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Eigenvalues and Eigenvectors in von Neumann and Related Growth Models: An Overview and Some Remarks

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Abstract

We take into consideration various relationships existing between eigenvalues and eigenvectors of suitable matrices or matrix pairs and the equilibrium solutions of the classical von Neumann growth model and of other related economic models.

Keywords: eigenvalues, eigenvectors, von Neumann growth model, Leontief-von Neumann models.

1. Introduction

The von Neumann economic growth model (von Neumann (1945-46)) is perhaps one of the most investigated models in economic growth theory and in mathematical economics in general. Indeed, this model was (together with the models presented by Sraffa (1960)) one of the first attempts to introduce in multi-sectoral production schemes, the possibility of joint production. Moreover, contrary to the models of Sraffa with joint production, the von Neumann model is described by inequalities, which permits considerations of optimality and efficiency of the production processes.

The aim of the present paper is to focalize some links existing between the solutions of the classical von Neumann model, together with some related models, and the eigenvalues and eigenvectors of (real) square matrices or of (non necessarily square) matrix pencils.

The paper is organized as follows.

Section 2 recalls the basic results concerning the classical von Neumann model.

Section 3 is concerned with solution properties, in terms of eigenvalues and eigenvectors, of a Leontief-von Neumann model.

Section 4 recall the main properties of the Leontief-von Neumann model, proposed by Morishima in the study of his "turnpike theorem".

Section 5 is concerned with various "regular" von Neumann models, in the sense of J. Łos (1971).

Section 6 contains some conclusive remarks.

Throughout the paper the notation $x \geqq y$ (x and y being two vectors of $\mathbb{R}^n$) means $x_i \geqq y_i$, $i = 1, ..., n$; $x \geq y$ means $x \geqq y$, but $x \neq y$; $x > y$ means $x_i > y_i$, $i = 1, ..., n$. If $y = [0]$ (zero vector), vector x is said to be, respectively, *nonnegative, semipositive, positive.* The notations $x \leqq y$, $x \leq y$, $x < y$ are evident. The same conventions are used to compare two matrices of the same order, say (m, n). We denote by $[0]$ the zero matrix, so that the notations $A \geqq [0]$, $A \geq [0]$, $A > [0]$ mean that A is, respectively, a *nonnegative matrix,* a *semipositive matrix,* a *positive matrix.* If A is a matrix of m rows and n columns, A_i, $i = 1, ..., m$, denotes its i-th row, whereas A^j, $j = 1, ..., n$, denotes its j-the column.

2. Basic Results on the Classical von Neumann Growth Model

The literature on the classical von Neumann growth model is abundant. We quote only Bruckmann and Weber (1971), Gale (1960), Howe (1960), Karlin (1959), J. and M. Łos (1974), Morgenstern and Thompson (1976), Morishima (1964), Murata (1977), Nikaido (1968, 1970), Takayama (1985), Woods (1978). However, we point out that many mathematical treatments of the von Neumann model are incomplete and unsatisfactory (see, e. g., Giorgi and Meriggi (1987, 1988)). Some other treatments are correct but quite long and complicate. We follow the description of the model and the conventions adopted by Kemeny, Morgenstern and Thompson (1956). We consider a finite set of m processes that produces a finite set of n different goods. Each process operates at an *intensity level* $x^\top = [x_1, ..., x_m]$, whereas $p^\top = [p_1, ..., p_n]$ is a *price vector.* The model is characterized by a pair of two nonnegative matrices (A, B), both of order (m, n) : the rows of

A represent the various activities, the columns of A describe the inputs; the rows of B represent the various activities and the columns of B describe the outputs. In other words, a_{ij} is the quantity of good j technologically required per unit of process i. The output coefficient b_{ij} simply represents the quantity of good j produced per unit of process i.

The basic assumptions on A and B are

$$A \geqq [0]\,, \qquad B \geqq [0]\,. \tag{1}$$

Then, following Kemeny, Morgenstern and Thompson, we precise better inequalitis (1), in the sense that we impose the following conditions:

$$A_i \geq [0]\,,\ \forall i = 1, ..., m; \tag{2}$$

$$B^j \geq [0]\,,\ \forall j = 1, ..., n. \tag{3}$$

Intuitively, (3) means that every good can be produced by some process, and (2) that every process uses some inputs.

We note that (2) is equivalent to

$$x \geq [0] \Longrightarrow x^{\top}A \geq [0] \tag{2'}$$

and that (3) is equivalent to

$$p \geq [0] \Longrightarrow Bp \geq [0]\,. \tag{3'}$$

The *expansion rate* or *growth rate* is denoted by α and the *interest rate* is denoted by β. Several authors call α the *expansion factor* and β the *interest factor*; then, in this case, $(\alpha - 1)$ is the expansion rate and $(\beta - 1)$ is the interest rate.

Definition 1. A quadruplet (x, p, α, β), $x \geq [0]\,,\ p \geq [0]\,,\ \alpha \geqq 0, \beta \geqq 0$ is an *equilibrium solution* for the von Neumann technology (A, B) if it satisfies the following system

$$x^{\top}B \geqq \alpha x^{\top}A \tag{4}$$

$$(x^{\top}B - \alpha x^{\top}A)p = 0 \tag{5}$$

$$Bp \leqq \beta Ap \tag{6}$$

$$x^{\top}(Bp - \beta Ap) = 0 \tag{7}$$

$$x^{\top}Bp > 0. \tag{8}$$

The following result is quite immediate.

Lemma 1. If $(\bar{x}, \bar{p}, \bar{\alpha}, \bar{\beta})$ is an equilibrium solution, then

$$\bar{\alpha} = \bar{\beta} = \bar{x}^{\top}B\bar{p}/\bar{x}^{\top}A\bar{p} > 0.$$

Proof. From (8) we get $\bar{x}^{\top}B\bar{p} > 0$ and from (5) and (7) we get $\bar{x}^{\top}B\bar{p} = \bar{\alpha}x^{\top}A\bar{p} = \bar{\beta}x^{\top}A\bar{p} > 0$. Therefore $x^{\top}A\bar{p} > 0$ and $\bar{\alpha} = \bar{\beta} = \bar{x}^{\top}B\bar{p}/x^{\top}A\bar{p} > 0$. □

We can therefore put $\bar{\alpha} = \bar{\beta} = \bar{\lambda}$. Then, (4) becomes

$$\bar{x}^{\top}B \geqq \bar{\lambda}\bar{x}^{\top}A$$

and from (6) we get

$$\bar{x}^{\top}B\bar{p} \leqq \bar{\lambda}\bar{x}^{\top}A\bar{p}.$$

From these two last relations we easily obtain the "complementarity conditions" (5) and (7). Therefore we can rewrite Definition 1 as follows.

Definition 2. A triplet (x, p, λ), with $x \geq [0]\,,\ p \geq [0]\,,\ \lambda > 0$, is an *equilibrium solution* for the von Neumann technology (A, B) if it satisfies the following system

$$x^{\top}B \geqq \lambda x^{\top}A \tag{9}$$

$$Bp \leqq \lambda Ap \tag{10}$$

$$x^{\top}Bp > 0. \tag{11}$$

The number λ is called by Łos (1971) *equilibrium level* and by Kemeny, Morgenstern and Thompson (1956) *allowable level.* The basic results on the von Neumann growth model described by (9)-(10)-(11) are contained in the following theorems.

Theorem 1. Let A, B, x and p be defined as above, and suppose that assumptions (2) and (3) are satisfied. Then:

(i) Relations (9)-(10)-(11) admit a solution where $x \geq [0]$, $p \geq [0]$, $\lambda > 0$ ("equilibrium solution").

(ii) There exists a number $\lambda_{\max} > 0$ solution of the problem ("technological expansion problem")

$$\max \lambda, \text{ subject to } x^\top (B - \lambda A) \geqq [0]\,,\ x \geq [0]\,.$$

(iii) There exists a number $\lambda_{\min} > 0$ solution of the problem ("economic expansion problem")

$$\min \lambda, \text{ subject to } (B - \lambda A)p \leqq [0]\,,\ p \geq [0]\,.$$

(iv) $\lambda_{\max} \geqq \lambda_{\min} > 0$.

(v) It is possible to find $x \geq [0]$ and $p \geq [0]$ such that the triplets $(x, p, \lambda_{\max})$ and $(x, p, \lambda_{\min})$ are equilibreium solutions, i. e. satisfy relations (9)-(10)-(11).

(vi) The set of equilibrium capital stock vectors x is a convex set; the set of equilibrium price vectors p is a convex set.

Curiously, the above results are scattered in several papers and books (see the works quoted at the beginning of the present section). There is not, as far as we are aware, a complete and self-contained proof of Theorem 1, which, however, contains classical results. For other questions concerning the classical von Neumann model, see Giorgi and Meriggi (1987, 1988).

Usually, in the current literature, the number $\lambda_{\max}$ is denoted by α^* and is called *maximum growth rate,* the number $\lambda_{\min}$ is denoted by β^* and is called *minimum interest rate.* In order to state the conditions which assure that $\alpha^* = \beta^*$ we need the following notions.

Definition 3. Given a vector $x \geqq [0]$, $x \in \mathbb{R}^n$, we call the *support of* x the set of indices corresponding to the nonzero components of x; formally:

Definition 4. The pair (A, B) which characterizes the von Neumann model is *technologically irreducible* (or *technologically indecomposable*) if for each semipositive vector $x \in \mathbb{R}^m$ such that $\mathsf{supp}(x^\top A) \subset \mathsf{supp}(x^\top B)$ we have $\mathsf{supp}(x^\top A) = \{1, ..., n\}$, i. e. $x^\top A > [0]$. Otherwise the model is *technologically reducible* (or *technologically decomposable*).

The above notion is essentially due to Gale (1960) who, however, introduced the notion of reducibility (for a von Neumann pair (A, B)) in the following equivalent way: the von Neumann pair (A, B) is technologically reducible if there exist two permutation matrices P and Q such that matrices PAQ and PBQ are decomposed in the following form

$$PAQ = \begin{bmatrix} A_{11} & [0] \\ A_{21} & A_{22} \end{bmatrix}, \quad PBQ = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where each column of B_{11} has at least one positive element. If such matrices P and Q do not exist, the model is technologically irreducible. Note that if $A > [0]$, obviously the pair (A, B) is technologically irreducible. Note, moreover, that if (A, B) is technologically irreducible, then it satisfies also assumption (2), because if there is a vector $y \geq [0]$ such that $y^\top A = [0]$, we would have $\mathsf{supp}(y^\top A) = \varnothing \subset \mathsf{supp}(y^\top B)$ without having $y^\top A > [0]$, contradicting the assumption that the model is technologically irreducible. If $m = n$ and $B = I$, i. e. the joint production is excluded, the pair (A, I) is technologically irreducible (respectively technologically reducible) whenever A is irreducible (respectively reducible) in the usual sense of the theory of matrices (see, e. g., Debreu and Herstein (1953), Gantmacher (1959) and Section 3 of the present paper).

Definition 5. The pair (A, B) is said to be *economically irreducible* (or *economically indecomposable*) if $(B^\top, A^\top)$ is technologically irreducible. Otherwise (A, B) is *economically reducible* (or *economically decomposable*).

The previous definition is due to Robinson (1973) and obviously is a dual property with respect to Definition 4. We can also say that (A, B) is economically reducible when there exist two permutation matrices P and Q such that PAQ and PBQ have the following form:

$$PAQ = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad PBQ = \begin{bmatrix} B_{11} & B_{12} \\ [0] & B_{22} \end{bmatrix},$$

where each row of A_{11} has at least one positive element. We can also say that (A, B) is economically irreducible if for all vectors $p \geq [0]$ such that $\mathsf{supp}(Bp) \subset \mathsf{supp}(Ap)$ we have $Bp > [0]$, i. e. $\mathsf{supp}(Bp) = \{1, 2, ..., m\}$. Note that if

$B > [0]$, then (A, B) is economically irreducible. Moreover, any irreducible pair (A, B) satisfies also assumption (3), because if there exists a vector $p \geq [0]$ such that $BP = [0]$ we would have $\mathsf{supp}(Bp) = \varnothing \subset \mathsf{supp}(Ap)$ without $Bp > [0]$ and this contradicts the assumption of economic irreducibility. Note that the square simple production model (A, I) is economically reducible and the square model (I, B) is economically reducible (respectively economically irreducible) whenever B is reducible (respectively irreducible) in the usual sense of the theory of matrices. Robinson (1973) has observed that technological and economic irreducibility are independent properties.

Theorem 2. Let (A, B) satisfy assumptions (2) and (3).

1. There are at most $\min(m, n)$ equilibrium levels (or allowable levels) λ, satisfying relations (9)-(10)-(11), where (m, n) is the dimension of A and B.
2. There is only one equilibrium level $\lambda = \lambda_{\min} = \lambda_{\max} = \beta^* = \alpha^*$, satisfying relations (9)-(10)-(11), if any one of the following conditions holds:

a) It results $A + B > [0]$.

b) The pair (A, B) is technologically irreducible.

c) The pair (A, B) is economically irreducible.

Also for what regards Theorem 2, as far as we know, there is not a complete and self-contained proof. The result sub 1) has been proved by Kemeny, Morgenstern and Thompson (1956); see also Murata (1977). Condition a) is the famous condition due to von Neumann; it has been criticized as an economically unrealistic assumption, as it implies that every good is involved either as input or as output. Moreover, in his original paper (translated into English in von Neumann (1945-46)), von Neumann did not assume (2) nor (3), so condition a) can give rise to absurd cases, such as $A > [0]$ and $B = [0]$ or $A = [0]$ and $B > [0]$. Moreover, the absence of (2) and (3) makes the von Neumann's proofs not always fully correct. See also Giorgi and Meriggi (1987, 1988)). For the proof of b) and c) see Gale (1960) and Robinson (1973). We note that a more general condition assuring $\alpha^* = \beta^*$ is: for all $x \geq [0]$ such that $\mathsf{supp}(x^\top A) \subset \mathsf{supp}(x^\top B)$ we have $x^\top Bp > [0]$ or equivalently $x^\top Ap > [0]$. Jaksch (1977) proposed a condition on the pair (A, B), both necessary and sufficient to have $\alpha^* = \beta^*$. It must be noted, however, that the condition of Jaksch is not, based, as Gale's and Robinson's conditions, on a qualitative structure of (A, B), but on "quantitative properties" of particular sub-matrices of A and B.

Theorems 1 and 2 assure the existence of a positive equilibrium level λ satisfying (9)-(10)-(11), and in particular $\lambda = \alpha^*$and $\lambda = \beta^*$ are two equilibrium levels; the same theorems, however, do not assure that it holds $\lambda > 1$. In this case β^* is usually called *minimum interest factor* and α^* is called *maximum growth factor.*

Definition 6. The pair (A, B) (or the von Neumann model described by A and B) is called *productive* if there exists a vector $x \geq [0]$ such that $x^\top(B - A) > [0]$.

From a strictly mathematical point of view we can say that (A, B) is productive if and only if $(B - A)^\top$ belongs to the $\mathcal{S}$-class, in the terminology of Fiedler and Pták (1966); see also Giorgi and Zuccotti (2014). We recall that a matrix A belongs to the $\mathcal{S}$-class if there exists a vector $x \geq [0]$ for which $Ax > [0]$. It is easy to see that a matrix A is in $\mathcal{S}$ is and only if there exists a vector $x > [0]$ such that $Ax > [0]$.

If A is square, a sufficient condition that $A \in \mathcal{S}$ is that A is a $\mathcal{P}$-matrix (Fiedler and Pták (1966)), i. e. A has all its principal minors positive. Equivalently, it can be proved that $A \in \mathcal{P}$ if and only if for every vector $x \neq [0]$ there exists an index i such that $x_i(Ax)_i > 0$. This last equivalence was discovered also by Gale and Nikaido (1965). Another sufficient condition for a square matrix A to be in $\mathcal{S}$ is that A is an *$\mathcal{N}$-matrix of the first category* (Nikaido (1968)). Following Inada (1971), a square matrix A is termed $\mathcal{N}$-matrix if all its principal minors are negative and it is said to be an $\mathcal{N}$-matrix of the first category if, moreover, A has at least one positive element. Giorgi and Meriggi (1987, 1988) proved the following result.

Theorem 3. Let (A, B) satisfy assumptions (2) and (3) and let one of the conditions a). b), c) of Theorem 2 be satisfied. Then $\beta^* = \alpha^* > 1$ if and only if (A, B) is productive.

For other considerations on mathematical properties of the classical von Neumann model, see Giorgi and Meriggi (1987, 1988).

3. A Leontief-von Neumann Model

Gale (1960) and Łos (1971) have considered a von Neumann-type model where the number of goods coincides with the number of production processes, i. e. there is no joint production and therefore $m = n$ and $B = I$. Here we shall complete the treatment of the said authors of this model, model we may call a *Leontief-von Neumann model.* According to Definition 2 a triplet (x, p, λ), $x \in \mathbb{R}^n, x \geq [0]$, $p \in \mathbb{R}^n, p \geq [0]$, $\lambda \in \mathbb{R}, \lambda > 0$, is an *equilibrium solution* of the Leontief-von Neumann model here considered, if the following relations hold true.

$$x^\top \geqq \lambda x^\top A \tag{12}$$

$$p \leqq \lambda A p \tag{13}$$

$$x^\top p > 0. \tag{14}$$

We continue to assume $A \geqq [0]$ and that A satisfies (2); obviously $B = I$ satisfies (3). There exist strict relationships between the equilibrium solutions of this model and eigenvalues and eigenvectors of the semipositive matrix A. It is well-known (see, e. g., Gantmacher (1959), Debreu and Herstein (1953)), that , given a semipositive square matrix A, there exists a real nonnegative maximum eigenvalue, i. e. the *dominant or Frobenius eigenvalue*, denoted $\lambda^*(A)$, such that $\lambda^*(A) \geqq \mid \lambda \mid$, λ being any other root of the characteristic equation of A. The Frobenius eigenvalue is associated to a right-hand eigenvector p which is semipositive; the same statement holds for the left-hand eigenvector $x^\top$.

We recall that a square matrix A, of order n, is said to be *decomposable* or *reducible* (in the usual sense of Linear Algebra) if, after suitable permutations of its rows and of its *corresponding* columns, can be put in the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{15}$$

where A_{11} and A_{22} are square and at least one of matrices A_{12}, A_{21} is the zero matrix $[0]$. The sub-matrices A_{11} and A_{22} in (15) may be themselves reducible matrices, so that in this case we obtain a so-called "block-diagonal form" or also a more general decomposed form, due to Gantmacher (1959) and called *Gantmacher normal form.* Conversely, a square matrix A is called an *indecomposable matrix or irreducible matrix*, if it is not possible, by interchanging its rows and the corresponding columns, to reduce it to the form (15) with the specified properties on its sub-matrices.

It is well-known that, if $A \geq [0]$ is indecomposable, the problems

$$\begin{cases} x^\top A = \lambda x^\top \\ x > [0] \end{cases} \tag{16}$$

$$\begin{cases} Ap = \lambda p \\ p > [0] \end{cases} \tag{17}$$

have a unique solution if and only if $\lambda = \lambda^*(A)$. Moreover, it can be shown (see Gantmacher (1959), vol. II) that if $A \geq [0]$ has a dominant eigenvalue which is a simple root of its characteristic equation and problems (16) and (17) have a solution with $\lambda = \lambda^*(A)$, then A is indecomposable. However, when $A \geqq [0]$ is decomposable the above results in general do not hold. It is possible, in this case, to obtain only "partial results". For example, if we suppose that $A \geqq [0]$ has been reduced to the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ [0] & A_{22} \end{bmatrix},$$

with A_{11} and A_{22} square irreducible matrices and every column of A_{12} a semipositive vector, then if the Frobenius eigenvalue of A_{11} is greater than the Frobenius eigenvalue of A_{22}, the left-hand Frobenius eigenvector of A (associated to $\lambda^*(A_{11})$) has all positive components. The right-hand Frobenius eigenvector will be only semipositive: more precisely, its first k components will be positive, k being the order of matrix A_{11}, and the other $(n-k)$ components will be zero. By way of symmetry, if the Frobenius eigenvalue of A_{22} is larger than the Frobenius eigenvalue of A_{11}, then the associated right-hand Frobenius eigenvector of A has all positive components. On the other hand, the associated left-hand Frobenius eigenvector will be semipositive, in such a way that its last $(n-k)$ components will be positive and its first k components will be zero.

If $A \geqq [0]$ is decomposable it may be possible to obtain other semipositive left-hand eigenvectors or semipositive right-hand eigenvectors, besides the eigenvectors associated to the Frobenius eigenvalue $\lambda^*(A)$. Let $A \geq [0]$ be square of order n; let us denote by

$$S^+(A) = \{\lambda \in \mathbb{R} : \lambda > 0 \text{ is an eigenvalue of } A \text{ to which it is possible to associate a semipositive eigenvector}\}$$

the so-called *semipositive spectrum* of A.

In the above definition we do not specify if the semipositive associated eigenvector is a left-hand eigenvector or a right-hand eigenvector. This will appear from the context of the applications of this definition.

Theorem 4. Let the pair (A, I) describe a Leontief-von Neumann model. Then:

(i) If (x, p, λ) is an equilibrium solution of the model (A, I), then $(1/\lambda)$ is an eigenvalue of A to which it is associated a left-hand eigenvector $\bar{x}^\top \geq [0]$. The triplet $(\bar{x}, p, \lambda)$ is an equilibrium solution of the model (A, I).

(ii) If $\bar{x}^\top$ is a semipositive left-hand eigenvector of A, associated to the eigenvalue $(1/\lambda) > 0$, then there exists an equilibrium triplet (x, p, λ), with $x = \bar{x}$.

(iii) The scalar $\lambda > 0$ is a Leontief-von Neumann equilibrium level if and only if $\lambda^{-1} \in S^+(A)$.

In order to prove Theorem 4 we need a previous result, proved by Łos (1971) in a general framework of topological spaces and given also by Gale (1972), without proof. This result, which is a theorem of the alternative, is an easy consequence of the well-known Farkas-Minkowski lemma (see Giorgi and Meriggi (1987)). For the reader's convenience we give a direct proof.

Lemma 2. Let $\lambda > 0$ and $\bar{x} \geq [0]$ such that $\bar{x}^\top(B - \lambda A) \geqq [0]$, where B and A are two $(m.n)$ matrices. A necessary and sufficient condition for the existence of a vector $\bar{p} \geq [0]$ such that

$$(B - \lambda A)\bar{p} \leqq [0] \tag{18}$$

$$\bar{x}^\top B\bar{p} > 0 \tag{19}$$

(i. e. such that the triplet $(\bar{x}, \bar{p}, \lambda)$ is an equilibrium solution of the von Neumann model) is that the inequalities

$$\bar{x}^\top B \leqq \bar{x}^\top(B - \lambda A),\ \bar{x} \geqq [0] \tag{20}$$

admit no solution.

Proof. Let us suppose that (20) has a solution $x \geqq [0]$ and that there exists a vector $\bar{p} \geq [0]$ ($\bar{p} \in \mathbb{R}^n$) which satisfies (18) and (19). We obtain at once a contradiction, as from (19) and (20) it results $0 < \bar{x}^\top B\bar{p} \leqq \bar{x}^\top(B - \lambda A)\bar{p}$, whereas from (18) we get $\bar{x}^\top(B - \lambda A)\bar{p} \leqq 0$. Now we prove that if (20) has no solution, then there exists $\bar{p} \geq [0]$ ($\bar{p} \in \mathbb{R}^n$) which satisfies (18) and (19). Let us consider the polyhedral cone

$$P = \left\{ y \in \mathbb{R}^n : y = x^\top(B - \lambda A),\ x \geqq [0] \right\}.$$

The set $P - \mathbb{R}^n_+$, being the algebraic sum of two polyhedral cones, is itself a polyhedral cone. It is easy to see that inequality (20) has no solution if and only if $\bar{x}^\top B \notin P - \mathbb{R}^n_+$. Vector $\bar{x}^\top B$ can be separated from the polyhedral cone $P - \mathbb{R}^n_+$. Hence there exists $\bar{p} \in \mathbb{R}^n$ such that $\bar{x}^\top B\bar{p} > 0$ and $y^\top\bar{p} \leqq 0$ for any $y \in P - \mathbb{R}^n_+$. From this, being $P \subset P - \mathbb{R}^n_+$ and $-\mathbb{R}^n_+ \subset P - \mathbb{R}^n_+$ it results

$$\bar{x}(B - \lambda A)\bar{p} \leqq 0,\ \forall x \in \mathbb{R}^m_+, \tag{21}$$

$$y^\top\bar{p} \geqq 0,\ \forall y \in \mathbb{R}^n_+. \tag{22}$$

Relation (21) implies $\lambda A\bar{p} \geqq B\bar{p}$, and at the same time from (22) we obtain $\bar{p} \geq [0]$. □

Proof of Theorem 4.

(i) Let (x, p, λ) an equilibrrium solution of a Leontief-von Neumann model (A, I), i. e. the said triplet satisfies relations (12), (13) and (14). We shall prove the existence of a vector $\bar{x} \geq [0]$ such that

$$\begin{array}{rcl} \lambda^{-1}\bar{x}^\top & = & \bar{x}^\top A \\ \bar{x}^\top p & > & 0. \end{array}$$

Denoting by $A^{(k)}$ the k-th power of the square matrix A, let us consider the following sequences:

$$x^\top,\ \lambda x^\top A,\ \lambda^2 x^\top A^{(2)}, ..., \lambda^k x^\top A^{(k)}, ... \tag{23}$$

$$p,\ \lambda Ap,\ \lambda^2 A^{(2)}p, ..., \lambda^k A^{(k)}p, ... \tag{24}$$

$$x^\top p,\ \lambda x^\top Ap,\ \lambda^2 x^\top A^{(2)}p, ..., \lambda^k x^\top A^{(k)}p, ... \tag{25}$$

From inequality (12) it results that sequence (23) is non-increasing; similarly from (13) we deduce that sequence (24) is non-decreasing. Sequence (25) is both non-increasing and non-decreasing, so it is a constant sequence. As sequence (23) has all nonnegative elements, it is convergent. Let us set

$$\bar{x}^\top = \lim_{k\longrightarrow\infty} \lambda^k x^\top A^{(k)}. \tag{26}$$

From (26) we get

$$\bar{x}^\top = \lim_{k\longrightarrow\infty} \lambda^k x^\top A^{(k)} = \lambda(\lim_{k\longrightarrow\infty} \lambda^{k-1} x^\top A^{(k-1)}) = \lambda\bar{x}^\top A, \tag{27}$$

that is

$$\bar{x}^\top A = \frac{1}{\lambda}\bar{x}^\top. \tag{28}$$

Taking the limit for $k \longrightarrow \infty$ in (25), taking (26) and (14) into account and recalling that sequence (26) is constant, we get

$$\bar{x}^\top p = x^\top p > 0, \tag{29}$$

that is $\bar{x} \geq [0]$. Therefore $(1/\lambda)$ is an eigenvalue of A, with a (left-hand) semipositive eigenvector $\bar{x}^\top$ associated. From (13), (28) and (29) we deduce that the triplet $(\bar{x}, p, \lambda)$ is an equilibrium solution of the model.

(*ii*) Let $\bar{x} \geq [0]$ be a (left-hand) eigenvector of A, associated to the eigenvalue $\lambda^{-1} > 0$, i. e. it holds $\bar{x}^\top A = \lambda^{-1}\bar{x}^\top$. Let us absurdly suppose that there exist no vectors $p \in \mathbb{R}^n_+$ such that the triplet $(\bar{x}, p, \lambda)$ is an equilibrium solution of the Leontief-von Neumann technology. Then, by Lemma 2, it will exist a vector $x \in \mathbb{R}^m_+$ such that

$$\bar{x}^\top \leqq x^\top - \lambda x^\top A,$$

that is

$$\bar{x}^\top + \lambda x^\top A \leqq x^\top. \tag{30}$$

By multiplying both sides of (30) by λA and by adding in both sides vector $\bar{x}^\top$, we obtain

$$\lambda^2 x^\top A^{(2)} + 2\bar{x}^\top \leqq x^\top.$$

By repeating k times the said operation, we get

$$\lambda^k x^\top A^{(k)} + k\bar{x}^\top \leqq x^\top.$$

Therefore it holds

$$k\bar{x}^\top \leqq x^\top$$

for each $k \in \mathbb{N}$.

But from this inequality it follows that $\bar{x}^\top = [0]$, which is in contradiction with the assumption that $\bar{x}^\top$ is a semipositive (left-hand) eigenvector of A.

(*iii*) From (*i*) and (*ii*) it follows equivalence (*iii*). □

Remark 1. We may note the "asymmetric" version of Theorem 4. The following example clarifies the results of the said theorem. Let

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}.$$

Then, the equilibrium levels of the related Leontief-von Neumann model are $\lambda_1 = 1/2$ and $\lambda_2 = 1/3$. We have $S^+(A) = \{2; 3\}$. The vector $x^1 = [1/2;\ 1/2]$ is a (left-hand) eigenvector associated to 3, whereas $p^1 = [0;\ 1]^\top$ is a (right-hand) eigenvector associated to 3, i. e. p^1 is a left-hand eigenvector associated to 3 for $A^\top$. Hence, the triplet $(1/3, x^1, p^1)$ is an equilibrium solution of this Leontief-von Neumann model.

On the other hand, we can associate the (left-hand) eigenvector $x^2 = [1;\ 0]$ to the other eigenvalue 2; however, in this case there exists no semipositive eigenvector p^2 associated to $(A^\top, 2)$. Indeed, in general it holds $S^+(A) \neq S^+(A^\top)$. As already remarked, if A is irreducible (i. e. indecomposable in the usual sense of Matrix Theory), the equality between

$S^+(A)$ and $S^+(A^\top)$ holds. In this case $\lambda = \lambda_{\min} = \lambda_{\max}$ and the equilibrium vectors x and p are both positive (and unique up to multipication by a positive scalar). In the example of the present remark A is reducible and we have $S^+(A) = \{2; 3\}$, whereas $S^+(A^\top) = \{3\}$.

4. A Leontief-Morishima-von Neumann Model

For the reader's convenience in the present section we summarize the results of Morishima (1961, 1964) on a Leontief-von Neumann model in which an industry i can choose between m_i different activities for producing good i. The notational convention of Morishima is opposite to the one of Kemeny, Morgenstern and Thompson followed in the previous sections. The total set of activities can be described by an (n, m) matrix $\hat{A}$ where

$$\hat{A} = \begin{bmatrix} a^1_{1_1} & ... & a^1_{m_1} & a^1_{1_2} & ... & a^1_{m_2} & ... & a^1_{1_n} & ... & a^1_{m_n} \\ a^2_{1_1} & ... & a^2_{m_1} & a^2_{1_2} & ... & a^2_{m_2} & ... & a^2_{1_n} & ... & a^2_{m_n} \\ \vdots & ... & \vdots & \vdots & ... & \vdots & ... & \vdots & ... & \vdots \\ a^n_{1_1} & ... & a^n_{m_1} & a^n_{1_2} & ... & a^n_{m_2} & ... & a^n_{1_n} & ... & a^n_{m_n} \end{bmatrix},$$

where $m = \sum_{i=1}^n m_i$ and n is the number of commodities. An activity, say the s_i-th activity of industry i, is defined by an n-dimensional column vector

$$A^{s_i} = \begin{bmatrix} a^1_{s_i} \\ a^2_{s_i} \\ \cdot \\ \cdot \\ \cdot \\ a^n_{s_i} \end{bmatrix}$$

stating the inputs of n commodities per unit output. As there is no joint production, the output matrix, of order (n, m) is written as:

$$\hat{I} = \begin{bmatrix} 1 & ... & 1 & 0 & ... & 0 & ... & 0 & ... & 0 \\ 0 & ... & 0 & 1 & ... & 1 & ... & 0 & ... & 0 \\ \vdots & ... & \vdots & \vdots & ... & \vdots & ... & \vdots & ... & \vdots \\ 0 & ... & 0 & 0 & ... & 0 & ... & 1 & ... & 1 \end{bmatrix}.$$

Next, let x_{s_i} be the output of good i produced by the s_i-th activity of industry i and p_i the price of good i. Let $x \in \mathbb{R}^m$ be the m-dimensional column vector

$$x = \begin{bmatrix} x_{1_1} \\ x_{2_1} \\ \cdot \\ x_{m_1} \\ \cdot \\ x_{1_n} \\ \cdot \\ x_{m_n} \end{bmatrix}$$

and $p = [p_1, p_2, ..., p_n]$, $p \in \mathbb{R}^n$. Morishima sets up his von Neumann-like model in terms of the usual inequalities

$$(\hat{I} - \lambda\hat{A})x \geqq [0] \tag{31}$$

$$p(\hat{I} - \lambda\hat{A}) \leqq [0] \tag{32}$$

$$p\hat{I}x > 0. \tag{33}$$

If each industry selects a single activity from among those available to it, there are $m_1 \times m_2 \times ... \times m_n$ possible sets of activities that could be adopted by the economy. They are arranged in a certain order and denoted by $\alpha, \beta, ..., \mu$, where $\mu = m_1 \times m_2 \times ... \times m_n$. Let σ be the activity set $(s_1, s_2, ..., s_n)$ in which industry i selects its s_i-th activity $(i = 1, ..., n)$ and define A_σ any (n, n) matrix which represents a particular set of activities adopted $(\sigma = \alpha, \beta, ..., \mu)$. Since A_σ is nonnegative, it has a dominant eigenvalue λ^*_σ that is nonnegative. In particular, let $A_\varepsilon = [A^{e_1}, A^{e_2}, ..., A^{e_n}]$ be an activity set such that $\lambda^*_\varepsilon \leqq \lambda^*_\sigma$ $(\sigma = \alpha, \beta, ..., \mu)$, and let the right-hand and left-hand eigenvectors of A_ε, associated with λ^*_ε, be denoted, respectively by x^*_ε and p^*_ε (i. e. we have $p^*_\varepsilon A_\varepsilon = \lambda^*_\varepsilon p^*_\varepsilon$; $A_\varepsilon x^*_\varepsilon = \lambda^*_\varepsilon x^*_\varepsilon$). Let $x^*_{\varepsilon i}$ be the i-th component of x^*_ε and let y^*_ε be an m-dimensional column vector such that its s_i-th component is $x^*_{\varepsilon i}$ when $s_i = e_i$ and zero, when $s_i \neq e_i$. Next Morishima makes the following two assumptions and proves the following theorem.

Assumption 1: A_ε is indecomposable.

Assumption 2: A_ε is unique, i. e. $\lambda^*_\varepsilon < \lambda^*_\theta$, for any other activity θ.

Theorem 5. System (31)-(33) has solutions such that $x = y^*_\varepsilon$, $p = p^*_\varepsilon$ and $\lambda = (\lambda^*_\varepsilon)^{-1}$; furthermore, if x, p, λ are solutions to (31)-(33), then $x = r_1 y^*_\varepsilon$, $p = r_2 p^*_\epsilon$ and $\lambda = (\lambda^*_\varepsilon)^{-1}$, where r_1 and r_2 are arbitrary positive numbers.

5. Regular von Neumann Models

Here we continue to adopt the Kemeny, Morgenstern and Thompson (1956) notations and conventions. Łos (1971) suggests that the results of Theorem 4 can be generalized also to the classical von Neumann model, under suitable assumptions and along similar lines of the proof of Theorem 4. Łos does not prove his assertion, but considers the assumptions under which the same holds:

(L1) $\forall x \in \mathbb{R}^m_+, \exists \bar{x} \in \mathbb{R}^m_+ : x^\top A = \bar{x}^\top B$.

(L2) $x \in \mathbb{R}^m,\ x^\top B \geqq [0] \Longrightarrow x^\top A \geqq [0]$.

In order to investigate on (L1) and (L2), we define the following polyhedral cone; if A is an (m, n) matrix, the *cone generated by* A, K_A, is defined as

$$K_A = \left\{ x^\top A : x \in \mathbb{R}^m_+ \right\}.$$

Condition (L1) means that

$$x \in \mathbb{R}^m_+ \Longrightarrow x^\top A \in K_A \text{ and } x^\top B \in K_B.$$

In other words, this condition says that for all $a \in K_a$ there exists $b \in K_b$ with $a = b$. Therefore (L1) is equivalent to

(L1)' $K_A \subseteq K_B$.

Condition (L2) means $x \in (K_{B^\top})^*$, where $(K_{B^\top})^*$ is the *dual cone* of $K_{B^\top}$ (see Nikaido (1968)). On the other hand, $x^\top A \geqq [0]$ means $x \in (K_{A^\top})^*$. Therefore it holds

$$(L2) \Longleftrightarrow ((K_{B^\top})^* \subseteq (K_{A^\top})^*).$$

But, as also $K_{B^\top}$ and $K_{A^\top}$ are closed cones, we can apply the duality theorem (see, e. g., Nikaido (1968)) in order to obtain

$$((K_{B^\top})^* \subseteq (K_{A^\top})^*) \Longleftrightarrow (K_{A^\top} \subseteq K_{B^\top}).$$

Therefore we can rewrite the Łos conditions (L1) and (L2) as follows:

(L1)' $K_A \subseteq K_B$.

(L2)' $K_{A^\top} \subseteq K_{B^\top}$.

Now, (L1)' is in turn equivalent (see Mangasarian (1971)) to the existence of a semipositive matrix H, of order (m, m), such that $A = HB$. This condition has been used by Mangasarian (1971) in order to generalize the classical Perron-Frobenius theorem to a pair of nonnegative matrices A, B, both of order (m, n). It has been used, together with other similar conditions, by Giorgi (2014), by Giorgi and Magnani (1978) and by Punzo (1980) in the study of mathematical properties of linear joint production models. It appears also in the book of J. Hicks (1965) and in the paper of Fujimoto and Krause (1988).

If A has semipositive rows, i. e. if (2) holds, also H will have semipositive rows, therefore its Frobenius eigenvalue is positive.

Likewise, (L2)' is equivalent to the existence of a semipositive matrix G, of order (n, n), such that $A = BG$. Therefore, (L1) and (L2) are, respectively, equivalent to:

(L1)" $A = HB, \quad H \geq [0]$;

(L2)" $A = BG, \quad G \geq [0]$.

Definition 7. A von Neumann model, where (2) and (3) hold, is called *regular* in the sense of Łos, when both conditions (L1) and (L2) hold.

We note that if $m = n$ (i. e. A and B are square) and B is non-singular, then (L1)" is equivalent to:

(L1)"' $AB^{-1} \geq [0]$;

and (L2)” is equivalent to:

(L2)’’’ $B^{-1}A \geq [0]$.

Before giving some comments on the (not proved) assertions of Łos, we remark that there exists a link between the maximum interest rate $\lambda_{\min} = \beta^*$ of a von Neumann model, where (L1)” holds, and the Frobenius root of the square semipositive matrix H. We denote by $v(A)$ the value of the matrix game A, where A is a (m, n) pay off matrix. That is, if

$$\Delta_m = \left\{ x : x_i \geqq 0, \quad \sum_{i=1}^{m} x_i = 1 \right\}$$

$$\Delta_n = \left\{ y : y_i \geqq 0, \quad \sum_{i=1}^{n} y_i = 1 \right\},$$

$v(A) = \max_{\Delta_m} \min_{\Delta_n} x^\top A y = \min_{\Delta_n} \max_{\Delta_m} x^\top A y.$

Theorem 6. Let be given a von Neumann model (A, B), where (2) and (3) hold. Let $A = HB$, with $H \geq [0]$, i. e. (L1)” holds. Then we have

$$(\lambda^*(H))^{-1} \leqq \lambda_{\min} = \beta^*.$$

Proof. Let $s < (\lambda^*(H))^{-1}$. Then, there exists a vector $x \geq [0]$ such that $x^\top(I - sH) > [0]$.

As B has semipositive columns, it follows $x^\top(B - sHB) > [0]$, that is $v(B - sA) < 0$ and $s < (\lambda^*(H))^{-1}$. Therefore it is proved that $s < (\lambda^*(H))^{-1} \Longrightarrow s < \lambda_{\min}$, which implies $(\lambda^*(H))^{-1} \leqq \lambda_{\min}$. □

The following assertion of Łos (1971) states that, under the regularity assumptions (K1) and (L2), it is possible to generalize to a von Neumann model, where (2) and (3) hold, the results of Theorem 4.

Theorem 7. Let (A, B) be the technology of a *regular* von Neumann model, with (2) and (3) satisfied. Then:

(i) The scalar $\lambda > 0$ is a von Neumann equilibrium level if and only if $\lambda^{-1} \in S^+(H)$.

(ii) If $\lambda > 0$ is an equilibrium level, then there exists a triplet (λ, x, p) which is an equilibrium solution of the von Neumann model and where x is a (left-hand) eigenvector of H, associated to its eigenvalue λ^{-1}.

(iii) Let $x \geq [0]$ be a (left-hand) eigenvector of H, associated to its eigenvalue $\lambda^{-1} \in S^+(H)$. Then, there exists a vector $p \geq [0]$ such that the triplet (λ, x, p) is an equilibrium solution of the von Neumann model.

Remark 2. A proof of Theorem 7, not short nor similar to the proof of Theorem 4, is given by Vahrenkamp (1980). A proof of results similar to the ones of Theorem 7, is given by Kogelschatz (1981). This author gives a more compact proof, which relies on definitions and properties related to the concept of *generalized inverse* of a matrix of order (m, n). We recall the following basic facts (see, e. g., Rao and Mitra (1971)).

Let A be an (m, n) matrix. An (n, m) matrix A^- is a *generalized inverse* of A or *g-inverse* of A if $AA^-A = A$. Usually A admits infinite generalized inverses, unless A is square and non-singular: in this case A admits one generalized inverse which coincides with the usual inverse A^{-1}.

The matrix A can admit a *right g-inverse* A_r^-, i. e. a matrix A_r^- of order (n, m) such that

$$AA_r^- = I,$$

where I is of order (m, m). The matrix A can admit a *left g-inverse* A_ℓ^-, i. e. a matrix A_ℓ^- (of order (n, m)) such that

$$A_\ell^- A = I,$$

with I of order (n, n).

The matrices A_r^- and A_ℓ^- are generalized inverses of A, as it holds

$$\begin{aligned} AA_r^-A &= (AA_r^-)A = IA = A; \\ AA_\ell^-A &= A(A_\ell^-A) = AI = A. \end{aligned}$$

Theorem 8. A matrix A of order (m, n) admits at least one right g-inverse if and only if $\mathsf{rank}(A) = m$. In this case it holds $\mathsf{rank}(A_r^-) = m$. A right g-inverse of A is given by

$$A_r^- = A^\top(AA^\top)^{-1}.$$

A_r^- is unique if and only if A is square and non-singular; then it holds $A_r^- = A^{-1}$.

Theorem 9. A matrix A of order (m, n) admits at least one left g-inverse if and only if $\mathsf{rank}(A) = n$. In this case it holds $\mathsf{rank}(A_\ell^-) = n$. A left g-inverse of A is given by

$$A_\ell^- = (A^\top A)^{-1} A^\top.$$

A_ℓ^- is unique if and only if A is square and non-singular; then it holds $A_\ell^- = A^{-1}$.

Kogelschatz (1981) proves the following results, in a sense more complete than the ones of Theorem 7.

Theorem 10. Let (A, B) be the technology of a von Neumann model. with (2) and (3) satisfied. Moreover, let the following conditions be satisfied:

(a) $\mathsf{rank}(B) = m$;

(b) There exists two generalized inverses $\dot{B}^-$ and $\ddot{B}^-$ such that $A\dot{B}^- \geq [0]$ and $\ddot{B}^- A \geq [0]$;

(c) It holds $A\dot{B}^- B = A$.

Then:

1) The scalar $\lambda > 0$ is a von Neumann equilibrium level if and only if $\lambda^{-1} \in S^+(A\dot{B}^-)$. Moreover, λ is an equilibrium level for the von Neumann model (A, B) if and only if λ is an equilibtrium level for the Leontief-von Neumann model $(A\dot{B}^-, I)$.

2) If the pair (λ^{-1}, x) solves the system $x^\top(\lambda^{-1} I - A\dot{B}^-)$, $x \geq [0]$, then there exists a vector $p \geq [0]$ such that the triplet (λ, x, p) is an equilibrium solution of the von Neumann model (A, B).

3) If the triplet (λ, x, p) is an equilibtrium solution of the von Neumann model (A, B), then there exists a pair $(\lambda^{-1}, \hat{x})$, such that $\hat{x}$ is a solution of the system $\hat{x}^\top(\lambda^{-1} I - A\dot{B}^-)$, $\hat{x} \geq [0]$, and such that the triplet $(\lambda, \hat{x}, p)$ is an equilibrium of the von Neumann model (A, B).

4) There exists a triplet $(\lambda_{\min}, x, p)$ which is an equilibrium solution of the von Neumann model (A, B), for which it holds

$$x^\top(B - \lambda_{\min} A) = [0] \, ; \tag{34}$$

$$(B - \lambda_{\min} A)p = [0] \, . \tag{35}$$

5) If $A\dot{B}^-$ is irreducible, then λ is unique (it holds $\lambda = \lambda_{\min} = \lambda_{\max}$) and there exists a triplet (λ, x, p), with $x > [0]$, which is an equilibrium solution of the von Neumann model (A, B) and where (34) and (35) hold.

On the previous theorem the following comments may be useful. Thanks to assumption (a), we can assert the existence of a right g-inverse B_r^-, for which it holds obviously $A = BB_r^- A$. If $B_r^- A \geq [0]$, then matrix $B_r^- A$ can be identified with matrix G in conditions (L2)" of Łos. On the other hand, from conditions (b) and (c) we deduce that matrix $A\dot{B}^-$ can be identified with matrix H of condition (L1)" of Łos.

Remark 3. When in a von Neumann model, where conditions (2) and (3) are verified, A and B are *square* and B is non-singular, then, as already remarked, (L1)" is expressed by $AB^{-1} \geq [0]$ and (L2)" is expressed by $B^{-1}A \geq [0]$. In this case, we can obtain in a simple way some results similar to the ones contained in Theorems 9 and 10. We need the following result (see Abraham-Frois and Berrebi(1979)).

Theorem 11. Let A be a semipositive decomposable square matrix and r be a given positive number. When A has a left-hand eigenvector $x > [0]$, we have $\lambda^*(A) \geqq r$, if there exists a vector $z \geq [0]$ such that $Az \geqq rz$. We have $\lambda^*(A) \leqq r$ if there exists a vector $z \geq [0]$ such that $z^\top A \leqq rz$.

Now, let us suppose that in the classical von Neumann model, where (2) and (3) are verified, A and B are square, B is non-singular, $B^{-1}A \geq [0]$ and $AB^{-1} \geq [0]$. From $\left[I - \beta B^{-1} A\right] p \leqq [0]$ we get $[B - \beta A]\, p \leqq [0]$, being $B \geq [0]$, and, similarly, from $x^\top \left[I - \alpha AB^{-1}\right] \geqq [0]$ we get $x^\top [B - \alpha A] \geqq [0]$. If we call

- β_1 the lowest β such that $\left[I - \beta B^{-1} A\right] p \leqq [0]$, where $p \geq [0]$;
- β^* the lowest β such that $[B - \beta A]\, p \leqq [0]$, where $p \geq [0]$;
- α_1 the highest α such that $x^\top \left[I - \alpha AB^{-1}\right] \geqq [0]$, where $x \geq [0]$;
- α^* the highest α such that $x^\top [B - \alpha A] \geqq [0]$, where $x \geq [0]$, we have $\beta^* \leqq \beta_1$ and $\alpha_1 \leqq \alpha^*$.

If the semipositive matrix $B^{-1}A$ is *indecomposable,* by the Perron-Frobenius theorem it results that $1/\beta_1$ is the dominant (Frobenius) eigenvalue of $B^{-1}A$. When the semipositive matrix $B^{-1}A$ is decomposable and has a strictly positive left-hand eigenvector, $1/\beta_1$ is the dominant eigenvalue of $B^{-1}A$ (see Theorem 11). Similarly, when the square matrix AB^{-1} is semipositive, $1/\alpha_1$ is the dominant eigenvalue of AB^{-1} if AB^{-1} is indecomposable or if it is decomposable and has a strictly positive right-hand eigenvector. We can distinguish the following cases.

a) Both matrices $B^{-1}A$ and AB^{-1} are semipositive and at least one of them is indecomposable: we have $\beta^* = \beta_1 = \alpha_1 = \alpha^*$ and $p^* > [0]$, $x^* \geq [0]$ (or $p^* \geq [0]$, $x^* > [0]$), where p^* and x^* are, respectively, the right-hand eigenvector of $B^{-1}A$ and the left-hand eigenvector of $B^{-1}A$, associated to the dominant eigenvalue $(1/\beta^*) = (1/\alpha^*)$.

b) Both matrices $B^{-1}A$ and AB^{-1} are semipositive and decomposable: if the first matrix has a strictly positive left-hand eigenvector and the second matrix has a strictly positive right-hand eigenvector, we have $\beta^* = \beta_1 = \alpha_1 = \alpha^*$ and the vectors $p^* \geq [0]$, $x^* \geq [0]$ are, respectively, the right-hand eigenvector and the left-hand eigenvector of AB^{-1}, associated to the dominant eigenvalue $(1/\beta^*) = (1/\alpha^*)$.

Finally, the relation $x^\top Ap > 0$ can be obtained as in Howe (1960) or Nikaido (1968).

6. Final Remarks

Links between equilibrium solutions of the von Neumann model and (generalized) eigenvalues and eigenvector are analyzed also by Thompson and Weil (1970, 1971, 1972). See, for an account, the book of Morgenstern and Thompson (1976). Thompson and Weil introduce the definition of *central solution triplet,* i. e. a triplet $(\bar{x}, \bar{p}, \bar{\alpha})$ solution of the von Neumann model with the additional property that $\bar{x}$ and $\bar{p}$ each have the maximum number of positive elements. Now, let be given a von Neumann technology (A, B) where (2) and (3) hold, let (x, p, λ) be an equilibrium solution and let (I, J) be a nonempty subset of $\{M \times N\}$, where $M = \{1, ..., m\}$ and $N = \{1, ..., n\}$. The triplet (x_I, p_J, λ) is a *generalized eigensystem triplet* if

$$\begin{aligned} x_I^\top (B - \lambda A)_{(I,J)} &= [0], \quad x_I \geq [0] \\ (B - \lambda A)_{(I,J)} p_J &= [0], \quad p_J \geq [0] . \end{aligned}$$

Thompson and Weil (1971) prove various results, among which the following ones.

Theorem 12. Let $(\bar{x}, \bar{p}, \bar{\alpha})$ be a central solution triplet to the von Neumann model (A, B), i. e. the said triplet satisfies relations (9)-(10)-(11). Let $\bar{I} = \{i \in M : \bar{x}_i > 0\}$, $\bar{J} = \left\{j \in N : \bar{p}_j > 0\right\}$. Then $(\bar{x}_{\bar{I}}, \bar{p}_{\bar{J}}, \bar{\alpha})$ is a generalized eigensystem triplet.

Theorem 13. Let λ be an equilibrium level of the von Neumann model (A, B). If $v(B - \lambda A) = 0$,

the triplet $(\bar{x}, \bar{p}, \lambda)$ is a generalized eigensystem triplet for suitable submatrices $(\bar{A}, \bar{B})$ of (A, B).

The existence of equilibrium solutions of a von Neumann model (A, B), in terms of generalized eigenvalues and eigenvectors, has been considered also by Drandakis (1966). This author assumes A and B square, of order n, and such that (2) and (3) are satisfied. A scalar λ is a generalized eigenvalue and a nonzero vector $y \in \mathbb{R}^n$ is a generalized right-hand eigenvector of (A, B) associated to λ if

$$By = \lambda Ay.$$

Similarly, a nonzero vector $x \in \mathbb{R}^n$ is a generalized left-hand eigenvector of (A, B), associated to λ, if

$$x^\top B = \lambda x^\top A.$$

(A, B) is an *F-transformation* if there exists a unique, simple, positive generalized eigenvalue λ of (A, B) with positive right-hand and left-hand eigenvectors y and x, respectively. Now, let $\alpha^* = \lambda_{\max}$ the maximum growth rate for the von Neumann model (A, B), with A and B square and with (2) and (3) satisfied. Drandakis proves the following result, which is a sufficient condition for a pair (A, B) to be an F-transformation.

Theorem 14. If $b_{ii} > 0$ for all $i = 1, ..., n$, if $b_{ij} < \alpha^* a_{ij}$ for all $i \neq j$ for which $a_{ij} > 0$ and if A is indecomposable, then:

(a) The equilibrium vectors x^* and p^* (associated to α^*) are positive and unique, up to a scalar multiplication. They are, respectively, left-hand and right-hand generalized eigenvectors of (A, B).

(b) α^* is a simple root of the characteristic equation

$$\det(B - \alpha^* A) = 0.$$

(**c**) No other eigenvalue of (A, B) has a nonnegative right-hand or left-hand eigenvector.

Another paper where the transformation $A = HB$ is used to deduce properties of the classical von Neumann model, is due to Steenge and Konjin (1992) (these authors adopt a convention opposite to ours in defining A and B). Finally, a paper specifically concerned with the determination of the equilibtrium solutions of a von Neumann model, by means of eigenvalues and eigenvectors of certain matrices, related to the input and output matrices of the von Neumann model, is due to Lesanovsky (1979). Unfortunately this paper is in Czech.

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