) = \pi_*(\iota_*\alpha[\Sigma_0]^2) = \alpha c_r(\mathcal{E}),$$

as required. $\square$

See Theorem 15.6 for a generalization.

In Chapter 7 we described the top Chern class of a vector bundle $\mathcal{E}$ as as the zero locus of a general section in the case when the bundle had "enough sections", a condition that implies that the zero locus of the general section is reduced and of the right codimension. We can now show that the top Chern class is given by the zero locus of a section σ under the more general and more natural hypothesis σ itself vanishes in the correct codimension:

Corollary 11.20. *Let $\mathcal{E}$ be a vector bundle of rank r on a smooth projective variety X. If σ is a global section of $\mathcal{E}$ such that the zero scheme $V(\sigma) \subset X$ has codimension r, then the class $[V(\sigma)] \in A^r(X)$ is the top Chern class $c_r(\mathcal{E})$.*

Proof. Writing Σ_σ for the image of σ, and $\iota : X \to \Sigma_0 \subset \overline{\mathcal{E}}$ for the inclusion of the zero-section, we have $\iota(V(\sigma)) = \Sigma_\sigma \cap \Sigma_0$. Under the hypothesis of the Corollary the intersection is dimensionally transverse so, by Proposition 11.19,

$$[V(\sigma)] = \pi_*([\Sigma_\sigma][\Sigma_0]) = \pi_*[\Sigma_0]^2 = c_r\mathcal{E}.$$

□

An analogous statement is proven for all the Chern classes of $\mathcal{E}$ in Corollary 14.2.

11.6 Brauer-Severi varieties

A *Brauer-Severi variety* is a variety Y together with a smooth map $\pi : Y \to X$ such that all the (scheme-theoretic) fibers of π are isomorphic to $\mathbb{P}^r$, for some fixed r. Thus any projective bundle $\pi : Y \to X$ is a Brauer-Severi variety. But the converse is false. It is in fact the case that such a morphism π will be trivial locally in the étale (or, in case the ground field is $\mathbb{C}$, the analytic topology) in the sense that every point $x \in X$ will have an étale or analytic neighborhood U such that $\pi^{-1}(U) \cong U \times \mathbb{P}^r$. This is a consequence of the fact that $\mathbb{P}^r$ has no nontrivial deformations. But it may not be trivial in the Zariski topology. Here is an example.

Example 11.21. Let $\mathbb{P}^5$ be the space of conics in $\mathbb{P}^2 = \mathbb{P}V$, and consider the universal conic

$$\mathbb{P}^5 \xleftarrow{\pi_1} \Phi = \{(C, p) \in \mathbb{P}^5 \times \mathbb{P}^2 \mid p \in C\} \xrightarrow{\pi_2} \mathbb{P}^2$$

with its projections π_i to the two factors. We can realize Φ as the total space of a $\mathbb{P}^4$-bundle over $\mathbb{P}^2$ via π_2: Indeed, Φ is the projectivization of the rank 5 subbundle $\mathcal{E} \subset \operatorname{Sym}^2 V^*$ whose fiber $\mathcal{E}_p$ at a point p is the subspace

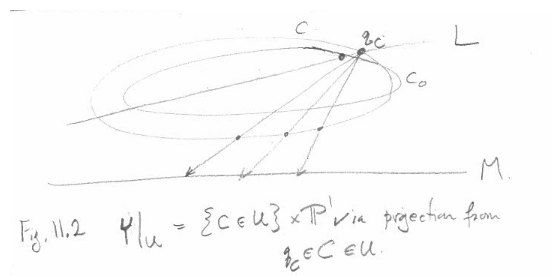

FIGURE 11.3. Local Analytic Trivality of the Universal Family of Conics in the Plane: $Y|_U \cong C \in U \times \mathbb{P}^1$ via projection from $q_C \in C \in U$.

of quadratic polynomials vanishing at p. (In particular, Φ is smooth.) In these terms the tautological class $\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)) \in A^1(\Phi)$ is the pullback of the hyperplane class $\pi_1^*(\mathcal{O}_{\mathbb{P}^5}(1))$. By Theorem 11.9, the divisor class group $A^1(\Phi) \cong \mathbb{Z}^2$ is generated by the pullbacks of the hyperplane classes from $\mathbb{P}^2$ and $\mathbb{P}^5$. Note that these classes restrict to classes of degrees 2 and 0 on any fiber of π_1. Thus the intersection of the fiber of π_1 with any divisor on Φ has even degree.

We now consider the projection π_1. To obtain a map whose fibers are all isomorphic to $\mathbb{P}^1$ we let $X \subset \mathbb{P}^5$ be the open subset corresponding to smooth conics, and let $\pi : Y = \Phi_X \to X$ be the restriction of π_1 to the preimage of X in Φ. By definition, the fibers of π are smooth conics, and in particular isomorphic to $\mathbb{P}^1$, so X is a Brauer-Severi variety.

We claim that $\pi : Y \to X$ is not a projective bundle. Indeed, if there were a nonempty Zariski open $U \subset X \subset \mathbb{P}^5$ such that $\pi : Y_U \to U$ were isomorphic to the projection to U of the product $U \times \mathbb{P}^1$ then we could take a section of Y_U and take its closure in Φ, obtaining a divisor in Φ meeting the general fiber of $\Phi \to \mathbb{P}^5$ in a reduced point. This contradicts the computation above. Thus $\pi : Y \to X$ is not a projective bundle.

If we work over the complex numbers we can see directly that π is locally trivial in the analytic topology (and the same argument would work more generally for the étale topology): Let $C_0 \in X$ be a smooth conic. Choose lines $L, M \subset \mathbb{P}^2$ such that L is transverse to C_0 and $M \cap L \cap C_0 = \varnothing$. Over a sufficiently small analytic neighborhood U of $C_0 \in X$ we can solve analytically for a point $q_C \in C \cap L$. The restriction of Y to U is isomorpphic to $U \times \mathbb{P}^1$ as U-schemes by the maps projecting a fiber C from q_C to M (see Figure 11.3).

The conclusion of this example may be interpreted as a theorem in polynomial algebra: it says that *there does not a exist a rational solution to the general quadratic polynomial.* In other words, there do not exist rational functions $X(a,b,c,d,e,f)$, $Y(a,b,c,d,e,f)$ and $Z(a,b,c,d,e,f)$ such that

$$aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ \equiv 0.$$

This is a generalization of the statement that the roots of a quadratic polynomial in one variable are not expressible as rational functions of its coefficients, though much stronger: polynomials in several variables have many more solutions than polynomials in one variable! The same is true of polynomials of any degree $d > 1$ in any number of variables (Exercise 11.56).

The set of Brauer-Severi varieties over a given variety X, modulo an equivalence relation that makes the projective bundles trivial, can be given the structure of a group, called the *Brauer group* of X. There is another avatar of this group, as the group of *Azumaya algebras* over $\mathcal{O}_X$ modulo those that are the endomorphism algebras of vector bundles. Understanding the Brauer groups of varieties is an important goal of arithmetic geometry. See for example ****find a good citation; maybe Serre's article in the class field theory book?****

11.7 Chow ring of a Grassmannian bundle

Suppose X is any variety, and $\mathcal{E}$ a vector bundle of rank n on X. Generalizing the projective bundle associated to $\mathcal{E}$ we can form the *Grassmann bundle* $G(k,E)$ of k-planes in the fibers of $\mathcal{E}$; that is,

$$G(k,E) = \{(x,V) : x \in X;\ V \subset E_x\} \xrightarrow{\pi} X.$$

(As with a single Grassmannian, we can realize $G(k,E)$ as a subvariety of the projectivization $\mathbb{P}(\wedge^k E)$.) There is a description of the Chow ring of $G(k,E)$ that extends both the description of the Chow ring of a projective bundle above, and the description of the Chow ring of $G(k,n)$ given in Theorem **??**; we'll explain it here without proof.

As in the projective bundle case, there is a *tautological subbundle* $S \subset \pi^*E$ defined on $G(k,E)$; this is a rank k bundle whose fiber over a point (x,V) is the vector space $V \subset E_x$. Let $Q = \pi^*(E)/S$ the *tautological quotient bundle*. As in the case of projective bundles, the Chow ring $A(G(k,E))$ is generated as an $A(X)$-algebra by the Chern classes $c_i(S)$, and also by the classes $c_i(Q)$. To understand the relations they satisfy, consider the exact sequence

$$0 \to S \to \pi^*E \to Q \to 0.$$

By the Whitney formula

$$c(Q) = \frac{c(E)}{c(S)}.$$

Since Q has rank $n-k$, the Chern classes $c_l(Q)$ vanish for $l > n-k$, and as in the projective bundle case (above) or the case of $G(k,n)$ (Theorem **??**) this gives all the relations:

Theorem 11.22. *Let X be a variety and let $\mathcal{E}$ be a vector bundle of rank n on X. If $G = G(k,E) \to X$ the bundle of k-planes in the fibers of $\mathcal{E}$ then*

$$A(G) = A(X)[\zeta_1, \dots \zeta_k]/\left(\left\{\frac{c(E)}{1+\zeta+\zeta^2+\dots}\right\}_l,\ l > \dim G - n + k\right)$$

where $\zeta = \zeta_1 + \dots + \zeta_k$ and $\{\eta\}_l$ denotes the component of η of codimension *l, an element of $A^l(G)$.*

One can go further and, fixing a sequence of ranks $0 < r_1 < \cdots r_m < \operatorname{rank} E$, consider the *flag bundle* $\mathbb{F}(r_1, \dots, r_m, E)$ whose fiber over a point of X is the set of all flags of subspaces of the given ranks in $\mathcal{E}$. There is again an analogous description of the Chow ring of this space. See Grayson and Stillman [$\geq$ 2013] for this result and an interesting proof that is in some ways more explicit than the one we have given, even in the case of $A(G(k,n))$. ****is there a published proof?****

11.8 Conics in $\mathbb{P}^3$ meeting eight lines

The family of conics contained in planes in $\mathbb{P}^3$ is naturally a projective bundle, and we will now apply the theory above to compute the number of such conics intersecting each of 8 general lines $L_1, \dots, L_8 \subset \mathbb{P}^3$.

We start by checking that we should expect a finite number. There is a 3-parameter family of planes in $\mathbb{P}^3$, and a five-parameter family of conics in each. Since two distinct planes intersect only in a line, the space of conics, whatever it is, should have dimension $3 + 5 = 8$.

Next, the locus D_L of conics meeting a given line $L \subset \mathbb{P}^3$ has codimension 1 in the space of conics: if $C \subset \mathbb{P}^3$ is the image of the map given by (F_0, F_1, F_2, F_3), the condition that C meet the line $Z_0 = Z_1 = 0$ is that F_0 and F_1 have a common zero. More geometrically: a one-parameter family of conics sweeps out a surface that meets L in a finite set, so a curve in the space of conics will intersect the locus of conics meeting L a finite number of times. It is reasonable, then, to ask whether there is only a finite number of conics that meet each of 8 general lines and, if so, how many there are.

As a parameter space $\mathcal{Q}$ for conics in $\mathbb{P}^3$ we will use a projective bundle, and this will allow us to use the theory developed earlier in this chapter to calculate in its Chow ring. In particular, we will identify the class $\delta \in A(\mathcal{Q})$ of the cycle $D_L \subset \mathcal{Q}$ of conics meeting a given line L, and compute the number $\deg \delta^8$, our candidate for the number of conics meeting 8 given general lines L_i. To prove that this number is correct, we must show that the cycles D_{L_i} meet transversely.

We will then identify $\mathcal{Q}$ with the Hilbert scheme $\mathcal{H} = \mathcal{H}_{2m+1}$ of subschemes of $\mathbb{P}^3$ having Hilbert polynomial $p(m) = 2m+1$. We will prove the necessary transversality by describing the tangent spaces to D_L in terms of the general description of the tangent spaces to Hilbert schemes from Theorem 8.21; this is a special case of an important general principle explaine in Exercise 11.66.

11.8.1 The Parameter space as projective bundle

Since the conics in a given plane naturally form a $\mathbb{P}^5$, and each conic is contained in a unique plane, it is plausible that the set of all conics in $\mathbb{P}^3$ is a $\mathbb{P}^5$-bundle over $\mathbb{P}^{3*}$, the projective space of planes in $\mathbb{P}^3$.

To make this structure explicit, consider the tautological exact sequence on $\mathbb{P}^{3*}$, which we may write as:

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{O}^4_{\mathbb{P}^{3*}} \xrightarrow{\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \end{pmatrix}} \mathcal{O}_{\mathbb{P}^3}(1) \to 0.$$

The projective bundle $\mathbb{P}\mathcal{S} \subset \mathbb{P}(\mathcal{O}^4_{\mathbb{P}^{3*}}) = \mathbb{P}^{3*} \times \mathbb{P}^3$ is the family of 2-planes in $\mathbb{P}^3$: the fiber of $\mathbb{P}\mathcal{S}$ over a point $a = (a_0, \dots, a_3) \in \mathbb{P}^{3*}$ is the plane $H_a \subset \mathbb{P}^3$ defined by $\sum a_i x_i = 0$. The dual $\mathcal{S}^*_a$ is thus the space of linear forms on this plane, and, setting $\mathcal{E} := \mathrm{Sym}_2(\mathcal{S}^*)$, the fiber of $\mathbb{P}\mathcal{E}$ over a may be identified with the set of conics in H_a. Thus there is a *tautological family of conics in* $\mathbb{P}^3$

$$\mathcal{X} \subset \mathbb{P}\mathcal{E} \times_{\mathbb{P}^{3*}} \mathbb{P}\mathcal{S} \subset \mathbb{P}\mathcal{E} \times \mathbb{P}^3$$

whose points are pairs consisting of a conic in 2-plane and a point on that conic, with projections both to $\mathbb{P}^{3*}$ and to $\mathbb{P}^3$.

From the dual of the exact sequence above we derive an exact sequence

$$0 \to \mathcal{O}^4_{\mathbb{P}^{3*}} \otimes \mathcal{O}_{\mathbb{P}^{3*}}(-1) \to \mathrm{Sym}^2(\mathcal{O}^4_{\mathbb{P}^{3*}}) \to \mathcal{E} \to 0.$$

If we denote by ω the tautological class on $\mathbb{P}^{3*}$ then Whitney's formula (Theorem **??**), taking into account that $\omega^4 = 0$, yields

$$c(\mathcal{E}) = 1/(1-\omega)^4 = 1 + 4\omega + 10\omega^2 + 20\omega^3.$$

We can now apply Theorem 11.9 to describe the Chow ring of $\mathcal{Q}$. Letting $\zeta \in A^1(\mathcal{Q})$ be the first Chern class of the tautological quotient $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1))$ of the pull-back of $\mathcal{E}^*$ to $\mathcal{Q}$ we get

$$\begin{aligned} A(\mathcal{Q}) &= A(\mathbb{P}^{3*})[\zeta]/(\zeta^6 + 4\omega\zeta^5 + 10\omega^2\zeta^4 + 20\omega^3\zeta^3) \\ &= \mathbb{Z}[\omega, \zeta]/(\omega^4, \zeta^6 + 4\omega\zeta^5 + 10\omega^2\zeta^4 + 20\omega^3\zeta^3) \end{aligned}$$

11.8.2 The class δ of the cycle of conics meeting a line

We next compute the class $\delta \in A^1(\mathcal{Q})$ of the divisor $D = D_L$ using the technique of undetermined coefficients. We know that $\delta = a\omega + b\zeta$ for some pair of integers a and b, and restricting to curves in $\mathcal{Q}$ gives us linear relations on a and b. Let $\Gamma \subset \mathcal{Q}$ be the curve corresponding to a general pencil $\{C_\lambda \subset H\}$ of conics in a general plane $H \subset \mathbb{P}^3$, let $\Phi \subset \mathcal{Q}$ be the curve consisting of a general pencil of plane sections $\{H_\lambda \cap Q\}$ of a fixed quadric Q. We denote their classes in $A_1(\mathcal{Q})$ by γ and φ respectively.

We claim that the following table gives the intersection numbers between our divisor classes ω, ζ, δ, and the curves Γ, Φ:

	ω	ζ	δ
γ	0	1	1
φ	1	0	2

The calculation of the five intersection numbers other than $\zeta\varphi$ is easy, and we leave to the reader the pleasure of working them out (Exercise 11.60).

We can compute $\zeta\varphi$ as the degree of the restriction of the bundle $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ to the curve Φ; equivalently, to show that $\zeta\varphi = 0$ we must show that $T = \mathcal{O}_{\mathbb{P}\mathcal{E}}(-1)$ is trivial on Φ. To see this, recall that a point of $\mathcal{Q}$ is a pair (H, ξ), with H a plane in $\mathbb{P}^3$ and ξ a one-dimensional subspace of $H^0(\mathcal{O}_H(2))$; the fiber of T over the point (H, ξ) is the vector space ξ. Now, if $F \in H^0(\mathcal{O}_{\mathbb{P}^3}(2))$ is the homogeneous quadratic polynomial defining Q, we see that the restrictions of F to the planes H_λ give an everywhere nonzero section of T over Φ, proving that $T|_\Phi$ is the trivial bundle, as required.

Given the intersection numbers in the table above, we conclude that

$$\delta = 2\omega + \zeta.$$

There is also a direct way to arrive at this class, which we'll describe in Exercise 11.61.

11.8.3 The degree of δ^8

To compute δ^8 we need to know the degreesof the monomials $\omega^i\zeta^j$ of degree 8. To start with, we have $\omega^4 = 0$; and since ω^3 is the class of a fiber of $\mathcal{Q} \to \mathbb{P}^{3*}$ and ζ restricts to the hyperplane class on this fiber, we have

$$\deg(\omega^3\zeta^5) = 1.$$

To evaluate the next monomial $\omega^2\zeta^6$, we use the relation

$$\zeta^6 = -4\omega\zeta^5 - 10\omega^2\zeta^4 - 20\omega^3\zeta^3$$

of Theorem 11.9, which gives

$$\begin{aligned}\deg \omega^2\zeta^6 &= \deg \omega^2(-4\omega\zeta^5 - 10\omega^2\zeta^4 - 20\omega^3\zeta^3)\\ &= -4.\end{aligned}$$

The same idea yields

$$\deg \omega\zeta^7 = 6; \quad \deg \zeta^8 = -4.$$

Putting these together we obtain

$$\begin{aligned}(2\omega+\zeta)^8 &= \zeta^8 + 2\binom{8}{1}\omega\zeta^7 + 4\binom{8}{2}\omega^2\zeta^6 + 8\binom{8}{3}\omega^3\zeta^8\\ &= 92\end{aligned}$$

Writing $\pi : \mathcal{Q} \to \mathbb{P}^3$ for the projection, the numbers $\omega^i\zeta^j$ computed above may be interpreted (via the push-pull formula) as the degrees of the classes $\pi_*\zeta^k$, which are called Segre classes of the bundle $\mathcal{E}$. See definition 12.1 and, for an alternate computation, Proposition 12.3.

11.8.4 The parameter space as Hilbert scheme

If $Q \subset \Lambda$ is a smooth plane conic then the Hilbert polynomial of Q is $p(m) = 2m + 1$. Let $\mathcal{H} := \mathcal{H}_{2m+1}$ be the Hilbert scheme of subschemes of $\mathbb{P}^3$ with this Hilbert polynomial, and let $\mathcal{C} \to \mathcal{H} \times \mathbb{P}^3$ be the universal family. We have already described the tautological family of plane conics $\mathcal{X} \to \mathcal{Q} \times \mathbb{P}^3$, and by the universal property of the Hilbert scheme, there is a unique map $\psi : \mathcal{Q} \to \mathcal{H}$ such that $\mathcal{X} = (\psi \times 1)^*\mathcal{C}$.

Theorem 11.23. *$\mathcal{Q}$ with its universal family $\mathcal{X} \to \mathcal{Q} \times \mathbb{P}^3$ is isomorphic to $\mathcal{H}$ with its universal family $\mathcal{C} \to \mathcal{H} \times \mathbb{P}^3$ via the map ψ.*

We postpone the proof to develop a few necessary facts about subschemes C with Hilbert polynomial $p(m) = 2m+1$. To show that C is really a conic, we first want to show that C is contained in a plane Λ—that is, there is a linear form vanishing on C. Since the number of independent linear forms on $\mathbb{P}^3$ is $4 = p(1)+1$, it suffices to show that the Hilbert function of C, that is the dimension $(S_C)_1$ of the degree 1 part of the homogeneous coordinate ring of C, is equal to $p(m)$.

Once this is established we must show that a nonzero quadratic form on Λ vanishes on C, and it suffices, for similar reasons as above, to show that $\dim(S_C)_2 = 5 = p(2)$. This is the idea behind the following result.

Proposition 11.24. *Let $C \subset \mathbb{P}^n$ be a subscheme, let $\mathcal{I}_C$ be its ideal sheaf, and let $S_C = K[x_0, \ldots, x_n]/I$ be its homogeneous coordinate ring.*

(a) *If the Hilbert polynomial of* S_C *is* $p_C(m) = 2m+1$ *then the Hilbert function of* S_C *is also equal to* $2m+1$.

(b) C *is the complete intersection of a uniqe 2-plane and a (non-unique) quadric hypersurface.*

(c) $H^1(\mathcal{I}(m)) = 0$ *for all* m.

Proof. The form of the Hilbert polynomial implies that C has dimension 1 and degree 2. Thus a general plane section $\Gamma = \{x = 0\} \cap C$ is a subscheme of degree 2 in the plane, either two distinct points or one double point. In either case, the Hilbert function of Γ is $h_\Gamma(m) = 2$ for all $m \geq 1$. Writing S_C for the homogeneous coordinate ring of C we have a surjective map $S_C \to S_\Gamma$ whose kernel contains xS_C, whence

$$h_C(m) - h_C(m-1) \geq h_\Gamma(m) = 2$$

for $m \geq 1$. Since $h_C(0) = 1$, it follows that $h_C(m) \geq 2m+1$ for all $m \geq 0$ and that a strict inequality for any value of m implies the same for all larger values. Since $h_C(m) = p_C(m) = 2m+1$ for large m, the inequality above must be an equality for all $m \geq 1$, proving the first statement.

The second statement follows. From $h_C(1) = 3$ we see that C is be contained in a unique plane Λ. From $h_C(2) = 5$ we see that C lies on five linearly independent quadrics; since at most four of these can contain Λ, we see that C lies on a quadric $Q \subset \mathbb{P}^3$ not containing Λ. The subscheme $C' := \Lambda \cap Q$ also has Hilbert function $2m+1$ and $C \subset C'$ they are equal.

To prove the last statement we use the long exact sequence in cohomology,

$$0 \to H^0\mathcal{I}_C(m) \to H^0\mathcal{O}_{\mathbb{P}^n}(m) \to H^0\mathcal{O}_C(m) \to H^1\mathcal{I}_C(m) \to H^1\mathcal{O}_{\mathbb{P}^n}(m).$$

Since the last term is zero and the cokernel of the map $H^0\mathcal{I}_C(m) \to H^0\mathcal{O}_\Lambda(m)$ is the component of degree m in S_C, it suffices to show that $h^0\mathcal{O}_C(m) = 2m+1$. But as C is defined in the plane by a quadratic hypersurface, we have also a sequence

$$0 \to H^0\mathcal{O}_{\mathbb{P}^2}(m-2) \to H^0\mathcal{O}_{\mathbb{P}^2}(m) \to H^0\mathcal{O}_C(m) \to H^1\mathcal{O}_{\mathbb{P}^2}(m-2),$$

and since the twists of $\mathcal{O}_{\mathbb{P}^2}$ have no intermediate cohomology, we get

$$h^0\mathcal{O}_C(m) = h^0\mathcal{O}_{\mathbb{P}^2}(m) - h^0\mathcal{O}_{\mathbb{P}^2}(m-2) = \binom{m+2}{2} - \binom{m}{2} = 2m+1$$

as required. □

Proof of Theorem 11.23. By Proposition 11.24, the fibers of $\mathcal{C} \to \mathcal{H} \times \mathbb{P}^3$ over closed points are precisely the distinct conics in $\mathbb{P}^3$. Since this is also true for $\mathcal{X} \to \mathcal{Q} \times \mathbb{P}^3$, the map $\psi : \mathcal{Q} \to \mathcal{H}$ is bijective on closed points.

Since $\mathcal{Q}$ is smooth, it now suffices to prove that $\mathcal{H}$ is smooth. From the bijectivity of ψ we see that $\dim \mathcal{H} = \dim \mathcal{Q} = 8$, so it suffices, in fact, to prove that the tangent space to $\mathcal{H}$ at each point $[C]$ has dimension 8. By Theorem 8.21 there is an isomorphism $T_{[C]/\mathcal{H}} \cong H^0(N_{C/\mathbb{P}^3})$. Using Proposition 11.24 again, we know that C is a complete intersection of a linear form and a quadric. Thus $N_{C/\mathbb{P}^3} = (\mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2))|_C$, and the dimension of the tangent space is $h^0\mathcal{O}_C(1) + h^0\mathcal{O}_C(2)$.

By Proposition 11.24, $H^1(\mathcal{I}_{C/\mathbb{P}^3}(m) = 0$ for all m, so the desired value is the sum of the values of the Hilbert function of C at 1 and at 2. Putting this together we get

$$\dim T_{[C]/\mathcal{H}} = (2 \cdot 1 + 1) + (2 \cdot 2 + 1) = 8$$

as required. □

11.8.5 Tangent spaces to incidence cycles

To prove that the D_{L_i} intersect transversely we need to compute their tangent spaces at the points of intersection. This task is made easier by the fact that, for general L_i, the intersection of the D_{L_i} takes place the locus U of smooth conics, as we shall now prove:

Lemma 11.25. *For a general choice of lines $L_1, \dots, L_8 \subset \mathbb{P}^3$, no singular conic meets all 8.*

Proof. The family of singular conics has dimension 7, and the family of lines meeting a line, or a singular conic, has dimension 3. Thus the family consisting of 8-tuples of lines meeting a singular conic has dimension $7 + 3 \cdot 8 = 31$, while the family of 8-tuples of lines has dimension $8 \cdot 4 = 32$. □

Next we describe the tangent spaces to the cycles D_L at points in U. Again we use the computation of the tangent space to $\mathcal{Q} \cong \mathcal{H}$ at a point $[C]$ corresponding to a conic C as $T_{[C]/\mathcal{H}} = H^0(N_{C/\mathbb{P}^3})$.

Proposition 11.26. *Let $L \subset \mathbb{P}^3$ be a line, and $D_L \subset \mathcal{H}$ the locus of conics meeting L. If $C \subset \mathbb{P}^3$ is a smooth plane conic such that $C \cap L = \{p\}$ is a single, reduced point, then D_L is smooth at $[C]$, and its tangent space at $[C]$ is the space of sections of the normal bundle whose value at p lies in the normal direction spanned by L; that is,*

$$T_{[C]}D_L = \left\{\sigma \in H^0(N_{C/\mathbb{P}^3}) : \sigma(p) \in \frac{T_pL + T_pC}{T_pC}\right\}.$$

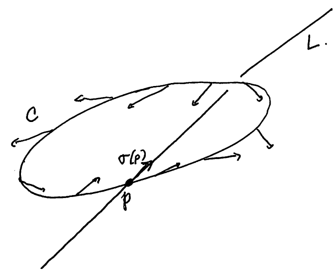

FIGURE 11.4. If C is a conic meeting a line L at a point p, then a deformation of C corresponding to a normal section σ remains in D_L if and only if $\sigma(p)$ is tangent to L.

Proof. We prove Proposition 11.26 by introducing an incidence correspondence: For $L \subset \mathbb{P}^3$ a line, we let

$$\Phi_L = \{(p, C) \in L \times \mathcal{H} \mid p \in C\}.$$

The image of Φ_L under the projection π_2 to the second factor is the cycle $D_L \subset \mathcal{H}$ of conics meeting L. By Lemma 8.23, the tangent space to Φ_L at the point (p, C) is

$$T_{(p,C)}\Phi_L = \big\{(\nu, \sigma) \in T_pL \times H^0(N_{C/\mathbb{P}^3}) \mid \sigma(p) \equiv \nu \bmod T_pC\big\}.$$

In particular, Φ_L will be smooth at (p, C) and the projection π_2 will carry its tangent space injectively to the space of sections $\sigma \in H^0(N_{C/\mathbb{P}^3})$ such that $\sigma(p) \in (T_pL + T_pC)/T_pC$. Since the map π_2 is one-to-one over p, it follows that D_L is smooth at $[C]$ with this tangent space. □

This argument applies to Hilbert schemes in a more general context; see Exercise 11.66.

Corollary 11.27. *Let C be a smooth conic in $\mathbb{P}^3$. If $L_1, \dots, L_8$ are general lines meeting C at general points, then the cycles $D_{L_1}, \dots D_{L_8} \subset \mathcal{Q} \cong \mathcal{H}$ meet transversely at $[C]$.*

Proof. By Proposition 11.26 it suffices to show that the 8 linear conditions specifying that a global section of the normal bundle of C lie in specified 1-dimensional subspaces at 8 points of C are independent, for a general choice of the points and the subspaces. Since the rank of the normal bundle is 2, this is a special case of Lemma 11.28. □

Next, we want to see that, for a given conic C and eight general lines meeting C the corresponding subspaces of $T_{[C]}\mathcal{H} = H^0(N_{C/\mathbb{P}^3})$ intersect transversely. This also follows from a more general statement. Note that we are not yet quite proving the transversality statement we wanted—this will be deduced in the next section.

Lemma 11.28. *Let $\mathcal{E}$ be a vector bundle on a projective variety X, and let $V \subset H^0\mathcal{E}$ be a space of global sections. If $p_1, \dots, p_k \in X$ are general points and $V_i \subset E_{p_i}$ a general linear subspace of codimension 1 in the fiber $\mathcal{E}_{p_i}$ of $\mathcal{E}$ at p_i then the subspace $W = \{\sigma \in V \mid \sigma(p_i) \in V_i$ has dimension*

$$\dim W = \max\{0, h^0(E) - k\}.$$

The obvious analog of this result fails if allow $\operatorname{codim} V_i > 1$; see Exercise 11.59.

Proof. Proceeding inductively, it suffices to do the case $k = 1$, and note that if the general section in V had value in every hyperplane $V_i \subset \mathcal{E}_p$ at a dense set of points $p \in X$, then $V = 0$. □

11.8.6 Proof of transversality

Proposition 11.29. *If $L_1, \dots, L_8 \subset \mathbb{P}^3$ are eight general lines, then the cycles $D_{L_i} \subset \mathcal{Q}$ intersect transversely.*

Proof. To start, we introduce the incidence correspondence

$$\Sigma = \left\{(L_1, \dots, L_8; C) \in \mathbb{G}(1,3)^8 \times \mathcal{Q} \mid C \cap L_i \neq \varnothing \ \forall i\right\}.$$

Since the locus of lines $L \subset \mathbb{P}^3$ meeting a given smooth conic C is an irreducible hypersurface in the Grassmannian $\mathbb{G}(1,3)$, we see via projection to $\mathcal{Q}$ that Σ is irreducible of dimension 32.

Now, let $\Sigma_0 \subset \Sigma$ be the locus of $(L_1, \dots, L_8; C)$ such that the cycles D_{L_i} fail to intersect transversely at $[C]$; this is a closed subset of Σ. By Corollary **??** $\Sigma_0 \neq \Sigma$, so $\dim \Sigma_0 < 32$. It follows that Σ_0 does not dominate $\mathbb{G}(1,3)^8$, so for a general point $(L_1, \dots, L_8) \in \mathbb{G}(1,3)^8$ the cycles D_{L_i} are transverse at every point of their intersection. □

In sum, we have proved:

Theorem 11.30. *There are exactly 92 distinct plane conics in $\mathbb{P}^3$ meeting eight general lines, and each of them is smooth.*

As with any enumerative formula that applies to the general form of a problem, the computation still tells us something in the case of 8 arbitrary lines. For one thing, it says that if $L_1, \dots, L_8 \subset \mathbb{P}^3$ are *any* eight lines, there will be at least one conic meeting all eight (here we have to include degenerate conics as well as smooth); and if we assume that the number of conics meeting all eight (again including degenerate ones) is finite, then assigning to each such conic C a multiplicity (equal to the scheme-theoretic degree of the component of the intersection $\cap D_{L_i} \subset \mathcal{H}$ supported at $[C]$), the total number of conics will be 92. In particular, as long as the number is finite, there can't be *more* than 92 distinct conics meeting all eight lines.

In Exercises 11.62-11.74 we'll look at some other problems involving conics in $\mathbb{P}^3$, including some problems involving calculations in $A(\mathcal{H})$, some other applications of the techniques we've developed here and some problems that require other parameter spaces for conics.

11.9 Exercises

Exercise 11.31. Equations for rational normal cuves. Choosing coordinates $x_0, x_1, \dots, x_a$ on $\mathbb{P}^a$ to correspond to the monomials $s^a, s^{a-1}, \dots, t^a$, show that the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \dots & x_{a-1} \\ x_1 & x_2 & \dots & x_a \end{pmatrix}$$

vanish identically on the rational normal curve $S(a)$. Show that the mapping taking (s,t) to the point where each of the linear forms in the vector

$$(s,t)\begin{pmatrix} x_0 & x_1 & \dots & x_{a-1} \\ x_1 & x_2 & \dots & x_a \end{pmatrix} = (sx_0 + tx_1, \dots, sx_{a-1} + tx_a$$

vanishes is an isomorphism. By working in local coordinates, show that the ideal I generated by the minors defines the curve scheme theoretically. Find a set of monomials that form a basis for the ring $K[x_0, x_1, \dots, x_a]/I$, and show that in degree d it has dimension $ad + 1$. By comparing this with the Hilbert function of $\mathbb{P}^1$, prove that I is the saturated ideal of the rational normal curve.

Exercise 11.32. Let X be a smooth projective variety, $\mathcal{E}$ a vector bundle on X and $\mathbb{P}E$ its projectivization. Let L be any line bundle on X; as we've seen, there is a natural isomorphism $\mathbb{P}\mathcal{E} \cong \mathbb{P}(E \otimes L)$.

(a) How does the class $c_1(\mathcal{O}_{\mathbb{P}(E\otimes L)}(1))$ relate to $c_1(\mathcal{O}_{\mathbb{P}E}(1))$?

(b) Using the results of Section 7.2.5, show that the two descriptions of the Chow ring of $\mathbb{P}E = \mathbb{P}(E \otimes L)$ agree.

Exercise 11.33. Let $\pi : Y \to X$ be a projective bundle.

(a) Show that the direct sum decomposition of the group $A(X)$ given in Theorem 11.9 *depends* on the choice of vector bundle $\mathcal{E}$ with $Y \cong \mathbb{P}\mathcal{E}$.

(b) Show that if we define group homomorphisms $\psi_i : A(Y) \to A(X)^{\oplus i+1}$ by

$$\psi_i : \alpha \mapsto \big(\pi_*(\alpha), \pi_*(\zeta\alpha), \dots, \pi_*(\zeta^i\alpha)\big)$$

then the filtration of $A(Y)$ given by

$$A(Y) \supset \mathrm{Ker}(\psi_0) \supset \mathrm{Ker}(\psi_1) \supset \cdots \supset \mathrm{Ker}(\psi_{r-1}) \supset \mathrm{Ker}(\psi_r) = 0$$

is *independent* of the choice of $\mathcal{E}$. (Hint: give a geometric characterization of the cycles in each subspace of $A(Y)$.)

Exercise 11.34. Let $X = \operatorname{Spec} D$, where D is a DVR, and let Q is the quotient field of D. Set $A = D \oplus Q$, and make A into a D-algebra by declaring that the product of any two elements in the second factor is zero. Show that the projectection $\operatorname{Spec}(D \oplus Q\epsilon) \to X$ is a flat map of finite type and that the closed fiber is a reduced point. This example shows that Proposition 11.3 requires something like the smoothness hypothesis.

Exercise 11.35. Show that the product of a base point free linear series with a very ample linear series is very ample. (Hint: prove that the product separates points and tangent vectors.)

Exercise 11.36. Let $\mathcal{F}$ be a vector bundle of rank r on a scheme X. Prove that $\mathcal{F}$ is very ample if and only if, for exach finite subscheme $Y \subset X$ of length 2 we have

$$\dim H^0(\mathcal{F}(-Y)) \leq \dim H^0(\mathcal{F}) - 2r, \text{ (in which case equality holds),}$$

where $H^0(\mathcal{F}(-Y))$ denotes the space of sections of $\mathcal{F}$ that vanish on Y (the section of the kernel of the map $\mathcal{F} \to \mathcal{F}/\mathcal{I}_{Y/X}\mathcal{F}$.

Solution: Let $\mathcal{E} = \mathcal{F}^*$, and let $\pi : \mathbb{P}\mathcal{E} \to X$ be the projection. Since every subscheme of length 2 in $\mathbb{P}(\mathcal{E})$ is contained in the preimage $\pi^{-1}Y = \mathbb{P}(\mathcal{E}|_Y)$ of some subscheme Y of length 2, the complete linear series $|\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)|$ is very ample if and only if its restriction to such preimages is very ample. Since every vector bundle on a finite scheme is trivial we have $H^0(\mathcal{F}|_Y) = \mathcal{O}_Y^r$, so $\mathbb{P}\mathcal{E}_Y = Y \times \mathbb{P}^{r-1}$, and the restriction of $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ to this scheme is $\mathcal{O}_Y \boxtimes \mathcal{O}_{\mathbb{P}^{r-1}}(1)$.

The complete linear series $|\mathcal{O}_Y \boxtimes \mathcal{O}_{\mathbb{P}^{r-1}}(1)|$ is very ample(for example by Proposition 11.6 where we take both $\mathcal{L}$ and $\mathcal{F}$ to be trivial bundles), and has dimension $2r$. Thus $\dim H^0(\mathcal{F}(-Y)) \leq \dim H^0(\mathcal{F}) - 2r$ if and only if the map $H^0(\mathcal{F}) \to H^0(\mathcal{F}|_Y)$ is surjective (in which case the inequality is an equality.) It thus suffices to show that no linear series of the form $\mathcal{O}_Y \boxtimes \mathcal{O}_{\mathbb{P}^{r-1}}(1), W$, with $W \subsetneq H^0\mathcal{O}_Y \boxtimes \mathcal{O}_{\mathbb{P}^{r-1}}(1)$, can be very ample on $Y \times \mathbb{P}^{r-1}$. Equivalently: there is no embedding of $Y \times \mathbb{P}^{r-1}$ in $\mathbb{P}^{2r-2}$ taking $Y_{red} \times \mathbb{P}^{r-1}$ to the disjoint union of (one or two) linear spaces.

Suppose, on the contrary, there were such an embedding. Since we assume that the ground field K is algebraically closed, there are only two possibilities for Y: two reduced points or one double point. If Y consists of two reduced points, so that the image of $Y_{red} \times \mathbb{P}^{r-1}$ is the union of two linear spaces, we get a contradiction simply from the fact that any two $r-1$-dimensional varieties in $\mathbb{P}^r$ meet.

On the other hand, if Y is a double point, $Y = \operatorname{Spec} K[\epsilon]/(\epsilon^2)$, then the normal bundle of Y_{red} is Y is trivial, and thus the normal bundle of $Y_{\mathrm{red}} \times \mathbb{P}^{r-1}$ in $Y \times \mathbb{P}^{r-1}$ is trivial. This bundle is a subbundle of the normal bundle of the linear space $Y_{\mathrm{red}} \times \mathbb{P}^{r-1}$ in $\mathbb{P}^{2r-2}$, so the top Chern class of this bundle is trivial. However, this normal bundle is isomorphic to $\mathcal{O}(1)^{r-2}$, and the degree of its top Chern class is actually 1, contradicting the existence of the embedding. (We could state this argument without mentioning Chern classes: the content is that any $r-2$ linear forms on $\mathbb{P}^{r-1}$ have a common zero.)

Exercise 11.37 (Vector Bundles on Elliptic Curves)**.** We will apply the criterion of Exercise 11.36 to prove the existence of certain embeddings of a scroll over an elliptic curve in Exercise **??**. To do this we need some facts about vector bundles on an elliptic curve from the rather complete theory of Atiyah [1957b]. We invite the reader to prove what we will need:

(a) Show that there is a unique indecomposable vectore bundle $\mathcal{E} := \mathcal{E}(2, \mathcal{L})$ of rank 2 on the genus 1 curve C with given determinant $\mathcal{L} := \wedge^2 \mathcal{E}$ of degree 1, and that there is a short exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{L} \to 0.$$

Similarly, show that there is a unique indecomposable rank 2 bundle with determinant $\mathcal{O}_C$.

(b) Deduce that there is a unique indecomposable bundle $\mathcal{E}(2, \mathcal{L})$ of rank 2 with any given determinant $\mathcal{L}$, and that for any line bundle $\mathcal{L}'$ of degree $\lfloor \deg \mathcal{L}/2 \rfloor$ there is an exact sequence

$$0 \to \mathcal{L}' \to \mathcal{E}(2, \mathcal{L}) \to \mathcal{L} \otimes \mathcal{L}'^{-1} \to 0.$$

If $d > 0$ then $h^0 \mathcal{E}(2, \mathcal{L}) = d$ and $h^1 \mathcal{E}(2, \mathcal{L}) = 0$.

Solution to Exercise 11.37. (a) Since $\operatorname{Ext}^1(\mathcal{L}, \mathcal{O}_C) = H^1(\mathcal{L}^{-1})$ is 1-dimensional, there is an extension of the form given, and the bundle $\mathcal{E}$ in the middle is unique up to isomorphism. We define $\mathcal{E}(2, \mathcal{L})$ to be this bundle.

Suppose that $\mathcal{E}(2, \mathcal{L})$ were decomposable; say $\mathcal{E}(2, \mathcal{L}) = \mathcal{L}' \oplus \mathcal{L}''$, with $\mathcal{L} = \mathcal{L}' \otimes \mathcal{L}''$. Since $\deg \mathcal{L}' + \deg \mathcal{L}'' = 1$, at least one of the two bundles has degree ≥ 1; suppose this is true of $\mathcal{L}'$. From the exact sequence defining $\mathcal{E}(2, \mathcal{L})$ we get an exact sequence

$$0 \to \mathit{Hom}(\mathcal{L}', \mathcal{O}_C) \to \mathit{Hom}(\mathcal{L}', \mathcal{E}(2, \mathcal{L})) \to \mathit{Hom}(\mathcal{L}', \mathcal{L}) \to \cdots .$$

The first term vanishes for degree reasons, so the inclusion map induces a nonzero homomorphism $\mathcal{L}' \to \mathcal{L}$. Since $\deg \mathcal{L}' \geq \deg \mathcal{L}$ it follows that $\mathcal{L}' \cong \mathcal{L}$. But this implies that the sequence defining $\mathcal{E}(2, \mathcal{L})$ is split, a contradiction.

Conversely, given an indecomposable rank 2 bundle $\mathcal{E}$ with determinant $\mathcal{L}$ of degree 1, The Riemann-Roch formula for vector bundles on a curve gives

$$h^0\mathcal{E} - h^1\mathcal{E} = \deg \mathcal{E} - (\operatorname{rank} \mathcal{E})(1-g) = 1,$$

so $\mathcal{E}$ has at least one global section, σ. Let $\mathcal{L}'$ be the preimage in $\mathcal{E}$ of the torsion in $\mathcal{E}/\mathcal{O}_C\sigma$. Since both $\mathcal{L}'' := \mathcal{E}/\mathcal{L}'$ and $\mathcal{L}'$ are torsion free, they are line bundles on C, and we have a short exact sequence

$$0 \to \mathcal{L}' \to \mathcal{E} \to \mathcal{L}'' \to 0.$$

Taking determinants, we see that $\mathcal{L}'' = \mathcal{L} \otimes \mathcal{L}'^{-1}$. Since $H^0\mathcal{L}'$ contains the section σ, we have $\deg \mathcal{L}' \geq 0$, and $\deg \mathcal{L}'' = \deg \mathcal{L} - \deg \mathcal{L}' = 1 - \deg \mathcal{L}'$. Since we supposed $\mathcal{E}$ indecomposable, it follows that the sequence above is not split; that is,

$$\operatorname{Ext}^1(\mathcal{L}', \mathcal{L}'') = H^1(\mathcal{L}' \otimes \mathcal{L}'^{-1}) \neq 0.$$

Since the degree of $\mathcal{L}' \otimes \mathcal{L}'^{-1}$ is $\deg \mathcal{L}' - (1 - \deg \mathcal{L}') = 2\deg \mathcal{L}' - 1$, we must have $\deg \mathcal{L}' = 0$, so $\mathcal{L}' = \mathcal{O}_C$ and $\mathcal{L}'' = \mathcal{L}$ as required. Thus $\mathcal{E} = \mathcal{E}(2, \mathcal{L})$.

(b) Let $\mathcal{L}'$ be a bundle of degree e, and set $\mathcal{L}_0 = \mathcal{L} \otimes \mathcal{L}'^{-2}$, a bundle of degree 1. Set $\mathcal{E}(2, d) = \mathcal{L}' \otimes \mathcal{E}(2, \mathcal{L}_0)$, which is indecomposable, has rank 2 and determinant $\mathcal{L}$. Given any other such bundle, the tensor product with $\mathcal{L}'^{-1}$ would give another indecomposable bundle of rank 2 and determinant $\mathcal{L}_0$; since we have shown that such a bundle is unique up to isomorphism, it follows that $\mathcal{E}(2, \mathcal{L})$ is the only indecomposable bundle with this rank and determinant. From the definitions there is a short exact sequence

$$0 \to \mathcal{L}' \to \mathcal{E}(2, \mathcal{L}) \to \mathcal{L} \otimes \mathcal{L}'^{-1} \to 0.$$

In case $d \geq 2$ the values for the $h^i\mathcal{E}(2, \mathcal{L})$ follow at once from the associated long exact sequence, using $H^1\mathcal{L}' = 0 = H^1\mathcal{L} \otimes \mathcal{L}'^{-1}$. If $d = 1$ the we tensor the defining sequence for $\mathcal{E}(2, \mathcal{L})$ with a nontrivial line bundle of degree 0 to get a sequence

$$0 \to \mathcal{L}' \to \mathcal{E}(2, \mathcal{L}) \to \mathcal{L} \otimes \mathcal{L}'^{-1} \to 0.$$

Since $H^0\mathcal{L}' = 0 = H^1\mathcal{L}'$, the desired result follows at once from the long exact sequence. (Using similar ideas, one can show that $h^0(\mathcal{E}(2, \mathcal{L}) = h^1(\mathcal{E}(2, \mathcal{L})) = 1$ when $\mathcal{L}$ has degree 0.)

$\square$

Exercise 11.38 (Elliptic Scrolls)**.** Let $\mathcal{L}$ be a line bundle of degree d on an elliptic curve C, and let $\mathcal{E} = \mathcal{E}(2, \mathcal{L})^*$. Show that $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ is very ample if and only if $d \geq 5$. In particular, taking $d = 5$, deduce that there is an embedding of $\mathbb{P}\mathcal{E}$ in $\mathbb{P}^4$ as a smooth surface of degree 5, an *elliptic quintic scroll.* (Hint: If $d \geq 5$ then for every divisor D of degree 2 on C, we have $h^0(\mathcal{E}(2, \mathcal{L})(-D)) = d - 2 * \operatorname{rank} \mathcal{E} = d - 4$ by Exercise 11.37, so we can use the criterion of Exercise 11.36. On the other hand, if $d = 4$ or less then there is some line bundle of degree 2 such that $h^0(\mathcal{E}(2, \mathcal{L})(-D)) \neq d - 4$.

One can decompose such a surface geometrically as follows. From Exercise 11.37 it follows that every line bundle of degree -3 can be embedded in $\mathcal{E}$, and gives rise to a section of the scroll. Since $\deg \mathcal{E} = 5 = 3 + 3 - 1$, any two of these sections, say X_1, X_2 meet in a single point p, and in fact lie in planes in $\mathbb{P}^4$ that meet only at p. The scroll S is the closure of the union of the lines connecting corresponding points of these two sections other than p. For more information see Lanteri [1980] and Ionescu and Toma [1997].

The elliptic quintic scrolls are closely related to the Horrocks-Mumford bundle on $\mathbb{P}^4$; see for example Hulek [1995]. The homogeneous coordinate rings of the scrolls are normal domains, and even nonsingular in codimension 2, but not Cohen-Macaulay (because H^1 of their structure sheaves is nonzero); in many ways they are the simplest examples of such rings.

Exercise 11.39. In Example 11.21 we used intersection theory to show that there does not a exist a rational solution to the general quadratic polynomial, that is, there do not exist rational functions $X(a, \dots, f), Y(a, \dots, f)$ and $Z(a, \dots, f)$ such that

$$aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ \equiv 0.$$

To gain some appreciation of the usefulness of intersection theory, give an elementary proof of this assertion.

Exercise 11.40. Let

$$\Phi = \left\{(L, p) \in \mathbb{G}(1, n) \times \mathbb{P}^n \mid p \in L\right\}$$

be the universal line in $\mathbb{P}^n$, and let σ_1, $\sigma_{1,1}$ and ζ be the pullbacks of the Schubert classes $\sigma_1 \in A^1(\mathbb{G}(1, n))$, $\sigma_{1,1} \in A^2(\mathbb{G}(1, n))$ and the hyperplane class $\zeta \in A^1(\mathbb{P}^n)$ respectively. Find the degree of all monomials $\sigma_1^a \sigma_{1,1}^b \zeta^c$ of top degree $a + 2b + c = \dim(\Phi) = 2n - 1$.

Exercise 11.41. Consider the flag variety of pairs consisting of a point $p \in \mathbb{P}^3$ and a line $L \subset \mathbb{P}^3$ containing p; that is,

$$\mathbb{F} = \{(p, L) \in \mathbb{P}^3 \times \mathbb{G}(1, 3) \mid p \in L \subset \mathbb{P}^3\}.$$

$\mathbb{F}$ may be viewed as a $\mathbb{P}^1$-bundle over $\mathbb{G}(1, 3)$, or as a $\mathbb{P}^2$-bundle over $\mathbb{P}^3$. Calculate the Chow ring $A(\mathbb{F})$ via each map, and show that the two descriptions agree.

Exercise 11.42. By Theorem 11.9, the Chow ring of the product $\mathbb{P}^3 \times \mathbb{G}(1, 3)$ is just the tensor product of their Chow rings; that is

$$A(\mathbb{P}^3 \times \mathbb{G}(1, 3)) = A(\mathbb{G}(1, 3))[\zeta]/(\zeta^4).$$

In these terms, find the class of the flag variety $\mathbb{F} \subset \mathbb{P}^3 \times \mathbb{G}(1, 3)$ of Exercise 11.41.

Exercise 11.43. Generalizing the preceding problem, let

$$\mathbb{F}(0,k,r) = \{(p,\Lambda) \in \mathbb{P}^r \times \mathbb{G}(k,r) \mid p \in \Lambda \subset \mathbb{P}^3\}.$$

Find the class of $\mathbb{F}(0,1,r) \subset \mathbb{P}^r \times \mathbb{G}(k,r)$.

Exercise 11.44. Generalizing Exercise 11.42 in a different direction, let

$$\Phi_r = \{(L,M) \in \mathbb{G}(1,r) \times \mathbb{G}(1,r) \mid L \cap M \neq \varnothing\}.$$

Given that, by Theorem 11.22 we have

$$A\big(\mathbb{G}(1,r) \times \mathbb{G}(1,r)\big) \cong A\big(\mathbb{G}(1,r)\big) \otimes A\big(\mathbb{G}(1,r)\big),$$

find the class of Φ_r in $A\big(\mathbb{G}(1,r) \times \mathbb{G}(1,r)\big)$

(a) $r = 3$;

(b) $r = 4$; and

(c) for general r.

Exercise 11.45. Let Z be the blowup of $\mathbb{P}^n$ along an $(r-1)$-plane, and let $E \subset Z$ be the exceptional divisor. Find the degree of the top power $e^n \in A(Z)$.

Exercise 11.46. Again let $Z = Bl_\Lambda \mathbb{P}^n$ be the blowup of $\mathbb{P}^n$ along an $(r-1)$-plane Λ. In terms of the description of the Chow ring of Z given in Section 11.4.2, find the classes of the following:

(a) the proper transform of a linear space $\mathbb{P}^s$ containing Λ, for each $s > r$;

(b) the proper transform of a linear space $\mathbb{P}^s$ in general position with respect to Λ (that is, disjoint from Λ if $s \leq n-r$; and transverse to Λ if $s > n-r$); and

(c) in general, the proper transform of a linear space $\mathbb{P}^s$ intersecting Λ is an l-plane.

Exercise 11.47. Let $Z = Bl_L \mathbb{P}^3$ be the blowup of $\mathbb{P}^3$ along a line. In terms of the description of the Chow ring of Z given in Section 11.4.2, find the classes of the proper transform of a smooth surface $S \subset \mathbb{P}^3$ of degree d containing L.

Exercise 11.48. Now let $Z = Bl_L \mathbb{P}^4$ be the blowup of $\mathbb{P}^4$ along a line, and let $S \subset \mathbb{P}^4$ be a smooth surface of degree d containing L. Show by example that the class of the proper transform of S in Z is not determined by this data. For example, try taking S a cubic scroll, with L either

(a) a line of the ruling of S; or

(b) the directrix of S

and seeing that you get different answers.

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Exercise 11.49. Let $Z = Bl_\Lambda \mathbb{P}^n$ be the blowup of $\mathbb{P}^n$ along an $(r-1)$-plane Λ; that is,

$$Z = \{(p, q) \in \mathbb{P}^n \times \Gamma \mid p \in \overline{\Lambda\, q}\}.$$

Using the description of the Chow ring of Z given in Section 11.4.2, find the class of $Z \subset \mathbb{P}^n \times \mathbb{P}^{n-r}$.

Exercise 11.50. Show that for very general $t \in \mathbb{P}^1$ (that is, for all but countably many t), the line bundle $\mathcal{O}_{C_t}(p_1 - p_2)$ is not torsion in $\operatorname{Pic}(C_t) = A^1(C_t)$.

Exercise 11.51. Now let S be the blow-up of the plane at the points $p_1, \dots, p_9$—that is, the graph of the rational map $\mathbb{P}^2 \to \mathbb{P}^1$ given by (F, G)—and let $E_1, \dots, E_9$ be the exceptional divisors. Show that there is a biregular automorphism $\varphi : S \to S$ that commutes with the projection $S \to \mathbb{P}^1$ and carries E_1 to E_2.

Exercise 11.52. Using the result of Exercise 11.50, show that the automorphism φ of Exercise 11.51 has infinite order, and deduce that the surface S contains infinitely many irreducible curves of negative self-intersection.

Exercise 11.53. Let C be a smooth curve of genus $g \geq 2$ over a field of characteristic $p > 0$; let $\varphi : C \to C$ be the Frobenius morphism. If $\Gamma_n \subset C \times C$ is the graph of φ^n and $\gamma_n = [\Gamma_n] \in A^1(C \times C)$ its class, show that the self-intersection $\deg(\gamma_n^2) \to -\infty$ as $n \to \infty$.

Exercise 11.54. An amusing enumerative problem: in the circumstances of the preceding exercises, for how many $t \in \mathbb{P}^1$ will it be the case that $\mathcal{O}_{C_t}(p_1 - p_2)$ is torsion of order 2—that is, that $\mathcal{O}_{C_t}(2p_1) \cong \mathcal{O}_{C_t}(2p_2)$?

Exercise 11.55. Show that if E is a vector bundle of rank 2 and degree e on a smooth projective curve X, and L and M sub-line bundles of degrees a and b corresponding to sections of $\mathbb{P}E$ with classes σ and τ, then

$$\sigma\tau = e - a - b$$

and

$$\sigma^2 + \tau^2 = 2e - 2a - 2b.$$

In particular, if L and M are distinct then $\deg \sigma^2 + \deg \tau^2 \geq 0$, with equality holding if and only if $E = L \oplus M$.

Exercise 11.56. Using the analysis of Example 11.21 as a template, show that for $d > 1$ the universal hypersurface

$$\Phi_{d,n} = \{(X, p) \in \mathbb{P}^N \times \mathbb{P}^n \mid p \in X\} \to \mathbb{P}^N$$

admits no rational section.

Exercise 11.57. Consider the flag variety of pairs consisting of a point $p \in \mathbb{P}^4$ and a 2-plane $\Lambda \subset \mathbb{P}^4$ containing p; that is,

$$\mathbb{F} = \{(p, L) : p \in \Lambda \subset \mathbb{P}^4\} \subset \mathbb{P}^4 \times \mathbb{G}(2, 4).$$

$\mathbb{F}$ may be viewed as a $\mathbb{P}^2$-bundle over $\mathbb{G}(2, 4)$, or as a $\mathbb{G}(1, 3)$-bundle over $\mathbb{P}^4$. Calculate the Chow ring $A(\mathbb{F})$ via each map, and show that the two descriptions agree.

Exercise 11.58. Using the proof of Proposition 11.26 as a template, prove Proposition **??**. Show also that the conclusion of Proposition **??** may be false if we replace $\mathbb{P}^n$ by an arbitrary smooth, projective variety of dimension n.

Exercise 11.59. Show that the analog of Lemma 11.28 is false if we allow the V_i to have codimension > 1: in other words, $V_i \subset E_{p_i}$ is a general linear subspace of codimension m_i, then the corresponding subspace $W \subset H^0(E)$ need not have dimension $\max\{0, h^0(E) - \sum m_i\}$. (Hint: Consider a bundle whose sections all lie in a proper subbundle.)

Exercise 11.60. Calculate the remaining five intersection numbers in the table of intersection numbers in Section 11.8.2.

Exercise 11.61. To find the class $\delta = [D_L] \in A^1(\mathcal{H})$ of the cycle of conics meeting a line directly, restrict to the open subset $U \subset \mathcal{H}$ of pairs $(H, \xi) \in \mathcal{H}$ such that H does not contain L (since the complement of this open subset of $\mathcal{H}$ has codimension 2, any relation among divisor classes that holds in U will hold in $\mathcal{H}$). Show that we have a map $\alpha : U \to L$ sending a pair (H, ξ) to the point $p = H \cap L$, and that in U the divisor D_L is the zero locus of the map of line bundles

$$T \to \alpha^* \mathcal{O}_L(2)$$

sending a quadric $Q \in \xi$ to $Q(p)$.

Exercise 11.62. Let $\Delta \subset \mathcal{H}$ be the locus of singular conics.

(a) Show that Δ is an irreducible divisor in $\mathcal{H}$.

(b) Express the class $\delta \in A^1(\mathcal{H})$ as a linear combination of ω and ζ.

(c) Use this to calculate the number of singular conics meeting each of 7 general lines in $\mathbb{P}^3$; and

(d) Verify your answer to the last part by calculating this number directly.

Exercise 11.63. Let $p \in \mathbb{P}^3$ be a point, and $F_p \subset \mathcal{H}$ the locus of conics containing the point p. Show that F_p is six-dimensional, and find its class in $A^2(\mathcal{H})$

Exercise 11.64. Use the result of the preceding exercise to find the number of conics passing through a point p and meeting each of 6 general lines in $\mathbb{P}^3$, the number of conics passing through two points p, q and meeting each of 4 general lines in $\mathbb{P}^3$, and the number of conics passing through three points p, q, r and meeting each of 2 general lines in $\mathbb{P}^3$. Verify your answers to the last two parts by direct examination.

Exercise 11.65. Find the class in $A^3(\mathcal{H})$ of the locus of double lines (note that this is five-dimensional, not four!)

Exercise 11.66. Suppose that $X \subset \mathbb{P}^n$ is a subscheme of pure dimension l, and $\mathcal{H}$ a component of the Hilbert scheme parametrizing subschemes of $\mathbb{P}^n$ of pure dimension $k < n - l$ in $\mathbb{P}^n$; let $[Y] \in \mathcal{H}$ be a smooth point corresponding to a subscheme $Y \subset \mathbb{P}^n$ such that $Y \cap X = \{p\}$ is a single reduced point, and suppose moreover that p is a smooth point of both X and Y. Finally, let $\Sigma_X \subset \mathcal{H}$ be the locus of subschemes meeting X.

Use the technique of Proposition 11.26 to show that $\Sigma_X \subset \mathcal{H}$ is smooth at $[Y]$, of the expected codimension $n - k - l$, with tangent space

$$T_{[Y]}\Sigma_X = \left\{\sigma \in H^0(N_{Y/\mathbb{P}^n}) : \sigma(p) \in \frac{T_pX + T_pY}{T_pY}\right\}.$$

The next few problems deal with an example of a phenomenon encountered in the preceding chapter: the possibility that the cycles in our parameter space corresponding to the conditions imposed in fact do not meet transversely, or even properly.

Exercise 11.67. Let $H \subset \mathbb{P}^3$ be a plane, and let $\mathcal{E}_H \subset \mathcal{H}$ be the closure of the locus of smooth conics $C \subset \mathbb{P}^3$ tangent to H. Show that this is a divisor, and find its fundamental class $\beta \in A^1(\mathcal{H})$.

Exercise 11.68. Find the number of smooth conics in $\mathbb{P}^3$ meeting each of 7 general lines $L_1, \dots, L_7 \subset \mathbb{P}^3$ and tangent to a general plane $H \subset \mathbb{P}^3$. More generally, find the number of smooth conics in $\mathbb{P}^3$ meeting each of $8 - k$ general lines $L_1, \dots, L_{8-k} \subset \mathbb{P}^3$ and tangent to a k general planes $H_1, \dots, H_k \subset \mathbb{P}^4$, for $k = 1, 2$ and 3.

Exercise 11.69. Why don't the methods developed here work to calculate the number of smooth conics in $\mathbb{P}^3$ meeting each of $8 - k$ general lines $L_1, \dots, L_{8-k} \subset \mathbb{P}^3$ and tangent to a k general planes $H_1, \dots, H_k \subset \mathbb{P}^4$, for $k \geq 4$? What can you do to find these numbers? (In fact, we have seen how to deal with this in Chapter 10)

Next, some problems involving conics in $\mathbb{P}^4$:

Exercise 11.70. Now let $\mathcal{K}$ be the space of conics in $\mathbb{P}^4$ (again, defined to be complete intersections of two hyperplanes and a quadric). Use the description of $\mathcal{K}$ as a $\mathbb{P}^5$-bundle over the Grassmannian $\mathbb{G}(2,4)$ to determine its Chow ring.

Exercise 11.71. In terms of your answer to the preceding problem, find the class of the locus D_Λ of conics meeting a 2-plane Λ, and of the locus $\mathcal{E}_L$ of conics meeting a line $L \subset \mathbb{P}^4$.

Exercise 11.72. Find the expected number of conics in $\mathbb{P}^4$ meeting each of 11 general 2-planes $\Lambda_1, \dots, \Lambda_{11} \subset \mathbb{P}^4$.

Exercise 11.73. Prove that your answer to the preceding problem is in fact the actual number of conics by showing that for general 2-planes $\Lambda_1, \dots, \Lambda_{11} \subset \mathbb{P}^4$ the corresponding cycles D_{Λ_i} intersect transversely.

Finally, here's a challenge problem:

Exercise 11.74. Let $\{S_t \subset \mathbb{P}^3\}_{t \in \mathbb{P}^1}$ be a general pencil of quartic surfaces (that is, take A and B general homogeneous quartic polynomials, and set $S_t = V(t_0 A + t_1 B) \subset \mathbb{P}^3$). How many of the surfaces S_t contain a conic?

12

Segre Classes and Varieties of Linear Spaces

Keynote Questions

(a) At how many points of $\mathbb{P}^n$ do $2n$ general tangent vector fields all annihilate the same nonzero cotangent vector?

(b) If f is a general polynomial of degree $d = 2k - 1$ in one variable over a field of characteristic 0, then there is a unique way to write f as a sum of k d^{th} powers of linear forms (Proposition **??**). If f and g are general polynomials of degree $d = 2k$ in one variable, how many linear combinations of f and g are expressible as a sum of k d^{th} powers of linear forms?

(c) If $C \subset \mathbb{P}^4$ is a general rational curve of degree d over a field of characteristic 0, how many 3-secant lines does C have?

(d) If $C \subset \mathbb{P}^3$ is a general rational curve of degree d over a field of characteristic 0, what is the degree of the surface swept out by the 3-secant lines to C?

12.1 Segre Classes

Our understanding of the Chow rings of projective bundles makes accessible the computation of the classes of another natural series of loci associated to a vector bundle. Suppose that $\mathcal{E}$ is a vector bundle on a scheme X that can be generated by global sections. How many global sections does it take to generate $\mathcal{E}$? More generally, what sort of locus is it where a given number of general global sections fail to generate $\mathcal{E}$ locally?

We can get a feeling for these questions as follows. First, consider the case where $\mathcal{E}$ is a line bundle. In this case each section corresponds to a divisor of class $c_1(\mathcal{E})$. If $\mathcal{E}$ is generated by its global sections, the linear series of these divisors is base-point-free, so a general collection of i of them will intersect in a codimension i locus of class $c_1(\mathcal{E})^i$. That is, the locus where i general sections of $\mathcal{E}$ fail to generate $\mathcal{E}$ has "expected" codimension i and class $c_1(\mathcal{E})^i$.

Now suppose that $\mathcal{E}$ has rank $r > 1$. It is clear that fewer than r sections can never generate $\mathcal{E}$ locally anywhere. On the other hand, since we have assumed that $\mathcal{E}$ is generated by global sections, we can choose r global sections that are minimal generators of the fiber of $\mathcal{E}$ at a given closed point $p \in X$, and by Nakayama's Lemma (Eisenbud [1995] Theorem ***) these sections will generate the stalk of $\mathcal{E}$ at p, and thus will generate $\mathcal{E}$ locally at every point of an open set containing p. Of course for r sections to generate $\mathcal{E}$ locally at a point p is the same as for the r sections to be independent at p, so we know that the locus where this fails (for r general sections) is the first Chern class, $c_1(\mathcal{E})$, as before; in particular it has codimension 1.

Given r general sections of $\mathcal{E}$, let X' be the codimension 1 subset of $\mathcal{E}$ consisting of points p where the sections do not generate $\mathcal{E}$. One can hope that at a general point of X' the sections have only one degeneracy relation, so that on some open set $U \subset X'$, they generate a co-rank 1 subbundle of $\mathcal{E}' \subset \mathcal{E}$, and the quotient $\mathcal{E}/\mathcal{E}'$ is a line bundle on U. The sections of $\mathcal{E}$ yield sections of $\mathcal{E}/\mathcal{E}'$, so if it is a line bundle they will vanish in codimension 1 in U; that is, we should expect $r+1$ general sections of $\mathcal{E}$ to generate $\mathcal{E}$ away from a codimension 2 subset of X. Continuing in this way (and assuming that $n \geq i$), it seems that $r+i-1$ sections of $\mathcal{E}$ might generate $\mathcal{E}$ away from a codimension i locus. In particular, $r+\dim X$ sections might generate $\mathcal{E}$ locally everywhere.

A case that is beloved of algebraists is that in which $\mathcal{E} = (\mathcal{O}_{\mathbb{P}V}(1))^r$. A collection of $r-i+1$ general sections is a general map

$$\mathcal{O}_{\mathbb{P}V}^{r-i+1} \xrightarrow{\varphi} (\mathcal{O}_{\mathbb{P}^n}(1))^r,$$

that is, a general $r \times (r-i+1)$ matrix of linear forms. The locus where the sections fail to generate is the support of the cokernel, which is defined by the $r \times r$ minors of the matrix. By the "Generalized Principle Ideal Theorem" of Eagon and Northcott (see Eisenbud [1995] Theorem ****), the codimension of this locus is at most i, and in fact equality holds (as we shall soon see) whenever $n+1 \geq i$. In fact, the support of the cokernel is exactly the scheme defined by the ideal of minors in this case (see Buchsbaum and Eisenbud [1977].)

It turns out that the construction of projective bundles gives us an effective way of reducing this question about vector bundles (and many others) to the case of line bundles, passing from $\mathcal{E}$ to the line bundle $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ on $\mathbb{P}\mathcal{E}$. To relate this line bundle to classes on X, we push forward its self intersections:

Definition 12.1. Let X be a smooth projective scheme; let $\mathcal{E}$ be a vector bundle of rank r on X, and let $\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1))$. The *i-th Segre class* of $\mathcal{E}$ is the class

$$s_i(\mathcal{E}) = \pi_*(\zeta^{r-1+i}) \in A^i(X).$$

and the the (total) *Segre class* of $\mathcal{E}$ is the sum

$$s(\mathcal{E}) = 1 + s_1(\mathcal{E}) + s_2(\mathcal{E}) + \dots.$$

(For a definition of the Segre classes in greater generality see Fulton [1984] Chapter 4.)

The Segre classes give the answer to our question about generating vector bundles:

Proposition 12.2. *If $\mathcal{E}$ is a vector bundle of rank r that is generated by global sections, and X_i is the locus where a given set of $r+i-1$ general global sections fail to generate $\mathcal{E}$, then every component of X_i has codimension exactly i, and the i-th Segre class $s_i(\mathcal{E}^*)$ of $\mathcal{E}^*$ is represented by a positive linear combination of the components of X_i.*

As we'll see in a moment, the Segre classes of a bundle and its dual satisfy the relation $s_i(\mathcal{E}^*) = (-1)^i s_i(\mathcal{E})$. Thus the Proposition shows an interesting parallel between the Chern classes and the Segre classes of a bundle:

- The i^{th} Chern class $c_i(\mathcal{E})$ is the locus of fibers where a suitably general bundle map
$$\mathcal{O}_X^{\oplus r+1-i} \to \mathcal{E}$$
fails to be injective.
- The i^{th} Segre class $s_i(\mathcal{E})$ is $(-1)^i$ times the locus of fibers where a suitably general bundle map
$$\mathcal{O}_X^{\oplus r-1+i} \to \mathcal{E}$$
fails to be surjective.

The Segre classes may seem to give a way of defining new cycle class invariants of a vector bundle; but in fact they are essentially a different way of packaging the information contained in the Chern classes. Postponing the proof of Proposition 12.2 for a moment, we explain the remarkable relationship. a special case of the Porteous formula (Chapter 14).

Proposition 12.3. *The Segre class and the Chern class of a bundle $\mathcal{E}$ on X are reciprocals of one another in the Chow ring of X:*

$$s(\mathcal{E})c(\mathcal{E}) = 1 \in A(X).$$

Using the formula $c_i(\mathcal{E}^*) = (-1)^i c_i(\mathcal{E})$ we deduce that

$$s_i(\mathcal{E}^*) = (-1)^i s_i(\mathcal{E}).$$

Also for any exact sequence $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ of vector bundles, the Whitney formula gives $c(\mathcal{F}) = c(\mathcal{E})c(\mathcal{G})$, whence

$$s(F) = s(\mathcal{E})s(\mathcal{G}).$$

Proof of Proposition 12.3. If S and Q are the tautological sub- and quotient bundles on $\mathbb{P}\mathcal{E}$, and $\zeta := c_1(S^*$ is the tautological class, then $c(S) = 1 - \zeta$ so, by the Whitney formula (Theorem **??**)

$$c(Q) = \frac{c(\pi^*\mathcal{E})}{c(S)} = c(\pi^*\mathcal{E})(1 + \zeta + \zeta^2 + \dots) \in A(\mathbb{P}\mathcal{E}).$$

We now push this equation forward to X. Considering first the left hand side, we see that for $i < r - 1$ the Chern class $c_i(Q)$ is represented by a cycle of dimension $> \dim X$, so it maps to 0, while the top Chern class $c_{r-1}(Q)$ maps to the fundamental class of X. Thus $\pi_*(c(Q)) = 1 \in A(X)$. On the other hand, the push-pull formula tells us that

$$\begin{aligned} \pi_*\big(c(\pi^*\mathcal{E})(1 + \zeta + \zeta^2 + \dots)\big) &= c(\mathcal{E}) \cdot \pi_*(1 + \zeta + \zeta^2 + \dots) \\ &= c(\mathcal{E})s(\mathcal{E}), \end{aligned}$$

completing the argument. $\square$

For example, if $X = \mathbb{P}^n$ and $\mathcal{E} = (\mathcal{O}_{\mathbb{P}^n}(1))^r$, then

$$s(\mathcal{E}) = \frac{1}{c(\mathcal{E})} = \frac{1}{(1+\zeta)^r} = 1 - r\zeta + \binom{r+1}{2}\zeta^2 - \binom{r+2}{3}\zeta^3 + \cdots.$$

Proof of Proposition 12.2. The locus X_i is given locally by the vanishing of the $r \times r$ minors of an $r \times (r+i-1)$ matrix. By the Generalized Principle Ideal Theorem (see for example Eisenbud [1995] Theorem ****), the locus X_i is either empty or everywhere of codimension at most i.

Now suppose that the section σ_i are general. To prove that the components of X_i are all of codimension precisely i and support the cycle $s_i(\mathcal{E}^*)$, we move to the projective bundle. Let $\pi : \mathbb{P}\mathcal{E}^* \to X$ be the projection morphism. Recall that the Theorem on Cohomology and Base Change, Theorem 6.14, implies that the natural map $\pi^*(\mathcal{E}) \to \mathcal{O}_{\mathbb{P}\mathcal{E}^*}(1)$ induces an isomorphism $\mathcal{E} = \pi_*\pi^*(\mathcal{E}) \to \pi_*\mathcal{O}_{\mathbb{P}\mathcal{E}^*}(1)$, and thus also induces an isomorphism on global sections

$$H^0(\mathcal{E}) \to H^0(\mathcal{O}_{\mathbb{P}\mathcal{E}^*}(1)).$$

The fiber of π over a point $p \in X$ is the projective space $\mathbb{P}\mathcal{E}_p^*$, and the points $q \in \pi^{-1}(p)$ correspond to the different one-dimensional quotients of $\mathcal{E}_p$. A collection of sections σ_i of $\mathcal{E}$ thus generates $\mathcal{E}_p$ if and only if the corresponding sections σ_i' of $\mathcal{O}_{\mathbb{P}\mathcal{E}^*}(1)$ generate that line bundle at every point $q \in \pi^{-1}(p)$. It follows that the locus in X where $\mathcal{E}$ is *not* generated by the σ_i is the image of the locus $Y_i \subset \mathbb{P}\mathcal{E}^*$ where $\mathcal{O}_{\mathbb{P}\mathcal{E}^*}(1)$ is not generated by the σ_i'. This last is the intersection of the $r-i+1$ divisors

$$D_i = \{q \in \mathbb{P}\mathcal{E}^* \mid \sigma_i'(q) = 0\}.$$

Since we have assumed that $\mathcal{E}$ is generated by global sections, the same will be true for $\pi^*(\mathcal{E})$, and thus of $\mathcal{O}_{\mathbb{P}\mathcal{E}^*}(1)$. By the Principal Ideal Theorem, every component of the intersection of j divisors corresponding to global sections of $\mathcal{O}_{\mathbb{P}\mathcal{E}^*}(1)$ will have codimension $\leq j$ in $\mathbb{P}\mathcal{E}^*$. Since the sections σ_i are general, every component of Y_i has codimension exactly $r+i-1$, and thus has dimension $= n-i$.

Each component $X_{i,j}$ of X_i is the image of some component of Y_i. Its dimension is thus $\leq n-i$, so its codimension is $\geq i$. The argument at the beginning of the proof gives the opposite inequality, so $X_{i,j}$ has codimension exactly i. It follows that π induces generically finite maps from some components of Y_i onto each $X_{i,j}$, so $\pi_*[Y_i]$ is a positive linear combination of the $[X_{i,j}]$ as required. □

We remark that, with notation as in the proof above, the locus in X where the fibers of the map $Y_{r-1+i} \to X_{r-1+i}$ have dimension k is precisely the locus in X where the rank of the map $\mathcal{O}_X^{r-1+i} \to \mathcal{E}$ induced by the σ_i drops rank by k.

We can now answer Keynote Question (a). A tangent vector field on $\mathbb{P}^n$ is a section of $T_{\mathbb{P}^n} = \Omega_{\mathbb{P}^n}^*$, so the question can be rephrased as: at how many points of $\mathbb{P}^n$ do $2n$ general sections of $T_{\mathbb{P}^n}$ fail to generate $T_{\mathbb{P}^n}$? By Proposition 12.2 this is $(-1)^n$ times the degree of the Segre class $s_n(T_{\mathbb{P}^n})$. By Proposition 12.3, $s(T_{\mathbb{P}^n}) = 1/c(T_{\mathbb{P}^n})$. From the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \to T_{\mathbb{P}^n} \to 0$$

and the Whitney formula (Theorem **??**) we get $c(T_{\mathbb{P}^n}) = (1+\zeta)^{n+1}$, where ζ is the hyperplane class on $\mathbb{P}^n$. Putting this together,

$$s(T_{\mathbb{P}^n}) = \frac{1}{(1+\zeta)^{n+1}} = 1 - (n+1)\zeta + \binom{n+2}{2}\zeta^2 + \cdots,$$

so the answer is $\binom{2n}{n}$.

12.2 Varieties swept out by linear spaces

We can use Segre classes to calculate the degrees of some interesting varieties "swept out" by linear spaces in the following sense. Let $B \subset \mathbb{G}(k,n)$ be a subvariety of dimension m in the Grassmannian, and let

$$X = \bigcup_{b \in B} \Lambda_b \ \subset \ \mathbb{P}^n$$

be the union of the planes corresponding to the points of B. If we let S be the universal subbundle on $\mathbb{G}(k,n)$, and

$$\Phi = \mathbb{P}S = \{(\Lambda, p) \in \mathbb{G}(k,n) \times \mathbb{P}^n \mid p \in \Lambda\}$$

the universal k-plane, with projection maps

$$\mathbb{G}(k,n) \xleftarrow{\ \pi\ } \Phi \xrightarrow{\ \eta\ } \mathbb{P}^n,$$

then we can write

$$X = \eta\big(\pi^{-1}(B)\big).$$

Since the preimage $\pi^{-1}(B) \subset \Phi$ is necessarily a variety of dimension $m+k$, we see from this that X will be a subvariety of $\mathbb{P}^n$ of dimension at most $m+k$. In case it has dimension equal to $m+k$—that is, the map η_B : $\alpha^{-1}(B) \to X$ is generically finite of some degree d—we will say that X is *swept out d times* by the planes Λ_b.

Assuming that $\dim X = m+k$, the degree of X is the number of times X meets a plane of dimension $n-m-k$. Recall that $\sigma_{1^m} = \sigma_{1,1,\dots,1} \in A^m(\mathbb{G}(k,n))$ is the class of the cycle of k-planes meeting a given $n-m-k$-plane. Thus, if the map η_B has degree d, then the degree of X is $1/d$ times $\deg(\sigma_{1^m} \cdot [B])$. The degree can also be conveniently expressed as the degree of a Segre class:

Proposition 12.4. *Let S be the tautological subbundle over $\mathbb{G}(k,n)$. Let $B \subset \mathbb{G}(k,n)$ be a smooth subvariety of dimension m, and let $\mathcal{E} = S_B$ be the restriction of S to B. If*

$$X = \bigcup_{b \in B} \Lambda_b \ \subset \ \mathbb{P}^n$$

is swept out d times by the planes corresponding to points of B then

$$\deg(X) = \deg(s_m(\mathcal{E}))/d.$$

Proof. If $L \in H^0(\mathcal{O}_{\mathbb{P}^n}(1))$ is a homogeneous linear form on $\mathbb{P}^n$ then L defines a section of $\mathcal{E}^*$ by restriction to each fiber of $\mathcal{E} = S_B$, and hence a section σ_L of $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$. The preimage $\eta_B^{-1}(H)$ of the hyperplane $H = V(L) \subset \mathbb{P}^n$ given by L is the zero locus of σ_L. Thus the pullback of the hyperplane class on $\mathbb{P}^n$ under the map η_B is the tautological class $\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1))$ on $\mathbb{P}\mathcal{E}$, and it follows that $d \cdot \deg(X) = \deg \zeta^{m+k} = \deg s_m(\mathcal{E})$ as required. $\square$

12.3 Secant varieties

The study of secant varieties to projective varieties $X \subset \mathbb{P}^n$ is a rich one with a substantial history and many fundamental open problems. In this section, we'll discuss some of the basic questions. In the following sections we will use Segre classes to compute the degrees of some varieties of secants of rational curves.

12.3.1 Symmetric powers

A k-secant m-plane to a variety X in $\mathbb{P}^r$ is a linear space of dimension m that meets X in k points, so it will be useful to introduce a classical construction of a variety whose points are (unordered) k-tuples of points of X: the *k-th symmetric power* $X^{(k)}$ of X.

Formally, we define $X^{(k)}$ to be the quotient of the ordinary k-fold product X^k of copies of X by the action of the symmetric group on k letters, $\mathfrak{S}_k$, acting on X^k by permuting the factors. If $X = \operatorname{Spec} A$ is any affine scheme, this means that

$$X^{(k)} := \operatorname{Spec}\big((A \otimes A \otimes \cdots \otimes A)^{\mathfrak{S}_k}\big).$$

(We will use this construction only for varieties, but see Exercise 12.24 for the case $A = \operatorname{Spec} K[t]/(t^n)$.) When X is quasiprojective $X^{(k)}$ is defined by patching together symmetric powers of affine open subsets of X. The main theorem of Galois theory shows that, when X is a variety, the extension of rational function fields $K(X^k)/K(X^{(k)})$ is Galois, and of degree $k!$.

One can show that such quotients are *categorical*; any morphism $X^{(k)} \to Y$ determines an $\mathfrak{S}_k$-invariant morphism $X^k \to Y$, and this is a one-to-one correspondence. Further, the closed points of X correspond naturally to the effective 0-cycles on X: they are usually denoted additively as $p_1 + \cdots + p_k$, where the $p_i \in X$ need not be distinct. For these results, see ****. **** need a reference; Fogarty's book on invariant theory??****

Since the natural map $X^k \to X^{(k)}$ is finite, $X^{(k)}$ is affine (respectively, projective) if and only if X is.

A familiar example is the case $X = \mathbb{A}^1$: then $X = \operatorname{Spec} K[t]$, so

$$(\mathbb{A}^1)^{(k)} = \operatorname{Spec}(K[t_1, \ldots, t_k]^{\mathfrak{S}_k}).$$

This ring of invariants is a polynomial ring on the k elementary symmetric functions (see for example Eisenbud [1995] Theorem *** for an algebraic proof), so $(\mathbb{A}^1)^{(k)} = \mathbb{A}^k$. Set-theoretically this is the statement that a polynomial is determined up to scalars by the set of its roots, counting multiplicity.

A similar result holds for $\mathbb{P}^1$. We could deduce it from the case of $\mathbb{A}^1$, but instead we give a geometric proof:

Proposition 12.5.

$$(\mathbb{P}^1)^{(k)} \cong \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^1}(k)) = \mathbb{P}^k.$$

Proof. We think of $\mathbb{P}^1$ as $\mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^1}(1))$ the space of linear forms in 2 variables, up to scalars. The product of k linear forms is a form of degree k, which is independent of the order in which the product is taken. Thus multiplication defines a morphism $\varphi : (\mathbb{P}^1)^k \to \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^1}(k))$ that is invariant under the group $\mathfrak{S}_k$. The morphism φ is finite and set-theoretically $k!$-to-one. Thus it has degree $\geq k!$. (In fact the degree is $k!$ and the map is separable, as one could show directly; but these things follow from the argument in the next paragraph.)

Since φ is invariant it factors through a morphism $\psi : (\mathbb{P}^1)^{(k)} \to \mathbb{P}^k$, and since the degree of the quotient map $(\mathbb{P}^1)^k \to (\mathbb{P}^1)^{(k)}$ is $k!$, we see that ψ is birational. Since $\mathbb{P}^k$ is normal and ψ is finite and birational, ψ is an isomorphism. $\square$

The construction of $X^{(k)}$ is most useful when X is a smooth curve. One reason is given by the following result.

Proposition 12.6. *If X is a variety and $k > 1$, then $X^{(k)}$ is smooth if and only if X is smooth and* $\dim X \leq 1$.

Proof. If $\dim X = 0$ then X consists of a single reduced point, and $X^{(k)}$ is also a single reduced point. Thus we may assume that $\dim X > 0$.

Away from the subsets where at least two factors are equal the quotient map $X^k \to X^{(k)}$ is an unramified covering. Thus if X is singular at a point p and $p, q_1, \dots, q_{k-1} \in X$ are distinct points, then near $p+q_1+\cdots+q_{k-1}$ the variety $X^{(k)}$ looks like the product X^k near $(p, q_1, \dots, q_{k-1})$; in particular, it is singular.

Now suppose that X is smooth. Imitating the argument above, we see that $X^{(k)}$ will be smooth if and only if points of the form $k'p = p + \cdots + p$ are smooth on $X^{k'}$ for all $k' \leq k$, so it suffices to show the smoothness of $X^{(k)}$ for all k at a point kp when X is smooth and 1-dimensional, and the singularity of $X^{(2)}$ when $\dim X > 1$. Together with the following Lemma, this leaves us with a problem in local algebra

Lemma 12.7. *Suppose that R is a normal Noetherian local domain. If G is a finite group of automorphisms of R, then the completion of the ring of invariants R^G is the ring of invariants of the completion of R under the induced action of G; that is,*

$$\widehat{R^G} = (\widehat{R})^G.$$

****is this true without normal? What's a reference?****

Proof. It is obvious that $\widehat{R^G} \subset (\widehat{R})^G$. The ring R is finite over R^G, and the quotient field extension $K(R)/K(R^G)$ has degree $|G|$. Thus $K(\widehat{R}) = \widehat{R^G} \otimes K(R)$ is of degree $|G|$ over $K(\widehat{R^G})$. It follows that $K(\widehat{R^G}) = K(\widehat{R}^G)$. Since $\widehat{R} = R \otimes \widehat{R^G}$ is finite over $\widehat{R^G}$, and $\widehat{R^G}$ is normal, we see that $\widehat{R^G} = (\widehat{R})^G$. $\square$

Continuing with the proof of Proposition 12.6, suppose first that $\dim X = 1$. In this case we see from the Lemma that the completion of the local ring at a point of the form kp is the ring of invariants $K[[x_1, \dots, x_k]]^{\mathfrak{S}_k}$. Since the action preserves degrees, the ring of invariants is the completion of the ring generated by the invariant homogeneous forms. This is is $K[[\sigma_1, \dots, \sigma_k]]$, where the σ_i are the elementary symmetric polynomials. Since the number of generators is equal to the dimension, this is a regular local ring as required.

On the other hand, if $\dim X = m > 1$ then the completion of the local ring of $X^{(2)}$ at $2p$ has the form

$$K[[x_{1,1}, \dots, x_{1,m}, x_{2,1}, \dots, x_{2,m}]].$$

Among the invariants are the $2m$ polynomials

$$\{f_i = x_{1,i} + x_{2,i}, g_i = x_{1,i}x_{2,i} \mid i = 1, \dots, m\}$$

and also the polynomials $h_{i,j} = x_{1,i}x_{2,j} + x_{1,j}x_{2,i}$ for $i \neq j$.

Since the action is homogeneous, the ring of invariants is generated by invariant homogeneous forms. It's easy to see that the only invariant linear forms $f_{i,j}$. If the ring of invariants were smooth then its maximal ideal would be generated by these and some m other forms. Modulo the f_i we may identify the two sets of variables, and write $g_i \equiv -x_i^2,\ h_{i,j} \equiv x_i x_j$. Since these are independent, we see that $X^{(2)}$ is singular as claimed. $\square$

The symmetric powers of a smooth curve C are central to the analysis of the geometry of C, as we'll see illustrated in Chapter 17. We can think of a point of $C^{(k)}$ as a subscheme $D \subset C$, and use notation such as $D \cup D'$ and $D \cap D'$ accordingly. In fact, $C^{(k)}$ is isomorphic to the Hilbert scheme of subschemes of C with constant Hilbert polynomial k —that is, zero-dimensional subschemes of degree d (see Arbarello et al. [1985] for a proof.) When $\dim X > 1$ a point on $X^{(k)}$ does *not* determine to a subscheme of X, and the Hilbert schemes $\mathcal{H}_k(X)$ are often more useful.

12.3.2 Secant varieties in general

Many results in the remainder of this chapter require that the ground field K has characteristic 0. As usual we write $\mathbb{P}^r$ for $\mathbb{P}^r_K$, and all our schemes are quasi-projective K-schemes.

In this subsection we will prove a basic result related to the dimension of secant varieties. Then we will state some general results that may help to orient the reader, without proof. In the following two sections we will prove a number of results about the secant varieties of rational curves.

Let $X \subset \mathbb{P}^r$ be a projective variety of dimension n, not contained in a hyperplane. Since r general points of X are linearly independent, for any $m \leq r$ we have a rational map

$$\tau : X^{(m)} \to \mathbb{G}(m-1, r),$$

called the *secant plane map*, sending a general m-tuple $p_1 + \cdots + p_m$ to the span $\overline{p_1 \ldots p_m} \cong \mathbb{P}^{m-1} \subset \mathbb{P}^r$. (In coordinates: if $p_i = (x_{i,0}, \ldots, x_{i,r})$, then τ is given by the maximal minors of the matrix $(x_{i,j})$.) The image $\Sigma\mathrm{ec}_m(X) \subset \mathbb{G}(m-1, r)$ of the rational map τ—that is, the closure in $\mathbb{G}(m-1, r)$ of the locus of $(m-1)$-planes spanned by m linearly independent points of X—is called the *locus of secant $(m-1)$-planes* to X. Finally, the variety

$$\mathrm{Sec}_m(X) = \bigcup_{\Lambda \in \Sigma\mathrm{ec}} \Lambda \ \subset \mathbb{P}^r$$

is called the m^{th} *secant variety of X*.

Caution: If $\Lambda \in \Sigma\mathrm{ec}_m$ and $\Lambda \cap X$ is finite, then $\deg(\Lambda \cap X) \geq m$, but the converse is false; Exercise 12.25 suggests an example of this.

If $n > 1$ and $m > 1$ then the secant plane map $\tau : X^{(m)} \to \mathbb{G}(m-1, r)$ is never regular: when a point $p \in X$ on a variety of dimension 2 or more approaches another point $q \in X$, the limiting position of the secant line $\overline{p\,q}$ necessarily depends on the direction of approach. (When X is a curve and q a smooth point of X, the limit is always the tangent line $\mathbb{T}_q X$.) This illustrates the point that the Hilbert scheme $\mathcal{H}_m(X)$ may be a better compactification of the space of unordered m-tuples of points on X than the symmetric power: when $m = 2$, for example, the map $\tilde{\tau} : \mathcal{H}_2(X) \to \mathbb{G}(1, r)$ sending a subscheme of length 2 to its span is always regular. Further, if we fix m and replace the embedding $X \subset \mathbb{P}^r$ by a sufficently high Veronese re-embedding, then every length m subscheme of X will span an $m-1$ plane; so the map $\mathcal{H}_m(X) \to \mathbb{G}(m-1, r)$ will be regular. In this chapter, we will care only about the image of τ, so it doesn't matter which we use.

We begin with the dimension of $\mathrm{Sec}_m(X)$.

Proposition 12.8. *Suppose that K has characteristic 0. If $m \le r - n$, then the map τ is birational onto its image; in particular, $\Sigma\mathrm{ec}_m(X)$ has dimension* $\dim X^{(m)} = mn$.

This is slightly more subtle than it might at first appear. The first case would be the statement that if $C \subset \mathbb{P}^3$ is a nondegenerate curve then the line joining two general points of C does not meet C a third time. Though intuitively plausible, this is tricky to prove, and requires the hypothesis of characteristic 0. For the proof we will use the following general position result:

Lemma 12.9 (General Position Lemma)**.** *Suppose that K has characteristic 0. If $X \subset \mathbb{P}^r$ is a nondegenerate variety of dimension n and $\Gamma \cong \mathbb{P}^{r-n} \subset \mathbb{P}^r$ a general linear subspace of complementary dimension, then the points of $\Gamma \cap X$ are in linear general position; that is, any $r-n+1$ of them span Γ.*

We won't prove this here; a good reference is the discussion of the Uniform Position Lemma in Arbarello et al. [1985].

Proof. The proposition amounts to the assertion that if $p_1, \dots, p_m \in X$ are general points, then the plane $\overline{p_1 \dots p_m}$ they span contains no other points of X.

To prove this, let $U \subset X^{(m)}$ be the open subset of m-tuples of distinct, linearly independent points, and consider the incidence correspondence

$$\Psi = \{(p_1 + \dots + p_m, \Gamma) \in U \times \mathbb{G}(r-n, r) \mid p_1, \dots, p_m \in \Gamma\}.$$

Via projection on the first factor, we see that Ψ is irreducible, and by Lemma 12.9 it dominates $\mathbb{G}(r-n, r)$; it follows that *a general $(r-n)$-plane Γ containing m general points $p_1, \dots, p_m \in X$ is a general $(n-r)$-plane in $\mathbb{P}^r$*, and applying Lemma 12.9 again we deduce that the $(m-1)$-plane $\overline{p_1 \dots p_m}$ contains no other points of X. □

Consider the universal sub vector bundle $S \subset$ over the Grassmannian $G(m, r+1) = \mathbb{G}(m-1, r)$. Let

$$\Phi = \mathbb{P}S = \{(\Lambda, p) \in \mathbb{G}(m-1, r) \times \mathbb{P}^r \mid p \in \Lambda\}$$

be the universal projective $(m-1)$-plane in $\mathbb{P}^r$, with projection maps

$$\mathbb{P}^r \xleftarrow{\eta} \Phi \xrightarrow{\pi} \mathbb{G}(k, r).$$

Set

$$\Phi_m(X) = \pi^{-1}(\Sigma\mathrm{ec}_m(X)) \quad \text{and} \quad \eta_X = \eta|_{\Phi_m(X)}$$

so that the m^{th} secant variety $\mathrm{Sec}_m(X)$ is the image of η_X. We'll call $\Phi_m(X)$ the *abstract secant variety.*

Projection on the first factor shows that $\Phi_m(X)$ is irreducible of dimension $mn + m - 1$, so that $\dim \operatorname{Sec}_m(X) \leq mn + m - 1$, with equality holding when a general point on $\operatorname{Sec}_m(X)$ lies on only finitely many m-secant $(m-1)$-planes to X. By way of language, if $X \subset \mathbb{P}^r$ has dimension n we'll call $\min(mn + m - 1, r)$ the *expected dimension* of the secant variety $\operatorname{Sec}_m(X)$; we'll say that X is *m-defective* if $\dim \operatorname{Sec}_m(X) < \min(mn + m - 1, r)$, and *defective* if it is m-defective for some m.

Everyone's favorite example of a defective variety is the Veronese surface in $\mathbb{P}^5$:

Proposition 12.10. *The Veronese surface $X := \nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$, given parametrically by*

$$\mathbb{P}^2 \ni (x_0, x_1, x_2) \mapsto (x_0^2, x_0x_1, \ldots, x_2^2) \in \mathbb{P}^5,$$

2-defective.

In fact (in any characteristic), the Veronese surface is the *only* 2-defective smooth projective surface! This much more difficult theorem was asserted, and partially proven, by Severi. The proof was completed by Moishezon in characteristic 0; see Dale [1985] for a modern treatment that works in all characteristics.

Proof. If we substitute x_ix_j for $z_{i,j}$ in the generic symmetric 3×3 matrix

$$M = \begin{pmatrix} z_{0,0} & z_{0,1} & z_{0,2} \\ z_{0,1} & z_{1,1} & z_{1,2} \\ z_{0,2} & z_{1,2} & z_{2,2} \end{pmatrix},$$

we see that this matrix has rank 1 at any point of X. (A more careful analysis would show that the 2×2 minors of M generate the ideal of X; see for example ****give ref to Hartshorne's exc or Joe's book exc****.) But if M has rank 1 at two points $p, q \in \mathbb{P}^5$, then M has rank at most 2 at any point of the form $\lambda p + \mu q$. Thus the determinant of M vanishes on the whole line spanned by p and q, so the the cubic form $\det M$ vanishes on secant locus $\operatorname{Sec}_2(X)$. Thus $\dim \operatorname{Sec}_2(X) \leq 5 - 1 = 4$, not $2 \times 2 + 1 = 5$. $\square$

We can give a more geometric proof in characteristic 0 using a basic result introduced by Terraccini [1911].

Proposition 12.11 (Terracini's Lemma)**.** *Suppose that K has characteristic 0. Let $X \subset \mathbb{P}^r$ be a variety and $p_1, \ldots, p_m \in X$ linearly independent smooth points of X. If $p \in \Gamma = \overline{p_1 \ldots p_m}$ is any point in their span not in the span of any proper subset, then the image of the differential $d\eta_X$ at the point $(\Gamma, p) \in \Phi_m(X)$ is the span*

$$\operatorname{Im} d\eta_X = \overline{\mathbb{T}_{p_1}X \ldots \mathbb{T}_{p_m}X}$$

of the tangent planes to X at the points p_i. In particular, if X has dimension n and $r \geq mn+m-1$ then X is m-defective if and only if its tangent spaces at m general points are dependent. □

For a proof, see Landsberg [2012].****need reference in Landsberg.****

We can use Terracini's Lemma to see that the Veronese variety X is degenerate as follows: A hyperplane H contains the tangent plane to X at a point p if and only if the curve $H \cap X$ is singular at p. Of course we can consider $H \cap X$ as a conic in $\mathbb{P}^2 \cong X$, and, from the definition of the Veronese we see that every conic appears in this way. Now two planes in $\mathbb{P}^5$ are dependent if and only if they are both contained in hyperplane. Putting this together with Terracini's Lemma, we see that to show that X is 2-defective we must show that, given any two points in $\mathbb{P}^2$, there is a conic in $\mathbb{P}^2$ that is singular at both these points: and of course the double line passing through the points is such a conic.

We can also use Terracini's Lemma to show that (in characteristic 0) there are no defective curves:

Proposition 12.12. *Suppose that K has characteristic 0. If $C \subset \mathbb{P}^r$ is a nondegenerate reduced irreducible curve, then* $\dim \mathrm{Sec}_m(X) = \min(2m-1, r)$ *for every m.*

Proof. By Teracini's Lemma it suffices to show that If $p_1, \ldots, p_m \in C$ are general points, then the tangent lines $\mathbb{T}_{p_i}C$ are linearly independent when $2m-1 \leq r$, and span $\mathbb{P}^r$ when $2m-1 \geq r$.

We have already seen in Section 9.6 that for a general point $p \in C$ the divisor $(r+1)p$ spans $\mathbb{P}^r$; it follows that the divisor $2m \cdot p$ spans a $\mathbb{P}^{2m-1}$ when $2m \leq r+1$ and spans $\mathbb{P}^r$ when $2m \geq r+1$. By upper-semicontinuity of rank, it follows that for general $p_1, \ldots, p_m$ the divisor $2p_1 + \cdots + 2p_m$ has the same dimension span. □

The general question of which nondegenerate varieties are defective is a fascinating one, with a long history. Perhaps because of Proposition **??** the case of Veronese embeddings of projective spaces has attracted a great deal of attention. Here is a result of Alexander and Hirschowitz [1995]. The proof was later simplified by Karen Chandler, and an exposition of this version, with a further simplification, can be found in Brambilla and Ottaviani [2008].

Theorem 12.13. *Suppose that K has characteristic 0. The only Veronese embeddings of projective spaces that are defective are:*

- $\nu_2(\mathbb{P}^n)$ *is 2-defective for any n;*
- $\nu_4(\mathbb{P}^2)$ *is 5-defective;*

- *$\nu_4(\mathbb{P}^3)$ is 9-defective; and*
- *$\nu_3(\mathbb{P}^4)$ is 7-defective.*
- *$\nu_3(\mathbb{P}^4)$ is 14-defective.*

$\square$

12.4 Secant varieties of rational normal curves

We turn now from secant varieties in general to the special case of rational curves. Every rational curve is the projection of a rational normal curve, and its secant varieties are correspondingly projections of the secant varieties of the rational normal curve, so we'll focus initially on that case.

12.4.1 Secants to rational normal curves

We begin with the observation that finite sets of points on a rational normal curve are always "as independent as possible". This property actually characterizes rational normal curves, as we invite the reader to show in Exercise 12.31.

Lemma 12.14. *Let $C \subset \mathbb{P}^d$ be the rational normal curve. If $D \subset C$ is a divisor of degree $m \leq d+1$, then D is not contained in any linear subspace of $\mathbb{P}^d$ of dimension $< m-1$.*

Informally: any finite subscheme $D \subset C$ of length $\leq d+1$ is linearly independent, in the sense that the map $H^0(\mathcal{O}_{\mathbb{P}^d}(1)) \to H^0(\mathcal{O}_D(1))$ is surjective. On an affine subset of $\mathbb{P}^1$ the parametrization of the rational normal curve looks like $t \mapsto (1, t, t^2, \dots t^d)$, so the independence of the images of any $d+1$ points $a_0, \dots, a_d$ is given by the nonvanising of the Vandermonde determinant

$$\det \begin{pmatrix} 1 & a_0 & \cdots & a_0^d \\ \vdots & \vdots & \dots & \vdots \\ 1 & a_d & \cdots & a_d^d \end{pmatrix} = \prod_{i \neq j} a_i - a_j.$$

Proof. If D were a subset of a linear subspace L of dimension $n < d$, then adding $d-n-1$ general points to d we would arrive at a divisor $D' \subset C$ of degree $m+d-n-1$ contained in a hyperplane H. Since C is not contained in a hyperplane, the intersection $H \cap C$ is finite, and we deduce the contradiction $\deg C > d$. $\square$

In Section 12.3.2 we described the secant map as a regular map on an open set, that is, as a rational map

$$\tau : C^{(m)} \to \mathbb{G}(m-1, d)$$

One consequence of Lemma 12.14 is that the secant plane map has a natural extension to an injective map of sets. It is not hard to show that τ is actually an embedding (See exercises 12.35 and 12.32. In case $d = m$ a computation of dimensions then shows that $\tau : C^{(d)} \to \mathbb{G}(d-1, d) = \mathbb{P}^{d*}$ is an isomorphism, giving another proof of Proposition 12.5.) We can thus regard the restriction Φ_C of the universal $\mathbb{P}^{m-1}$ bundle over $\mathbb{G}(m-1, d)$ as a $\mathbb{P}^{m-1}$-bundle over $(\mathbb{P}^1)^{(m)} = \mathbb{P}^m$, and $\mathrm{Sec}_m(C)$ is the image of this bundle.

Proposition 12.15. *Let $C \subset \mathbb{P}^d$ be a rational normal curve. When $2m - 1 \leq d$, the map $\eta_C : \Phi_m(C) \to \mathbb{P}^d$ is birational onto its image $\mathrm{Sec}_m(C)$; more precisely, it is one-to-one over the complement of $\mathrm{Sec}_{m-1}(C)$ in $\mathrm{Sec}_m(C)$.*

Proof. Suppose a point $p \in \mathbb{P}^d$ is the image of two different points of $\Phi_m(C)$, say $(\overline{D}\, p)$ and $(\overline{D}'\, p)$. Let $k = \dim(\overline{D} \cap \overline{D}')$; note that $0 < k < m-1$. Since the span of $\overline{D}$ and $\overline{D}'$ has dimension $2m-2-k$, by Lemma 12.14 the union (as subschemes of C) of D and D' can have degree at most $2m-k-1$. It follows that the intersection $D \cap D'$ (again, as subschemes of C) has degree at least $k+1$. Thus $\overline{D} \cap \overline{D}'$ is a secant k-plane, and $p \in S_k(C) \subset S_{m-1}(C)$. □

Proposition 12.15 is not particularly remarkable in case $2m-1 < d$: at least in characteristic 0, all curves $C \subset \mathbb{P}^d$ have the property that when $2m-1 < d$ a general point on the m-secant variety $\mathrm{Sec}_m(C)$ lies on a unique m-secant $(m-1)$-plane to C. ****Proof: Sec_m has right dim, so a gen pt on the var lies only on proper secants. If the general point lay on 2 m-secants, this would give too many 2m secant 2m-2 planes because of the Castelnuovo lemma that a genl hyp does not contain special secant planes.****

In case $2m-1 = d$, however, it is striking. For example, the twisted cubic curve $C \subset \mathbb{P}^3$ is the *unique* nondegenerate space curve whose secant lines sweep out $\mathbb{P}^3$ only once (see Exercise 12.33).

12.4.2 Degrees of the secant varieties

Let $C \subset \mathbb{P}^d$ be a rational normal curve, and m any integer with $2m-1 \leq d$. Since the secant plane map $\tau : C^{(m)} \to \mathbb{G}(m-1, d)$ is regular, and $\Phi_C \to \mathrm{Sec}_m(C)$ is birational, it is reasonable to hope that we can answer enumerative questions about the geometry of the varieties $\mathrm{Sec}_m(C)$. We will do this in the remainder of this section and the next, starting with the calculation of the degree of $\mathrm{Sec}_m(C)$.

There are a few cases that we can do without any machinery: For example $S_1(C) = C$, so $\deg S_1(C) = d$. The variety $S_2(C)$ has dimension 3, so its degree is the number of points in which it intersects a general $(d-3)$-plane Λ. The points of the intersection correspond to the nodes of the image of the projection of C from Λ to $\mathbb{P}^2$. Since a plane curve of degree d has arithmetic genus $N := \binom{d-1}{2}$, the number of nodes in the projection of C is N, so $\deg S_2(C) = N$. Finally, if d is odd and $m = (d+1)/2$, then the secant locus is all of $\mathbb{P}^d$, so $\deg S_m(C) = 1$.

In order to go further, we use the Segre class technique of Proposition 12.4. Now

$$\Phi_m(C) \xrightarrow{\ \pi\ } \tau((\mathbb{P}^1)^{(m)}) \cong (\mathbb{P}^1)^{(m)} \cong \mathbb{P}^m.$$

has the form $\mathbb{P}\mathcal{E}$ where $\mathcal{E}$ is the restriction of the Tautological sub bundle on $G(m, d+1)$ to $\mathbb{P}^m$. Thus we want to compute the Segre or Chern classes of this bundle.

Let $V = H^0(\mathcal{O}_{\mathbb{P}^1}(d))$, and let $G = G(m, d+1)$ be the Grassmannian of m-dimensional subspaces of V^*. By definition, the rational normal curve is the image in $\mathbb{P}(V^*)$ of the map sending a point $p \in \mathbb{P}^1$ to the functional given by evaluation at p; that is, to the one dimensional space of linear functionals on the 1-dimensional quotient $H^0\mathcal{O}_{\{p\}}(d)$ of V, considered as linear functionals on V.

The bundle $\Phi_m(C)$ is the restriction to $\tau((\mathbb{P}^1)^{(m)}) \subset G$ of the universal P^{m-1}-bundle $\mathbb{P}S \subset G \times \mathbb{P}(V^*)$. If $D \subset \mathbb{P}^1 \cong C$ is a divisor of degree $m \leq d$ then $H^1\mathcal{I}_D(d) = H^1\mathcal{O}_{\mathbb{P}^1}(d-m) = 0$, so the restriction map $V \to H^0\mathcal{O}_D(d)$ is a surjection. Thus $\Phi = \mathbb{P}\mathcal{E}$, where $\mathcal{E}$ is the restriction of S to the subvariety of G whose points correspond to the spans of divisors $D \subset C$. The fiber of $\mathcal{E}$ over the span of D is the space of linear functionals on the quotient $H^0\mathcal{O}_D(d)$ of V, regarded as functional on V.

Theorem 12.16. *If $C \subset \mathbb{P}^d$ is the rational normal curve of degree d, and $2m-1 \leq d$ then*

$$\deg \mathrm{Sec}_m(C) = \binom{d-m+1}{m}.$$

Moreover, suppose that $m \leq d$ and let $\mathcal{E}$ be the restriction of the universal subbundle S on the Grassmannian G of m-dimensional subspaces of $(H^0(\mathcal{O}_{\mathbb{P}^1}(d)))^*$ to the subvariety X of G whose points correspond to spans of divisors $D \subset C$ of degree m. We have $X \cong \mathbb{P}^m$, and if ζ denotes the hyperplane class on $\mathbb{P}^m$, then

$$s(\mathcal{E}) = (1+\zeta)^{d-m+1}$$

and

$$c(\mathcal{E}) = \frac{1}{s(\mathcal{E})} = \sum_i (-1)^i \binom{d-m+i}{i} \zeta^i \in A(\mathbb{P}^m).$$

Proof. We will deduce the first statement from the second, and the second from the third. By Propositions 12.4 and 12.15, the degree of $\mathrm{Sec}_m(C)$ is the degree of the m-th Segre class $s_m\mathcal{E}$. According to the formula given in the Theorem, this is $\binom{d-m+1}{m}$.

The formula for the Segre class follows from that for the Chern class, together with the basic formula $c(\mathcal{E})s(\mathcal{E}) = 1$.

Since $c_i(\mathcal{E}) = (-1)^i c_i(\mathcal{E}^*)$, it suffices for the Chern class formula to prove that $c_i(\mathcal{E}^*) = \binom{d-m+i}{i}\zeta^i$. Any element $F \in V$, regarded as a section of the trivial bundle, defines a section σ_F of $\mathcal{E}^*$ by restriction. These sections generate $\mathcal{E}^*$. We will describe the Chern classes of $\mathcal{E}^*$ as degeneracy loci of appropriate collections of these global sections.

Let $F_1, \dots, F_{m-i+1} \in H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ be general polynomials of degree d. The subscheme of $\mathbb{P}^m$ where the corresponding sections $\sigma_{F_1}, \dots, \sigma_{F_{m-i+1}}$ of $\mathcal{E}$ are dependent (defined as the vanishing locus of the section $\sigma := \sigma_{F_1} \wedge \cdots \wedge \sigma_{F_{m-i+1}}$ of $\wedge^{m-i+1}\mathcal{E}$) is supported on the set of divisors D of degree m on C such that some nonzero linear combination of $F_1, \dots, F_{m-i+1}$ vanishes on D. If this scheme has codimension i, then the associated cycle represents $c_i(\mathcal{E})$.

Since $c_i(\mathcal{E}) \in A^i(\mathbb{P}^m)$ is necessarily a multiple $\alpha\zeta^i$ of the i^{th} power of the hyperplane class $\zeta \in A^1(\mathbb{P}^m)$, the coefficient α is the degree of the restriction of this cycle to an i-plane $\Lambda \subset \mathbb{P}^m$. Rather than choosing Λ to be a general i-plane, we will take $p_1, \dots, p_{m-i} \in C$ general points, and use the i-plane

$$\Lambda = \{D \in \mathbb{P}^m \mid p_1, \dots, p_{m-i} \in D\}.$$

The intersection $\Lambda \cap V(\sigma_{F_1} \wedge \cdots \wedge \sigma_{F_{m-i+1}})$ of this plane Λ with the degeneracy locus of $\sigma_{F_1}, \dots, \sigma_{F_{m-i+1}}$ is the locus of divisors D such that

(a) some nonzero linear combination $\sum a_j F_j$ of $F_1, \dots, F_{m-i+1}$ vanishes on D; and

(b) $p_1, \dots, p_{m-i} \in D$.

These two conditions imply that the sum $\sum a_j F_j$ vanishes at the points $p_1, \dots, p_{m-i}$. But the points p_α being general, there is, up to a scalar, only one linear combination $F = \sum a_j F_j$ that vanishes on them all. We claim that the divisor of F is a sum of d *distinct* points

$$(F) = p_1 + \cdots + p_{m-i} + q_1 + \cdots + q_{d-m+i}.$$

From this it follows that D is of the form

$$D = p_1 + \cdots + p_{m-i} + q_{\alpha_1} + \cdots + q_{\alpha_i},$$

where $\{\alpha_1, \dots, \alpha_i\} \subset \{1, \dots, d-m+i\}$ is a subset of cardinality i.

Conversely, any divisor D of this form is a point of intersection of Λ with the degeneracy locus of $\sigma_{F_1}, \dots, \sigma_{F_{m-i+1}}$. The cardinality of this set is $\binom{d-m+i}{i}$, so to complete the proof we must verify that, for general points $p_1, \dots, p_{m-i} \in \mathbb{P}^1$, general forms $F_1, \dots, F_{m-i+1} \in H^0(\mathcal{O}_{\mathbb{P}^1}(d))$, and the subspace $\Lambda \subset \mathbb{P}^m$ as above, two things are true:

(a) the scheme

$$\Lambda \cap V(\sigma_{F_1} \wedge \cdots \wedge \sigma_{F_{m-i+1}})$$

is reduced of dimension 0; and

(b) if $F = \sum a_j F_j$ is the unique linear combination of $F_1, \dots, F_{m-i+1}$ vanishing at $p_1, \dots, p_{m-i}$, then the remaining zeros of F are all simple.

For part (a), we'll do the case $i = m$, and leave the remaining cases as Exercise 12.36. ****either add a good hint to the Exercise or actually do the proof here! Can it be done as an exercise in GRR?**** Here we have a single polynomial $F = F_1$, which we'll write in affine coordinates as

$$F = \prod_{i=1}^{d} (t - a_i);$$

F being general, the a_i are distinct. Now let $D \in V(\sigma_F)$ be a zero of σ_F, that is, a divisor contained in (F); say, $D = a_1 + \cdots + a_m$. In an étale neighborhood of $D \in C^{(m)}$, the coordinates $a_1, \dots, a_m$ are a system of local coordinates. In terms of a suitable trivialization of $\mathcal{E}^*$ the section σ_F is given by

$$\sigma_F(a_1 + \cdots + a_m) = (F(a_1), \dots, F(a_m)).$$

Now, the matrix of partial derivatives

$$\frac{\partial F}{\partial a_i}(a_j) = \begin{cases} -\prod_{m \neq i}(a_m - a_i), & \text{if } i = j; \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

is diagonal, with nonzero entries on the diagonal, hence nonsingular; and we can conclude that the scheme $V(\sigma_F)$ is zero-dimensional and reduced at D.

Finally, for part (b), let $\varphi : \mathbb{P}^1 \to C \subset \mathbb{P}^{m-i}$ be the map given by the polynomials $F_1, \dots, F_{m-i+1}$. The divisor $V(F)$ is then the preimage $\varphi^{-1}(H)$ of the hyperplane H spanned by the points $\varphi(p_1), \dots, \varphi(p_{m-i}) \in \mathbb{P}^{m-i}$. Since the plane spanned by $m-i$ general points on a nondegenerate curve $C \subset \mathbb{P}^{m-i}$ is a general hyperplane in $\mathbb{P}^{m-i}$, the preimage $\varphi^{-1}(H)$ is reduced. $\square$

12.4.3 Expression of a form as a sum of powers

We now undertake to answer Keynote Question (b): if f and g are general polynomials of degree $d = 2m$ in one variable over a field of characteristic 0, how many linear combinations of f and g are expressible as a sum of m d^{th} powers of linear forms?

This question is related to secants of rational normal curves because, if we realize $\mathbb{P}^d$ as the projective space of forms of degree d on $\mathbb{P}^1$, then (in characteristic 0) the curve of pure d-th powers is isomorphic to the rational normal curve. Indeed, this set is the image of the morphism

$$\mu : \mathbb{P}^1 \ni (s,t) \mapsto \left(s^d, ds^{d-1}t, \binom{d}{2}s^{d-2}t, \dots, t^d\right) \in \mathbb{P}^d.$$

The map μ differs from the standard parametrization of the rational normal curve by monomial by the diagonal map $\mathbb{P}^d \to \mathbb{P}^d$ multiplying the i-th coordinate by the scalar $\binom{d}{i}$, an isomorphism in characteristic is 0 (or when the characteristic doesn't divide any $\binom{d}{i}$). (If, for example, d is equal to the characteristic, then μ is a purely inseparable map whose image is a line!)

In this section we will henceforward assume that the characteristic of the ground field K is zero, and we will consider the rational normal curve to be *the curve of pure d-th powers in* $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(d))$.

A point $p \in \mathbb{P}^d$ lies on the plane spanned by distinct points $q_1, \dots, q_m \in C$ if and only if the homogeneous coordinates of p can be expressed as a linear combination of the homogeneous coordinates of $q_1, \dots, q_m$. Thus a form of degree d is a linear combination of m d-th powers of linear forms if and only if the corresponding point in $\mathbb{P}^d$ lies in the union of the m-secant $(m-1)$-planes to $\mu(\mathbb{P}^1)$, and questions about the expression of a polynomial as a sum of powers become questions about the secants.

There is an important subtlety: it is *not* the case that every point of $\mathrm{Sec}_m(C)$ corresponds to a polynomial that is expressible expressible as a sum of m d-th powers! For example, the tangent lines to C are contained in $\mathrm{Sec}_2(C)$. If $d \geq 3$ then no 2-plane in $\mathbb{P}^d$ meets the rational normal curve in 4 points, so a tangent line to C cannot meet any other secant line at a point off C. Thus the points on the tangent lines away from C are points of $\mathrm{Sec}_2(C)$ that cannot be expressed as the sum of two pure d-th powers.

(The points on the tangent lines do have an interesting characterization, however: at the point corresponding to the polynomial $f(t) = (t-\lambda)^d$ is the set of linear combinations of f and $\partial f/\partial t$, or equivalently the set of polynomials that have $d-1$ roots equal to λ.)

By definition, $\mathrm{Sec}_m(C)$ has an open set consisting of points on the secant $(m-1)$-planes spanned by m distinct points of C. Further by Proposition 12.15, a point in the open subset $\mathrm{Sec}_m(C) \setminus \mathrm{Sec}_{m-1}(C$ of $\mathrm{Sec}_m(C)$ lies on the span of a unique divisor of degree m. Thus Theorem 12.16 yields the answer to Keynote Question (b), and even a generalization:

Corollary 12.17. *Suppose that the characteristic of K is zero. $d \geq 2m-1$, then the number of linear combinations of $d-2m+2$ general forms of degree d that can be expressed as the sum of m pure d-th powers is* $\deg Sec_m(C) = \binom{d-m+1}{m}$.

12.5 Special secant planes

For a curve $C \subset \mathbb{P}^r$ other than a rational normal curve, it is interesting to consider the subspaces that meet C in a dependent set of points; these are called *special secant planes.* Examples of this that we'll investigate below include trisecant and quadrisecant lines to a curve $C \subset \mathbb{P}^3$, and trisecant lines to a curve $C \subset \mathbb{P}^4$.

We start, as usual, with the question of dimension: when would we expect a curve $C \subset \mathbb{P}^r$ to contain m points lying in a $\mathbb{P}^{m-1-k}$-plane? What would be the expected dimension of the locus $C_k^{(m)} \subset C^{(m)}$ of such m-tuples?

There are many ways to set this up. One would be to express the locus of such m-tuples as a determinantal variety: if the coordinates of points $p_1, \dots, p_m \in C$ are the rows of the matrix

$$M = \begin{pmatrix} x_{1,0} & x_{1,1} & \dots & x_{1,r} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m,0} & x_{m,1} & \dots & x_{m,r} \end{pmatrix},$$

then $C_k^{(m)}$ is just the locus where this matrix has rank $m-k$ or less. Now, in the space of $m \times (r+1)$ matrices, those of rank $m-k$ or less have codimension $k(r+1-m+k)$, so we'd expect the locus $C_k^{(m)}$ of m-tuples spanning only a $\mathbb{P}^{m-1-k}$ have dimension

$$m - k(r+1-m+k) = (k+1)(m-r-k) + r.$$

An alternative in the case C is a rational curve would be to express C as the projection $\pi_\Lambda : \tilde{C} \to C$ of a rational normal curve $\tilde{C} \subset \mathbb{P}^d$ from a plane $\Lambda \cong \mathbb{P}^{d-r-1}$. The m-secant $(m-1-k)$-planes to C then correspond to the m-secant $(m-1)$-planes to $\tilde{C}$ that intersect Λ in a $(k-1)$-plane; that is, the preimage under the secant plane map $\tau : C^{(m)} \to \mathbb{G}(m-1, d)$ of the Schubert cycle

$$\Sigma_{(r+1-m+k)^k}(\Lambda) = \{\Gamma \in \mathbb{G}(m-1,d) \mid \dim(\Gamma \cap \Lambda) \geq k-1\}.$$

This Schubert cycle has codimension $k(r+1-m+k)$, so again we'd expect the preimage to have dimension $m - k(r+1-m+k)$.

The first three cases (with $k > 0$) are trisecants to a curve $C \subset \mathbb{P}^3$ (that is, $r = 3$, $m = 3$ and $k = 1$), where we expect a one-parameter family; quadrisecants to a curve $C \subset \mathbb{P}^3$ (that is, $r = 3$, $m = 4$ and $k = 2$), where we expect finitely many, and trisecants to a curve $C \subset \mathbb{P}^4$ (that is, $r = 4$, $m = 3$ and $k = 1$) where, again, we expect finitely many. We'll show how to count the quadrisecants to a general rational curve in $\mathbb{P}^4$ (That this is the case for general rational curves is shown in Exercise 12.37, though it's *not* necessarily true of a general point on any component of the Hilbert scheme of curves of higher genus, as shown in Exercise 12.38.) And we'll determine the degree of the trisecant surface of a general rational curve in $\mathbb{P}^3$. We leave the case of quadrisecants to a rational curve in $\mathbb{P}^3$ to Exercise 12.39 for now; it will also be a direct application of Porteous' formula in Chapter 14.

12.5.1 The class of the locus of secant planes

In case the curve C is rational, the answers to all of the above questions come directly from the answer to a question we haven't yet addressed directly: if $C \subset \mathbb{P}^d$ is a rational normal curve, $\tau : C^{(m)} \cong \mathbb{P}^m \to \mathbb{G}(m-1, d)$ the secant plane map and $\Sigma\mathrm{ec}_m(X) \subset \mathbb{G}(m-1, d)$ the image of τ, *what is the class* $[\Sigma\mathrm{ec}_m(X)] \in A_m(\mathbb{G}(m-1, d))$*?*

We have all the tools to answer this question at hand: we know that the pullback $\tau^* S^*$ of the dual of the universal subbundle S on $\mathbb{G}(m-1, d)$ is the bundle $\mathcal{E}^*$ whose Chern classes we gave in Section 12.4.2. We know that $c_i(S^*) = \sigma_{1^i}$, so this says that

$$\tau^*\sigma_{1^i} = \binom{d-m+i}{i}\zeta^i \in A^i(\mathbb{P}^m)$$

where ζ as usual is the hyperplane class in $C^{(m)} \cong \mathbb{P}^m$. Equivalently, since we also know that the Segre class of S is $s(S) = 1 + \sigma_1 + \sigma_2 + \cdots + \sigma_{d-m}$ we have

$$\tau^*\sigma_i = \binom{d-m+1}{i}\zeta^i \in A^i(\mathbb{P}^m). \tag{12.1}$$

Since the classes σ_i generate the Chow ring of $\mathbb{G}(m-1, d)$ (and τ is an embedding), this determines the class of the image. We'll use this idea to compute $[\Sigma\mathrm{ec}_m(X)]$ explicitly in the cases below.

Trisecants to a rational curve in $\mathbb{P}^4$. How many trisecant lines does a general rational curve of degree d in $\mathbb{P}^4$ possess? We already know the answer in at least two cases: First, there are no trisecant lines to a rational normal curve, the case $d = 4$. If $d = 5$, (Proposition **??**) that a general point $p \in \mathbb{P}^5$ lies on a unique 3-secant 2-plane to a rational normal curve $\tilde{C} \subset \mathbb{P}^5$, and thus the projection of that curve from a general point has just one trisecant line.

Using the tools above, we can treat the general case: A rational curve $C \subset \mathbb{P}^4$ of degree d is the projection of a rational normal curve $\tilde{C} \subset \mathbb{P}^d$ from a $(d-5)$-plane Λ; the trisecant lines to C correspond to 3-secant 2-planes to $\tilde{C}$ of degree d that meet Λ. The trisecant lines to C, correspond to the intersection of the Schubert cycle $\Sigma_3(\Lambda) \subset \mathbb{G}(2,d)$ of 2-planes meeting Λ with the cycle $\Sigma\mathrm{ec}_3(\tilde{C}) \subset \mathbb{G}(2,3)$ of 3-secant 2-planes to $\tilde{C}$.

This gives the answer to Keynote Question (c):

Proposition 12.18. *Suppose that the ground field K has characteristic 0. If $C \subset \mathbb{P}^4$ is a general rational curve of degree d, then C has $\binom{d-2}{3}$ trisecant lines.*

Proof. Since C is general, it is the projection of the rational normal curve $\tilde{C}$ from a general $(d-5)$-plane Λ. The number of trisecant lines is the number of points in which $\Sigma_3(\Lambda)$ meets $\Sigma\mathrm{ec}_3(\tilde{C})$. By Kleiman's transversality Theorem this is the degree of the intersection class $[\Sigma\mathrm{ec}_m(X)]\sigma_3$, or equivalently the degree of the pullback $\tau^*\sigma_3$; by the above, this is $\binom{d-2}{3}$. □

Trisecants to a rational curve in $\mathbb{P}^3$. We next turn to Keynote Question (d): If $C \subset \mathbb{P}^3$ is a general rational curve of degree d, what is the degree of the surface $S \subset \mathbb{P}^3$ swept out by the 3-secant lines to C?

Again we already know the answer in the simplest cases: 0 in the case $d = 3$ (a rational normal curve has no trisecants); and 2 in the case $d = 4$, since a rational quartic is a curve of type $(1,3)$ on a quadric surface $Q \subset \mathbb{P}^3$, and the trisecants of C comprise one ruling of Q.

To set up the general case, let C be the projection $\pi_\Lambda(\tilde{C})$ of a rational normal curve $\tilde{C} \subset \mathbb{P}^d$ from a general plane $\Lambda \cong \mathbb{P}^{d-4}$; let $L \subset \mathbb{P}^3$ be a general line, and let $\Gamma = \pi_\Lambda^{-1}(L) \subset \mathbb{P}^d$ be the corresponding the $(d-2)$-plane containing Λ. The points of intersection of L with S correspond to 3-secant 2-planes to $\tilde{C} \subset \mathbb{P}^d$ that

(a) meet Λ; and

(b) intersect Γ in a line.

These are the points of intersection of $\Sigma\mathrm{ec}_3(\tilde{C})$ with the Schubert cycle $\Sigma_{2,1}(\Lambda, \Gamma)$. In characteristic 0, Kleiman's transversality theorem shows that the cardinality of this intersection is the degree of the pullback $\tau^*(\sigma_{2,1})$.

To evaluate this we express $\sigma_{2,1}$ as a polynomial in $\sigma_1, \sigma_2, \ldots$ and evaluate each term using (12.1). Giambelli's formula **??** tells us that

$$\sigma_{2,1} = \begin{vmatrix} \sigma_1 & \sigma_0 \\ \sigma_3 & \sigma_2 \end{vmatrix} = \sigma_1\sigma_2 - \sigma_3,$$

(an equality we could readily derive by hand.) By (12.1),

$$\deg \tau^*(\sigma_1\sigma_2) = \binom{d-2}{1}\binom{d-2}{2} \quad \text{and} \quad \deg \tau^*(\sigma_3) = \binom{d-2}{3}.$$

Putting these things together, we have the answer to the question:

Proposition 12.19. *If $C \subset \mathbb{P}^3$ is a general rational curve of degree d, then the degree of the surface swept out by trisecant lines to C is*

$$\binom{d-2}{1}\binom{d-2}{2} - \binom{d-2}{3} = 2\binom{d-1}{3}.$$

12.5.2 Secants to curves of positive genus

It is instructive to ask whether we could extend the computations of this section and the preceding one to curves other than rational normal curves. For nonsingular rational curves this is easy: any rational curve $C \subset \mathbb{P}^r$ of degree d is the linear projection $\pi(\tilde{C})$ of a rational normal curve $\tilde{C} \subset \mathbb{P}^d$, and its secant varieties $S_k(C)$ are correspondingly the projections of the corresponding $S_k(\tilde{C})$. If the center of projection doesn't meet $S_k(\tilde{C})$, and if the projection map $\pi : S_k(\tilde{C}) \to S_k(C)$ is birational, the formula above for the degree of $S_k(\tilde{C})$ applies to C as well.

A more difficult problem arises with a curve C of positive genus. In the case of $g = 1$ this is not too hard; it is the content of Exercises 12.47 and 12.48

But for the general case we must proceed differently. In treating rational curves, we used the fact that the space of effective divisors of degree k on $\mathbb{P}^1$ is the variety $\mathbb{P}^k$, whose Chow ring we know. But when C is neither rational nor elliptic, the Chow rings $A(C^{(k)})$ of symmetric powers of C are unknown.

Nevertheless one can describe a subring of the algebraic cohomology ring in which a calculation analogous to the one above can be made. This is explained in (Arbarello et al. [1985] Chapter 8), where there are explicit formulas generalizing all the above formulas to arbitrary genus.

12.6 Dual varieties and conormal varieties

We next turn to a remarkable property of projective varieties called *reflexivity*, which holds in characteristic 0 or, more generally, under a certain separability hypothesis. A corollary of reflexivity is the deep fact that the dual of the dual of a variety is the variety itself. See Kleiman [1986] for a comprehensive account of the history of these matters. Our account is based on that in Kleiman [1984].

Let $X \subset \mathbb{P}^n$ be a subvariety of dimension k. If X is smooth, we define the *conormal variety* $CX \subset \mathbb{P}^n \times \mathbb{P}^{n*}$ to be the incidence correspondence

$$CX = \{(p, H) \in \mathbb{P}^n \times \mathbb{P}^{n*} \mid p \in X \text{ and } \mathbb{T}_p X \subset H\}.$$

If X is singular, we define CX to be the closure in $\mathbb{P}^n \times \mathbb{P}^{n*}$ of the locus CX° of such pairs (p, H) where p is a smooth point of X. Whatever the dimension of X, the conormal variety CX will have dimension $n-1$: it is the closure of the locus CX°, which maps onto the smooth locus of X with fibers of dimension $n-k-1$. The *dual variety* $X^* \subset \mathbb{P}^{n*}$ of X is the image of CX under projection on the second factor.

****$\mathcal{X} \to X$ is automatically smooth over smooth points, since CX is defined by equations linear in the coeffs of H (dot with the partials of each eqn of X it's zero.)****

Theorem 12.20 (Reflexivity). *Let $X \subset \mathbb{P}^n$ be any variety and $X^* \subset \mathbb{P}^{n*}$ its dual. If the characteristic of the ground field is zero, or more generally if the maps from $CX \to X^*$ and $C(X^*) \to X$ are separable, then the conormal variety $CX \subset \mathbb{P}^n \times \mathbb{P}^{n*}$ is equal to $C(X^*) \subset \mathbb{P}^{n*} \times \mathbb{P}^n$ with the factors reversed. In such cases $(X^{**}) = X$—that is, the dual of the dual of X is X.*

If X is a plane curve in characteristic 0 then the statement $X^{**} = X$ says, heuristically, that the family of tangent lines at points near a smooth point $x \in X$ (that is, the family of points of X^* near the point y corresponding to the tangent line of X) is, to first order, the family of lines through x. More picturesquely put, the tangent lines to points near $x \in X$ "roll" on the point x. ****insert figure****. It is true more generally that the osculating k-planes to a curve $X \subset \mathbb{P}^n$ at points near $x \in X$ move, to first order, by "rolling" on the osculating $(k-1)$-plane to X at x. See ****reference? Is this in G&H? or in Joe's "Lectures"? Or should this be an exercise?****

This picture, for plane curves, can be made precise as follows. Observe that if $p \in X$ is a smooth point, then the tangent line $\mathbb{T}_p X \subset \mathbb{P}^2$ is the limit of the secant lines $\overline{p\,q}$ as $q \in X$ approaches p. Applied to the dual curve $X^* \subset \mathbb{P}^{2*}$, this says that the tangent line $\mathbb{T}_L X^* \subset \mathbb{P}^{2*}$ to the curve X^* at a point L is the limit of the secant lines $\overline{L\,M}$ as $M \in X^*$ approaches L. But the line $\overline{L\,M} \subset \mathbb{P}^{2*}$ joining two points $L, M \in \mathbb{P}^{2*}$ corresponding to lines $L, M \subset \mathbb{P}^2$ is the line in $\mathbb{P}^{2*}$ dual to the point $L \cap M$ in $\mathbb{P}^2$.

Now $X^{**} = X$ means that the tangent line to X^* at the point $L = \mathbb{T}_p X$ corresponding to $p \in X$ is p itself; this amounts to saying that the limit, as $q \in X$ approaches p, of the point of intersection $r = \mathbb{T}_p X \cap \mathbb{T}_q X$ is just the point p itself, which is clear from Figure 12-7.

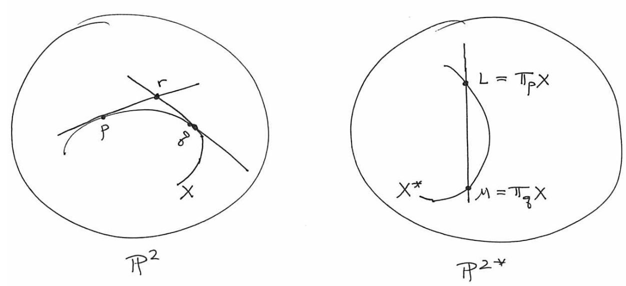

Fig 12.5 : The tangent line to $X^* \subset \mathbb{P}^{2*}$ at $L = \mathbb{T}_pX$ is dual to p.

FIGURE 12-7. The tangent line to $X^* \subset \mathbb{P}^{2*}$ at $L = \mathbb{T}_p X$ is the line dual to p.

An immediate consequence of Theorem **??** is that the Gauss map of a hypersurface in characteristic 0 is either birational map or has positive dimensional fibers: ****can we improve this to say "has connected fibers"?****

Corollary 12.21. *If X is a smooth hypersurface, or more generally a hypersurface whose dual is also a hypersurface, and the characteristic of the ground field is zero, then the Gauss map $\mathcal{G}_X : X \to X^*$ is birational, with inverse $\mathcal{G}_{X^*} : X^* \to X$. Thus if $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree d then X^* is a hypersurface of degree $d(d-1)^{n-1}$.*

Proof of Corollary 12.21. By Section 1.2.2, the dual of a smooth hypersurface is always a hypersurface.

If X and X^* are both hypersurfaces then both $\mathcal{G}_X$ and $\mathcal{G}_{X^*}$ are well-defined rational maps. Since the graphs of CX and $C(X^*)$ are equal after exchanging factors, the two rational maps are inverse to each other, and are thus birational. As already noted in Chapter 1.2.2, the degree computation follows from the birationality of $\mathcal{G}_X$. $\square$

One aspect of Theorem 12.20 may seem puzzling. The only way the dual of a variety $X \subset \mathbb{P}^n$ can fail to be a hypersurface is if the map $CX \to X^*$ has positive-dimensional fibers—in other words, if every singular hyperplane section of X has positive-dimensional singular locus. This is a rare circumstance—as we'll see in Exercise 12.42, it can never be the case for a smooth complete intersection, and as we'll see in Exercise 12.44 it can only happen for X swept out by positive-dimensional linear spaces. But if we have a one-to-one correspondence between varieties $X \subset \mathbb{P}^n$ and their dual varieties, we seem to be suggesting that there are as many hypersurfaces as varieties of arbitrary dimension in $\mathbb{P}^n$! The discrepancy is due to the fact that the duals of smooth varieties tend to be highly singular—see for example Exercise 12.45.

There are many fascinating results about the geometry of dual varieties and conormal varieties. We recommend in particular Kleiman [1986], and the surprising and beautiful theorems of Ein and Landman (Ein [1986]) and Zak (Zak [1991]). Ein and Landman prove, for example, that for any smooth variety $X \subset \mathbb{P}^n$ of dimension d in characteristic 0, the difference $(n-1) - \dim X^*$ is congruent to $\dim X$ modulo 2.

As we mentioned earlier, it's relatively rare for the dual X^* of a smooth variety $X \subset \mathbb{P}^n$ to be other than a hypersurface. Exercises 12.43 and 12.46 give two circumstances where it does happen.

12.6.1 The universal hyperplane as projectivized cotangent bundle

The proof of the reflexivity theorem will make use of the "universal hyperplane", also called the Fano variety of hyperplanes in $\mathbb{P}^n = \mathbb{P}V$, which we introduced in Section 8.1.1. We defined it there to be the divisor in $\mathbb{P}^n \times \mathbb{P}^{n*}$ defined by the equation $x_i y_i = 0$, where $\{x_i\}$ and $\{y_i\}$ are dual coordinate systems; more invariantly, for a vector space V and its dual $W := V^*$ we can write:

$$\Psi = \{(v, w) \in \mathbb{P}V \times \mathbb{P}W \mid w(v) = 0.$$

It is obvious that the fiber of Ψ over a point $w \in \mathbb{P}W$ is the hyperplane defined by the vanishing of w as a functional on v.

Write the tautological sequence on $\mathbb{P}V$ as

$$0 \to \mathcal{O}_{\mathbb{P}V}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}V} \to Q \to 0.$$

Thus $Q^* \subset W \otimes \mathcal{O}_{\mathbb{P}V}$. From the inclusion it follows that the line bundle $\mathcal{O}_{\mathbb{P}Q^*}(-1)$ on $\mathbb{P}Q^*$ is the restriction from $\mathbb{P}^n \times \mathbb{P}^{n*}$ of the bundle $\pi_2^* \mathcal{O}_{\mathbb{P}^{n*}}(-1)$. In Section 8.1.1 we observed that $\Psi \subset \mathbb{P}V \times \mathbb{P}W$ may be regarded as the projectivization $\mathbb{P}Q^*$ inside $\mathbb{P}W \otimes \mathcal{O}_{\mathbb{P}V} = \mathbb{P}V \times \mathbb{P}W$.

To simplify the notation, we write π_V, π_W for the projections from $Z := \mathbb{P}V \times \mathbb{P}W$ to $\mathbb{P}V$ and $\mathbb{P}W$ respectively, and we set $\mathcal{O}_Z(a, b) := \pi_V^* \mathcal{O}_{\mathbb{P}V}(a) \otimes \pi_W^* \mathcal{O}_{\mathbb{P}W}(b)$. In this language, $\Psi = \pi_V^*(Q^*)$ and $\mathcal{O}_{\mathbb{P}Q^*}(-1)$ is the restriction of $\mathcal{O}_Z(0, -1)$. For our present purpose, we want to give a more symmetric description.

Proposition 12.22. *$\pi_V : \Psi \to \mathbb{P}V$ may be described as the projectivization of the cotangent bundle $\mathbb{P}\Omega_{\mathbb{P}V}$ of $\mathbb{P}V$. The tautological subbundle*

$$\mathcal{O}_{\mathbb{P}\Omega_{\mathbb{P}V}}(-1) \subset \pi_V^*(\Omega_{\mathbb{P}V})$$

is the restriction to $\Psi \subset \mathbb{P}V \times \mathbb{P}W = Z$ *of* $\mathcal{O}_Z(-1,-1)$.

Proof. The Euler sequence

$$0 \to \Omega_{\mathbb{P}V} \to W \otimes \mathcal{O}_{\mathbb{P}V}(-1) \to \mathcal{O}_{\mathbb{P}V} \to 0$$

that may be taken as the definition of $\Omega_{\mathbb{P}V}$ is the tautological sequence twisted by $\mathcal{O}_{\mathbb{P}V}(-1)$, and in particular $\Omega_{\mathbb{P}V} = Q^* \otimes \mathcal{O}_{\mathbb{P}V}(-1)$. By Theorem **??** we have $\mathbb{P}Q^* \cong \mathbb{P}\Omega$, with

$$\mathcal{O}_{\mathbb{P}\Omega}(-1) = \mathcal{O}_{\mathbb{P}Q^*} \otimes \pi_V^* \mathcal{O}_{\mathbb{P}V}(-1)$$

and this is the restriction to Ψ of $\mathcal{O}_Z(0,-1) \otimes \pi_V^* \mathcal{O}_{\mathbb{P}V}(-1) = \mathcal{O}_Z(-1,-1)$, as required. $\square$

Proof of Theorem 12.20. If $X \subset \mathbb{P}V$ is any subvariety then, over the open set where X is smooth, the conormal variety $CX \subset \Psi = \mathbb{P}\Omega_{\mathbb{P}V} = \operatorname{Proj}\operatorname{Sym} T_{\mathbb{P}V}$ is the projectivized conormal bundle $\mathbb{P}K = \operatorname{Proj}\operatorname{Sym} K^*$, where

$$K := \operatorname{Ker}\bigl(\Omega_{\mathbb{P}V}|_X \to \Omega_X\bigr).$$

The conormal variety of X itself is defined as the closure of this set. Over the open set where X is smooth, K^* is the cokernel of the map of bundles

$$T_X \to T_{\mathbb{P}V}|_X$$

so $\operatorname{Sym} K^*$ is equal to $\operatorname{Sym} T_{\mathbb{P}V}|_X$ modulo the ideal generated by T_X, thought of as contained in the degree 1 part $T_{\mathbb{P}V}|_X$ of the graded algebra $\operatorname{Sym} T_{\mathbb{P}V}|_X$. Sheafifying, this means that the ideal sheaf of CX in $\mathcal{O}_{\mathbb{P}\Omega_{\mathbb{P}V}}$ is the image of the composite map u in the following diagram:

$$\begin{tikzcd}
\pi_V^* T_X \otimes \mathcal{O}_{\Omega_{\mathbb{P}^n}}(-1) \arrow[r] \arrow[dr, "u"'] & \pi_V^* T_{\mathbb{P}^n} \otimes \mathcal{O}_{\Omega_{\mathbb{P}^n}}(-1) \arrow[d] \\
 & \mathcal{O}_{\mathbb{P}\Omega_{\mathbb{P}^n}},
\end{tikzcd}$$

where the vertical map is the dual of

$$\mathcal{O}_{\mathbb{P}\Omega_{\mathbb{P}^n}}(-1) \xrightarrow{\sum A_i dZ_i} \pi_V^* \Omega_{\mathbb{P}^n},$$

the tautological inclusion, tensored with $\mathcal{O}_{\Omega\mathbb{P}^n}(-1)$.

Having the equations, we can tell whether a given subvariety C of $\Psi \cap \pi_V^{-1}(X)$ is a subset of CX. Let $\iota : C \to \Psi$ be the inclusion. Set

$$v = u^* \otimes \mathcal{O}_{\mathbb{P}\Omega_{\mathbb{P}^n}}(-1) : \mathcal{O}_{\mathbb{P}\Omega_{\mathbb{P}^n}}(-1) \to \pi_V^* \Omega_X,$$

and consider the diagram

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}\Omega_{\mathbb{P}^n}}(-1)|_C \arrow[r, "\sum A_i dZ_i"] \arrow[dr, "v/_C"'] & \pi_V^* \Omega_{\mathbb{P}^n}|_C \arrow[r, "d\pi_V"] \arrow[d] & \Omega_\Psi|_C \arrow[d, "d\iota"] \\
 & \pi_V^* \Omega_X|_C \arrow[r, "d(\pi_1|_C)"'] & \Omega_C.
\end{tikzcd}$$

From what we have said about the equations of the conormal variety we see that $C \subset CX$ if and only if $v|_C = 0$. If the projection $\pi_V|_C : C \to X$ is separable, then $d\pi_V|_C$ is generically injective. Thus $C \subset CX$ if and only if the composition $d\pi_V|_C \circ v|_C$ is zero.

We will show that if C is separable over X^* then this condition is symmetric, so that $C \subset CX$ if and only if $C \subset C(X^*)$ (with the factors reversed). Since Ψ is defined by a hypersurface of bi-degree $(1,1)$, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}W}(-1,-1) \xrightarrow{\varphi} \Omega_{\mathbb{P}^n \times \mathbb{P}W}|_\Psi \longrightarrow \Omega_\Psi \longrightarrow 0.$$

In coordinates, using the decomposition

$$\Omega_{\mathbb{P}^n \times \mathbb{P}W} = \pi_V^*(\Omega_{\mathbb{P}^n}) \oplus \pi_W^*(\Omega_{\mathbb{P}W})$$

this becomes

$$\begin{aligned}0 \longrightarrow \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}W}(-1,-1)|_\Psi &\xrightarrow{\sum A_i dZ_i, \sum Z_i dA_i} \pi_V^*(\Omega_{\mathbb{P}^n})|_\Psi \oplus \pi_W^*(\Omega_{\mathbb{P}W})|_\Psi \\ &\xrightarrow{(d\pi_V, d\pi_W)} \Omega_\Psi \longrightarrow 0.\end{aligned}$$

Noting that $\mathcal{O}_{\mathbb{P}\Omega_{\mathbb{P}^n}}(-1) = \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}W}(-1,-1)|_\Psi$, we see that if $\iota : C \to \Psi$ is the inclusion of any subvariety then the composition

$$\mathcal{O}_{\mathbb{P}\Omega_{\mathbb{P}^n}}(-1) \xrightarrow{\sum A_i dZ_i} \pi_V^*(\Omega_{\mathbb{P}^n}) \xrightarrow{d\pi_V} \Omega_\Psi$$

is the negative of the composition

$$\mathcal{O}_{\mathbb{P}\Omega_{\mathbb{P}^n}}(-1) \xrightarrow{\sum Z_i dA_i} \pi_W^*(\Omega_{\mathbb{P}W}) \xrightarrow{d\pi_W} \Omega_\Psi.$$

It follows that the composition

$$\begin{tikzcd} \mathcal{O}_{\mathbb{P}\Omega_{\mathbb{P}^n}}(-1)|_C \arrow[r, "\sum A_i dZ_i"] & \pi_V^* \Omega_{\mathbb{P}^n}|_C \arrow[r, "d\pi_V"] & \Omega_\Psi|_C \arrow[d, "d\iota"] \\ & & \Omega_C. \end{tikzcd}$$

is zero if and only if the composition

$$\begin{tikzcd} \mathcal{O}_{\mathbb{P}\Omega_{\mathbb{P}W}}(-1)|_C \arrow[r, "\sum Z_i dA_i"] & \pi_W^* \Omega_{\mathbb{P}W}|_C \arrow[r, "d\pi_W"] & \Omega_\Psi|_C \arrow[d, "d\iota"] \\ & & \Omega_C. \end{tikzcd}$$

is zero.

If $C \to X$ is separable and $C \subset C(X^*)$ then the composite above is zero, and it follows that $C \subset CX$. If $C \subset \Psi$ and $C \to X^*$ is also separable, then $C \subset CX$ if and only if $C \subset CX'$. Applying this argument to $C = CX$ and $C = C(X^*)$, we obtain the desired equality.

$\square$

12.7 Exercises

Exercise 12.23. Use the result of Exercise 11.43 (describing the class of the universal k-plane in $\mathbb{P}^r \times \mathbb{G}(k, r)$) to give an alternative proof of Proposition 12.4.

Exercise 12.24. Let X be the scheme $\operatorname{Spec} k[t]/(t^n)$, and $X^{(k)}$ its k^{th} symmetric power.

(a) Show that the Zariski tangent space to $X^{(k)}$ at its unique closed point has dimension k.

(b) What is the degree of $X^{(k)}$?

Exercise 12.25. Let $X \subset \mathbb{P}^r$ be a variety, and $\Sigma\mathrm{ec}_m(X) \subset \mathbb{G}(2, r)$ the image of the secant plane map $\tau : X^{(m)} \to \mathbb{G}(m-1, r)$. Show by example that not every $(m-1)$-plane Λ such that $\deg(\Lambda \cap X) \geq m$ lies in $\Sigma\mathrm{ec}_m(X)$. (For example, try X a curve in $\mathbb{P}^5$ with a trisecant line, with $m = 3$.)

Exercise 12.26. Prove Proposition 12.8 in the case of a nondegenerate space curve $C \subset \mathbb{P}^3$—that is, that the line joining two general points of C does not meet the curve a third time—without using the General Position Lemma 12.9.

Exercise 12.27. Show that for $p, q \in \mathbb{P}^n$, the subspace $H^0(\mathcal{I}_p^2\mathcal{I}_q^2(2)) \subset H^0(\mathcal{O}_{\mathbb{P}^n}(2))$ of quadrics singular at p and q has codimension $2n+1$ (rather than the expected $2n+2$). Deduce that any two tangent planes to the quadratic Veronese variety $\nu_2(\mathbb{P}^n)$ meet, and thus that $\nu_2(\mathbb{P}^n)$ is 2-defective for any n.

Exercise 12.28. Show that for any five points $p_1, \dots, p_5 \in \mathbb{P}^2$, there exists a quartic curve double at all five; deduce that the tangent planes $\mathbb{T}_{p_i}S$ to the quartic Veronese surface $S = \nu_4(\mathbb{P}^2) \subset \mathbb{P}^{14}$ are dependent (equivalently, fail to span $\mathbb{P}^{14}$), and hence that S is 5-defective.

Exercise 12.29. Show that for any nine points $p_1, \dots, p_9 \in \mathbb{P}^3$, there exists a quartic surface double at all nine; deduce that the tangent planes $\mathbb{T}_{p_i}X$ to the quartic Veronese threefold $X = \nu_4(\mathbb{P}^3) \subset \mathbb{P}^{34}$ fail to span $\mathbb{P}^{34}$), and hence that X is 9-defective.

Exercise 12.30. Finally, show that for any seven points $p_1, \dots, p_7 \in \mathbb{P}^4$, there exists a cubic threefold double at all seven; deduce that the tangent planes $\mathbb{T}_{p_i}X$ to the cubic Veronese fourfold $X = \nu_3(\mathbb{P}^4) \subset \mathbb{P}^{34}$ are dependent (equivalently, fail to span $\mathbb{P}^{34}$), and hence that X is 7-defective. (Hint: this problem is harder than the preceding three; you have to use the fact that through seven general points in $\mathbb{P}^4$ there passes a rational normal quartic curve.)

Exercise 12.31. Show that the rational normal curve is the only nondegenerate curve $C \subset \mathbb{P}^d$ with the property that every divisor of degree d on C spans a hyperplane.

Exercise 12.32. Let $C \subset \mathbb{P}^d$ be a rational normal curve. Show that for any $m \leq d$ the secant plane map

$$\tau : C^{(m)} \to \mathbb{G}(m-1, d)$$

is actually an embedding, by verifying that the differential in injective everywhere.

Exercise 12.33. Show that the twisted cubic curve is the unique nondegenerate curve $C \subset \mathbb{P}^3$ such that a general point $p \in \mathbb{P}^3$ lies on a unique secant line to C.

Exercise 12.34. Let C be a smooth curve, and $C^{(d)}$ its d^{th} symmetric power. Show that there is a *universal family of divisors* of degree on C over $C^{(d)}$; that is, a subscheme $\mathcal{D} \subset C^{(d)} \times C$, flat over $C^{(d)}$, whose fiber over a point $D \in C^{(d)}$ is the subscheme $D \subset C$

Exercise 12.35. Let $C \subset \mathbb{P}^d$ be a rational normal curve, and for $m \leq d$ let $\mathcal{D} \subset C^{(m)} \times C$ be the universal family of divisors of degree m on C introduced in Exercise 12.34. Let $\alpha : C^{(m)} \times C \to C^{(m)}$ and $\beta : C^{(m)} \times C \to C$ be the projections. Show that the map of sheaves on $C^{(m)} \times C$

$$\beta^* \mathcal{O}_C(1) \to \beta^* \mathcal{O}_C(1) \otimes \mathcal{O}_{\mathcal{D}}$$

pushes forward under α to give a surjective map of vector bundles from the trivial bundle with fiber $H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ to a vector bundle of rank m; and that this induces a regular map $C^{(m)} \to \mathbb{G}(m-1, d)$ extending the secant plane map.

Exercise 12.36. ********add a solid hint to this (or do the result in the text!******** In the setting of Theorem 12.16, show in general that, for general points $p_1, \ldots, p_m$ and general polynomials $F_1, \ldots, F_{m-i+1}$, the scheme

$$\Lambda \cap V(\sigma_{F_1} \wedge \cdots \wedge \sigma_{F_{m-i+1}}),$$

where $\Lambda = \{D \in \mathbb{P}^m \mid p_1, \ldots, p_{m-i} \in D\}$, is reduced of dimension 0.

Exercise 12.37. Show that if $C \subset \mathbb{P}^r$ is a general rational curve, then the locus of m-secant $(m-k-1)$-planes has the expected dimension $m - k(r + 1 - m + k)$.

Exercise 12.38. By contrast with the preceding exercise, show that there exist components $\mathcal{H}$ of the Hilbert scheme of curves in $\mathbb{P}^3$ whose general point corresponds to a smooth, nondegenerate curve $C \subset \mathbb{P}^3$ with a positive-dimensional family of quadrisecant lines, or with a quintisecant line.

Exercise 12.39. Compute the number of quadrisecant lines to a general rational curve $C \subset \mathbb{P}^3$ of degree d (Hint: in the notation of Section 12.5, the answer is the degree of the class $\deg \tau^*(\sigma_{2,2}) \in A^4(\mathbb{P}^4)$. Express the class $\sigma_{2,2}$ in terms of the special Scubert classes σ_i and use (12.1) to evaluate it.)

Exercise 12.40. Let $C \subset \mathbb{P}^3$ be a smooth curve of degree n and genus g, and S and $T \subset \mathbb{P}^3$ two smooth surfaces containing C, of degrees d and e. At how many points of C are S and T tangent?

Exercise 12.41. Show that the conclusion of Corollary 12.21 fails in characteristic $p > 0$:

(a) Let K be a field of characteristic 2, and consider the plane curve

$$C = V(X^2 - YZ) \subset \mathbb{P}^2.$$

Show that C is smooth, but that the dual curve $C^* \subset \mathbb{P}^{2*}$ is a line, so that $C^{**} \neq C$.

(b) Now let K be a field of characteristic p, set $q = p^e$ and consider the plane curve

$$C = V(YZ^q + Y^qZ - X^{q+1}) \subset \mathbb{P}^2.$$

Show that C is smooth, and that the dual curve $C^{**} = C$, but that $\mathcal{G}_C : C \to C^*$ is not birational!

Exercise 12.42. We saw in Section **??** that if $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree $d > 1$ then the dual variety $X^* \subset \mathbb{P}^{n*}$ must again be a hypersurface. Show more generally that if $X \subset \mathbb{P}^n$ is any smooth complete intersection of degree $d > 1$ then X^* will be a hypersurface.

Exercise 12.43. Let $X \subset \mathbb{P}^n$ be a k-dimensional *scroll*, that is, a variety given as the union

$$X = \bigcup \Lambda_b$$

of a one-parameter family of $(k-1)$-planes $\{\Lambda_b \cong \mathbb{P}^{k-1} \subset \mathbb{P}^n\}$.

(a) Show that if $H \subset \mathbb{P}^n$ is a hyperplane containing the tangent plane $\mathbb{T}_pX$ to X at a smooth point p then the hyperplane section $H \cap X$ is reducible; and

(b) Deduce that $\dim X^* \leq n - k + 2$.

Exercise 12.44. This is a sort of partial converse to Exercise 12.43 above. Let $X \subset \mathbb{P}^n$ be any variety. Use Theorem 12.20 to deduce that if the dual X^* is not a hypersurface, X must be swept out by positive-dimensional linear spaces.

Exercise 12.45. Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d > 2$. Show that the dual variety X^* is necessarily singular.

Exercise 12.46. Let $X = \mathbb{G}(1,4) \subset \mathbb{P}^9$ be the Grassmannian of lines in $\mathbb{P}^4$, embedded in $\mathbb{P}^9$ by the Plücker embedding. Show that the dual of X is again projectively equivalent to X itself!

Exercise 12.47. If E is a smooth elliptic curve (over an algebraically closed field this means a curve of genus 1 with a chosen point), the addition law on E expresses the k^{th} symmetric power E_k as a $\mathbb{P}^{k-1}$ bundle over E. Verify this, and use it to give a description of $A(E_k)$.

Exercise 12.48. Using the preceding exercise, find the degrees of the secant varieties of an elliptic normal curve $E \subset \mathbb{P}^d$.

Exercise 12.49. We saw in Chapter **??** that

Exercise 12.50. Using the preceding exercise, find the degrees of the secant varieties of an elliptic normal curve $E \subset \mathbb{P}^d$.

13

Contact Problems and Bundles of Relative Principal Parts

(a) Given a general quintic surface $S \subset \mathbb{P}^3$, how many lines $L \subset \mathbb{P}^3$ meet S in only one point? (Answer on page 501)

(b) If $\mathcal{D}$ is a net of cubic plane curves, how many of the curves $C \in \mathcal{D}$ will have cusps? (Answer on page 529)

(c) If $\{C_t = V(t_0F + t_1G) \subset \mathbb{P}^2\}$ is a general pencil of quartic plane curves, how many of the curves C_t will have hyperflexes? (Answer on page 517)

(d) If $\{C_t\}$ is again a general pencil of quartic plane curves, what is the degree and genus of the curve traced out by flexes of members of the pencil? (Answer in Section 13.3.2)

Problems such as these are known as *contact problems*. They can often be reduced to computations of the Chern classes of associated bundles. The most important of the bundles involved is a relative version of the bundle of principle parts introduced in Chapter 9, and described by Theorem 9.4. We will begin with an illustration showing how these arise.

Two notes before we start. First, many of the results derived here are characteristic-dependent: some hold only in characteristic 0; some if we exclude specific characteristics (typically 2 and 3), and others hold in arbitrary characteristic. In the interest of clarity, *we will assume throughout this chapter that the characteristic of the ground field is 0.*

Secondly, we define the *order of contact* of a curve $C \subset X$ with a Cartier divisor $D \subset X$ at $p \in C$ to be the length of the scheme of intersection $C \cap D$, or (equivalently) if $\nu : \tilde{C} \to C$ is the normalization, the sum of the orders of vanishing of the defining equation of D at points of $\tilde{C}$ lying over p. If p is an isolated point of $C \cap D$, this is the same as the intersection multiplicity $m_p(C \cdot D)$, and we will use this to denote the order of contact; but we'll also adopt the convention that if $C \subset D$ the order of contact is ∞, so that the condition $m_p(C \cdot D) \geq m$ is a closed condition on C, D and p.

13.1 Lines meeting a surface to high order

Consider a general quintic surface $S \subset \mathbb{P}^3$. A general line meets S in 5 points; to require them all to coincide is 4 conditions, and there is a 4-dimensional family of lines in $\mathbb{P}^3$. Thus we would "expect" there to be just finitely many lines meeting S in just one point. On this basis we would, more generally, expect that for a general surface $S \subset \mathbb{P}^3$ of any degree $d \geq 5$ there will be a finite number of lines having a point of contact of order 5 with S.

As we shall show, this expectation is fulfilled, and we can compute the number. To verify the dimension statement, we introduce the flag variety

$$\Phi = \{(L, p) \in \mathbb{G}(1,3) \times \mathbb{P}^3 \mid p \in L\},$$

which we think of as the universal line over $\mathbb{G}(1,3)$; we can also realize Φ as the projectivization $\mathbb{P}\mathcal{S}$ of the universal subbundle $\mathcal{S}$ on $\mathbb{G}(1,3)$. Next, we fix $d \geq 4$ and look at pairs consisting of a point $(L,p) \in \Phi$ and a surface $S \subset \mathbb{P}^3$ of degree d such that the line L has contact of order at least 5 with S at p (or is contained in S):

$$\Gamma = \{(L, p, S) \in \Phi \times \mathbb{P}^N \mid m_p(S \cdot L) \geq 5\},$$

where $\mathbb{P}^N$ is the space of surfaces of degree d in $\mathbb{P}^3$.

The fiber of Γ over any point $(L,p) \in \Phi$ is a linear subspace of codimension 5 in the space $\mathbb{P}^N$ of surfaces of degree d. Since Φ is irreducible of dimension 5 it follows that Γ is irreducible of dimension N, and hence that the fiber of Γ over a general point $[S] \in \mathbb{P}^N$ is finite. Note that this includes the possibility that the fiber over a general point is empty, as in fact will be the case when $d = 4$: any line with a point of contact of order 5 with a quartic surface S must lie in S, but as we've seen, a general quartic surface contains no lines. In case $d = 4$, correspondingly, Γ projects with one-dimensional fibers to the hypersurface $\Sigma \subset \mathbb{P}^{34}$ of quartics that do contain a line. By contrast, we'll see (as a consequence of Theorem 13.1) that if $d \geq 5$ then the projection $\Gamma \to \mathbb{P}^N$ is generically finite, and surjective.

To linearize the problem we consider, for each pair $(L, p) \in \Phi$, the 5-dimensional vector space

$$E_{(L,p)} = \frac{\{\text{germs of sections of } \mathcal{O}_L(d) \text{ at } p\}}{\{\text{germs vanishing to order} \geq 5 \text{ at } p\}}$$

of germs of sections of $\mathcal{O}_L(d)$ at p, modulo those vanishing to order at least 5 at p. To say that $m_p(S \cdot L) \geq 5$ means exactly that the defining equation F of S is in the kernel of the map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(d)) \to E_{(L,p)}$$

given by restriction of F to a neighborhood of p in L.

To compute the number of lines with 5-fold contact, we will define a vector bundle $\mathcal{E}$ on Φ whose fiber at a point $(L, p) \in \Phi$ is the vector space $E_{(L,p)} = H^0(\mathcal{O}_L(d)/\mathfrak{m}^5_{L,p}(d))$ above, so that a polynomial F of degree d on $\mathbb{P}^3$ will give a global section σ_F of $\mathcal{E}$ by restriction in turn to each line L. The zeros of the section σ_F will then be the points $(L, p) \in \Phi$ such that $m_p(S \cdot L) \geq 5$, and—assuming that there are no unforeseen multiplicities—the answer to our enumerative problem will be the degree of the top Chern class $c_5(\mathcal{E}) \in A^5(\Phi)$. The necessary theory and computation will occupy the next two sections, and will prove:

Theorem 13.1. *If S is a general quintic surface, then there are exactly 575 lines meeting S in only one point. More generally, if $S \subset \mathbb{P}^3$ is a general surface of degree $d \geq 4$ then there are exactly $35d^3 - 200d^2 + 240d$ lines having a point of contact of order 5 with S.*

Note that this does return the correct answer 0 in the case $d = 4$.

13.1.1 Bundles of relative principle parts

The desired bundle $\mathcal{E}$ on Φ is a *bundle of relative principle parts* associated to the map

$$\pi : \Phi \to \mathbb{G}(1, 3); \quad (L, p) \mapsto [L].$$

The construction is a relative version of that of Section 9.3; the reader may wish to review that section before proceeding. The facts we need are the analogs of some of the properties spelled out in Theorem 9.4.

Suppose, more generally, that $\pi : Y \to X$ is a proper smooth map of schemes, and let $\mathcal{L}$ be a vector bundle on Y. Set $Z = Y \times_X Y$, the fiber product of Y with itself over X, with projection maps $\pi_1, \pi_2 : Z \to Y$, and let $\Delta \subset Z$ be the diagonal, so that we have a diagram

$$\begin{tikzcd} \Delta \arrow[r] & Z = Y \times_X Y \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & Y \arrow[d, "\pi"] \\ & Y \arrow[r, "\pi"] & X \end{tikzcd}$$

The *bundle of relative* m^{th} *order principle parts* $\mathcal{P}^m_{Y/X}(\mathcal{L})$ is, by definition,

$$\mathcal{P}^m_{Y/X}(\mathcal{L}) = \pi_{2*}\left(\pi_1^*\mathcal{L} \otimes \mathcal{O}_Z/\mathcal{I}_\Delta^{m+1}\right).$$

Theorem 13.2. *With $\pi : Y \to X$ and projections $\pi_i : Y \times_X Y \to Y$ as above:*

(a) *The sheaf $\mathcal{P}^m_{Y/X}(\mathcal{L})$ is a vector bundle on Y, and its fiber at a point $y \in Y$ is the vector space of germs of sections of $\mathcal{L}$ on the fiber $F_y = \pi^{-1}(\pi(y)) \subset Y$ of π at y, modulo those vanishing to order at least $m+1$. That is,*

$$\mathcal{P}^m_{Y/X}(\mathcal{L})_y = \frac{\{\textit{germs of sections of } \mathcal{L}|_{F_y} \textit{ at } y\}}{\{\textit{germs vanishing to order} \geq m+1 \textit{ at } y\}}.$$

(b) *We have an isomorphism $\pi^*\pi_*\mathcal{L} \cong \pi_{2*}\pi_1^*\mathcal{L}$.*

(c) *The quotient map $\pi_1^*\mathcal{L} \to \pi_1^*\mathcal{L} \otimes \mathcal{O}_Z/\mathcal{I}_\Delta^{m+1}$ pushes forward to give a map*

$$\pi^*\pi_*\mathcal{L} \cong \pi_{2*}\pi_1^*\mathcal{L} \to \mathcal{P}^m_{Y/X}(\mathcal{L}),$$

and the image of a global section $G \in H^0(\mathcal{L})$ is the section σ_G of $\mathcal{P}^m_{Y/X}(\mathcal{L})$ whose value at a point $y \in Y$ is the restriction of G to a neighborhood of y in F_y.

(d) *$\mathcal{P}^0_{Y/X}\mathcal{L} = \mathcal{L}$. For $m > 1$ the filtration of the fiber of $(\mathcal{P}^m_{Y/X}(\mathcal{L}))_y$ by order of vanishing at y corresponds to a filtration of $\mathcal{P}^m_{Y/X}(\mathcal{L})$ by subbundles that are the kernels of surjections $\mathcal{P}^m_{Y/X}(\mathcal{L}) \to \mathcal{P}^{m-1}_{Y/X}(\mathcal{L})$. The graded pieces of this filtration are identified by the exact sequences*

$$0 \to \mathcal{L} \otimes \mathrm{Sym}^m(\Omega_{Y/X}) \to \mathcal{P}^m_{Y/X}(\mathcal{L}) \to \mathcal{P}^{m-1}_{Y/X}(\mathcal{L}) \to 0. \tag{13.1}$$

The exact sequences (13.1) allow us to express the Chern classes of the bundles $\mathcal{P}^m_{Y/X}(\mathcal{L})$ in terms of the Chern classes of $\mathcal{L}$ and those of $\Omega_{Y/X}$. We will compute the latter in the case where Y is a projectivized vector bundle over X in the next section, and this will allow us to complete the answer to Keynote Question (a).

Proof. Part (a) is an application of the Theorem on cohomology and base change, just as in the absolute case (Theorem 9.4).

For part (b), informally speaking the bundles $\pi^*\pi_*\mathcal{L}$ and $\pi_{2*}\pi_1^*\mathcal{L}$ are isomorphic because they are both "the bundle on Y whose fiber over a point $y \in Y$ is the vector space $H^0(\mathcal{L}|_{F_y})$." To make this rigorous, observe first that the natural evaluation map $\pi^*\pi_*\mathcal{L} \to \mathcal{L}$ pulls back and pushes forward to a map $\pi_{2*}\pi_1^*\pi^*\pi_*\mathcal{L} \to \pi_{2*}\pi_1^*\mathcal{L}$. But $\pi\pi_1 = \pi\pi_2$, so we can rewrite the source of this map as $\pi_{2*}\pi_2^*\pi^*\pi_*\mathcal{L}$; composing, we have a map

$$\pi^*\pi_*\mathcal{L} \to \pi_{2*}\pi_2^*\pi^*\pi_*\mathcal{L} = \pi_{2*}\pi_1^*\pi^*\pi_*\mathcal{L} \to \pi_{2*}\pi_1^*\mathcal{L}.$$

By the theorem on cohomology and base change, this map is an isomorphism on each fiber, and hence an isomorphism of sheaves.

Part (c) is also a direct analog of the absolute case. For (d), consider the diagonal $\Delta := \Delta_{Y/X} \subset Y \times_X Y$, and its ideal sheaf $\mathcal{I}_\Delta$. As in the absolute case we have $\pi_{1*}(\mathcal{I}_\Delta/\mathcal{I}_\Delta^2) = \Omega_{Y/X}$ (See Eisenbud [1995] Theorem *** for the affine case, to which the problem reduces.) The sheaf $\Omega_{Y/X}$ is a vector bundle on Y because π is smooth. Since Δ is locally a complete intersection in $Y \times_X Y$, it follows that

$$\mathcal{I}_\Delta^m/\mathcal{I}_\Delta^{m+1} = \mathrm{Sym}_m(\mathcal{I}_\Delta/\mathcal{I}_\Delta^2).$$

The desired exact sequences are derived from this exactly as in the absolute case. □

13.1.2 Relative tangent bundles of projective bundles

To use the sequences (13.1) to calculate the Chern class of $\mathcal{P}^m_{Y/X}(\mathcal{L})$ we need to understand the relative tangent bundle $\mathcal{T}_{Y/X}$. Recall first the definition: if $\pi : Y \to X$ is a smooth map, then the differential $d\pi : T_Y \to \pi^*T_X$ is surjective. Its kernel is called the *relative tangent bundle* of π, denoted T_π or, when there is no ambiguity, as $T_{Y/X}$; its local sections are the vector fields on Y that are everywhere tangent to a fiber. Thus, for example, if $x \in X$ then the restriction $T_{Y/X}|_{\pi^{-1}}(x)$ is the tangent bundle to the smooth variety $\pi^{-1}(x)$.

One special case in which we can describe the relative tangent bundle explicitly is when $\pi : Y = \mathbb{P}\mathcal{E} \to X$ is a projective bundle (as was the case in the example above); in this section we'll show how.

Recall from Section 2.2.4 that if $\xi \in \mathbb{P}V$ is a point in the projectivization $\mathbb{P}V$ of a vector space V, then we can identify the tangent space $T_\xi\mathbb{P}V$ with the vector space $\mathrm{Hom}(\xi, V/\xi)$. As we showed, these identifications fit together to give an identification of bundles

$$\mathcal{T}_{\mathbb{P}V} = \mathrm{Hom}(\mathcal{S}, \mathcal{Q}),$$

where $\mathcal{S} = \mathcal{O}_{\mathbb{P}V}(-1)$ and $\mathcal{Q}$ are the universal sub- and quotient bundles.

This identification extends to families of projective spaces. Explicitly, suppose $\mathcal{E}$ is a vector bundle on X and $\mathbb{P}\mathcal{E}$ its projectivization, with universal sub- and quotient bundles $\mathcal{S} = \mathcal{O}_{\mathbb{P}\mathcal{E}}(-1)$ and $\mathcal{Q}$. At every point

$(x, \xi) \in \mathbb{P}\mathcal{E}$, with $x \in X$ and $\xi \subset \mathcal{E}_x$, we have an identification $T_\xi \mathbb{P}\mathcal{E}_x = \operatorname{Hom}(\xi, \mathcal{E}_x/\xi) = \operatorname{Hom}(\mathcal{S}_{(x,\xi)}, \mathcal{Q}_{(x,\xi)})$, and these agree on overlaps of such open sets to give a global isomorphism:

Proposition 13.3.

$$\mathcal{T}_{\mathbb{P}\mathcal{E}/X} \cong \operatorname{Hom}(\mathcal{S}, \mathcal{Q}).$$

Proof. This is a special case of the statement that, with notation as in the Theorem, the relative tangent bundle of the Grassmannian bundle $G(k, \mathcal{E}) \to X$ is

$$\mathcal{T}_{Gr(k,\mathcal{E})/X} = \operatorname{Hom}(\mathcal{S}, \mathcal{Q}).$$

Over an open subset where $\mathcal{E}$ is trivial this is an immediate consequence of the isomorphism described in Section 2.2.4 between the tangent bundle of a Grassmannian and the bundle $\operatorname{Hom}(\mathcal{S}, \mathcal{Q})$; and as in that setting the fact that these isomorphisms are independent of choices says they fit together to give the desired isomorphism $\mathcal{T}_{Gr(k,\mathcal{E})/X} = \operatorname{Hom}(\mathcal{S}, \mathcal{Q})$. □

Using the exact sequence

$$0 \to \mathcal{S} \to \pi^*\mathcal{E} \to \mathcal{Q} \to 0,$$

Proposition 13.3 yields an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}\mathcal{E}} \to \pi^*\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}\mathcal{E}}(1) \to \mathcal{T}_{\mathbb{P}\mathcal{E}/X} \to 0$$

Applying the formula for the Chern classes of the tensor product of a vector bundle with a line bundle (Proposition 7.10), we arrive at the following Theorem:

Theorem 13.4. *If $\mathcal{E}$ is a vector bundle of rank $r+1$ on the smooth variety X then the Chern classes of the relative tangent bundle $\mathcal{T}_{\mathbb{P}\mathcal{E}/X}$ are*

$$c_k(\mathcal{T}_{\mathbb{P}\mathcal{E}/X}) = \sum_{i=0}^{k} \binom{r+1-i}{k-i} c_i(\mathcal{E})\zeta^{k-i}$$

where $\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)) \in A^1(\mathbb{P}\mathcal{E})$, and we identify $A(X)$ with its image in $A(\mathbb{P}\mathcal{E})$ via the pullback map.

13.1.3 Chern classes of contact bundles

Returning to Keynote Question (a), we again let

$$\Phi = \{(L, p) \in \mathbb{G}(1,3) \times \mathbb{P}^3 \mid p \in L\},$$

be the universal line over $\mathbb{G}(1,3)$. Via the projection $\pi : \Phi \to \mathbb{G}(1,3)$ this is the projectivization $\mathbb{P}\mathcal{S}$ of the universal subbundle $\mathcal{S}$ on $\mathbb{G}(1,3)$. Let $\mathcal{E}$ be the bundle on Φ given by

$$\mathcal{E} = \mathcal{P}^4_{\Phi/\mathbb{G}(1,3)}(\beta^*\mathcal{O}_{\mathbb{P}^3}(d))$$

where $\beta : \Phi \to \mathbb{P}^3$ is the projection $(L,p) \mapsto p$ on the second factor. By the theorem on cohomology and base change (Theorem 6.6), this has fiber $\mathcal{E}_{(L,p)} = H^0(\mathcal{O}_L(d)/\mathfrak{m}^5_{L,p}(d))$ at a point $(L,p) \in \Phi$. Thus, counting multiplicities, the number of lines having a point of contact of order at least 5 with a general surface of degree d is the degree of the Chern class $c_5(\mathcal{E})$.

To find the degree of $c_5(\mathcal{E})$, we recall first the description of the Chow ring of Φ given in Section 11.4.1: since

$$\Phi = \mathbb{P}\mathcal{S} \to \mathbb{G}(1,3)$$

is the projectivization of the universal subbundle on $\mathbb{G}(1,3)$, and

$$c_1(\mathcal{S}) = -\sigma_1 \quad \text{and} \quad c_2(\mathcal{S}) = \sigma_{11},$$

Theorem 11.9 yields

$$A(\Phi) = A(\mathbb{G}(1,3))[\zeta]/(\zeta^2 - \sigma_1\zeta + \sigma_{11}),$$

where $\zeta \in A^1(\Phi)$ is the first Chern class of the line bundle $\mathcal{O}_{\mathbb{P}\mathcal{S}}(1)$. Recall moreover that the class ζ can also be realized as the pullback $\zeta = \beta^*\omega$, where $\beta : \Phi \to \mathbb{P}^3$ is the projection $(L,p) \mapsto p$ on the second factor, and $\omega \in A^1(\mathbb{P}^3)$ is the hyperplane class.

The relative tangent bundle $\mathcal{T}_{\Phi/\mathbb{G}(1,3)}$ is a line bundle on Φ. By Theorem 13.4, its Chern class is

$$c_1(\mathcal{T}_{\Phi/\mathbb{G}(1,3)}) = 2\zeta - \sigma_1.$$

By Theorem 13.2 the bundle $\mathcal{E} = \mathcal{P}^4_{\Phi/\mathbb{G}(1,3)}(\beta^*\mathcal{O}_{\mathbb{P}^3}(d))$ has a filtration with successive quotients

$$\beta^*\mathcal{O}_{\mathbb{P}^3}(d),\ \beta^*\mathcal{O}_{\mathbb{P}^3}(d) \otimes \Omega_{\Phi/\mathbb{G}(1,3)},\ \dots,\ \beta^*\mathcal{O}_{\mathbb{P}^3}(d) \otimes \operatorname{Sym}^4 \Omega_{\Phi/\mathbb{G}(1,3)}.$$

The bundle $\Omega_{\Phi/\mathbb{G}(1,3)}$ is dual to the relative tangent bundle $\mathcal{T}_{\Phi/\mathbb{G}(1,3)}$, so its m-th symmetric power has Chern class

$$c(\operatorname{Sym}^m \Omega_{\Phi/\mathbb{G}(1,3)}) = 1 + m(\sigma_1 - 2\zeta).$$

With the formula $c_1(\beta^*\mathcal{O}_{\mathbb{P}^3}(d)) = d\zeta$, this gives

$$c(\beta^*\mathcal{O}_{\mathbb{P}^3}(d) \otimes \operatorname{Sym}^m \Omega_{\Phi/\mathbb{G}(1,3)}) = 1 + (d-2m)\zeta + m\sigma_1$$

and altogether

$$c(\mathcal{E}) = \prod_{m=0}^{4} \big(1 + (d-2m)\zeta + m\sigma_1\big).$$

In particular,

$$c_5(\mathcal{E}) = d\zeta \cdot ((d-2)\zeta + \sigma_1) \cdot ((d-4)\zeta + 2\sigma_1) \cdot ((d-6)\zeta + 3\sigma_1) \cdot ((d-8)\zeta + 4\sigma_1).$$

We can evaluate the degrees of monomials of degree 5 in ζ and σ_1 by using the Segre classes introduced in Section 12.1, and in particular Proposition 12.3: we have

$$\begin{aligned}\deg(\zeta^a\sigma_1^b) &= \deg \pi_*(\zeta^a\sigma_1^b)\\ &= \deg(s_{a-1}(\mathcal{S})\sigma_1^b)),\end{aligned}$$

where $s_k(\mathcal{S})$ is the k^{th} Segre class of $\mathcal{S}$. Combining Proposition 12.3 and the Whitney formula, we have

$$s(\mathcal{S}) = \frac{1}{c(\mathcal{S})} = c(\mathcal{Q}) = 1 + \sigma_1 + \sigma_2$$

and so we have

$$\deg(\zeta\sigma_1^4)_\Phi = \deg(\sigma_1^4)_{\mathbb{G}(1,3)} = 2; \quad \deg(\zeta^2\sigma_1^3)_\Phi = \deg(\sigma_1^4)_{\mathbb{G}(1,3)} = 2; \quad \text{and}$$

$$\deg(\zeta^3\sigma_1^2)_\Phi = \deg(\sigma_2\sigma_1^2)_{\mathbb{G}(1,3)} = 1.$$

The remaining monomials of degree 5 in ζ and σ_1 are all zero: $\sigma_1^5 = 0$ since the Grassmannian $\mathbb{G}(1,3)$ is four-dimensional, while $\zeta^4\sigma_1 = \zeta^5 = 0$ because the Segre classes of $\mathcal{S}$ vanish above degree 2 (alternatively, since $\zeta = \beta^*\omega$ is the pullback of a class on $\mathbb{P}^3$ we see immediately that $\zeta^4 = 0$).

Putting this together with the formula above for $c_5(\mathcal{E})$ we get

$$\begin{aligned}&\deg c_5(\mathcal{E}) =\\ &\deg\big(d\zeta((d-2)\zeta+\sigma_1)((d-4)\zeta+2\sigma_1)((d-6)\zeta+3\sigma_1)((d-8)\zeta+4\sigma_1)\big)\\ &\qquad = \deg\big(\, 24d\zeta\sigma_1^4\\ &\qquad\qquad + d(50d-192)\zeta^2\sigma_1^3\\ &\qquad\qquad + d(35d^2-200d+240)\zeta^3\sigma_1^2\big)\\ &\qquad = 35d^3 - 200d^2 + 240d.\end{aligned}$$

This is, assuming there are only finitely many and counting multiplicities, the number of lines having a point of contact of order at least 5 with S.

To answer the keynote question we need to know in addition that for a general surface $S \subset \mathbb{P}^3$ of degree $d \geq 5$ all the lines having a point of contact of order 5 with S "count with multiplicity 1"—that is, all the zeros of the corresponding section of the bundle $\mathcal{E}$ on Φ are simple zeros. To do this, we invoke the irreducibility of the incidence correspondence

$$\Gamma = \{(L,p,S) : m_p(S \cdot L) \geq 5\} \subset \Phi \times \mathbb{P}^N$$

introduced earlier in the discussion. By virtue of the irreducibility of Γ, it's enough to show that at just one point $(L,p,S) \in \Gamma$ the section of $\mathcal{E}$ corresponding to S has a simple zero at $(L,p) \in \Phi$: given this, the locus of (L,p,S) for which this is not the case, being a proper closed subvariety of Γ, will have strictly smaller dimension, and so cannot dominate $\mathbb{P}^N$. As for locating such a triple (L, p, S), Exercise 13.17 suggests one. We should also check that for S general, no line has a point of contact of order at least 6 with S, or more than one point of contact of order at least 5; this is implied by Exercise 13.18. This completes the proof of Theorem 13.1.

13.2 The case of negative expected dimension

In this section, we'll describe a rather different application of the contact calculus developed so far: we will use it—in characteristic 0—to bound the maximum number of occurrences of some phenomena that occur in negative "expected dimension."

We begin by explaining an example. We don't expect a surface $S \subset \mathbb{P}^3$ of degree $d \geq 4$ to contain any lines. But some smooth quartics do contain a line and some contain several. Thus we can ask: how many lines can a smooth surface of degree d contain?

We observe to begin with that the number of lines a smooth surface of degree d can contain is certainly bounded: If we let $\mathbb{P}^N$ be the space of surfaces of degree $d \geq 4$, and write

$$\Sigma = \{(S, L) \in \mathbb{P}^N \times \mathbb{G}(1, 3) \mid L \subset S\}$$

for the incidence correspondence, then the set of points of $\mathbb{P}^N$ over which the fiber of the map $\pi : \Sigma \to \mathbb{P}^N$ has cardinality greater than m is a closed subset of $\mathbb{P}^N$ for any m. Since, as we saw in Section 1.4.2, a smooth surface in $\mathbb{P}^3$ of degree > 2 cannot contain infinitely many lines, the cardinalities of the fibers over the open set $U \subset \mathbb{P}^N$ of smooth surfaces are bounded. We can thus ask:

Question 13.5. *What is the largest number $M(d)$ of lines that a smooth surface $S \subset \mathbb{P}^3$ of degree d can have?*

Remarkably, we don't know the answer to this in general!

The situation here is typical: there is a large range of quasi-enumerative problems where the actual number is indeterminate because the expected dimension of the solution set is negative. In general, almost every time we have an enumerative problem there are analogous "negative-expected-dimension" variants—how many conics in $\mathbb{P}^2$ can be tangent to each of 6 conics; how many conics in $\mathbb{P}^3$ can meet each of 9 lines, and so on. For example, we can ask

Question 13.6. (a) *How many isolated singular points can a hypersurface $X \subset \mathbb{P}^n$ of degree d have?*

(b) *How many tritangents can a plane curve $C \subset \mathbb{P}^2$ of degree d have? How many hyperflexes?*

(c) *How many cuspidal curves can a pencil of plane curves of degree d have? How many reducible ones? How many totally reducible ones (that is, unions of lines)?*

We can even go all the way back to Bezout, and ask:

Question 13.7. *How many isolated points of intersection can $n+k$ linearly independent hypersurfaces of degree d in $\mathbb{P}^n$ have?*

Here there is at least a conjecture, described in Eisenbud et al. [1996] and proved in case $k = 1$ by Lazarsfeld (**?**]) ****wrote lazarsfeld to ask him for a reference****. We'll come back to this in subsection **??**.

Another negative-expected-dimension variant of Bézout's Theorem is the question:

Question 13.8. *How many isolated points of intersection can non degenerate curves in $\mathbb{P}^3$ of degrees d and e have? How about in $\mathbb{P}^n$?*

(There are known bounds in this case: see Diaz [1986] and Giuffrida [1988].)

All of these problems are attractive (especially Question 13.7). But we won't pursue them here; rather, we'll focus on the original problem of bounding the number of lines on a smooth surface in $\mathbb{P}^3$, in order to illustrate how we can use enumerative methods to find such a bound.

13.2.1 Lines on smooth surfaces in $\mathbb{P}^3$

Since the number of lines on a smooth surface S is variable, it cannot be the solution to an enumerative problem of the sort we have been considering. But we can still use enumerative geometry to bound the number. What we'll do is to find a curve F on S whose degree is determined enumeratively, and such that F contains all the lines on S.

A natural approach is to relax the condition that a line L be contained in S to the condition that L meets S with high multiplicity at some point $p \in S$. We can adjust the multiplicity so that the set of points p for which some line satisfies this condition has expected dimension 1, defining a curve on the surface. Since this curve must contain all the lines lying on the surface, its degree—which we can compute by enumerative means—is a bound for the number of such lines. It turns out that the right multiplicity is 4, as we shall now explain.

First of all, we say that a point $p \in S$ is *flecnodal* if there exists a line $L \subset \mathbb{P}^3$ having contact of order 4 or more with S at p; let $F \subset S$ be the locus of such points. (The reason for the name comes from another characterization of such points: for a general surface S, a general flecnodal point $p \in S$ will be one such that the intersection $S \cap \mathbb{T}_p S$ has a *flecnode* at p, that is, a node with one branch of the node having a flex.) As we'll show in Proposition 13.9, the flecnodal locus $F \subset S$ of a smooth surface of degree $d \geq 3$ will always have dimension 1.

As we have observed, any line lying in S is contained in the flecnodal locus F. Of course when $d = 3$ any line meeting S with multiplicity ≥ 4 must lie in S, so the flecnodal locus is exactly the union of the 27 lines in S. To describe the locus of flecnodes on S more generally, we again write Φ for the incidence correspondence

$$\Phi = \{(L, p) \in \mathbb{G}(1,3) \times \mathbb{P}^3 \mid p \in L\};$$

and we let ζ and $\sigma_1 \in A^1(\Phi)$ be the pullbacks of the corresponding classes on $\mathbb{P}^3$ and $\mathbb{G}(1,3)$. Given a surface S, we wish to find the class of the locus

$$\Gamma = \{(L, p) \in \Phi \mid m_p(L \cdot S) \geq 4\}.$$

Since the flecnodal locus $F \subset S$ is the image of Γ under the projection of Φ to $\mathbb{P}^3$, knowledge of this class will determine in particular the degree of F.

To compute the class of Γ, consider the bundle $\mathcal{F} = \mathcal{P}^3_{\Phi/\mathbb{G}(1,3)}(\pi_2^*\mathcal{O}_{\mathbb{P}^3}(d))$ of third order relative principal parts of $\pi_2^*\mathcal{O}_{\mathbb{P}^3}(d)$. It is a bundle of rank 4 on Φ whose fiber at a point (L, p) is the vector space of germs of sections of $\mathcal{O}_L(d)$ at p, modulo those vanishing to order at least 4 at p:

$$\mathcal{F}_{(L,p)} = H^0(\mathcal{O}_L(d)/\mathcal{I}^4_{p,L}(d)).$$

If $A \in H^0(\mathcal{O}_{\mathbb{P}^3}(d))$ is a homogeneous polynomial of degree d defining a surface S, then the restrictions of A to each line $L \subset \mathbb{P}^3$ yield a global section σ_A of the bundle F, whose zeros are the pairs (L, p) such that L meets S with multiplicity ≥ 4 at p.

Since $\dim \Phi = 5$ and $\mathcal{F}$ has rank 4, the locus Γ (if not empty) is at least one-dimensional; if it has dimension exactly 1 then its class is the top Chern class

$$[\Gamma] = c_4(\mathcal{F}) \in A^4(\Phi).$$

We can calculate this class as before: we can filter the bundle $\mathcal{F}$ by order of vanishing—that is, invoke the exact sequences (13.1)—and apply the Whitney formula to arrive at

$$c_4(\mathcal{F}) = d\zeta \cdot ((d-2)\zeta + \sigma_1) \cdot ((d-4)\zeta + 2\sigma_1) \cdot ((d-6)\zeta + 3\sigma_1).$$

Of course, none of this will help us bound the number of lines on S if every point of S is a flecnode! The following result is thus crucial for this approach:

Proposition 13.9. *If $S \subset \mathbb{P}^3$ is a smooth surface of degree $d \geq 3$ over a field of characteristic 0, the locus*

$$\Gamma = \{(L, p) \in \Phi \mid m_p(L \cdot S) \geq 4\}$$

has dimension 1. In particular, the general point of S is not flecnodal.

We'll defer the proof of this proposition to the next section, and continue to derive our bound on the number of lines. By the proposition, the flecnodal locus $F \subset S$ of S is a curve, whose degree is the degree of the intersection of Γ with the class ζ. We can evaluate this as before:

$$\begin{aligned}\deg(F) &= d\zeta^2 \cdot ((d-2)\zeta + \sigma_1) \cdot ((d-4)\zeta + 2\sigma_1) \cdot ((d-6)\zeta + 3\sigma_1) \\ &= d(11d - 24).\end{aligned}$$

Putting this together, we have proven a bound on the number of lines in S:

Proposition 13.10. *If the ground field has characteristic 0, then the maximum number $M(d)$ of lines lying on a smooth surface $S \subset \mathbb{P}^3$ of degree $d \geq 3$ is at most $d(11d - 24)$.*

In the case $d = 3$ this gives the exact answer since $d(11d-24) = 27$. But for $d \geq 4$ the bound is not sharp: Segre [1943] proves the slightly better bound $M(d) \leq d(11d - 28) + 12$. Even this is not sharp; for example with $d = 4$ we have $d(11d-28)+12 = 76$, but Segre also shows that the maximum number of lines on a smooth quartic surface is exactly $M(4) = 64$.

Of course, we can give a lower bound for $M(d)$ simply by exhibiting a surface with a large number of lines. The Fermat surface $V(x^d + y^d + z^d + w^d)$, for example, has exactly $3d^2$ lines (Exercise 13.26), whence $M(d) \geq 3d^2$. This is still the record-holder for general d. More is known for some particular values of d; Exercises 13.27 and 13.28 exhibit surfaces with more lines in case $d = 4, 6, 8, 12$ and 20 (respectively, 64, 180, 256, 864, and 1600), and Boissière and Sarti find an octic with 352 (the current champion!) in Boissière and Sarti [2007].

13.2.2 The flecnodal locus

It remains to prove that for any smooth surface $S \subset \mathbb{P}^3$ of degree $d \geq 3$, the locus $\Gamma \subset \Phi$ of pairs (L, p) with $m_p(L \cdot S) \geq 4$ has dimension 1.

Proof of Proposition 13.9. Suppose on the contrary that the locus $\Gamma \subset \Phi$ has a component Γ_0 of dimension 2 or more, and let (L_0, p_0) be a general point of Γ_0. Since the map $\Gamma \to S$ has fibers of dimension at most 1, and only finitely many positive-dimensional fibers (Exercise 13.54), p_0 will be a general point of S, and it follows (Exercise 13.55) that the tangent plane section $S \cap \mathbb{T}_{p_0} S$ has a node at p_0.

We will proceed by introducing local coordinates on Φ, and writing down the defining equations of the subset Γ. To start with, we can find an affine open $\mathbb{A}^3 \subset \mathbb{P}^3$ and choose coordinates (x, y, z) on $\mathbb{A}^3$ so that the point p_0 is the origin $(0,0,0) \in \mathbb{A}^3$, and the line $L_0 = \{(x,0,0)\}$ is the x-axis; we can also take the tangent plane $\mathbb{T}_{p_0} S$ to be the plane $z = 0$, and, given that the tangent plane section $S \cap \mathbb{T}_{p_0} S$ has a node at p_0, we can take the tangent cone at p to the intersection $S \cap \mathbb{T}_{p_0} S$ is the union $V(z, xy)$ of the x- and y-axes.

We can take coordinates (a, b, c, d, e) in a neighborhood U of $(L_0, p_0) \in \Phi$ so that if (L, p) is the pair corresponding to (a, b, c, d, e), then

$$p = (a, b, c) \quad \text{and} \quad L = \{(a + t, b + dt, c + et)\}.$$

Let $f(x, y, z)$ be the defining equation of S in $\mathbb{A}^3$. If we write the restriction of f to L as

$$f|_L = f(a + t, b + dt, c + et) = \sum_{i \geq 0} \alpha_i(a, b, c, d, e) t^i,$$

the four functions $\alpha_0, \alpha_1, \alpha_2$ and α_3 will be the defining equations of Γ in U. We want to show their common zero locus has codimension 4 in Φ; we'll in fact prove the epsilonically stronger fact that their differentials at (L_0, p_0) are independent.

By the specifications above of p_0, L_0, $\mathbb{T}_{p_0} S$ and $\mathbb{T}C_{p_0}(S \cap \mathbb{T}_{p_0} S)$, we can write

$$f(x, y, z) = z \cdot u(x, y, z) + xy \cdot v(x, y) + y^3 \cdot l(y) + x^4 \cdot m(x).$$

Note that since S is smooth at p_0 we have $u(0,0,0) \neq 0$, and since the tangent plane section $S \cap \mathbb{T}_{p_0} S$ has multiplicity 2 at p_0 we have $v(0,0) \neq 0$; rescaling the coordinates, we can assume $u(0,0,0) = v(0,0) = 1$. Note by contrast that we may have $m(0) = 0$; this will be the case exactly when $m_{p_0}(L_0 \cdot S) \geq 5$.

Now, we can just plug $(a + t, b + dt, c + et)$ in for (x, y, z) in this expression to write out $f|_L$, and hence the coefficients $\alpha_i(a, b, c, d, e)$. This is potentially messy, but in fact it will be enough to evaluate the differentials of the α_i at $(L_0.p_0)$—that is, at $(a, b, c, d, e) = (0, 0, 0, 0, 0)$—and so we can work modulo the ideal $(a, b, c, d, e)^2$. That said, we have

$$\begin{aligned} f|_L = f(a + t, b + dt, c + et) = {} & (c + et)u \\ & + (a + t)(b + dt)v \\ & + (b + dt)^3 l(b + dt) \\ & + (a + t)^4 m(a + t) \end{aligned}$$

and thus

$$\alpha_0 \equiv c \mod (a,b,c,d,e)^2,$$

$$\alpha_1 \equiv e+b \mod (c)+(a,b,c,d,e)^2,$$

$$\alpha_2 \equiv d \mod (b,c,e)+(a,b,c,d,e)^2$$

and finally

$$\alpha_3 \equiv 4a \cdot m(0) \mod (b,c,d,e)+(a,b,c,d,e)^2.$$

What we see from this is that *the differentials of* $\alpha_0, \dots, \alpha_3$ *at* (L_0, p_0) *are linearly independent, unless* $m(0) = 0$; or in other words, *if there is a 2-dimensional family of pairs* (L,p) *such that* $m_p(L \cdot S) \geq 4$*, then in fact we must have* $m_p(L \cdot S) \geq 5$ *for all such* (L,p). But we can carry out exactly the same argument again to show that if there is a 2-dimensional family of lines having contact of order 5 or more with S, then all these lines in fact have contact of order 6 or more with S, and so on. We conclude that if Γ has dimension 2 or more, then S must be ruled by lines; in other words, S must be a plane or a quadric.[1] □

13.3 The Cartesian approach to flexes

In our initial discussion of flexes in Section 9.6, we gave the curve $C \subset \mathbb{P}^2$ in question *parametrically*—that is, as the image of a map $\nu : \tilde{C} \to \mathbb{P}^2$ from a smooth curve $\tilde{C}$, the normalization of C, to $\mathbb{P}^2$. We defined flexes as the points $p \in \tilde{C}$ such that for some line $L \subset \mathbb{P}^2$ the multiplicity $m_p(\nu^* L) \geq 3$.

This definition does not work well in families of curves. As we shall see, when a smooth plane curve degenerates to one with a node, a certain number of the flexes approach the node; but according to the definition in Section 9.6, the nodal point will generally not be a flex, since in general neither branch of the node will have contact of order three or more with its tangent line. (A node such that one branch is flexed is called a *flecnode*; se Figure **??**.) Thus to track the behavior of flexes in families we need a different way of describing them. This necessitates a different name: we'll call the objects described here "flex lines" rather than flexes.

We define a *flex line* of $C \subset \mathbb{P}^2$ to be a pair (L,p) with $L \subset \mathbb{P}^2$ a line and a point $p \in L$ a point such that C and L intersect at p with multiplicity ≥ 3: that is, the set Γ of flex lines is the locus

$$\Gamma = \{(L,p) \in \mathbb{P}^{2^*} \times \mathbb{P}^2 \mid m_p(C \cdot L) \geq 3\}.$$

[1] This argument was shown to us by Francesco Cavazzani

Thus if C is the vanishing locus of a polynomial F, then (L, p) is a flex line if and only if the scheme cut out by the restriction of F to L has multiplicity at least 3 at p (or is equal to L). For example, if C is a general curve with a node at p, the tangent lines to the two branches are flex lines at the node. (We called this section the "Cartesian approach to flexes" because, as we shall see below, the equation F of C plays the central role rather than the parametrization.)

We hasten to remark that if $C \subset \mathbb{P}^2$ is smooth at p, then the two definitions will coincide, since C is—locally near p—its own normalization.

To compute the number of flexes on a curve defined by a homogeneous form F, we define Ψ to be the incidence correspondence

$$\Psi = \{(L, p) \in \mathbb{P}^{2*} \times \mathbb{P}^2 \mid p \in L\},$$

thought of as the universal line over $\mathbb{P}^{2*}$, and consider

$$\mathcal{E} = \mathcal{P}^2_{\Psi/\mathbb{P}^{2*}}(\pi_2^*\mathcal{O}_{\mathbb{P}^2}(d)),$$

a rank 3 vector bundle on the three-dimensional variety Ψ, whose fiber at a point (L, p) is

$$\mathcal{E}_{(L,p)} = H^0(\mathcal{O}_L(d)/\mathcal{I}^3_{p,L}(d)).$$

The homogeneous polynomial F gives rise to a section σ_F of $\mathcal{E}$ and the zeros of this section correspond to the flex lines of the corresponding plane curve $C = V(F)$. Thus the number of flex lines—when this number is finite, as is the case in characteristic 0 whenever C is reduced and does not contain a line—is the degree of $c_3(\mathcal{E}) \in A^3(\Psi)$.

Since the projection on the first factor expresses Ψ as the projectivization

$$\Psi = \mathbb{P}\mathcal{S} \to \mathbb{P}^{2^*}$$

of the universal subbundle $\mathcal{S}$ on $\mathbb{P}^{2^*}$, we can give a presentation of the Chow ring exactly as in the case of the universal line Φ over $\mathbb{G}(1,3)$ in Section 13.1.3. Letting $\sigma \in A^1(\mathbb{P}^{2*})$ be the hyperplane class, we have

$$A(\Psi) = A(\mathbb{P}^{2*})[\zeta]/(\zeta^2 - \sigma\zeta + \sigma^2),$$

where $\zeta \in A^1(\Phi)$ is the first Chern class of the line bundle $\mathcal{O}_{\mathbb{P}\mathcal{S}}(1)$. Recall moreover that the class ζ can also be realized as the pullback $\zeta = \beta^*\omega$, where $\beta : \Phi \to \mathbb{P}^2$ is the projection $(L, p) \mapsto p$ on the second factor, and $\omega \in A^1(\mathbb{P}^2)$ is the hyperplane class.

We can also evaluate the degrees of monomials of degree 3 in ζ and σ as before by using the Segre classes introduced in Section **??**, and in particular Proposition 12.3: we have

$$\deg(\zeta\sigma^2) = \deg(\zeta^2\sigma) = 1; \quad \text{and} \quad \deg(\zeta^3) = \deg(\sigma^3) = 0.$$

(We could also see these directly by observing that $\Psi \subset \mathbb{P}^{2*} \times \mathbb{P}^2$ is a hypersurface of bidegree $(1, 1)$, and the classes σ and ζ are the pullbacks of the hyperplane classes in the two factors.)

We can now calculate the Chern classes of $\mathcal{E}$ by applying the exact sequences (13.1) and using Whitney's Formula, and we get

$$c_3(\mathcal{E}) = d\zeta \cdot ((d-2)\zeta + \sigma) \cdot ((d-4)\zeta + 2\sigma).$$

Hence

$$\begin{aligned}\deg(c_3(\mathcal{E})) &= 2d(d-2) + d(d-4) + 2d \\ &= 3d(d-2).\end{aligned}$$

This shows that the number of flex lines, counted with multiplicity, is the same in the singular case as in the smooth case, whenever the number is finite. (Note that if $F = 0$ defines a non-reduced curve, or a curve containing a straight line as a component, the section defined by F vanishes in the wrong codimension.) The present derivation allows us to go further in two ways, both having to do with the behavior of flexes in families. In particular, it will permit us to solve Keynote Question (c) above.

13.3.1 Hyperflexes

We define a *hyperflex line* to a plane curve C similarly: it is a pair (L, p) such that L and C meet with multiplicity at least 4 at p. As with ordinary flex lines (and for the same reason) this definition is equivalent to the definition of a hyperflex given in Section 9.6 when the point p is a smooth point of C, but not in general: if a curve $C \subset \mathbb{P}^2$ has an ordinary flecnode at p (that is, two branches, one not a flex and the other a flex that is not a hyperflex) then the tangent line to the flexed branch of C at p will be a hyperflex line, though p is not a hyperflex in the sense of Section 9.6. Since a general pencil of plane curves will not include any elements possessing a flecnode (Exercise 13.31), this won't affect our answer to Keynote Question (c).

To describe the locus of hyperflex lines in a family of curves, we denote by $\mathbb{P}^N$ the space of plane curves of degree d, and consider the incidence correspondence

$$\Sigma = \{(L, p, C) \in \Psi \times \mathbb{P}^N \mid m_p(L \cdot C) \geq 4\}.$$

When $d \geq 3$, the fibers of the projection $\Sigma \to \Psi$ are linear spaces of dimension $N - 4$, from which we see that Σ is irreducible of dimension $N - 1$; in particular, it follows that a general curve $C \subset \mathbb{P}^2$ of degree $d > 1$ has no hyperflexes. Furthermore, since for $d \geq 4$ the general fiber of the projection $\Sigma \to \mathbb{P}^N$ is finite (see Exercise 13.30) the locus $\Xi \subset \mathbb{P}^N$ of curves that do admit a hyperflex is a hypersurface in $\mathbb{P}^N$ in this case. Keynote Question (c) is equivalent to asking for the degree of this hypersurface in case $d = 4$. We'll actually compute it for all d.

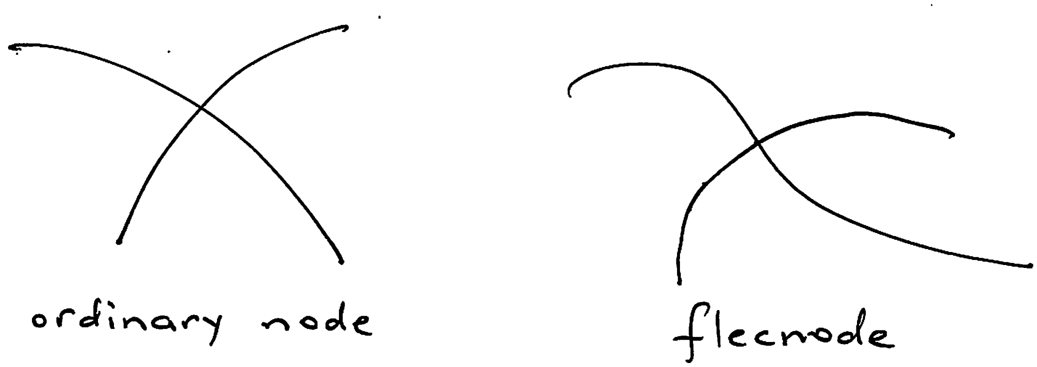

FIGURE 13.1. A flecnode is a node in which one branch has a flex at the node.

To do this we consider the three-dimensional variety $\Psi \subset \mathbb{P}^{2*} \times \mathbb{P}^2$ as above, and introduce the rank 4 bundle

$$\mathcal{H} = \mathcal{P}^3_{\Psi/\mathbb{P}^{2*}}(\pi_2^* \mathcal{O}^2_{\mathbb{P}}(d)),$$

whose fiber at a point (L, p) is

$$\mathcal{H}_{(L,p)} = H^0(\mathcal{O}_L(d)/\mathcal{I}^4_{p,L}(d)).$$

With this definition in hand, we consider a general pencil $\lambda F + \mu G$ of homogeneous polynomials of degree d on $\mathbb{P}^2$. The polynomials F, G give rise to sections σ_F, σ_G of $\mathcal{H}$, and the set of pairs (L, p) that are hyperflexes of some element of our pencil is the locus where these sections fail to be linearly independent. Thus the number of hyperflex lines, possibly counted with some multiplicities, is the degree of $c_3(\mathcal{H}) \in A^3(\Psi)$.

We do this as before: filtering the bundle $\mathcal{H}$ by order of vanishing, we arrive at the expression

$$c(H) = \big(1 + d\zeta\big)\big(1 + (d-2)\zeta + \sigma\big)\big(1 + (d-4)\zeta + 2\sigma\big)\big(1 + (d-6)\zeta + 3\sigma\big).$$

Thus

$$\begin{aligned} c_3(H) &= \big(18d^2 - 88d + 72\big)\zeta^2\sigma + \big(22d - 36\big)\zeta\sigma^2 \\ &= 18d^2 - 66d + 36 \\ &= 6(d-3)(3d-2). \end{aligned}$$

This gives zero when $d = 3$, as it should: a cubic with a hyperflex is necessarily reducible, and a general pencil of plane cubics won't include any reducible ones. We also remark that the number is meaningless in case $d = 1$ or $d = 2$, since every point on a line is a hyperflex, and a pencil of conics will contain a reducible conic equal to the union of two lines.

To show that the actual number of elements of a general pencil possessing hyperflexes is equal to the number predicted we have to verify that for general polynomials F and G the degeneracy locus $V(\sigma_F \wedge \sigma_G) \subset \Psi$ is reduced. We do this, as in the argument carried out in Section 9.4.1, in two steps: we first use an irreducibility argument to reduce the problem to exhibiting a single pair F, G of polynomials and a point $(L, p) \in \Psi$ such that $V(\sigma_F \wedge \sigma_G)$ is reduced at (L, p); then use a local calculation to show that there do indeed exist such F, G and (L, p).

For the first, a standard incidence correspondence suffices: we let $\mathbb{P}^N$ be the space of plane curves of degree d, and $\mathbb{G} = \mathbb{G}(1, N)$ the Grassmannian of pencils of such curves, and consider the locus

$$\Upsilon = \{(\mathcal{D}, L, p) \in \mathbb{G} \times \Psi \mid \text{some } C \in \mathcal{D} \text{ has a hyperflex line at } (L, p)\}.$$

The fiber of Υ over (L, p) is irreducible of dimension $2N - 5$: it's the Schubert cycle $\Sigma_3(\Lambda) \subset \mathbb{G}$, where $\Lambda = \{C \in \mathbb{P}^N \mid m_p(L \cdot C) \geq 4\}$ is the codimension 4 subspace of $\mathbb{P}^N$ consisting of curves with a hyperflex line at (L, p). It follows that Υ is irreducible of dimension $2N - 2 = \dim \mathbb{G}$. Now, if $\Upsilon' \subset \Upsilon$ is the locus of $(\mathcal{D}, L, p)$ such that $V(\sigma_F \wedge \sigma_G)$ is *not* reduced of dimension 0 at (L, p) (where $\mathcal{D}$ is the pencil spanned by F and G), then

$$\Upsilon' \neq \Upsilon \implies \dim \Upsilon' < 2N - 2$$

and it follows that Υ' cannot dominate $\mathbb{G}$.

Thus we need only exhibit a single F, G and (L, p) such that $V(\sigma_F \wedge \sigma_G)$ is reduced at (L, p). We do this using local coordinates. Choose $\mathbb{A}^2 \subset \mathbb{P}^2$ with coordinates x, y so that $p = (0, 0)$ is the origin and $L \subset \mathbb{A}^2$ is the line $y = 0$. Set $f(x, y) = F(x, y, 1)$ and $g(x, y) = G(x, y, 1)$.

As local coordinates on Ψ in a neighborhood of the point (L, p) we can take the functions x, y and b, where

$$p = (x, y) \quad \text{and} \quad L = \{(x + t, y + bt)\}_{t \in K}.$$

We can trivialize the bundle H in this neighborhood of (L, p), so that the section σ_F of H is given by the first four terms in the Taylor expansion of the polynomial $f(x + t, y + bt)$ around $t = 0$. Thus, for example, the section associated to the polynomial $f(x, y) = y + x^4$ (that is, $F(x, y, z) = yz^{d-1} + x^4 z^{d-4}$) is represented by the first four terms in the expansion of $y + bt + (x + t)^4$:

$$\sigma_F = (y + x^4, b + 4x^3, 6x^2, 4x)$$

and the general polynomial $g(x, y) = \sum a_{i,j} x^i y^j$ gives rise to the section σ_G represented by the vector

$$(a_{0,0} + a_{1,0}x + a_{0,1}y + \dots, a_{1,0} + a_{0,1}b + a_{1,1}y + 2a_{2,0}x + \dots, a_{2,0} + \dots, a_{3,0} + \dots).$$

(Here we're omitting terms in the ideal $(x, y, b)^2$.) The section $\sigma_F \wedge \sigma_G$ is given by the 2×2 minors of the matrix

$$\begin{pmatrix} y + x^4 & b + 4x^3 & 6x^2 & 4x \\ a_{0,0} + \dots & a_{1,0} + \dots & a_{2,0} + \dots & a_{3,0} + \dots \end{pmatrix}.$$

We have minors with linear terms $a_{1,0}y - a_{0,0}b$, $a_{3,0}y - 4a_{0,0}x$ and $a_{3,0}b - 4a_{2,0}x$, and for general values of the $a_{i,j}$ these are independent. This shows that the section $\sigma_F \wedge \sigma_G$ vanishes simply at p, as required. Thus:

Proposition 13.11. *In a general pencil of plane curves of degree d, exactly $6(d-3)(3d-2)$ will have hyperflexes; in particular, in a general pencil of plane quartic curves, exactly 60 members will have hyperflexes.*

We remark that the approach to counting flexes on a plane curve C taken in Chapter 9—introducing a bundle on C whose fiber at a point $p \in C$ is the space $H^0(\mathcal{O}_C(d)/\mathcal{I}^3_{p,C}(d))$—cannot be directly adapted to prove Proposition 13.11. We can certainly construct a relative version of the sheaf used in Chapter 9—that is, a sheaf $\mathcal{F}$ on the total space

$$\mathcal{C} = \{(t, p) \in \mathbb{P}^1 \times \mathbb{P}^2 \mid p \in C_t\}$$

whose fiber at a point $(t, p) \in \mathcal{C}$ with C_t smooth at p is the space

$$\mathcal{F}_{(t,p)} = H^0(\mathcal{O}_{C_t}(d)/\mathcal{I}^4_{p,C_t}(d));$$

that would be the sheaf $\mathcal{P}^3_{\mathcal{C}/\mathbb{P}^1}(\pi_2^*\mathcal{O}^2_{\mathbb{P}}(d))$. But this bundle isn't locally free at points $(t, p) \in \mathcal{C}$ where p is a singular point of C_t, and so we can't use this approach directly. There is a way around this problem: in Ran [2005a] and Ran [2005b], Ziv Ran shows that $\mathcal{F}$ extends to a locally free sheaf on a blow-up of $\mathcal{C}$, realized as a subscheme of the relative Hilbert scheme of $\mathcal{C}$ over $\mathbb{P}^1$. This yields an alternative treatment to the one given here.

13.3.2 Flexes on families of curves

We can also use the Cartesian approach to answer another question about flexes in pencils, one that sheds some more light on how flexes behave in families. Again, suppose that $\{C_t = V(t_0F + t_1G)\}_{[t_0,t_1] \in \mathbb{P}^1}$ is a general pencil of plane curves of degree d. The general member C_t of the pencil will have, as we've seen, $3d(d-2)$ flex points, and as t varies these points will sweep out another curve B in the plane. We can ask: what are the degree and genus of this curve? What is the geometry of this curve around singular points of curves in the pencil? We will answer these questions in this section and the next.

To this end, we again write

$$\Psi = \{(L, p) \in \mathbb{P}^{2*} \times \mathbb{P}^2 \mid p \in L\},$$

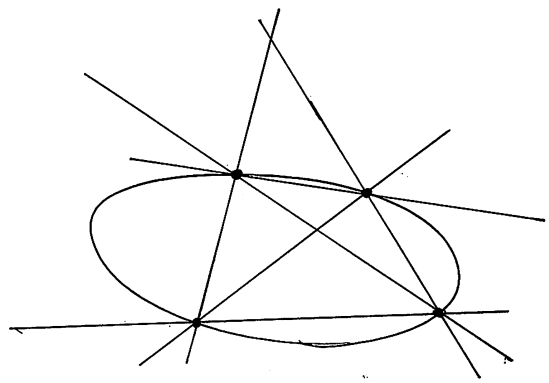

FIGURE 13.2. The singular elements of a pencil of conics are the pairs of lines joining the four base points.

and set

$$\Gamma = \{(t, L, p) \in \mathbb{P}^1 \times \Psi \mid m_p(L \cdot C_t) \geq 3\}.$$

We will describe Γ as the zero locus of a section of a rank 3 vector bundle on the four-dimensional variety $\mathbb{P}^1 \times \Psi$. For $d \geq 2$ we will show that Γ has the expected dimension 1, and we will ask the reader to show that in fact Γ is smooth by completing the sketch given in Exercise 13.34. This will allow us to determine not only the class of Γ (which will give us the degree of its image B under the projection $\mathbb{P}^1 \times \Psi \to \Psi \to \mathbb{P}^2$) but its genus as well. We will also describe the projection of Γ to $\mathbb{P}^1$, which tells how the flexes may come together as the curve moves in the pencil.

The case of a pencil of conics, $d = 2$, is easy to analyze directly, and already exhibits some of the phenomena involved. As we saw in Proposition 9.6 and the discussion immediately following, a general pencil of conics will have 3 singular elements, each consisting of two of the straight lines through 2 of the 4 base points of the pencil.

A smooth conic has no flexes, while the flex lines of a singular conic C are the pairs (L, p) with $p \in L \subset C$. Thus the curve B, consisting of points $p \in \mathbb{P}^2$ such that some (L, p) is a flex line, is the union of the singular members of the pencil—that is, the union of the 6 lines joining two of the four base points. As such it has degree 6, four triple points, and 3 additional double points. However, the points of the curve Γ "remember" the flex line to which they belong, so Γ is the *disjoint* union of the six lines—a smooth curve, which is the normalization of B. The singularities of B are typical of the situation of pencils of curves of higher degree, as we shall see: in general, B will have triple points at the base points of the pencil, and nodes at the nodes of the singular elements of the pencil. In the case of conics, the projection map $\Gamma \to \mathbb{P}^1$ has three nonempty fibers, each consisting of one of the singular members of the pencil. For general pencils of degree > 2 we shall see that the projection is a finite cover.

Returning to the general case, we again write $\mathcal{E}$ for the rank 3 vector bundle

$$\mathcal{E} = \mathcal{P}^2_{\Psi/\mathbb{P}^{2*}}(\pi_2^*\mathcal{O}_{\mathbb{P}^2}(d)).$$

Writing V for the two-dimensional vector space spanned by F and G, the sections σ_F and σ_G define a map of bundles

$$V \otimes \mathcal{O}_\Psi \to \mathcal{E}.$$

We now pull this map back to $\mathbb{P}^1 \times \Psi$ via the projection $\nu : \mathbb{P}^1 \times \Psi \to \Psi$. If $\mathbb{P}^1 = \mathbb{P}V$ is the projective line parametrizing our pencil, we also have a natural inclusion

$$\mathcal{O}_{\mathbb{P}V}(-1) \hookrightarrow V \otimes \mathcal{O}_{\mathbb{P}V}$$

which we can pull back to the product $\mathbb{P}^1 \times \Psi$ via the projection $\mu : \mathbb{P}^1 \times \Psi \to \mathbb{P}^1$. Composing these, we arrive at a map

$$\rho : \mu^*\mathcal{O}_{\mathbb{P}^1}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}^1 \times \Psi} \to \nu^*\mathcal{E};$$

over the point $(t, L, p) \in \mathbb{P}^1 \times \Psi$, this is the map that takes a scalar multiple of $t_0F + t_1G$ to its restriction to L (mod sections of $\mathcal{O}_L(d)$ vanishing to order 3 at p). In particular, *the zero locus of this map is the incidence correspondence* Γ.

Tensoring with the line bundle $\mu^*\mathcal{O}_{\mathbb{P}^1}(1)$, we can think of ρ as a section of the bundle $\mu^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \nu^*\mathcal{E}$; the class of Γ is thus given by the Chern class

$$[\Gamma] = c_3\big(\mu^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \nu^*\mathcal{E}\big) \in A^3(\mathbb{P}^1 \times \Psi).$$

Denote by η the clase of a point in $A^1(\mathbb{P}^1)$, or its pull back to $\mathbb{P}^1 \times \Psi$. Similarly, we use the notations η and σ, introduced as classes in $A(\Psi)$ above, to denote the pullbacks of these classes to $\mathbb{P}^1 \times \Psi$. With this notation we have

$$c\big(\mu^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \nu^*\mathcal{E}\big) = \big(1+\eta+d\zeta\big)\big(1+\eta+(d-2)\zeta+\sigma\big)\big(1+\eta+(d-4)\zeta+2\sigma\big).$$

Collecting the terms of degree 3 we get

$$\begin{aligned} c_3(\mu^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \nu^*\mathcal{E}) &= (3d^2 - 8d)\zeta^2\sigma + 2d\zeta\sigma^2 \\ &\quad + \eta\big((3d^2 - 12d + 8)\zeta^2 + (6d - 8)\zeta\sigma + 2\sigma^2\big). \end{aligned}$$

To find the degree of the curve $B \subset \mathbb{P}^2$ swept out by the flex points of members of the family, we intersect with the (pullback of the) class ζ of a line $L \subset \mathbb{P}^2$; we get

$$\begin{aligned} \deg(B) &= \zeta \cdot c_3(\mu^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \nu^*\mathcal{E}) \\ &= 6d - 6 \end{aligned}$$

Note that this yields the answer 6 in case $d = 2$, consistent with our previous analysis.

We can use the same constructions to find the geometric genus of the curve Γ. As we observed in Proposition 8.15, the normal bundle to Γ in the product $\mathbb{P}^1 \times \Psi$ is the restriction to Γ of the bundle $\mu^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \nu^*\mathcal{E}$. Since $\Psi \subset \mathbb{P}^{2^*} \times \mathbb{P}^2$ is a hypersurface of bidegree $(1,1)$, its canonical class is

$$-c_1(\mathcal{T}_\Psi) = K_\Psi = K_{\mathbb{P}^{2*} \times \mathbb{P}^2} + \zeta + \sigma = -2\zeta - 2\sigma;$$

it follows that

$$K_{\mathbb{P}^1 \times \Psi} = -2\eta - 2\zeta - 2\sigma.$$

By the calculation above,

$$c_1(\mathcal{E}) = 3\eta + (3d-6)\zeta + 3\sigma$$

and so we have

$$K_\Gamma = \big(\eta + (3d-8)\zeta + \sigma\big)|_\Gamma.$$

We've seen that the degree of $\eta|_\Gamma$ is $3d(d-2)$, and $\deg(\zeta|_\Gamma) = 6d - 6$; similarly, we can calculate

$$\deg(\sigma|_\Gamma) = (3d^2 - 12d + 8) + (6d - 8) = 3d^2 - 6d$$

Altogether, we have

$$\begin{aligned} 2g(\Gamma) - 2 &= \deg(K_\Gamma) \\ &= 24d^2 - 78d + 48 \end{aligned}$$

and so

$$g(\Gamma) = 12d^2 - 39d + 25.$$

Note that when $d = 2$ this yields $g(\Gamma) = -5$, as it should: as we saw, in this case Γ consists of the disjoint union of 6 copies of $\mathbb{P}^1$.

13.3.3 Geometry of the curve of flex lines

We will leave the proofs of most of the assertions in this section to Exercises 13.35-13.39, simply outlining the main points of the analysis.

We begin with the geometry of the plane curve B traced out by the flex points of the curves C_t—that is, the image of the curve Γ under projection to $\mathbb{P}^2$. We have already seen that the degree of B is $6d - 6$.

The singularities of B can be located as follows: at each base point p of the pencil, all members of the pencil are smooth. We will see in Exercise 13.35 that three members of the pencil have flexes at p, so that B has a triple point at each base point of the pencil. The only other singularities of B occur at points $p \in \mathbb{P}^2$ that are nodes of the curve C_t containing them.

As we've seen, at such a point the tangent lines to the two branches are each flex lines to C_t, so that map $\Gamma \to B$ is 2-1 there; as we'll verify in Exercise 13.36 the curve B will have a node there.

Since the projection $\Gamma \to B$ is the normalization, these observations give another way to derive the formula for the genus of Γ: There are in general d^2 base points of the pencil, and as we saw in Chapter 9 there will be $3(d-1)^2$ nodes of elements C_t of our pencil, so that the genus of Γ is

$$\begin{aligned} g(\Gamma) &= p_a(B) - 3d^2 - 3(d-1)^2 \\ &= \frac{(6d-7)(6d-8)}{2} - 3d^2 - 3(d-1)^2 \\ &= 12d^2 - 39d + 25. \end{aligned}$$

We can study the geometry of the curve Γ in another way as well, via the projection $\Gamma \to \mathbb{P}^1$ on the first factor. Since a general member of our pencil has $3d(d-2)$ flexes, Γ is a degree $3d(d-2)$ cover of the line $\mathbb{P}^1$ parametrizing our pencil. Where is this cover branched? The Plücker formula of Section 9.6.2 shows that if C_t is smooth it can fail to have exactly $3d(d-2)$ flexes only if it has a hyperflex, in which case the hyperflex counts as two ordinary flexes. Such hyperflexes are thus ordinary ramification points of the cover $\Gamma \to \mathbb{P}^1$.

That leaves only the singular elements of the pencil to consider, and this is where it gets interesting. By the formula of Section 9.6.2, a curve of degree d with a node has genus one less, and hence 6 fewer flexes (in the sense of that section) than a smooth curve of the same degree. If C_{t_0} is a singular element of a general pencil of plane curves then as $t \to t_0$ three of the flex lines of the curves C_t approach each of the tangent lines to the branches of C_{t_0} at the node (Exercise 13.39). Thus each of the tangent lines to the branches of C_{t_0} at the node is a ramification point of index 2 of the cover $\Gamma \to \mathbb{P}^1$.

We can put this all together with the Riemann-Hurwitz formula to compute the genus of Γ yet again: since there are $6(d-3)(3d-2)$ hyperflexes in the pencil, and $3(d-1)^2$ singular elements,

$$2g(\Gamma) - 2 = -2 \cdot 3d(d-2) + 6(d-3)(3d-2) + 4 \cdot 3(d-1)^2$$

and so

$$\begin{aligned} g(\Gamma) &= -3d(d-2) + 3(d-3)(3d-2) + 2 \cdot 3(d-1)^2 + 1 \\ &= 12d^2 - 39d + 25. \end{aligned}$$

13.4 Cusps of plane curves

As a final application we will answer the second keynote question of this chapter (in characteristic 0 only): How many curves in a general net of cubics in $\mathbb{P}^2$ have cusps? This will finally complete our calculation, begun in Section 1.2.5, of the degrees of loci in the space $\mathbb{P}^9$ of plane cubics corresponding to isomorphism classes of cubic curves. Solving this problem requires the introduction of a new class of vector bundles that represent a further generalization of the idea of the bundles of principle parts.

We should start by saying what we mean by a cusp. Recall that an *ordinary cusp* of a plane curve C over the complex numbers is a point p such that in an analytic neighborhood of p in the plane, the equation of C can be written as $y^2 - x^3 = 0$ in suitable (analytic) coordinates. If we were working over an algebraically closed field K other than the complex numbers, we could say instead that the completion of the local ring of C at p is isomorphic to $K[[x, y]]/(y^2 - x^3)$, and this is equivalent when $K = \mathbb{C}$. Similar generalizations can be made for many of the remarks below.

It is inconvenient to do enumerative geometry with ordinary cusps directly, because the locus of ordinary cusps in a family of curves is not closed: ordinary cusps can degenerate to various other sorts of singularities (as in the family $y^2 - tx^3 + x^n$, as $t \to 0$). For this reason we will define a *cusp* of a plane curve to be point where the Taylor expansion of the equation of the curve has no constant or linear terms, and where the quadratic term is a square (possibly zero). As will become clearer in the next section, this means that a cusp is a point at which the completion of the local ring of the curve, in some local analytic coordinate system, has the form

$$\widehat{\mathcal{O}}_{C,p} = K[[x, y]]/(ay^2 + \text{terms of degree at least 3}),$$

where a is a constant that may be equal to 0. From the point of view of a general net of curves of degree at least 3, the difference between an ordinary cusp and a general cusp, in our sense, is immaterial: Proposition 13.13 will show that no cusps other than ordinary ones appear.

It is interesting to ask questions about curves on other smooth surfaces besides $\mathbb{P}^2$. Most of the results of this section can be carried over to general nets of curves in any sufficiently ample linear series on any smooth surface, but we will not pursue this generalization.

13.4.1 Plane curve singularities

Before plunging into the enumerative geometry of cusps we pause to explain a little of the general picture of curve singularities.

Let $p \in C$ be a point on a reduced curve. In an analytic neighborhood of the point, C looks like the union of finitely many branches, each parametrized by a one-to-one map from a disc. Over the complex numbers these maps can be taken to be parametrizations by holomorphic functions of one variable; in general, this statement should be interpreted to mean simply that the completion $\widehat{\mathcal{O}}_{C,p}$ of the local ring $\mathcal{O}_{C,p}$ of C at p is reduced and the normalization of each of its irreducible components (the branches) has the form $K[[t]]$, where K is the ground field (if our ground field were not algebraically closed then the coefficient field might be a finite extension of the ground field.) These statements are part of the theory of completions; see Eisenbud [1995] Chapter ****.

It is a consequence of the Weierstrass preparation theorem that, over the complex numbers, two reduced germs of analytic curves are isomorphic if and only if the completions of their local rings are isomorphic, so we will use the analytic language although we will work with the completions. See Greuel et al. [2007] for more details.

The case of singularities of plane curves has an additional rich structure, and we will now focus on this case.

We say that curve $C \subset \mathbb{P}^2$ has a *double point* at a point p if the completion $\hat{\mathcal{O}}_{C,p}$ of the local ring $\mathcal{O}_{C,p}$ of C at p has the form $K[[x,y]]/g(x,y)$ where g has leading term of order exactly 2. Thus we will say that C has a double point if C is defined locally analytically near p by an equation with nonzero quadratic leading term. We will say that a double point is an A_n-singularity if, locally analytically in suitable (analytic) coordinates, it has equation $y^2 - x^{n+1} = 0$.

In characteristic 0 double points are easy to classify:

Proposition 13.12. *Over an algebraically closed field of characteristic 0 any double point of a plane curve C is an A_n singularity for some $n \geq 1$. If $n = 2m+1$ is odd, then C consists of two smooth branches meeting with multiplicity $m+1$, while if n is even then C is analytically irreducible at p.*

For example a double point $p \in C$ is an ordinary node (C has two smooth branches meeting transversely at p) if and only if it has, in suitable analytic coordinates, equation $y^2 - x^2 = 0$, and is thus an A_1 singularity. Similarly, an *ordinary cusp* is a point $p \in C$ with local analytic equation $y^2 - x^3 = 0$ (A_2 singularity), and an *ordinary tacnode*, is a point with local analytic equation $y^2 - x^4 = 0$ (A_3 singularity); this looks like two smooth curves simply tangent to one another at p.

Proof. We work in the power series ring $\mathbb{C}[[x,y]]$, and we must show that if a power series f has nonzero leading term then, after multiplication by a unit of $\mathbb{C}[[x,y]]$ and a change of variables, it can be written in the form

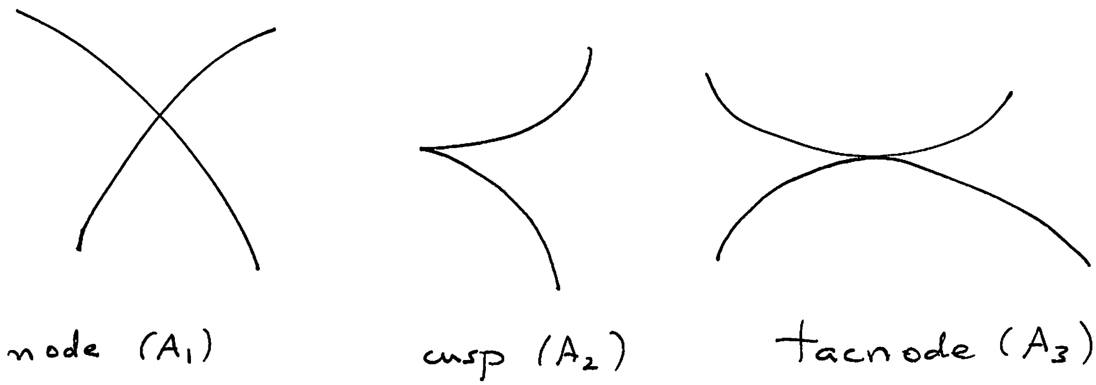

FIGURE 13.3. The simplest double points.

$y^2 - x^n$. Since any nonzero quadratic form over $\mathbb{C}$ may be written as $y^2 + ax^2$ with $a \in CC$, we may assume that f has the form $f = y^2 + g(x) + yg_1(x) + y^2 g_2(x, y)$. Multiplying f by the unit $1 - g_2(x, y)$, we reduce to the case $g_2 = 0$. Making a change of variable of the form $y' = y - g_1(x)$ (called a *Tschirnhausen transformation*) we can raise the order of vanishing of g_1; repeating this operation and taking the limit we may assume that $g_1 = 0$ as well. But if g has order n, then by Hensel's Lemma (Eisenbud [1995], Theorem ****), g has an n-th root of the form $x + ax^2 + \ldots$. We may take this power series to be a new variable, and after this change of variables we get $f(x, y) = y^2 - x^n$ as required.

If $n = 2m$ then $f = (y - x^m)(y + x^m)$, so C is the union of two smooth branches, and the multiplicity of their intersection, which may be defined as the length of

$$\mathbb{C}[[x, y]]/(y - x^m, y + x^m) = \mathbb{C}[[x, y]]/(y, x^m)$$

is m. On the other hand, if n is odd, then an elementary computation shows that we cannot write f as the product of two power series: that is, C is analytically irreducible. $\square$

There is much more to say about the classification of plane curve singularities over $\mathbb{C}$: For example, the intersection of C with a small ball $B \subset \mathbb{C}^2 = \mathbb{R}^4$ around p is, up to homeomorphism, a union of disks, one for each branch of C. Moreover the boundary of B, the 3-sphere S^3, meets C transversely. Since the intersection $C \cap S^3$ is a 1-dimensional compact manifold, it is a union of (knotted, linked) circles corresponding to the branches of C. This union is called the *link* of C. The knotting of one of the circles in the link is a measure of the singularity of the corresponding branch; the linking number of two of the circles is the intersection number of the two corresponding branches. The links that can appear in this way are precisely the *iterated torus links* satisfying a certain positivity condition coming from the canonical orientation of the complex numbers. The set $C \cap B$ is topologically embedded in B as the cone over the link, so the link determines C topologically in a neighborhood of p. Plane curves with a given topological type may still have analytic moduli, but these moduli spaces are still only partly understood. For these stories and much more, see for example Milnor [1968], Eisenbud and Neumann [1985] and Zariski [1986].

Returning to the family of plane curves, we can estimate the dimension of the locus of curves having certain types of singularities, at least when the degree of the curves is large compared with the complexity of the singularity (this is an open problem when the degree is small; see Greuel et al. [2007] for more information):

Proposition 13.13. *Let $\mathbb{P}^N$ be the space of plane curves of degree $d \geq k$. The set $\Delta_k \subset \mathbb{P}^N$ of curves with an A_k singularity has codimension k in $\mathbb{P}^N$. Its closure is irreducible, and contains in addition the loci $\Phi \subset \mathbb{P}^N$ of curves with a point of multiplicity 3 or more and $\Xi \subset \mathbb{P}^N$ of curves with a multiple component; in fact:*

$$\overline{\Delta_k} = \Phi \cup \Xi \cup \bigcup_{l \geq k} \Delta_l.$$

Thus we expect to see nodes in codimension 1, cusps in codimension 2 and tacnodes in codimension 3; all other singularities should occur in codimension 4 and higher.

The proposition explains the index k in the name A_k: plane curves of large degree d with an A_k singularity have codimension k in the space of all plane curves of degree d. There are also deep connections to the Dynkin diagrams "A_k" (which appear, for example, when one makes a resolution of singularities of the surface $z^2 = y^2 - x^{k+1}$, which is a double cover of the plane branched over a curve with an A_k singularity) and even a direct connection to the family of Lie groups of the same name. Moreover, these connections extend to other families of singularities (the "A,D,E" singularities) and beyond. For the beginnings of this huge theory, see Greuel et al. [2007].

13.4.2 Characterizing cusps

As in the case of the simpler problem of counting singular elements of a pencil of curves, the first thing we need to do to study the cusps in a net of plane curves is to linearize the problem. The difficulty arises from the fact that even after we specify a point $p \in \mathbb{P}^2$, it's not a linear condition on the curves in our linear system to have a cusp at p. It becomes linear, though, if we specify both the point p and a line $L \subset \mathbb{P}^2$ through p with which we require our curve to have intersection multiplicity at least 3. Thus we'll work on the universal line over $\mathbb{P}^{2*}$

$$\Psi = \{(L,p) \in \mathbb{P}^{2*} \times \mathbb{P}^2 \mid p \in L\},$$

which we used in Section 13.3 above. In the present circumstances, we also want to think of Ψ as a subscheme of the Hilbert scheme $\mathcal{H}_2(\mathbb{P}^2)$ parametrizing subschemes of $\mathbb{P}^2$ of degree 2. Specifically, it's the locus in $\mathcal{H}_2(\mathbb{P}^2)$ of subschemes of $\mathbb{P}^2$ supported at a single point: we associate to $(L,p) \in \Psi$ the subscheme $\Gamma = \Gamma_{L,p} \subset \mathbb{P}^2$ supported at p with tangent line $\mathbb{T}_p\Gamma = \mathbb{T}_pL \subset T_p\mathbb{P}^2$.

For a given point $(L,p) \in \Psi$, we want to express the condition that the curve $C = V(\sigma)$ associated to a section σ of a line bundle $\mathcal{L} := \mathcal{O}_{\mathbb{P}^2}(d)$ on $\mathbb{P}^2$ have a cusp at p with $m_p(C \cdot L) \geq 3$. This suggests that we introduce for each (L,p) the ideal $J_{L,p}$ of functions whose zero locus has such a cusp; that is, we set

$$J_{L,p} = m_p^3 + I_\Gamma^2$$

where $\Gamma = \Gamma_{L,p} \subset \mathbb{P}^2$ is the subscheme of degree 2 supported at p with tangent line L.

We want to construct a vector bundle $\mathcal{E}$ on Ψ whose fiber at a point (L,p) is

$$\mathcal{E}_{(L,p)} = H^0(\mathcal{L}/\mathcal{L} \otimes J_{L,p}).$$

To do this, consider the product $\Psi \times \mathbb{P}^2$, with projection maps π_1 and π_2 to Ψ and $\mathbb{P}^2$. Let $\Delta \subset \Psi \times \mathbb{P}^2$ be the graph of the projection map $\Psi \subset \mathbb{P}^{2*} \times \mathbb{P}^2 \to \mathbb{P}^2$—in other words,

$$\Delta = \{((L,p),q) \in \Psi \times \mathbb{P}^2 \mid p = q\}.$$

Likewise, let $\Gamma \subset B \times \mathbb{P}^2$ be the universal scheme of degree 2 over $\Psi \subset \mathcal{H}_2(\mathbb{P}^2)$. We then take

$$\mathcal{E} = \pi_{1*}\left(\pi_2^*\mathcal{L}/\pi_2^*\mathcal{L} \otimes (\mathcal{I}^3_{\Delta/\Psi\times\mathbb{P}^2} + \mathcal{I}^2_{\Gamma/\Psi\times\mathbb{P}^2})\right);$$

by the theorem on cohomology and base change (Theorem 6.6), this is the bundle we want.

A global section of the line bundle $\mathcal{L}$ gives rise to a section of $\mathcal{E}$ by restriction. Given a net $\mathcal{D}$ corresponding to a three-dimensional vector space $V \subset H^0(\mathcal{L})$ we get three sections of $\mathcal{E}$, and the locus in B where they fail to be independent—that is, where some linear combination is zero—is the locus of (p,ξ) such that some element of the net has a cusp at p in the direction ξ. In sum, observing that two elements of our net can't have cusps at the same point, and that a general cuspidal curve has a unique cusp, the (enumerative) answer to our question is the degree of the Chern class $c_3(\mathcal{E})$. In the remainder of this section we'll calculate this.

13.4.3 Solution to the enumerative problem

We start by recalling the description of the Chow ring $A(\Psi)$ of $\Psi \subset \mathbb{P}^{2*} \times \mathbb{P}^2$ from Section 13.3: we have

$$A(\Psi) = A(\mathbb{P}^{2*})[\zeta]/(\zeta^2 - \sigma\zeta + \sigma^2) = \mathbb{Z}[\sigma, \zeta]/(\sigma^3, \zeta^2 - \sigma\zeta + \sigma^2)$$

where $\sigma \in \Psi$ is the pullback of the hyperplane class in $\mathbb{P}^{2*}$ and ζ is the pullback of the hyperplane class in $\mathbb{P}^2$ (equivalently, if we view Ψ as the projectivization of the universal subbundle $\mathcal{S}$ on $\mathbb{P}^{2*}$, the first Chern class of the line bundle $\mathcal{O}_{\mathbb{P}\mathcal{S}}(1)$). The degrees of monomials of top degree 3 in σ and ζ are

$$\deg(\zeta\sigma^2) = \deg(\zeta^2\sigma) = 1; \quad \text{and} \quad \deg(\zeta^3) = \deg(\sigma^3) = 0.$$

Now, in order to find the Chern class of $\mathcal{E}$ we want to relate it to more familiar bundles. To this end, we observe that the inclusions

$$m_p^3 \hookrightarrow J_{L,p} \hookrightarrow m_p^2$$

and the corresponding quotients

$$\frac{\mathcal{L}_p}{m_p^2\mathcal{L}_p} \to \frac{\mathcal{L}_p}{J_{L,p}\mathcal{L}_p} \to \frac{\mathcal{L}_p}{m_p^3\mathcal{L}_p}$$

globalize to give us surjections of sheaves

$$\beta^*\mathcal{P}^2_{\mathbb{P}^2}(\mathcal{L}) \to \mathcal{E} \to \beta^*\mathcal{P}^1_{\mathbb{P}^2}(\mathcal{L}) \to 0$$

and a corresponding inclusion

$$\frac{\beta^*\mathcal{P}^2_{\mathbb{P}^2}(\mathcal{L})}{\mathcal{E}} \hookrightarrow \frac{\beta^*\mathcal{P}^2_{\mathbb{P}^2}(\mathcal{L})}{\beta^*\mathcal{P}^1_{\mathbb{P}^2}(\mathcal{L})} = \beta^*(\operatorname{Sym}^2\mathcal{T}^*_{\mathbb{P}^2} \otimes \mathcal{L}).$$

What is the image? It's the tensor product of $\mathcal{L}$ with the sub-line bundle of $\beta^*\operatorname{Sym}^2\mathcal{T}^*_{\mathbb{P}^2}$ whose fiber at each point (L,p) is the square of the linear form on $T_p\mathbb{P}^2$ vanishing on $T_pL \subset T_p\mathbb{P}^2$. In other words, the inclusion $T_pL \hookrightarrow T_p\mathbb{P}^2$ at each point $(L,p) \in \Psi$ gives rise to a sequence

$$0 \to \mathcal{N} \to \beta^*\mathcal{T}^*_{\mathbb{P}^2} \to \mathcal{T}^*_{\Psi/\mathbb{P}^{2*}} \to 0 \tag{13.2}$$

where $\mathcal{N}$ is the sub-line bundle of $\beta^*\mathcal{T}^*_{\mathbb{P}^2}$ whose fiber at (L,p) is the linear form on $T_p\mathbb{P}^2$ vanishing on $T_pL \subset T_p\mathbb{P}^2$ (we can think of $\mathcal{N}$ as the "relative conormal bundle" of the family $\Psi \subset \mathbb{P}^{2*} \times \mathbb{P}^2 \to \mathbb{P}^{2*}$). We have then an inclusion

$$\operatorname{Sym}^2\mathcal{N} \hookrightarrow \beta^*\operatorname{Sym}^2\mathcal{T}^*_{\mathbb{P}^2}$$

and correspondingly

$$\operatorname{Sym}^2 \mathcal{N} \otimes \beta^* \mathcal{L} \hookrightarrow \beta^*(\operatorname{Sym}^2 \mathcal{T}^*_{\mathbb{P}^2} \otimes \mathcal{L})$$

whose image is exactly $\beta^* \mathcal{P}^2_{\mathbb{P}^2}(\mathcal{L})/\mathcal{E}$.

We can put this all together to calculate the Chern class $c(\mathcal{E})$. To begin with, we know the classes of the bundle $\mathcal{P}^2_{\mathbb{P}^2}(\mathcal{L})$ from Proposition 9.7: we have

$$c(\beta^* \mathcal{P}^2_{\mathbb{P}^2}(\mathcal{L})) = (1 + (d-2)\zeta)^6 = 1 + 6(d-2)\zeta + 15(d-2)^2\zeta^2$$

Next, the Chern class of the line bundle $\mathcal{N}$ can be found from the sequence (13.2): we have

$$\begin{aligned} c_1(\mathcal{N}) &= c_1(\beta^* \mathcal{T}^*_{\mathbb{P}^2}) - c_1(\mathcal{T}^*_{\Psi/\mathbb{P}^{2*}}) \\ &= -3\zeta - (-2\zeta + \sigma) \\ &= -\sigma - \zeta. \end{aligned}$$

where the equality $c_1(\mathcal{T}^*_{\Psi/\mathbb{P}^{2*}}) = -2\zeta + \sigma$ comes from Theorem 13.4. Thus

$$c(\operatorname{Sym}^2 \mathcal{N} \otimes \beta^* \mathcal{L}) = 1 + (d-2)\zeta - 2\sigma$$

and since $\beta^* \mathcal{P}^2_{\mathbb{P}^2}(\mathcal{L})/\mathcal{E} \cong \operatorname{Sym}^2 \mathcal{N}$, the Whitney formula gives

$$\begin{aligned} c(\mathcal{E}) &= \frac{c\big(\beta^* \mathcal{P}^2_{\mathbb{P}^2}(\mathcal{L})\big)}{c(\operatorname{Sym}^2 \mathcal{N})} \\ &= \frac{1 + 6(d-2)\zeta + 15(d-2)^2\zeta^2}{1 - (2\sigma - (d-2)\zeta)} \\ &= \big(1 + 6(d-2)\zeta + 15(d-2)^2\zeta^2\big) \sum_{k=0}^{3} (2\sigma - (d-2)\zeta)^k. \end{aligned}$$

In particular, the third Chern class $c_3(\mathcal{E})$ is

$$c_3(\mathcal{E}) = (2\sigma - (d-2)\zeta)^3 + 6(d-2)\zeta(2\sigma - (d-2)\zeta)^2 + 15(d-2)^2\zeta^2(2\sigma - (d-2)\zeta)$$

and taking degrees we have

$$\begin{aligned} \deg c_3(\mathcal{E}) &= 30(d-2)^2 + 24(d-2) - 24(d-2)^2 - 12(d-2) + 6(d-2)^2 \\ &= 12d^2 - 36d + 24. \end{aligned}$$

We have thus proven the enumerative formula:

Proposition 13.14. *The number of cuspidal elements of a net $\mathcal{D}$ of curves of degree d on $\mathbb{P}^2$, assuming there are only finitely many and counting multiplicities, is*

$$12d^2 - 36d + 24 = 12(d-1)(d-2)$$

Of course, to answer Keynote Question (b), we have to verify that for a general net there are indeed only finitely many cusps, and that they all count with multiplicity 1. The first of these statements follows easily from the dimension count of Proposition 13.13. The second can be verified by explicit calculation in local coordinates, analogous to what we did to verify for example that hyperflexes in a general pencil occur with multiplicity 1; alternatively, we can use the method described in Section 13.4.4 below.

Note that the formula yields 0 in the cases $d = 1$ and 2, as it should. And in case $d = 3$, we see that a general net of plane quartics will have 72 cuspidal members, answering Keynote Question (b). Equivalently, we see that the locus of cuspidal plane cubics has degree 24 in the space $\mathbb{P}^9$ of all plane cubics, completing the analysis begun in Section 1.2.5.

A similar analysis could be made for the number of cusps (possibly with multiplicities) in a sufficiently general net of divisors associated to a sufficiently ample line bundle $\mathcal{L}$ on any surface S, yielding the answer

$$\deg(12\lambda^2 - 12\lambda c_1 + 2c_1^2 + 2c_2)$$

where $\lambda = c_1(\mathcal{L})$ and $c_i = c_i(\mathcal{T}_S)$. As always, this number is subject to the usual caveats: it is meaningful only if the number of cuspidal curves in the net is in fact finite; and in this case it represents the number of cuspidal curves counted with multiplicity (with multiplicity defined as the degree of the component of the zero-scheme of the corresponding section of $\mathcal{E}$ supported at (p, ξ)).

13.4.4 Another approach to the cusp problem

There is another approach to the problem of counting cuspidal curves in a linear system, one that gives a beautiful picture of the geometry of nets. It's not part of the overall logical structure of this book, so we'll run through the sequence of steps involved without proof; the reader who's interested can view supplying the verifications as an extended exercise.

To begin with, let S be a smooth projective surface and $\mathcal{L}$ a very ample line bundle; let $\mathcal{D} \subset |\mathcal{L}|$ be a general two-dimensional subseries, corresponding to the 3-dimensional vector subspace $V \subset H^0(\mathcal{L})$. We have a natural map

$$\varphi : S \to \mathbb{P}^2 = \mathbb{P}V^*$$

to the projectivization of the dual V^*; the preimages $\varphi^{-1} \subset S$ of the lines $L \subset \mathbb{P}V^*$ are the divisors $C \subset S$ of the linear system $\mathcal{D}$. If we want, we can think of the complete linear system $|\mathcal{L}|$ as giving an embedding of S in the larger projective space $\mathbb{P}^n = \mathbb{P}H^0(\mathcal{L})^*$, and the map φ as the projection of S corresponding to a general $(n-3)$-plane.

Now, the geometry of generic projections of smooth varieties is well-understood in low dimensions; in this case, it's the case ****need reference**** that

- The ramification divisor $R \subset S$ of the map φ is a smooth curve; and
- The branch divisor $B \subset \mathbb{P}V^*$ is the birational image of R, and has only nodes and ordinary cusps as singularities.

In fact, étale locally around any point $p \in S$, one of three things is true: either

i. the map is étale (if $p \notin R$);

ii. the map is simply ramified, that is, of the form $(x, y) \mapsto (x, y^2)$ (if p is a point of R not lying over a cusp of B); or

iii. the surface S is given, in terms of local coordinates (x, y) on $\mathbb{P}V^*$ around $\varphi(p)$, by the equation

$$z^3 - xz - y = 0.$$

(This is the picture around a point where 3 sheets of the cover come together; in a neighborhood of $\varphi(p)$ the branch curve is the zero locus of the discriminant $4x^3 - 27y^2$, and in particular has a cusp at $\varphi(p)$.)

The interesting thing about this set-up is that we have two plane curves associated to it, lying in dual projective planes: we have

(a) the branch curve $B \subset \mathbb{P}V^*$ of the map φ, and

(b) in the dual space $\mathbb{P}V$ parametrizing the divisors in the net $\mathcal{D}$, we have the *discriminant curve* $\Delta \subset \mathbb{P}V$, that is, the locus of singular elements of the net.

What ties everything together is the observation that *the discriminant curve* $\Delta \subset \mathbb{P}V$ *is the dual curve of the branch curve* $B \subset \mathbb{P}V^*$. To see this, note that if $L \subset \mathbb{P}V^*$ is a line transverse to B (in particular, not passing through any of the singular points of B), then the preimage $\varphi^{-1}(L) \subset S$ will be smooth: this is certainly true away from points of B, where the map φ is étale; and at a point $p \in L \cap B$ we can take local coordinates (x, y) on $\mathbb{P}V^*$ with L given by $y = 0$ and B by $x = 0$; at a point of $\varphi^{-1}(p)$ the cover $S \to \mathbb{P}V^*$ will either be étale or given by $z^2 = x$. A similar calculation shows conversely that if L is tangent to B at a smooth point, then $\pi^{-1}(L)$ will be singular.

At this point, we invoke the classical *Plücker formulas for plane curves*. These say that, if $C \subset \mathbb{P}^2$ is a plane curve of degree $d > 1$ and geometric genus g having δ nodes and κ cusps as its only singularities, and the dual curve C^* has degree d^* and δ^* nodes and κ^* cusps as singularities, then

$$d^* = d(d-1) - 2\delta - 3\kappa; \quad d = d^*(d^*-1) - 2\delta^* - 3\kappa^*, \text{ and}$$

$$g = \frac{(d-1)(d-2)}{2} - \delta - \kappa = \frac{(d^*-1)(d^*-2)}{2} - \delta^* - \kappa^*.$$

See, for example, Griffiths and Harris [1978], page 277 and ff. Given these, all we have to do is write down everything we know about the curves R, B and Δ. To begin with, we invoke the Riemann-Hurwitz formula for finite covers $f : X \to Y$: if η is a rational canonical form on Y with divisor D, the divisor of the pullback $f^*\eta$ will be the preimage of D, plus the ramification divisor $R \subset X$; thus

$$K_X = f^*K_Y + R \in A^1(X).$$

In our present circumstances, this says that

$$K_S = \varphi^* K_{\mathbb{P}V^*} \otimes \mathcal{O}_S(R);$$

since the pullback $\varphi^*\mathcal{O}_{\mathbb{P}V^*}(1) = \mathcal{L}$, we can write this as

$$K_S = \mathcal{L}^{-3}(R),$$

or in terms of the notation $c_1 = c_1(\mathcal{T}_S^*)$ and $\lambda = c_1(\mathcal{L})$, the class of R is

$$[R] = c_1 + 3\lambda \in A^1(S).$$

Among other things, this tells us the genus g of the curve R: since R is smooth, by adjunction we have

$$\begin{aligned} g &= \frac{R \cdot (R + K_S)}{2} + 1 \\ &= \frac{(c_1 + 3\lambda)(2c_1 + 3\lambda)}{2} + 1 \\ &= \frac{9\lambda^2 + 9\lambda c_1 + 2c_1^2}{2} + 1. \end{aligned}$$

It also tells us the degree d of the branch curve $B = \varphi(R) \subset \mathbb{P}V^*$: this is the intersection of R with the preimage of a line, so that

$$d = \lambda(c_1 + 3\lambda) = 3\lambda^2 + \lambda c_1.$$

Finally, we also know the degree e of the discriminant curve $\Delta \subset \mathbb{P}V$: this is the number of singular elements in a pencil, which we calculated back in Chapter 9; we have

$$e = 3\lambda^2 + 2\lambda c_1 + c_2.$$

We now have enough information to determine the number of cusps of Δ. Let δ and κ denote the number of nodes and cusps of Δ respectively. First off, the geometric genus of Δ is given by

$$g = \frac{(e-1)(e-2)}{2} - \delta - \kappa;$$

and the degree d of the dual curve is

$$d = e(e-1) - 2\delta - 3\kappa.$$

Subtracting twice the first equation from the second yields

$$\begin{aligned}\kappa &= 2g - d + 2(e-1)\\ &= 9\lambda^2 + 9\lambda c_1 + 2c_1^2 + 2 - (3\lambda^2 + \lambda c_1) + 2(3\lambda^2 + 2\lambda c_1 + c_2 - 1)\\ &= 12\lambda^2 + 12\lambda c_1 + 2c_1^2 + 2c_2,\end{aligned}$$

agreeing with our previous calculation. Note that this method also gives us a geometric sense of when a cusp "counts with multiplicity one;" in particular, if all the hypotheses above about the geometry of the map φ are satisfied, the count is exact.

Note that this also gives us a formula for the number of curves C in the net with two nodes. This is the number δ of nodes of the curve Δ, which we get by subtracting three times the equation for g above from the equation for d: this yields

$$\delta = d - 3g - e(e-1) + \frac{3}{2}(e-1)(e-2)$$

where d, e and g are given in terms of the classes λ, c_1 and c_2 by the equations above.

Note that the formula returns 0 in case $d = 1$ and $d = 2$, as it should; and in case $d = 3$ it gives 21—the degree of the locus of reducible cubics in the $\mathbb{P}^9$ of all cubics, as calculated in Section 1.2.5.

Exercises 13.51-13.53 describe an alternative (and perhaps cleaner) way of deriving the formula for the number of binodal curves in a net, via linearization.

13.5 Exercises

Exercise 13.15. Let $X \subset \mathbb{P}^4$ be a general hypersurface of degree $d \geq 6$. How many lines $L \subset \mathbb{P}^4$ will have a point of contact of order 7 with X?

Exercise 13.16. Let $S \subset \mathbb{P}^3$ be a general surface of degree $d \geq 2$. Using the dimension counts of Proposition 13.13 and incidence correspondences, show that

(a) For p in a dense open subset $U \subset S$, the intersection $S \cap \mathbb{T}_p S$ has an ordinary double point (a node) at p;

(b) There is a one-dimensional locally closed locus $Q \subset S$ such that for $p \in Q$ the intersection $S \cap \mathbb{T}_p S$ has a cusp at p;

(c) There will be a finite set Γ of points $p \in S$, lying in the closure of Q, such that the intersection $S \cap \mathbb{T}_p S$ has a tacnode at p; and

(d) S is the disjoint union of U, Q and Γ; that is, no singularities other than nodes, cusps and tacnodes appear among the plane sections of S.

Exercise 13.17. Let Φ be the universal line over $\mathbb{G}(1,3)$, and $\mathcal{E}$ the bundle on Φ introduced in Section 13.1. Let $L \subset \mathbb{P}^3$ be the line $X_2 = X_3 = 0$, and let $p \in L$ be the point $[1,0,0,0]$. By trivializing the bundle $\mathcal{E}$ in a neighborhood of $(L,p) \in \Phi$ and writing everything in local coordinates, show that the section of $\mathcal{E}$ coming from the polynomial $X_1^5 + X_0^4 X_2 + X_0^2 X_1^2 X_3$ has a simple zero at (L,p).

Exercise 13.18. Let $S \subset \mathbb{P}^3$ be a general surface of degree $d \geq 4$. Show that for any line $L \subset \mathbb{P}^3$ and any pair of distinct points $p, q \in L$,

(a) $m_p(S \cdot L) \leq 5$; and
(b) $m_p(S \cdot L) + m_q(S \cdot L) \leq 6$

Exercise 13.19. A point p on a smooth surface $S \subset \mathbb{P}^3$ is called an *Eckhart point* of S if the intersection $S \cap \mathbb{T}_p S$ has a triple point at p. Recall that in Exercise 9.51 we saw that a general surface $S \subset \mathbb{P}^3$ of degree d has no Eckhart points.

(a) Show that the locus of smooth surfaces that do have an Eckhart point is an open subset of an irreducible hypersurface $\Psi \subset \mathbb{P}^{\binom{d+3}{3}-1}$ in the space of all surfaces.
(b) Show that a general surface $S \subset \mathbb{P}^3$ that does have an Eckhart point has only one.
(c) Find the degree of the hypersurface Ψ.

Exercise 13.20. Consider a smooth surface $S \subset \mathbb{P}^4$. Show that we would expect there to be a finite number of hyperplane sections $H \cap S$ of S with triple points, and count the number in terms of the hyperplane class $\zeta \in A^1(S)$ and the Chern classes of the tangent bundle to S.

Exercise 13.21. Applying your answer to the preceding exercise, find the number of hyperplane sections of $S \subset \mathbb{P}^4$ with triple points in each of the following cases:

(a) S is a complete intersection of two quadrics in $\mathbb{P}^4$;
(b) S is a cubic scroll; and
(c) S is a general projection of the Veronese surface $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$.

In each case, can you check your answer directly?

Exercise 13.22. Consider a smooth surface $S \subset \mathbb{P}^5$. Show that we would expect there to be a one-dimensional family of hyperplane sections $H \cap S$ of S with triple points, and determine (again in terms of the hyperplane class

$\zeta \in A^1(S)$ and the Chern classes of the tangent bundle to S) the degree and genus of the curve $C \subset S$ traced out by these triple points.

Exercise 13.23. For $S \subset \mathbb{P}^3$ a general surface of degree d, find the degree of the surface swept out by the lines in $\mathbb{P}^3$ having a point of contact of order at least 4 with S.

Exercise 13.24. Let $S \subset \mathbb{P}^3$ be a general surface of degree d.

(a) Find the first Chern class of the bundle $\mathcal{F}$ in Section 13.2.1.

(b) Show that the curve Γ is smooth, and that the projection $\Gamma \to C$ is generically one-to-one.

(c) Using the preceding parts, find the genus of the curve Γ.

(d) Show, on the other hand, that the flecnodal curve of S is the intersection of S with a surface of degree $11d - 24$, and use this to calculate the arithmetic genus of C.

(e) Can you describe the singularities of the curve C? Do these account for the discrepancy between the genera of Γ and of C?

Exercise 13.25. Let $\mathbb{P}^N$ be the space of surfaces of degree $d \geq 4$ and $\Psi \subset \mathbb{P}^N$ the locus of surfaces containing a line. Show that the maximum possible number $M(d)$ of lines on a smooth surface $S \subset \mathbb{P}^3$ of degree d is at most the degree of Ψ, by considering the pencil spanned by S and a general second surface T. Is this bound better or worse than the one derived in Section 13.2.1?

Exercise 13.26. Show that for $d \geq 3$ the Fermat surface $S_d = V(x^d + y^d + z^d + w^d) \subset \mathbb{P}^3$ contains exactly $3d^2$ lines.

Exercise 13.27. For $F(x, y)$ any homogenenous polynomial of degree d, consider the surface $S \subset \mathbb{P}^3$ given by the equation

$$F(x, y) - F(z, w) = 0.$$

If α is the order of the group of automorphisms of $\mathbb{P}^1$ preserving the polynomial F (that is, carrying the set of roots of F to itself), show that S contains at least $d^2 + \alpha d$ lines. Hint: if L_1 and L_2 are the lines $z = w = 0$ and $x = y = 0$ respectively, and $\varphi : L_1 \to L_2$ any isomorphism carrying the zero locus $F(x, y) = 0$ to $F(z, w) = 0$, consider the intersection of S with the quadric

$$Q_\varphi = \bigcup_{p \in L_1} \overline{p\,\varphi(p)}.$$

Exercise 13.28. Using the preceding exercise, exhibit smooth surfaces $S \subset \mathbb{P}^3$ of degrees 4, 6, 8, 12 and 20 having at least 64, 180, 256, 864 and 1600 lines respectively.

Exercise 13.29. Verify that the Fermat quartic curve $C = V(x^4 + y^4 + z^4) \subset \mathbb{P}^2$ has 12 hyperflexes and no ordinary flexes.

Exercise 13.30. (a) Show that there exists a smooth curve with a hyperflex, and deduce that a general curve with a hyperflex has only finitely many. (Used in Section 13.3.1)

(b) Show that in fact a general curve with a hyperflex has only one.

Exercise 13.31. Recall that a node $p \in C$ of a plane curve is called a *flecnode* if one of the branches of C at p has contact of order 3 or more with its tangent line. Show that the locus, in the space $\mathbb{P}^N$ of all plane curves of degree $d \geq 4$, of curves with a flecnode is locally closed and irreducible of dimension $N - 2$.

Exercise 13.32. Verify that for a general pencil $\{C_t = V(t_0F + t_1G)\}$ of plane curves of degree d, if (L, p) is a hyperflex of some element C_t of the pencil then

(a) $m_p(C_t \cdot L) = 4$; that is, no line has a point of contact of order 5 or more with any element of the pencil;

(b) p is a smooth point of C_t; and

(c) p is not a base point of the pencil.

Using these facts, show that the degeneracy locus of the sections σ_F and σ_G of the bundle $\mathcal{H}$ introduced in Section 13.3.1 is reduced.

Exercise 13.33. Let $\{C_t = V(t_0F + t_1G)\}$ be a general pencil of plane curves of degree d. If $p \in \mathbb{P}^2$ is a general point, how many flex lines to members of the pencil $\{C_t\}$ pass through p?

For Exercises 13.34-13.39, we let $\{C_t = V(t_0F+t_1G)\}$ be a general pencil of plane curves of degree d,

$$\Psi = \{(L, p) \in \mathbb{P}^{2*} \times \mathbb{P}^2 \mid p \in L\}$$

the universal line and

$$\Gamma = \{(t, L, p) \in \mathbb{P}^1 \times \Phi \mid m_p(L \cdot C_t) \geq 3\}.$$

Let $B \subset \mathbb{P}^2$ be the image of Γ under the projection $\Gamma \to \Phi \to \mathbb{P}^2$; that is, the curve traced out by flex points of members of the pencil.

Exercise 13.34. First, show that Γ is indeed smooth, by showing that the "universal flex"

$$\Sigma = \{(C, L, p) : m_p(C \cdot L) \geq 3\} \subset \mathbb{P}^N \times \Phi$$

(where $\mathbb{P}^N$ is the space parametrizing all plane curves of degree d) is smooth, and invoking Bertini. Can you give explicit conditions on the pencil equivalent to the smoothness of Γ?

Exercise 13.35. If $p \in \mathbb{P}^2$ is a base point of the pencil, show that exactly three members of the pencil have a flex point at p, and that that the curve B has an ordinary triple point at p.

Exercise 13.36. If a point $p \in \mathbb{P}^2$ is a node of the curve C_t containing it, the tangent lines to the two branches are each flex lines to C_t, so that map $\Gamma \to B$ is 2-1 there. Show that the curve B has correspondingly a node at p.

Exercise 13.37. Finally, show that the triple points and nodes of B described in the preceding two exercises are the only singularities of B.

Exercise 13.38. Let C_t be an element of our pencil with a hyperflex (L, p). Show that the map $\Gamma \to \mathbb{P}^1$ is simply ramified at (t, L, p), and simply branched at t.

Exercise 13.39. Let C_t be an element of our pencil with a node p; let L_1 and L_2 be the tangent lines to the two branches of C_t at p. Show that $(t, p, L_i) \in \Gamma$, and that these are ramification points of weight 2 of the map $\Gamma \to \mathbb{P}^1$ (that is, each of the lines L_i is a limit of three flex lines of nearby smooth curves in our pencil, and these three are cyclically permuted by the monodromy in the family). Conclude that t is a branch point of multiplicity 4 for the cover $\Gamma \to \mathbb{P}^1$.

Exercise 13.40. Let $\{C_t\}$ be a general pencil of plane curves of degree d including a cuspidal curve C_0. (That is, let $C_0 = V(F)$ be a general cuspidal curve, $C_\infty = V(G)$ a general curve and $\{C_t = V(F + tG)\}$ the pencil they span.) As $t \to 0$, how many flexes of C_t approach the cusp of C_0? How about if C_0 has a tacnode?

Exercise 13.41. How many elements of a general net of plane curves of degree d will have flecnodes?

The following series of exercises (13.42-13.47) sketches a proof of Proposition 13.13.

Exercise 13.42. Suppose that $p \in C$ is an A_n singularity, for $n \geq 3$. Show that the blowup $C' = Bl_p C$ of C at p has a unique point q lying over p, and that $q \in C'$ is an A_{n-2} singularity. Conclude in particular that the normalization $\tilde{C} \to C$ of C at p has genus

$$p_a(\tilde{C}) = p_a(C) - \lfloor \frac{n+1}{2} \rfloor.$$

Exercise 13.43. Let S be a smooth surface and $C \subset S$ a curve with an A_{2n-1} singularity at p.

(a) Show that there is a unique curvilinear subscheme $\Gamma \subset S$ of degree n supported at p such that a local defining equation of $C \subset S$ at p lies in the ideal $\mathcal{I}_\Gamma^2$.

(b) If $\tilde{S} = Bl_\Gamma S$ is the blow-up of S along Γ, show that the proper transform $\tilde{C}$ of C in $\tilde{S}$ is smooth over p and intersects the exceptional divisor E transversely twice at smooth points of $\tilde{S}$.

(c) Conversely, show that if $D \subset \tilde{S}$ is any such curve, then the image of D in S has an A_{2n-1} singularity at p.

Exercise 13.44. Prove the analogue of Exercise 13.43 for A_{2n} singularities. This is the same statement, except that in the second and third parts that phrase "intersects the exceptional divisor E transversely twice at smooth points of $\tilde{S}$" should be replaced with "is simply tangent to the exceptional divisor E at a smooth point of $\tilde{S}$ and does not meet E otherwise".

Exercise 13.45. Let $\mathcal{L}$ be a line bundle on a smooth surface S, and assume that for any curvilinear subscheme $\Gamma \subset S$ of degree n supported at a single point, we have

$$H^1(L \otimes \mathcal{I}_\Gamma^2) = 0.$$

Show that the locus $\Delta_k \subset \mathbb{P}H^0(L)$ of curves in the linear series $|\mathcal{L}|$ with an A_k singularity is locally closed and irreducible of codimension k in $\mathbb{P}H^0(\mathcal{L})$ for all $k \leq 2n-2$

Exercise 13.46. Deduce from the above exercises the statement of Proposition 13.13.

Exercise 13.47. Show that if L is the n^{th} power of a very ample line bundle, then the condition $H^1(L \otimes \mathcal{I}_\Gamma^2) = 0$ is satisfied for any curvilinear subscheme $\Gamma \subset S$ of degree $n/2$ or less. Conclude in particular that if $\mathcal{D} \subset |L|$ is a general net in the complete linear series $|L|$ associated to the fourth or higher power of a very ample bundle then no curves $C \in \mathcal{D}$ has singularities other than nodes and ordinary cusps.

Exercise 13.48. Let S be a smooth surface, and L a line bundle on S. Let $B = \mathbb{P}\mathcal{T}_S$ be the projectivization of the tangent bundle of S, which we may think of as a parameter space for subschemes $\Gamma \subset S$ of degree 2 supported at a single point. Construct a vector bundle $\mathcal{E}$ on B whose fiber at a point $\Gamma \in B$ may be naturally identified with the vector space

$$\mathcal{E}_\Gamma = H^0(L/L \otimes \mathcal{I}_\Gamma^2)$$

Exercise 13.49. In terms of the description of the Chow ring $A(B)$ of $B = \mathbb{P}\mathcal{T}_S$ given in Section 13.4.2, calculate the top Chern class of the bundle constructed in Exercise 13.48 above.

Exercise 13.50. Using the preceding two exercises, find an enumerative formula for the number of curves in a 3-dimensional linear series $\mathcal{D} \subset |\mathcal{L}|$ that have a tacnode. If $S \subset \mathbb{P}^3$ is a smooth surface of degree d, apply this to find the number of plane sections with a tacnode.

The following three exercises describe a way of deriving the formula for the number of binodal curves in a net via linearization. We begin by introducing a smooth, projective compactification of the space of unordered pairs of points $p, q \in \mathbb{P}^2$: we set

$$\tilde{\Phi} = \{(L, p, q) : p, q \in L\} \subset \mathbb{P}^{2*} \times \mathbb{P}^2 \times \mathbb{P}^2$$

and let Φ be the quotient of $\tilde{\Phi}$ by the involution $(L, p, q) \mapsto (L, q, p)$. To put it differently, Φ consists of pairs (L, D) with $L \subset \mathbb{P}^2$ a line and $D \subset L$ a subscheme of degree 2; or differently still, Φ is the Hilbert scheme of subschemes of $\mathbb{P}^2$ with Hilbert polynomial 2. (Compare this with the description in Section 11.8.4 of the Hilbert scheme of conic curves in $\mathbb{P}^3$—this is the same thing, one dimension lower.)

Exercise 13.51. Observe that the projection $\Phi \to \mathbb{P}^{2*}$ expresses Φ as a projective bundle over $\mathbb{P}^{2*}$, and use this to calculate its Chow ring.

Exercise 13.52. Viewing Φ as the Hilbert scheme of subschemes of $\mathbb{P}^2$ of dimension 0 and degree 2, construct a vector bundle $\mathcal{E}$ on Φ whose fiber at a point D is the space

$$\mathcal{E}_{(L,p,q)} = H^0(\mathcal{O}_{\mathbb{P}^2}(d)/\mathcal{I}_D^2(d)).$$

(What would go wrong if, instead of using the Hilbert scheme Φ as our parameter space, we used the Chow variety—that is, the symmetric square of $\mathbb{P}^2$?) Express the condition that a curve $C = V(F) \subset \mathbb{P}^2$ be singular at p and q in terms of the vanishing of an associated section σ_F of $\mathcal{E}$ on $\mathcal{H}$ at (L, p, q)

Exercise 13.53. Calculate the Chern classes of this bundle, and derive accordingly the formula for the number of binodal curves in a net.

The last two exercises can be done by using the *second fundamental form* of a smooth variety $X \subset \mathbb{P}^n$; for more on this, see Harris [1992].

Exercise 13.54. Let $S \subset \mathbb{P}^3$ be an arbitrary smooth surface. Show that there can only be finitely many points $p \in S$ such that the tangent plane section $S \cap \mathbb{T}_p S$ has a point of multiplicity 3 or more at p.

Exercise 13.55. Let $S \subset \mathbb{P}^3$ be an arbitrary smooth surface and $p \in S$ a general point. Show that the tangent plane section $S \cap \mathbb{T}_p S$ has a node at p

****last version before rewrite = 1099****

14

Porteous' Formula

Keynote Questions

(a) Let $M \cong \mathbb{P}^{ef-1}$ be the space of $e \times f$ matrices, and let $M_k \subset M$ the locus of matrices of rank k or less. What is the degree of M_k?

(b) Let $Z \subset \mathbb{P}^{ef-1}$ be the Segre embedding of $\mathbb{P}^{e-1} \times \mathbb{P}^{f-1}. And let \pi : Z \to$ $\mathbb{P}^t$ be a general linear projection. For which $t > \dim Z$ is the image $\pi(Z)$ nonsingular? If t is the minimal such value, then the general projection to $\mathbb{P}^{t-1}$ will have only ordinary double points. How many?

(c) Let S be a smooth surface and $f : S \to \mathbb{P}^3$ a suitably general map. At how many points $p \in S$ will the map f fail to be an immersion?

(d) Let $C \subset \mathbb{P}^3$ be a smooth rational curve of degree d. How many lines $L \subset \mathbb{P}^3$ meet C four times?

14.1 Degeneracy loci

We saw in Chapters 7 and 12 that the Chern and Segre classes of a vector bundle $\mathcal{F}$ on a variety X can be characterized, when $\mathcal{F}$ is generated by global sections, as classes of the loci where a (suitably general) collection of sections of $\mathcal{F}$ fail to be independent, or fail to span $\mathcal{F}$. In other words, they are the classes of loci where a bundle map

$$\varphi : \mathcal{O}_X^{\oplus e} \to \mathcal{F}$$

determined by e general global sections of $\mathcal{F}$ fails to have maximal rank.

There are two ways to extend the usefulness of this characterization. First, we can replace $\mathcal{O}_X^{\oplus e}$ with an arbitrary vector bundle $\mathcal{E}$ and consider a map $\varphi : \mathcal{E} \to \mathcal{F}$. Second, instead of considering the locus where φ has less than maximal rank, we can consider the locus $M_k := M_k(\varphi)$ where φ has rank $\leq k$ for some given k. Such loci are called *degeneracy loci*.

We give $M_k(\varphi)$ the the structure of subscheme of X defined locally by the vanishing of the $(k+1) \times (k+1)$ minors of a matrix representation of φ. We will show that whenever the degeneracy locus has the "expected" codimension—which we define to be $(\operatorname{rank} \mathcal{E} - k)(\operatorname{rank} \mathcal{F} - k)$—the class of $[M_k(\varphi)]$ does not depend on φ, but can be written as a polynomial in the Chern classes of $\mathcal{E}$ and $\mathcal{F}$ in an elegant way. The result, from Porteous [1971] and Kempf and Laksov [1974], is called *Porteous' Formula*.

Because elements of positive degree are nilpotent in the Chow ring $A(X)$, the total Chern class $c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + \cdots$ is invertible. The formula is written in terms of the components of the ratio $c(\mathcal{F})/c(\mathcal{E}) \in A(X)$, a fact that we will explain below.

14.1.1 The general Porteous formula

Theorem 14.1 (Porteous' formula)**.** *Let $\varphi : \mathcal{E} \to \mathcal{F}$ be a map of vector bundles of ranks e and f on a smooth variety X, and write c_i for the degree i component of $c(\mathcal{F})/c(\mathcal{E}) \in A(X)$. If the scheme $M_k(\varphi) \subset X$ has codimension $(e-k)(f-k)$, then its class is given by*

$$[M_k(\varphi)] = \det \begin{pmatrix} c_{f-k} & c_{f-k+1} & \cdots & c_{e+f-2k-1} \\ c_{f-k1} & c_{f-k} & \cdots & c_{e+f-2k-2} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ c_{f-e+1} & \cdots & \cdots & c_{f-k} \end{pmatrix}.$$

To see why there should be a formula for $[M_k(\varphi)]$ in terms of the classes of $\mathcal{E}$ and $\mathcal{F}$ we linearize the problem by introducing more data. To say that the the map $\varphi_x : \mathcal{E}_x \to \mathcal{F}_x$ has rank $\leq k$ at a point $x \in X$ means that there is some k-dimensional subspace of $\mathcal{F}_x$ that contains the image. If we introduce the Grassmannian $\pi : G(k, \mathcal{F}) \to X$, this means that, at least set-theoretically, $M_k\varphi$ is the image under π of the locus where $\pi^*(\varphi) : \pi^*\mathcal{E} \to \pi^*\mathcal{F}$ factors through the tautological rank k subbundle $\mathcal{S}_k \subset \pi^*\mathcal{F}$; equivalently, it is the locus of points x where the composite map $\mu : \pi^*\mathcal{E} \to \pi^*\mathcal{F} \to \mathcal{Q} := (\pi^*\mathcal{F})/\mathcal{S}_k$ vanishes.

To simplify notation we will simply write $\mathcal{E}, \mathcal{F}$ instead of $\pi^*\mathcal{E}, \pi^*\mathcal{F}$. The composite map μ may be regarded as a section of the bundle $\mathcal{E}^* \otimes \mathcal{Q}$. If μ vanishes in codimension equal to $\operatorname{rank} \mathcal{E}^* \otimes \mathcal{Q} = e(f-k)$, then by **** its zero locus will be the top Chern class of $\mathcal{E}^* \otimes \mathcal{Q}$. Since the dimension of the Grassmannian is $\dim X + k(f-k)$, saying that the zero locus of μ has codimension $e(f-k)$ says that the dimension of the zero locus is $\dim X + k(f-k) - e(f-k) = \dim X - (e-k)(f-k)$, the expected dimension of $M_k\varphi$.

It follows that $\pi_*\left(c_{e(f-k)}(\mathcal{E}^* \otimes \mathcal{Q})\right)$ is at least supported on the same set as $[M_k\varphi]$, and we will see that they are equal as cycles. We will also see, in Section 14.2, how to compute the Chern class $c_{e(f-k)}(\mathcal{E}^* \otimes \mathcal{Q})$ in terms of the Chern classes of $\mathcal{E}$ and $\mathcal{Q}$, and thus in terms of the Chern classes of $\mathcal{E}^*$ and $\mathcal{F}$. In Section 14.3 we will explain how to push this expression down from the Grassmann bundle $G(k, \mathcal{F})$ to X. The result will be Porteous' formula.

This argument does not make it clear why the formula for $[M_k(\varphi)]$ depends only on the components of $c(\mathcal{F})/c(\mathcal{E})$, and doesn't require a full knowledge of $c(\mathcal{E})$ and $c(\mathcal{F})$. Indeed, in the proof we will give the formula emerges rather magically in this form. However, there is a simple heuristic explanation. Assume that the dual bundle $\mathcal{E}^*$ is generated by global sections and choose a surjection $\psi^* : \mathcal{O}_X^m \to \mathcal{E}^*$. It follows that

$$\widetilde{\varphi} : \mathcal{E} \xrightarrow{\begin{pmatrix} \psi \\ \varphi \end{pmatrix}} \mathcal{O}_X^m \oplus \mathcal{F}$$

is an injection of bundles. Let $\mathcal{Q} = \operatorname{coker} \widetilde{\varphi}$, and note that $c(\mathcal{Q}) = c(\mathcal{F})/c(E)$ by Whitney's formula. The original map $\varphi : \mathcal{E} \to \mathcal{F}$ is the composition of $\widetilde{\varphi}$ with the projection to $\mathcal{F}$, so it has rank $\leq k$ at the points where the fiber of $\widetilde{\varphi}(\mathcal{E})$ meets the fiber of $\mathcal{O}_X^m$ in dimension $\geq m-k$, or, equivalently, where the induced map $\mathcal{O}_X^m \to \mathcal{Q}$ drops rank by at least $m-k$. By the argument given above, the class of this locus depends only on the Chern classes of the bundles $\mathcal{O}_X^m$ and $\mathcal{Q}$—that is, only on the components of the ratio $c(\mathcal{F})/c(\mathcal{E})$.

There are two difficulties with the derivation of Porteus' formula outlined above. The first is that even when the scheme $M_k\varphi$ has the expected codimension the zero locus of the section σ may not. The second is that $M_k\varphi$ is defined as the cycle associated to the scheme defined (locally) by minors of a certain matrix, and it is far from clear why this gives the same cycle class as the pushforward of the class of the zero locus of σ.

We will overcome both these difficulties by pulling everything back from a "generic" situation, where both the scheme $M_k\varphi$ and the one defined by $\sigma = 0$ are reduced and irreducible of the expected codimension, and where the morphism between them is birational. To understand the passage to this generic situation, let

$$\varphi' : \mathcal{E} \xrightarrow{\begin{pmatrix} 1 \\ \varphi \end{pmatrix}} \mathcal{E} \oplus \mathcal{F}$$

be the map taking $\mathcal{E}$ onto the graph $\Gamma_\varphi \subset \mathcal{E} \oplus \mathcal{F}$ of φ. The original map φ is obviously the composition of φ' with the projection to $\mathcal{F}$. Since φ' is an inclusion of bundles, the universal property of the Grassmannian guarantees that there is a unique map $\alpha : X \to G(e, \mathcal{E} \oplus \mathcal{F})$ such that the pullback of the universal sub bundle $\mathcal{S}_e \subset \mathcal{E} \oplus \mathcal{F}$ on the Grassmannian is $\mathcal{E}$, and the map φ' is the pullback under α of the map $\psi : \mathcal{S}_e \to \mathcal{F}$ that is the composite of the tautological inclusion $\mathcal{S}_e \to \mathcal{E} \oplus \mathcal{F}$ with the projection $\mathcal{E} \oplus \mathcal{F}$ to $\mathcal{F}$. Thus $M_k\varphi = \alpha^{-1}(M_k\psi)$.

The map ψ is generic in the sense that over any open set in the base where $\mathcal{E}$ and $\mathcal{F}$ are trivial the Grassmannian becomes the product of X with the Grassmannian $G(e, E \oplus F)$, where E, F are vector spaces of dimensions e, f respectively. A basic open set of $G(e, E \oplus F)$ as defined in Chapter 3 is itself isomorphic to an affine space of dimension ef, and the map ψ is represented by a matrix whose entries are ef variables $x_{i,j}$—a generic matrix. In Section 14.4 we will explain the necessary facts about the ideals of minors of such generic matrices.

The points of $M_k(\psi)$ are the points $x \in G(e, \mathcal{E} \oplus \mathcal{F})$ such that the fiber of $\mathcal{S}_e$ meets the subbundle $\mathcal{E} \subset \mathcal{E} \oplus \mathcal{F}$, which is the kernel of the projection, in dimension at least $e - k$. This is the global version of the Schubert cycle $\Sigma := \Sigma_{(e-k)^{f-k}}(\mathcal{E})$ in notation parallel to that of Chapter 3; it is defined over any open subset of X where the bundles in question are trivial by the same determinantal condition that defines the corresponding Schubert cycle in the case of vector spaces, and a look at this formula shows that $\alpha^{-1}(\Sigma) = M_k(\varphi)$ as schemes.

We will see that we can apply Theorem 5.12, and conclude that $[M_k(\varphi)] = \alpha^*[\Sigma]$. Since elements of the Chow ring of the Grassmannian are polynomials in the classes of the universal sub and quotient bundles, and these pull back to $\mathcal{E}$ and $\mathcal{F}$ on X, there is a formula for $M_k(\varphi)$ in terms of $c(\mathcal{E})$ and $c(\mathcal{F})$ obtained by pulling back a universal formula from the Grassmannian.

Before proving Porteous' formula, we thus need three items: a formula for the top Chern class of a tensor product; a way of handling the pushforward map to $A(X)$ from the Chow ring of a Grassmann bundle over X; and an understanding of generic determinantal varieties. Each of these items is important in its own right, and we will spend the next three sections exploring them. But first we pause to point out an important special case.

14.1.2 Chern classes of a bundle with some sections

In Chapter 7, where we first defined Chern classes, we saw that if a bundle $\mathcal{E}$ of rank r has enough global sections, and if $\sigma_1, \dots, \sigma_{r-k+1}$ is a general collection of $r-k+1$ sections of $\mathcal{E}$, then the k^{th} Chern class $c_k(\mathcal{E})$ is the class of the degeneracy locus $V(\sigma_1 \wedge \cdots \wedge \sigma_r - k + 1) \subset X$. We showed more generally in Section **??** that if σ is *any* global section of $\mathcal{E}$ vanishing in codimension r, the top Chern class $c_r(\mathcal{E}) = [V(\sigma)]$—we don't need to assume $\mathcal{E}$ has any other sections, or that the section σ itself is generically transverse, as long as its zero locus has the expected dimension. Using Porteous' formula we can complete the story:

Corollary 14.2. *If $\mathcal{E}$ is a bundle of rank k on X and if $\sigma_1, \dots, \sigma_{r-k+1}$ is any collection of sections of $\mathcal{E}$ such that the degeneracy locus $V(\sigma_1 \wedge \cdots \wedge \sigma_{r-k+1}) \subset X$ has codimension k, then*

$$c_k(\mathcal{E}) = [V(\sigma_1 \wedge \cdots \wedge \sigma_{r-k+1})]\,. \quad \square$$

An immediate extension tells us about the locus where a map $\varphi : \mathcal{E} \to \mathcal{F}$ of bundles of rank e and $f \geq e$ respectively fails to be an inclusion of bundles: that is, the locus $M_{e-1}(\varphi)$.

Corollary 14.3. *Suppose that $\varphi : \mathcal{E} \to \mathcal{F}$ is a map of vector bundles on a smooth variety X, with $e := \operatorname{rank} \mathcal{E} \leq \operatorname{rank} \mathcal{F} =: f$. If φ fails to be a monomorphism in the expected codimension $f - e + 1$, then the degeneracy scheme $M_{e-1}(\varphi)$ has class equal to the component of degree $f - e + 1$ of the element*

$$\frac{c(\mathcal{F})}{c(\mathcal{E})} \in A(X).$$

$\square$

The formula in the Corollary is sometimes written as

$$[M_{e-1}] = \big[\frac{c(\mathcal{F})}{c(\mathcal{E})}\big]_{f-e+1}.$$

One way to interpret this formula is to note that if the degeneracy locus was empty then $\mathcal{F}/\mathcal{E}$ would be a vector bundle of rank equal to $\operatorname{rank} \mathcal{F} - \operatorname{rank} \mathcal{E} = f - e$, so the $(f-e+1)$-st Chern class of $\mathcal{F}/\mathcal{E}$ would be zero. By the Whitney formula, this class is the component of degree $f - e + 1$ of

$$c_{f-e+1}(\mathcal{F}/\mathcal{E}) = \left[\frac{c(\mathcal{F})}{c(\mathcal{E})}\right]_{f-e+1}.$$

Thus we can think of this class Corollary as measuring the failure of $\mathcal{F}/\mathcal{E}$ to be a vector bundle, and Porteous' formula identifies this measurement as the class of the degeneracy scheme.

14.2 The top Chern class of a tensor product

Let $\mathcal{E}, \mathcal{F}$ be vector bundles of ranks e, f on a smooth variety X as before. To carry out the proof of Porteous' formula we need a formula for the top Chern class of the tensor product $\mathcal{E} \otimes \mathcal{F}$. a If $\mathcal{E} = \sum_i \mathcal{L}_i$ and $\mathcal{F} = \sum_i \mathcal{M}_i$ are sums of line bundles then $\mathcal{E} \otimes \mathcal{F}$ is the sum of the $\mathcal{L}_i \otimes \mathcal{M}_j$. If we write $(\mathcal{L}_i) = 1 + e_i$ and $(\mathcal{M}_i) = 1 + f_i$ then $c(\mathcal{L}_i \otimes \mathcal{M}_j) = 1 + e_i + f_j$ so, by Whitney's formula

$$c(\mathcal{E} \otimes \mathcal{F}) = \prod_{i,j}(1 + e_i + f_j).$$

Using symmetric functions one could derive an expression for $c(\mathcal{E} \otimes \mathcal{F})$ terms of those of $\mathcal{E}$ and $\mathcal{F}$ alone, and, by the splitting principle, such an expression must hold for general vector bundles $\mathcal{E}, \mathcal{F}$ as well.

After tensoring the Chow ring with the rational numbers we could re-encode the Chern classes of $\mathcal{E}, \mathcal{F}$ in the Chern characters, $ch(\mathcal{E}), ch(\mathcal{F})$ and $ch(\mathcal{E} \otimes \mathcal{F}) = ch(\mathcal{E}) \cdot ch(\mathcal{F})$—the Chern character has already absorbed the work of the symmetric functions! However, to find the expressions directly, without denominators, is quite complicated except when e or f is 1 (treated already in Section ****) and in the cases of c_1 and c_{ef}, the first and the top Chern classes or

In the case of c_1, the work is trivial: the degree 1 component of $\prod_{i,j}(1 + e_i + f_j$ is $e(\sum_j f_j)) + f(\sum_i e_i)$, so

$$c_1(\mathcal{E} \otimes \mathcal{F}) = e \cdot c_1\mathcal{F} + f \cdot c_1\mathcal{E}.$$

The case of the top Chern class will occupy us for the rest of this section.

The key observation is that the degree ef component of

$$c(\mathcal{E} \otimes \mathcal{F}) = \prod_{1 \le i \le e, 1 \le j \le f} (1 + e_i + f_j)$$

is also the degree ef component of $\prod_{1 \le i \le e, 1 \le j \le f}(e_i + f_j)$, and this vanishes precisely when some f_j is equal to some $-e_i$. Thus we may think of the expression as the condition that the two monic polynomials $A = \prod_{i=1}^{e}(x + e_i)$ and $B = \prod_{j=1}^{f}(x - f_j)$ have a root in common.

This condition is given by the vanishing a single polynomial in the coefficients of A and B called the *resultant*, denoted $R(A, B)$. The resultant has been studied for a very long time—in fact Leibniz originally introduced determinants in the 17th century in order to compute the resultant of two quadratic polynomials. Moreover, the coefficients of B are the elementary symmetric functions in the f_i, which are the Chern classes of $\mathcal{F}$. Similarly, the coeffients of A are the negatives of the Chern classes of $\mathcal{E}$ (or equivalently and conveniently for us, the Chern classes of the dual bundle $\mathcal{E}^*$. Thus $R(A, B) = c_{ef}(\mathcal{E}^* \otimes \mathcal{F})$, at least up to a nonzero scalar.

We will use the expression for the resultant given by Sylvester in 1840—the "Sylvester determinant." It is discussed—along with its significance for elimination theory—at the beginning of Chapter 14 of Eisenbud [1995], and in many other places. The idea is that polynomials $A = x^e + a_1x^{e-1} + \cdots$ and $B = x^f + b_1x^{f-1} + \cdots$ have a common factor if and only if they have least common multiple of degree $< e + f$; that is, if and only if the $e + f$ polynomials obtained by multiplying A by $1, x, \ldots, x^{f-1}$ and multiplying B by $1, x, \ldots, x^{e-1}$ are linearly dependent. This condition (say over the field of complex numbers) is given by the vanishing of the determinant of the matrix

$$Syl = \begin{pmatrix} 1 & a_1 & \dots & \dots & a_e & 0 & 0 & \dots & \dots & 0 \\ 0 & 1 & a_1 & \dots & a_{e-1} & a_e & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & \dots & 1 & a_1 & \dots & \dots & \dots & a_e \\ 1 & b_1 & \dots & \dots & b_{f-1} & b_f & 0 & 0 & \dots & 0 \\ 0 & 1 & b_1 & \dots & b_f & b_{f-1} & b_f & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & \dots & \dots & b_{f-e+1} & \dots & \dots & & \dots b_f \end{pmatrix}.$$

Using the $f \times f$ identity block in the upper left corner we can simplify the determinant. An efficient trick for doing this is to form the inverse power series

$$\frac{1}{A(x)} = 1 + d_1x + \cdots,$$

and then multiply $Syl(A, B)$ on the right by the matrix

$$D = \begin{pmatrix} 1 & d_1 & d_2 & d_3 & d_4 & \dots & \dots & d_{e+f-1} \\ 0 & 1 & d_1 & d_2 & d_3 & \dots & \dots & d_{e+f-2} \\ 0 & 0 & 1 & d_1 & d_2 & \dots & \dots & d_{e+f-3} \\ 0 & 0 & 0 & 1 & d_1 & \dots & \dots & d_{e+f-4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & \dots & & & 1 & d_1 \\ 0 & 0 & \dots & \dots & & & 0 & 1 \end{pmatrix}.$$

obtaining

$$Syl \cdot D = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \dots & 1 & 0 & 0 & \dots & 0 \\ 1 & c_1 & \dots & \dots & \dots & c_f & c_{f+1} & \dots & c_{e+f-1} \\ 0 & 1 & c_1 & \dots & \dots & c_{f-1} & c_f & \dots & c_{e+f-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \dots & 0 & c_{f-e+1} & \dots & \dots & c_f \end{pmatrix}$$

where

$$\sum_{i=0}^{\infty} c_i x^i = \frac{B(x)}{A(x)}$$

as formal power series. Since $\det D = 1$, we see that $R(A, B)$ may be written as

$$\Delta_{e,f}(c) := \det \begin{pmatrix} c_f & c_{f+1} & \dots & c_{e+f-1} \\ c_{f-1} & c_f & \dots & c_{e+f-2} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ c_{f-e+1} & \dots & \dots & c_f \end{pmatrix}$$

Thus

$$\Delta_{e,f}(c) = \lambda \prod_{1 \le i \le e,\ 1 \le j \le f} (a_i + b_j)$$

for some scalar λ; and comparing the coefficients of $(\prod_i a_i)^f$ we see that $\lambda = 1$. Though we carried out the computation in the ring of power series over some field, the formula above is an identity in $\mathbb{Z}[x]$, and thus is valid for a_i, b_j in *any* ring; here the c_i are interpreted as certain polynomials in the symmetric functions of the a_i and of the b_j.

Putting all this together we have proven:

Theorem 14.4. *If $\mathcal{E}$ and $\mathcal{F}$ are vector bundles of ranks e and f on a variety X, then*

$$\begin{aligned} c_{ef}(\mathcal{E}^* \otimes \mathcal{F}) &= \Delta_{f,e}\left(\frac{c(\mathcal{F})}{c(\mathcal{E})}\right) \\ &= (-1)^{ef}\Delta_{e,f}\left(\frac{c(\mathcal{E})}{c(\mathcal{F})}\right). \end{aligned}$$

Here the second equality holds because $\mathcal{F}^* \otimes \mathcal{E} = (\mathcal{E}^* \otimes \mathcal{F})^*$. If $\mathcal{E} \subset \mathcal{F}$ were a sub bundle then, by Whitney's theorem, we have

$$c(\frac{\mathcal{F}}{\mathcal{E}}) = \frac{c(\mathcal{F})}{c(\mathcal{E})},$$

For this reason the first of the formulas above is sometimes written as $c_{ef}(\mathcal{E}^* \otimes \mathcal{F}) = \Delta_{f,e}(c(\mathcal{F}/\mathcal{E}))$, where now $c(\mathcal{F}/\mathcal{E})$ is simply defined as the sequence of coefficients of $c(\mathcal{F})/c(\mathcal{E}) \in A(X)$.

14.3 The direct image of a cycle on a Grassmannian bundle

For the proof of Porteous' formula we also need to evaluate the map $\pi_* : A(G(k,\mathcal{F})) \to A(X)$. From Section 11.7 we have a description of the Chow ring of the Grassmann bundle G:

$$A(G) = A(X)[\zeta_1, \dots \zeta_k]/\left(\left[\frac{c(\mathcal{F})}{1+\zeta+\zeta^2+\dots}\right]_l, l > n-k\right)$$

where $\zeta_1, \dots, \zeta_k$ are the Chern classes of the tautological subbundle S on G. From this we can deduce the pushforward of any monomial in the ζ_i, in much the same way as we did for projective bundles in the same chapter. We only need a small part of this information:

Proposition 14.5. *Let $\pi : G = G(k,\mathcal{F}) \to X$ be a Grassmann bundle, and $\zeta_i = c_i(S) \in A^i(G)$ the Chern classes of the universal subbundle S on G.*

(a) *If $\zeta^A = \zeta_1^{a_1} \cdot \dots \cdot \zeta_k^{a_k} \in A(G)$ is a monomial of weighted degree*

$$|A| = \sum ia_i < k(n-k),$$

then $\pi_*(\zeta^A) = 0$; and

(b) *If ζ^A is a monomial of weighted degree $|A| = k(n-k)$, then*

$$\pi_*(\zeta^A) = d \cdot [X]$$

where d is the degree of the intersection of ζ^A with a fiber of π; that is, the product σ^A of the corresponding Schubert cycles $(\sigma_{1^i})^{a_i}$ in the Grassmannian $G(k,n)$.

Proof. The first statement is clear from degree considerations: a cycle of codimension less than $k(n-k)$ in G has dimension strictly bigger than $\dim X$, so its pushforward is necessarily 0. Likewise, if $\alpha \in A^{k(n-k)}(G)$ is any class of codimension equal to $k(n-k)$, its pushforward can only be a multiple of the fundamental class of X; and by the push-pull formula this multiple is the degree of the restriction of α to a fiber of $G \to X$. $\square$

14.4 Generic determinantal varieties

In this section we present some facts about the commutative algebra of minors of matrix. We will give specific references for the results we need. For more on this rich subject, see for example de Concini et al. [1980], Bruns and Vetter [1988], and Bruns and Herzog [1993].

We have defined the "expected" codimension of $M_k(\varphi)$ to be $(\operatorname{rank}\mathcal{E}-k+1)(\operatorname{rank}\mathcal{F}-k+1)$. To explain this number, set $e = \operatorname{rank}\mathcal{E}$ and $f = \operatorname{rank}\mathcal{F}$, and consider the "generic" $f \times e$ matrix $\Phi = (x_{i,j})$ defined over a polynomial ring in $e \times f$ variables $x_{i,j}$. We may think of Φ as defining the "generic map of vector bundles"

$$\mathcal{O}_{\mathbb{P}^N}^{\operatorname{rank}\mathcal{E}}(-1) \xrightarrow{\Phi} \mathcal{O}_{\mathbb{P}^N}^{\operatorname{rank}\mathcal{F}}$$

with $N = (\operatorname{rank}\mathcal{E})(\operatorname{rank}\mathcal{F})-1$. The scheme $M_k(\Phi)$ is the scheme associated to the ideal of $(k+1)\times(k+1)$ minors of Φ. In fact this ideal is prime, and thus $M_k(\Phi)$ is a variety. It's closed points over the algebraically closed field K correspond to the set of matrices of rank $\leq k$ over K, so it is easy to analyze set-theoretically, and as we'll see below this immediately yields its irreducibility and codimension:

****move the following to Ch 7? We should admit that the same proof shows that all the Schubert cycles are CM****

Proposition 14.6. *Let $\Phi = (x_{i,j})$ be a generic $f \times e$ matrix as above. For each $k = 1, \ldots, \min(e, f)$ the ideal generated by the $(k+1)\times(k+1)$ minors of Φ is prime of codimension $(e-k)(f-k)$. If $\varphi : \mathcal{E} \to \mathcal{F}$ is any map of vector bundles of ranks e and f respectively then the codimension of any irreducible (isolated) component of $M_k(\varphi)$ has codimension $\leq (e-k)(f-k)$. If $M_k(\varphi)$ has the expected codimension, $(e-k)(f-k)$, and X is smooth (or even Cohen-Macaulay) then $M_k(\varphi)$ is Cohen-Macaulay.*

Notes on the proof. We can easily see at least that $M_k(\Phi)$ is irreducible and of the given codimension: Regarding it as the set of matrices of rank $\leq k$, we see that it is the closure of the set of matrices of rank exactly k, and is thus the closure of a single orbit of the group $GL(e, K) \times GL(f, K)$ acting by left and right multiplication on the vector space of matrices. This shows that $M_k(\Phi)$ is irreducible. Moreover, a generic matrix in $M_k(\Phi)$ is determined by it image, which is a subspace of K^f of dimension k—that is, an element of the Grassmannian $G(k,f)$, which has dimension $k(f-k)$—together with an element of $Hom(K^e,K^k)$, a space of dimension ek. It follows that the dimension of $M_k(\Phi)$ is $k(f-k)+ek = ef-(e-k)(f-k)$, proving that $M_k(\Phi)$ has codimension k.

The proof that the scheme $M_k(\Phi)$ is reduced is harder. The simplest proofs work by constructing a basis of the polynomial ring modulo I—see for examplede Concini et al. [1980] or Bruns and Herzog [1993] ****give Thm ref**** —and we will not reproduce them here The bound on the codimension of the components of $M_k(\varphi)$ follows from the ones before together with Serre's Generalized Principle Ideal Theorem 0.2, but the original proof, given by Eagon and Northcott [1962], is much simpler. A do-it-yourself version with hints may be found in **??**, Exercise 10.9.

The last statement, the Cohen-Macaulay property, is the deepest part of the Proposition. It was first proven in Hochster and Eagon [1971] in the context of invariant theory, but there is an easier proof using ideas related to Gröbner bases; see for example De Concini et al. [1982]. □

The variety $M_1(\Phi)$ defined by the 2×2 minors of Φ is already familiar: regarded as a projective variety, it is the Segre variety $Z \subset \mathbb{P}^{ef-1}$, the product of $\mathbb{P}^{e-1}$ and $\mathbb{P}^{f-1}$ embedded by the line bundle $\mathcal{L} := \pi_1^*(\mathcal{O}_{\mathbb{P}^{e-1}}(1)) \otimes \pi_2^*(\mathcal{O}_{\mathbb{P}^{f-1}}(1))$. Indeed, as a projective variety, $M_1(\Phi)$ has dimension $(ef-1)-(e-1)(f-1) = (e-1)+(f-1) = \dim \mathbb{P}^{e-1}\times\mathbb{P}^{f-1}$, so it is enough to show that $M_1(\Phi)$ contains the Segre variety. But if y_1, y_2 are sections of $\mathcal{O}_{\mathbb{P}^{e-1}}(1)$ and z_1, z_2 are sections of $\mathcal{O}_{\mathbb{P}^{f-1}}(1)$, then by commutativity

$$(y_1\otimes z_1)(y_2\otimes z_2) = y_1y_2\otimes z_1z_2 = (y_1\otimes z_2)(y_2\otimes z_1).$$

as sections of

$$\mathcal{L}^2 = \mathcal{L} := \pi_1^*(\mathcal{O}_{\mathbb{P}^{e-1}}(2)) \otimes \pi_2^*(\mathcal{O}_{\mathbb{P}^{f-1}}(2)).$$

Written in terms of the basis $x_{i,j} = y_i\otimes z_j$ of $H^0(\mathcal{L})$, this gives the vanishing on Z of the 2×2 minor $x_{1,1}x_{2,2} - x_{1,2}x_{2,1}$.

The other varieties $M_k(\Phi)$ have a related interpretation. A matrix of rank exactly k can be written as the sum of k distinct rank 1 matrices (to see this, choose bases so that the matrix is diagonal). Conversely, the sum of k rank 1 matrices has rank at most k. But a sum of vectors v_i, considered as a point in projective space, represents a point in the linear span of the v_i. This shows that, set-theoretically at least, $M_k(\Phi)$ is the closure of the union of the $(k-1)$-secant k-planes to the Segre variety Z.

There is one more result about generic matrices that we will need.

Proposition 14.7. *If*

$$S^r \xrightarrow{B=(b_{i,j})} S^q \xrightarrow{A=(a_{i,j})} S^p$$

are maps defined by matrices of indeterminates over a polynomial ring $S = K[\{a_{i,j}\},\{b_{i,j}\}]$ *then the entries of the product matrix* AB *generate a radical ideal* I*, with irreducible components* $I + I(M_{p'}(A)) + I(M_{q'}(B)$ *where* $0 \le p' \le p,\ 0 \le r' \le r$*, and* $p' + r' = q$*. In particular, after inverting any* $p \times p$ *minor of* A *the ideal* I *becomes prime.*

Notes on the proof. Note that this is clear set-theoretically: if A, B had entries in the ground field K, and $AB = 0$, then $\operatorname{rank} A + \operatorname{rank} B \le q$. The variety defined by the condition $AB = 0$ (that is, by the ideal generated by the entries of the matrix AB) is called a *variety of complexes of length 2*, since, restricted to this variety, the sequence of modules in the Proposition is the generic complex of free modules of the given ranks. For the full statement see for example Tchernev [2001]. $\square$

14.5 Proof of Porteous' formula

Again let $\varphi : \mathcal{E} \to \mathcal{F}$ be a map of vector bundles of ranks e and f on a smooth variety X, and suppose that the codimension of $M_k(\varphi)$ is $(e-k)(f-k)$. Let $\varphi' : \mathcal{E} \to \mathcal{E} \oplus \mathcal{F}$ be the map sending $\mathcal{E}$ to the graph of φ, and let $\alpha : X \to G(e, \mathcal{E} \oplus \mathcal{F})$ be the unique section of the Grassmann bundle such that φ' is the pull-back of the inclusion of the tautological bundle subbundle $\mathcal{S}_e$ on $G(e, \mathcal{E} \oplus \mathcal{F})$. Let $\psi : \mathcal{S}_e \to \mathcal{F}$ be the composite of the tautological inclusion $\mathcal{S}_e \subset \mathcal{E} \oplus \mathcal{F}$ with the projection $\mathcal{E} \oplus \mathcal{F} \to \mathcal{F}$.

Since $\alpha^*(\psi) = \varphi$ we have $\alpha^{-1}(M_k(\psi)) = M_k(\varphi)$, and since these are Cohen-Macaulay varieties, it follows by Theorem 5.12 that $\alpha^*[M_k(\psi)] = [\alpha^{-1}(M_k(\psi)]$. Thus it suffices to prove Porteous' formula for the class of $M_k(\psi)$ in the variety $X := G(e, \mathcal{E} \oplus \mathcal{F})$ with the same value of k as before.

We now consider the further Grassmann bundle $\pi : G(k, \mathcal{F}) \to X$, and we write $\mathcal{Q}$ for the tautological quotient bundle (of rank $f-k$) on $G(k, \mathcal{F})$. Let $\mu \in \operatorname{Hom}(\pi^*\mathcal{E}, Q) = \mathcal{E}^* \otimes \mathcal{Q}$ be the section corresponding to the composite of the map ψ pulled back to $G(k, \mathcal{F})$ and the tautological map $\nu : \mathcal{F} \to C\mathbb{Q}$. As explained in Section **??**, the locus $M_k(\psi) \subset X$ is the image $\pi(Z)$ of the scheme $Z \subset G(k, \mathcal{F})$ defined by the vanishing of μ.

Since we are in the situation with $X = G(e, \mathcal{E} \oplus \mathcal{F})$, the map ψ is locally given by a generic matrix, and thus $M_k(\psi)$ is reduced and irreducible. and the locus $M_{k-1}(\psi)$ is a proper subvariety (it has codimension $(e-k+1)(f-k+1) - (e-k)(f-k) = e+f-2k$ in $M_k(\psi)$.) Over a point $x \in M_k(\psi) \backslash M_{k-1}(\psi)$ there is a unique subbundle of rank k containing the image of ψ, and thus a unique point of Z lying over x. Since $M_k(\psi) \setminus M_{k-1}(\psi)$ is dense in $M_k(\psi)$, it follows that the map $Z \to M_k(\psi)$ is, set-theortically, generically one to one. In particular, the codimension of Z is

$$\begin{aligned}\dim G(k, \mathcal{F}) - \dim Z &= \dim G(k, \mathcal{F}) - \dim M_k(\psi)\\ &= \dim G(k, \mathcal{F}) - (\dim X - (e-k)(f-k)\\ &= \dim X + k(f-k) - \dim X + (e-k)(f-k)\\ &= e(f-k)\\ &= \operatorname{rank}(\mathcal{E}^* \otimes \mathcal{Q}).\end{aligned}$$

Thus $[Z]$ is $c_{e(f-k)}(\mathcal{E}^* \otimes \mathcal{Q})$, the top Chern class of this bundle.

We now wish to show that the variety $M_k(\psi)$ has class equal to $\pi_*[Z]$. Since $M_k(\psi)$ is reduced and irreducible, and the map π is generically one-to-one on Z, it suffices to show that Z is reduced and irreducible.

If we restrict to an open subset of X over which the bundles $\mathcal{E}$ and $\mathcal{F}$ become trivial, and then to an open subset $U \cong X \times \mathbb{A}^{k(f-k)}$, the map $\nu : \mathcal{F} \to \mathcal{Q}$ is given by a second generic $k \times (f-k)$ matrix with one of its $k \times k$ minors inverted. By Proposition 14.7, the entries of the composite matrix $\nu\psi$ define a reduced irreducible variety, proving that $M_k(\psi) = \pi_*[Z]$ as desired.

From Theorem 14.4 we see that

$$\begin{aligned}[Z] &= c_{e(f-k)}(\pi^*\mathcal{E}^* \otimes \mathcal{Q})\\ &= \Delta_{f-k,e}\left(\frac{c(\mathcal{Q})}{c(\pi^*\mathcal{E})}\right)\\ &= (-1)^{e(f-k)}\Delta_{e,f-k}\left(\frac{c(\pi^*\mathcal{E})}{c(\mathcal{Q})}\right).\end{aligned}$$

We must now evaluate $\pi_*([Z])$. At first glance this seems daunting: when we expand either of the determinantal expressions above for z, we get monomials of degree bigger than $k(n-k)$ in the Chern classes of the tautological bundles $\mathcal{S}_k \subset \mathcal{F}$ and $\mathcal{Q} = \mathcal{F}/\mathcal{S}_k$ on $G(k, \mathcal{F})$, and we have not given a computation for their direct images, which is complicated. But we're in luck: the exact sequence

$$0 \to \mathcal{S}_k \to \pi^*\mathcal{F} \to \mathcal{Q} \to 0$$

gives

$$c(\mathcal{Q}) = \frac{c(\pi^*\mathcal{F})}{c(S)},$$

and applying this to the second expression for $[Z]$ above we have

$$z = (-1)^{e(f-k)}\Delta_{e,f-k}\left(\frac{c(\pi^*\mathcal{E})}{c(\pi^*\mathcal{F})}c(S)\right).$$

The key observation is that this is the determinant of an $(n-k) \times (n-k)$ matrix, and each entry of the matrix is a linear combination, with coefficients in $A(X)$, of the Chern classes $\zeta_1, \dots, \zeta_k$ of S. When we expand the determinant the degree of each product in $A(X)$ is at most $k(n-k)$. Moreover, the term of degree $k(n-k)$ comes from taking the ζ_k term in each entry. By the first part of Proposition 14.5 any term of degree strictly less than $k(n-k)$ has direct image equal to zero in $A(X)$; so only the ζ_k term in each entry contributes to the direct image of the determinant.

If $c \in A(X)$ we will write $[c]_m$ for the component of c of degree m. Also, if $B = (b_{i,j})$ is a square matrix, we will write $|b_{ij}|$ for $\det(b_{i,j})$.

We can apply the push-pull formula $\pi_*(\pi^*\alpha \cdot \beta) = \alpha \cdot \pi_*\beta$ to write

$$\pi_* \begin{vmatrix} \left[\frac{c(\pi^*\mathcal{E})}{c(\pi^*\mathcal{F})}c(S)\right]_e & \dots & \dots & \left[\frac{c(\pi^*\mathcal{E})}{c(\pi^*\mathcal{F})}c(S)\right]_{e+f-k+1} \\ \vdots & \dots & \dots & \vdots \\ \vdots & \dots & \dots & \vdots \\ \left[\frac{c(\pi^*\mathcal{E})}{c(\pi^*\mathcal{F})}c(S)\right]_{e-f+k+1} & \dots & \dots & \left[\frac{c(\pi^*\mathcal{E})}{c(\pi^*\mathcal{F})}c(S)\right]_e \end{vmatrix}$$

$$= \pi_* \begin{vmatrix} \left[\frac{c(\pi^*\mathcal{E})}{c(\pi^*\mathcal{F})}\right]_{e-k} \zeta_k & \dots & \dots & \left[\frac{c(\pi^*\mathcal{E})}{c(\pi^*\mathcal{F})}\right]_{e+f-2k+1} \zeta_k \\ \vdots & \dots & \dots & \vdots \\ \vdots & \dots & \dots & \vdots \\ \left[\frac{c(\pi^*\mathcal{E})}{c(\pi^*\mathcal{F})}\right]_{e-f+1} \zeta_k & \dots & \dots & \left[\frac{c(\pi^*\mathcal{E})}{c(\pi^*\mathcal{F})}\right]_{e-k} \zeta_k \end{vmatrix}$$

$$= \begin{vmatrix} \left[\frac{c(\mathcal{E})}{c(\mathcal{F})}\right]_{e-k} & \dots & \dots & \left[\frac{c(\mathcal{E})}{c(\mathcal{F})}\right]_{e+f-2k+1} \\ \vdots & \dots & \dots & \vdots \\ \vdots & \dots & \dots & \vdots \\ \left[\frac{c(\mathcal{E})}{c(\mathcal{F})}\right]_{e-f+1} & \dots & \dots & \left[\frac{c(\mathcal{E})}{c(\mathcal{F})}\right]_{e-k} \end{vmatrix} \cdot \pi_*(\zeta_k^{f-k})$$

$$= \Delta_{e-k,f-k}\left[\frac{c(\mathcal{E})}{c(\mathcal{F})}\right] \cdot \pi_*(\zeta_k^{f-k}).$$

To evaluate $\pi_*(\zeta_k^{f-k})$, recall that the k^{th} Chern class $c_k(S^*)$ of the dual of the universal subbundle on a single Grassmannian $G(k,f)$ is the class $\sigma_{1,1,\dots,1}$ of the Schubert cycle of k-planes lying in a hyperplane; the $(f-k)^{\text{th}}$ power of this class is thus the class of a single point. By the second part of

Proposition 14.5 we have

$$\pi_*(\zeta_k^{f-k}) = (-1)^{k(f-k)}[X]$$

and we conclude finally that

$$\begin{aligned}\pi_*(z) &= (-1)^{(e-k)(f-k)}\Delta_{e-k,f-k}\left(\frac{c(\mathcal{E})}{c(\mathcal{F})}\right)\\ &= \Delta_{f-k,e-k}\left(\frac{c(\mathcal{F})}{c(\mathcal{E})}\right).\end{aligned}$$

This is Porteous' formula. □

14.6 Geometric applications

14.6.1 Degrees of determinantal varieties

The primeness of the ideal of minors will be used in the proof of Porteous' formula. But once proved, the formula allows us to compute the degree of the $M_k(\Phi)$, and indeed to do so in a more general case:

Theorem 14.8. *Let A be an $e \times f$ matrix of linear forms on $\mathbb{P}^r$, and let $M_k := M_k(A) \subset \mathbb{P}^r$ be the scheme defined by its $(k+1) \times (k+1)$ minors. If M_k has the expected codimension $(e-k)(f-k)$ in $\mathbb{P}^r$, its degree is*

$$\deg(M_k) = \prod_{i=0}^{e-k-1} \frac{i!(f+i)!}{(k+i)!(f-k+i)!}.$$

The formula telescopes in the case $k = e-1$, with $f \geq e$. In this case we see that

$$\deg(M_{e-1}) = \frac{0!(f)!}{(e-1)!(f-(e-1))!} = \binom{f}{e-1}.$$

Naturally, in case $e = f$ we get f, the degree of the determinant.

The product also telescopes nicely in the case $k = 1$, recovering the degree of the Segre variety (calculated by other means in Section 1.2.3: we have

$$\prod_{i=0}^{m-2} \frac{i!}{(i+1)!} = \frac{1}{(m-1)!}$$

and

$$\prod_{i=0}^{m-2} \frac{(n+i)!}{(n+i-1)!} = \frac{(m+n-2)!}{(n-1)!}$$

so the formula yields

$$\deg(M_1) = \frac{1}{(m-1)!} \cdot \frac{(m+n-2)!}{(n-1)!} = \binom{m+n-2}{m-1}.$$

Proof of Theorem 14.8. Multiplication by the matrix A defines a vector bundle map

$$(\mathcal{O}_{\mathbb{P}^r})^{\oplus e} \to (\mathcal{O}_{\mathbb{P}^r}(1))^{\oplus f},$$

and we are asking for the class of the locus where this map has rank k or less. Letting $\zeta \in A^1(\mathbb{P}^r)$ be the hyperplane class, we have

$$c(\mathcal{F}) = (1+\zeta)^f = \sum_{r=0}^{f} \binom{f}{r} \zeta^r$$

from which we conclude that the class of M_k is given by

$$[M_k] = \begin{vmatrix} \binom{f}{f-k}\zeta^{f-k} & \cdots & \binom{f}{f+e-2k-1}\zeta^{f+e-2k-1} \\ \vdots & & \vdots \\ \binom{f}{f-e+1}\zeta^{f-e+1} & \cdots & \binom{f}{f-k}\zeta^{f-k} \end{vmatrix}$$

$$= \begin{vmatrix} \binom{f}{f-k} & \cdots & \binom{f}{f+e-2k-1} \\ \vdots & & \vdots \\ \binom{f}{f-e+1} & \cdots & \binom{f}{f-k} \end{vmatrix} \zeta^{(e-k)(f-k)}.$$

The degree of M_k is thus the last determinant.

To verify the theorem, we simplify this expression. To begin with, we make a series of column operations: first, we replace each column, starting with the second, with the sum of it and the column to its left, to arrive at

$$\begin{vmatrix} \binom{f}{f-k} & \binom{f}{f-k+1} & \cdots & \binom{f}{f+e-2k-1} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \binom{f}{f-e+1} & \binom{f}{f-e+2} & \cdots & \binom{f}{f-k} \end{vmatrix}$$

$$= \begin{vmatrix} \binom{f}{f-k} & \binom{f+1}{f-k+1} & \cdots & \binom{f+1}{f+e-2k-1} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \binom{f}{f-e+1} & \binom{f+1}{f-e+2} & \cdots & \binom{f+1}{f-k} \end{vmatrix}$$

Now we do the same thing again, this time starting with the third column; then again, starting with the fourth, and so on, to arrive at the determinant

$$\begin{vmatrix} \binom{f}{f-k} & \binom{f+1}{f-k+1} & \cdots & \binom{f+e-k-1}{f+e-2k-1} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \binom{f}{f-e+1} & \binom{f+1}{f-e+2} & \cdots & \binom{f+e-k-1}{f-k} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{f!}{k!(f-k)!} & \frac{(f+1)!}{k!(f-k+1)!} & \cdots & \frac{(f+e-k-1)!}{k!(f+e-2k-1)!} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \frac{f!}{(e-1)!(f-e+1)!} & \frac{(f+1)!}{(e-1)!(f-e+2)!} & \cdots & \frac{(f+e-k-1)!}{(e-1)!(f-k)!} \end{vmatrix}$$

We can pull a factor of $f!$ from the first column, $(f+1)!$ from the second, and so on; similarly, we can pull a $k!$ from the denominators in the first row, a $(k+1)!$ from the denominators in the second row, and so on. We arrive at the product

$$\prod_{i=0}^{e-k-1} \frac{(f+i)!}{(k+i)!} \begin{vmatrix} \frac{1}{(f-k)!} & \frac{1}{(f-k+1)!} & \cdots & \frac{1}{(f+e-2k-1)!} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \frac{1}{(f-e+1)!} & \frac{1}{(f-e+2)!} & \cdots & \frac{1}{(f-k)!} \end{vmatrix}.$$

Next, we multiply the first column by $(f-k)!$, the second by $(f-k+1)!$, and so on, to arrive at the expression

$$\prod_{i=0}^{e-k-1} \frac{(f+i)!}{(k+i)!(f-k+i)!} \begin{vmatrix} 1 & 1 & \cdots \\ f-k & f-k+1 & \cdots \\ (f-k)(f-k-1) & (f-k+1)(f-k) & \cdots \\ \vdots & \vdots & \end{vmatrix}$$

Finally, we can recognize the columns of this matrix as the series of monic polynomials $1, x, x(x-1), x(x-1)(x-2), \ldots,$ of degrees $0, 1, 2, \ldots, m-k-1$, applied to the integers $f-k, f-k+1, \ldots, f+e-2k-1$. Its determinant is thus equal to the van der Monde determinant

$$\begin{vmatrix} 1 & 1 & 1 & \dots \\ f-k & f-k+1 & f-k+2 & \dots \\ (f-k)^2 & (f-k+1)^2 & (f-k+2)^2 & \dots \\ \vdots & \vdots & \vdots & \end{vmatrix},$$

which is equal to $\prod_{i=0}^{e-k-1} i!$. Putting this all together, we have established Theorem 14.8. □

14.6.2 Pinch points of surfaces

Let $C \subset \mathbb{P}^n$ be a smooth curve, and $\pi = \pi_\Lambda : C \to \mathbb{P}^2$ the projection from a general $(n-3)$-plane. It's a standard exercise that the map π will be an immersion, and that the image will have only ordinary nodes as singularities.

Is there a similar description for surfaces? In other words, if $\pi : S \to \mathbb{P}^3$ is a general projection of a smooth surface $S \subset \mathbb{P}^n$ to $\mathbb{P}^3$, can we say what singularities occur on the image surface $S_0 = \pi(S)$? The answer is yes (see for example [GH] for a fuller discussion): what we find is that there will be a curve in the image over which the map is 2-1, and at a general such point on this curve the surface S_0 is the union of two smooth sheets crossing transversely. There will also be a finite number of points where the map is 3-1, and at each such point the image surface will be the union of three smooth sheets intersecting transversely.

There is only one additional type of singularity that $\pi(S)$ can have. At a finite number of points of S, the map π will fail to be an immersion, and the images of these points are called *pinch points*. In suitable local analytic coordinates the equation of $\pi(S)$ near a pinch point is

$$z^2 = xy^2.$$

In these coordinates the double curve is be the x-axis, and the two sheets at the point $(x_0, 0, 0)$ will have tangent planes $z = \pm\sqrt{x_0} \cdot y$. The geometry of the map, and of the image, is beautiful: pinch points are points of the double curve of $\overline{S}$ where the local monodromy interchanges the two sheets.

We now ask the enumerative question: in terms of the standard invariants of the surface $S \subset \mathbb{P}^n$, *how many pinch points will* S_0 *have?* We can answer this question with Porteous' formula. The differential of the map π is a vector bundle map

$$d\pi : T_S \to \pi^* T_{\mathbb{P}^3}$$

and the formula tells us that the number of points where this map fails to be injective, counted with multiplicities, is the degree 2 piece of the quotient

$$\frac{c(\pi^* T_{\mathbb{P}^3})}{c(T_S)}.$$

Denote by $\zeta = c_1(\mathcal{O}_S(1))$ the pullback to S of the hyperplane class on $\mathbb{P}^3$ (equivalently, the restriction to S of the hyperplane class on $\mathbb{P}^n$), and write c_1 and c_2 for the Chern classes $c_1(T_S^*)$ and $c_2(T_S^*)$. We thus have

$$\begin{aligned}\frac{c(\pi^* T_{\mathbb{P}^3})}{c(T_S)} &= \frac{1+4\zeta+6\zeta^2}{1-c_1+c_2}\\ &= (1+4\zeta+6\zeta^2)(1+c_1+(c_1^2-c_2))\end{aligned}$$

The degree 2 part of this expression gives our answer:

Proposition 14.9. *The number of pinch points of a general projection of a smooth surface $S \subset \mathbb{P}^n$ to $\mathbb{P}^3$, counted with multiplicities, is*

$$6\zeta^2 + 4\zeta c_1 + c_1^2 - c_2.$$

There is another way to interpret this last formula. Recall that, for a smooth surface $S \subset \mathbb{P}^n$, the *tangential variety* X of S is defined to be the union of the tangent planes $\mathbb{T}_p(S) \subset \mathbb{P}^n$ to S in $\mathbb{P}^n$; if $n \geq 4$ this is always a fourfold ****proof or reference****. Now, if $\pi = \pi_\Lambda : S \to \mathbb{P}^3$ is the projection from a general $(n-4)$-plane $\Lambda \subset \mathbb{P}^n$, the points where the map fails to be an immersion are the points p such that $\mathbb{T}_p(S) \cap \Lambda \neq \varnothing$. If the tangent planes to S sweep out the tangential variety X just once—that is, if a general point on X lies on just one tangent plane to S—then the number of such points is the degree of X. If the multiplicities with which the pinch points occur are also equal to 1 then we can view the formula above as a formula for the degree of the tangential variety of a surface $S \subset \mathbb{P}^n$.

What about the singularities of a general projection of a smooth variety $X \subset \mathbb{P}^n$ of dimension k to $\mathbb{P}^{k+1}$? For $k \leq 14$ powerful results of Mather provide a complete picture, but for large k we don't even know what the maximum multiplicity of a point on the image $\overline{X} = \pi(X) \subset \mathbb{P}^{k+1}$ will be, beyond the fact that it grows exponentially with k. One way to see this is to use the logic suggested by Exercise 14.18: we expect that there will be places where the corank of the differential $d\pi$ is roughly $\sqrt{k}$, and over the image of such points the degree of the fiber of π will be at least on the order of $2^{\sqrt{k}}$. For a recent discussion of such questions see Beheshti and Eisenbud [2010].

14.6.3 Quadrisecants to rational curves

We come now to the last keynote problem: counting the quadrisecant lines to a space curve. We start with a qualitative discussion. Consider a smooth, nondegenerate space curve $C \subset \mathbb{P}^3$, and ask: how would we expect lines $L \subset \mathbb{P}^3$ to intersect C?

To begin with, a general line is disjoint from C, while the lines that do meet C form an irreducible, 3-dimensional family (that is, an irreducible divisor in the Grassmannian $\mathbb{G}(1,3)$). (We can see this by considering the incidence correspondence $\Sigma = \{(L,p) : p \in L \cap C\} \subset \mathbb{G}(1,3) \times C$.) The locus of secant lines—that is, lines such that $\deg(L \cap C) \geq 2$—is likewise an irreducible surface in $\mathbb{G}(1,3)$, being the image of the product $C \times C$ under the regular map $(p,q) \mapsto \overline{pq}$ (this is at first glance defined only on the complement of the diagonal in $C \times C$; it extends to all of $C \times C$ by sending (p,p) to the tangent line $\mathbb{T}_p(C)$).

What about trisecant lines, which we define to be lines L such that $\deg(L \cap C) \geq 3$? By Exercise 14.19 the general secant line to C is not trisecant, and we can deduce from this that the locus of trisecant lines to C, if nonempty, has dimension 1. In fact, the locus is empty only in case C is a twisted cubic or an elliptic quartic.

We remark that the locus of trisecant lines need not be irreducible; an example where it's not would be the intersection $C = Q \cap S$ of a smooth quadric Q and a cubic surface S.

Continuing in this way, we might expect that there will be a finite number of 4-secant lines to C, and no 5-secant lines. Neither of these statements is true for an arbitrary curve, or even a general curve (meaning a curve corresponding to a general point of a component of the Hilbert scheme): look, for example, at complete intersections of quadric with quintic surfaces. Nonetheless, we may ask: is there an enumerative formula for the number of 4-secant lines to C, say in terms of the degree d and genus g of C?

We will derive such a formula in case C is smooth and rational by using Porteous' formula. We could derive the general case as well; we discuss how to do this at the end of the subsection. But for now, we assume that $C \cong \mathbb{P}^1$.

The key idea is to turn the problem around: instead of looking at all lines in $\mathbb{P}^3$ and imposing the condition of meeting C four times, we'll look at 4-tuples of points on C and impose the condition that they span only a line. We will use the setup of Section 12.4.2: we identify the space of subschemes of degree 4 in $C \cong \mathbb{P}^1$ with the symmetric product $C^{(4)} \cong \mathbb{P}^4$ of C, and introduce the bundle $\mathcal{E}^*$ on $C^{(4)}$ with fibers

$$\mathcal{E}^*_\Gamma = H^0(\mathcal{O}_\Gamma(d)).$$

As in Section 12.4.2, a global section of $\mathcal{O}_C(1) = \mathcal{O}_{\mathbb{P}^1}(d)$ gives rise to a global section of $\mathcal{E}^*$ by restriction to each subscheme of C in turn. The restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \hookrightarrow H^0(\mathcal{O}_{\mathbb{P}^1}(d)) \longrightarrow H^0(\mathcal{O}_\Gamma(d))$$

gives us a map $\varphi : \mathcal{F} \to \mathcal{E}^*$ of vector bundles on $C^{(4)}$, where $\mathcal{F}$ is the trivial bundle with fiber $H^0(\mathcal{O}_{\mathbb{P}^3}(1))$; and we see that *the locus* $M_2(\varphi) \subset \mathbb{P}^4$ *of subschemes* $\Gamma \subset \mathbb{P}^1$ *of degree 4 with* $\dim \overline{\Gamma} = 1$ *is the locus where the map* φ *has rank 2*. (Note that φ can never have rank < 2.) Porteous will then give us a formula for the class of this locus, in case it has the expected dimension

$$4 - 2 \times 2 = 0.$$

To carry this out, recall from Proposition 12.16 the Chern classes of $\mathcal{E}^*$ are

$$c_i(\mathcal{E}^*) = \binom{d-4+i}{i} \zeta^i \in A^i(\mathbb{P}^4).$$

Now we apply Porteous' formula, which tells us that the class of the locus of subschemes $\Gamma \subset \mathbb{P}^1$ contained in a line is

$$\begin{aligned}
[M_2(\varphi)] &= \begin{vmatrix} c_2(\mathcal{E}^*) & c_3(\mathcal{E}^*) \\ c_1(\mathcal{E}^*) & c_2(\mathcal{E}^*) \end{vmatrix} \\
&= \begin{vmatrix} \binom{d-2}{2}\omega^2 & \binom{d-1}{3}\omega^3 \\ \binom{d-3}{1}\omega & \binom{d-2}{2}\omega^2 \end{vmatrix} \\
&= \begin{vmatrix} \binom{d-2}{2} & \binom{d-1}{3} \\ \binom{d-3}{1} & \binom{d-2}{2} \end{vmatrix} \omega^4 \\
&= \left(\frac{(d-2)^2(d-3)^2}{4} - \frac{(d-1)(d-2)(d-3)^2}{6} \right) \omega^4 \\
&= \frac{(d-3)^2(d-2)}{12} \left(3(d-2) - 2(d-1)\right) \omega^4 \\
&= \frac{(d-2)(d-3)^2(d-4)}{12} \omega^4.
\end{aligned}$$

This gives us the enumerative formula:

Proposition 14.10. *If* $C \subset \mathbb{P}^3$ *is a rational space curve of degree* d *possessing only finitely many quadrisecant lines, the number of such lines,* *properly counted, is*

$$\frac{(d-2)(d-3)^2(d-4)}{12}.$$

Note as a check that this number is 0 in case $d = 2$, 3 or 4, as it should be.

We will see in Exercise 14.22 a condition for a given quadrisecant line to be simple, that is, to count with multiplicity one. We will also see in Exercise 14.23 that for $C \subset \mathbb{P}^3$ a *general* rational curve of degree d—that is, a general projection of a rational normal curve from $\mathbb{P}^d$ to $\mathbb{P}^3$—all quadrisecants are simple, so this is the actual number of quadrisecant lines.

Quadrisecants to curves of higher genus If we try to generalize the arguments above to the case where C has higher genus, a new issue arises. The Hilbert scheme parametrizing subschemes of degree 4 in $\mathbb{P}^1$ is $\mathbb{P}^4$, whose Chow ring we know, the space of subschemes of degree 4 of a smooth curve C of higher genus (again, the fourth symmetric power $C^{(4)}$ of C) is more complex; in particular, we don't know its Chow ring explicitly. It is possible, however, to say enough about the ring $A(C^{(4)})$ to carry out the same calculation there, and in fact we'll derive the relevant information in the final Chapter of this book. The most general formula, for the number of d-secant $(d-r-1)$-planes to a curve of degree n and genus g in $\mathbb{P}^s$, is derived in this way in Chapter 8 of Arbarello et al. [1985]; the formula is on page 350 and the specialization to 4-secant lines on page 351.

We could also approach the problem of counting quadrisecant lines to a space curve via the classical theory of correspondences. This is described in Chapter 2 of Griffiths and Harris [1978].

14.7 Exercises

Exercise 14.11. Let $A = (P_{i,j})$ be a 2×3 matrix whose entries $P_{i,j}$ are general polynomials of degree $a_{i,j}$ on $\mathbb{P}^3$. Assuming that $a_{1,j} + a_{2,k} = a_{1,k} + a_{2,j}$ for all j and k—so that the minors of A are homogeneous—what is the degree of the curve $M_1(A)$ where A has rank 1?

Exercise 14.12. In Exercise **??** of the first chapter, we introduced the variety of triples of collinear points, that is,

$$\Psi = \{(p, q, r) \in \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n \mid p, q \text{ and } r \text{ are collinear in } \mathbb{P}^n\}.$$

Calculate the class $\psi = [\Psi] \in A^{n-1}(\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n)$ by applying Porteous to the evaluation map $\mathcal{E} \to \mathcal{F}$, where $\mathcal{E}$ is the trivial bundle on $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$ with fiber $H^0(\mathcal{O}_{\mathbb{P}^n}(1))$ and

$$\mathcal{F} = \pi_1^*\mathcal{O}_{\mathbb{P}^n}(1) \oplus \pi_2^*\mathcal{O}_{\mathbb{P}^n}(1) \oplus \pi_3^*\mathcal{O}_{\mathbb{P}^n}(1),$$

with $\pi_i : \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ projection on the i^{th} factor.

Exercise 14.13. Verify Proposition 14.9 directly in case S is a smooth surface in $\mathbb{P}^3$ to begin with.

Exercise 14.14. Verify Proposition 14.9 directly in case $S \subset \mathbb{P}^4$ is a cubic scroll. What is the double curve of the image S_0?

Exercise 14.15. Verify Proposition 14.9 directly in case S is the quadratic Veronese surface. What does the double curve of S_0 look like in this case, and how many triple points will S_0 have?

Exercise 14.16. Let $S \subset \mathbb{P}^n$ be a smooth surface.

(a) Show that we have a map from the projective bundle $\mathbb{P}(T_S \oplus \mathcal{O}_S)$ to $\mathbb{P}^n$ with image the tangential variety X of S (specifically, carrying the fiber over p to the tangent plane $\mathbb{T}_p(S)$);

(b) Show that the pullback of $\mathcal{O}_{\mathbb{P}^n}(1)$ under this map is the line bundle $\mathcal{O}_{\mathbb{P}(T_S \oplus \mathcal{O}_S)}(1) \otimes \mathcal{O}_S(1)$; and

(c) Use this and our description of the Chow ring of the projective bundle $\mathbb{P}(T_S \oplus \mathcal{O}_S)$ to rederive the formula of Proposition 14.9 for the degree of X.

Exercise 14.17. Let $X \subset \mathbb{P}^n$ be a smooth threefold, and $\pi : X \to \mathbb{P}^5$ a general projection. Find the number of points where the map π fails to be an immersion.

Exercise 14.18. Let $X \subset \mathbb{P}^n$ be a smooth 6-fold, and $\pi : X \to \mathbb{P}^7$ a general projection. Find the number of points where the differential $d\pi$ has rank 4 or less.

Exercise 14.19. Let $C \subset \mathbb{P}^3$ be an smooth, nondegenerate curve. Show that the general secant line to C is not trisecant, and deduce that the locus of trisecant lines to C, if nonempty, has dimension 1. (For extra credit, show that it is empty only in case C is a twisted cubic or an elliptic quartic.)

Exercise 14.20. Check the conclusion of Proposition 14.10 in cases $d = 5$ and 6 by independently counting the number of quadrisecant lines to a general rational quintic and sextic. (Hint: such curves will lie on smooth cubic surfaces.)

Exercise 14.21. Let $C \subset \mathbb{P}^4$ be a smooth, nondegenerate rational curve of degree d. Use similar methods to find the expected number of

(a) trisecant lines to C; and

(b) 6-secant 2-planes to C.

Exercise 14.22. Let $C \subset \mathbb{P}^3$ be a smooth curve, $L \subset \mathbb{P}^3$ a line meeting C in exactly 4 points $p_1, \dots, p_4$ and not tangent to C at any of them. Suppose that the tangent lines $\mathbb{T}_{p_i}(C)$ to C at the p_i are independent mod L (that is, they span distinct planes with L), and that the cross-ratio of the four points $p_1, \dots, p_4 \in L$ is *not* equal to the cross-ratio of the four planes $\mathbb{T}_{p_1}(C) + L, \dots, \mathbb{T}_{p_4}(C) + L$. Show that $\Gamma = p_1 + \cdots + p_4 \in \mathbb{P}^4$ counts as a quadrisecant line with multiplicity 1 (that is, the 3×3 minors of a matrix representative of φ near Γ generate the maximal ideal $m_\Gamma \subset \mathcal{O}_{\mathbb{P}^4,\Gamma}$).

Exercise 14.23. Let C now be a general rational curve of degree d in $\mathbb{P}^3$. (Note that the family of rational curves $C \subset \mathbb{P}^3$ of degree d is irreducible, so this makes sense.)

(a) Show that C has no 5-secant lines.

(b) Show that if $L \subset \mathbb{P}^3$ is any quadrisecant line to C, then L meets C in four distinct points.

(c) Finally, show that every quadrisecant line to C satisfies the conditions of the preceding exercise, and deduce that the number of quadrisecant lines to C is exactly $\frac{(d-2)(d-3)^2(d-4)}{12}$. (Note: the first two items are straightforward dimension counts; this one is a little more subtle.)

Exercise 14.24. Let $\{Q_\mu \subset \mathbb{P}^3\}_{\mu \in \mathbb{P}^3}$ be a general web of quadrics in $\mathbb{P}^3$; that is, the three-dimensional linear series corresponding to a general four-dimensional vector space $V \subset H^0(\mathcal{O}_{\mathbb{P}^3}(2))$.

(a) Find the number of 2-planes $\Lambda \subset \mathbb{P}^3$ that are contained in some quadric of the net.

(b) A line $L \subset \mathbb{P}^3$ is said to be a *special line* for the web if it lies on a pencil of quadrics in the web. Find the class of the locus $\Sigma \subset \mathbb{G}(1,3)$ of special lines.

15

Excess Intersections and Blowups

Keynote Questions:

(a) Suppose that S, T and $U \subset \mathbb{P}^3$ are three surfaces, of degrees s, t and u, whose intersection consists of the disjoint union of a reduced line L and a zero-dimensional scheme $\Gamma \subset \mathbb{P}^3$. What is the degree of Γ?

(b) Suppose that S and $T \subset \mathbb{P}^4$ are two surfaces, whose intersection consists of the disjoint union of a reduced line L and a zero-dimensional scheme $\Gamma \subset \mathbb{P}^4$. In terms of the geometry of S and T, can we say what is the degree of Γ? Can we say what the degree of Γ is in terms of the degrees of S and T alone?

(c) Let $\Lambda \cong \mathbb{P}^{n-2} \subset \mathbb{P}^n$ be a codimension 2 linear subspace, and let $Q_1, \dots, Q_k \subset \mathbb{P}^n$ be k general quadric hypersurfaces containing Λ. If we write the intersection $\cap Q_i$ as the union

$$\bigcap_{i=1}^{k} Q_i = \Lambda \cup Z,$$

what is the degree of Z?

(d) Let $C \subset \mathbb{P}^3$ be a smooth curve of degree d and genus g. If $S, T \subset \mathbb{P}^3$ are smooth surfaces of degrees s and t containing C, at how many points of C are S and T tangent?

The first three keynote questions of this Chapter are simple examples of what are called *excess intersection* problems: situations in which we wish to describe the intersection of cycles Z_i on an ambient variety X, but where the intersection of those cycles has components of greater dimension than the expected.

In these circumstances, knowledge of the Chow ring of the ambient space X is seemingly irrelevant; intersection theory, on the face of it, tells us something about the actual intersection of the cycles Z_i only under the hypothesis that the intersection has the expected dimension. But examples, like the one worked out in Section 15.1.2 below in response to keynote question (a), suggest that even when the intersection of cycles has components of too large a dimension, the class of the components that do have the expected dimension is not arbitrary. The question is, can we in fact say something about the class of these components?

One way to think about this question, which we'll explore more fully in Section 15.1.4 below, is to imagine that the cycles Z_i we wish to intersect are the limits of cycles $Z_i(t)$ that do intersect transversely. For example, in case the expected dimension of the intersection is 0, we could rephrase the question by asking: how many of the points of the nearby transverse intersection $\cap Z_i(t)$ have limits in the components of positive dimension? In general, can we say what is the class of those components of the limit of the transverse intersection $\cap Z_i(t)$ that lie in the components of too large dimension? Are these classes determined by some aspect of the local geometry near the non-transverse intersection?

The answer is "yes," in a remarkably strong sense: there is a very general formula, called the *excess intersection formula*, which does exactly this. More specifically, what the excess intersection formula does, under relatively mild hypotheses on the singularities of the cycles being intersected, is to associate to each connected component of the intersection a cycle class of the correct dimension, in such a way that the sum of all the classes indeed equals the product of the classes of the cycles.

This has two consequences, one practical and one foundational. The practical one is that it allows us to answer questions like the ones above: in both of the first two keynote questions, the expected dimension of the cycles is zero, so the excess intersection formula associates to the component of excess intersection—in each case, the line L—a cycle class of dimension 0. If we then subtract the degree of this class from the product of the classes of the cycles, what is left will be the degree of the zero-dimensional part of the intersection.

The foundational consequence is that we can—in contrast to the approach taken here—use this to *define* intersection products. Traditionally, intersection theory was based on the moving lemma of Chapter 5, which says that for any two classes $\alpha, \beta \in A(X)$, we can find cycles A and B representing those classes that intersected generically transversely, and that the class of the intersection didn't depend on the choice of representative cycles; we then define the product $\alpha\beta \in A(X)$ to be the class of the intersection $A \cap B$. As an alternative to this, given the excess intersection formula, we can say: for any pair of cycles A and B representing the classes α and β (at least one of which is a linear combination of local complete intersection subvarieties), take $\alpha\beta$ to be the sum of the classes associated to the various components of the intersection $A \cap B$. This has aesthetic virtues—since the moving lemma is onerous to prove and ultimately has very little to do with how we calculate intersection products in practice, it's pleasing to be able to dispense with it—and also real ones: for example, it allows us to define intersection products in their greatest possible generality, in particular on singular varieties X.

In the present text, we have long ago committed to the traditional approach. Nevertheless, the excess intersection formula is a tool of tremendous importance in modern intersection theory, and we want to take some time here to describe its manifestations and applications. We also strongly urge the reader to study Fulton's book—for geometers who work in intersection theory, as well as those for whom it's simply a tool for use in the study of other questions, this is without question the way things should be done.

What we'll do here is to work out a few relatively simple examples, including the keynote questions above; these will both illustrate approaches to deriving excess intersection formulas and (hopefully) suggest why we might expect such a formula to hold in the first place.

One warning before we start, though: given our approaches to the problem, we are going to require a number of hypotheses in each case about the smoothness of the cycles we're intersecting, and on their intersection. In fact, *almost none of these assumptions are actually necessary for the existence of an excess intersection formula.* (We'll discuss the general form of the excess intersection formula in Section 15.6.) To prove the excess intersection formula in such generality, however, requires that we set up intersection theory differently from the beginning, in the manner of Fulton's book Fulton [1984]. Thus we can only suggest that the reader who wants to see the full strength of these formulas consult Fulton.

15.1 Examples of excess intersection formulas

15.1.1 Intersection of divisors and curves

Our first example is the simplest possible situation in which excess intersection can occur: we take X to be a smooth projective variety, $D \subset X$ an irreducible divisor and $C \subset X$ an irreducible curve; we ask for the intersection number of D and C in X. If C intersects D transversely, this is just the cardinality of the intersection; and more generally as long as C is not contained in D, this will be the sum, over all points p of the intersection $C \cap D$, of the locally defined intersection multiplicity $m_p(C \cdot D)$. But what if C *is* contained in D?

The answer in this case is not hard. Associated to the divisor D is a line bundle $L = \mathcal{O}_X(D)$ on X, which has first Chern class $c_1(L) = [D] \in A^1(X)$ the class of D. By the functoriality of Chern classes, then,

$$[D] \cdot [C] = [D]|_C = c_1(L)|_C = c_1(L|_C).$$

Now, we can write

$$\begin{aligned} L^*|_C &= \mathcal{O}_X(-D)|_C \\ &= (\mathcal{O}_X(-D) \otimes \mathcal{O}_X/\mathcal{I}_D) \otimes \mathcal{O}_X/I_C \\ &= \left(\mathcal{I}_D/\mathcal{I}_D^2\right) \otimes \mathcal{O}_X/I_C; \end{aligned}$$

and recognizing the term in parentheses in the last expression as the conormal bundle of D in X, we conclude that $L|_C$ is the normal bundle of D in X, restricted to C. Note that this makes sense even if D and X are singular, as long as

(a) D is a Cartier divisor on X; and

(b) By the "normal bundle" $N_{D/X}$ we mean the sheaf $\mathrm{Hom}(\mathcal{I}_D/\mathcal{I}_D^2, \mathcal{O}_D)$ (which is locally free of rank 1 in this case).

In sum, we have the

Proposition 15.1. *Let X be smooth and projective; let D be a divisor on X and $C \subset X$ a curve in X contained in D. The intersection number $(C \cdot D)$ is the degree of the restriction to C of the normal bundle $N_{D/X}$ of D in X. More precisely, the restriction to C of the class $[D] \in A^1(X)$ is the first Chern class of the restriction of $N_{D/X}$ to C.*

There is a geometric way of seeing this, at least in some cases, that will suggest an approach to the more general problem. Suppose for the moment that you could deform the divisor $D \subset X$—that is, you could find a flat family of divisors $\mathcal{D} \subset \Delta \times X$, parametrized by a smooth rational curve Δ with marked point $0 \in \Delta$, such that the special fiber $D_0 = D$. Suppose moreover that a general member D_λ of the deformation was transverse to C; that is,

$$D_\lambda \cap C = \{p_1(\lambda), \ldots, p_k(\lambda)\} \quad \text{for} \quad \lambda \neq 0$$

with $k = [D_\lambda] \cdot [C] = [D] \cdot [C]$. Finally, suppose that the deformation is nontrivial to first order: that is, its restriction to the scheme $\operatorname{Spec} k[\epsilon]/(\epsilon)^2 \subset \Delta$ supported at 0 corresponds to a nonzero section σ of the normal bundle $N_{D/X}$.

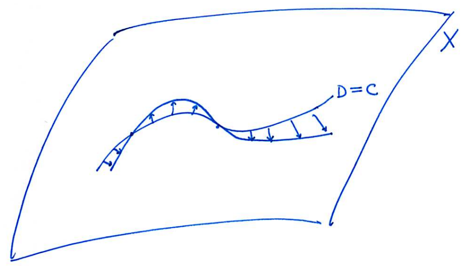

FIGURE 15.1. The limits of the points of intersection of C with the deformed curve are zeros of the corresponding section of the normal bundle.

In this situation, we can ask: what are the limits as $\lambda \to 0$ of the points $\{p_1(\lambda), \dots, p_k(\lambda)\}$? The answer, we claim, is that *they are the zeros of the section σ of the normal bundle*. This in intuitively reasonable: in classical language, away from zeros of σ, the deformation is moving D away itself (and hence away from C), while the points where σ is zero are "staying put," at least to first order. This is easiest to visualize in case X is a surface (so that $D = C$); higher-dimensional cases are essentially the same.

To see this explicitly, take local analytic coordinates $(z_1, \dots, z_n)$ on X such that

$$D = (z_n = 0) \quad \text{and} \quad C = (z_2 = \dots = z_n = 0).$$

We can then write the family $\mathcal{D} \subset \Delta \times X$ as the zero locus of the function

$$z_n + \lambda f_1 + \lambda^2 f_2 + \dots,$$

with f_1 the corresponding section of the normal bundle. Restricting to $\Delta \times C$, this becomes

$$\lambda (f_1)|_C + \lambda^2 (f_2)_C + \dots = \lambda \left((f_1)|_C + \lambda (f_2)_C + \dots \right),$$

from which we see that a zero of f_1 of order m will be a limit of exactly m points of intersection of C with D_λ as $\lambda \to 0$.

Before we go on, note that the same logic that led us to Proposition 15.1 works for the intersection of any cycle with a divisor: by the same argument as initially given above, if $Z \subset X$ is any subvariety of codimension k and $D \subset X$ a divisor containing Z, then we have

Proposition 15.2. *The restriction to Z of the class $[D] \in A^1(X)$ is the first Chern class $c_1(N_{D/X}|_Z) \in A^1(Z)$ of the restriction to Z of the normal bundle of D in E; and correspondingly the product of $[Z]$ and $[D]$ in $A(X)$ is given by*

$$[Z] \cdot [D] = i_* c_1(N_{D/X}|_Z) \in A^{k+1}(X)$$

where $i : Z \hookrightarrow X$ is the inclusion.

15.1.2 Intersection of three surfaces in $\mathbb{P}^3$ containing a curve

Next, let's consider the first of the keynote questions of this Chapter: we're given three surfaces S, T and $U \subset \mathbb{P}^3$, of degrees s, t and u, whose intersection consists of the disjoint union of a reduced line L and a zero-dimensional scheme Γ; the problem is to find the degree of Γ.

We'll start by approaching it naively, solving it the way it probably would have been solved in the 19^{th} century. To set up, we claim first that we can take S to be smooth: after renaming, we can assume that the degree s of S is the largest of the three integers s, t and u, and then replace S by the zero locus of a general linear combination $F+AG+BH$, where F, G and H are the defining equations of S, T and U respectively, and A and B are general polynomials of degrees $s-t$ and $s-u$. The surface S is then the general element of a linear system in $\mathbb{P}^3$ whose base locus is smooth of dimension less than $3/2$, and by the amped up Bertini theorem (Proposition 7.4) it follows that S is smooth.

Given this, consider the intersection of the two surfaces S and T, viewed as a divisor on S; write it as

$$S \cap T = L + D \in Z_1(S)$$

for some divisor D on S. Similarly, write

$$S \cap U = L + E \in Z_1(S)$$

for some divisor E on S. The subscheme Γ of the intersection $S \cap T \cap U$ will then be the scheme of intersection of D and E. Moreover, all the points of intersection of D and E occur away from L (if D and E met at a point $p \in L$, the intersection $S \cap T \cap U$ would be singular at p). Thus the degree of Γ is the intersection number of the classes of D and E in $A(S)$, which we'll now compute.

We start by applying the adjunction formula to the curve $L \subset S$. This says that

$$-2 = 2g(L) - 2 = (L \cdot L)_S + (L \cdot K_S).$$

We know that $K_S = \mathcal{O}_S(s-4)$, and so

$$(L \cdot K_S) = s - 4;$$

it follows that

$$(L \cdot L)_S = 2 - s.$$

Now, denote by $H \in A^1(S)$ the hyperplane class on S; since $S \cap T \sim tH$ and $S \cap U \sim uH$, we have

$$D \sim tH - L \quad \text{and} \quad E \sim uH - L \ \in A^1(S).$$

Thus, in $A(S)$ we have

$$\begin{aligned}(D \cdot E) &= (tH - L) \cdot (uH - L)\\ &= tu(H \cdot H) - (t+u)(H \cdot L) + (L \cdot L)\\ &= stu - s - t - u + 2,\end{aligned}$$

and this is the answer to our first keynote question.

(As always, check! If $s = t = u = 1$, we should get zero, and we do; more generally, if $s = 1$ the residual curves D and E are plane curves of degrees $t-1$ and $u-1$ respectively, so we should get $(t-1)(u-1)$, and we do.)

We should remark that the statement here is unnecessarily restrictive: we can replace the line L by any smooth curve C of known degree d and genus g, and apply exactly the same approach. We have $(C \cdot K_S) = d(s-4)$; from adjunction, we deduce that

$$(C \cdot C)_S = 2g - 2 - d(s-4).$$

We have then

$$\begin{aligned}(D \cdot E) &= (tH - C) \cdot (uH - C)\\ &= tu(H \cdot H) - (t+u)(H \cdot C) + (C \cdot C)\\ &= stu - d(t+u) + 2g - 2 - d(s-4)\\ &= stu - d(s+t+u-4) + 2g - 2.\end{aligned}$$

There is another way to write this formula that may seem strange at first, but generalizes to the excess intersection formula. Note that from the exact sequence

$$0 \to T_C \to T_{\mathbb{P}^3}|_C \to N_{C/\mathbb{P}^3} \to 0$$

we see that

$$\deg(N_{C/\mathbb{P}^3}) = 4d + 2g - 2,$$

so we can rewrite this in terms of degrees of normal bundles as

$$\begin{aligned}\deg(\Gamma) = stu &- \deg(N_{S/\mathbb{P}^3}|_C) - \deg(N_{T/\mathbb{P}^3}|_C) - \deg(N_{U/\mathbb{P}^3}|_C)\\ &+ \deg(N_{C/\mathbb{P}^3}).\end{aligned} \tag{15.1}$$

Yet another way to express this, if S, T and U are all smooth, is as

$$\deg(\Gamma) = stu + \deg(N_{C/S}) + \deg(N_{C/T}) + \deg(N_{C/U}) - 2\deg(N_{C/\mathbb{P}^3}).$$

In any event, we have proved the

Proposition 15.3. *Let S, T and $U \subset \mathbb{P}^3$ be surfaces of degrees s, t and u, whose intersection consists of the disjoint union of a smooth curve C of degree d and genus g and a zero-dimensional scheme Γ. In this situation, the degree of Γ is given by*

$$\begin{aligned}\deg(\Gamma) &= stu - d(s+t+u-4) + 2g - 2\\ &= stu - \deg(N_{S/\mathbb{P}^3}|_C) - \deg(N_{T/\mathbb{P}^3}|_C) - \deg(N_{U/\mathbb{P}^3}|_C) + \deg(N_{C/\mathbb{P}^3}).\end{aligned}$$

There's a cute application of this formula to a qualitative question: if $C \subset \mathbb{P}^3$ is a smooth curve, can C always be expressed as the (scheme-theoretic) intersection of three surfaces? The question was answered in Peskine and Szpiro [1974]; the solution here was given in Fulton [1984], Example 9.1.2.

The answer can be shown to be "no" by an application of Proposition 15.3: by that formula, the degrees s, t and u of the three surfaces would have to satisfy the equality

$$stu - d(s + t + u - 4) + 2g - 2 = 0$$

and we can find examples of smooth curves C of degree d and genus g such that no solution of this equation exists in integers s, t and u among the degrees of surfaces containing C. For example, if C is a elliptic quintic curve, C lies on no planes or quadrics (this follows from the genus formula for smooth curves on a plane and quadric, among other things); but the quantity $stu - 5(s+t+u-4)$ is positive for all $s, t, u \geq 3$. (In Exercise 15.16, we'll answer the corresponding question for quintic curves $C \subset \mathbb{P}^3$ of genus 0 and 2.)

15.1.3 Intersections of hypersurfaces in general

We want to go on and consider other approaches to excess intersection situations. Before we do, though, it's worth taking a moment out to discuss in what generality the approach we just tried works.

Briefly, suppose that X is a smooth, n-dimensional projective variety and $D_1, \dots, D_k$ a collection of hypersurfaces in X. If the intersection $Y = \cap D_i$ is proper, then of course its class (if we assign to each component the appropriate multiplicity) is the product $\prod \delta_i \in A^k(X)$ of the classes $\delta_i = [D_i]$ of the divisors D_i. But what if the intersection has mixed dimension—that is, it has components of dimension strictly greater than $n - k$ as well as components of the expected dimension $n - k$. For example, suppose that

$$\bigcap_{i=1}^{k} D_i = \Phi \cup \Gamma$$

with Φ and Γ smooth and disjoint, Φ of pure dimension $n - k + m$ and $\dim \Gamma = n - k$. Can we find the class of the sum Γ of the components of the expected dimension $n - k$?

In fact, we can, if we know something about the geometry of the divisors D_i and their intersection. What we want to do here is intersect the hypersurfaces D_i one at a time, and focus on the first stage where the intersection fails to have the expected dimension; if we allow ourselves to change the order of the divisors D_i, we can assume that this occurs after we have intersected $k-m+1$ of the divisors, at which point Φ appears as a component of excess intersection. The point is, if we back up one step, we see that *the previous intersection* $D_1 \cap \cdots \cap D_{k-m}$ *must have been reducible*: if $\Gamma \neq \varnothing$, then we must have

$$D_1 \cap \cdots \cap D_{k-m} = \Phi \cup B_0.$$

Now back up one further step, and consider the intersection $S = D_1 \cap \cdots \cap D_{k-m-1}$, which has dimension $n-k+m+1$. We have an equation of divisors on S:

$$D_{k-m} \cap S = A + B_0$$

and similarly we can write

$$D_{k-m+\alpha} \cap S = A + B_\alpha$$

for each $\alpha = 0, \dots, m$. We can then express the cycle Γ as a proper intersection of $m+1$ divisors on the variety S:

$$\Gamma = B_0 \cap \cdots \cap B_m$$

and if we know enough about the geometry of Φ—specifically, if we can evaluate the products of powers $[\Phi]^l \in A^l(S)$ with products of the classes $[D_\alpha]|_S \in A^1(S)$—we can then evaluate the class of this intersection as

$$[\Gamma] = \prod_{\alpha=0}^{m} ([D_{k-m+\alpha}]|_S - [\Phi]) \in A^{m+1}(S) \to A^{n-k}(X).$$

To see some examples of this approach, try Exercises 15.17 and 15.18.

15.1.4 Intersection of two surfaces in $\mathbb{P}^4$ containing a curve

We come now to the second of our keynote questions: we're given two smooth surfaces S and $T \subset \mathbb{P}^4$, say of degrees d and e, whose intersection consists of a reduced curve C and a collection Γ of reduced points; and we ask, as before, for the degree of Γ. To answer this question, we're going to introduce a new approach. This approach, though it is furthest removed from the methods developed in earlier chapters, gives the best geometric picture of excess intersection, and is also the most broadly applicable; it is also closest in spirit to the way in which Fulton's general excess intersection formula is derived.

The key to this approach is to *deform the surfaces S and T in $\mathbb{P}^4$ to ones that do intersect transversely*: that is, find families $\{S_\lambda\}$ and $\{T_\lambda\}$, parametrized by a smooth rational curve Δ, with $S = S_0$ and $T = T_0$, and such that the intersection $S_\lambda \cap T_\lambda$ is transverse for $\lambda \neq 0$. We can then think of the intersection $S_\lambda \cap T_\lambda$ as a collection of de points $p_1(\lambda), \dots, p_{de}(\lambda)$ varying with the parameter λ, and the question becomes: *as $\lambda \to 0$, how*

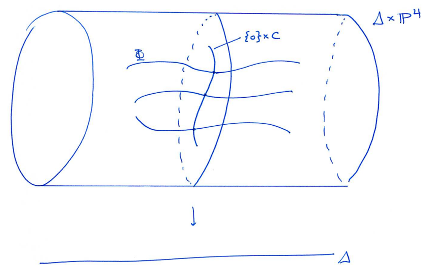

FIGURE 15.2. Limits of points of intersection of the deformed surfaces S_λ, T_λ are singular points of $\mathcal{S}\cap\mathcal{T}$.

many of the points $p_i(\lambda)$ *approach the curve* C*, and how many approach* Γ*?*

To set this up, note first that such families are easy to find: for example, we could let A and $B : \Delta \to PGL_5$ be two general arcs in the group PGL_5 of automorphisms of $\mathbb{P}^4$ with $A(0) = B(0) = I$ the identity, and take S_λ the image of S under the map $A_\lambda : \mathbb{P}^4 \to \mathbb{P}^4$—that is,

$$\mathcal{S} = \{(\lambda, p) \in \Delta\times\mathbb{P}^4 \mid p \in A_\lambda(S)\}$$

and

$$\mathcal{T} = \{(\lambda, p) \in \Delta\times\mathbb{P}^4 \mid p \in A_\lambda(T)\}.$$

By Kleiman's transversality theorem (Theorem 5.15), then, S_λ and T_λ will intersect transversely in de points for $\lambda\neq 0$, at least in some neighborhood of $0\in\Delta$.

Consider now the intersection $\mathcal{S}\cap\mathcal{T}$ of the threefolds $\mathcal{S}$ and $\mathcal{T}$ in the fivefold $\Delta\times\mathbb{P}^4$. We can express this intersection as a union

$$\mathcal{S}\cap\mathcal{T} = \Phi\cup D$$

where Φ is flat over Δ, consisting of the points of intersection $S_\lambda\cap T_\lambda$ for $\lambda\neq 0$ and their limits—that is, the closure in $\Delta\times\mathbb{P}^4$ of the intersection of $\mathcal{S}\cap\mathcal{T}$ with $(\Delta\setminus\{0\})\times\mathbb{P}^4$—and $D = \{0\}\times C$ is the positive-dimensional component of the intersection $S_0\cap T_0$.

Now, we know that $\Phi\cap(\{0\}\times\mathbb{P}^4)$ is zero-dimensional, of degree de, and consists of the union of the scheme Γ with the intersection $\Phi\cap D$; to find the degree of Γ, accordingly, we have to figure out the degree of the intersection $\Phi\cap D$.

How do we do this? Well, one way to characterize the points of $\Phi\cap D$ is to observe that *they are singular points of the intersection* $\mathcal{S}\cap\mathcal{T}$; in fact, they are exactly the singular points of $\mathcal{S}\cap\mathcal{T}$ occurring along D. Now, if $\mathcal{S}$ and $\mathcal{T}$ met transversely everywhere along D, we would have

$$T_D = T_{\mathcal{S}}|_D \cap T_{\mathcal{T}}|_D \quad\text{in}\quad T_{\Delta\times\mathbb{P}^4}|_D$$

and hence a direct sum decomposition

$$N_{D/\Delta\times\mathbb{P}^4} = N_{\mathcal{S}/\Delta\times\mathbb{P}^4}|_D \oplus N_{\mathcal{T}/\Delta\times\mathbb{P}^4}|_D.$$

In general, we see that we have a map

$$N_{D/\Delta\times\mathbb{P}^4} \to N_{\mathcal{S}/\Delta\times\mathbb{P}^4}|_D \oplus N_{\mathcal{T}/\Delta\times\mathbb{P}^4}|_D$$

between vector bundles of rank 4 on D, and the locus where this map fails to be surjective is exactly the singular locus of $\mathcal{S}\cap\mathcal{T}$ along D.

To find the degree of this locus, note first that since $D = \{0\}\times C$,

$$N_{D/\Delta\times\mathbb{P}^4} = N_{C/\mathbb{P}^4} \oplus \mathcal{O}_C,$$

and hence

$$c_1(N_{D/\Delta\times\mathbb{P}^4}) = c_1(N_{C/\mathbb{P}^4}).$$

Next, since S_0 is the transverse intersection of $\mathcal{S}$ with the fiber $\{0\}\times\mathbb{P}^4$,

$$N_{\mathcal{S}/\Delta\times\mathbb{P}^4}|_D = N_{S/\mathbb{P}^4}|_C$$

and likewise

$$N_{\mathcal{T}/\Delta\times\mathbb{P}^4}|_D = N_{T/\mathbb{P}^4}|_C.$$

Putting this all together, we deduce that the degree of $\Phi\cap D$ is the difference

$$\deg(\Phi\cap D) = c_1(N_{S/\mathbb{P}^4}|_C) + c_1(N_{T/\mathbb{P}^4}|_C) - c_1(N_{C/\mathbb{P}^4})$$

and hence

$$\deg(\Gamma) = de - c_1(N_{S/\mathbb{P}^4}|_C) - c_1(N_{T/\mathbb{P}^4}|_C) + c_1(N_{C/\mathbb{P}^4}).$$

Note the similarity with formula (15.1) in the previous case of three surfaces!

One restatement of this: from the sequence

$$0 \to N_{C/S} \to N_{C/\mathbb{P}^4} \to (N_{S/\mathbb{P}^4})|_C \to 0$$

we have

$$c_1(N_{S/\mathbb{P}^4}|_C) = c_1(N_{C/\mathbb{P}^4}) - c_1(N_{C/S})$$

and similarly for T; so we can rewrite this as

$$\deg(\Gamma) = de + c_1(N_{C/S}) + c_1(N_{C/T}) - c_1(N_{C/\mathbb{P}^4}).$$

If C has degree a and genus g, we can also write this as

$$\deg(\Gamma) = de + (C\cdot C)_S + (C\cdot C)_T - 5a - 2g + 2.$$

In particular, if we take $C = L$ a line in $\mathbb{P}^4$, we arrive at the answer to our second keynote question:

$$\deg(\Gamma) = de + (L\cdot L)_S + (L\cdot L)_T - 3.$$

An immediate check: if S and T are 2-planes, this formula returns 0, as it should. For other examples, see Exercise 15.19.

Note that, in contrast to the answer to our first keynote question, this does not depend only on the degrees of S and T (that is, their classes in $A(\mathbb{P}^4)$), but on their geometry; the simplest example of this is described in Exercise 15.20 below.

15.2 The excess intersection formula in general dimensions

As we indicated, one virtue of the last approach to excess intersection is that it can be applied, with little change, to excess intersections in arbitrary dimensions on arbitrary ambient spaces X (though, at least in our version, we will still have to make some fairly strong smoothness hypotheses). To set this up, we suppose the following:

- X will be a smooth, projective variety of dimension n;
- S and $T \subset X$ will be smooth subvarieties of codimensions k and l respectively;
- The intersection $C = S \cap T$ will be smooth, with connected components C_α of codimension $k + l - m_\alpha$.

As in the examples above, we want to assign to each C_α a cycle class $\gamma_\alpha \in A^{m_\alpha}(C_\alpha)$ of dimension $k + l - n$, which represents the contribution of C_α to the total intersection $S \cap T$; that is, such that if we denote by $i_\alpha : C_\alpha \to X$ the inclusion, then

$$\sum_\alpha (i_\alpha)_*(\gamma_\alpha) = [S] \cdot [T] \in A^{k+l}(X).$$

As in the last case, we'll do this by imagining that we can deform S and T to cycles S_λ and T_λ on X intersecting transversely, and ask for the limiting position of the intersection $S_\lambda \cap T_\lambda$ as $\lambda \to 0$. We therefore suppose further that we can find families $\mathcal{S}$ and $\mathcal{T} \subset \Delta \times X$ over a smooth rational curve Δ such that

- $\mathcal{S}$ and $\mathcal{T} \subset \Delta \times X$ will be flat over Δ, with fibers $S_0 = S$ and $T_0 = T$, and $S_\lambda \cap T_\lambda$ is transverse for $\lambda \neq 0$;

and from here on in the argument flows exactly as before. Specifically, we can express the intersection $\mathcal{S} \cap \mathcal{T}$ of the $(k+1)$- and $(l+1)$-folds $\mathcal{S}$ and $\mathcal{T}$ in the $(n+1)$-fold $\Delta \times X$ as a union

$$\mathcal{S} \cap \mathcal{T} = \Phi \cup D$$

where Φ is flat over Δ, consisting of the points of intersection $S_\lambda \cap T_\lambda$ for $\lambda \neq 0$ and their limits—that is, the closure in $\Delta \times X$ of the intersection of $\mathcal{S} \cap \mathcal{T}$ with $(\Delta \setminus \{0\}) \times X$—and $D = \{0\} \times C$, as in Figure 15.2 above.

Now, we know that the cycle $\Xi = \Phi \cap (\{0\} \times X)$ has dimension $k+l-n$, and has class $[S] \cdot [T] \in A^{k+l}(X)$ as a cycle on X. Moreover, Ξ consists of the sum of its intersections $\Xi_\alpha = \Phi \cap D_\alpha$ with the connected components $D_\alpha = \{0\} \times C_\alpha$ of $S \cap T$. To find the class $[S] \cdot [T]$, accordingly, we have to figure out the class of the intersection $\Phi \cap D_\alpha$ for each connected component.

Finally, as before one way to characterize the points of $\Phi \cap D_\alpha$ is to observe that *they are the points $p \in D_\alpha$ where the tangent spaces $T_p\mathcal{S}$ and $T_p\mathcal{T}$ fail to intersect in T_pD_α*. Now, if we had

$$T_{D_\alpha} = (T_\mathcal{S})|_{D_\alpha} \cap (T_\mathcal{T})|_{D_\alpha} \quad \text{in} \quad (T_{\Delta \times X})|_{D_\alpha}$$

we would have a direct sum decomposition

$$N_{D_\alpha/\Delta \times X} = N_{\mathcal{S}/\Delta \times X}|_{D_\alpha} \oplus N_{\mathcal{T}/\Delta \times X}|_{D_\alpha}.$$

In general, we see that we have a map

$$N_{D_\alpha/\Delta \times X} \to N_{\mathcal{S}/\Delta \times X}|_{D_\alpha} \oplus N_{\mathcal{T}/\Delta \times X}|_{D_\alpha}$$

between vector bundles on D_α of ranks $k+l-m_\alpha+1$ and $k+l$, and the locus where this map fails to be injective (as a map to vector bundles) is exactly the cycle Ξ_α.

We have as before

$$c(N_{D_\alpha/\Delta \times X}) = c(N_{C_\alpha/X}).$$

and

$$N_{\mathcal{S}/\Delta \times X}|_{D_\alpha} = N_{S/X}|_{C_\alpha}$$

and likewise

$$N_{\mathcal{T}/\Delta \times X}|_D = N_{T/X}|_{C_\alpha}.$$

Now we can apply Porteous, to deduce that the class of $\Xi = \Phi \cap D_\alpha$ is

$$\gamma_\alpha = [\Xi_\alpha] = \left[\frac{c(N_{S/X}|_{C_\alpha}) \cdot c(N_{T/X}|_{C_\alpha})}{c(N_{C_\alpha/X})}\right]_{m_\alpha} \in A^{m_\alpha}(D_\alpha).$$

One restatement of this: from the sequence

$$0 \to N_{C_\alpha/S} \to N_{C_\alpha/X} \to (N_{S/X})|_{C_\alpha} \to 0$$

we have

$$c(N_{S/X}|_{C_\alpha}) = \frac{c(N_{C_\alpha/X})}{c(N_{C_\alpha/S})}$$

so we can rewrite this as

$$\gamma_\alpha = \left[\frac{c(N_{T/X}|_{C_\alpha})}{c(N_{C_\alpha/S})}\right]_{m_\alpha} \in A^{m_\alpha}(D_\alpha).$$

Applying the same identity to the normal bundle to T in X we have another alternative:

$$\gamma_\alpha = \left[\frac{c(N_{C_\alpha/S}) \cdot c(N_{C_\alpha/T})}{c(N_{C_\alpha/X})}\right]_{m_\alpha}.$$

In sum, we have the

Proposition 15.4 (Excess intersection formula)**.** *Let S and T be smooth subvarieties of the smooth n-dimensional variety X, of codimensions k and l respectively. Suppose that the intersection $S \cap T$ is a disjoint union of smooth varieties C_α, of codimension $k + l - m_\alpha$. Then subject to the hypothesis above—that there exist deformations of S and T intersecting transversely—we have*

$$\begin{aligned}
[S] \cdot [T] &= \sum_\alpha (i_\alpha)_* \left(\left[\frac{c(N_{S/X}|_{C_\alpha}) \cdot c(N_{T/X}|_{C_\alpha})}{c(N_{C_\alpha/X})}\right]_{m_\alpha} \right) \\
&= \sum_\alpha (i_\alpha)_* \left(\left[\frac{c(N_{T/X}|_{C_\alpha})}{c(N_{C_\alpha/S})}\right]_{m_\alpha} \right) \\
&= \sum_\alpha (i_\alpha)_* \left(\left[\frac{c(N_{C_\alpha/X})}{c(N_{C_\alpha/S}) \cdot c(N_{C_\alpha/T})}\right]_{m_\alpha} \right) \in A^{k+l}(X)
\end{aligned}$$

Finally (for now!), there is a formula analogous to Proposition 15.4 for the intersection of more than 2 subvarieties of X:

Proposition 15.5. *Let X be a smooth projective variety and let $S_1, \dots, S_l \subset X$ be subvarieties of codimensions k_i; let $k = \sum k_i$ be the expected codimension of their intersection. If the intersection $\cap S_\alpha$ is a disjoint union of smooth varieties C_α of codimension $k - m_\alpha$, then assuming the S_α admit deformations intersecting transversely,*

$$\begin{aligned}
\prod_i [S_i] &= \sum_\alpha (i_\alpha)_* \left(\left[\frac{\prod_i c(N_{S_i/X}|_{C_\alpha})}{c(N_{C_\alpha/X})}\right]_{m_\alpha} \right) \\
&= \sum_\alpha (i_\alpha)_* \left(\left[\frac{c(N_{C_\alpha/X})^{l-1}}{\prod_i c(N_{C_\alpha/S_i})}\right]_{m_\alpha} \right) \in A^k(X).
\end{aligned}$$

This, it should be said, is far from the definitive version of the excess intersection formula; see Section 15.6 for a discussion of its extensions and generalization.

15.2.1 Quadrics containing a linear space

As an application of Proposition 15.5, let's do keynote question (c). To recall, we're given k general quadric hypersurfaces $Q_1, \dots, Q_k \subset \mathbb{P}^n$ containing a codimension 2 plane $\Lambda \cong \mathbb{P}^{n-2} \subset \mathbb{P}^n$; we write

$$\bigcap_{i=1}^{k} Q_i = \Lambda \cup Z,$$

and we ask for the degree of Z.

To start, we can reduce to the case $k = n$ by intersecting with general hyperplanes; this ensures that the hypothesis $\Lambda \cap X = \varnothing$ is satisfied. Now, letting $\zeta \in A^1(\Lambda)$ be as usual the hyperplane class, we have

$$c(N_{Q_i/\mathbb{P}^n}|_\Lambda) = 1 + 2\zeta$$

and since the normal bundle $N_{\Lambda/\mathbb{P}^n} \cong \mathcal{O}_\Lambda(1)^{\oplus 2}$, we have

$$c(N_{\Lambda/\mathbb{P}^n}) = (1+\zeta)^2.$$

Thus the contribution of the component Λ to the degree of the intersection is given by

$$\alpha = \deg \left[\frac{(1+2\zeta)^n}{(1+\zeta)^2}\right]_{n-2} \in A^{n-2}(\Lambda) \cong \mathbb{Z}$$

To evaluate this expression, we may realize it as the residue at $z = 0$ of the meromorphic differential

$$\omega = \frac{(1+2z)^n}{(1+z)^2} \frac{dz}{z^{n-1}}$$

on the Riemann sphere. This is a differential with three poles: at $z = 0$, at $z = -1$ and at $z = \infty$; since the sum of the residues must be zero, in order to calculate $Res_0(\omega)$ it will suffice to find the residues at the other two poles. This is simpler: at $z = \infty$, in terms of the local coordinate $w = 1/z$ we can write

$$\omega = -\frac{(w+2)^n}{(w+1)^2} \frac{dw}{w}$$

from which we see that $Res_\infty(\omega) = -2^n$. Similarly, around the point $z = -1$ we take $u = z + 1$ and write

$$\begin{aligned}\omega &= \frac{(2u-1)^n}{(u-1)^{n-1}} \frac{du}{u^2}\\ &= -(1 - 2nu + \dots)(1 + (n-1)u + \dots)\frac{du}{u^2}\\ &= (-1 + (n+1)u + \dots)\frac{du}{u^2},\end{aligned}$$

from which we see that $Res_{-1}(\omega) = n + 1$. Altogether, then, we have

$$\alpha = 2^n - n - 1,$$

and correspondingly

$$\deg(Z) = 2^n - \alpha = n + 1$$

when $k = n$. In the general case, as we said, we can intersect with $n - k$ hyperplanes to reduce to the case $\dim Z = 0$ and deduce the answer to keynote question (c):

$$\deg(Z) = k + 1.$$

As a check, we can describe the residual intersection Z of k quadrics $Q_i \subset \mathbb{P}^n$ containing an $(n-2)$-plane Λ directly, as follows. Suppose Λ is the zero locus of the linear forms X_0 and X_1. Then each of the quadrics Q_i can be written as the zero locus of a linear combination of X_0 and X_1:

$$Q_i = V(F_i) \quad \text{where} \quad F_i = X_0 L_i + X_1 M_i$$

for some linear forms L_i and M_i. Now consider the $2 \times (k+1)$ matrix

$$\Phi = \begin{pmatrix} X_0 & M_1 & M_2 & \dots & M_k \\ X_1 & L_1 & L_2 & \dots & L_k \end{pmatrix}.$$

Away from the locus $\Lambda = V(X_0, X_1)$ where X_0 and X_1 both vanish, the rank 1 locus of Φ is just the intersection of the quadrics Q_i; in other words,

$$Z = \{X \in \mathbb{P}^n \mid \operatorname{rank}(\Phi(X)) \leq 1\}.$$

The degree of Z is then given by Porteous' formula: as we worked out in Section 14, this has degree $k + 1$.

Note, finally, one issue that we have swept under the rug here: if $n \geq 4$, no quadric $Q \subset \mathbb{P}^n$ containing an $(n-2)$-plane can be smooth, and so the quadrics Q_i cannot satisfy the hypotheses of (15.5). In fact, given that the $Q_i \subset \mathbb{P}^n$ are local complete intersections, this hypothesis is not necessary, as we'll discuss in Section 15.6.

15.3 Intersections in a subvariety

There is a special case of the excess intersection situation that arises quite frequently: when two cycles A and B on a smooth, projective variety X fail to intersect properly because they happen to both be contained in a proper subvariety. In this setting we can both state the theorem in a particularly simple and useful form, and give a proof independent of the constructions above.

To set this up, let X as usual be a smooth projective variety; suppose $i : Z \hookrightarrow X$ is a smooth subvariety of codimension m, and A and $B \subset Z$ subvarieties of codimensions a and b in Z respectively. We'll assume moreover that A and B intersect transversely in Z, so that the intersection $A \cap B$ is smooth of codimension $a+b$ in Z (and thus codimension $a+b+m$ in X). The question is, what can we say about the intersection product of the cycles A and B in X? Or, to put it differently, if $\alpha \in A^a(Z)$ and $\beta \in A^b(Z)$ are the classes of A and B as cycles in Z, how can we relate the product $\alpha\beta = [A \cap B] \in A^{a+b}(Z)$ to the product $i_*(\alpha)i_*(\beta) \in A^{a+b+2m}(X)$? The answer is expressed in the following Theorem:

Theorem 15.6. *Let $i : Z \to X$ be an inclusion of smooth projective varieties of codimension m, and let $N = N_{Z/X}$ be the normal bundle of Z in X.*

(a) *For any class $\alpha \in A(Z)$ we have*
$$i^*\big(i_*\alpha\big) = \alpha \cdot c_m(N) \in A^{a+m}(Z); \text{ and}$$
(b) *If $\alpha \in A^a(Z)$ and $\beta \in A^b(Z)$ then*
$$i_*\alpha \cdot i_*\beta = i_*\big(\alpha \cdot \beta \cdot c_m(N_{Z/X})\big) \in A^{a+b+2m}(X).$$

As we shall see, part (b) of Theorem 15.6 follows easily from part (a), but the first one requires a new idea. Recall that we have already proven the formula for the case when X is the total space of the compactification $\mathbb{P}(N_{Z/X} \oplus \mathcal{O}_X)$ of the normal bundle of Z (Proposition 11.19). The intuition behind part (a) of Theorem 15.6 in general is that a neighborhood of Z in X "looks like" a neighborhood of Z in its normal bundle. Indeed, if we were working with $\mathcal{C}^\infty$ manifolds a neighborhood in the classical topology would be $\mathcal{C}^\infty$ equivalent to a neighborhood of Z in its normal bundle. This $\mathcal{C}^\infty$ equivalence is, at least, enough to make i^*i_* behave the same in homology.

This idea doesn't give a proof of Theorem 15.6, even in the complex analytic case: rational equivalence is more subtle than homological equivalence, and the tubular neighborhood theorem that we used is false in the category of complex analytic or algebraic varieties. (For example, no analytic neighborhood of a conic curve $C \subset \mathbb{P}^2$ is biholomorphic to any neighborhood of the zero section in the normal bundle $N_{C/\mathbb{P}^2}$; see Exercise 15.25). However, there is a way out: a construction called "deformation to the normal cone" provides a flat degeneration from the neighborhood of Z in X to the neighborhood of Z in its normal bundle. This is one of the beautiful and important techniques integrated into the subject in Fulton [1984]. We first explain the construction.

15.3.1 Deformation to the normal cone

Suppose that X is a smooth projective variety of dimension n, and $Z \subset X$ a smooth subvariety of codimension m. Let
$$\mu : \mathcal{X} = Bl_{Z\times\{0\}}(X \times \mathbb{P}^1) \to X \times \mathbb{P}^1,$$
be the blowup of $X \times \mathbb{P}^1$ along the subvariety $Z \times \{0\}$, and write $E \subset \mathcal{X}$ for the exceptional divisor. As a variety, E is the projectivization of the

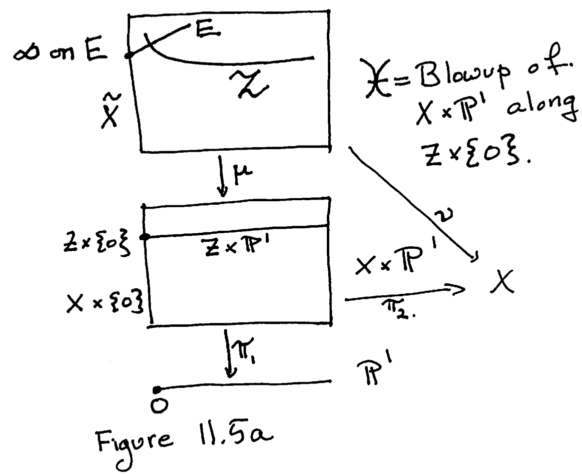

FIGURE 15.3. Deformation to the Normal Cone

normal bundle

$$N_{Z\times\{0\}/X\times\mathbb{P}^1} \cong N_{Z/X} \oplus \mathcal{O}_Z.$$

Thus E is the compactification of the total space of the normal bundle $N = N_{Z/X}$ described in Section **??**.

We think of $\mathcal{X}$ as a family of projective varieties over $\mathbb{P}^1$ via the composition $\pi_2 \circ \mu : \mathcal{X} \to X \times \mathbb{P}^1 \to \mathbb{P}^1$. The fibers X_t of $\mathcal{X}$ over $t \neq 0$ are all isomorphic to X, while the fiber X_0 of $\mathcal{X}$ over $t = 0$ consists of two irreducible components: the proper transform $\tilde{X}$ of $X \times \{0\}$ in $\mathcal{X}$ (isomorphic to the blow-up $Bl_Z(X)$), and the exceptional divisor $E \cong \mathbb{P}(N \oplus \mathcal{O}_Z)$, with the two intersecting along the "hyperplane at infinity" $\mathbb{P}N \subset \mathbb{P}(N \oplus \mathcal{O}_X)$ in E, which is the exceptional divisor in $\tilde{X} \cong Bl_Z(X)$.

Now consider the subvariety $Z \times \mathbb{P}^1 \subset X \times \mathbb{P}^1$, and let $\mathcal{Z}$ be its proper transform in $\mathcal{X}$. Because $Z \times \mathbb{P}^1$ intersects $X \times \{0\}$ transversely in $Z \times \{0\}$, the intersection $\mathcal{Z} \cap X_0$ consists exactly of the zero section $N_0 \subset |N| \subset \mathbb{P}(N \oplus \mathcal{O}_Z)$ corresponding to the sub-line bundle $\mathcal{O}_Z \subset N \oplus \mathcal{O}_Z$. In particular, all the fibers $\mathcal{Z}_t$ of $\mathcal{Z}$ are isomorphic to Z, and $\mathcal{Z}$ goes nowhere near the component $\tilde{X}$ of X_0.

Also, let

$$\nu = \pi_1 \circ \mu : \mathcal{X} \to X$$

be the composition of the blow-up map $\mu : \mathcal{X} \to X \times \mathbb{P}^1$ with the projection on the first factor. For $t \in \mathbb{P}^1$ other than 0, ν carries X_t isomorphically to X. As for the fiber X_0, on the component $\tilde{X} \subset X_0$ the map ν is the blow-down map $\tilde{X} = Bl_Z X \to X$, while on the component E the map ν is the composition $i \circ \pi$ of the bundle map $\pi : E \cong \mathbb{P}(N \oplus \mathcal{O}_Z) \to Z$ with the inclusion $i : Z \hookrightarrow X$. The situation is summarized in Figure 15.3.

The point of this construction is that as $t \in \mathbb{P}^1$ approaches 0, the neighborhood of Z in the fibers of the family $\mathcal{X} \to \mathbb{P}^1$ degenerates from the neighborhood of Z in X to the neighborhood of Z in the total space of its normal bundle in X. If we have a family $\{A_t\}_{t \in \mathbb{P}^1}$ of cycles in X that we'd like to intersect with Z, we can use this construction to transform the intersection of A_0 with Z into the intersection of the fiber of the proper transform of $\mathcal{A}$ in $\mathcal{X}$ with the zero section in the compactified normal bundle $E \cong \mathbb{P}(N \oplus \mathcal{O}_Z)$. We can use our knowledge of the Chow rings of projective bundles to analyze this intersection.

The idea of the following proof is that under the deformation to the normal cone the class $i^* i_* \alpha$ is deformed into the rationally equivalent class $i_N^* i_{N*} \alpha$, where

$$i_N : Z \to \mathbb{P}(N_{Z/X} \oplus \mathcal{O}_Z) = E$$

is the section sending Z to the zero locus $N_0 \subset |N| \subset \overline{N} := \mathbb{P}(N_{Z/X} \oplus \mathcal{O}_Z)$. By Proposition 11.19, $i_N^* i_{N*} \alpha = \alpha \cdot c_m(N)$

Proof of Theorem 15.6. We will use the notation introduced in the construction of the deformation to the normal cone, above.

We begin with the first formula of the Theorem. We may assume that α is the class of an irreducible subvariety $A \subset Z$. Let $\mathcal{A}$ and $\mathcal{Z}$ be the proper transforms of the subvarieties $A \times \mathbb{P}^1$ and $Z \times \mathbb{P}^1$. Since $Z \times \mathbb{P}^1$ meets $X \times \{0\}$ transversely, $\mathcal{Z} \cong Z \times \mathbb{P}^1$ via the projection, and this isomorphism also induces an isomorphism $\mathcal{A} \cong A \times \mathbb{P}^1$. To simplify notation, we will write $A_t \subset Z_t \subset X_t$ for the copies in the general fiber of $\mathcal{X}$ but we will write $A \subset Z \subset E$ instead of $A_0 \subset Z_0 \subset E$ for the fibers $A \subset Z$ contained in $E \subset X_0$.

By the moving lemma we can find a cycle $\mathcal{C}$ on $\mathcal{X}$ linearly equivalent to $\mathcal{A}$ and generically transverse to $\mathcal{Z}$, to Z, to E to X_t and to Z_t. The family $\mathcal{A}$ meets X_t and E generically transversely in A_t and A, respectively, so the equality $[\mathcal{C}] = [\mathcal{A}] \in A(\mathcal{X})$ restricts to equalities $i_*[A_t] = i_*[\mathcal{A} \cap X_t] = [\mathcal{C} \cap X_t] \in A(X_t)$, and $i_{N*}[A] = [\mathcal{C} \cap E] \in A(E)$. Since $\mathcal{C}$ meets Z_t and Z generically transversely as well, we have $i^* i_*[A_t] = [\mathcal{C} \cap Z_t]$ and $i_N^* i_{N*}[A] = [\mathcal{C} \cap E]$.

By generic transversality, neither Z_t nor Z can be contained in $\mathcal{C}$. It follows that, after removing any components that do not dominate $\mathbb{P}^1$, the cycle $\mathcal{C}$ in $\mathcal{Z} \cong Z \times \mathbb{P}^1$ is a rational equivalence between $i^* i_*[A]$ and $i_N^* i_{N*}[A]$. By Proposition 11.19, $i_N^* i_{N*}[A] = [A] \cdot c_m(N)$ as required.

Now it is easy to deduce the second formula of the Theorem. If $\beta = [Z]$, the formula follows from part (a) of Theorem 15.6 by taking the pushforward i_* of both sides. On the other hand, if $\beta \in A(Z)$ is arbitrary, we can use the push-pull formula: since

$$i^*(i_*\alpha) = \alpha \cdot c_m(N),$$

we can write

$$\begin{aligned} i_*\big(\alpha \cdot \beta \cdot c_m(N)\big) &= i_*\big(\beta \cdot i^*(i_*\alpha)\big) \\ &= i_*\alpha \cdot i_*\beta. \end{aligned}$$

□

15.4 Pullbacks

A variant of the excess intersection formula (more properly, a generalization of it) is the following. Suppose we have a morphism $f : X \to Y$ of smooth projective varieties, and $A \subset Y$ a subvariety of codimension a. As we've seen, if the preimage $f^{-1}(A) \subset X$ is generically reduced of codimension a in X, then

$$[f^{-1}(A)] = f^*([A]);$$

that is, the class of $f^{-1}(A)$ is the pullback of the class $\alpha = [A] \in A(X)$. But what if $f^{-1}(A)$ has codimension strictly less than a in X?

Of course, the moving lemma tells us that in this case we can find a cycle A' rationally equivalent to A and such that $f^{-1}(A')$ is generically reduced of the expected dimension. But it will not surprise the reader to learn that when $f^{-1}(A)$ has the wrong dimension, there is also a formula expressing the pullback class $f^*[A]$ in terms of the geometry of A and its preimage.

We'll derive this formula, as usual, under the additional hypothesis that both A and $f^{-1}(A)$ are smooth.

Now, our definition of the pullback $f^*\alpha$ in Chapter 2 was to take the cycles $X \times A$ and $\Gamma_f \subset X \times Y$, where Γ_f is the graph of f; we took the product of their classes in the Chow ring $A(X \times Y)$ and defined $f^*\alpha$ to be the pushforward of this product under the projection map $X \times Y \to X$ on the first factor. In that setting, if the cycles $X \times A$ and Γ_f failed to intersect generically transversely—that is, if the preimage $f^{-1}(A)$ were not generically reduced of codimension a—our prescription was to find another cycle in $X \times Y$ rationally equivalent to $X \times A$ that did, intersect that with Γ_f, and take the pushforward of that cycle. Given the content of this chapter, however, we don't need to: if we assume that A and $f^{-1}(A)$ are smooth, we can apply the excess intersection formula to the intersection of $X \times A$ with Γ_f.

Specifically, suppose that B is any connected component of $(X \times A) \cap \Gamma_f$ (that is, via the identification $\Gamma_f \cong X$, a connected component of $f^{-1}(A)$);

suppose that the codimension of B in $\Gamma_f \cong X$ is $a-m$. We associate to B the class

$$\gamma_B = \left[\frac{c(N_{X\times A/X\times Y})}{c(N_{B/\Gamma_f})}\right]_m$$
$$= \left[\frac{c(f^*N_{A/Y})}{c(N_{B/X})}\right]_m \in A^m(B).$$

By the excess intersection formula, then, the sum of the classes γ_B (or rather their images in $A(X)$) over all components B of $f^{-1}(A)$ will be the pullback class $f^*\alpha$. In other words, we have the

Proposition 15.7 (Excess intersection formula for pullbacks)**.** *Let $f : X \to Y$ be a morphism of smooth projective varieties and $A \subset Y$ a smooth subvariety of codimension a. Assuming $f^{-1}(A)$ is smooth, we have*

$$f^*[A] = \sum_B i_* \left[\frac{c(f^*N_{A/Y})}{c(N_{B/X})}\right]_{a-\mathrm{codim}(B\subset X)} \in A^a(X)$$

where the sum is taken over all connected components B of $f^{-1}(A)$, and $i : B \to X$ is the inclusion.

As with the excess intersection formula itself, we have proved this only under the assumption that suitable deformations of $A \subset Y$ exist; but the formula holds without this hypothesis; likewise, suitably interpreted, it applies more generally to local complete intersection cycles $A \subset Y$.

A classic case of this (and one of the most commonly applied) is when $f : X \to Y$ is a generically finite surjective map and we want to calculate its degree. If we can find a point $q \in Y$ such that f is étale over a neighborhood of q—that is, such that the differential df_p is an isomorphism for all $p \in f^{-1}(q)$—then all we have to do is to count the number of points in the fiber $f^{-1}(q)$. More generally, if we can find a point $q \in Y$ over which f is finite, we can count the points of $f^{-1}(q)$ with multiplicity to arrive at the degree of f. What we are saying now is that we can in fact express the degree of f in terms of the local geometry of f around a fiber $f^{-1}(q)$, *even when that fiber is positive-dimensional*: taking $A = \{q\}$ in the above, we have

Corollary 15.8. *Let $f : X \to Y$ be a generically finite surjective map of smooth projective varieties. If $q \in Y$ is any point such that $f^{-1}(q)$ is smooth, then*

$$\deg(f) = \sum_B s_{\dim B}(N_{B/X}),$$

where the sum is over connected components of $f^{-1}(q)$ and s denotes the Segre class.

For example, suppose $f : X = Bl_qY \to Y$ is the blow-up of an n-dimensional variety Y at a point q. Since the normal bundle to the exceptional divisor $E = f^{-1}(q) \cong \mathbb{P}^{n-1}$ is $\mathcal{O}_E(-1)$, we have

$$s(N_{E/X}) = \frac{1}{1-\zeta} = 1 + \zeta + \cdots + \zeta^{n-1};$$

we conclude the degree of f is 1.

It may seem unnecessary to use Proposition 15.8 in this case: after all, if we were truly ignorant of the fact that a blow-up map has degree 1, we could verify it immediately by looking at any point at all on the target other than the one point we blew up. But in fact there are many cases in real life where we know more about the geometry of a map in a neighborhood of a special fiber than we do about a general one, and a formula like this is essential. An absolutely beautiful example of this is the calculation by Donagi and Smith (Donagi and Smith [1980]) of the degree of the Prym map in genus 6. This is a map from the space R_6 of unramified covers of curves of genus 6 to the space A_5 of abelian varieties of dimension 5 (both of dimension 15) defined by the Prym construction; while it does not seem possible to enumerate the points of a general fiber, Donagi and Smith are able to calculate its degree by looking at a very special point (the Prym of a double cover of a smooth plane quintic) over which the fiber has three components: a point, a curve and a surface!

15.4.1 The double point formula

One particularly nice application of Proposition 15.7 is the *double point formula*, a formula describing the locus where a map $f : X \to Y$ fails to be one-to-one. The situation is this: we consider a map $f : X \to Y$ of smooth projective varieties, with $m = \dim X \le n = \dim Y$; we assume that f maps X birationally onto its image in Y. We'd expect the locus of pairs $p, q \in X$ with $p \neq q$ but $f(p) = f(q)$ to have dimension $2m - n$ (if this is not apparent, it will be soon); we'd like to know its class if it does.

At very first glance, this may seem completely straightforward: the locus Φ of pairs $(p, q) \in X \times X$ such that $f(p) = f(q)$ is the preimage, under the map

$$f \times f : X \times X \to Y \times Y,$$

of the diagonal $\Delta_Y \subset Y \times Y$. The problem is, the preimage $\Phi = (f \times f)^{-1}(\Delta_Y)$ won't have the expected dimension when $m < n$: it's supposed to have codimension n in $X \times X$, but contains the diagonal $\Delta_X \subset X \times X$ as a component of codimension m. We're interested in the union of the remaining components, that is, the closure Φ_0 of the locus $(f \times f)^{-1}(\Delta_Y) \cap (X \times X \setminus \Delta_X)$.

This is exactly the situation that Proposition 15.7 is designed to deal with. The normal bundle of the diagonal $\Delta_Y \subset Y \times Y$ is just the tangent bundle to $\Delta_Y \cong Y$, and likewise for X. The class $\gamma_{\Delta_X} \in A^{n-m}(\Delta_X)$ associated to the component Δ_X of $(f \times f)^{-1}(\Delta_Y)$ by Proposition 15.7 is thus

$$\gamma_{\Delta_X} = \left[\frac{f^* c(T_Y)}{c(T_X)}\right]_{n-m}.$$

Assuming the union Φ_0 of the remaining components of $(f \times f)^{-1}(\Delta_Y)$ are generically reduced of the expected dimension $2m - n$, then, we have the *double point formula*:

$$[\Phi_0] = (f \times f)^*(\delta_Y) - \left[\frac{f^* c(T_Y)}{c(T_X)}\right]_{n-m}.$$

Let's see how this plays out in a simple case: a map f from a smooth curve C of genus g to a nodal plane curve $C_0 \subset \mathbb{P}^2$ of degree d. (Here we take $X = C$, $Y = \mathbb{P}^2$; hence $m = 1$ and $n = 2$.) Let $\zeta \in A^1(\mathbb{P}^2)$ be the class of a line. We have to begin with

$$\deg(c_1(T_C)) = 2 - 2g \quad \text{and} \quad \deg(f^* c_1(T_{\mathbb{P}^2})) = 3d.$$

We know the class $\delta_{\mathbb{P}^2}$ of the diagonal $\Delta_{\mathbb{P}^2} \subset \mathbb{P}^2 \times \mathbb{P}^2$: it's

$$\delta_{\mathbb{P}^2} = \pi_1^*\zeta^2 + \pi_1^*\zeta\pi_2^*\zeta + \pi_2^*\zeta^2,$$

and since $f^*(\zeta^2) = 0$, so that

$$\deg\big((f \times f)^*(\pi_i^*\zeta^2)\big) = 0,$$

this reduces to

$$\deg\big((f \times f)^*(\delta_Y)\big) = d^2$$

Altogether, then, the number of pairs $(p, q) \in C \times C$ with $p \neq q$ but $f(p) = f(q)$ is

$$\deg(\Phi_0) = d^2 - 3d + 2 - 2g$$

and the number δ of nodes of C_0 is half that, that is

$$\delta = \binom{d-1}{2} - g.$$

There is an instructive alternative interpretation of this formula, based on the two notions of the normal bundle. To begin with, since $C_0 \subset \mathbb{P}^2$ is a Cartier divisor, it has a normal bundle

$$N = N_{C_0/\mathbb{P}^2} = \mathrm{Hom}(\mathcal{I}_{C_0}, \mathcal{O}_{C_0})^* \cong \mathcal{O}_{C_0}(d).$$

Sections of N correspond to first-order deformations of C_0 as a subscheme of $\mathbb{P}^2$, and zeros of such sections correspond to points of intersection of C_0 with the deformed curve. (More precisely, if $\{C_t\}$ is an arc in the space $\mathbb{P}^N$ of all

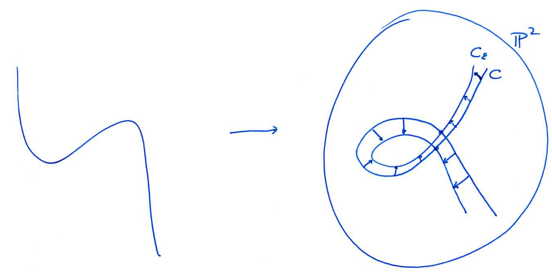

FIGURE 15.4. The deformed curve C_ϵ meets C twice near the node $p \in C$, even though the corresponding section of the normal bundle of the map is nonzero.

plane curves of degree d, the zeros of the section σ of N corresponding to the restriction of this family to $\operatorname{Spec} K[t]/(t^2)$ are the limit points $\lim_{t\to 0}(C_0 \cap C_t)$.) From this description, or just the fact that $N \cong \mathcal{O}_{C_0}(d)$, we see that the degree of N is d^2.

On the other hand, we have the normal bundle N_f of the map f: this is the line bundle on C defined as the cokernel of the differential $df : T_C \to f^*T_{\mathbb{P}^2}$. From the exact sequence

$$0 \to T_C \to f^*T_{\mathbb{P}^2} \to N_f \to 0$$

we conclude that the degree of N_f is

$$\deg(N_f) = c_1(f^*T_{\mathbb{P}^2}) - c_1(T_C) = 3d + 2g - 2.$$

Now, sections of N_f correspond to deformations of the map $f : C \to \mathbb{P}^2$, and again the zeros of such a section correspond to points of intersection of the deformed curve with C_0. The difference is at the nodes of C_0: if we have a deformation of the map f corresponding to a section σ of N_f nonzero at the points $q, r \in C$ lying over a node $p \in C_0$, the local picture of the deformation is as shown in Figure (15.4). Specifically, the deformed curve C_ϵ intersects C_0 twice in a small neighborhood of p_0 even though σ is nonzero at q and r; these two points of intersection count toward the degree $d^2 = [C]^2$ of the normal bundle $N = N_{C_0/\mathbb{P}^2}$, but not toward the degree of N_f. The difference of the degrees is thus twice the number δ of nodes on C_0, from which we conclude that

$$2\delta = d^2 - 3d - (2g - 2)$$

as before.

This is in fact a general way of viewing the double point formula applied to a map $f : X \to Y$: for example, in case $\dim Y = 2 \dim X$, so that we expect a finite number δ of double points, this number will be one-half the difference between the self-intersection of the image $f(X) \subset Y$, and the top Chern class of the normal bundle of the map f.

Finally, it's interesting to see what happens when the map f isn't an immersion—for example, if $f : \tilde{C} \to C_0$ is the normalization of a curve $C_0 \subset \mathbb{P}^2$ with a cusp p. What we see is that the scheme structure on the preimage $(f \times f)^{-1}(\Delta_Y)$ reflects the fact that f is not an immersion at p: the scheme $(f \times f)^{-1}(\Delta_Y)$ has an embedded point of multiplicity 2 at (p, p).

A good way to see this is to realize $f : \tilde{C} \to C_0$ as the limit of a one-parameter family of maps $f_t : \{\tilde{C}_t \to C_t\}$ with C_t smooth $\forall t$, such that for $t \neq 0$ the curve C_t has a node at a point p_t, with $\lim_{t \to 0} p_t = p$—for example, the family of maps $f_t : \mathbb{P}^1 \to \mathbb{P}^2$ given in affine coordinates by

$$f_t : \lambda \mapsto (\lambda^2 + t, \lambda^3 + \lambda t)$$

(The image curve C_t is the cubic $y^2 = x^2(x - t)$.) Here, for each $t \neq 0$, the scheme $(f_t \times f_t)^{-1}(\Delta_{\mathbb{P}^2})$ contains the two points $(\sqrt{t}, -\sqrt{t})$ and $(-\sqrt{t}, \sqrt{t})$, from which we see that $(f \times f)^{-1}(\Delta_{\mathbb{P}^2})$ has an embedded point at $(0, 0)$.

In fact, in this sort of situation it is still possible to define a "double point scheme" in a way that both reflects the failure of f to be an embedding, and such that its class (assuming it has the expected codimension) is given by the formula above. We let $\pi : Z = Bl_{\Delta_X}(X \times X) \to X \times X$ be the blow-up along the diagonal, with $E \subset Z$ the exceptional divisor—that is, the inverse image of the diagonal. We now take the preimage W of Δ_Y under the composition $\pi \circ f$, and since W contains E, we divide by the ideal of E: that is, we take $\tilde{D}_f$ to be the scheme defined by the ideal

$$\mathcal{I}_{\tilde{D}_f} = (\mathcal{I}_E, \mathcal{I}_W).$$

The *double point scheme* D_f of f is then defined to be the image of $\tilde{D}_f$ under π. Moreover, the class of D_f is, as we said, given by the double point formula (see Fulton [1984] for a proof).

15.5 Chow ring of a blow-up

We now have all the necessary ingredients to describe the Chow ring of a blow up—or at any rate the blow-up of a smooth projective variety along a smooth subvariety. We start with a partial description—we'll give a set of generators for the Chow ring of a blow up, and say how to calculate their products—and then illustrate this with some examples in Section 15.5.2. Finally, in Section 15.5.3 we'll complete our description by giving all relations among our generators.

We begin by establishing the notation we'll use throughout the remainder of this section. Let X be a smooth projective variety and $Z \subset X$ a smooth subvariety of codimension m; we'll denote by $i : Z \hookrightarrow X$ the inclusion.

Let $\pi : W = Bl_Z X \to X$ be the blow-up of X along Z. We'll denote by $E \subset W$ the exceptional divisor, and let $j : E \hookrightarrow W$ be the inclusion. Finally, if $N = N_{Z/X}$ is the normal bundle of Z in X, then in terms of the identification $E \cong \mathbb{P}N$ we'll let $\zeta \in A^1(E)$ be as usual the first Chern class of the line bundle $\mathcal{O}_{\mathbb{P}N}(1)$. Altogether, we have the diagram

$$\begin{tikzcd} W = Bl_Z X \arrow[d, "\pi"'] & E \arrow[l, "j"'] \arrow[d, "\pi_E"] \\ X & Z \arrow[l, "i"'] \end{tikzcd}$$

15.5.1 Preliminary description

We start with a couple basic facts about the geometry of the exceptional divisor of a blow up:

Lemma 15.9. *Let X, Z, W and E be as above. We have an identifiction*

$$E = \mathbb{P}N_{Z/X},$$

and in terms of this we have

$$N_{E/W} = \mathcal{O}_{\mathbb{P}N_{Z/X}}(-1).$$

In other words, the normal bundle of E in W is the tautological subbundle on $\mathbb{P}N_{Z/Y}$.

Proof. We start by recalling the characterization of the blowup given in Hartshorne [1977] or in Theorem IV-23 of Eisenbud and Harris [2000]: if we let $\mathcal{O} = \mathcal{O}_X$ be the structure sheaf of X and $\mathcal{I} = \mathcal{I}_Z \subset \mathcal{O}_X$ the ideal sheaf of Z, then the blowup W is Proj of the *Rees algebra*

$$\mathcal{A} = \mathcal{O} \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \dots .$$

The preimage E of $Z = V(\mathcal{I}) \subset X$ is then

$$E = \mathrm{Proj}(\mathcal{A} \otimes \mathcal{O}/\mathcal{I}) = \mathrm{Proj}\big(\mathcal{O}/\mathcal{I} \oplus \mathcal{I}/\mathcal{I}^2 \oplus \mathcal{I}^2/\mathcal{I}^3 \oplus \dots\big);$$

and if $Z \subset X$ is smooth (or more generally a local complete intersection), so that

$$\mathcal{I}^k/\mathcal{I}^{k+1} = \mathrm{Sym}^k N^*_{Z/X},$$

this says that

$$E = \mathrm{Proj}(\mathrm{Sym}\, N^*_{Z/X}) = \mathbb{P}N_{Z/X}$$

as desired. Similarly, the conormal bundle

$$N^*_{E/W} = \mathcal{I}_E/\mathcal{I}_E^2$$

of $E \in W$ is the line bundle associated to the module

$$\mathcal{I}/\mathcal{I}^2 \oplus \mathcal{I}^2/\mathcal{I}^3 \oplus \cdots = \mathcal{O}_{\mathbb{P}N_{Z/X}}(1),$$

so that

$$N^*_{E/W} = \mathcal{O}_{\mathbb{P}N_{Z/X}}(1)$$

as stated. □

It's also useful to see the statement of Lemma 15.9 on the level of vector spaces. To do this, let $p \in Z$ be a point, and let $q \in \pi^{-1}(p)$ be a point in the fiber of E over p; let $L \subset (N_{Z/Y})_p$ be the one-dimensional subspace of the normal space to Z at p corresponding to q. We claim that

(a) the differential $d\pi_q$ of π at q has rank exactly $k+1$;

(b) the kernel of $d\pi_q$ is the tangent space $T_q(\pi^{-1}(p)) \subset T_qX$ to the fiber of π through q, and

(c) the image of $d\pi_q$ is the subspace $\tilde{L} = L + T_pZ \subset T_pY$.

Given these, Lemma 15.9 follows, at least on the level of fibers: the image $d\pi_q(T_qE)$ of the tangent space to E is the tangent space $T_pZ \subset T_pY$; and taking the quotient by this the differential $d\pi_q$ induces an isomorphism between the normal space $(N_{E/X})_q = T_qX/T_qE$ and the subspace $L = \tilde{L}/T_pZ$.

As for the claim, observe that since the tangent space $T_q(\pi^{-1}(p)) \subset T_qX$ to the fiber of π through q is clearly in the kernel of $d\pi_q$, the rank of $d\pi_q$ is at most $k+1$; to prove the claim we need only show that its image contains $\tilde{L}$. For this, simply take a smooth arc γ in Y passing through the point p, with normal direction L to Z at p; the proper transform of the arc will then pass through q and map isomorphically to γ, showing that $L \subset \operatorname{Im}(d\pi_q)$.

We return to the description of the Chow ring of the blowup. With X, Z, W and E still as above, let $N = N_{Z/X}$ be the normal bundle of Z in X. In terms of the identification $E \cong \mathbb{P}N$ let $\zeta \in A^1(E)$ be the first Chern class of the line bundle $\mathcal{O}_{\mathbb{P}N}(1)$, so that the first Chern class of the normal bundle of E in W is $-\zeta$. In these terms, we have the following preliminary description of the Chow ring of W:

Proposition 15.10. *The Chow ring $A(W)$ is generated by $\pi^*A(X)$ and $j_*A(E)$, that is, classes pulled back from X and classes supported on E. The rules for multiplication are:*

$$\begin{aligned} \pi^*\alpha \cdot \pi^*\beta &= \pi^*(\alpha\beta), \quad \textit{for } \alpha, \beta \in A(X); \\ \pi^*\alpha \cdot j_*\gamma &= j_*(\gamma \cdot \pi_E^* i^*\alpha), \quad \textit{for } \alpha \in A(X) \textit{ and } \gamma \in A(E); \textit{ and} \\ j_*\gamma \cdot j_*\delta &= -j_*(\gamma \cdot \delta \cdot \zeta), \quad \textit{for } \gamma, \delta \in A(E) \end{aligned}$$

Proof. The first two multiplication formulas are elementary: the first simply says that the pullback $\pi^* : A(X) \to A(W)$ is a ring homomorphism, and the second follows by choosing a cycle representing the class α and generically transverse to Z. The only formula that may seem strange at first glance is the third; and this is just Proposition 15.6, given that the normal bundle of E in W has first Chern class $c_1(N_{E/W}) = c_1(\mathcal{O}_{\mathbb{P}N}(-1)) = -\zeta$.

As for the fact that $A(W)$ is generated by $\pi^*A(X)$ and $j_*A(E)$, suppose that $A \subset W$ is any irreducible subvariety not contained in E, with class α; we claim that the difference $\alpha - \pi^*\pi_*\alpha$ can be represented by a cycle supported on E. To see this, invoke the moving lemma and let $\{B_t\}$ be a one-parameter family of cycles in X with $B_0 = \pi(A)$ and B_t generically transverse to Z for $t \neq 0$. Let $A_t = \pi^*B_t$ for general t, and let A_0 be the limit of the cycles A_t as $t \to 0$. Since π is an isomorphism away from E and Z, the cycle A_0 agrees with A on $W \setminus E$, so that the difference $A - A_0$ is supported on E. $\square$

Given our knowledge of the Chow rings of projective bundles, Proposition 15.10 says that if we know four things—the Chow rings of X and Z, the pullback/restriction map $i^* : A(X) \to A(Z)$ and the Chern classes of the normal bundle $N = N_{Z/X}$—then we can calculate in the Chow ring of W.

15.5.2 Example: the blowup of $\mathbb{P}^3$ along a curve

The first case of a blow-up of a positive-dimensional variety is already interesting. For the following, let $C \subset \mathbb{P}^3$ be a smooth curve of degree d and genus g, and let $\pi : W \to \mathbb{P}^3$ be the blow-up of $\mathbb{P}^3$ along C. We'll let E, i and j be as in the general discussion above, so that we have the diagram

$$\begin{tikzcd} W = Bl_C\mathbb{P}^3 \arrow[d, "\pi"'] & E \cong \mathbb{P}N \arrow[l, "j"'] \arrow[d, "\pi_E"] \\ \mathbb{P}^3 & C \arrow[l, "i"'] \end{tikzcd}$$

In addition, we'll let $h \in A^1(\mathbb{P}^3)$ be the class of a plane, and $l = h^2 \in A^2(\mathbb{P}^3)$ the class of a line; we'll denote by $\tilde{h} = \pi^*h \in A^1(W)$ and $\tilde{l} = \pi^*l \in A^2(W)$ their pullbacks to W. Finally, for $D \in Z^1(C)$ any divisor, we'll denote by $F_D = \pi_E^*D \in Z^1(E)$ the corresponding linear combination of fibers of $E \to C$, and similarly for divisor classes.

In these terms, we can describe $A(W)$ as follows. To begin with, $A^0(W)$ and $A^3(W) \cong \mathbb{Z}$, generated by the fundamental class of W and the class of a point, respectively. The group $A^1(W)$ is also not hard to determine:

it's generated by the classes $\tilde{h}$ and e, which are independent (π_* sends $\tilde{h}$ to h and e to 0). Lastly, by Proposition 15.10, the group $A^2(W)$ is generated by the classes $\tilde{l}$ (the pullback of $A^2(\mathbb{P}^3)$), and the classes $j_*\zeta$ and j_*F_D for $D \in A^1(C)$.

Proposition 15.10 also tells us how to multiply these generators. To begin with, the intersection products on $A^1(W)$ are given by

$$\tilde{h}^2 = \tilde{l}, \quad \tilde{h} \cdot e = j_*F_H \quad \text{and} \quad e^2 = -j_*\zeta,$$

where H is the hyperplane divisor class on C. Note that the first two are clear on a cycle level, if we represent the class $\tilde{h}$ by the preimage of a general plane in $\mathbb{P}^3$; the third is less straightforward.

The pairing between $A^1(W)$ and $A^2(W)$ is similarly easy to write down: we have

$$\deg(\tilde{h} \cdot l) = 1; \quad \deg(\tilde{h} \cdot j_*f_D) = 0$$

for any divisor class D on C, and

$$\deg(\tilde{h} \cdot j_*\zeta) = d.$$

(The first two can be seen on a cycle level; the last can be seen by observing that $\pi_*(j_*\zeta) = [C] = dl \in A^2(\mathbb{P}^3)$ and invoking push-pull.) Likewise,

$$\deg(e \cdot l) = 0; \quad \deg(e \cdot j_*f_D) = -\deg D$$

for any divisor class D on C, and

$$\deg(e \cdot j_*\zeta) = -\deg c_1(N_{C/\mathbb{P}^3}) = -4d - 2g + 2.$$

We can combine these to describe the triple products of the generators $\tilde{h}$ and e of $A^1(W)$:

$$\begin{aligned} &\deg(\tilde{h}^3) = 1; \\ &\deg(\tilde{h}^2 e) = 0; \\ &\deg(\tilde{h} e^2) = -d; \quad \text{and} \\ &\deg(e^3) = -4d - 2g + 2 \end{aligned} \tag{15.2}$$

Three surfaces in $\mathbb{P}^3$ revisited. As a first application of this description, we want to go back and revisit the first keynote question of this chapter, or rather its generalization to Proposition 15.3. To recall, we have three surfaces S, T and $U \subset \mathbb{P}^3$, whose intersection consists of the disjoint union of a smooth curve C of degree d and genus g, and a zero-dimensional scheme Γ; we want to determine the degree of Γ.

Now, to a classical geometer, the problem of how to get rid of unwanted components of the intersection of subvarieties of a given variety has an immediate answer: blow up. In this case, that means blowing up the curve $C \subset \mathbb{P}^3$, and indeed our hypotheses ensure that this works: if we let $\tilde{S}$,

$\tilde{T}$ and $\tilde{U}$ be the proper transforms of the three surfaces we started with, then the fact that the intersection $S \cap T \cap U$ is reduced along C says that the proper transforms $\tilde{S}$, $\tilde{T}$ and $\tilde{U}$ have no intersection along E. In other words,

$$\tilde{S} \cap \tilde{T} \cap \tilde{U} = \pi^{-1}(\Gamma)$$

and in particular,

$$\deg(\Gamma) = \deg\big([\tilde{S}] \cdot [\tilde{T}] \cdot [\tilde{U}]\big).$$

Moreover, in terms of the generators of $A(W)$ above, we have

$$[\tilde{S}] = s\tilde{h} - e; \quad [\tilde{T}] = t\tilde{h} - e \quad \text{and} \quad [\tilde{U}] = u\tilde{h} - e$$

and so our answer is

$$\deg(\Gamma) = \deg\big((s\tilde{h} - e)(t\tilde{h} - e)(u\tilde{h} - e)\big) \in A^3(X) \to \mathbb{Z}.$$

By our calculation (15.2) of triple products in $A(W)$, this yields

$$\deg(\Gamma) = stu - d(s + t + u) + 4d + 2g - 2$$

as before.

Tangencies along a curve. We can also use the setup here to answer keynote question (d) of this chapter: if $C \subset \mathbb{P}^3$ is a smooth curve of degree d and genus g, and $S, T \subset \mathbb{P}^3$ smooth surfaces of degrees s and t containing C, at how many points of C are S and T tangent?

In fact, what we're asking here is simply the intersection of the proper transforms $\tilde{S}$ and $\tilde{T}$ with the exceptional divisor; and the intersection number is given by

$$\begin{aligned}\deg\big([\tilde{S}] \cdot [\tilde{T}] \cdot [E]\big) &= \deg\big((s\tilde{h} - e)(t\tilde{h} - e)e\big) \\ &= -(s + t)\deg(\tilde{h}e^2) + \deg(e^3) \\ &= (s + t)d - 4d - 2g + 2.\end{aligned}$$

Thus, for example, two planes meeting along a line are nowhere tangent; but two quadrics Q_1, Q_2 containing a twisted cubic curve C will be tangent twice along C—as we can see directly, since the intersection $Q_1 \cap Q_2$ will consist of the union of C and a line meeting C twice.

15.5.3 Complete description of the Chow ring of a blowup

At the outset of this section, we gave a set of generators for the Chow ring of a blowup, and showed how to calculate their products. To complete our description of the Chow ring, we'll state the following:

Theorem 15.11. *Let $i : Z \to X$ be the inclusion of a smooth subvariety of codimension m in a smooth variety X, $\pi : W \to X$ the blowup of X along Z and E the exceptional divisor, with inclusion $j : E \to W$. If Q is the universal quotient bundle on $E \cong \mathbb{P}N_{Z/X}$, there is a split exact sequence of additive groups, preserving the grading by dimension:*

$$0 \longrightarrow A(Z) \xrightarrow{\begin{pmatrix} i_* \\ h \end{pmatrix}} A(X) \oplus A(E) \xrightarrow{\begin{pmatrix} \pi^* & j_* \end{pmatrix}} A(W) \longrightarrow 0$$

where $h : A(Z) \to A(E)$ is defined by $h(\alpha) = -c_{m-1}(Q)\pi_E^*(\alpha)$.

Proof. Perhaps the most mysterious element in this statement at first glance is the presence of the factor $c_{m-1}(Q)$ in the definition of h. In fact, this comes from the excess intersection formula for pullbacks, that is, Proposition 15.7: Q should be regarded as the the pull-back of the normal bundle of Z in X modulo the normal bundle $\mathcal{O}_{\mathbb{P}E}(-1)$ of E in W, and $m-1$ is the difference $\operatorname{codim}_X Z - \operatorname{codim}_W E$. Proposition 15.7 then tells us exactly that the two maps in the exact sequence compose to zero. As for the surjectivity of the right-hand map, this was part of our preliminary description of $A(W)$ in Proposition 15.10.

Finally, to prove that the left-hand map is a split monomorphism, it is enough to prove that h is a split monomorphism. We will show that $\pi_E \circ h = 1$ on $A(Z)$. To this end, we compute the Chern class of Q in terms of ζ and the Chern class of N:

$$c(Q) = \frac{c(N)}{1-\zeta}; \quad \text{so} \quad c_{m-1}(Q) = \zeta^{m-1} + c_1(N)\zeta^{m-2} + \cdots + c_{m-1}(N),$$

where we are regarding the $c_i(N)$ as elements of $A(E)$ via the ring homomorphism π_E^*. Since $c_{m-1}(Q)$ is monic in ζ, Lemma 11.10 shows that

$$\pi_{E*}(h(\alpha)) = \alpha$$

as required. □

15.6 The excess intersection formula in general

In Sections 15.1 and 15.2, we tried to indicate why we might hope that an excess intersection formula exists, and why it should have the particular form that it does. To do this, we made unnecessarily restrictive hypotheses, such as the smoothness of the cycles intersected and their deformability. In fact, suitably generalized it holds far more broadly; and while we are not in a position to prove it in this generality we should at least discuss the correct statement of the theorem.

To begin, recall the situation of Proposition 15.5: we have a smooth projective variety X and smooth subvarieties $S_1, \ldots, S_l \subset X$ of codimensions k_i; we let $k = \sum k_i$ be the expected codimension of their intersection. We suppose that the intersection $\cap S_\alpha$ is a disjoint union of smooth varieties C_α of codimension $k - m_\alpha$; assuming the S_α admit deformations intersecting transversely, we have

$$\prod_i [S_i] = \sum_\alpha (i_\alpha)_* \left(\left[\frac{\prod_i c(N_{S_i/X}|_{C_\alpha})}{c(N_{C_\alpha/X})} \right]_{m_\alpha} \right). \tag{15.3}$$

To start with, one remarkable fact we can state immediately is that this formula *holds without the assumption that the cycles S_i admit any deformations in X.*

Even more importantly, in the context of Proposition (15.5) *we do not need to assume that any of X, the S_i or their intersection $\cap S_i$ are smooth.* This may seem a strange assertion for four reasons:

(a) The statement of Proposition (15.5) involves Chern classes of normal bundles, which are not locally free for arbitrary subvarieties $S \subset X$.

(b) Even if the normal bundles of the subvarieties $S \subset X$ are locally free, we haven't defined Chern classes for bundles on singular varieties.

(c) The products in the right hand side of this expression take place in the Chow groups of the components C_α of the intersection; and as we've pointed out (e.g., in Example 5.19), there is no intersection pairing on the Chow groups of arbitrary varieties.

(d) The left hand side of this formula is a product of Chow classes in $A(X)$—and, as we just pointed out, this is not defined when X is singular.

Thus, to extend Proposition (15.5) to the more general setting, we need first to say what we mean by the terms in the formula. We'll consider each issue in turn.

The first issue is resolved in part if we make the milder restriction that the subvarieties S_i being intersected are locally complete intersection subvarieties of X (so that, for example, if X is smooth they can be arbitrary hypersurfaces). In this setting, we can define

$$N_{S/X} = \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{I}_S/\mathcal{I}_S^2, \mathcal{O}_S) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_S, \mathcal{O}_S),$$

a vector bundle of rank equal to the codimension of S_i in X.

As for the second objection, this is simply a defect of our chosen definition of Chern classes: in fact, it's possible to extend the definition to vector bundles on arbitrary schemes, without losing any of their essential properties. For example, we could take the formula of Theorem 11.9 as a *definition*: that is, we prove first that if E is a vector bundle of rank $r+1$ over an arbitrary scheme X and $\mathbb{P}E = \operatorname{Proj}\operatorname{Sym} E^*$ its associated projective bundle, then we have a well-defined line bundle $\mathcal{O}_{\mathbb{P}E}(1)$ on $\mathbb{P}E$ such that $\zeta = c_1(\mathcal{O}_{\mathbb{P}E}(1))$ restricts to the hyperplane class on each fiber, and we can show that

$$A(\mathbb{P}\mathcal{E}) \cong \bigoplus_{i=0}^{r} \zeta^i A(X)$$

just as in the smooth case (see Fulton [1984] Chapter 3). We can then use this result to *define* the Chern classes of E as the coefficients in the expression of $-\zeta^{r+1}$ as a linear combination of the classes $1, \zeta, \dots, \zeta^r$.

Finally, what about the issue that on a singular variety, the Chow groups do not form a ring? The answer here is that we are not multiplying arbitrary classes, but only the Chern classes of vector bundles, and it's possible to define the intersection of the Chern classes of a bundle E on a scheme X with the class of an arbitrary subvariety $A \subset X$ simply as the Chern classes of the restriction $E|_A$.

Using these ideas we can make sense of formula (15.3) whenever the S_i and C_α are locally complete intersections. But even when the S_i satisfy this, there is usually no reason for the components C_α of their intersection to do so, so it is worthwhile weakening the hypothesis on C. To do so, we single out one of the S_i, say $S = S_l$, and for each of the components $C = C_\alpha$ of the intersection consider the fraction $c(N_{S/X}|_C)/c(N_{C/X})$. If we assume for a moment that S and C are both smooth, then from the exact sequence

$$0 \longrightarrow \mathcal{I}_{S/X} \longrightarrow \mathcal{I}_{C/X} \longrightarrow \mathcal{I}_{C/S} \longrightarrow 0$$

we get an exact sequence of vector bundles on C,

$$0 \longrightarrow N_{C/S} \longrightarrow N_{C/X} \longrightarrow N_{S/X}|_C \longrightarrow 0,$$

whence

$$\frac{c(N_{S/X}|_C)}{c(N_{C/X})} = 1/c(N_{C/S}),$$

and we recall that this latter is the Segre class $s(N_{C/S})$. Thus we could modify formula (15.3) to read

$$\prod_i [S_i] = \sum_\alpha (i_\alpha)_* \left(\left[\prod_{i=1}^{l-1} c(N_{S_i/X}|_{C_\alpha}) s(N_{C_\alpha, S_l}) \right]_{m_\alpha} \right).$$

This might be a pointless exercise in relabeling, but for one crucial fact: *the definition of the Segre class $s(N_{C/S})$ of the normal bundle of C in S can be generalized to any pair of schemes $C \subset S$.*

The case where A is a projective variety will suffice for our purposes:

Definition 15.12. Let $C \subset S$ be a closed subscheme of a projective variety S. Let $W = Bl_C S$ be the blowup of S along C, and let $E \subset W$ be the exceptional divisor, so that

$$E = \operatorname{Proj} \bigoplus_{i \geq 0} \mathcal{I}^i_{C/S}/\mathcal{I}^{i+1}_{C/S},$$

with tautological line bundle $\mathcal{O}_E(1)$. Let $\pi : E \to Z$ be the projection. The Segre class $s(C, S)$ of C in S is defined to be

$$s(C, S) = \sum_j \pi_*\big(c_1(\mathcal{O}_E(1))^j\big) \in A(C);$$

equivalently, if we denote by e the class of E in $A(W)$,

$$s(C, S) = \sum_j \pi_*\big((-1)^j (e|_E)^j\big) \in A(C).$$

The point is, even if C, S and E are singular, the powers of the first Chern class of a line bundle on E are well-defined, as are their push-forwards. (Note, however, that we can't define the Chern class of the normal bundle $N_{C/S}$ as the inverse of $s(C, S)$: that would require taking products of the graded pieces of $s(C, S)$, and unless C is smooth we have no assurance that we can do so.) Observe that in case $C \subset S$ is locally a complete intersection, the Segre class $s(C, S)$ is indeed the reciprocal of the Chern class of the normal bundle $N_{C/S}$.

Segre classes of subschemes satisfy many of the relations that Chern classes of normal bundles do, at least in situations where those relations are well-defined. For example, if we have inclusions of schemes $C \subset S \subset X$ and $S \subset X$ is locally a complete intersection, we have

$$s(C, S) = s(C, X)c(N_{S/X}|_C),$$

as would follow from Whitney's formula in case $C \subset S$ were locally a complete intersection as well.

We have now arrived at something close to the form of the excess intersection formula given in Fulton and MacPherson [1977] and Fulton and MacPherson [1978]:

Theorem 15.13 (Excess intersection theorem—general version)**.** *Suppose $S_1, \dots S_{l-1}$ are locally complete intersection subvarieties of a projective variety X, and let S_l be an arbitrary subvariety. Suppose that $\cap_{i=1}^l S_i = \cup_\alpha C_\alpha$ is a disjoint union of subschemes C_α. Write $\iota_\alpha : C_\alpha \to X$ for the inclusion morphism. If d is the "expected dimension" of the intersection (that is, $d = \dim X - \sum_{i=1}^l \operatorname{codim} S_i$), then*

$$\prod_{i=1}^l [S_i] = \sum_\alpha (\iota_\alpha)_* \left(\left\{ \big(\prod_{i=1}^{l-1} c(N_{S_i/X}|_{C_\alpha})\big) s(C_\alpha, S_l) \right\}_d \right) \in A_m(X),$$

where for any Chow class $\gamma \in A(X)$, we denote by $\{\gamma\}_d$ the part of γ of dimension d. If, moreover, the subvariety $S_l \subset X$ is locally a complete intersection as well, we can write this as

$$\prod_{i=1}^{l}[S_i] = \sum_{\alpha}(\iota_\alpha)_*\left(\left\{(\prod_{i=1}^{l} c(N_{S_i/X}|_{C_\alpha}))s(C_\alpha, X)\right\}_d\right) \in A_m(X),$$

Note that the formula in Theorem 15.13 gives what is called a *refined intersection*: that is, it expresses the intersection class $\prod[S_i]$ as the pushforward of a class defined on the actual intersection $C = \cap S_i$. This is a substantial improvement even when X is smooth and the dimension of C is the expected dimension of the intersection.

Now, at the outset of this discussion we mentioned four problems with the naive generalization of Theorem 15.5 to the intersection of arbitrary subvarieties. We have addressed the first three, but not the fourth: if X is singular, the right hand side of the formula in Theorem 15.13 is not even defined a priori.

This is, as they say, not a bug but a feature! In fact, *this idea can be used to define intersection products with locally complete intersection cycles on an arbitrary variety, and thence of any cycles on a smooth variety*, without the Moving Lemma. This is the approach taken in Fulton [1984], Chapters 6 and 8.

To understand how we can go from intersections with locally complete intersection cycles on an arbitrary variety to intersections of arbitrary cycles on a smooth variety X, note first that the intersection of subvarieties S, T in X is the same as the intersection of the direct product $S \times T$ in $X \times X$ with the diagonal $X \cong \Delta \subset X \times X$. Next observe that when X is smooth, Δ is locally a complete intersection: for if $\dim X = n$, then the completion of the local ring of X at a point p is isomorphic to $K[[x_1, \dots, x_n]]$, so the completion of the local ring of $X \times X$ at (p, p) is isomorphic to $K[[x_1, \dots, x_n, y_1, \dots y_n]]$ and the ideal of Δ at this point is generated by $x_1 - y_1, \dots, x_n - y_n$, a regular sequence. Putting this together, we get the promised refined intersections:

Corollary 15.14. *Suppose that X is a smooth projective variety, and S, T are arbitrary subvarieties. Set $d = \dim X - \operatorname{codim} S - \operatorname{codim} T$. The intersection product $[S][T]$ is the pushforward into $A_d(X)$ of the class*

$$\left\{c(N_{\Delta/X \times X}|_{S \cap T})s(S \cap T, S \times T)\right\}_m \in A_d(S \cap T).$$

Corollary 15.14 is essentially the definition of intersection products in Fulton [1984].

15.6.1 Application: the five conic problem revisited

As an application of the general form of excess intersection formula, we revisit the five-conic problem of Chapter 10 and see another way to solve it. (This was first done in Fulton and MacPherson [1978]; a short version appears in Fulton [1984], p. 158.) Other applications of Theorem 15.13 appear in Exercises 15.32 and 15.38.

To recall briefly the circumstances, the problem asks how many conics $C \subset \mathbb{P}^2$ are tangent to each of five given conics $C_1, \dots, C_5$. To answer it, we may observe first that for any conic C_i, the set of conics tangent to it is a sextic hypersurface $Z_i \subset \mathbb{P}^5$ in the space $\mathbb{P}^5$ parametrizing all plane conics; by Bezout, then, a naive first guess at the answer would be $6^5 = 7,776$.

This fails, however, because the hypersurfaces $Z_1, \dots, Z_5 \subset \mathbb{P}^5$ do not intersect properly; rather, they all contain the Veronese surface $S \subset \mathbb{P}^5$ corresponding to double lines. In other words, if we let T be the component of the (scheme-theoretic) intersection $\cap Z_i$ supported on S, we can write

$$\bigcap_{i=1}^{5} Z_i = T \cup \Gamma,$$

and the problem is to determine the cardinality of Γ. In Chapter 10, we did this by replacing $\mathbb{P}^5$ by the space of *complete conics*. Now, however, we might view this as an opportunity to apply the excess intersection formula, and that's what we'll do here.

Of course, to apply Theorem 15.13 we need to know something about the local structure of the varieties we're intersecting. Here's what we need in this case:

(a) The Chern class of the restriction to S of the normal bundle of $Z_i \subset \mathbb{P}^5$. This is the easiest part: let $\zeta \in A^1(S)$ be the hyperplane class in $S \cong \mathbb{P}^2$, and let $\eta \in A^1(\mathbb{P}^5)$ be the hyperplane class on $\mathbb{P}^5$; note that the restriction of η to S is 2ζ. Since the Z_i are sextic hypersurfaces, $N_{Z_i/\mathbb{P}^5} = \mathcal{O}_{Z_i}(6)$ and so

$$c(N_{Z_i/\mathbb{P}^5}|_S) = 1 + 12\zeta.$$

(b) The multiplicity of Z_i along S. By Riemann-Hurwitz, a general pencil of plane conics including a double line $2L$ will have 4 other elements tangent to C_i, so that $\mathrm{mult}_S(Z_i) = 2$. (See Exercise 15.37.)

(c) The Chern classes of the normal bundle of $S \subset \mathbb{P}^5$. In terms of the hyperplane classes $\eta \in A^1(\mathbb{P}^5)$ and $\zeta \in A^1(S)$, we have

$$c(T_S) = (1+\zeta)^3 = 1 + 3\zeta + 3\zeta^2,$$

and

$$\begin{aligned} c(T_{\mathbb{P}^5}|_S) &= (1+\eta)^6|_S \\ &= 1 + 12\zeta + 60\zeta^2. \end{aligned}$$

Applying the Whitney formula to the sequence

$$0 \to T_S \to T_{\mathbb{P}^5}|_S \to N_{S/\mathbb{P}^5} \to 0,$$

we conclude that

$$\begin{aligned} c(N_{S/\mathbb{P}^5}) &= \frac{1+12\zeta+60\zeta^2}{1+3\zeta+3\zeta^2} \\ &= 1 + 9\zeta + 30\zeta^2 \end{aligned}$$

and inverting this we have

$$s(N_{S/\mathbb{P}^5}) = 1 - 9\zeta + 51\zeta^2.$$

(d) The scheme-theoretic intersection of the hypersurfaces Z_i. This is easy to state: the component of $\cap Z_i$ supported on S is exactly the scheme $T = V(\mathcal{I}^2_{S/\mathbb{P}^5})$ defined by the square of the ideal $\mathcal{I}_{S/\mathbb{P}^5}$. We won't give a proof here; given part (b) above, the statement is equivalent to the statement that the proper transforms of the Z_i in the blowup of $\mathbb{P}^5$ along S have no common intersection in the exceptional divisor, which is proved in Chapter 6 of Griffiths and Harris [1978]. Alternatively, via the isomorphism of the blowup with the space of complete conics it's tantamount to the statement, proved in Section **??**, that every complete conic tangent to each of $C_1, \dots, C_5$ is smooth.

Given part (d), the blowup of $\mathbb{P}^5$ along T is the same as the blowup along S, but with the exceptional divisor doubled. Applying Definition 15.12, the k^{th} graded piece of the Segre class $s(T, \mathbb{P}^5)$ is 2^{k+3} times the corresponding graded piece of $s(S, \mathbb{P}^5)$, so that

$$s(T, \mathbb{P}^5) = 8 - 144\zeta + 1632\zeta^2.$$

We have then that the contribution of S to the degree of the intersection $\cap Z_i$ is

$$\begin{aligned} \deg\Big(\prod c(N_{Z_i/\mathbb{P}^5}|_S) \cdot s(T, \mathbb{P}^5)\Big) &= \deg\big((1+12\zeta)^5(8 - 144\zeta + 1632\zeta^2)\big) \\ &= 1632 - 60 \cdot 144 + 1440 \cdot 8) \\ &= 4512, \end{aligned}$$

and the degree of Γ is correspondingly $7776 - 4512 = 3264$.

15.7 Exercises

Exercise 15.15. Show that the formula of Proposition 15.3 applies more generally if we replace $\mathbb{P}^3$ by an arbitrary smooth projective threefold X—that is, under the hypotheses of the Proposition, we have

$$\begin{aligned}\deg(\Gamma) = \deg\bigl([S]\cdot[T]\cdot[U]\bigr) - \deg(N_{S/X}|_L) - \deg(N_{T/X}|_L)\\ - \deg(N_{U/X}|_L) + \deg(N_{L/X}).\end{aligned}$$

Exercise 15.16. (a) Show that a smooth quintic curve $C \subset \mathbb{P}^3$ of genus 2 is the scheme-theoretic intersection of three surfaces in $\mathbb{P}^3$; and

(b) Show that a smooth rational quintic curve $C \subset \mathbb{P}^3$ is the scheme-theoretic intersection of three surfaces in $\mathbb{P}^3$ if and only if it lies on a quadric surface; conclude that some rational quintics are expressible as such intersections and some are not.

Exercise 15.17. Let S, T, U and $V \subset \mathbb{P}^4$ be smooth hypersurfaces of degrees d, e, f and g respectively, and suppose that the intersection

$$S \cap T \cap U \cap V = C \cup \Gamma$$

with C a smooth curve of degree a and genus g, and Γ a zero-dimensional scheme disjoint from C. What is the degree of Γ?

Exercise 15.18. Let S, T, U, V and $W \subset \mathbb{P}^5$ be smooth hypersurfaces of degrees d, and suppose that the intersection

$$S \cap T \cap U \cap V \cap W = \Lambda \cup \Gamma$$

with Λ a 2-plane and Γ a zero-dimensional scheme disjoint from C. What is the degree of Γ?

Exercise 15.19. Check the answer to keynote question (b) given in Section 15.1.4 in the following cases:

(a) S is a smooth surface of degree d in a hyperplane $\mathbb{P}^3 \subset \mathbb{P}^4$ containing a line L, and T is a general 2-plane in $\mathbb{P}^4$ containing L; and

(b) S and T are smooth quadric surfaces.

Exercise 15.20. Let $S = S(1,2) \subset \mathbb{P}^4$ be a cubic scroll, as in Section 11.1.1. Show directly that a general 2-plane $T \subset \mathbb{P}^4$ containing a line of the ruling of S meets S in one more point, but a general 2-plane containing the directrix $S(1)$ of S (that is, the line of S transverse to the ruling) does not meet S anywhere else.

Exercise 15.21. Derive the formula of Proposition 15.5, by an argument analogous to that given for Proposition 15.4

Exercise 15.22. Verify the formula of Proposition 15.5 by using it to derive again the answer to the first keynote question of this chapter.

Exercise 15.23. Let $\mathbb{P}^8$ be the space of 3×3 matrices, and $\mathbb{P}^5 \subset \mathbb{P}^8$ the subspace of symmetric matrices. Show that the Veronese surface in $\mathbb{P}^5$ is the intersection of $\mathbb{P}^5$ with the Segre variety $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$, and verify the excess intersection formula in this case.

Exercise 15.24. Let $X \subset \mathbb{P}^5$ be a smooth hypersurface of degree d, and $\Lambda \cong \mathbb{P}^2 \subset X$ a 2-plane contained in X. What is the degree of the self-intersection $[\Lambda]^2 \in A^4(X)$ of Λ in X? Check this in case $d = 1$ and 2.

Exercise 15.25. Let $X = 2C \subset \mathbb{P}^2$ be a double conic; that is, the subscheme defined by the square of a quadratic polynomial whose zero locus is a smooth conic curve $C \subset \mathbb{P}^2$. Show that the dualizing sheaf $\omega_X \cong \mathcal{O}_X(1)$, and applying Riemann-Roch deduce that X is not hyperelliptic; that is, it does not admit a degree 2 map to $\mathbb{P}^1$. Conclude that, as asserted in Section 15.3, no analytic neighborhood of C in $\mathbb{P}^2$ is biholomorphic to an anaytic neightborhood of the zero section in the normal bundle $N_{C/\mathbb{P}^2}$. (See Bayer and Eisenbud [1995] for more about this.)

In Exercises (15.26)-(15.28), we adopt the notation of Section 15.5.2; in particular, we let $C \subset \mathbb{P}^3$ be a smooth curve of degree d and genus g, $W = Bl_C\mathbb{P}^3$ the blowup of $\mathbb{P}^3$ along C and $E \subset W$ the exceptional divisor.

Exercise 15.26. Let $q \in \mathbb{P}^3$ by any point not on the tangential surface of C, and let $\Gamma \subset E \subset W$ be the curve of intersections with E of the proper transforms of lines $\overline{pq}$ for $p \in C$. Find the class of Γ in $A(W)$.

Exercise 15.27. Let $B \subset \mathbb{P}^3$ be another curve, of degree m, and suppose that B meets C in the points of a divisor D. Show that the class of the proper transform $\tilde{B} \subset W$ of B is

$$[\tilde{B}] = ml - F_D.$$

Exercise 15.28. Let $S \subset \mathbb{P}^3$ be a smooth surface of degree e containing C, $\tilde{S} \subset W$ its proper transform and $\Sigma_S = \tilde{S} \cap E$ the curve on E consisting of normal vectors to C contained in the tangent space to S. Find the class $[\Sigma_S] \in A(W)$ of Σ_S in the blow-up

(a) by applying Proposition **??** of Chapter 11; and

(b) by multiplying the class $[\Sigma_S]$ by the class $e = [E]$ in $A(W)$.

Exercise 15.29. In Section 11.4.2 of Chapter 11, we observed that the blow-up $X = Bl_{\mathbb{P}^k}\mathbb{P}^n$ of $\mathbb{P}^n$ along a k-plane was a $\mathbb{P}^{k+1}$-bundle over $\mathbb{P}^{n-k-1}$, and used this to describe the Chow ring of X. We now have another description of the Chow ring of X. Compare the two, and in particular

(a) Express the generators of $A(X)$ given in this chapter in terms of the generators given in Section 11.4.2; and thus

(b) Verify the relations among the generators given here.

Exercise 15.30. Redo Exercise 15.17 by the method of Section 15.5.2, that is, by blowing up the positive-dimensional component of the intersection.

Exercise 15.31. To generalize the solution to keynote question (c) given in Section 15.2.1, let $\Lambda \subset \mathbb{P}^n$ be a k-plane, and suppose that $Q_1, \dots, Q_n \subset \mathbb{P}^n$ are general quadrics containing Λ. If we write

$$\bigcap_{i=1}^{n} Q_i = \Lambda \cup Z,$$

what is the degree of Z?

Exercise 15.32. Consider a simpler version of the five-conic problem: given 5 general lines $L_1, \dots, L_5 \subset \mathbb{P}^2$ in the plane, how many conics are tangent to all 5? Answer this by applying the excess intersection formula (15.5) to the hypersurfaces $\Sigma_i \subset \mathbb{P}^5$ of conics tangent to the lines L_i, whose intersection consists of the union of the Veronese surface S and the locus of actual solutions. (As we saw in Chapter 10, the answer is known to be 1—by passing to the dual plane, the problem translates into the question, how many plane conics pass thought 5 general points?—but the point is to see another application of formula (15.5.)

Exercise 15.33. Let $Q \subset \mathbb{P}^3$ be a quadric cone with vertex p, and $X = Bl_pQ \cong \mathbb{F}_2$ its blow-up at p. Let $\pi : Q \to \mathbb{P}^2$ be a general projection, and $f : X \to \mathbb{P}^2$ the composition of the blow-up map with the projection. Find the degree of f by looking at the fiber over $q = \pi(p)$.

Exercise 15.34 (degree of the secant variety to a surface)**.** Let $X \subset \mathbb{P}^n$ be a smooth surface of degree d, with $n \geq 5$; let $\zeta = c_1(\mathcal{O}_X(1)) \in A^1(X)$ be the hyperplane class and ω and χ the first and second Chern classes of the cotangent bundle T_X^*.

(a) Show that if the dimension of the secant variety $S_2(X)$ of X is 5, as expected, its degree is the number of double points of a general projection $\pi : X \to \mathbb{P}^4$; and
(b) Use the double point formula to conclude that

$$\deg S_2(X) = \frac{1}{2}\big(d^2 - 10d - 5\omega\zeta - \omega^2 + \chi\big).$$

Exercise 15.35. Let $f : X \to Y$ be a map of smooth projective varieties, with $\dim Y = 2 \dim X$. Show that f is an embedding if and only if the double point scheme $D_f = \varnothing$.

Exercise 15.36. Prove that if $Z \subset A$ is a locally complete intersection subscheme inside the projective variety A, then $s(Z, A) = s(N_{Z/A})$.

Exercise 15.37. Let $\mathbb{P}^N$ be the space of plane curves of degree d, and $X \subset \mathbb{P}^N$ the locus of d-fold lines dL. Let $C \subset \mathbb{P}^2$ be a smooth curve of degree m, and let $\Sigma \subset \mathbb{P}^N$ be the locus of curves tangent to C (that is, intersecting C in fewer than dm distinct points).

(a) Let $\mathcal{D} \subset \mathbb{P}^N$ be a general line. Show that every curve $D \in \mathcal{D}$ is either transverse to C or meets C in exactly $dm-1$ points, and use Riemann-Hurwitz to conclude that

$$\deg(\Sigma) = 2md + m(m-3).$$

(b) Now suppose that $\mathcal{D} \subset \mathbb{P}^N$ is a general line meeting the locus X of d-fold lines. Show that $\mathcal{D}$ meets Σ in $2md + m(m-3) - m(d-1)$ other points, and conclude that

$$\operatorname{mult}_X(\Sigma) = m(d-1).$$

Exercise 15.38. Recall that a *sphere* $Q \subset \mathbb{P}^3$ is a quadric containing the "circle at infinity" $W = X^2 + Y^2 + Z^2 = 0$. Let $Q_1, \ldots, Q_4 \subset \mathbb{P}^3$ be four general spheres, and let $S_i \subset \mathbb{G}(1,3)$ be the locus of lines tangent to S_i. Using Theorem 15.13 applied to the intersection $\cap S_i$, find the number of lines tangent to all 4.

16

The Grothendieck-Riemann-Roch Theorem

Convention: To simplify notation in this chapter we often identify a class in $A_0(X)$ with its degree when X is a projective algebraic variety.

16.1 The Riemann-Roch formula for curves and surfaces

16.1.1 Nineteenth century Riemann-Roch

Riemann-Roch formulas relate the dimension of the space of solutions of an analytic or algebraic problem—typically realized as the space of global sections of a coherent sheaf on a compact analytic or projective algebraic variety—to topological invariants such as the topological Chern classes of the sheaf. Our goal in this chapter is to state, explain and apply a version of this theorem proved by Grothendieck. To clarify its context, we start this chapter with a review of some classical versions. Although these were first proven in an analytic context, we will stick with the category of projective algebraic varieties.

The original Riemann-Roch formula deals with a smooth projective curve C. It says in particular that the dimension $h^0(K_C)$ of the space of regular 1-forms on C, an algebraic invariant, is equal to the topological genus $g(C) = 1 - \chi_{\text{top}}(C)/2$. To express this in modern language and suggest the generalizations to come, we invoke Serre duality, which says that

$$h^0(K_C) = h^1(\mathcal{O}_C),$$

and the Hopf index theorem for the topological Euler characteristic, which says that $\chi_{\text{top}}(C) = c_1(T_C)$. In these terms, we can state the Riemann-Roch Theorem as the formula

$$\chi(\mathcal{O}_C) := h^0(\mathcal{O}_C) - h^1(\mathcal{O}_C) = \frac{c_1(T_C)}{2}.$$

Once we know this, the usual Riemann-Roch formula for line bundles on a curve follows: if $L = \mathcal{O}_C(D)$ for some effective divisor D of degree $c_1(L) = d$, then from the sequence

$$0 \to \mathcal{O}_C \to L \to L|_D \to 0$$

we see that

$$\begin{aligned} \chi(L) &= c_1(L) + \frac{c_1(T_C)}{2} \\ &= d + 1 - g. \end{aligned}$$

(See Section 17.1.1 for a fuller discussion of this theorem and its consequences.)

It's not hard to go from this to the version for an arbitrary sheaf $\mathcal{F}$ on C,

$$\chi(\mathcal{F}) = c_1(\mathcal{F}) + \operatorname{rank}(\mathcal{F})\frac{c_1(T_C)}{2} :$$

any sheaf $\mathcal{F}$ of positive rank admits a sub-line-bundle $\mathcal{L} \subset \mathcal{F}$, and from the relation

$$\chi(\mathcal{F}) = \chi(\mathcal{L}) + \chi(\mathcal{F}/\mathcal{L})$$

and an induction on $\operatorname{rank}(\mathcal{F})$, we are reduced to the case of rank 1. Here, the cokernel $\mathcal{F}/\mathcal{L}$ is supported on a finite set, and the relation follows.

To state a Riemann-Roch Theorem for a smooth projective surface S, we start again from a special case,

$$\chi(\mathcal{O}_S) = \frac{c_1(T_S)^2 + c_2(T_S)}{12},$$

usually referred to as *Noether's formula*. From this, the prior Riemann-Roch for curves, and sequences of the form

$$0 \to L \to L(D) \to L(D)|_D \to 0$$

for smooth effective divisors $D \subset S$, we can deduce the version for line bundles:

$$\chi(L) = \frac{c_1(L)^2 + c_1(L)c_1(T_C)}{2} + \frac{c_1(T_S)^2 + c_2(T_S)}{12}.$$

For example, from the sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(D) \to \mathcal{O}_S(D)|_D \to 0$$

we have

$$\begin{aligned}\chi(\mathcal{O}_S(D)) &= \chi(\mathcal{O}_S) + \chi(\mathcal{O}_S(D)|_D)\\ &= \frac{c_1(T_S)^2 + c_2(T_S)}{12} + \chi(\mathcal{O}_S(D)|_D).\end{aligned}$$

To evaluate the last term, observe that $\mathcal{O}_S(D)|_D$ is a line bundle of degree $D \cdot D$ on the curve D, which by adjunction has genus

$$g(D) = \frac{D \cdot D + D \cdot K_S}{2} + 1;$$

by the Riemann-Roch for curves, then, we have

$$\begin{aligned}\chi(\mathcal{O}_S(D)|_D) &= D \cdot D - \frac{D \cdot D + D \cdot K_S}{2}\\ &= \frac{D \cdot D + D \cdot c_1(T_S)}{2}\end{aligned}$$

and the Riemann-Roch formula above follows for $L = \mathcal{O}_S(D)$.

More generally if $\mathcal{F}$ is any coherent sheaf on S then

$$\chi(\mathcal{F}) = \frac{c_1(\mathcal{F})^2 - 2c_2(\mathcal{F}) + c_1(\mathcal{F})c_1(T_C)}{2} + \operatorname{rank}(\mathcal{F})\frac{c_1(T_S)^2 + c_2(T_S)}{12}.$$

An important step in the proof is to show that both sides of the formula are additive on exact sequences. For the left hand side, this follows from the long exact sequence in cohomology, and for the right hand side it comes from the properties of Chern characters and Todd classes, which we'll describe next.

Using this additivity, and resolving a sheaf by vector bundles, we reduce the proof to the case when $\mathcal{F}$ is a vector bundle. By the push-pull formula for direct images from Proposition 6.8, the Euler characteristic doesn't change when we pull a sheaf back to a projective bundle, so (as with the "splitting principle" we used to compute Chern classes) we can use additivity again to reduce to the case of a line bundle.

16.2 The Chern character and the Todd class

In order to generalize the various Riemann-Roch theorems stated so far to higher dimensions, we need to introduce two further invariants of vector bundles/coherent sheaves on a smooth projective variety X: the *Chern character* and the *Todd class*. These, it should be said, do not represent new invariants of a vector bundle; rather, they're ways of repackaging the data in the Chern class that are much more convenient for some purposes. The first of these in particular will allow us to relate the Chow ring to the Grothendieck K-ring of vector bundles on X, as described in section 16.2.3 below.

16.2.1 Chern character

To define the Chern character, suppose E is a vector bundle/locally free sheaf of rank n on a smooth variety X, and formally factor its Chern class:

$$c(E) = \prod_{i=1}^{n}(1+\alpha_i)$$

We define the *Chern character* to be

$$\mathrm{Ch}(E) = \sum_{i=1}^{n} e^{\alpha_i} \in A(X)\otimes\mathbb{Q}.$$

In other words, the k^{th} graded piece $\sum \frac{\alpha_i^k}{k!}$ of the power series $\sum e^{\alpha_i}$ is symmetric in the variables $\alpha_1, \dots, \alpha_n$, and hence expressible uniquely as a polynomial in the elementary symmetric polynomials in the α_i; we take the k^{th} graded piece $\mathrm{Ch}_k(E)$ of the Chern character to be this polynomial, applied to the Chern classes of E. We have immediately

$$\mathrm{Ch}_0(E) = \mathrm{rank}(E)[X] \quad \text{and} \quad \mathrm{Ch}_1(E) = c_1(E).$$

Further, it is not hard to compute that

$$\begin{aligned}\mathrm{Ch}_2(E) &= \frac{1}{2}\sum \alpha_i^2\\ &= \frac{1}{2}\Big((\sum \alpha_i)^2 - 2\sum \alpha_i\alpha_j\Big)\\ &= \frac{c_1(E)^2 - 2c_2(E)}{2}.\end{aligned}$$

Since the Chern roots of E^* are the negatives of the Chern roots of E the Chern character of E^* satisfies a similar formula to the Chern class,

$$\mathrm{Ch}_i(E^*) = (-1)^i\,\mathrm{Ch}_i(E).$$

Because of the denominators in the power series for e^{α}, the Chern character necessarily takes values in the tensor product $A(X)\otimes\mathbb{Q}$ rather than in $A(X)$. In fact, since the power sums $\sum \alpha_i^k$ generate the ring of symmetric polynomials over $\mathbb{Q}$, the Chern character encodes exactly the same information as the *rational Chern class*—that is, the image $c(E)_{\mathbb{Q}}$ of $c(E)$ under the map $A(X) \to A(X)\otimes\mathbb{Q}$.

Suppose that

$$0 \to F \to G \to H \to 0$$

is an exact sequence of sheaves. If we formally factor the Chern classes of F and H:

$$c(F) = \prod_{i=1}^{m}(1+\alpha_i) \quad \text{and} \quad c(H) = \prod_{j=1}^{n}(1+\beta_j)$$

then the Chern class of G will be given by

$$c(G) = \prod_{i=1}^{m}(1+\alpha_i)\prod_{j=1}^{n}(1+\beta_j).$$

Thus

$$\mathrm{Ch}(G) = \sum_{i=1}^{m} e^{\alpha_i} + \sum_{j=1}^{n} e^{\beta_j}.$$

It follows that the Chern characters satisfy the Whitney formula

$$\mathrm{Ch}(G) = \mathrm{Ch}(F) + \mathrm{Ch}(H)$$

(now the name is unambiguous!)

Unlike Chern classes, Chern characters behave in a simple way with respect to tensor products of vector bundles: if the Chern classes of the vector bundles F and H are formally factored as above then, by the splitting principle,

$$c(F \otimes H) = \prod_{i=1}^{m}\prod_{j=1}^{n}(1+\alpha_i+\beta_j).$$

Thus

$$\mathrm{Ch}(F \otimes H) = \sum_{i,j} e^{\alpha_i+\beta_j};$$

that is, the Chern character satisfies the identity

$$\mathrm{Ch}(F \otimes H) = \mathrm{Ch}(F)\,\mathrm{Ch}(H).$$

Since the Chern character is equivalent data to the rational Chern class, this yields a formula for the rational Chern class of a tensor product. The result is quite convenient for machine computation, but the conversion of polynomials in the power sums to polynomials in the elementary symmetric polynomials is complicated enough that it is not so useful for computation by hand; see Exercise 16.8 for an example.

16.2.2 Chern classes and Chern characters of coherent sheaves

Let X be a smooth projective variety, and $\mathcal{F}$ a coherent sheaf of X. By the Hilbert Syzygy Theorem, we can resolve $\mathcal{F}$ by locally free sheaves: that is, we can find an exact sequence

$$0 \to \mathcal{E}_n \to \mathcal{E}_{n-1} \to \cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$$

in which all the sheaves $\mathcal{E}_i$ are locally free, that is, sheaves of sections of vector bundles E_i. We can in turn use this to extend the definitions of Chern classes and Chern characters to all coherent sheaves.

If $\mathcal{F}$ were locally free, by the Whitney formula its Chern class would be the alternating product of the Chern classes of the other bundles E_i in the sequence, and we adopt this as our definition in general: we set

$$c(\mathcal{F}) = \prod_{i=0}^{n} c(E_i)^{(-1)^i}.$$

Similarly, we can appeal to the Whitney formula and define the Chern character

$$\mathrm{Ch}(\mathcal{F}) = \sum_{i=0}^{n} (-1)^i \,\mathrm{Ch}(\mathcal{E}_i).$$

We will suggest a proof that this is well-defined in Exercise 16.9.

Caution: If $Y \subset X$ is a subvariety of codimension c of a smooth variety, and we view its structure sheaf $\mathcal{O}_Y$ as a sheaf on X, then the Chern class $c(\mathcal{O}_Y) \in A(X)$ is in general *not* equal to 1 plus the cycle class $[Y]$. In general the Chern class of $\mathcal{O}_Y$ may have components of codimensions greater than c, and even the component in A^c differs from $[Y]$ by a factor of $(-1)^{c-1}c!$. This is in fact a consequence of the Grothendieck Riemann-Roch Theorem below; for examples, see Exercises 16.10-16.11.

16.2.3 K-theory

Let X be any variety. The *Grothendieck ring* of finite rank vector bundles on X, written $K^0(X)$, is the group of formal finite linear combinations of vector bundles of finite rank on X, modulo the relations $E - F + G = 0$ whenever there exists an exact sequence

$$0 \to E \to F \to G \to 0.$$

The product in this ring is defined by setting the product of the classes of bundles E and F equal to the class of the tensor product bundle $E \otimes F$, and extending by linearity. This operation preserves the equivalence relation because tensoring with a vector bundle preserves exact sequences.

We define the *Grothendieck group of coherent sheaves* on X, written $K_0(X)$, by taking formal combinations of coherent sheaves in place of vector bundles. Tensor product with a vector bundle preserves short exact sequences, so the group $K_0(X)$ is a module over the ring $K^0(X)$. Since a vector bundle of finite rank may be regarded as a coherent sheaf, there is a natural map $K^0(X) \to K_0(X)$. If X is smooth and projective then any coherent sheaf has a finite resolution by vector bundles, so this map is an isomorphism.

In these terms, we can express the Whitney formula and the analogous relation for the Chern character of a tensor product by saying that *the Chern character is a ring homomorphism from the K-ring $K^0(X)$ to the rational Chow ring $A(X)$*. In fact, we can say much more:

Theorem 16.1 (Grothendieck)**.** *If X is a smooth projective variety, then the map sending a bundle E to* $\mathrm{Ch}(E)$*, is an isomorphism of rings from $\mathbb{Q} \otimes K^0(X)$ to $\mathbb{Q} \otimes A(X)$.*

See Fulton [1984] Theorem 15.2.16b.

Strikingly, there is an analogous statement in the category of differentiable manifolds: If we define the topological K-group of a manifold M to be the group of formal linear combinations of $\mathcal{C}^\infty$ vector bundles, with ring structure given as above by tensor products, then the Chern character gives an isomorphism

$$\mathbb{Q} \otimes \mathrm{gr}\ K(M) \cong H^{2*}(X, \mathbb{Q})$$

where the term on the right is the ring of even-degree rational cohomology classes (see Griffiths [1974]).

16.2.4 The Todd class

To define the Todd class, we proceed much as in the definition of the Chern character: suppose E is a vector bundle/locally free sheaf of rank n on a smooth variety X, and formally factor its Chern class:

$$c(E) = \prod_{i=1}^{n} (1 + \alpha_i).$$

We define the *Todd character* of E to be

$$\mathrm{td}(E) = \prod_{i=1}^{n} \frac{\alpha_i}{1 - e^{-\alpha_i}},$$

written as a power series in the elementary symmetric polynomials $c_i(E)$ of the α_i. As with the Chern character, the n^{th} Todd class $\mathrm{td}_n(E)$ of E is just the n^{th} graded piece of the Todd class.

To calculate the first few terms of the Todd class, write

$$1 - e^{-\alpha} = \alpha - \frac{\alpha^2}{2} + \frac{\alpha^3}{6} - \frac{\alpha^4}{24} + \dots$$

so

$$\frac{1-e^{-\alpha}}{\alpha} = 1 - \frac{\alpha}{2} + \frac{\alpha^2}{6} - \frac{\alpha^3}{24} + \frac{\alpha^4}{120} - \dots;$$

inverting this, we get

$$\frac{\alpha}{1-e^{-\alpha}} = 1 + \frac{\alpha}{2} + \frac{\alpha^2}{12} - \frac{\alpha^4}{720} + \dots$$

so

$$\mathrm{td}(E) = \prod_{i=1}^{n} (1 + \frac{\alpha_i}{2} + \frac{\alpha_i^2}{12} - \frac{\alpha_i^4}{720} + \dots).$$

Rewriting the first few of these in terms of the symmetric polynomials of the α_i—that is, the Chern classes of E—we get

$$\mathrm{td}_0(E) = 1,$$
$$\mathrm{td}_1(E) = \sum \frac{\alpha_i}{2} = \frac{c_1(E)}{2}$$

and

$$\begin{aligned}\mathrm{td}_2(E) &= \frac{1}{12}\sum \alpha_i^2 + \frac{1}{4}\sum_{i<j} \alpha_i\alpha_j \\ &= \frac{c_1^2(E) + c_2(E)}{12}.\end{aligned}$$

Note that by the first two cases of the Riemann-Roch theorem, the Euler characteristics of the structure sheaves of a curve C and surface S are just the degrees of the Todd classes $\mathrm{td}_1(T_C)$ and $\mathrm{td}_2(T_S)$ of their tangent bundles. As we'll see in a moment, this is true in general, and is the property of Todd classes that led Todd to their definition (see Exercise 16.12).

16.3 Hirzebruch-Riemann-Roch

Much of the content of the formulas in Section 16.1 above was known to 19th century algebraic geometers, although the formulas were expressed without cohomology, and only for line bundles (represented by divisors). For curves, any cohomology group is either an H^0, or dual to one; for surfaces, what we would write as h^2 was expressed in terms of the dual h^0, and h^1 was left as an "error term," yielding an inequality in place of the equality. (The term $h^1(\mathcal{F})$ was called the *superabundance* of $\mathcal{F}$, in view of the fact that $h^0(\mathcal{F})$ could be that much greater than "expected.")

In the 20th century these formulas were extended to sheaves on varieties of arbitrary dimension. One key to this extension, of course, was the introduction of cohomology groups in general, and the recognition that the left hand side of all the classical formulations of Riemann-Roch represented Euler characteristics of sheaves. Equally important was understanding how to express the polynomials in the Chern classes that appear in the right hand side of these formulas in a way that generalized to arbitrary dimensions.

The answer is beautifully simple: first, the Euler characteristic of the structure sheaf of a smooth projective variety X of dimension n is given by the formula

$$\chi(\mathcal{O}_X) = \mathrm{Td}_n(T_X),$$

and, for an arbitrary coherent sheaf $\mathcal{F}$:

Theorem 16.2 (Hirzebruch [1966]). *If X is a smooth projective variety of dimension n and $\mathcal{F}$ a coherent sheaf on X, then*

$$\chi(\mathcal{F}) = \big(\mathrm{Ch}(\mathcal{F})\,\mathrm{Td}(T_X)\big)_n.$$

This formula was first stated and proved in the setting of algebraic varieties by Hirzebruch **ref**; it was then generalized to the differentiable setting by Atiyah and Singer **ref**.

16.4 Grothendieck-Riemann-Roch

Grothendieck's version of the Riemann-Roch Theorem introduces a fundamental new idea into the mix. Briefly, suppose we have a family $\{X_b\}_{b\in B}$ of schemes, and a family of sheaves $\mathcal{F}_b$ on X_b—in other words, a morphism $\pi : X \to B$ and a sheaf $\mathcal{F}$ on X. As we've seen, the vector spaces $H^0(\mathcal{F}_b)$ form, at least for b in an open subset $U \subset B$, the fibers of a sheaf on B; this is the direct image $\pi_*\mathcal{F}$. We can think of the "classical" Hirzebruch Riemann-Roch theorem, applied to the sheaf $\mathcal{F}_b$ on the general fiber X_b, as a formula for the dimension $h^0(\mathcal{F}_b)$ (that is, the rank of the sheaf $\pi_*\mathcal{F}$), with "error terms" coming from the dimensions $h^i(\mathcal{F}_b)$ of the higher cohomology groups (i.e., the ranks of the direct images $R^i\pi_*\mathcal{F}$). Grothendieck asks, reasonably enough, if we can say how the spaces $H^0(\mathcal{F}_b)$ fit together as b varies—in other words, can we describe the twisting of the sheaf $\pi_*\mathcal{F}$, as measured by its Chern classes?

This is exactly what the Grothendieck-Riemann-Roch formula does: it's a formula for the Chern character of the direct image $\pi_*\mathcal{F}$, with additional terms coming from the Chern characters of the higher direct images $R^i\pi_*\mathcal{F}$.

Theorem 16.3 (Grothendieck's Riemann-Roch Formula). *If $\pi : X \to B$ is a projective morphism and $\mathcal{F}$ is a coherent sheaf on X then*

$$\sum_{i=0}^{n}(-1)^i\,\mathrm{Ch}(R^i\pi_*\mathcal{F}) = \pi_*\Big[\frac{\mathrm{Ch}(\mathcal{F})\cdot\mathrm{Td}(T_X)}{\pi^*\,\mathrm{Td}(T_B)}\Big].$$

A very readable proof can be found in Borel and Serre [1958].

Note that the Hirzebruch Riemann-Roch Theorem is simply the equality of the degree 0 terms in the Grothendieck-Riemann-Roch. Equivalently, Hirzebruch Riemann-Roch is the Grothendieck-Riemann-Roch in the special case when B is a single point.

There are other, equivalent formulations of the Grothendieck-Riemann-Roch. For example, using the push-pull formula we can rewrite it in the form

$$\left(\sum_{i=0}^{n}(-1)^i \operatorname{Ch}(R^i\pi_*\mathcal{F})\right)\operatorname{Td}(T_B) = \pi_*\Big[\operatorname{Ch}(\mathcal{F})\cdot\operatorname{Td}(T_X)\Big].$$

Also, in case the map π is a submersion—that is, the differential $d\pi$ is surjective everywhere—then from the short exact sequence

$$0 \to T^v_{X/B} \to T_X \to \pi^*T_B \to 0$$

for the relative tangent bundle of π and the multiplicativity of the Todd class we have

Corollary 16.4. *If $\pi : X \to B$ is a projective morphism and a submersion, and $\mathcal{F}$ is a coherent sheaf on X, then*

$$\sum_{i=0}^{n}(-1)^i \operatorname{Ch}(R^i\pi_*\mathcal{F}) = \pi_*\Big[\operatorname{Ch}(\mathcal{F})\cdot\operatorname{Td}(T^v_{X/B})\Big].$$

When applying these formulas it is crucial to know when the sheaves $R^i\pi_*(\mathcal{F})$ have fibers at an arbitrary point b equal to $H^i(\mathcal{F}|_{\pi^{-1}(b)})$. The Theorem on Cohomology and Base Change proved in Chapter 6 gives conditions under which this happens. Given this, it's possible to use GRR to calculate the Chern classes of many of the bundles we've encountered earlier in the book, and whose Chern classes we calculated by ad hoc methods like filtrations and the splitting principle. Indeed, virtually all of the bundles we've introduced and analyzed in the preceding chapters are defined as direct images, and consequently we can use the Grothendieck-Riemann-Roch formula to calculate their Chern classes.

In the following subsection we'll give an example of this, by way of illustrating the application of GRR in practice; other examples are suggested in Exercises 16.14-16.16. One warning before we get started: while it's nice to have a systematic approach to these calculations, in practice the sort of ad-hoc methods we used in earlier chapters are almost always easier and faster than hauling out the GRR.

Following this example, the remainder of this chapter will be concerned with two applications of GRR in situations where we don't have alternative ways of calculating the Chern classes of the bundles in question. In

Section 16.5, we'll describe an application of GRR to the geometry of vector bundles on projective space; and in Section 16.6 an application to the geometry of families of curves.

16.4.1 Example: the bundle $\mathrm{Sym}^3 \mathcal{S}^*$ *on* $\mathbb{G}(1,3)$

Recall that in Chapter 8, where we wanted to find the number of lines on a smooth cubic surface $S \subset \mathbb{P}^3$, we introduced a vector bundle E on the Grassmannian $G = \mathbb{G}(1,3)$ of lines in $\mathbb{P}^3$; informally, we described E by saying that for each line $L \subset \mathbb{P}^3$, the fiber of E at the point $[L] \in G$ was the vector space of homogeneous cubic polynomials on L: that is,

$$E_{[L]} = H^0(\mathcal{O}_L(3)).$$

Back then, we calculated the Chern classes of E by realizing it as the third symmetric power of the dual of the universal subbundle on G and using the splitting principle. But as we saw in that chapter (and as is the case with virtually every vector bundle we introduced, and whose Chern classes we calculated), E may be realized as a direct image, and its Chern classes calculated from the Grothendieck-Riemann-Roch formula. We'll recall here the setup, and then go on to see how Grothendieck-Riemann-Roch is applied in this instance.

To do this, we introduce the universal family of lines over G, that is, the incidence correspondence

$$\Phi = \{(L, p) : p \in L\} \subset G \times \mathbb{P}^3;$$

we'll let $\alpha : \Phi \to G$ and $\beta : \Phi \to \mathbb{P}^3$ be the projection maps. Now, we want a vector bundle on G whose fiber $E_{[L]}$ is the space of global sections of $\mathcal{O}_L(3)$; this suggests that we find a line bundle on Φ whose restriction to each fiber L of α is $\mathcal{O}_L(3)$, and take its direct image. The first is easy: we simply take the line bundle $\mathcal{O}_{\mathbb{P}^3}(3)$ on $\mathbb{P}^3$, pull it back to the product $G \times \mathbb{P}^3$, and restrict to Φ—in other words, the line bundle $\mathcal{L} = \beta^*\mathcal{O}_{\mathbb{P}^3}(3)$.

What happens when we take the direct image of $\mathcal{L}$? Well, things are about as nice as they're ever going to get. First of all, $\mathcal{L}$ is flat over G: the projection $\alpha : \Phi \to G$ is flat, and $\mathcal{L}$ is a line bundle on Φ. Moreover, all the higher cohomology of the restriction $\mathcal{O}_L(3)$ of $\mathcal{L}$ to the fiber $\Phi_{[L]} = \alpha^{-1}([L]) = L$ vanishes, and the dimension of $H^0(\mathcal{L}|_{\Phi_{[L]}})$ is constant. Thus the direct image $\alpha_*\mathcal{L}$ is locally free, with fiber $H^0(\mathcal{O}_L(3))$ at $[L]$. In short, we can take

$$E = \alpha_*\mathcal{L} = \alpha_*\big(\beta^*\mathcal{O}_{\mathbb{P}^3}(3)\big).$$

Moreover, because of the vanishing of the higher cohomology of $\mathcal{L}$ on the fiber of α, Grothendieck-Riemann-Roch gives us a complete description of the Chern class of E: we have

$$\mathrm{Ch}(E) = \alpha_*\big(\mathrm{Ch}(\mathcal{L}) \cdot \mathrm{Td}(T^v_{\Phi/G})\big).$$

To evaluate this explicitly, we have to

(a) describe the Chow rings $A(G)$ and $A(\Phi)$, and the Gysin map $\alpha_* : A(\Phi) \to A(G)$;

(b) calculate the Chern classes of $\mathcal{L}$ and the relative tangent bundle $T^v_{\Phi/G}$;

(c) convert these to the Chern character and Todd class, respectively;

(d) take the Gysin image of their product, to arrive at $\mathrm{Ch}(E)$; and finally

(e) convert this back into the Chern classes of E.

****consider stopping this section here!****

For the first, we've seen how to do this in Chapters ** and **: $\Phi = \mathbb{P}S$ is the projectivization of the universal subbundle on G, so that we have

$$A(\Phi) = A(G)[\zeta]/(\zeta^2 - \sigma_1\zeta + \sigma_{1,1})$$

where $\zeta = c_1(\mathcal{O}_{\mathbb{P}S}(1))$. As for the Gysin map $\alpha_* : A(\Phi) \to A(G)$, we have $\alpha_*(\zeta) = [G]$; the Gysin image of higher powers of ζ may be evaluated via applications of the relation $\zeta^2 = \sigma_1\zeta - \sigma_{1,1}$ satisfied by ζ and the image of any product of a power of ζ with the pullback of a class from G by the push-pull formula. Explicitly, we have

$$\begin{aligned}\alpha_*(\zeta^2) &= \alpha_*(\sigma_1\zeta - \sigma_{1,1})\\ &= \sigma_1,\end{aligned}$$

$$\begin{aligned}\alpha_*(\zeta^3) &= \alpha_*(\sigma_1\zeta^2 - \sigma_{1,1}\zeta)\\ &= \sigma_1^2 - \sigma_{1,1}\\ &= \sigma_2,\end{aligned}$$

and

$$\begin{aligned}\alpha_*(\zeta^4) &= \alpha_*(\sigma_1\zeta^3 - \sigma_{1,1}\zeta^2)\\ &= \sigma_1\sigma_2 - \sigma_{1.1}\sigma_1\\ &= 0.\end{aligned}$$

(The last one is just a check: we know that $\zeta^4 = 0$ in $A(\Phi)$, since ζ is the pullback of the hyperplane class on $\mathbb{P}^3$.)

One other remark we should make is that, since the fiber of the line bundle $\mathcal{O}_{\mathbb{P}S}(1)$ at a point $(L, p) \in \Phi$ is just the dual of the one-dimensional vector subspace of $\mathbb{C}^4$ corresponding to p, we also have

$$\zeta = c_1(\mathcal{O}_{\mathbb{P}S}(1)) = \beta^* c_1(\mathcal{O}_{\mathbb{P}^3}(1).$$

In particular, it follows that

$$c_1(\mathcal{L}) = 3\zeta$$

and so

$$\mathrm{Ch}(\mathcal{L}) = 1 + 3\zeta + \frac{9}{2}\zeta^2 + \frac{27}{6}\zeta^3$$

(again, higher powers of ζ must vanish.

As for the relative tangent bundle of Φ over G, we will see how to find its Chern classes in Section 13.1.2: if we denote by U the tautological sub-bundle on $\Phi = \mathbb{P}S$, and by Q the tautological quotient bundle, we have

$$T^v_{\Phi/G} = U^* \otimes Q.$$

From the exact sequence

$$0 \to U \to \alpha^* S \to Q \to 0,$$

moreover, we see that

$$c_1(Q) = c_1(\alpha^* S) - c_1(U) = -\sigma_1 + \zeta$$

and hence

$$c_1(T^v_{\Phi/G}) = c_1(U^* \otimes Q) = \zeta + c_1(Q) = -\sigma_1 + 2\zeta.$$

Plugging this into the formula for the Todd class, we have

$$\mathrm{Td}(T^v_{\Phi/G}) = 1 + \frac{2\zeta - \sigma_1}{2} + \frac{(2\zeta - \sigma_1)^2}{12} - \frac{(2\zeta - \sigma_1)^4}{720}.$$

Taking the product, we calculate

$$\begin{aligned}
\mathrm{Ch}(\mathcal{L})\,\mathrm{Td}(T^v_{\Phi/G}) = 1 &+ \frac{8\zeta - \sigma_1}{2} + \frac{1}{12}\big(94\zeta^2 - 22\sigma_1\zeta + \sigma_1^2\big) \\
&+ \frac{1}{12}\big(120\zeta^3 - 39\sigma_1\zeta^2 + 3\sigma_1^2\zeta\big) \\
&+ \frac{1}{720}\big(-2668\sigma_1\zeta^3 + 246\sigma_1^2\zeta^2 + 8\sigma_1^3\zeta - \sigma_1^4\big) \\
&+ \frac{1}{720}\big(198\sigma_1^2\zeta^3 + 24\sigma_1^3\zeta^2 - 3\sigma_1^4\zeta\big)
\end{aligned}$$

and applying the Gysin map we find that, by Grothendieck-Riemann-Roch,

$$\mathrm{Ch}(E) = 4 + 6\sigma_1 + (7\sigma_2 - 3\sigma_{1,1}) - 3\sigma_{2,1} + \frac{1}{3}\sigma_{2,2}.$$

It remains to convert this to the Chern class of E. Once more, this can be messy: we have

$$c_1(E) = \mathrm{Ch}_1(E) = 6\sigma_1$$

and

$$\begin{aligned} c_2(E) &= \frac{\mathrm{Ch}_1(E)^2}{2} - \mathrm{Ch}_2(E) \\ &= 18\sigma_1^2 - (7\sigma_2 - 3\sigma_{1,1}) \\ &= 11\sigma_2 + 21\sigma_{1,1}. \end{aligned}$$

Similarly,

$$\begin{aligned} c_3(E) &= \frac{\mathrm{Ch}_1(E)^3}{6} - \mathrm{Ch}_1(E)\,\mathrm{Ch}_2(E) + 2\,\mathrm{Ch}_3(E) \\ &= 36\sigma_1^3 - 6\sigma_1(7\sigma_2 - 3\sigma_{1,1}) - 6\sigma_{2,1} \\ &= 72\sigma_{2,1} - 24\sigma_{2,1} - 6\sigma_{2,1} \\ &= 42\sigma_{2,1} \end{aligned}$$

and finally—the payoff, at last!—

$$\begin{aligned} c_4(E) &= \frac{\mathrm{Ch}_1(E)^4}{24} - \frac{\mathrm{Ch}_1(E)^2\,\mathrm{Ch}_2(E)}{2} + \frac{\mathrm{Ch}_2(E)^2}{2} + 2\,\mathrm{Ch}_1(E)\,\mathrm{Ch}_3(E) - 6\,\mathrm{Ch}_4(E) \\ &= 54\sigma_1^4 - 18\sigma_1^2(7\sigma_2 - 3\sigma_{1,1}) + \frac{1}{2}(7\sigma_2 - 3\sigma_{1,1})^2 - 36\sigma_1\sigma_{2,1} - 2\sigma_{2,2} \\ &= (108 - 72 + 29 - 36 - 2)\sigma_{2,2} \\ &= 27\sigma_{2,2}. \end{aligned}$$

We have thus both calculated the number of lines on a cubic surface, and illustrated a fact well-known to practicing algebraic geometers: one should almost never use the Grothendieck-Riemann-Roch to calculate the Chern classes of a bundle if there is any alternative.

16.5 Application: Jumping lines

In this section, we'll describe the notion of *jumping lines*, an invariant used to study the geometry of vector bundles on projective space. To keep the notation relatively simple, we'll deal here just with the case of vector bundles of rank 2 on $\mathbb{P}^2$; but as indicated in Exercises 16.17-16.19, the generalization to bundles on $\mathbb{P}^n$ is straightforward.

We start by recalling Theorem 8.29, which says that every vector bundle E on $\mathbb{P}^1$ is a direct sum

$$E = \bigoplus \mathcal{O}_{\mathbb{P}^1}(a_i)$$

of line bundles, with

$$\sum a_i = c_1(E).$$

Recall that analogous statement is clearly false in general for bundles on projective space $\mathbb{P}^n$ of dimension $n \geq 2$ (see Exercise 8.53 for some simple examples).

Now, while the classification of bundles on $\mathbb{P}^1$ doesn't generalize to $\mathbb{P}^n$, it does provide a useful tool for analyzing bundles on $\mathbb{P}^n$, based on the fact that while a bundle E on $\mathbb{P}^n$ may not split, its restriction to each line $L \subset \mathbb{P}^n$ does. For every collection $a = (a_1, \ldots, a_r)$ of $r = \operatorname{rank}(E)$ integers with $\sum a_i = c_1(E)$, then, we may define a subset $\Gamma_a \subset \mathbb{G}(1,n)$ of the Grassmannian of lines in $\mathbb{P}^n$ by

$$\Gamma_a = \{L \in \mathbb{G}(1,n) : E|_L \cong \oplus \mathcal{O}_L(a_i)\}.$$

As we'll see in a moment, these will be locally closed subsets of $\mathbb{G}(1,n)$. In particular, the decomposition of $E|_L$ will be constant for L in an open dense subset of $\mathbb{G}(1,n)$; the lines outside this open are called *jumping lines*. Together the loci Σ_a give a stratification of $\mathbb{G}(1,n)$ whose geometry is an important invariant of E; the closures of the strata Γ_a are called *loci of jumping lines*.

At this point, we need to introduce some basic facts about how vector bundles on $\mathbb{P}^1$ behave in families; we'll do that in the following subsection and then return to the discussion of jumping lines.

16.5.1 Families of vector bundles on $\mathbb{P}^1$

We'll be dealing here with families of vector bundles on $\mathbb{P}^1$ over a connected base B, by which we'll mean a vector bundle $\mathcal{E}$ on the product $B \times \mathbb{P}^1$; we'll denote by r and d the rank and degree $\deg c_1(E_t)$ of the restriction E_t of $\mathcal{E}$ to a fiber $\{t\} \times \mathbb{P}^1$, which are independent of $t \in B$. The actual decomposition $E_t \cong \oplus \mathcal{O}_{\mathbb{P}^1}(a_i(t))$, however, may vary with t; we want to describe here the possible variation.

We start by introducing a partial ordering on the set of sequences

$$a = (a_1, \ldots, a_r) \quad \text{with} \quad a_1 \leq a_2 \leq \cdots \leq a_r \text{ and } \sum a_i = d :$$

we'll say that

$$a \leq b \quad \text{if} \quad \sum_{i=1}^{j} a_i \leq \sum_{i=1}^{j} b_i \text{ for all } j = 1, \ldots, r.$$

Thus, the "largest" sequence with given r and d is the most balanced; that is, the unique sequence of the form $(a, \ldots, a, a+1, \ldots, a+1)$, or equivalently the unique such sequence with $|a_i - a_j| \leq 1 \ \forall\ i, j$. As a measure of the deviation of a given sequence a from the most balanced, we'll set

$$u(a) = \sum_{i<j} \min(a_j - a_i - 1, 0).$$

Our main foundational result is the following theorem:

Theorem 16.5. *Let $\mathcal{E}$ be as above a vector bundle on $B \times \mathbb{P}^1$. If for each sequence a we set*

$$\Gamma_a = \{t \in B \mid E_t \cong \oplus \mathcal{O}_{\mathbb{P}^1}(a_i)\},$$

then

(a) *For each a, the locus*

$$\overline{\Gamma}_a = \bigcup_{a' \leq a} \Gamma_{a'} \subset B$$

is closed in B (so that in particular Γ_a is a locally closed subset of B); and

(b) *The codimension of Γ_a in B is at most $u(a)$.*

Proof. For the first part, consider the function on B

$$\mu(t) = \max\{m \mid h^0(E_t(-m)) > 0\}.$$

Since $h^0(E_t(-m))$ is upper semicontinuous in t, the function μ is as well; this shows that the degree $a_r(t)$ of the largest summand of E_t is upper semicontinuous. Applying this to the bundle $\wedge^k(E_t)$, we see that the function

$$\mu_k(t) = \max\{m \mid h^0((\bigwedge^k E_t)(-m)) > 0\} = a_r(t) + \cdots + a_{r-k+1}(t)$$

is likewise upper semicontinuous, and correspondingly

$$d - \mu_{r-k}(t) = a_1(t) + \cdots + a_k(t)$$

is lower semicontinuous, establishing part (a).

The second part of the Theorem is more subtle; we'll prove it here just in the case $r = 2$ of bundles of rank 2 (which is where we'll apply it). Suppose then that $\mathcal{E}$ is such a bundle, and let $t_0 \in B$ be any point. The function $u(a)$ is invariant under the addition of a fixed quantity to all the a_i, so we can twist the bundle $\mathcal{E}$ by the pullback of a line bundle on $\mathbb{P}^1$ without affecting the truth of the statement; after such a twist we can assume that the bundle $E_{t_0} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)$ for some n. Since $h^1(E_{t_0}) = 0$, the section $(1, 0)$ of E_{t_0} extends to a nowhere zero section of $\mathcal{E}$ in a neighborhood of the fiber $\{t_0\} \times \mathbb{P}^1$. Replacing B by a suitably small open neighborhood of $t_0 \in B$, then, we have a sequence

$$0 \longrightarrow \mathcal{O}_{B \times \mathbb{P}^1} \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{B \times \mathbb{P}^1}(n) \longrightarrow 0$$

.

Now, a sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow E \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^1}(n) \longrightarrow 0$$

splits if and only if there exists a bundle map $\varphi : \mathcal{O}_{\mathbb{P}^1}(n) \to E$ such that $\alpha \circ \varphi$ is the identity on $\mathcal{O}_{\mathbb{P}^1}(n)$. Accordingly, consider the exact sequence of bundles on $B \times \mathbb{P}^1$

$$\begin{aligned} 0 \to \operatorname{Hom}(\mathcal{O}_{B\times\mathbb{P}^1}(n), \mathcal{O}_{B\times\mathbb{P}^1}) \to \operatorname{Hom}(\mathcal{O}_{B\times\mathbb{P}^1}(n), \mathcal{E}) \\ \to \operatorname{Hom}(\mathcal{O}_{B\times\mathbb{P}^1}(n), \mathcal{O}_{B\times\mathbb{P}^1}(n)) \to 0 \end{aligned}$$

and the coboundary map

$$\pi_*\big(\operatorname{Hom}(\mathcal{O}_{B\times\mathbb{P}^1}(n), \mathcal{O}_{B\times\mathbb{P}^1}(n))\big) \xrightarrow{\delta} R^1\pi_*\big(\operatorname{Hom}(\mathcal{O}_{B\times\mathbb{P}^1}(n), \mathcal{O}_{B\times\mathbb{P}^1})\big)$$

appearing in the associated long exact sequence of sheaves on B. If we let $\sigma = \delta(id)$ be the image in $R^1\pi_*\big(\operatorname{Hom}(\mathcal{O}_{B\times\mathbb{P}^1}(n), \mathcal{O}_{B\times\mathbb{P}^1})\big)$ of the identity section of $\pi_*\big(\operatorname{Hom}(\mathcal{O}_{B\times\mathbb{P}^1}(n), \mathcal{O}_{B\times\mathbb{P}^1}(n))\big)$, then the zero locus $(\sigma) \subset B$ of σ will be contained in the stratum $\Gamma_{(0,n)}$; since $R^1\pi_*\big(\operatorname{Hom}(\mathcal{O}_{B\times\mathbb{P}^1}(n), \mathcal{O}_{B\times\mathbb{P}^1})\big)$ is locally free of rank $n-1$, it follows that the codimension of $\Gamma_{(0,n)}$ in B is at most $u(0,n) = n-1$. $\square$

In general, given a family of vector bundles on $\mathbb{P}^1$, we think of $u(a)$ as the "expected" codimension of the locus $\Gamma_a \subset B$.

16.5.2 Bundles of rank 2 on $\mathbb{P}^2$

To keep the notation relatively simple, we'll deal here just with the case of vector bundles of rank 2 on $\mathbb{P}^2$; but as indicated in Exercises 16.17-16.19, the generalizations to bundles on $\mathbb{P}^n$ is straightforward.

So: let E be a vector bundle of rank 2 on $\mathbb{P}^2$. As we'll see, the locus of jumping lines depends very much on the parity of $c_1(E) \in A^1(\mathbb{P}^2) \cong \mathbb{Z}$; we'll consider the case of $c_1(E)$ even and odd in turn.

Vector bundles with even first Chern class. Assume first that $c_1(E) = 2k\alpha$, where $\alpha \in A^1(\mathbb{P}^2)$ is the class of a line. We want to think of the restrictions of E to each line in $\mathbb{P}^2$ in turn as a family of vector bundles on $\mathbb{P}^1$, parametrized by the Grassmannian $\mathbb{G}(1,2)$ (otherwise known as the dual plane $\mathbb{P}^{2*}$). To do this, we introduce the by now familiar universal line

$$\Phi = \{(L,p) : p \in L\} \subset \mathbb{P}^{2*} \times \mathbb{P}^2;$$

let $\pi_1 : \Phi \to \mathbb{P}^{2*}$ and $\pi_2 : \Phi \to \mathbb{P}^2$ be the projections. We can view $\pi_1 : \Phi \to \mathbb{P}^{2*}$ as the projectivization of the universal subbundle S on $\mathbb{P}^{2*} = \mathbb{G}(1,2)$; if we denote by

$$\zeta = c_1(\mathcal{O}_{\mathbb{P}S}(1)) \in A^1(\Phi)$$

the tautological class, and by α both the hyperplane class on $\mathbb{P}^{2*}$ and its pullback to Φ, then the Chow ring of Φ is given by

$$A(\Phi) = \mathbb{Z}[\alpha,\zeta]/(\alpha^3, \zeta^2 - \alpha\zeta + \alpha^2).$$

Note that ζ can also be realized as the pullback of the hyperplane class from $A^1(\mathbb{P}^{2*})$; we can see either from this or the representation of $A(\Phi)$ above that $\zeta^3 = 0$. We see also that the pushforward map $(\pi_1)_* : A(\Phi) \to A(\mathbb{P}^{2*})$ is given by

$$\begin{aligned} \alpha^2 &\mapsto 0 \\ \alpha\zeta &\mapsto \alpha \\ \zeta^2 &\mapsto \alpha \end{aligned}$$

Now, to realize the restrictions of E to the lines in $\mathbb{P}^2$ as a family, we want to consider simply the pullback bundle

$$F = \pi_2^*(E)$$

on Φ. From Theorem 16.5, we would expect that, for an open dense subset U of $L \in \mathbb{P}^{2*}$, the restriction of F to the fiber over L would split evenly, that is, as

$$E|_L = F|_{\pi_2^{-1}(L)} \cong \mathcal{O}_L(k) \oplus \mathcal{O}_L(k);$$

that there will be a curve $C \subset \mathbb{P}^{2*}$ of lines L such that

$$E|_L = F|_{\pi_2^{-1}(L)} \cong \mathcal{O}_L(k+1) \oplus \mathcal{O}_L(k-1),$$

and that these are the only two splitting types that will occur. The first question we might ask, then, is just: what is the degree of the curve C?

We'd like to answer this by applying the Grothendieck-Riemann-Roch formula to the direct image of the bundle F under the map π_1. It's not clear, though, what this will tell us: if $k \geq 0$, the direct image $(\pi_1)_*F$ will be the bundle on $\mathbb{P}^{2*}$ whose fiber at each point $L \in \mathbb{P}^{2*}$ is the space of sections of $E|_L$, and it's not clear how knowing the Chern classes of this bundle reflect the different possible splittings of $E|_L$.

The trick, in fact, is to use the theorem on cohomology and base change. If we replace E by $E' = E \otimes \mathcal{O}_{\mathbb{P}^2}(-k-1)$ and F by the corresponding $F' = \pi_2^* E'$, then the restriction of the bundle E' to L will split as

$$E'|_L \cong \begin{cases} \mathcal{O}_L(-1)^{\oplus 2} & \text{if } L \in U, \text{ and} \\ \mathcal{O}_L \oplus \mathcal{O}_L(-2) & \text{if } L \in C. \end{cases}$$

The point is, we now have

$$h^0(E'|_L) = h^1(E'|_L) = \begin{cases} 0 & \text{if } L \in U, \text{ and} \\ 1 & \text{if } L \in C. \end{cases}$$

What happens now when we take the direct image $(\pi_1)_*(F')$? First of all, the direct image itself will be 0: over the open subset $U \subset \mathbb{P}^{2*}$ there are no sections, and since $(\pi_1)_*(F')$ is torsion free, it follows that $(\pi_1)_*(F') = 0$. Where the jump in the cohomology groups $H^i(E'|_L)$ will be reflected is in the higher direct image $R^1(\pi_1)_*(F')$; this will be a sheaf supported on the curve C. It follows that

$$c_1(R^1(\pi_1)_*(F')) = m[C]$$

for some multiplicity m; and the class $c_1(R^1(\pi_1)_*(F'))$ is something we can calculate from Grothendieck-Riemann-Roch.

To set up, we first need the Todd class of the relative tangent bundle of $\pi_1 : \Phi \to \mathbb{P}^{2*}$. In fact, we've already calculated the Chern class of this bundle: we have, from Theorem 13.4,

$$c_1(T^v_{\Phi/\mathbb{P}^{2*}}) = -\alpha + 2\zeta$$

and correspondingly

$$\mathrm{Td}(T^v_{\Phi/\mathbb{P}^{2*}}) = 1 + \frac{-\alpha + 2\zeta}{2} + \frac{(-\alpha + 2\zeta)^2}{12}$$

(note that since $T^v_{\Phi/\mathbb{P}^{2*}}$ is a line bundle, there's no cubic term). If we write the Chern class of the bundle F' as

$$c(F') = -2\zeta + e\zeta^2$$

then we have

$$\begin{aligned}\mathrm{Ch}(F') &= \mathrm{rank}(F') + c_1(F') + \frac{c_1^2(F') - 2c_2(F')}{2}\\ &= 2 - 2\zeta + (2-e)\zeta^2.\end{aligned}$$

Altogether, then, Grothendieck-Riemann-Roch tells us that

$$\begin{aligned}[C] &= \mathrm{Ch}_1\left(R^1(\pi_1)_*(F')\right)\\ &= -(\pi_1)_*\left\{\left(2 - 2\zeta + (2-e)\zeta^2\right)\left(1 + \frac{-\alpha+2\zeta}{2} + \frac{(-\alpha+2\zeta)^2}{12}\right)\right\}_2\\ &= -(\pi_1)_*\left(\frac{(-\alpha+2\zeta)^2}{6} - \zeta(-\alpha+2\zeta) + (2-e)\zeta^2\right)\\ &= (e-1)\zeta;\end{aligned}$$

or, in other words, the degree of the curve of jumping lines is $e-1$. Given that $E' = E \otimes \mathcal{O}_{\mathbb{P}^2}(-k-1)$, we have

$$e = c_2(E') = c_2(E) - (k^2 - 1)$$

and so for our original bundle E, we have

Proposition 16.6. *If E is a vector bundle of rank 2 on $\mathbb{P}^2$ with even first Chern class $2k\alpha$, and the restriction of E to a general line $L \subset \mathbb{P}^2$ is balanced (that is, $E|_L \cong \mathcal{O}_L(k) \oplus \mathcal{O}_L(k)$), then the degree of the curve of jumping lines, counted with appropriate multiplicity, is $c_2(E) - k^2 = c_2(E) - c_1^2(E)/4$.*

Note that we have not described the multiplicity, or given conditions for it to be 1; and indeed there may be bundles for which the curve of jumping lines has multiplicity $m > 1$. (To give a precise formula for m would require a discussion of *Fitting ideals*, which would take us too far afield; see ****ref**** for a discussion.) Thus the formula remains a purely enumerative one. Nonetheless, it has content: for example, we may deduce that the degree of C is at most $c_2(E)-c_1^2(E)/4$, and if $c_2(E)-c_1^2(E)/4 \neq 0$ we may conclude that the curve of jumping lines is nonempty.

Vector bundles with odd first Chern class. Assume now that $c_1(E) = (2k+1)\alpha$. The basic set-up is the same as in the preceding case, but here our expectations are qualitatively different. As before, we would expect that, for an open dense subset U of $L \in \mathbb{P}^{2*}$, the restriction of F to the fiber over L would split "as evenly as possible," that is, as

$$E|_L = F|_{\pi_2^{-1}(L)} \cong \mathcal{O}_L(k+1) \oplus \mathcal{O}_L(k).$$

But in this case we'd expect that the locus where the splitting is more unbalanced—specifically, the set of lines L such that

$$E|_L = F|_{\pi_2^{-1}(L)} \cong \mathcal{O}_L(k+2) \oplus \mathcal{O}_L(k-1)\text{—}$$

will be not a curve but a finite set Γ of points. (Again, we'd expect that these are the only two splitting types that will occur.) In this case we ask: what is the degree of the finite set Γ?

Our approach to this problem starts out similarly: we begin by twisting the vector bundle E so that the jump in splitting type is reflected in the ranks of the cohomology groups of its restriction to lines. Specifically, we replace E by $E' = E \otimes \mathcal{O}_{\mathbb{P}^2}(-k-1)$, so that the restriction of the bundle E' to L will split as

$$E'|_L \cong \begin{cases} \mathcal{O}_L \oplus \mathcal{O}_L(-1) & \text{if } L \in U, \text{ and} \\ \mathcal{O}_L(1) \oplus \mathcal{O}_L(-2) & \text{if } L \in \Gamma. \end{cases}$$

The point is, we now have

$$h^0(E'|_L) = \begin{cases} 1 & \text{if } L \in U, \text{ and} \\ 2 & \text{if } L \in \Gamma. \end{cases}$$

and

$$h^1(E'|_L) = \begin{cases} 0 & \text{if } L \in U, \text{ and} \\ 1 & \text{if } L \in \Gamma. \end{cases}$$

As before, this means that the sheaf $R^1\pi_*(F')$ is supported on the exceptional locus, in this case Γ; and in the transverse case the analysis of families of vector bundles shows that it is isomorphic to the structure sheaf $\mathcal{O}_\Gamma$.

The big difference is that the direct image $\pi_*(F')$ is nonzero; in fact, away from Γ it's locally free of rank 1. In the transverse case, moreover, we've seen that it's also locally free in a neighborhood of a point $p \in \Gamma \subset \mathbb{P}^{2*}$ corresponding to a line $L \subset \mathbb{P}^2$, though the comparison map

$$\pi_*(F')|_p \to H_0(F'|_L)$$

is zero: none of the sections of $F'|_L$ extend to a neighborhood of L. In other words, we have

$$\pi_*(F') \cong \mathcal{O}_(m)$$

for some m; the value of m will also emerge in the calculation.

We can now get to work and apply GRR. To begin with, write the Chern class of the bundle E' as

$$c(E') = 1 - \zeta + e\zeta^2;$$

the Chern character of E', and of its pullback F' to Φ, is then

$$\mathrm{Ch}(F') = 2 - \zeta + (\frac{1}{2} - e)\zeta^2.$$

The Grothendieck Riemann Roch formula then tells us that

$$\begin{aligned}\mathrm{Ch}&(\pi_*F') - \mathrm{Ch}(R^1\pi_*F')\\ &= \pi_*\left[\left(2 - \zeta + (\frac{1}{2} - e)\zeta^2\right)\left(1 + \frac{-\alpha + 2\zeta}{2} + \frac{(-\alpha+2\zeta)^2}{12}\right)\right]\\ &= 1 - e\alpha + \frac{e}{2}\alpha^2.\end{aligned}$$

Now, from the isomorphism $\pi_*(F') \cong \mathcal{I}_\Gamma(m)$ we have

$$\mathrm{Ch}(\pi_*F') = 1 + m\alpha + \frac{m^2}{2}\alpha^2;$$

since

$$\mathrm{Ch}(R^1\pi_*F') = \mathrm{Ch}(\mathcal{O}_\Gamma) = [\Gamma] = \gamma\alpha^2,$$

all in all we have

$$1 - e\alpha + \frac{e}{2}\alpha^2 = 1 + m\alpha + (\frac{m^2}{2} - 2\gamma)\alpha^2.$$

From the degree 1 terms, we see that $m = -e$; and then equating the degree 2 terms we arrive at our answer: the degree of Γ is given by

$$\gamma = \frac{e^2 - e}{2}.$$

Finally, to express this in terms of the Chern classes of our original bundle E, we observe that by the splitting principle, for an arbitrary bundle E of rank 2 on $\mathbb{P}^2$ with first Chern class $c_1(E) = (2k+1)\zeta$ the degree of the second Chern class of the twist $E' = E(-k-1)$ is

$$\begin{aligned} e &= c_2(E) - (k+1)c_1(E) + (k+1)^2 \\ &= c_2(E) - k(k+1). \end{aligned}$$

We thus have the

Proposition 16.7. *Let E be a vector bundle of rank 2 on $\mathbb{P}^2$ with odd first Chern class $(2k+1)\alpha$, and suppose that the restriction of E to a general line $L \subset \mathbb{P}^2$ is balanced (that is, $E|_L \cong \mathcal{O}_L(k) \oplus \mathcal{O}_L(k+1)$). If the locus $\Gamma \subset \mathbb{P}^{2*}$ of jumping lines has dimension zero, then it has degree $(e^2 - e)/2$, where $e = c_2(E) - k(k+1)$.*

As in the case of Proposition 16.6, this statement implicitly invokes a scheme structure on Γ that we have not described; again, however, even in the absence of this we can deduce that the number of jumping lines is at most $(e^2 - e)/2$, and if $e^2 - e \neq 0$ we may conclude that the locus of jumping lines is nonempty.

16.5.3 Examples

Some examples would be in order. We start with the simplest examples of vector bundles of rank 2 on $\mathbb{P}^2$, the universal quotient bundle Q and the tangent bundle $T_{\mathbb{P}^2} \cong Q(1)$. In either case, when we twist the bundle to make the second Chern class equal to $-\zeta$, the bundle $E' \cong Q(-1) \cong T_{\mathbb{P}^2}(-2)$ will have Chern class

$$c(E') = 1 - \zeta + \zeta^2.$$

Our formula then says that these bundles will have no jumping lines, which is clear in any event from homogeneity: since the automorphism group of $\mathbb{P}^2$ carries each of these bundles to itself and acts transitively on lines, the splitting type of their restrictions to lines can't vary.

For another example, suppose now that F_0, F_1 and F_2 are general homogeneous polynomials of degree d on $\mathbb{P}^2$. We can then define a vector bundle of rank 2 on $\mathbb{P}^2$ by associating to each point p the quotient space

$$E_p = \mathbb{C}^3 / \langle (F_0(p), F_1(p), F_2(p)) \rangle;$$

in other words, this is the pullback of the universal quotient bundle Q by the map $\mathbb{P}^2 \to \mathbb{P}^2$ given by the triple $[F_0, F_1, F_2]$. Since $c(Q) = 1 + \zeta + \zeta^2$, we have

$$c(E) = 1 + d\zeta + d^2\zeta^2.$$

When the degree $d = 2k$ is even, the bundle $E' = E(-k-1)$ will have Chern class

$$c(E') = 1 - 2\zeta + (3k^2 + 1)\zeta^2,$$

and so the curve of jumping lines will have degree $3k^2$, or $\frac{3}{4}d^2$. When the degree $d = 2k+1$ is odd, we have

$$c(E') = 1 - \zeta + (3k^2 + 3k + 1)\zeta^2,$$

so the number of jumping lines will be the binomial coefficient $\binom{3k^2+3k+1}{2}$.

We'll see how to verify this count in case $d = 3$ in Exercise 16.20.

16.6 Application: Invariants of families of curves

One of the most important applications of GRR is to the geometry of algebraic curves, and we close this chapter by describing this application.

To set this up, consider one-parameter families of curves of genus g: that is, flat morphisms $f : X \to B$ from a surface X to a curve B, with fibers curves of genus g. The question we want to discuss is, *How can we measure how much the isomorphism class of X_b is varying with $b \in B$?*

In order to address this meaningfully, we have to make some restrictions of the type of curves that appear as fibers of f. For simplicity, in the present discussion we'll assume that X and B are smooth, and that the fibers X_b of f are irreducible with at worst nodes as singularities. These conditions can be relaxed—for example, we can drop the requirement that X and B be smooth, and weaken the hypothesis that X_b be irreducible to require only that X_b be *stable*—but our goal here is not to achieve maximal generality but just to acquaint the reader with the basic ideas. A far more precise version of this is given in Harris and Morrison [1998].

That said, there are three natural ways of quantifying the nontriviality of the family:

(a) *The number of singular fibers.* A singular fiber is not isomorphic to a smooth fiber; therefore, the presence of singular fibers is a reflection of variation in the isomorphism class of X_b.[1] Thus, we can associate to a family $f : X \to B$ as above the number $\delta(f)$ of nodes appearing in the fibers of f.

(b) *The degree of the Hodge bundle.* To every smooth fiber X_b we can associate the g-dimensional vector space $H^0(K_{X_b})$, and over the open set $U \subset B$ where the fibers are smooth these vector spaces fit together

[1] This would not be true if we did not assume that the fibers were irreducible: you could take a trivial family $B \times C \to B$ and blow up a point in $B \times C$ to arrive at a family of curves of constant modulus with one singular fiber. The logic would be valid if we assumed all fibers to be *stable curves*; in the present context we avoid getting into a discussion of this notion by assuming all fibers irreducible.

into a vector bundle, called the *Hodge bundle* E of f. This bundle has a natural extension to all of B, described in the following paragraph. If the family were the trivial family $B \times C \to B$, this would be the trivial bundle; so we can think of the twisting of the Hodge bundle, as measured by its first Chern class, as a reflection of the nontriviality of the family. Accordingly we take our second invariant of the family to be

$$\lambda(f) = c_1(E).$$

To extend the Hodge bundle over all of B, we have to introduce what's called the *relative dualizing sheaf* $\omega_{X/B}$ of the family. This has many characterizations (again, see Harris and Morrison [1998]); for our purposes, though, it will suffice to define $\omega_{X/B}$ to be the unique line bundle on X whose restriction to the smooth locus X° of f—that is, the complement of the finite set of nodes of fibers of f—is the vertical cotangent bundle. (Any line bundle on an open subset U of a smooth variety X can be extended to all of X; if the complement $X \setminus U$ has codimension 2 or more in X, this extension is unique.) More concretely, we have

$$\omega_{X/B} = K_X \otimes f^* K_B^{-1},$$

since the sheaf on the right is locally free of rank 1, and agrees with the vertical cotangent bundle on X°. Having defined the relative dualizing sheaf of f, we can then define the Hodge bundle E to be its direct image

$$E = f_*(\omega_{X/B}).$$

(c) For our third invariant of f, we reason (if that's the word) as follows: again, if f were the trivial family $X = B \times C \to B$, the relative cotangent bundle $\omega_{X/B}$ would be simply the pullback of K_C via projection on the second factor; in particular, its self-intersection would be 0. We may thus think of this self-intersection as measuring the variation in the family, and accordingly we define our third invariant to be

$$\kappa(f) = \big(c_1(\omega_{X/B})\big)^2$$

We thus have three invariants of a family of curves as above, all of which may be thought of as measuring the variation in the family $\{X_b\}$ of curves. The question we ask now is, *what relations (if any) hold in general among these three?*

16.6.1 Examples

Before we launch into the answer, let's consider some examples in genus 3, to get an idea of what we might expect.

Pencils of quartics in the plane. Let's start with a family whose geometry we've already studied in earlier chapters: a general pencil $\{C_t = V(t_0F + t_1G) \subset \mathbb{P}^2\}$ of plane quartic curves. In this case, the total space of the family is the locus

$$X = \{(t, p) : p \in C_t\} \subset \mathbb{P}^1 \times \mathbb{P}^2.$$

We observe that X is smooth—the projection $\beta : X \to \mathbb{P}^2$ on the second factor expresses X as the blow-up of the plane at the 16 base points $F = G = 0$ of the pencil. Moreover, we've already seen that every fiber C_t of $\alpha : X \to \mathbb{P}^1$ is either smooth, or is irreducible with a single node; thus the family $\alpha : X \to \mathbb{P}^1$ satisfies the conditions of our discussion.

What are the invariants δ, λ and κ of f? Well, one we've already calculated in Chapter 9: the number of nodes in the fibers of f is

$$\delta(\alpha) = 27.$$

As for κ, this is a straightforward calculation in the Chow ring of X. To set it up, let $L \in A^1(X)$ be the pullback of the class of a line in $\mathbb{P}^2$, and let $E \in A^1(X)$ be the sum of the classes of the 16 exceptional divisors of β. In these terms, the class of the fibers C_t of the projection α—that is, the pullback under α of the class η of a point in $\mathbb{P}^1$—is given by

$$[C_t] = \alpha^*(\eta) = 4L - E.$$

In particular, this implies that

$$\alpha^* K_{\mathbb{P}^1} = -2\alpha^*(\eta) = -8L + 2E.$$

On the other hand, from the blow-up map $\beta : X \to \mathbb{P}^2$ we see that

$$K_X = \beta^*(K_{\mathbb{P}^2}) + E = -3L + E.$$

Thus, the class of the relative dualizing sheaf $\omega_{X/\mathbb{P}^1}$ is

$$c_1(\omega_{X/\mathbb{P}^1}) = K_X - \alpha^* K_{\mathbb{P}^1} = 5L - E$$

and the invariant $\kappa(\alpha)$ is given by

$$\kappa(\alpha) = \big(c_1(\omega_{X/\mathbb{P}^1})\big)^2 = 25 - 16 = 9.$$

Next, the Hodge bundle can be described explicitly: since $5L - E = (4L - E) + L$, we can write the dualizing sheaf as

$$\begin{aligned} \omega_{X/\mathbb{P}^1} &= \mathcal{O}_X(5L - E) \\ &= \alpha^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \beta^*\mathcal{O}_{\mathbb{P}^2}(1). \end{aligned}$$

Now, the pushforward $\alpha_*(\beta^*\mathcal{O}_{\mathbb{P}^2}(1))$ is just the trivial bundle on $\mathbb{P}^1$ with fiber $H^0(\mathcal{O}_{\mathbb{P}^2}(1))$; we have then

$$\begin{aligned} \alpha_*\big(\omega_{X/\mathbb{P}^1}\big) &= \alpha_*\big(\alpha^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \beta^*\mathcal{O}_{\mathbb{P}^2}(1)\big) \\ &= \mathcal{O}_{\mathbb{P}^1}(1) \otimes H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \\ &\cong \mathcal{O}_{\mathbb{P}^1}(1)^3, \end{aligned}$$

and correspondingly

$$\lambda(\alpha) = c_1\big(\alpha_*(\omega_{X/\mathbb{P}^1})\big) = 3.$$

To see this more concretely, choose affine coordinates t on $\mathbb{A}^1 \subset \mathbb{P}^1$ and (x, y) on $\mathbb{A}^2 \subset \mathbb{P}^2$. We write the equation of C_t as $f_t(x, y)$, where f_t is a quartic polynomial in x and y whose coefficients are linear in t. In these terms, we can write down explicitly a basis for the space $H^0(K_{C_t})$ of regular differentials on C_t: we observe first that in the finite plane, the differential

$$\varphi_t = \frac{dx}{\frac{\partial}{\partial y} f_t(x, y)}$$

is regular and nowhere zero on the smooth locus of $C_t \cap \mathbb{A}^2$: since

$$df_t = \frac{\partial f_t}{\partial x} dx + \frac{\partial f_t}{\partial y} dy \equiv 0 \text{ on } C$$

and at a smooth point of C_t the partials $\frac{\partial f_t}{\partial x}$ and $\frac{\partial f_t}{\partial y}$ don't simultaneously vanish, the numerator vanishes to exactly the same order as the denominator. Moreover, changing coordinates on $\mathbb{P}^2$ we see that the differential φ will have zeros at the points of intersection of C_t with the line $L_\infty = \mathbb{P}^2 \setminus \mathbb{A}^2$ at infinity; so that the differentials $x\varphi_t$ and $y\varphi_t$ are also regular on C_t. Finally, since φ_t, $x\varphi_t$ and $y\varphi_t$ are regular sections of the relative dualizing sheaf $\omega_{X/\mathbb{P}^1}$ on the smooth locus of α over $\mathbb{A}^1$, they extend to regular sections of $\omega_{X/\mathbb{P}^1}$ on all of $X_{\mathbb{A}^1}$. In sum, on $\mathbb{A}^1$, *the sections*

$$\varphi,\ x\varphi \ \text{ and } \ y\varphi \in \alpha_*(\omega_{X/\mathbb{P}^1})(\mathbb{A}^1)$$

give a trivialization of $\omega_{X/\mathbb{P}^1}$ *over* $\mathbb{A}^1$.

What happens over $t = \infty$? If we look again at the expression above for φ_t, we see that since the denominator $\frac{\partial f_t(x,y)}{\partial y}$ is linear in t, the section φ of $\alpha_*(\omega_{X/\mathbb{P}^1})$ extends to a regular section of $\alpha_*(\omega_{X/\mathbb{P}^1})$ over all of $\mathbb{P}^1$, having a simple zero at $t = \infty$, and the same is true of $x\varphi$ and $y\varphi$ as well. Moreover, making a change of coordinates on $\mathbb{P}^1$, we see that the multiples $t\varphi$, $tx\varphi$ and $ty\varphi$ form a frame for the bundle $\alpha_*(\omega_{X/\mathbb{P}^1})$ over $\mathbb{P}^1 \setminus \{0\}$; so that we have

$$\alpha_*(\omega_{X/\mathbb{P}^1}) \cong \mathcal{O}_{\mathbb{P}^1}(1)^3$$

and correspondingly

$$\lambda(\alpha) = c_1\big(\alpha_*(\omega_{X/\mathbb{P}^1})\big) = 3.$$

To summarize, we have

$$\delta(\alpha) = 27, \quad \kappa(\alpha) = 9 \quad \text{and} \quad \lambda(\alpha) = 3.$$

Pencils of plane sections of a quartic surface. This is very similar to the preceding example; we'll just sketch the calculation. To set up, let $S \subset \mathbb{P}^3$ be a general quartic surface, and $\{H_t \subset \mathbb{P}^3\}_{t \in \mathbb{P}^1}$ a general pencil of planes; we let $C_t = S \cap H_t$. As we saw in Chapter 9, the curves C_t are all either smooth or irreducible with a single node; as we also saw there , the number of singular elements of the pencil $\{C_t\}$ is the degree of the dual surface $S^* \subset \mathbb{P}^{3*}$, which is

$$\delta = 36.$$

To calculate the other invariants, note first that the total space of our family is

$$X = \{(t, p) : p \in C_t\} \subset \mathbb{P}^1 \times S.$$

Via the projection $\beta : X \to S$, this is the blow-up of S at the four base points of the pencil. We'll let H and $E \in A^1(X)$ be the pullback of the hyperplane class on S and the sum of the classes of the exceptional divisors, respectively; we note that

$$H^2 = 4, \quad HE = 0 \quad \text{and} \quad E^2 = -4.$$

Since the canonical bundle K_S is trivial by adjunction, we have

$$K_X = E;$$

and since the fibers C_t of the projection map $\alpha : X \to \mathbb{P}^1$ on the first factor have class $H - E$, we have

$$c_1(\omega_{X/\mathbb{P}^1}) = K_X - f^*K_{\mathbb{P}^1} = E - (-2(H - E)) = 2H - E.$$

It follows that

$$\kappa(f) = (2H - E)^2 = 12.$$

To determine $\lambda(f)$, we use a variant of the method used above in the case of a pencil of plane quartics. Writing $2H - E = (H - E) + H$, we have

$$\begin{aligned} \omega_{X/\mathbb{P}^1} &= \mathcal{O}_X(2H - E) \\ &= \alpha^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \beta^*\mathcal{O}_S(1). \end{aligned}$$

Now, if $\mathbb{P}^3$ is the projectivization of a four-dimensional vector space V, and we think of the pencil of planes $\{H_t\}$ as a line $\mathbb{P}^1 \hookrightarrow \mathbb{P}^{3*}$ in the dual projective space $\mathbb{P}^{3*} = \mathbb{P}(V^*)$, the pushforward $\alpha_*(\beta^*\mathcal{O}_S(1))$ is just the restriction to $\mathbb{P}^1$ of the universal quotient bundle Q. As we've seen many times, the first Chern class of the Q is the hyperplane class, whose restriction to a line $\mathbb{P}^1 \subset \mathbb{P}^{3*}$ has degree 1; it follows that

$$\begin{aligned} \alpha_*\big(\omega_{X/\mathbb{P}^1} &= \mathcal{O}_X(2H - E)\big) \\ &= \mathcal{O}_{\mathbb{P}^1}(1) \otimes \alpha_*\big(\beta^*\mathcal{O}_S(1)\big) \\ &= \mathcal{O}_{\mathbb{P}^1}(1) \otimes Q. \end{aligned}$$

Finally, applying our formula for the Chern class of a tensor product, we have

$$\lambda(\alpha) = c_1(\alpha_*(\omega_{X/\mathbb{P}^1}) = 4.$$

In sum, then, we have

$$\delta(\alpha) = 36, \quad \kappa(\alpha) = 12 \quad \text{and} \quad \lambda(\alpha) = 4.$$

Families of hyperelliptic curves. Finally, for a change let's consider a family of curves of genus 3 all of which are hyperelliptic. Such families are easy to write down: a hyperelliptic curve of genus 3 has equation

$$y^2 = f(x),$$

with f of degree 8; and we can vary the coefficients of f any way we like—for example, we can take them to be quadratic polynomials of a parameter $t \in \mathbb{P}^1$. In modern language, we are taking the double cover $X \to \mathbb{P}^1 \times \mathbb{P}^1$ branched over a general curve $B \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(2, 8)$, and considering the composition

$$\eta : X \to \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$$

with the projection on the first factor. We note that X is smooth if B is; and if B has no more than simple tangency with any fiber of $\mathbb{P}^1 \times \mathbb{P}^1$ over $\mathbb{P}^1$ the singularities of the fibers of η will be no worse than nodes; both conditions are satisfied for B general.

The numerical invariants of the family $\eta : X \to \mathbb{P}^1$ can be calculated readily. To begin with, the number of singular fibers is just the number of branch points of the projection $B \to \mathbb{P}^1$ on the first factor. By adjunction, the genus of B is

$$g(B) = (2-1)(8-1) = 7,$$

so by Riemann-Hurwitz the number of branch points of the 8-sheeted cover $B \to \mathbb{P}^1$ is

$$\delta = 2d + 2g - 2 = 28.$$

Alternatively, if we think of the defining equation $f(t, x)$ as an octic polynomial in x whose coefficients are quadratic polynomials in t, and recall that the discriminant of a polynomial of degree n is a polynomial of degree $2n - 2$ in its coefficients, we see that the discriminant of $f(t, x)$ is a polynomial of degree $2 \cdot 14 = 28$ in t.

To evaluate $\kappa(\eta)$, let E_0 and $F_0 \in A^1(\mathbb{P}^1 \times \mathbb{P}^1)$ be the classes of the fibers of the projections $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, and E and $F \in A^1(X)$ their pullbacks to X, so that

$$E^2 = F^2 = 0 \quad \text{and} \quad EF = 2.$$

Let $R \in A^1(X)$ be the class of the ramification divisor; note that since the preimage of the divisor $B \subset \mathbb{P}^1 \times \mathbb{P}^1$ is $2R$, and $B \sim 2E_0 + 8F_0$, we have

$$2R = 2E + 8F,$$

so that

$$RE = 8, \quad RF = 2 \quad \text{and} \quad R^2 = 16.$$

Now, by Riemann-Hurwitz applied to the map $\pi : X \to \mathbb{P}^1 \times \mathbb{P}^1$, we have

$$K_X = \pi^* K_{\mathbb{P}^1 \times \mathbb{P}^1} + R = -2E - 2F + R$$

and so

$$c_1(\omega_{X/\mathbb{P}^1}) = -2F + R;$$

it follows that

$$\kappa(\eta) = \big(c_1(\omega_{X/\mathbb{P}^1})\big)^2 = 8.$$

Finally, from the original equation $y^2 = f(t, x)$ of a fiber C_t, we see that a basis for the space of regular differentials is given by

$$\frac{dx}{y}, \quad x\frac{dx}{y} \quad \text{and} \quad x^2\frac{dx}{y};$$

these sections of the Hodge bundle $\eta_*(\omega_{X/\mathbb{P}^1})$ thus give a trivialization of this bundle over the affine open $t \neq \infty$ in $\mathbb{P}^1$. Moreover, since the coefficients of f are quadratic in t, and $y^2 = f$, these sections all vanish simply at the point $t = \infty$; and as before this gives an identification

$$\eta_*((\omega_{X/\mathbb{P}^1})) = \mathcal{O}_{\mathbb{P}^1}(1)^3.$$

In particular, we see that

$$\lambda(\eta) = c_1\big((\omega_{X/\mathbb{P}^1})\big) = 3.$$

In sum, then, the invariants of the family $\eta : X \to \mathbb{P}^1$ are

$$\delta(\eta) = 28, \quad \kappa(\eta) = 8 \quad \text{and} \quad \lambda(\eta) = 3.$$

We see from this in particular that the invariants δ, κ and λ are not in general fixed scalar multiples of each other.

16.6.2 The Mumford relation

It's time now to see how the Grothendieck-Riemann-Roch theorem applies in this setting. Again, we assume that $f : X \to B$ is a morphism with X a smooth projective surface and B a smooth projective curve, with fibers irreducible curves of (arithmetic) genus g having at most nodes as singularities. Given that one of invariants we've introduced is the first Chern class of a pushforward, it's natural to see what GRR can tell us about it.

Before carrying out this calculation, we want to make two remarks. Recall that the statement of Kodaira-Serre duality for smooth curves says that for any invertible sheaf $\mathcal{F}$ on a smooth curve C, we have a natural identification

$$H^1(\mathcal{F}) = H^0(K_C \otimes \mathcal{F}^*)^*.$$

Here the word "natural" means in particular that the identification applies simultaneously for all fibers in a family: that is, if $f : X \to B$ is a family of smooth, projective curves of genus g and $\mathcal{F}$ an invertible sheaf on X the ranks of whose cohomology groups on the fibers of f is constant, then we have an identification of sheaves

$$R^1 f_*(\mathcal{F}) = f_*(\omega_{X/B} \otimes \mathcal{F}^*)^*.$$

(In fact, a version of this identification holds far more generally, but this is all we'll need for the following, and we want to keep this simple; for the general statement, see Hartshorne [1977].) In particular, if we apply this to the case of $\mathcal{F} = \omega_{X/B}$, we see that

$$R^1 f_*(\omega_{X/B}) = f_*(\mathcal{O}_X) = \mathcal{O}_B.$$

Now, the classical form of Kodaira-Serre duality applies just to smooth curves and families of smooth curves. But in fact we can extend it to our present context: if $f : X \to B$ is as above a family of curves with at most nodes as singularities, and $\omega_{X/B}$ is the relative dualizing sheaf as defined above, then it is still true that

$$R^1 f_*(\omega_{X/B}) = \mathcal{O}_B.$$

In the following calculation, we'll use this fact; for a proof, again see Hartshorne [1977].

The second remark is simply a preliminary calculation. We recall first the *Hopf Index Theorem* (Theorem 1.45): that the degree of the top Chern class $c_2(T_X)$ of the tangent bundle of the smooth projective surface X is the topological Euler characteristic of X. Moreover, the topological Hurwitz formula (introduced in Section 9.8) says that the topological Euler characteristic of X is the product of the Euler characteristics $2-2g$ of the general fiber of f and the Euler characteristic of the base curve B, plus the total number δ of nodes in the fibers of f. Combining these, we see that

$$\begin{aligned} c_2(T_X) &= (2-2g)c_1(T_B) + \delta \\ &= -c_1(\omega_{X/B}) \cdot f^*c_1(T_B) + \delta. \end{aligned}$$

Abbreviating the expression $\omega_{X/B}$ for the relative dualizing sheaf to simply ω, we can thus express the ratio $c(T_X)/f^*c(T_B)$ as

$$\begin{aligned} \frac{c(T_X)}{f^*c(T_B)} &= \big(1 + c_1(T_X) + c_2(T_X)\big)\big(1 - f^*c_1(T_B)\big) \\ &= 1 - c_1(\omega) + \delta, \end{aligned}$$

and correspondingly

$$\frac{Td(T_X)}{f^*Td(T_B)} = 1 - \frac{c_1(\omega)}{2} + \frac{c_1(\omega)^2 + \delta}{12}.$$

We can now apply GRR to the pushforward of the sheaf $\omega_{X/B}$. Taking just the terms of degree 1 in $A(B)$ (and recalling that $c_1(R^1 f_* \omega_{X/B}) = 0$), we conclude that

$$\begin{aligned}\lambda(f) =& c_1(f_*\omega)\\ &= \left\{ f_*\big((1 + c_1(\omega) + \frac{c_1(\omega)^2}{2})(1 - \frac{c_1(\omega)}{2} + \frac{c_1(\omega)^2 + \delta}{12})\big) \right\}_{\dim B - 1}\\ &= f_*\left(\frac{c_1(\omega)^2}{2} - \frac{c_1(\omega)^2}{2} + \frac{c_1(\omega)^2 + \delta}{12} \right)\\ &= \frac{\kappa(f) + \delta(f)}{12}.\end{aligned}$$

This fundamental relation among the invariants λ, κ and δ is called the *Mumford relation*; as is suggested by the examples we calculated above in genus 3, it is the only linear relation satisfied by these three invariants in general.

Further examples of one-parameter families of curves for which we can verify the Mumford relation are given in Exercises 16.22-16.24.

16.6.3 Inequalities

At the outset of this discussion, we asked for ways of measuring the variation in a family $X \to B$ of curves, with X and B smooth and projective and the fibers irreducible with at worst nodes as singularities. While we can't prove this, we should at least indicate to what extent the invariants δ, κ and λ we introduced do this.

The answers are:

(a) The invariant δ does not necessarily detect variation in the fibers of $X \to B$: there exist families of curves with varying modulus, but with no singular fibers at all. By contrast,

(b) The invariants κ and λ do: if $f : X \to B$ is a family as above and either $\kappa(f) = 0$ or $\lambda(f) = 0$, then all the fibers are in fact isomorphic.

As we indicated, it's not hard to see by constructing examples that the Mumford relation above is the only one satisfied universally by δ, κ and λ. The inequalities satisfied by these invariants, however, are more subtle. For example, if $X \to B$ is a one-parameter family of irreducible nodal curves of genus g, generically smooth, we have the inequality

$$\delta \leq \big(8 + \frac{4}{g}\big)\lambda;$$

this is sharp, as shown by the example of Exercise 16.25. For a discussion of these questions and references, see Harris and Morrison [1998].

16.7 Exercises

Exercise 16.8. (a) Find the Chern characters of the universal bundles S and Q on the Grassmannian $\mathbb{G} = \mathbb{G}(1,3)$.

(b) Use this to find the Chern character of the tangent bundle $T_{\mathbb{G}} = S^* \otimes Q$.

(c) Use this in turn to find the Chern class of $T_{\mathbb{G}}$.

Exercise 16.9. Verify that this is well-defined; that is, $c(\mathcal{F})$ does not depend on the choice of resolution. ****this should have a hint.****

Exercise 16.10. Let $p \in \mathbb{P}^n$ be a point. Using the Koszul complex, show that the Chern class of the structure sheaf $\mathcal{O}_p$, viewed as a coherent sheaf on $\mathbb{P}^n$, is

$$c(\mathcal{O}_p) = 1 + (-1)^{n-1}(n-1)![p].$$

Exercise 16.11. Let $C \subset \mathbb{P}^3$ be a smooth curve. Find the Chern class of the structure sheaf $\mathcal{O}_C$ in case

(a) C is a twisted cubic;

(b) C is an elliptic quartic curve, and

(c) C is a rational quartic curve.

Exercise 16.12. Consider the three varieties $X_1 = \mathbb{P}^3$, $X_2 = \mathbb{P}^1 \times \mathbb{P}^2$ and $X_3 = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

(a) In each case, calculate the degrees of the classes $c_3(T_{X_i})$, $c_1(T_{X_i})c_2(T_{X_i})$ and $c_1(T_{X_i})^3$.

(b) Show that the resulting 3×3 matrix is nonsingular.

(c) Show that the Euler characteristic $\chi(\mathcal{O}_{X_i}) = 1$ for each i.

(d) Given that the Euler characteristic of the structure sheaf of a smooth projective threefold X is expressible as a polynomial of degree 3 in the Chern classes of its tangent bundle, show from the above examples that the polynomial must be

$$\mathrm{Td}_3(c_1, c_2, c_3) = \frac{c_1 c_2}{24}.$$

Exercise 16.13. Verify that the formula of Theorem 16.2 gives the classical Riemann-Roch formula in case $n = 1$ or 2, and write down the analogous formula in case $n = 3$.

Exercise 16.14. In Section 9.3 we introduced a vector bundle E on $\mathbb{P}^2$ whose fiber at a point $p \in \mathbb{P}^2$ is the space $H^0(\mathcal{O}_{\mathbb{P}^2}(d)/\mathcal{I}_p^2(d))$ of homogenenous polynomials of degree d, modulo those vanishing to order 2 or more at p; we also described E as a direct image. Use this and the Grothendieck-Riemann-Roch formula to calculate the Chern classes of E.

Exercise 16.15. In Chapter 13 we introduced a vector bundle E on the universal line

$$\Phi = \{(L, p) \in \mathbb{G}(1,3) \times \mathbb{P}^3 \mid p \in L\}$$

whose fiber at a point $(L, p) \in \Phi$ is the space $H^0(\mathcal{O}_L(d)/\mathcal{I}_p^5(d))$ of homogenenous polynomials of degree d on L, modulo those vanishing to order 5 or more at p; we also described E as a direct image. Use this and the Grothendieck-Riemann-Roch formula to calculate the Chern classes of E.

Exercise 16.16. Let $C \subset \mathbb{P}^2$ be an irreducible plane curve of degree d. Construct a bundle E on the dual plane $\mathbb{P}^{2*}$ whose fiber at a point L is the space of sections of the structure sheaf $\mathcal{O}_\Gamma$ of the intersection $\Gamma = C \cap L$, and use the Grothendieck-Riemann-Roch formula to calculate the Chern classes of E.

Exercise 16.17. Let E be an indecomposable vector bundle of rank 2 on $\mathbb{P}^3$ with first Chern class 0, and let

$$\Sigma_1 = \{L \in \mathbb{G}(1,3) \mid E|_L \cong \mathcal{O}_L(k) \oplus \mathcal{O}_L(-k) \text{ with } k \geq 1\}.$$

Find the class of the divisor $\Sigma_1 \subset \mathbb{G}(1,3)$.

Exercise 16.18. With E as in the preceding problem, let

$$\Sigma_2 = \{L \in \mathbb{G}(1,3) \mid E|_L \cong \mathcal{O}_L(k) \oplus \mathcal{O}_L(-k) \text{ with } k \geq 2\}.$$

Find the class of the locus $\Sigma_2 \subset \mathbb{G}(1,3)$, assuming it has the expected dimension 1.

Exercise 16.19. Now let E be a vector bundle of rank 2 on $\mathbb{P}^3$ with first Chern class $c_1(E) = \zeta$ the hyperplane class, and let

$$\Sigma_i = \{L \in \mathbb{G}(1,3) \mid E|_L \cong \mathcal{O}_L(k+1) \oplus \mathcal{O}_L(-k) \text{ with } k \geq i\}$$

for $i = 1, 2$ Find the classes of the loci $\Sigma_i \subset \mathbb{G}(1,3)$, assuming they have the expected codimension $2i$.

Exercise 16.20. Let F_0, F_1 and F_2 be general homogeneous polynomials of degree 3 on $\mathbb{P}^2$, and as in Section 16.5.3 define a vector bundle of rank 2 on $\mathbb{P}^2$ by associating to each point p the quotient space

$$E_p = \mathbb{C}^3/\langle (F_0(p), F_1(p), F_2(p)) \rangle;$$

in other words, E is the pullback of the universal quotient bundle Q by the map $\mathbb{P}^2 \to \mathbb{P}^2$ given by the triple $[F_0, F_1, F_2]$. Show that the jumping lines of E are exactly the 21 lines in $\mathbb{P}^2$ contained in some element of the net.

Exercise 16.21. As an example, let $q : \mathbb{C}^4 \times \mathbb{C}^4 \to \mathbb{C}$ be a nondegenerate skew-symmetric bilinear form. We can define a bundle of rank 2 E on $\mathbb{P}^3$ by setting

$$E_p = \langle p \rangle^\perp / \langle p \rangle.$$

Describe the locus of jumping lines for such a bundle.

Exercise 16.22. Let $\{C_t \subset \mathbb{P}^2\}_{t\in\mathbb{P}^1}$ be a general pencil of plane curves of degree d. Calculate the numerical invariants δ, κ and λ for the family, and verify the Mumford relation.

Exercise 16.23. Let $\{C_t \subset S\}_{t\in\mathbb{P}^1}$ be a general pencil of plane sections of a smooth surface $S \subset \mathbb{P}^3$ of degree d. Calculate the numerical invariants δ, κ and λ for the family, and verify the Mumford relation.

Exercise 16.24. Let $\{C_t \subset \mathbb{P}^1 \times \mathbb{P}^1\}_{t\in\mathbb{P}^1}$ be a general pencil of curves of bidegree (a, b) in $\mathbb{P}^1 \times \mathbb{P}^1$. Calculate the numerical invariants δ, κ and λ for the family, and verify the Mumford relation.

Exercise 16.25. Let $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a general curve of bidegree $(2, 2g+2)$, and let $X \to \mathbb{P}^1 \times \mathbb{P}^1$ be the double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched over C. Viewing $X \to \mathbb{P}^1$ as a family of hyperelliptic curves of genus g via projection on the first factor, calculate the invariants δ, κ and λ for the family; verify the Mumford relation, and also show that the inequality

$$\delta \leq \left(8 + \frac{4}{g}\right)\lambda$$

stated in Section 16.6.3 is sharp.

17

Maps from curves to projective space

Keynote Questions

(a) What is the smallest degree of a non-constant map from a general curve of genus g to $\mathbb{P}^1$? 652)

(b) In how many ways can a general curve C of genus 4 be expressed as a 3-sheeted cover of $\mathbb{P}^1$? (Answer on page 652)

(c) What is the smallest degree of a birational map from a general curve of genus g in $\mathbb{P}^2$? 652)

(d) In how many ways can a general curve C of genus 6 be expressed as curve of degree 6 in $\mathbb{P}^2$, up to automorphisms of $\mathbb{P}^2$? (Answer on page 652)

(e) What is the smallest degree of an embedding of a general curve of genus g in $\mathbb{P}^3$? 652)

(f) In how many ways can a general curve C of genus 8 be embedded in $\mathbb{P}^3$, up to automorphisms of $\mathbb{P}^3$? 652)

In the final chapter of this book we'll describe, and prove half of, a foundational result in the theory of algebraic curves: the *Brill-Noether theorem*, whose statement we'll give below. Here the methods of enumerative geometry will lead to a qualitative result that answers a natural non-enumerative question. In the book so far, we've treated problems in enumerative geometry as interesting for the aspects of algebraic geometry that they illuminate, and for their own sake—there's a certain fascination with being able to enumerate solutions to a geometric problem, even when we can't find those solutions explicitly. There are, however, many times when the methods of enumerative geometry turn out to be crucial in the analysis of qualitative questions in geometry, and it seems to us fitting to close this book with such an example.

The proof given here also motivates the need for an equivalence relation on cycles less fine than rational equivalence, since the equalities we need do not hold in the Chow ring. Instead, for technical simplicity, we will work with homology and cohomology in the classical topology over the complex numbers. The cohomology ring is generally a much simpler object than the Chow ring, and many tools such as the Lefschetz hyperplane theorem, described in Appendix **??**, allow us to make computations. In this chapter we'll be dealing with intersections on abelian varieties; and, as we'll explain, the cohomology ring of an abelian variety is easy to describe explicitly, while the Chow ring is not. The techniques developed can be extended to all characteristics using étale cohomology (Kleiman and Laksov [1972]). or numerical equivalence (Kleiman and Laksov [1974]).

To work with cohomology we need to know that there is a map from the Chow ring of a smooth variety to the cohomology ring—essentially this is the theorem that any algebraic subvariety has a topological fundamental class. One can see this, for example, from the fact that any subvariety can be realized as a sub-simplicial complex in an appropriate triangulation; see for example Hironaka [1975], and Chapter 0 of Griffiths and Harris [1978] for a simpler approach in de Rham cohomology. In particular, we will work with the images in cohomology of the Chern classes of algebraic vector bundles—we will call them Chern classes too. All this is discussed further in Appendix **??**.

17.1 What maps to projective space do curves have?

Until the 20th century, varieties were defined as subsets of projective space. In this respect, algebraic geometry was much like other fields in mathematics: for example, in the 19-th century a group was by definition a subset of GL_n or S_n closed under the operations of composition and inversion; the modern definition of an abstract group did not appear until well into the 20-th century. But in about 1860 Riemann's work introduced a way of talking about curves that crystallized, over the next hundred years, into our notion of an *abstract variety*—a geometric object defined independently of any particular embedding in projective space.

The basic problem of classifying all curves in projective space was thus broken down into two parts: the description of the family of abstract curves (the study of *moduli spaces* of curves); and the problem of describing all the ways in which a given curve C might be embedded in or, more generally, mapped to a projective space. To continue our analogy with group theory, the latter question is the analog of *representation theory*, that is, the study of the ways in which a given abstract group G can be mapped to GL_n.

Among the most basic questions we can pose along these lines is, "What maps to projective space does *every* curve of genus g have?" To focus on the objects of principal interest and avoid redundancies, we consider only *nondegenerate* maps $\varphi : C \to \mathbb{P}^r$—that is, maps whose image does not lie in any hyperplane. We define the *degree* of such a map to be the degree of the line bundle $\varphi^*\mathcal{O}_{\mathbb{P}^r}(1)$, or equivalently the cardinality of the preimage $\varphi^{-1}(H)$ of a general hyperplane $H \subset \mathbb{P}^r$, and ask, "For which d, g and r is it the case that every curve of genus g admits a nondegenerate map of degree d or less to $\mathbb{P}^r$?"

It was actually not long after Riemann's work that the correct answer to this question was given by Brill and Noether [1874]. However, it took about 100 years before the first correct proof was given! Here is a weak form of the result:

Theorem 17.1. *Every smooth projective curve of genus g over $\mathbb{C}$ admits a nondegenerate map of degree d or less to $\mathbb{P}^r$ if and only if*

$$\rho(g, r, d) := g - (r+1)(g - d + r) \geq 0$$

Thus, for example, we see that curves of genus 1 and 2 are expressible as 2-sheeted covers of $\mathbb{P}^1$; curves of genus 3 and 4 are expressible as covers of $\mathbb{P}^1$ of degree 3 or less, and, more generally, any curve of genus g is expressible as a cover of $\mathbb{P}^1$ of degree $\lceil \frac{g+2}{2} \rceil$ or less. Likewise, curves of genus 2 and 3 admit maps of degree 4 to $\mathbb{P}^2$; curves of genus 4 admit maps of degree 5 to $\mathbb{P}^2$, and curves of genus 5 and 6 admit maps to $\mathbb{P}^2$ of degree 6 or less; and generally a curve of genus g will admit a map to $\mathbb{P}^2$ of degree $\lceil \frac{2g+6}{3} \rceil$ or less.

The "if" part of Theorem 17.1, the existence, was first proved by Kleiman and Laksov [1972] and Kempf [1972] and the "only if" by Griffiths and Harris [1980]. The two statements require quite different methods, the "if" part using enumerative geometry and the "only if" involving specialization techniques. In this chapter we'll prove a strong form of the existence statement: when $\rho(g, r, d) \geq 0$ then *every* curve of genus g admits a nondegenerate map of degree d or less to $\mathbb{P}^r$.

Before beginning with the Brill-Noether theorem we review three classic but more elementary results that provide limitations on what maps to projective space a curve can possess: theorems of Riemann-Roch, Clifford, and Castelnuovo.

Maps $\varphi : C \to \mathbb{P}^r$ (modulo the group PGL_{r+1} of automorphisms of the target $\mathbb{P}^r$) correspond one-to-one to pairs (L, V) where L is a line bundle of degree d on C and $V \subset H^0(L)$ an $(r+1)$-dimensional vector space of sections without common zeroes (base locus). If we drop the requirement that the sections of V have no common zeroes, such an object is called a *linear series of degree d and dimension r*; classically, it was referred to as a g^r_d. (Note that if a g^r_d has a nonempty base locus then we can subtract that locus and get a map $\varphi : C \to \mathbb{P}^r$, with degree smaller than d.)

17.1.1 The Riemann-Roch Theorem

As we saw in Chapter 16, the Riemann-Roch theorem for curves says that for a line bundle L on a smooth curve of genus g,

$$h^0(\mathcal{L}) = d - g + 1 + h^0(\omega_C \otimes \mathcal{L}^{-1}),$$

where ω_C denotes the sheaf of differential forms on C. We will often exploit the equivalence between the notions of line bundle and divisor on a smooth curve, and write a divisor D in place of the line bundle $\mathcal{O}(D)$. Thus we allow ourselves to rewrite the Riemann-Roch theorem as

$$h^0(D) = d - g + 1 + h^0(K - D),$$

where K denotes a canonical divisor.

For line bundles L of degree $d > 2g - 2$, the last term is zero, and so Riemann-Roch tells us the dimension precisely:

$$h^0(\mathcal{L}) = d - g + 1.$$

For line bundles of degree close to $2g-2$ it gives us approximate information: for example, if $d = 2g - 2$, Riemann-Roch says that

$$h^0(\mathcal{L}) = \begin{cases} g, & \text{if } \mathcal{L} = \omega_C; \\ g - 1, & \text{otherwise} \end{cases} \tag{17.1}$$

and if $g > 0$ and $d = 2g - 3$, it says that

$$h^0(\mathcal{L}) = \begin{cases} g - 1, & \text{if } \mathcal{L} = \omega_C(-p) \text{ for some point } p \in C; \\ g - 2, & \text{otherwise.} \end{cases} \tag{17.2}$$

It also tells us that for any line bundle,

$$d + 1 \geq h^0(\mathcal{L}) \geq d - g + 1.$$

Beyond these values the Riemann-Roch theorem gives less precise information.

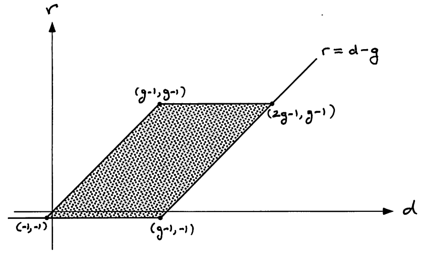

FIGURE 17.1. A point in the shaded area of this chart represents a pair (d, r) such that Riemann-Roch allows for the existence of a line bundle $\mathcal{L}$ on a curve C of genus g with $h^0(\mathcal{L}) = r + 1$.

We may summarize what the Riemann-Roch Theorem tells us about the possible existence of g^r_ds on curves of genus g in Figure 17.1, which shows the possible pairs (d, r) where $\mathcal{L}$ is a line bundle of degree d on C and $r = h^0(\mathcal{L}) - 1$.

****Silvio: We want to change this figure and the next to be similar to the third and fourth – that is, g=1000 – which will make the part under the line $y = 0$ invisible. We also want the pictures to be cumulative – or perhaps just each one appears superimposed on the one before? Let's see what looks best....****

17.1.2 Clifford's Theorem

Clifford's theorem says that if C is a curve of genus g and D is a divisor of degree $d \leq 2g$ on C then

$$h^0(D) \leq \frac{d}{2} + 1.$$

In the case when $h^0(K - D) = 0$, this inequality follows at once from the Riemann-Roch theorem, so its import is for divisors D such that $h^0(K - D) \neq 0$—these are called *special divisors.* An extension of Clifford's Theorem says that if moreover equality holds then either $\mathcal{L} = \mathcal{O}$, $\mathcal{L} = \omega_C$ or C is hyperelliptic and $\mathcal{L}$ is a multiple of the g^1_2 on C. If we exclude the cases of degree $d = 0$ and $d \geq 2g - 2$ (where after all Clifford is not telling us anything new), we may state this as

Theorem 17.2 (Clifford)**.** *If C is a curve of genus g and $\mathcal{L}$ a line bundle of degree d on C with $0 < d < 2g - 2$, then*

$$h^0(\mathcal{L}) \leq \frac{d}{2} + 1,$$

with equality holding only if C is hyperelliptic and $\mathcal{L}$ a multiple of the g^1_2.

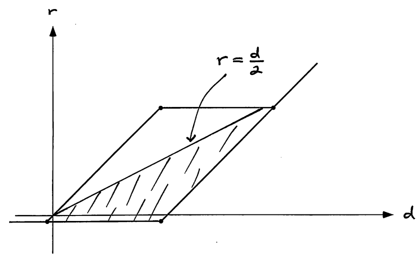

FIGURE 17.2. Values of $d = c_1(\mathcal{L})$ and $r = h^0(\mathcal{L}) - 1$ allowed by Clifford

This cuts the above chart of allowed values of d and r essentially in half, as shown in Figure 17.2.

Clifford's theorem is sharp: for every d, g and r allowed by Theorem 17.2, there exist curves of genus g and g^r_ds on them. This does not represent a satisfactory answer to our basic problem, for two reasons:

• Given our motivation for studying g^r_ds on curves—the classification of curves in projective space—you may say that our real object of interest is not g^r_ds in general but those whose associated maps give birational embeddings in $\mathbb{P}^r$. But the linear systems satisfying equality in Clifford's theorem, and, as we'll see in a moment, those that are close to this, are not birationally very ample. Thus, we may refine our original question and ask: *what birationally very ample linear series exist on curves of genus g*—in other words, for which d, g and r do there exist irreducible, nondegenerate curves of degree d and geometric genus g in $\mathbb{P}^r$?

• As we have seen, interesting linear series that achieve equality in Clifford's Theorem exist only on hyperelliptic curves, which are very special. We may ask what linear series exist on all curves of genus g?

17.1.3 Castelnuovo's Theorem

The issue of which linear series can embed a curve is partially dealt with in a classical theorem of Castelnuovo:

Theorem 17.3 (Castelnuovo). *Let $C \subset \mathbb{P}^r$ be an irreducible, nondegenerate curve of degree d and geometric genus g. Then*

$$g \leq \pi(d, r) := \binom{m}{2}(r-1) + m\epsilon$$

where $d = m(r-1) + \epsilon + 1$ and $0 \leq \epsilon \leq r-2$.

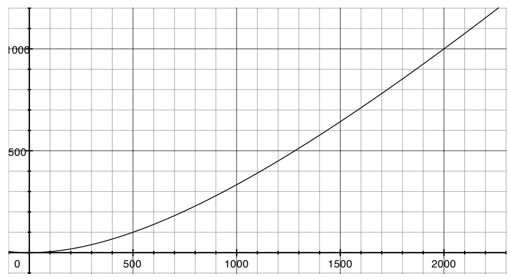

FIGURE 17.3. Values of d and r allowed by Castelnuovo's bound in case $g = 1000$

Castelnuovo showed that this bound is sharp, and using his analysis it's not hard to see that curves of all geometric genera between 0 and the bound do occur. Figure 17.3 shows the values of d and r allowed by Castelnuovo's bound in case $g = 1000$.

It should be said that Castelnuovo's theorem addresses the question of what birationally very ample linear series a curve *may* have. It does not, however, tell us what linear series are present on every curve. To put it another way, smooth projective curves of genus g are parametrized by a variety M_g, and inside that space the locus of "Castelnuovo curves"—that is, curves $C \subset \mathbb{P}^r$ of degree d with $g = \pi(d, r)$—is contained in a subvariety of high codimension. Thus, the question remains of what linear series exist on a *general* curve of genus g. This is the question addressed by the Brill-Noether theorem of which Theorem 17.1 a weak version. To state it in a strong form we will have to analyze the geometry of curves in greater depth.

17.2 Families of divisors

17.2.1 The Jacobian

A deeper study of linear series requires us to make sense of the set of linear systems as a variety. For this purpose we introduce the *Jacobian*.

One of the early motivations for studying algebraic curves came from calculus, specifically the desire to make sense of integrals of algebraic functions. In modern terms, this means integrals

$$\int_{p_0}^{p} \omega$$

where ω is a holomorphic (or more generally meromorphic) differential on a smooth, projective curve C over the complex numbers, p_0 and p are points of C and the integral is taken along a path from p_0 to p on C.

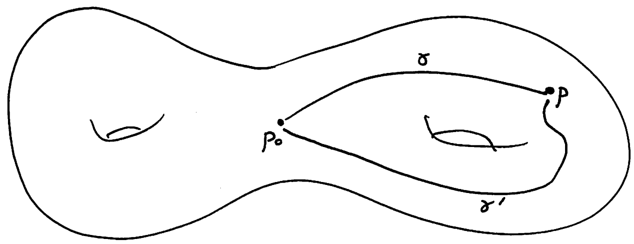

FIGURE 17.4. The integral $\int_{p_0}^{p} \omega$ may depend on the choice of path.

One problem in trying to define such integrals is that if the curve in question has positive genus then the value of the integral depends on the choice of the path. We can incorporate this by viewing the expression $\int_{p_0}^{p}$ as a linear functional on the space $H^0(\omega_C)$ of holomorphic differentials on C, defined *modulo the subgroup of those linear functions obtained by integration over closed loops on* C. Now, the integral of holomorphic (and hence closed) forms over a closed loop γ depends only on the homology class of γ; we thus have a map

$$H_1(C,\mathbb{Z}) \to H^0(\omega_C)^*$$

which embeds $H_1(C,\mathbb{Z}) \cong \mathbb{Z}^{2g}$ as a discrete lattice in $H^1(\omega_C)^* \cong \mathbb{C}^g$ (see for example Griffiths and Harris [1978], page 228). We may thus view the integral $\int_{p_0}^{p}$ as an element of the quotient

$$\int_{p_0}^{p} \in H^0(\omega_C)^*/H_1(C,\mathbb{Z})$$

We define the *Jacobian* $J(C)$ of the curve C to be this quotient. By our construction, $J(C)$ is a complex torus, and in particular a compact complex manifold; in fact, it is a projective variety over $\mathbb{C}$, and the map $p \to \int_{p_0}^{p}$ is a map of projective varieties. We will use this often; see Griffiths and Harris [1978], §2.2 and §2.3, for a treatment.

Having defined the Jacobian, we see that after choosing a base point $p_0 \in C$ integration defines a map $C \to J(C)$, and more generally maps

$$u = u_d : C_d \to J(C)$$

from the symmetric products $C_d = C^d/\mathfrak{S}_d$ of C to its Jacobian, defined by

$$D = \sum p_i \mapsto \sum \int_{p_0}^{p_i} .$$

These maps are called *Abel-Jacobi* maps.

17.2.2 Abel's theorem

Theorem 17.4. *Let $u : C_d \to J(C)$ be the Abel-Jacobi map. Divisors D, E on C of the same degree are linearly equivalent if and only if $u(D) = u(E)$.* $\square$

One implication of Abel's theorem is relatively easy to prove: if D and E are linearly equivalent, there is a pencil of divisors, parameterized by $\mathbb{P}^1$, interpolating between them. But if $f : \mathbb{P}^1 \to A$ is any map from $\mathbb{P}^1$ to a torus, the pullbacks $f^*\eta$ of holomorphic 1-forms on A vanish identically. Since these 1-forms generate the cotangent space at every point of A, it follows that the differential $df \equiv 0$ and hence that f is constant; it follows that $u(D) = u(E)$. The hard part (which was in fact proved by Clebsch) is the converse. See Griffiths and Harris [1978], page 235 for a treatment.

The import of Abel's theorem is that we may, for each d, identify the set of linear equivalence classes of divisors of degree d on C with the Jacobian $J(C)$. The identification is not canonical; it depends on the choice of a base point $p_0 \in C$. In Section 17.2.3 we will use this correspondence to show that there exists a moduli space $\mathrm{Pic}^d(C)$ for line bundles of degree d on C, isomorphic (again, non-canonically) to $J(C)$.

Abel's theorem tells us that the fiber $u^{-1}(u(D))$ of u through a point $D \in C_d$ is the complete linear system $|D| = \{E \in C_d \mid E \sim D\}$—set-theoretically, at least, a projective space. (We will see in Lemma 17.5 that it is indeed isomorphic to $\mathbb{P}^{r(D)}$.) Beyond this, the behavior of the map u depends very much on d.

To begin with, if $p_1, \dots, p_d \in C$ are general points with $d \leq g$ then the conditions of vanishing at the p_i are independent linear conditions on differential forms. Writing $D = p_1 + \cdots + p_d$ we get $h^0(\omega - D) = g - d$. From the Riemann-Roch theorem we see that $h^0(D) = 1$; that is, no other effective divisor of degree d is linearly equivalent to D. It follows from Abel's theorem that the map $u : C_d \to J(C)$ is one-to-one near D, and in particular that the image $W_d := u(C_d) \subset J(C)$ is again d-dimensional.

In particular the map $u : C_g \to J(C)$ is birational (this statement is called the *Jacobi inversion theorem*; see Exercise 17.18 for a more classical version). Further, the image of $u : C_{g-1} \to J(C)$ is a divisor in $J(C)$, called the *theta divisor*, and written Θ.

When $d \geq g$ the same argument shows that the map $u : C_d \to J(C)$ is surjective. When $g \leq d \leq 2g-2$ the dimensions of the fibers will vary, but when $d > 2g-2$, the picture becomes regular: the fibers of $u : C_d \to J(C)$ are all of dimension $d - g$. We'll see in this case that $u : C_d \to J(C)$ is a projective bundle, and in Section 17.5.2 we will identify the vector bundle

$\mathcal{E}$ on $J(C)$ such that $C_d \cong \mathbb{P}\mathcal{E}$ and the divisor Θ corresponds to the line bundle $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$.

Note that the subset of line bundles $\mathcal{L}$ of degree d such that $h^0(\mathcal{L}) \geq r+1$ is Zariski closed in $\mathrm{Pic}^d(C)$: it is the locus where the fiber dimension of u is r or greater. We will denote this locus by $W^r_d(C)$. In these terms, we can interpret the problem "describe the set of g^r_ds on C" to mean: "describe the locus $W^r_d(C)$—its dimension, irreducible components, smooth and singular loci. . . "

Here is the first step toward proving that for large d the Abel-Jacobi map is the projection from a bundle:

Lemma 17.5. *For $d > 2g-2$, the map u is a submersion; that is, the differential $du : T_D C_n \to T_{u(D)} \mathrm{Pic}^n$ is surjective everywhere. More generally, for any d, the scheme-theoretic fibers of u are smooth, and hence isomorphic to projective spaces.*

Proof. We start with the case $d > 2g-2$, and consider a point $D \in C_d$ corresponding to a reduced divisor; that is, $D = p_1 + \cdots + p_d \in C_d$ with the points p_i distinct. Recall first that we have an identification

$$T^*_{u(D)} J(C) = H^0(\omega_C)$$

of the cotangent space to the Jacobian at any point with the space of regular differentials on C. We can similarly identify the cotangent space to C_d at D: we have

$$T^*_D C_n = \oplus T^*_{p_i} C = H^0(\omega_C/\omega_C(-D)).$$

Differentiating the Abel-Jacobi map

$$D = \sum p_i \mapsto \sum \int_{p_0}^{p_i};$$

with respect to the points p_i, we see that in terms of these identifications, the differential du_D of u at a point $D \in C_d$ is given as the transpose of the evaluation map

$$\begin{aligned} H^0(\omega_C) &\to \oplus T^*_{p_i} C \\ \omega &\mapsto \big(\omega(p_1), \ldots, \omega(p_d)\big); \end{aligned}$$

in particular, the cokernel of the differential du_D of u at a point $D \in C_d$ is the annihilator of the subspace $H^0(\omega_C(-D)) \subset H^0(\omega_C)$ of differentials vanishing along D. Since we are working in the range $d \geq 2g-1$, there are no such differentials, and we are done.

To extend this argument to the case of an arbitrary divisor $D \in C_d$, we invoke the fact (see for example Arbarello et al. [1985], §IV.1) that the identification $T^*_D C_n = H^0(\omega_C/\omega_C(-D))$ extends over all of C_d, and that in these terms the differential is again the transpose of the evaluation map

$$H^0(\omega_C) \to H^0(\omega_C/\omega_C(-D));$$

thus the same logic applies.

Finally, in case $d \leq 2g-2$ Riemann-Roch tells us that the dimension of the kernel of the differential du at any point D—that is, the dimension of the cokernel of the evaluation map $H^0(\omega_C) \to H^0(\omega_C/\omega_C(-D))$—is exactly the dimension $r(D)$ of the fiber of u through D; thus the fibers of u are smooth in this case as well, even though they're not all of the same dimension. $\square$

17.2.3 Moduli spaces of divisors and line bundles

Abel's theorem gives us a bijection between points of the Jacobian $J(C)$ and the set of line bundles $\mathcal{L}$ of degree d on C, suggesting the existence of a moduli space of such bundles. This is in fact the case:

Proposition 17.6. *Let C be a smooth, projective curve of genus g and d any integer. There exists a projective scheme $\operatorname{Pic}^d(C)$ and a line bundle $\mathcal{P}$ on $\operatorname{Pic}^d(C) \times C$ such that for any scheme B and any line bundle $\mathcal{M}$ on $B \times C$ of relative degree d, there exists a unique map $\varphi : B \to \operatorname{Pic}^d(C)$ such that*

$$\mathcal{M} = (\varphi \times \mathrm{id}_C)^* \mathcal{P}.$$

In fact, as suggested by Abels Theorem, $\operatorname{Pic}^d(C)$ is isomorphic to $J(C)$, but non-canonically; for clarity, it's best to think of the $\operatorname{Pic}^d(C)$ as distinct schemes. As for the "universal line bundle" $\mathcal{P}$ on $\operatorname{Pic}^d(C) \times C$, this is called a *Poincaré bundle*, and we'll see how to construct it in Section 17.4.1.

There is a similar characterization of the symmetric product C_d of C as the Hilbert scheme of subschemes of degree d on C:

Proposition 17.7. *There exists a divisor $\mathcal{D} \subset C_d \times C$ such that for any scheme B and any effective divisor $\Delta \subset B \times C$ of relative degree d, there exists a unique map $\varphi : B \to C_d$ such that*

$$\Delta = (\varphi \times \mathrm{id}_C)^{-1} \mathcal{D}.$$

In other words, *C_d is the Hilbert scheme of subschemes of degree d in C*. In this case, it's easy to see what the "universal divisor" is: it's just the reduced divisor

$$\mathcal{D} = \{(D, p) : p \in D\},$$

which we'll encounter repeatedly in what follows.

For proofs of Propositions 17.6 and 17.7, see for example Arbarello et al. [1985], §IV.2.

A few words are in order about the construction of $\mathrm{Pic}^d(C)$ in the algebraic setting. When the ground field is $\mathbb{C}$, the construction above of the Jacobian works, and this is what was done classically. But in the 20-th century, this became increasingly unsatisfactory: when the ground field K is, for example, a number field, the construction produces a variety over $\mathbb{C}$ but not over K; and when K has positive characteristic it doesn't work at all. One of the challenges of algebraic geometers in the first half of the 20-th century was thus to give an algebraic construction of $\mathrm{Pic}^d(C)$.

This was first done by André Weil. He observed that, via the birational map $u : C_g \to J(C)$, an open subset of $J(C)$ was isomorphic to an open subset of C_g. Composing u with translations, we see that $J(C)$ may be covered by such open sets, and these can be glued together to construct $J(C)$. (Indeed, it was the desire to carry out this construction that led Weil to the definition of an abstract variety.) In even greater generality, Grothendieck applied his theory of *étale equivalence relations* to construct $\mathrm{Pic}^d(C)$ as the quotient of C_d by linear equivalence for large d; see Milne [2008] for a description.

17.3 The Brill-Noether theorem

As we indicated, the Brill-Noether theorem answers the question, "For which values of d, g and r is it the case that every curve of genus g has a g^r_d?" We gave an answer in the initial section of this chapter; here is a more refined version:

Theorem 17.8. *If C is a general curve of genus g, the dimension of $W^r_d(C)$ is*

$$\rho(g, r, d) = g - (r+1)(g - d + r).$$

In particular, every curve has a g^r_d if and only if $\rho \geq 0$.

We may parse the formula in the Brill-Noether theorem as follows: g is the dimension of the Jacobian of C, which may be thought of as the space $\mathrm{Pic}^d(C)$ of all line bundles of degree d on C. Furthermore, if a line bundle $\mathcal{L} \in \mathrm{Pic}^d(C)$ has $h^0(\mathcal{L}) = r+1$, so that $\mathcal{L} \in W^r_d(C) \setminus W^{r+1}_d(C)$, the Riemann-Roch theorem asserts that $h^1(\mathcal{L}) = g - d + r$. Thus the Brill-Noether Theorem asserts that, if C is a general curve of genus g then the codimension of $W^r_d \subset \mathrm{Pic}^d(C)$ is $h^0(\mathcal{L})h^1(\mathcal{L})$. Indeed, as we shall see, W^r_d can be thought of as the rank k locus of a map between vector bundles of ranks $k + h^0(\mathcal{L})$ and $k + h^1(\mathcal{L})$ (for any large k!), so this is the "expected" codimension in the sense of Chapter **??**.

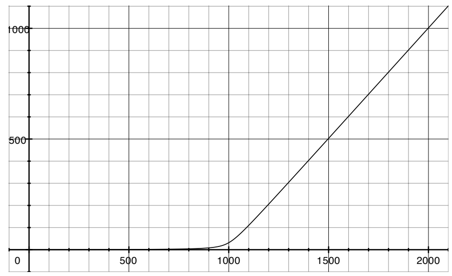

FIGURE 17.5. Values of d and r for linear series on a general curve in case $g = 1000$ (allowed values correspond to points below the curve).

In general, the formula of Theorem 17.8 is far more restrictive than Castenouvo's bound. For example, Figure 17.5 shows the values of d and r allowed in case $g = 1000$; we see that only a tiny fraction of the complete linear series allowed by Castelnuovo actually occur on a general curve.

In the final section of this chapter we will give an enumerative proof of the existence half of Theorem 17.8, namely:

Theorem 17.9. *If $\rho = g-(r+1)(g-d+r) \geq 0$, then every smooth curve C of genus g has*

$$\dim W^r_d(C) \geq \rho.$$

In particular, there exist linear systems on C of degree d and dimension r.

The heart of our proof of Theorem 17.9 is an enumerative formula for the class of $W^r_d(C)$, given in the stronger Theorem 17.17. When $g = (r+1)(g-d+r)$—that is, $\rho = 0$—this formula becomes a number:

Corollary 17.10. *If C is a smooth curve of genus $g = (r+1)(g-d+r)$ and if W^r_d is finite then C possesses*

$$g! \prod_{i=1}^{r} \frac{i!}{(g-d+r+i)!}$$

linear series of degree d and dimension r, counted with multiplicity. When $r = 1$ and $g = 2(g-d+1) = 2k$ this number is the Catalan number

$$\frac{1}{k+1}\binom{2k}{k}.$$

Proof. Replace the factor θ^g by $g!$ in the formula of Theorem 17.17. □

For example, in the first case that is not answered by the Riemann-Roch theorem, we can ask if a general curve of genus 4 is expressible as a 3-sheeted cover of $\mathbb{P}^1$, and if so in how many ways; this is the content of

Keynote Question (b). Corollary 17.10 gives an answer: it says that C will admit two such maps. Indeed, we can see this directly: the canonical model of a non-hyperelliptic curve of genus 4 is the complete intersection of a quadric and a cubic surface in $\mathbb{P}^3$, and if the quadric is smooth its two rulings will each cut out a g^1_3 on C. (Note that we see in this example a case where multiplicities may arise: if the quadric surface in question is a cone, the curve C will possess only one g^1_3, counted with multiplicity 2.)

Similarly, Corollary 17.10 says that a general curve C of genus 6 will possess 5 g^1_4s, and dually 5 g^2_6s. Again, we can see these linear series explicitly: by the extension below, any one of the g^2_6s will give a birational embedding of C as a plane sextic $C_0 \subset \mathbb{P}^2$ with four nodes as singularities. The five g^1_4s will then be the pencils cut out on C by the pencil of lines through each node, and the pencil cut by conics passing through all four. (See Exercises 17.19 and g14s on sextic for a proof that these are all the g^1_4s on C.)

In general, however, we are not able to identify the g^r_ds on a curve C whose existence is asserted by Theorem 17.8 and Corollary 17.10. For example, in the next case—g^1_5s on a general curve of genus 8—Corollary 17.10 asserts that on a general such C there are 14. But according to Theorem 17.8, the simplest plane model of C is a plane octic with 13 nodes, and there is no obvious way to identify the 14 g^1_5s on such a curve; likewise, the simplest embedding of C in $\mathbb{P}^3$ is as a curve of degree 9, and again the g^1_5s are not visible in this model. (See Exercise 17.21 for more on this.)

There are various extensions of this theorem for general curves C of genus g: Fulton and Lazarsfeld show that if $\rho > 0$ then $W^r_d(C)$ is irreducible; Gieseker proves that the singular locus of $W^r_d(C)$ is exactly $W^{r+1}_d(C)$, and hence (given Exercise 17.23) $W^r_d(C)$ is reduced. In particular, we see that for a general curve there are in fact no multiplicities in the formula in Corollary 17.10.

It is also shown in Griffiths and Harris [1980] that a general linear series on a general curve is as well-behaved as possible. The following Proposition answers Keynote Questions a, c, e.

Proposition 17.11. *Let C be a general curve of genus g.*

- $r = 1$*: The map $\varphi : C \to \mathbb{P}^1$ given by a general g^1_d is simply branched;*
- $r = 2$*: The map $\varphi : C \to \mathbb{P}^2$ given by a general g^2_d is birational onto a plane curve $C_0 \subset \mathbb{P}^2$ having only nodes as singularities; and*
- $r \geq 3$*: For $r \geq 3$ the map $\varphi : C \to \mathbb{P}^r$ given by a general g^r_d is an embedding.*

Thus, for example, a general curve of genus g is birational to a plane curve of degree $\lceil \frac{2g+6}{3} \rceil$, and is embeddable in projective space as a curve of degree $\lceil \frac{3g+12}{4} \rceil$.

17.3.1 How to guess the Brill-Noether theorem and prove existence

Here is one way to describe the locus $W^r_d(C)$. Fix m distinct points

$$p_1, \dots, p_m \in C.$$

For any line bundle $\mathcal{L}$ on C there is an exact sequence

$$0 \to \mathcal{L} \to \mathcal{L}(p_1 + \cdots + p_m) \xrightarrow{b_{\mathcal{L}}} \oplus_{i=1}^m \mathcal{L}(p_1 + \cdots + p_m)_{p_i} \cong \oplus_{i=1}^m \mathcal{O}_{p_i} \to 0,$$

and taking cohomology we see that

$$H^0(\mathcal{L}) = \operatorname{Ker}\big(\mathcal{L}(p_1 + \cdots + p_m) \xrightarrow{b_{\mathcal{L}*}} \oplus_{i=1}^m \mathbb{C}\big).$$

If the degree of $\mathcal{L}$ is d and the number m is large—say $m > -\deg \mathcal{L} + 2g - 2$—then the Riemann-Roch formula tells us that $h^0(\mathcal{L}(p_1 + \cdots + p_m)) = m + d - g + 1$, independently of $\mathcal{L}$. Thus as $\mathcal{L}$ varies over the set $\operatorname{Pic}^d(C)$ of line bundles of degree d, the locus $W^r_d(C) \subset \operatorname{Pic}^d(C)$ is the locus where the $m \times (m + d - g + 1)$ matrix $b_{\mathcal{L}*}$ has rank at most $(m + d - g + 1) - (r + 1)$. The expected codimension of the locus where an $s \times t$ matrix with $s \leq t$ has rank u, in the sense of Chapter **??**, is $(s - u)(t - u)$. Thus the "expected" codimension of W^r_d in $\operatorname{Pic}^d(C)$ is $(r + 1)(m - (m + d - g + 1 - r - 1)) = (r + 1)(g - d + r)$, exactly the codimension predicted for a general curve by the Brill-Noether theorem.

As we'll see, the maps $b_{\mathcal{L}*}$ vary algebraically with $\mathcal{L} \in \operatorname{Pic}^d(C)$. It follows that *if* $W^r_d(C)$ *is nonempty* for a given curve C then its dimension is at least $\rho(g, r, d)$, and given the existence of one curve C_0 for which W^r_d is really nonempty and of dimension $\rho(g, r, d)$ it would follow that this is true for an open set of curves in any family containing C_0. Brill and Noether must have known many cases where these conditions were all satisfied; but they lacked the tools to give a proof of the theorem.

In Section 17.4.2 we will identify the map $b_{\mathcal{L}}$ as the fiber of a map of vector bundles $b : \mathcal{F} \to \mathcal{G}$ over $\operatorname{Pic}^d(C)$, which is isomorphic to the Jacobian $J(C)$ of C (this implies that the map $b_{\mathcal{L}*}$ varies algebraically with $\mathcal{L}$). As remarked above, this implies that $W^r_d(C)$ has dimension at least $\rho(g, r, d)$ provided that it is non-empty.

To prove that $W^r_d(C)$ is nonempty when $\rho(g, r, d) \geq 0$ we will compute the Chern classes of the vector bundles $\mathcal{F}, \mathcal{G}$. Porteous' formula allows us to compute the class $\alpha \in H^{2(r+1)(g-d+r)}(J(C))$ that the locus W^r_d *would* have *if* it had dimension $\rho(g, r, d)$. We will show that this class is nonzero

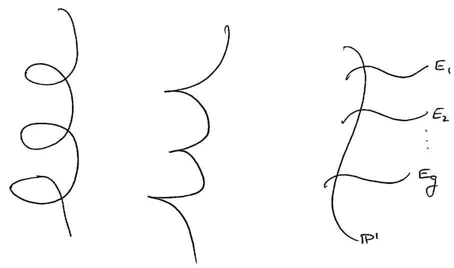

FIGURE 17.6. Three singular curves used in specialization arguments for the noneistence half of Brill-Noether

when $0 \leq (r+1)(g-d+r) \leq g$, and this suffices to prove the desired existence.

17.3.2 How the other half is proven

The proof of the other half of the Brill-Noether theorem—the statement that for a general curve C, the dimension $\dim W^r_d(C) \leq \rho(g,r,d)$, and in particular that C possesses no g^r_ds when $\rho < 0$—requires very different ideas. One could prove it by exhibiting, for each g, r and d, a smooth curve C of genus g with $\dim W^r_d(C) = \rho(g,r,d)$ (or with $W^r_d(C) = \varnothing$ if $\rho < 0$); but no one has ever succeeded in doing this explicitly in higher genera. The known proofs fall into two families:

Degeneration to singular curves. One approach to this problem is to consider a one-parameter family of curves $\{C_t\}$ specializing from a smooth curve to a singular one C_0. What needs to be done in this setting is first of all to describe the limit, as $t \to 0$, of a g^r_d on C_t, and then to prove that such limits don't exist on C_0 when $\rho < 0$. This was done in the original proof, with C_0 a general g-nodal curve (that is, $\mathbb{P}^1$ with g pairs of general points identified): the possible limits of a g^r_d on C_t were identified in Kleiman [1976], and the proof that no such limit existed when $\rho < 0$ was given in Griffiths and Harris [1980]. Another proof (Eisenbud and Harris [1983a]) used a g-cuspidal curve as C_0, and in Eisenbud and Harris [1983b] the role of C_0 was played by a curve consisting of a copy of $\mathbb{P}^1$ with g elliptic tails attached. Much more recently, a proof was given using the methods of tropical geometry in Cools et al. [2012].

Curves on a very general K3 surface. A completely different proof was given by Lazarsfeld [1986], who showed that a smooth curve C embeddable in a very general K3 surface—specifically, one whose Picard group was generated by the class of the curve C—necessarily satisfied the statement of the Brill-Noether theorem. The theorem was thus proved by specializing to a smooth curve, rather than a singular one (though the smooth curve in question could still not be explicitly given, inasmuch as we have no way to produce explicit $K3$ surfaces with Picard number 1 and arbitrary degree).

17.4 W_d^r as a degeneracy locus

In the remainder of this chapter we will deal with a fixed curve C. To simplify notation we will write J for the Jacobian $J(C)$ and Pic^d for the Picard variety $\mathrm{Pic}^d(C)$ parametrizing line bundles of degree d on C.

In this section we will explain how to construct the family of all line bundles of given degree, and how to put the maps $b_{\mathcal{L}}$ of Section 17.3.1 together into a map of bundles. To do this, we first need to construct the *Poincaré bundle*, a fundamental object in the theory.

17.4.1 The universal line bundle

Choose a base point $p_0 \in C$. The Poincaré bundle is a line bundle on the product $\mathrm{Pic}^d \times C$ whose restriction to the fiber $\{\mathcal{L}\} \times C$ over $\mathcal{L} \in \mathrm{Pic}^d$ is isomorphic to L and whose restriction to the cross-section $\mathrm{Pic}^d \times \{p_0\}$ is trivial.

Without the normalizing condition of triviality on $\mathrm{Pic}^d \times \{p_0\}$ the bundle $\mathcal{P}$ would not be determined uniquely: we could tensor with the pullback of any line bundle on $\mathrm{Pic}^d(C)$ and get another. But with the normalizing condition, Corollary 6.7(b) shows that the Poincaré bundle is unique—if it exists.

We will construct the Poincaré bundle as the direct image of a line bundle $\mathcal{M}$ on $C_d \times C$ under the map

$$\eta = u \times Id : C_d \times C \to \mathrm{Pic}^d \times C.$$

To describe $\mathcal{M}$, let $\mathcal{D}_0$ be the divisor $\mathcal{D}_0 = X_{p_0} \times C \subset C_d \times C$, and let $\mathcal{D} \subset C_d \times C$ be the *universal divisor* of degree d as in Proposition 17.7: that is,

$$\mathcal{D} = \{(D, p) : D - p \geq 0\}.$$

Thus the restriction of $\mathcal{D}$ to a fiber $\{D\} \times C$ of the projection to C_d is the divisor D, and the restriction of $\mathcal{D}$ to the fiber $C_d \times \{p\}$ of the projection to C is the divisor $X_p = C_{d-1} + p \subset C_d$. Finally, define

$$\mathcal{M} = \mathcal{O}_{C_d \times C}(\mathcal{D} - \mathcal{D}_0).$$

and set $\mathcal{P} = \eta_* \mathcal{M}$.

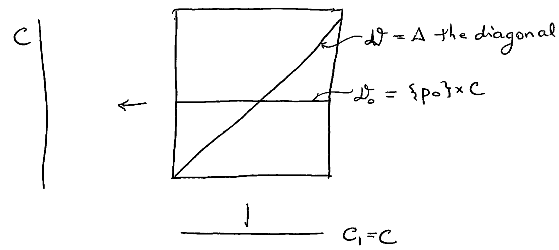

FIGURE 17.7. The divisors $\mathcal{D}$ and $\mathcal{D}_0$ in case $d=1$. ****in the text it's $C_d \times C$, so it should be $C_1 \times \{p_0\}$ in the figure.****

Proposition 17.12. *$\mathcal{P} = \eta_*(\mathcal{M})$ is a Poincaré bundle on* $\operatorname{Pic}^d \times C$*; that is,*

$$\mathcal{P}|_{\{\mathcal{L}\}\times C} \cong \mathcal{L}$$

for any point $\mathcal{L} \in \operatorname{Pic}^d$, and

$$\mathcal{P}_{\operatorname{Pic}^d \times \{p_0\}} \cong \mathcal{O}_{\operatorname{Pic}^d}.$$

Proof. Since the restriction of $\mathcal{M}$ to any fiber $\mathbb{P}^r$ of η is trivial (both divisors $\mathcal{D}$ and $\mathcal{D}_0$ intersect $\mathbb{P}^r$ in a hyperplane), the theorem on cohomology and base change, Theorem 6.14, shows that the direct image $\eta_*\mathcal{M}$ is a line bundle and the formation of this direct image commutes with base change.

The proof that $\mathcal{P}_{\operatorname{Pic}^d \times \{p_0\}} \cong \mathcal{O}_{\operatorname{Pic}^d}$ is immediate: if we restrict to the preimage

$$\eta^{-1}(\operatorname{Pic}^d \times \{p_0\})$$

the divisors $\mathcal{D}$ and $\mathcal{D}_0$ agree, so that $\mathcal{M}|_{\eta^{-1}(\operatorname{Pic}^d \times \{p_0\})}$ is trivial and so is its direct image.

To prove that $\mathcal{P}|_{\{\mathcal{L}\}\times C} \cong \mathcal{L}$ we use the theorem on cohomology and base change. It implies that the formation of the direct image $\eta_*(\mathcal{M})$ commutes with base change, so we can first restrict to the preimage

$$|\mathcal{L}| \times C = \eta^{-1}(\{\mathcal{L}\} \times C)$$

where $|\mathcal{L}| \cong \mathbb{P}^r \subset C_d$ is the linear system of effective divisors D on C with $\mathcal{O}_C(D) \cong \mathcal{L}$. The restriction of η to $|\mathcal{L}| \times C$ is projection on the second factor.

As observed, the line bundle $\mathcal{M} = \mathcal{O}_{C_d \times C}(\mathcal{D} - \mathcal{D}_0)$ is trivial on each fiber of $\eta : |\mathcal{L}| \times C \to C$, so that the restriction $\mathcal{M}|_{|\mathcal{L}|\times C}$ must be a pullback of some line bundle on C; to prove that $\eta_*(\mathcal{M})|_{\{\mathcal{L}\}\times C} \cong \mathcal{L}$ amounts to showing that this line bundle is $\mathcal{L}$. In other words, we have to show that

$$\mathcal{M}|_{|\mathcal{L}|\times C} = \eta^*\mathcal{L};$$

and to do this it is enough to choose any divisor $D \in |\mathcal{L}|$ and verify that

$$\mathcal{M}|_{\{D\}\times C} \cong \mathcal{L}.$$

But this is immediate, if we just choose D not containing the point p_0: by definition, the divisor $\mathcal{D}_0 = X_{p_0} \times C \subset C_d \times C$ is disjoint from $\{D\} \times C$, while the divisor $\mathcal{D}$ intersects $\{D\} \times C$ in the divisor D. $\square$

17.4.2 The evaluation map

Fix a reduced divisor $D = p_1 + \cdots + p_m$ of degree $m \geq 2g - 1 - d$ on C and set $n = m + d$. Choosing D gives us an identification of Pic^d with Pic^n. We will describe the locus $W^r_d + D \subset \mathrm{Pic}^n$ as the degeneracy locus of an evaluation map.

On the product $\mathrm{Pic}^n \times C$, consider the evaluation map

$$\mathcal{P} \to \mathcal{P}|_\Gamma$$

where

$$\Gamma = \mathrm{Pic}^n \times D = \bigcup_{i=1}^{m} (\mathrm{Pic}^n \times \{p_i\})$$

is the union of the horizontal sections of $\mathrm{Pic}^n \times C$ over Pic^n corresponding to the points p_i. Taking the direct image of this map under the projection $\pi : \mathrm{Pic}^n \times C \to \mathrm{Pic}^n$, we have a map of vector bundles

$$\rho : \mathcal{E} := \pi_*(\mathcal{P}) \to \mathcal{F} = \bigoplus_{i=1}^{m} \mathcal{L}_i,$$

where $\mathcal{L}_i$ is the restriction of $\mathcal{P}$ to the cross-section $\mathrm{Pic}^n \times \{p_i\}$. For each point $\mathcal{L} \in \mathrm{Pic}^n$, this is the map

$$\mathcal{E}_\mathcal{L} = H^0(\mathcal{L}) \to \bigoplus \mathcal{L}_{p_i}$$

obtained by evaluation sections of $\mathcal{L}$ at the points p_i. In particular, the kernel of this map is the vector space $H^0(\mathcal{L}(-D)) \subset H^0(\mathcal{L})$ of sections vanishing along D. We have now proven that *the locus $W^r_d + D \subset \mathrm{Pic}^n$ is the locus where the map ρ has rank $n - g - r$ or less.*

It remains to show that this locus is nonempty. To do so we will compute the Chern classes of the bundles $\mathcal{E}$ and $\mathcal{F}$ and apply Porteous' Theorem. Before we do so, however, we must develop some basic information about the cohomology ring of the Jacobian, where these Chern classes live, and we must also identify the bundle $\mathcal{E}$ in a more useful way.

17.5 Natural classes in the cohomology ring of the Jacobian

Why are we working with the cohomology ring of the Jacobian rather than with the Chow ring? It turns out that Jacobians, simple as their cohomology is, are among the spaces whose Chow groups (beyond codimension 1) seem to be essentially unknowable. To give one example of this, it's not hard to see that a cycle $Z \subset J$ will not in general be rationally equivalent to its translate $Z_a = Z + a$ by an element $a \in J$: for example, the group law on J gives a map from the space of effective 0-cycles on J to J, and since any map from $\mathbb{P}^1$ to J is constant, rationally equivalent 0-cycles must "add up" to the same sum in J. But in fact this merely scratches the surface of our ignorance: for any two points $a, b \in J$ and any cycle Z on J, we can ask if the sum

$$Z - Z_a - Z_b + Z_{a+b}$$

is rationally equivalent to 0. It is true for cycles in codimension 1, but beyond this the answer is unknown, even for 0-cycles. ****check this**** And our ignorance of when linear combinations of translates of cycles are rationally equivalent to zero is very much an issue here: most of the cycles whose classes we might hope to determine are in fact only defined after a choice of base point—in effect, only up to translation.

Analogues of all the basic properties of Chow rings—pullback maps in general, pushforward (or *Gysin*) maps when the spaces involved are oriented manifolds, and the push-pull formula—hold for Cohomology rings. The basic formula (Theorem 11.9) for projective bundles also holds in this context; see for example Bott and Tu [1982].

The Jacobian J is the quotient of the contractible space $H^0(\omega_C)^* \cong \mathbb{C}^g$ by the subgroup $H_1(C, \mathbb{Z}) \cong \mathbb{Z}^{2g}$, so there is a natural identification

$$H_1(J, \mathbb{Z}) = H_1(C, \mathbb{Z}).$$

The first cohomology $H^1(J, \mathbb{Z})$ is similarly identified with $H^1(C, \mathbb{Z})$. These identifications are induced by the Abel-Jacobi map $u : C \to J$ because the integral takes a closed path $\gamma : [0, 1] \to \mathbb{C}$ to the path $\tilde{\gamma} : [0, 1] \to H^0(\omega_C)^*$ defined by

$$\tilde{\gamma}(t) = \int_{\gamma([0,t])} \in H^0(\omega_C)^*,$$

which joins the origin to the lattice point corresponding to the homology class of γ.

****revised to here at 5:20pm on March 22****

We choose a basis $\alpha_1, \dots \alpha_g, \beta_1, \dots, \beta_g$ for $H^1(C, \mathbb{Z})$, normalized so that the cup product is skew-diagonal: that is,

$$(\alpha_i \cup \beta_j)[C] = \begin{cases} 1, & \text{if } i = j; \text{ and} \\ 0, & \text{otherwise;} \end{cases}$$

and

$$\alpha_i \cup \alpha_j = \beta_i \cup \beta_j = 0 \quad \text{for all } i, j.$$

By abuse of notation, we will also use the symbols α_i and β_i to denote the corresponding cohomology classes in $H^1(J, \mathbb{Z})$.

Topologically, $J \cong (S^1)^{2g}$ is a product of S^1s, so by the Künneth formula we have

$$H^*(J, \mathbb{Z}) = \bigwedge^* H^1(J, \mathbb{Z}) = \bigwedge^* H^1(C, \mathbb{Z}),$$

with cup product on the left corresponding to wedge product on the right. For a multiindex $I = (i_1, \dots, i_k)$ with $i_1, \dots i_k \in \{1, \dots, g\}$, we will write α_I and β_I for the classes

$$\alpha_I = \alpha_{i_1} \cup \cdots \cup \alpha_{i_k} \quad \text{and} \quad \beta_I = \beta_{i_1} \cup \cdots \cup \beta_{i_k} \in H^k(J, \mathbb{Z}).$$

The classes

$$\{\alpha_I \cup \beta_J : I, J \subset \{1, \dots, g\}\}$$

form a basis for $H^*(J, \mathbb{Z})$. The cup product in complementary dimension is given, for $I, J, K, L \subset \{1, \dots, g\}$, by

$$((\alpha_I \cup \beta_J) \cup (\alpha_K \cup \beta_L))[J] = \begin{cases} \pm 1, & \text{if } K = I^\circ \text{ and } L = J^\circ; \text{ and} \\ 0 & \text{otherwise,} \end{cases} \tag{17.3}$$

where I° denotes the complement of I. To determine the signs, we can pull back to the direct product of d copies of C, and use formula 17.9 below to prove that

$$(\alpha_1 \cup \beta_1 \cup \alpha_2 \cup \beta_2 \cup \cdots \cup \alpha_g \cup b_g)[J] = +1;$$

the signs of the other products above are determined by skew-symmetry.

Of special interest are the classes $\eta_i \in H^2(J, \mathbb{Z})$, defined as

$$\eta_i = \alpha_i \cup \beta_i \quad \text{for } i = 1, \dots, g.$$

For a multiindex $I = (i_1, \dots, i_k)$ with $i_1, \dots i_k \in \{1, \dots, g\}$, we will write η_I for the class

$$\eta_I = \eta_{i_1} \cup \cdots \cup \eta_{i_k} \in H^{2k}(J, \mathbb{Z}).$$

Rearranging the terms we see that $\eta_I = (-1)^{\binom{d}{2}} \alpha_I \cup \beta_I$. In particular, $\eta_{(1,\dots,g)} = 1 \in H^{2g}(J) = \mathbb{Z}$.

The cup product in complementary dimension is easy to compute: for $I, J \subset \{1, \dots, g\}$ we have:

$$(\eta_I \cup \eta_J)[J] = \begin{cases} 1, & \text{if } J = I^\circ\text{; and} \\ 0 & \text{otherwise.} \end{cases} \tag{17.4}$$

17.5.1 Poincaré's formula

The objects of primary interest to us are the classes of the subvarieties $W_d \subset \mathrm{Pic}^d$ parametrizing divisor effective classes of degree d, that is, the images of the maps $u : C_d \to J \cong \mathrm{Pic}^d$. Like most of the objects in our treatment, the map ud depends on the choice of base point $p_0 \in C$, so the subvarieties $W_d \subset J$ are really only defined up to translation. However, their classes in $H^*(J)$ are well-defined. Here is the basic result:

Proposition 17.13 (Poincaré's formula)**.**

$$[W_d] = \sum_{\substack{I \subset \{1,\dots,g\} \\ |I| = g-d}} \eta_I \ \in \ H^{2g-2d}(J, \mathbb{Z}).$$

In particular, the divisor $W_{g-1} \subset J$ has class

$$[W_{g-1}] = [\Theta] = \eta_1 + \cdots + \eta_g.$$

This class occurs often, and we will denote it by θ. With this notation we can restate the Proposition as

$$[W_d] = \frac{\theta^{g-d}}{(g-d)!}$$

The formula $[W_d] = [W_{g-1}]^{g-d}/(g-d)!$ makes sense in the ring of cycles on J mod numerical equivalence—we don't need to introduce the topological cohomology of J to state it—and indeed it was proven for curves and their Jacobians over arbitrary fields in Kleiman and Laksov [1974]. We do not know if there is an analogous formula in a finer cycle theory such as the group of cycles modulo algebraic equivalence.

Proof of Proposition 17.13. By Poincaré duality, it suffices to take the product of both sides of the formula with an arbitrary element $\alpha_I \cup \beta_J$ with $|I| = |J| = d$, evaluate on the fundamental class of J and show that they are the same. In view of the formulas 17.3 and 17.4 above, in case $I \neq J$ we have $\alpha_I \cup \beta_J \cup \eta_K = 0$ for any η_K of complementary degree, while if $I = J$ we have $(\alpha_I \cup \beta_I \cup \eta_K)[J] = (-1)^{\binom{d}{2}}$ if $K = I^\circ$ and 0 otherwise. Thus to establish Proposition 17.13 we need to prove that

$$(u^*(\alpha_I \cup \beta_J))[C_d] = \begin{cases} (-1)^{\binom{d}{2}}, & \text{if } I = J\text{; and} \\ 0 & \text{otherwise} \end{cases} \tag{17.5}$$

To evaluate the expression on the left, it's useful to pull back from the symmetric product C_d of the curve to C^d, the ordinary d-fold product. Let $\pi : C^d \to C_d$ be the quotient map, and let $\nu = u \circ \pi : C^d \to J$ be the composition. Since π is a $d!$-fold cover

$$\pi_*[C^d] = d! \cdot [C_d],$$

so (17.5) is equivalent to

$$(\nu^*(\alpha_I \cup \beta_J))[C^d] = \begin{cases} (-1)^{\binom{d}{2}} d!, & \text{if } I = J; \text{ and} \\ 0 & \text{otherwise;} \end{cases} \tag{17.6}$$

that is,

$$(\nu^*(\alpha_I \cup \beta_J))[C^d] = 0 \quad \text{if } I \neq J; \tag{17.7}$$

and

$$(\nu^* \eta_I)[C^d] = d! \quad \text{for all } I. \tag{17.8}$$

Let $\rho_k : C^d \to C$ be projection on the k^{th} factor, and set $\alpha_i^k = \rho_k^* \alpha_i$ and $\beta_i^k = \rho_k^* \beta_i$. By the Künneth formula, $H^1(C^d) = \oplus_k \rho_k^*(H^1(C))$. Writing $\iota_k : C \to C^d$ for the inclusion sending C to

$$\{p_0\} \times \cdots \times \{p_0\} \times C \times \{p_0\} \times \cdots \times \{p_0\}$$

with C in the k-th position, we see that $\rho^k \iota^k : C \to C$ is the identity, while $\rho^j \iota^k$ is the constant map when $j \neq k$. It follows that if $\gamma \in H^1(C^d)$, then $\gamma = \sum_k \rho^{k*}(\iota^{k*}(\gamma))$. Applying this to $\nu^* \alpha_i$ and $\nu^* \beta_i$ we see that

$$\nu^* \alpha_i = \alpha_i^1 + \cdots + \alpha_i^d \quad \text{and} \quad \nu^* \beta_i = \beta_i^1 + \cdots + \beta_i^d. \tag{17.9}$$

By symmetry we may assume that $I = \{1, \ldots, d\}$. Applying formula 17.9 to

$$\nu^* \eta_I = \nu^* \alpha_1 \cup \nu^* \beta_1 \cup \cdots \cup \nu^* \alpha_d \cup \nu^* \beta_d$$

we get the sum of all products of the form

$$\alpha_1^{j_1} \cup \beta_1^{k_1} \cup \cdots \cup \alpha_d^{j_d} \cup \beta_d^{k_d}.$$

This product is zero unless each $j_i = k_i$ and the set $\{j_1 \ldots, j_d\}$ is equal to $\{1, \ldots, d\}$. For these $d!$ terms,

$$\alpha_1^{j_1} \cup \beta_1^{j_1} \cup \cdots \cup \alpha_d^{j_d} \cup \beta_d^{j_d} = 1$$

because $\alpha_i^{j_i} \cup \beta_i^{j_i}$ is the pullback of the class of a point under ρ_{j_i}. $\square$

17.5.2 Symmetric products as projective space bundles

Recall that we need to compute the Chern classes of $\mathcal{E} = \pi_*\mathcal{P}$, where $\mathcal{P}$ is the Poincaré bundle on $\operatorname{Pic}^n \times C$ for some $n > 2g-2$ and $\pi : \operatorname{Pic}^n \times C \to \operatorname{Pic}^n$ is the projection. To do this we will identify $\mathcal{E}$ in another way:

Theorem 17.14. *With notation as above, the map $u : C_n \to \operatorname{Pic}^n$ is isomorphic to the projective bundle $\mathbb{P}\mathcal{E} \to \operatorname{Pic}^n$, via an isomorphism sending $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ to $\mathcal{O}_{C_n}(X_{p_0})$.*

Proof. It should not be surprising that $C_n \cong \mathbb{P}\mathcal{E} = \mathbb{P}(\pi_*\mathcal{P})$; after all, the fiber of the bundle $\pi_*\mathcal{P}$ at a point $\mathcal{L} \in \operatorname{Pic}^n$ is naturally identified with the vector space $H^0(\mathcal{L})$ of sections of the line bundle $\mathcal{L}$, so that the fiber of its projectivization may be identified in turn with the fiber $|\mathcal{L}| = u^{-1}\{\mathcal{L}\}$ of of the map u. Of course this fiber-by-fiber argument does not constitute a proof, but it suggests how we might go about giving one.

Recall first that by Lemma 17.5, the scheme-theoretic fibers of the map $u : C_n \to \operatorname{Pic}^n$ are projective spaces $\mathbb{P}^{n-g}$ (in other words, $u : C_n \to \operatorname{Pic}^n$ is a projective bundle in the analytic topology). By Proposition 11.3, the fact that we have a divisor $X_{p_0} \subset C_n$ whose restriction to each fiber is a hyperplane then says that $C_n \to \operatorname{Pic}^n$ is in fact a projective bundle in the Zariski topology; that is, the projectivization of some vector bundle $\mathcal{G}$ on Pic^n. Moreover, by Proposition 11.3 there is a unique such bundle $\mathcal{G}$ such that in terms of this identification $\mathcal{O}_{\mathbb{P}\mathcal{G}}(1) = \mathcal{O}_{C_n}(X_{p_0})$; and this bundle $\mathcal{G}$ may be realized as the dual of the direct image $u_*\mathcal{O}_{C_n}(X_{p_0})$.

Our goal is thus to show that the direct image $\mathcal{E} = \pi_*\mathcal{P}$ is isomorphic to $u_*\mathcal{O}_{C_n}(X_{p_0})$. The key is to consider the direct image $\pi_*\mathcal{P}$ as the direct image of the line bundle $\mathcal{M} = \mathcal{O}_{C_d \times C}(\mathcal{D} - \mathcal{D}_0)$ in two ways: by definition $\pi_*\mathcal{P} = \pi_*(\eta_*\mathcal{M})$, and since $\pi \circ \eta = u \circ \pi_1$ that means we can also write it as $u_*(\pi_{1*}\mathcal{M})$. The essential step is the following Lemma:

Lemma 17.15. *With notation as above,*

$$\pi_{1*}\mathcal{M} \cong u^*(\pi_*\mathcal{P}).$$

Proof. It will be helpful to have a diagram of the relevant spaces and maps:

$$\begin{tikzcd} & C_n \times C \arrow[dl, "\pi_1"'] \arrow[dr, "\eta"] & \\ C_n \arrow[dr, "u"'] & & \operatorname{Pic}^n \times C \arrow[dl, "\pi"] \\ & \operatorname{Pic}^n & \end{tikzcd}$$

Now, we have a natural evaluation map

$$u^*u_*(\pi_{1*}\mathcal{M}) \to \pi_{1*}\mathcal{M};$$

since $u_*\pi_{1*}\mathcal{M} = \pi_*\mathcal{P}$, this gives a map

$$u^*(\pi_*\mathcal{P}) \to \pi_{1*}\mathcal{M}.$$

We claim this map is an isomorphism. This follows by looking at the map on fibers at a point $D \in C_n$. Since the sheaves involved have no higher cohomology of the fibers of the morphisms, the theorem on cohomology and base change allows us to identify the fibers of $u^*(\pi_*\mathcal{P})$ and $\pi_{1*}\mathcal{M}$ at D with the spaces

$$u^*(\pi_*\mathcal{P})_D = H^0(\mathcal{M}|_{|D|\times C}) \quad \text{and} \quad (\pi_{1*}\mathcal{M})_D = H^0(\mathcal{M}|_{\{D\}\times C}),$$

and in these terms the induced map $u^*(\pi_*\mathcal{P})_D \to (\pi_{1*}\mathcal{M})_D$ is the restriction map $H^0(\mathcal{M}|_{|D|\times C}) \to H^0(\mathcal{M}|_{\{D\}\times C})$. But as we've seen, the bundle $\mathcal{M}|_{|D|\times C}$ on $|D|\times C$ is the pullback of a bundle on C, and so the restriction map is an isomorphism on global sections. $\square$

Given the identification of Lemma 17.15, the proof of Proposition 17.14 is straightforward. Since

$$\mathcal{M} = \mathcal{O}_{C_d\times C}(\mathcal{D} - \mathcal{D}_0) = \mathcal{O}_{C_d\times C}(\mathcal{D}) \otimes \pi_1^*\mathcal{O}_{C_d}(X_{p_0})$$

we have

$$\pi_{1*}\mathcal{M} = \pi_{1*}\mathcal{O}_{C_d\times C}(\mathcal{D}) \otimes \mathcal{O}_{C_d}(-X_{p_0}).$$

We have an inclusion $\mathcal{O}_{C_d\times C} \hookrightarrow \mathcal{O}_{C_d\times C}(\mathcal{D})$ coming from the effective divisor $\mathcal{D}$; taking the direct image gives an inclusion

$$\mathcal{O}_{C_d} \hookrightarrow \pi_{1*}\mathcal{O}_{C_d\times C}(\mathcal{D})$$

and hence an inclusion

$$\mathcal{O}_{C_d}(-X_{p_0}) \hookrightarrow \pi_{1*}\mathcal{M}$$

and a surjection

$$(\pi_{1*}\mathcal{M})^* \to \mathcal{O}_{C_d}(X_{p_0})$$

inducing an isomorphism of direct images. $\square$

17.5.3 Chern classes from the symmetric product

We can now calculate the Chern class of $\mathcal{E}$:

Theorem 17.16. *For $n \geq 2g-1$, the pushforward $\mathcal{E}$ of the Poincaré bundle $\mathcal{P}$ from $\operatorname{Pic}^n \times C$ to Pic^n has Chern class $c(\mathcal{E}) = e^{-\theta}$; that is, $c_i(\mathcal{E}) = (-1)^i\theta^i/i!$ for each i.*

Proof. Computing the Chern class is equivalent to computing the the Segre class $s(\mathcal{E}) = \sum s_i(\mathcal{E})$ since $c(\mathcal{E}) = 1/s(\mathcal{E})$ by Proposition 12.3. Recall that

$$s_k(\mathcal{E}) = u_*(\zeta^{k+n-g}),$$

where $\zeta = [X_p] \in H^2(C_n)$.

Since we're working in $H^*(C_n)$ rather than $A(C_n)$, the class ζ is also the class of the divisor $X_q = C_{n-1} + q \subset C_d$ for any point $q \in C$. To represent the class ζ^{k+n-g} we can just choose distinct points $p_1, \dots, p_{k+n-g} \in C$ and consider the intersection

$$\begin{aligned} \bigcap_i X_{p_i} &= \{D \in C_n : D - p_i \geq 0 \ \forall i\} \\ &= C_{g-k} + E \ \subset \ C_d \end{aligned}$$

where $E = p_1 + \dots + p_{k+n-g}$. This intersection is generically transverse—it is visibly transverse at a point $E + D'$ where D' consists of $g - k$ distinct points distinct from $p_1, \dots, p_{k+n-g}$, and no component of the intersection is contained in the complement of this locus—and so we have

$$\zeta^{k+n-g} = \big[C_{g-k} + E\big] \ \in \ H^{2k+2n-2g}(C_d).$$

We have

$$\begin{aligned} s_k(\mathcal{E}) &= u_*\big[C_{g-k} + E\big] \\ &= \big[W_{g-k}\big] \ \in \ H^{2k}(J). \end{aligned}$$

Applying Poincaré's formula, this yields

$$s_k(\mathcal{E}) = \frac{\theta^k}{k!}.$$

We can express this compactly as

$$s(\mathcal{E}) = e^\theta,$$

from which the Theorem follows. □

Theorem 17.16 allows us to give a description of the cohomology ring of C_d; though we won't use it in what follows, we state it here. By the analog of Theorem 11.9 for topological cohomology, we have for $n \geq 2g - 1$

$$H^*(C_n) = H^*(J)[\zeta]/\big(\zeta^{n-g+1} - \theta\zeta^{n-g} + \frac{\theta^2}{2}\zeta^{n-g-1} - \cdots\big).$$

Finally, we remark that there is an alternative way to derive the Chern classes of $\mathcal{E}$: given that $\mathcal{E} = \eta_*\mathcal{M}$, we can apply Grothendieck-Riemann-Roch to the morphism η to arrive at Theorem 17.16. This approach involves a larger initial investment—we have to have more knowledge of products in $H^*(C_d)$ than we currently do—but is also much more broadly applicable. This approach is carried out in Arbarello et al. [1985], Chapter 8, where many other applications are given.

17.5.4 The class of W_d^r

We can now use Porteous' formula to calculate the class of W_d^r as a degeneracy locus of the map $\mathcal{E} \to \mathcal{F}$ of vector bundles on Pic^n obtained by pushing forward the evaluation map $\mathcal{P} \to \oplus \mathcal{L}_i$. Here is the result:

Theorem 17.17. *If C is a smooth curve of genus g such that $W_d^r(C)$ has dimension $\rho(g, r, d) = g - (r+1)(g-d+r)$, then the class of $W_d^r(C)$ in the cohomology ring of the Jacobian $J(C)$ is*

$$[W_d^r] = \prod_{i=1}^{r} \frac{i!}{(g-d+r+i)!} \theta^{(r+1)(g-d+r)}$$

If $\rho \geq 0$ then this class is nonzero and thus W_d^r is nonempty and of dimension $\geq \rho(g, r, d)$.

The last statement is the content of Theorem 17.9.

Proof. The necessary ingredients are the Chern class of $\mathcal{E}$, computed in the previous section, and the Chern class of $\mathcal{F}$. The line bundles $\mathcal{L}_i$ can all be continuously deformed to the bundle $\mathcal{P}_{p_0} = \mathcal{P}|_{\mathrm{Pic}^n \times \{p_0\}}$, which is trivial by our normalization of $\mathcal{P}$. The Chern classes $c_1(\mathcal{L}_i) \in H^2(\mathrm{Pic}^n)$ are thus all 0, so that

$$c(\mathcal{F}) = 1 \in H^*(\mathrm{Pic}^n).$$

We have

$$\frac{c(\mathcal{F})}{c(\mathcal{E})} = \frac{1}{e^{-\theta}} = e^{\theta}$$

and so Porteous' formula tells us that if W_d^r has pure dimension $\rho = g - (r+1)(g-d+r)$, then its class is the determinant

$$\begin{aligned}[W_d^r] &= \begin{vmatrix} \frac{\theta^{g-d+r}}{(g-d+r)!} & \frac{\theta^{g-d+r+1}}{(g-d+r+1)!} & \cdots & \frac{\theta^{g-d+2r}}{(g-d+2r)!} \\ \frac{\theta^{g-d+r-1}}{(g-d+r-1)!} & \frac{\theta^{g-d+r}}{(g-d+r)!} & \cdots & \frac{\theta^{g-d+2r-1}}{(g-d+2r-1)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\theta^{g-d}}{(g-d)!} & \frac{\theta^{g-d+1}}{(g-d+1)!} & \cdots & \frac{\theta^{g-d+r}}{(g-d+r)!} \end{vmatrix} \\ &= D_{a,r} \cdot \theta^{(r+1)(g-d+r)} \end{aligned}$$

where $D_{a,r}$ is the $(r+1) \times (r+1)$ determinant

$$D_{a,r} = \begin{vmatrix} \frac{1}{a!} & \frac{1}{(a+1)!} & \cdots & \frac{1}{(a+r)!} \\ \frac{1}{(a-1)!} & \frac{1}{a!} & \cdots & \frac{1}{(a+r-1)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(a-r)!} & \frac{1}{(a-r+1)!} & \cdots & \frac{1}{a!} \end{vmatrix}$$

It remains to evaluate $D_{a,r}$. To do this, we clear denominators by multiplying the first column by $a!$, the second column by $(a+1)!$, and so on; we arrive at the expression

$$D_{a,r} = \prod_{i=0}^{r} \frac{1}{(a+i)!} \cdot M$$

where M is the determinant

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ a & a+1 & \dots & a+r \\ a(a-1) & (a+1)a & \dots & (a+r)(a+r-1) \\ \vdots & \vdots & \ddots & \vdots \\ a\cdots(a-r+1) & (a+1)\cdots(a-r+2) & \dots & (a+r)\cdots(a+1) \end{vmatrix}.$$

Since the columns of M all consist of the same sequence of monic polynomials, applied to the argumets $a, \dots, a+r$, the determinant is equivalent to the van der Monde determinant, and thus has value

$$\prod_{0 \le i < j \le r} (j-i) = \prod_{i=0}^{r} i!.$$

Thus

$$D_{a,r} = \prod_{i=1}^{r} \frac{i!}{(a+i)!}.$$

It follows that if the dimension of $W_d^r(C)$ is ρ, then it has the class given in the theorem. In particular, if it were empty, then it would have this class, which is nonzero, a contradiction. Thus it must be nonempty. Since it is defined as a degeneracy locus, it must have dimension at least the "expected dimension" locally at each of its points. This completes the proof of both Theorems 17.17 and 17.9. □

17.6 Exercises

Exercise 17.18. Use the statements of Section 17.2.2 to prove the original form of Jacobi inversion: that, given two g-tuples of points $p_1, \dots, p_g$ and $q_1, \dots, q_g \in C$ on a smooth curve C of genus g, there exists a g-tuple of points $r_1, \dots, r_g \in C$, whose coordinates are rational functions of the coordinates of the p_i and q_i, such that

$$\sum \int_{p_0}^{p_i} + \sum \int_{p_0}^{q_i} = \sum \int_{p_0}^{r_i}$$

Exercise 17.19. Let $C_0 \subset \mathbb{P}^2$ be a plane sextic with four nodes as singularities, whose normalization is a general curve of genus 6. Show that no three of the nodes are collinear. (Hint: use a dimension count.)

Exercise 17.20. Let $C_0 \subset \mathbb{P}^2$ be a plane sextic with four nodes as singularities, whose normalization is a general curve of genus 6. We have seen that there are five g^1_4s on C: the pencils cut out on C by the pencil of lines through each node, and the pencil cut by conics passing through all four. Show that there are no others.

Exercise 17.21. Let C be a curve of genus 8, embedded in $\mathbb{P}^3$ by one of the g^3_8s on C. Show that if C is general then the image curve $C_0 \subset \mathbb{P}^3$ does not lie on a cubic surface. In case it does, can you locate the 14 g^1_5s on C?

Exercise 17.22. Let C be a general curve of genus 9. How many plane octic curves $C_0 \subset \mathbb{P}^2$ are birational to C?

Exercise 17.23. Show that $W^r_d(C) \setminus W^{r+1}_d(C)$ is dense in $W^r_d(C)$. (Hint: for any point $\mathcal{L} \in W^{r+1}_d(C)$, consider the line bundle $\mathcal{L}(p-q)$ for general points $p, q \in C$.)

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