A curve and its abstract Jacobian

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Abstract

Let $C(K)$ be the K-points of a smooth projective curve C of genus $g > 1$ and $J(K)$ its Jacobian. Fixing a point on the curve one has a canonical embedding of $C(K)$ into $J(K)$ with the point identified as 0 of the group. Consider $\langle J(K); +, C(K)\rangle$ as an abstract structure of a group with a distiguished subset. Using model theory we prove that, for K algebraically closed, one can recover from this data the field K and the curve C, up to isomorphisms of fields and a bijective isogeny of J. In characteristic 0 the bijective isogeny is just a regular isomorphism, and in positive characteristic such an isogeny can be seen as a "Frobenius twisting". In such an interpretation our theorem, in particular, proves a conjecture posed by F.Bogomolov, M.Korotaev and Yu.Tschinkel.

1 Introduction and preliminaries

1.1 Let $C(K)$ be the K-points of a smooth projective curve C of genus $g > 1$ and $J(K)$ its Jacobian, the abelian group of degree 0 cycles on C. We assume throughout that K is algebraically closed and denote k the algebraic closure of the minimal field of definition of C.

We consider the structure on the set $C(K)$ defined by the $(4g+2)$-ary relation

$$R(u_1, \ldots, u_g, u_{g+1}, v_1, \ldots, v_g, v_{g+1}, t_1, \ldots, t_g, s_1, \ldots, s_g)$$

interpreted as

$$u_1 + \ldots + u_g + u_{g+1} + s_1, \ldots, s_g \equiv_{\text{linear}} v_1 + \ldots + v_g + v_{g+1} + t_1 + \ldots + t_g,$$

the linear equivalence of divisors. This relation can be written in the form

$$x_1 + \ldots + x_g + x_{g+1} = y_1 + \ldots + y_g$$

for 0-degree divisors of the form $x_i = [u_i - v_i]$, $y_j = [t_j - s_j]$. This is sufficient for defining the relation $z_1 + z_2 = z_3$ for arbitrary 0-cycles $z_k = x_{1,k} + \ldots + x_{g,k} \in J$, hence for defining the group structure $(J, +)$.

We denote the structure $(C(K), R)$ by $C^J(K)$ or simply C^J. We have observed above that $(J, +)$ is interpretable in C^J.

Fix a point $c^0 \in C(k)$ and let $j : C(K) \to J(K)$ be the embedding $j : x \mapsto [x - c^0]$. We consider the pair $(J(K), C(K))$ as a structure with the universe $J(K)$, the group operation $+$ and a unary predicate that distingushes the subset $j(C(K))$ in $J(K)$. We write it in a model-theretic way as $(J; C, +)$. Clearly, $(J; C, +)$ is definable in C^J using the parameter c^0 and vice versa.

We study the situation when two such structures over corresponding fields K_1 and K_2 are isomorphic via an isomorphism α,

$$C_1^J \cong_\alpha C_2^J. \tag{1}$$

By the above this is equivalent to

$$(J_1; C_1, +) \cong_\alpha (J_2; C_2, +) \tag{2}$$

for some choices for c_0 on the corresponding curves.

Examples. 1. Let $K_1 \cong_\beta K_2$ be an isomorphism of fields, C_1 a smooth projective curve over k_1, J_1 its Jacobian and C_2 a curve over k_2 such that $C_2 = \beta(C_1)$ by the induced bijection. It is easy to see that the image $J_2 = \beta(J_1)$ is the Jacobian of C_2. In this situation we have (1).

2. Let $K_1 = K_2 = K$ and C_1, C_2, J_1, J_2 be as above, and let $\psi : J_1 \to J_2$ be an isogeny with trivial kernel such that $\psi(C_1) = C_2$. Then we have (1). (Here the kernel is trivial group-theoretically)

3. Clearly, the composition of β of example 1 and ψ of example 2 gives an example of an isomorphism α of (1). In particular, when β is of the form Frob^m, some $m \in \mathbb{Z}$, then α is an isogeny $J_1 \to J_2$ with trivial kernel.

Remark. If char $K = 0$ then ψ in example 2 has to be an isomorphism of the varieties. This follows from the more general facts in 2.3 below.

1.2 In the recent paper [1], F.Bogomolov, M.Korotaev anf Yu.Tschinkel proved the following.

Theorem. Let $K_1 = K_2 = \mathbb{F}_p^{alg}$ be the algebraic closure of a field of p elements, $p > 3$ and (1) holds. Then J_1 and J_2 are isogenous.

They also conjectured that *under the above assumptions C_1 and C_2 are isomorphic as algebraic varieties, modulo Frobenius twisting.*

The proof in [1] is specific to the situation in locally finite fields $\mathbb{F}_p^{alg}$ and uses the theory of profinite groups of automorphisms.

Our goal is to give a model-theoretic proof of the following.

1.3 Theorem. *For any algebraically closed fields K_1 and K_2, any instance of (1) is as in example 3 above:*

There is a field-isomorphism $\beta : K_1 \to K_2$ inducing an isomorphism of pairs, and a bijective isogeny $\psi : \beta(J_1) \to J_2$ such that

$$\alpha = \psi \circ \beta : \ (J_1; C_1, +) \to (J_2; C_2, +).$$

The bijective isogeny ψ is a composition of a bijective morphism of algebraic groups with a Frobenius isomorphism.

In particular, when $K_1 = K_2 = K$ and C_1 is defined over a finite field, we have that $\beta(J_1)$ is isogenous to J_1 via a map induced by Frob^m, *for some $m \in \mathbb{Z}$, and so J_1 is bijectively isogenous to J_2. In particular, C_1 is in bijective correspondence to $\beta(C_1)$ via a map induced by* Frob^{-m}, *and the regular morphism ψ sends $\beta(C_1)$ onto C_2.*

When the characteristic of the field is 0, *the isogeny ψ is an isomorphism of algebraic varieties.*

Remark. In [1] the authors do not make precise what an "isomorphism modulo Frobenius twisting" must be. Our theorem says that when the field of definition is finite, J_1 is isomorphic to J_2 as abstract groups by $\psi \circ \mathrm{Frob}^{-m}$, which is a composition of a bijective isogeny with a power of Frobenius. A bijective isogeny between Jacobians (and the map induced by it on the curves) is probably the best that can stand for an "isomorphism modulo Frobenius twisting". For example, recall that it is possible to have a split Jacobian for some curves C in positive characteristic (see e.g. discussion in [2]). In particular, one can have a Jacobian of the form $J(C) = E_1 \oplus E_2$, the sum of two elliptic curves. One has an isogeny $\psi : E_1 \oplus E_2 \to E_1 \oplus E_2$ defined as an identity on E_1, and as a power of Frobenius on E_2.

We must note that in this example we do not know whether the curve $C' = \psi(C)$ is smooth, and whether J is the Jacobian variety for C'.

1.4 The proof of 1.3 follows the scheme outlined in [9], and is based on the now well-known model-theoretic result, the Rabinovich theorem.

This technically hard theorem proved by E.Rabinovich [8], is a partial answer to the Restricted Trichotomy Conjecture by the present author, which states that *assuming a strongly minimal structure M is interpretable in an algebraically closed field K and M is not locally modular, a field isomorphic to K is interpretable in M.* The Rabinovich theorem proves this statement under additional assumption that M as a set can be identified with $C(K)$, for a rational curve C over K.

(Note that the case when M is locally modular is well-understood and basically classifiable.)

Once we find that, by Rabinovich theorem, the field K is definable in our structure $C^J(K)$, we proceed to recover $C^J(K)$ as a Jacobian variety for a curve C. This provides the required isomorphism and the proof of 1.3.

This method is quite standard in model theory. E.g. we used it in [11] to describe abstract automorphisms of $G(K)$ for simple algebraic groups G, without the use of the structural theory of such groups. Similar ideas with a technically different analysis was adapted by Yu.Manin in [12].

The Rabinovich theorem and earlier attempts in this direction preceded and inspired the work [10], which introduced and classified Zariski geometries. The latter had found many applications in Diophantine geometry, and in its extended versions has become one of the main tools of the "fine" classification theory.

It is therefore natural to aim for a new proof of Rabinovich's theorem, or even a full proof of the Restrited Trichotomy along the lines of the classification theorem of [10], or by other modern methods. This is a challenge for the model-theoretic community.

2 Main Construction

2.1 We study here the structure C^J and the related structure $(J; C, +)$, where C is defined over an algebraically closed field k, $C = C(K)$.

We also use "definable" as a model-theoretic term. We provide the main definitions below, but recommend [3] for more detail and examples.

2.2 We say that a **subset** $S \subseteq M^n$ **is definable** in the structure $(M; L)$, where L stands for a collection of relations on M, if there are parameters $a_1, \ldots, a_k \in M$ and a first-order L-formula $\Phi(x_1, \ldots, x_n, a_1, \ldots, a_k)$ such that

$$S = \{\langle b_1, \ldots, b_n \rangle \in M^n : \ (M; L) \models \Phi(b_1, \ldots, b_n, a_1, \ldots, a_k)\}.$$

In more precise terms, one says that S is definable in $(M; L)$ over $A = \{a_1, \ldots, a_k\}$, or simply A-definable.

One may also call such an S a definable n-ary relation on M.

More generally, we say that a **set** S **is interpretable** in $(M; L)$ if there is a definable subset $T \subseteq M^n$, and a definable equivalence relation $E \subset T \times T$ (so $E \subset M^{2n}$) and such that $S = T/E$.

An **interpretation of** S **in** $(M; L)$ is the pair of formulas that define T and E.

The same definition is used for an interpretable (definable) structure, which is an interpretable set with interpretable relations on it. An interpretation of a structure $(N; L_N)$ in $(M; L)$ is the collection of formulas used to interprete N along with the L_N-relations on N.

We say 0-definable if no parameters are used.

Note that when $(M; L)$ is an algebraically closed field, being interpretable is equivalent to being definable because of *elimination of imaginaries* in algebraically closed fields, see [3].

Following this terminology, C^J is k-defined in the field K (that is in $(K; +, \cdot)$) via an interpretation i. We will write this fact as

$$K \sqsupset_k^i C^J.$$

If we extend K to a larger algebraically closed field K^* then we still have

$$K^* \sqsupset_k^i C^J(K^*).$$

That is, the same interpretation i defines K^*-points of the curve and its Jacobian. Moreover, since $K \preccurlyeq K^*$ (as fields) the extention of the whole structure (including the field K and the interpretation i) is elementary.

The main results of this section prove existence of certain definable objects in $C^J(K^*)$, and in the $C^J(K^*)$ together with K^*. By elementary equivalence these results pass to the original setting. So we simply assume in this section, without loss of generality, that transcendence degree of K over k is infinite.

2.3 Lemma. *Let $f : K^n \to K$ be a definable function. Then there is a nonempty open set $U \subseteq K^n$ such that:*

i) If K has characteristic 0, then there is a rational function r such that $f_{|U} = r$.

ii) If K has characteristic $p > 0$, then there is a natural number m and a rational function r such that $f_{|U} = \mathrm{Frob}^{-m} \circ r$, where Frob *is the Frobenious automorphism $x \mapsto x^p$.*

This is a well-known corollary of quantifier-elimination for algebraically closed fields. See [3], Theorem 1.11.□

Corollary. *Let G_1 and G_2 be algebraic groups over K, and $h : G_1 \to G_2$ a definable bijection which is a group-isomorphism. Then there is $m \geq 0$, and a bijective morphism of algebraic groups $h' : G_1 \to \mathrm{Frob}^m(G_2)$ such that $h = \mathrm{Frob}^{-m} \circ h'$.*

If G_1 and G_2 are abelian varieties, then h is an isogeny.

If $\mathrm{char}\, K = 0$, h is an isomorphism of algebraic groups.

Proof. By the Lemma above applied to each co-ordinate, there are affine open subsets $V_1 \subseteq G_1$ and $V_2 \subseteq G_2$ and the restriction $h_V : V_1 \to V_2$ such that

$$h_V = \mathrm{Frob}^{-m} \circ r_V, \quad r_V \text{ regular on } V_1,$$

for some $m \geq 0$. Let $G = \mathrm{Frob}^m(G_2)$. Then $r_V : V_1 \to G$ is a regular injective map defined on V_1 satisfying

$$r_V(x_1 \cdot x_2) = r_V(x_1) \cdot r_V(x_2), \;\; r_V(x_1^{-1}) = r_V(x_1)^{-1}, \text{ for a generic pair } x_1, x_2 \in V_1.$$

By Weil's group chunk argument, r_V can be uniquely extended to a morphism $h' : G_1 \to G$ of algebraic groups that has to be bijective, since it is bijective on an open set.

In case of abelian varieties, h' and Frob^{-m} are isogenies. □

3 The Rabinovich theorem and its corollaries

Theorem (E.D.Rabinovich [8]) *Assume M is a strongly minimal structure definable in an algebraically closed field K in such a way that the universe of M is a rational curve over K. Also assume that M is not locally modular. Then a field F isomorphic to K is interpretable in M.*

So the theorem settles the restricted trichotomy conjecture, under the extra assumption of rationality of the curve on which M is defined. This assumption can be weakened: it is enough to assume that a *rational curve M' is interpretable in M*. Indeed, since the combinatorial geometry of M is isomorphic to that of any strongly minimal set in M, we have that the structure induced on M' from M is not locally modular. Finally, note that by transitivity, a field F is interpretable in M', that is interpretable in M.

3.1 Lemma. In C^J, a set isomorphic (as algebraic variety) to the projective line $\mathbf{P}^1$ is definable.

Proof. Fix a class $[D_0]$ of a very ample effective divisor. The divisors of $[D_0]$ form a complete linear system. Equivalently it corresponds to an embedding $C \subset \mathbf{P}^n$, with the property that divisors $D \in [D_0]$ correspond to intersections $D = C \cap H$ for hyperplanes $H \subset \mathbf{P}^n$, and vice versa.

For $a_1, \ldots, a_m \in C$, denote by $L_{a_1,\ldots,a_m}$ the set of hyperplanes in $\mathbf{P}^n$ passing through $a_1, \ldots, a_m$. For $m = 0$, $L_{a_1,\ldots,a_m}$ is the set of all hyperplanes L^n, which is isomorphic to $\mathbf{P}^n$. By definition, $L_{a_1,\ldots,a_m}$ is a linear subspace (the intersection of hyperplanes) of L^n, and $\dim L_{a_1,\ldots,a_m,a_{m+1}} < \dim L_{a_1,\ldots,a_m}$ provided a_{m+1} is not in the linear subspace of $\mathbf{P}^n$ spanned by $a_1, \ldots, a_m$. The latter condition is satisfied for some $a_{m+1} \in C$, as long as the linear subspace of $\mathbf{P}^n$ spanned by $a_1, \ldots, a_m$ is not the whole $\mathbf{P}^n$ (note that by assumptions $C \not\subseteq H$ for a hyperplane $H \subset \mathbf{P}^n$). It follows that for some distinct $a_1, \ldots, a_{n-1} \in C$, $\dim L_{a_1,\ldots,a_{n-1}} = 1$, and so it is isomorphic to $\mathbf{P}^1$ as algebraic variety. It remains to observe that $L_{a_1,\ldots,a_{n-1}}$ can be identified with

$$\{D \in [D_0] : \ a_1, \ldots, a_{n-1} \in D\},$$

and the latter is equal to

$$\{\langle x_1, \ldots, x_{d-n+1}\rangle \in C^{d-n+1} : a_1 + \ldots + a_{n-1} + x_1 + \ldots + x_{d-n+1} = D_0\},$$

where $+$ is the group operation on the Jacobian, $d = \deg D_0$, and points x on C are identified with $j(x)$ as defined in subsection 1.1. These are definable in C^J using the parameter c^0, so $\mathbf{P}^1$ is definable in C^J using parameters in $C(k)$. $\square$

3.2 Theorem. *There exists an algebraically closed field F, $C(k)$-defined in the structure C^J along with a non-constant $C(k)$-definable map $h : C \to F$. In other words,*

$$K \quad \sqsupset_k \ C^J \quad \sqsupset^h_{C(k)} \ F.$$

Moreover, there is an isomorphism of fields $\phi : K \to F$, definable in the field K. The isomorphism ϕ is determined uniquely up to Frobenius automorphisms of K.

The map

$$\phi^{-1}h : C(K) \to K$$

coincides on an open subset of C with a rational map defined over k.

Proof. We claim that C^J is not locally modular. This is immediate from the fact that in any group definable in a locally modular structure any definable subset is a coset of a definable subgroup. See [4] for the most general statement of this type. The subset $C \subset J$ contradicts such a condition.

Now the Rabinovich theorem along with Lemma 3.1, tells us that a strongly minimal (and algebraically closed) field F is definable in C^J.

Next, we claim that there is a map $h : C \to F$, definable in C^J such that h is non-constant on any infinite subset of C. Indeed, since F is non-orthogonal to C in the structure C^J, there is a finite-to-finite correspondence $S \subset C \times F$ between C and F. Given a generic $x \in C$, we get $y_1, \ldots, y_m \in F$ corresponding to x via S. Let $s_1, \ldots, s_m$ be the symmetric functions in m variables, and let $h_i(x) := s_i(y_1, \ldots, y_m)$ for x and the y_k satisfying $\bigwedge_k S(x, y_k)$. At least one of the functions has to be non-constant since S is a finite-to-finite correspondence.

Note that the restriction of the structure to k-points is an elementary substructure since $k \preccurlyeq K$. It follows that we can choose F and h definable using parameters in $C(k)$ only.

The isomorphism ϕ is given by [5] (but is also implicit in the proof of Rabinovich's theorem).

Note that the map $\phi^{-1}h$ is definable in the field K over k by construction. By 2.3, there exists a non-negative integer m such that the map $\mathrm{Frob}^m \phi^{-1} h$ is rational on an open subset of C. Redefining $\phi := \phi \circ \mathrm{Frob}^{-m}$ we get the last statement. □

4 Representing the curve in F.

Below we continue to work in the structure C^J, or equivalently in $(J; C, +)$. We use the notion of dimension in the model-theretic sense, where $\dim S$ is understood as the Morley rank of a definable (in C^J) set S, but note that this notion of dimension coincides with that of algebraic geometry. This follows easily from the fact that the universe C of the structure C^J is 1-dimensional and irreducible in both senses.

We say that a tuple $\langle s_1, \ldots s_n \rangle \in S^n$ is generic in an A-definable set S, if every A-definable subset $R \subseteq S^n$ containing $\langle s_1, \ldots s_n \rangle$ is of dimension equal to $n \cdot \dim S$.

The above is applicable to interpretable sets, and to tuples inside them also.

4.1 Proposition. *There exists $n \in \mathbb{N}$, and a map $f : J \to F^n$ with finite fibres, $C(k)$-definable in C^J with domain $\mathrm{Dom}\, f$ Zariski open in J.*

Moreover, for any n the following two conditions are equivalent:

- *one can choose f to be generically injective on J (i.e. injective on an open subset J^0 of J)*
- *one can choose f to be generically injective on the shift $y^0 + C$ of the curve C, for some $y^0 \in J(k)$,*

Proof. Once a point $c_0 \in C(k)$ is fixed, we may identify a point $x \in C$ with a point $x - c_0 \in J$. Then a generic element y of J can be represented as $y = x_1 + \ldots + x_g$ for some generic g-tuple $x_1, \ldots, x_g \in C$, where g is the genus of C. Moreover, this representation is unique up to the permutation of the $x_1, \ldots, x_g$.

We now have a well-defined map $x_1 + \ldots + x_g \mapsto \{h(x_1), \ldots, h(x_g)\}$, from an open subset of J to $F^{(g)}$, the set of g-element subsets of F. On the other hand, there is an injective map

$$F^{(g)} \to F^g;\ \{z_1, \ldots, z_g\} \mapsto \langle s_1(z_1, \ldots, z_g), \ldots, s_g(z_1, \ldots, z_g) \rangle,$$

where s_i are the symmetric functions in g variables. The composition of these two maps is the required map associated with h.

Now assume that f is injective on an open subset J^0 of J. By dimensional considerations, for some $y^0 \in J$ of the form $y^0 = x_1 + \ldots + x_{g-1}$, up to finitely many points the curve $y^0 + C$ is a subset of J^0. Since k is algebraically closed we may choose $y^0 \in J(k)$. Hence, $x \mapsto f(y^0 + x)$ is a generically injective map $f : C \to F^n$. □

4.2 Given $f : J \to F^n$ as in 4.1, define for $y \in \mathrm{Dom}\, f$

$$\mathrm{fibre}_y f := \{y' \in J : f(y) = f(y')\}.$$

Let $n_0 \in \mathbb{N}$ and $f^0 : J \to F^{n_0}$ be such that for a generic y, $\mathrm{fibre}_y f^0$ is of minimal size among the maps of 4.1.

Lemma. *For every generic pair $y, t \in J$, and every $y' \in J$*

$$f^0(y) = f^0(y') \Rightarrow f^0(y + t) = f^0(y' + t).$$

Proof. Choose an arbitrary $a \in J(k)$ and consider the map $f_a^0 : J \to F^{n_0}$, $y \mapsto f^0(y + a)$ along with the map $\langle f^0, f_a^0\rangle : J \to F^{2n}$, $y \mapsto \langle f^0(y), f_a^0(y)\rangle$. Clearly

$$\mathrm{fibre}_y\langle f^0, f_a^0\rangle = \mathrm{fibre}_y f^0 \cap \mathrm{fibre}_y f_a^0.$$

By minimality we get $\mathrm{fibre}_y f^0 = \mathrm{fibre}_y f_a^0$. Hence, the statement holds for $t = a$, for all $a \in J(k)$. The lemma now follows since k is algebraically closed.□

Consider the $C(k)$-definable equivalence relation on $\mathrm{Dom}\, f^0$,

$$y \sim y' \Leftrightarrow f^0(y) = f^0(y').$$

Corollary. *For any $f : J \to F^n$ as in 4.1, on an open subset of J (depending on f),*

$$y \sim y' \Rightarrow f(y) = f(y').$$

4.3 Lemma. *The set*

$$A = \{a \in J : f^0(y + a) = f^0(y) \text{ for all } y \text{ in an open subset of } J\}$$

is a finite subgroup of $J(k)$.

The generic fibre is a coset of A,

$$\mathrm{fibre}_y f^0 = y + A.$$

A is trivial iff f^0 is generically injective.

Proof. Let $Z = f^0(J) \subset F^n$ and consider the map $h : Z \times J \to K^n$ defined on an open subset of $Z \times J$ as follows:

$$h(z, t) = w \Leftrightarrow \exists y \in J\, f^0(y) = z \;\&f^0(y + t) = w.$$

By Lemma 4.2 this is well-defined.

Let $t, t' \in J$, t generic over k, $t \sim t'$ and fix $z_0 \in J$ generic over $k(t, t')$. We have by the Corollary to Lemma 4.2 $h(z_0, t) = h(z_0, t')$.

It follows that for $y_0 \in J$ generic over $k(t,t')$, we have $f^0(y_0+t) = f^0(y_0+t')$. This can be rewritten as

$$f^0(y+a) = f^0(y)$$

where $a = t - t'$ and $y = y_0 + t'$. By genericity this holds for all y in an open subset of J, and hence a belongs to the subgroup A.

Since the fibres of f^0 are finite, A is finite, and so $A \subset J(k)$. By construction, every t' in the equivalence class of t is of the form $t+a$ for $a \in A$, so the f^0-fibre containing t is of the form $t+A$. Hence all generic fibres are of this form. □

4.4 Proposition. *The A of 4.3 is trivial, f^0 is injective on a $C(k)$-definable open subset J^0 of J, and there is a $C(k)$-definable $h^0 : C \to F^{n_0}$ generically injective on C.*

Proof. Aiming for a contradiction, assume that A is non-trivial. Then by 4.3 f^0 is not generically injective, and so by 4.1 f^0 is not generically injective on $y^0 + C \subset J^0$ (up to finitely many points), where J^0 is the domain of f^0 and $y^0 \in J(k)$. So for some generic $x \in C$, there is $x' \in C$, $x' \neq x$, $f^0(y^0+x) = f^0(y^0+x')$. It follows by 4.3 that for some non-zero $a \in A$, $x' = x + a \in C$. Since x is generic, the latter holds for any $x \in C$, i.e. $a + C = C$. But this is not possible unless $a = 0$, by Lemma 2.1 of [1]. □

4.5 Model-theoretic generalisation of Weil's group chunk theorem.

Consider again the definable injection

$$J^0 \quad \to_{f^0} \quad F^{n_0}$$

of an open definable subset J^0 into the affine space F^{n_0}, and denote $G^0 = f^0(J^0)$. The map f^0 transfers the definable subsets and relations on J^0 to ones on G^0. Note that $(J^0; +, -)$, where $+$ is a the partial operation on J^0 induced from J, and $-$ is the corresponding partial unary operation, is Weil's group chunk (a Weil pre-group) introduced in [6]. It follows that its image $(G^0; +, -)$ is a *definable group chunk*. This means that $+$ and $-$ are definable partial operations such that $-z$ and $z_1 + z_2$ is defined for any generic z and any generic pair z_1, z_2 in G^0, and also for any generic triple $(z_1+z_2)+z_3 = z_1+(z_2+z_3)$.

There have been various generalisations of Weil's group chunk theorem to the definable context, the most general one by E.Hrushovski, see a detailed exposition of this in [7]. We need the following corollary of these results.

Fact. *There is a group G definable in C^J, and a definable injective morphism of pre-groups $f^1 : G^0 \to G$ such that for a generic pair $z_1, z_2 \in J^0$, $f^1(z_1+z_2) = f^1(z_1) + f^1(z_2)$ and G^0 generates G. Moreover, the embedding*

$$f^1 \circ f^0 : J^0 \to G$$

can be extended to a definable isomorphism

$$j : J \to G.$$

5 Proof of the Main Theorem

5.1 We now come back to the assumption (1), and represent it using the results of section 2 schematically as follows (dropping the assumption of infinite transcendence degree K/k).

$$\begin{array}{llllll} K_1 & \sqsupset_k^{i_1} & C_1^J \sqsupset J_1 & \to_{f^0} & F_1^{n_0}; & \phi_1 : K_1 \to F_1 \\ & & \updownarrow \alpha & & \updownarrow \bar{\alpha} & \\ K_2 & \sqsupset_k^{i_2} & C_2^J \sqsupset J_2 & \to_{f^0} & F_2^{n_0}; & \phi_2 : K_2 \to F_2 \end{array}$$

Here, i_1 and i_2 are two interpretations of the curves and the Jacobians in $K_1$and K_2, and f^0 is the definable injective map of section 2 which is defined by the same formula in both structures C_1^J and C_2^J. The map α is the isomorophism given by (1) which induces via the interpretation h of 3.2 an isomorphism $\bar{\alpha} : F_1 \to F_2$ of fields. We apply the same notation to the bijective map $F_1^{n_0} \to F_2^{n_0}$ defined as $\bar{\alpha}$ coordinatewise.

In addition the picture also shows definable isomorphisms between fields $\phi_1 : K_1 \to F_1$ and $\phi_2 : K_2 \to F_2$.

The data above implies

5.2 Claim 1. *In (1) the fields are isomorphic,*

$$\breve{\alpha} : K_1 \cong K_2, \text{ where } \breve{\alpha} = \phi_2^{-1} \bar{\alpha} \phi_1.$$

In particular, when $K_1 = K_2 = K$ we will have the following diagram

$$\begin{array}{llllll} K & \sqsupset_k^{i_1} & C_1^J \sqsupset J_1 & \to_{f^0} & F_1^{n_0}; & \phi_1 : K \to F_1 \\ \updownarrow \breve{\alpha} & & \updownarrow \alpha & & \updownarrow \bar{\alpha} & \\ K & \sqsupset_k^{i_2} & C_2^J \sqsupset J_2 & \to_{f^0} & F_2^{n_0}; & \phi_2 : K \to F_2 \end{array}$$

Claim 2. *If $\breve{\alpha} = \mathrm{id}$ then the structures C_1^J, C_2^J, F_1 and F_2 are interpretable in the field K, and $\bar{\alpha}$ is a definable in the field K isomorphism of fields.*

Indeed, the interpretability of the structures is by interpretations i_1, i_2, and the definability of $\bar{\alpha}$ follows from the definability of ϕ_1 and ϕ_2 by Claim 1.

5.3 Now we rewrite the diagram above, replacing the affine spaces $F_1^{n_0}$ and $F_2^{n_0}$ with (definably equivalent) groups G_1 and G_2, correspondingly constructed in 4.5,

$$\begin{array}{llllll} K & \sqsupset_k^{i_1} & C_1^J \sqsupset J_1 & \to_{j_1} & G_1(F_1); & \phi_1 : K \to F_1 \\ \updownarrow \breve{\alpha} & & \updownarrow \alpha & & \updownarrow \hat{\alpha} & \\ K & \sqsupset_k^{i_2} & C_2^J \sqsupset J_2 & \to_{j_2} & G_2(F_2); & \phi_2 : K \to F_2 \end{array},$$

with the isomorphism $\hat{\alpha}$ between the definable groups induced by $\bar{\alpha}$. Here j_1 and j_2 correspond to the definable isomorphism j established in the Fact of 4.5.

5.4 Finally, we apply a transformation to the diagram 5.3 by applying a field automorphism $\breve{\alpha}^{-1}$ to the bottom-left field K. This automorphism clearly induces an abstract isomorphism $\beta : C_2^J \to C_3^J$ onto a new curve and its Jacobian. Set $\hat{\beta}$ analogously to the isomorphism $\hat{\alpha}$ of 5.3. By denoting $\breve{\beta} := \breve{\alpha}^{-1}$, we get

$$
\begin{array}{llll}
K \ \sqsupset_k^{i_1} & C_1^J \sqsupset J_1 \ \rightarrow_{j_1} & G_1(F_1); & \phi_1 : K \rightarrow F_1 \\
\updownarrow \check{\alpha} & \updownarrow \alpha & \updownarrow \hat{\alpha} & \\
K \ \sqsupset_k^{i_2} & C_2^J \sqsupset J_2 \ \rightarrow_{j_2} & G_2(F_2); & \phi_2 : K \rightarrow F_2 \\
\updownarrow \check{\beta} & \updownarrow \beta & \updownarrow \hat{\beta} & \\
K \ \sqsupset_k^{i_2} & C_3^J \sqsupset J_3 \ \rightarrow_{j_3} & G_3(F_3); & \phi_3 : K \rightarrow F_3
\end{array}
$$

Since $\check{\alpha}\check{\beta} = \mathrm{id}$, by Claim 2 of 5.2 we have the chain of isomorphisms definable in the field K of definable groups with distinguished curves

$$J_1 \rightarrow_{j_1} G_1(F_1) \rightarrow_{\hat{\beta}\circ\hat{\alpha}} G_3(F_3) \rightarrow_{j_3^{-1}} J_3.$$

Set

$$\psi = j_3^{-1} \circ \check{\beta} \circ \check{\alpha} \circ j_1.$$

By Corollary 2.3 the definable isomorphism of abelian varieties is an isogeny. Composing the isomorphisms we get a definable isomorphism, so ψ is a bijective isogeny between J_1 and J_3 which also respects the curves,

$$\psi : (J_1; C_1, +) \rightarrow (J_3; C_3, +).$$

Recall that $J_3 = \beta(J_2)$, where β is induced by an isomorphism of fields. This finishes the proof of the main part of the main theorem 1.3.

Consider now the case when the field of definition of C is finite, call the field k^0. Then $\beta_{|k^0} = \mathrm{Frob}^m$ for some $m \in \mathbb{Z}$. Now we have $J_3 = \mathrm{Frob}^m(J_2)$, and by definition Frob^m is an isogeny between J_2 and J_3. □

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