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The Boundary Value Problem with Haseman-shift for *k*-regular Functions on Unbounded Domains in Clifford Analysis

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Abstract

In this paper, we introduce the boundary value problem with Haseman shift for k-regular function on unbounded domains, and give the unique solution for this problem by integral equation method and fixed-point theorem.

Keywords: k-regular function, unbounded domains, Clifford analysis

1. Introduction

The boundary value problem is one of the important aspects in Clifford analysis. This problem on bounded domains has seen great achievements. But the one on unbounded domains has not, which is widely used in practical applications. So it is necessary to discuss the properties and boundary value problems of functions on unbounded domains. [Wen, 1991; Huang, 1996; Zhang et al., 2001] have discussed Riemann-Hilbert boundary value problems of regular function on bounded domains. [Li, 2007] characterized boundary value problems of k- regular functions. In this paper, we introduce the *Haseman-shift*'s boundary value problem of k-regular functions on unbounded domains, and give an unique solution to this problem by integral equation method and fixed-point theorem.

Let n be a positive integer, and $\{e_0, e_1, \cdots, e_n\}$ be basis for the Euclidean space $\mathbb{R}^{n+1}$. We denote by $\mathcal{A}$ the 2^n dimensional real Clifford algebra, which is generated by $\mathbb{R}^{n+1}$; denote the basis of $\mathcal{A}$ by $e_A = e_{\alpha_1,\alpha_2,\cdots,\alpha_h}$, $A = \{\alpha_1, \alpha_2, \cdots, \alpha_h\} \subseteq \{1, 2, \cdots, n\}$, $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_h \leq n$. In particular, if $A = \emptyset$, $e_\phi = e_0$. So, for an arbitrary $u \in \mathcal{A}$, we have $u = \sum_A u_A e_A$ with $u_A \in \mathbb{R}$. In $\mathcal{A}$, we have

$$e_i^2 = -1, e_i e_j = -e_j e_i \text{ for } i \neq j, i, j = 1, 2, \cdots, n,$$

that is so-called combinative and incommutable multiplication rule of Clifford algebra. For $u \in \mathcal{A}$, we write $u^* = \sum_A (-1)^{\frac{|A|(|A|-1)}{2}} u_A e_A$, $u' = \sum_A (-1)^{\frac{|A|}{2}} u_A e_A$ and $|u|$ for its module, where $|A|$ is the cardinality of index set A. Define $|u|^2 = \sum_A |u_A|^2$; $\bar{u}$ its conjugate with $\bar{u} = (u^*)'$, where $u^* = \sum_A (-1)^{\frac{|A|(|A|-1)}{2}} u_A e_A$, and $u' = \sum_A (-1)^{|A|} u_A e_A$. For u, $v \in \mathcal{A}$, we have

$$|u + v| \leq |u| + |v|, |uv| \leq 2^n |u||v|.$$

Let D be a region in $\mathbb{R}^{n+1}$. For a differentiable function $f : D \to \mathcal{A}$ with $f(x) = \sum_A f_A(x) e_A$, we say f is a *regular function* if

$$\bar{\partial} f = \sum_{i=0}^{n} e_i \frac{\partial f}{\partial x_i} = \sum_{i=0}^{n} \sum_{A} e_i e_A \frac{\partial f_A}{\partial x_i} = 0,$$

and a *k-regular function* if $\bar{\partial}^k f = 0$, where the operator $\bar{\partial} = \sum_{j=0}^{n} \frac{\partial}{\partial_{x_j}} e_j$. Let $\Omega \subset \mathbb{R}^{n+1}$ be a unbounded domain with smooth oriented Liapunove boundary $\partial\Omega$, and Ω^c, the complementary set of Ω containing a non-empty open set. We denote the bounded Hölder continuous function on $\partial\Omega$ in order of $\beta (0 < \beta < 1)$ by $H(\partial\Omega, \beta)$. For $f \in H(\partial\Omega, \beta)$, we define its norm by

$$\|f\|_\beta = \sup_{t \in \partial\Omega} |f(t)| + \sup_{t_1 \neq t_2} |\frac{f(t_1) - f(t_2)}{t_1 - t_2}|.$$

Then $H(\partial\Omega, \|\cdot\|_\beta)$ is a Banach space. And for f, $g \in H(\partial\Omega, \|\cdot\|_\beta)$, we have

$$\|f + g\|_\beta \leq \|f\|_\beta + \|g\|_\beta, \ \|fg\|_\beta \leq 2^n \|f\|_\beta \|g\|_\beta.$$

2. Main Result

In what follows, we denote by Ω a unbounded domain in $\mathbb{R}^{n+1}$ with smooth oriented Liapunove boundary $\partial\Omega$, and a non-empty open set in Ω^c. We first give the boundary value problem with Haseman shift for k-regular function.

Definition 2.1. Let $a(t), b(t), c(t), d(t)$, $g(t)$ be functions on $\partial\Omega$, and $d(t)$ an isomorphic map on $\partial\Omega$. Write $\Omega^+ = \Omega$, $\Omega^- = \mathbb{R}^{n+1}\backslash\bar{\Omega}$ with $\bar{\Omega} = \Omega \cup \partial\Omega$. If there exists some function ϕ such that

1) ϕ is a k-regular function on $\Omega^{\pm}$;

2) ϕ is continuous on $\mathbb{R}^{n+1}$, and $\phi^-(y) = 0$;

3)

$$\begin{cases} a(t)\phi^+(t) + b(t)\phi^+(d(t)) + c(t)\phi^-(t) = g(t) \\ a(t)(\dfrac{\bar{\partial}\phi(t)}{\bar{\partial}t})^+ + b(t)(\dfrac{\bar{\partial}\phi(d(t))}{\bar{\partial}t})^+ + c(t)(\dfrac{\bar{\partial}\phi(t)}{\bar{\partial}t})^- = g(t) \\ \quad\vdots \\ a(t)(\dfrac{\bar{\partial}^{k-1}\phi(t)}{\bar{\partial}t^{k-1}})^+ + b(t)(\dfrac{\bar{\partial}^{k-1}\phi(d(t))}{\bar{\partial}t^{k-1}})^+ + c(t)(\dfrac{\bar{\partial}^{k-1}\phi(t)}{\bar{\partial}t^{k-1}})^- = g(t). \end{cases} \tag{1}$$

Then we say ϕ is a solution to the boundary problem with Haseman-shift. And this problem is also called boundary problem with Haseman-shift for k-regular function on unbounded domains.

The following lemmas are borrowed from [Yang et , 2008]:

Lemma 2.1. Let f be a k-regular function on $\partial\Omega$. Then for $x \in \Omega$, we have

$$f(x) = \int_{\partial\Omega} L(u,x)n(u)[\sum_{j=0}^{k-1} \frac{(-1)^j}{j!}(u_0 - x_0)^j \bar{\partial}_u^j f(u)]d_{s(u)},$$

where $u = (u_0, u_1, \cdots, u_n)$, $d_{s(u)}$ is Lebesgue measure on $\partial\Omega$, and $n(u)$ is the unit vector in $\partial\Omega$'s normal direction,

$$L(\xi, x) = \frac{1}{w_{n+1}}(\frac{\bar{\xi} - \bar{x}}{|\xi - x|^{n+1}} - \frac{\bar{\xi} - \bar{y}}{|\xi - y|^{n+1}}), \quad y \in \bar{\Omega}^c, \xi \in \partial\Omega$$

is the modified Cauchy kernel on unbounded domain $\partial\Omega$.

Lemma 2.2 Let ϕ be a k-regular function on Ω. Then the Cauchy type integral of k-regular function is given by

$$\phi(x) = \int_{\partial\Omega} L(u,x)n(u)[\sum_{j=0}^{k-1} \frac{(-1)^j}{j!}(u_0 - x_0)^j \varphi_j(u)]d_{s(u)}, \tag{2}$$

where $\varphi_j \in L^\infty(|x-y|^{-s})$, that is, the weighted Hardy space, which can be referred to [Xu et al., 2008], and

$$\frac{(\bar{\partial}^m\phi)^+}{\bar{\partial}^m t} = \frac{\varphi_m(t)}{2} + \int_{\partial\Omega} L(u,x)n(u) \times [\sum_{j=m}^{k-1} \frac{(-1)^{j+m}}{(j-m)!}(u_0 - t_0)^{j-m}\varphi_j(u)]d_{s(u)}, \tag{3}$$

$$\frac{(\bar{\partial}^m\phi)^-}{\bar{\partial}^m t} = -\frac{\varphi_m(t)}{2} + \int_{\partial\Omega} L(u,x)n(u) \times [\sum_{j=m}^{k-1} \frac{(-1)^{j+m}}{(j-m)!}(u_0 - t_0)^{j-m}\varphi_j(u)]d_{s(u)}, \tag{4}$$

for $m = 0, 1, \cdots, k-1$.

Similar to the Theorem 1 in [Xu et al., 2008], we have

Lemma 2.3. Let $m = 0, 1, \cdots, k-1$. For φ_m in Lemma 2.2, write $\theta_m\varphi_m = \frac{\varphi_m}{2} - P_m\varphi_m$ with

$$P_m\varphi_m(x) = \int_{\partial\Omega} L(u,x)n(u) \times [\sum_{j=m}^{k-1} \frac{(-1)^{j+m}}{(j-m)!}(u_0 - t_0)^{j-m}\varphi_j(u)]d_{s(u)}.$$

Then there exists some positive constant C_m such that

$$||\theta_m||_\beta \le C_m.$$

Theorem 2.1. Let $\Omega \subset \mathbb{R}^{n+1}$, $d = d(t)$ in Definition 2.1 satisfy Lipschitz condition, and $a(t), b(t), c(t), g(t) \in H(\partial\Omega, \beta)$. If

$$\|a + c\|_\beta < \varepsilon < 1, \|b\|_\beta < \varepsilon < 1, \|1 + a\|_\beta < \varepsilon < 1, 0 < u = 2^{n+1}(\max\{C_m : m = 0,\ 1\ \cdots,\ k-1\} + 1)\varepsilon < 1,$$

$$\|g\|_\beta < M(1-u),$$

where C_m is in Lemma 2.3, then the solution of the m-th equation in (1) is given by

$$\phi_m(x) = \int_{\partial\Omega} L(u, x)n(u) \times [\sum_{j=m}^{k-1} \frac{(-1)^{j+m}}{(j-m)!}(u_0 - t_0)^{j-m}\varphi_j(x)]ds_u,$$

for $m = 0,\ 2,\ \cdots,\ k-1$.

Proof. Write

$$\widetilde{P_m}\varphi_m(x) = P_m\varphi_m(d(x)) \text{ and } \widetilde{\varphi_m}(x) = \varphi(d(x)).$$

Substituting (3) and (4) into (1), we have

$$a(\frac{\varphi_m}{2} + P_m\varphi_m) + b(\frac{\widetilde{\varphi}_m}{2} + \widetilde{P}_m\varphi_m) + c(-\frac{\varphi_m}{2} + P_m\varphi_m) = g(t),$$

for $m = 0,\ 1\ \cdots, k-1$, that is,

$$(a + c)(-\frac{\varphi_m}{2} + P_m\varphi_m) + b(-\frac{\widetilde{\varphi}_m}{2} + \widetilde{P}_m\varphi_m) + (1 + a)\varphi_m + b\widetilde{\varphi}_m - g(t) = \varphi_m. \tag{5}$$

Write

$$F\varphi_m = (a + c)(-\frac{\varphi_m}{2} + P_m\varphi_m) + b(-\frac{\widetilde{\varphi}_m}{2} + \widetilde{P}_m\varphi_m) + (1 + a)\varphi_m + b\tilde{\varphi}_m - g,$$

then (5) is

$$F\varphi_m = \varphi_m.$$

Let

$$T = \{\varphi_m, m = 0,\ 1\ \cdots, k-1 | \varphi_m \in H(\partial\Omega, \beta), \|\varphi_m\|_\beta \le M, \}.$$

Then T is a closed subspace of Banach space $H(\partial\Omega, \beta)$. Since

$$\begin{aligned}
\|F\varphi_m\|_\beta \quad & \le 2^n\|a + c\|_\beta\|\theta_m\varphi_m\|_\beta + 2^n\|1 + a\|_\beta\|\varphi_m\|_\beta + 2^n\|b\|_\beta\|\widetilde{\theta_m}\varphi_m\|_\beta + 2^n\|b\|_\beta\|\tilde{\varphi}_m\|_\beta + \|g\|_\beta \\
& \le 2^nC_m\varepsilon\|\varphi_m\|_\beta + 2^n\varepsilon\|\phi_m\|_\beta + 2^nC_m\varepsilon\|\varphi_m\|_\beta + 2^n\varepsilon\|\phi_m\|_\beta + \|g\|_\beta \\
& = 2^{n+1}(C_m + 1)\varepsilon\|\varphi_m\|_\beta + \|g\|_\beta \\
& \le u\|\varphi_m\|_\beta + \|g\|_\beta \\
& \le M,
\end{aligned} \tag{6}$$

F is a map on T. For $\varphi'_m, \varphi''_m \in H(\partial\Omega, \beta)$, we have

$$\begin{aligned}
\|F\varphi'_m - F\varphi''_m\|_\beta \quad \le \quad & \|(a + c)[-\frac{1}{2}(\varphi'_m - \varphi''_m) + P(\varphi'_m - \varphi''_m)]\|_\beta + \|b[-\frac{1}{2}(\widetilde{\varphi_m}' - \widetilde{\varphi_m}'') \\
+ \quad & \widetilde{P_m}(\varphi'_m - \varphi''_m)]\|_\beta + \|(1 + a)(\varphi'_m - \varphi''_m)\|_\beta + \|b(\widetilde{\varphi_m}' - \widetilde{\varphi_m}'')\|_\beta \\
\le \quad & u\|\varphi'_m - \varphi''_m\|_\beta
\end{aligned} \tag{7}$$

with $0 < u < 1$, and thus T is a compression map on T. So, there is an unique fixed φ_m such that $F\varphi_m = \varphi_m$ by fixed point theorem, which implies that

$$\phi_m(x) = \int_{\partial\Omega} L(u, x)n(u) \times [\sum_{j=m}^{k-1} \frac{(-1)^{j+m}}{(j-m)!}(u_0 - t_0)^{j-m}\varphi_j(x)]ds_u.$$

is the unique solution for the m-th equation in (1). This gives the proof.

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