Asymptotic Analysis of Differential Semblance for Layered Acoustics

William. W. Symes *

ABSTRACT

Differential semblance velocity estimators have well-defined and smooth high frequency asymptotics. A version appropriate for analysis of CMP gathers and layered acoustic models has no secondary minima within a sublevel set of velocity profiles, independent of data frequency content, unlike other estimators such as stack power and output least squares. The sublevel set is characterized by the interaction between reflector density and velocity variability.

Introduction

The inverse problem of reflection seismology includes as an important subproblem the estimation of compressional wave velocity as a function of position, on a scale of hundreds of meters or kilometers. This estimation, often called "velocity analysis", is critical in seismic data processing. Attempts to treat it as a data-fitting problem have largely foundered on the strongly nonlinear relation between seismic data and the long scale component of wave velocity. Various objective functions, including notably output least squares, have proven to have great numbers of spurious stationary points, both close to and far away from a reasonable model estimate, so that only global optimization methods are likely to be successful (Gauthier et al., 1986; Scales et al., 1991; Sen and Stoffa, 1991b; Sen and Stoffa, 1991a; Chauris, 2000). Since global optimization on the scale of field seismograms and reasonable subsurface models entails infeasible computational cost, these approaches have had little impact on seismic data processing practice.

Differential semblance (Symes, 1986) is an alternate measure of the fit of a long-scale velocity model to data, which appears not to have the pathological mathematical features of other misfit measures, in particular spurious stationary points. It has been used to extract reasonable estimates of long scale wave velocity from both synthetic and

*The Rice Inversion Project, Department of Computational and Applied Mathematics, Rice University, Houston TX 77251-1892 USA, email symes@caam.rice.edu

field seismograms, using only local (Newton-like) optimization, and appears in principle feasible on field scale (Symes and Carazzone, 1991a; Symes and Versteeg, 1993; Kern and Symes, 1994; Araya et al., 1996; Symes, 1997; Minkoff and Symes, 1997; Chauris et al., 1998; Chauris and Noble, 2001; Mulder and ten Kroode, 2001).

The purpose of this paper is to give an asymptotic analysis of the version of differential semblance based on layered acoustics. This is the simplest specialization of elastodynamics able to describe principal features of field data, at least in some cases. The principal result is roughly this: if the velocity is restricted to vary on a scale sufficiently long relative to the density of reflected wave energy in time, and if the data is consistent with this assumption, then (i) the only zero of the differential semblance function occurs at the correct velocity, and (ii) the only stationary point with a sufficiently small value of the objective occurs at the correct velocity.

This paper approximates the seismic response of a layered model by linearization about the long-scale model, and uses high frequency asymptotics to approximate the reflection seismogram and many associated quantities. A related investigation using high frequency asymptotics of the linearized response and approximate kinematics (the "hyperbolic moveout" approximation) was able to establish a stronger result, without any assumption about a relation of length scales between data and velocity distribution: that the only stationary points are global minima (Symes, 1999). The approach of (Symes, 1999) is essentially followed here, with amendments due to the use of exact rather than approximate kinematics. On the other hand the paper (Symes, 1991) reached very similar conclusions to those presented here, without linearization or high frequency asymptotics, but using plane wave reflection data and theory available only for plane waves in layered media. The development presented here avoid these restrictions, and hopefully will serve as a model for an analysis of multidimensional (nonlayered) differential semblance, which has so far been investigated only numerically (Symes and Carazzone, 1991a; Symes and Versteeg, 1993; Kern and Symes, 1994; Chauris et al., 1998; Chauris and Noble, 2001; Mulder and ten Kroode, 2001).

The paper begins by defining the convolutional (linearized asymptotic) model for layered acoustics, and discussing various types of error inherent in this approximation, the domain of the data, and natural admissible model sets. This groundwork supports an asymptotic analysis of the differential semblance objective, which reveals that in the case of noise free data it is essentially a data-weighted mean square error in RMS ray parameter (a geometric optics quantity associated with traveltimes). This observation leads directly to the main results, and demonstrates the essentially tomographic nature of differential semblance velocity estimation. A crucial step is the derivation of hyperbolic systems linking various geometric optics quantities directly to the wave equation coefficients; these systems may have some interest in themselves.

The Convolutional Model for Laterally Homogenous Acoustics

Linearization of the acoustic model for a layered fluid and application of high frequency asymtotics leads to the *convolutional model* of reflection seismograms. The convolutional model is one of the simplest models of the reflection process within which to pose the velocity analysis problem. This model has been the basis for much data processing since at least the 1950's, and is described and used extensively in every textbook on the subject (Dobrin, 1976; Robinson and Treitel, 1980). I present it here as the result of various approximations to the basic physics of seismic waves. The few derivations of the convolutional model from first principles to have appeared in print are unfortunately marred by misprints or outright mistakes, so I have included a complete derivation in an appendix.

Remark: A similar model for plane wave traces is almost equally simple, and was the subject of earlier work on differential semblance (Symes and Carazzone, 1991b; Minkoff and Symes, 1997). However synthesis of accurate plane wave traces is a nontrivial task. Accordingly the version of the model developed here uses offset domain data.

To some approximation, seismic waves obey the laws of linear elastodynamics, with material properties (density, Hooke tensor,...) depending on position $\mathbf{x} \in \mathbf{R}^3$. A *layered* elastic material has material properties depending only on the depth $x_3 = z \geq 0$. Constant density layered acoustic materials (fluids) have vanishing shear moduli and are characterized completely by the velocity $v(z)$. The acoustic wave equation governs pressure fluctuations $p(t, \mathbf{x})$ about equilibrium in such a material:

$$\frac{1}{v^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = f$$

in which $f(t, \mathbf{x})$ is the divergence of a body force density representing energy inputs ("sources"). An *isotropic point* source takes the form $f(t, \mathbf{x}) = w(t)\delta(\mathbf{x} - \mathbf{x}_s)$, and is a crude approximation to actual source dynamics. The *source position* $\mathbf{x}_s$ is an auxiliary datum of the seismic experiment, as is the *source time function* $w(t)$.

While the surface of the Earth acts as a significant boundary for seismic waves, it is possible to some extent to process seismic measurements to remove the effects of this boundary, so this paper will treat the seismic wavefield as defined in all of $\mathbf{R}^3$.

Idealized "2D" seismic data is the measurement of the pressure fluctuations at a set of *receiver positions* $\mathbf{x}_r$ at or near the surface, which are also auxiliary data of the seismic experiment. The rotation and translation invariance of layered materials leads to the conclusion that the simulated seismic data can be a function only of the *offset*, i.e. the source - receiver distance. Therefore a complete data set consists of measurements at receiver positions along the half x-axis through the source position: $\mathbf{x}_r = \mathbf{x}_s + (x, 0, 0)^T$ for $x \geq 0$. The acoustic model predicts that this idealized "2D" seismic data depends on velocity v:

$$\mathcal{F}[v](t, x) = p(t, \mathbf{x}_s + (x, 0, 0)^T)$$

The *forward map* $\mathcal{F}$ relating velocity profile to data is nonlinear, as it expresses the relation between the coefficients of a linear equation and its solution. Linearization expresses the velocity profile as $v(1+r)$, in terms of a *reference velocity*, still denoted by v, and a relative perturbation $r = \frac{\delta v}{v}$ called the *reflectivity*. A perturbation calculation leads to a formal derivative or *linearized forward map*

$$D\mathcal{F}[v]r \simeq \mathcal{F}[v(1+r)] - \mathcal{F}[v]$$

which is related to the solution of a *perturbational wave equation* in the same way that $\mathcal{F}$ is related to the pressure fluctuation field. It is widely believed that this approximation is accurate when (i) v is relatively smooth, and (ii) r is oscillatory. So far only numerical evidence exists to support this presumption, except for the rigorous treatment of layered acoustics in (Lewis and Symes, 1991).

Application of yet another layer of perturbation theory, this time in the form of high-frequency asymptotics, leads directly to the convolutional model. Fortunately, the frequency or scale dichotomy apparently responsible for the success of the linearization approximation is also a necessary condition for success of high-frequency asymptotics.

The *two-way time* function $T(z,x)$ plays a critical role in the description of this approximation. It is related to the solution of the point source problem for the eikonal equation

$$\|\nabla\tau(z,x)\| = \frac{1}{v(z)},\ \tau(z,x) - \frac{\sqrt{z^2+x^2}}{v(0)} = o(\sqrt{z^2+x^2})$$

by $T(z,x) = 2\tau(z,x/2)$.

Denote by $t_{\max}$ and $x_{\max}$ the maximum time and offset respectively. That is, data will be regarded as functions on $[0,t_{\max}]\times[0,x_{\max}]$. We assume the existance of a function $t_m : [0,x_{\max}] \to \mathbf{R}$ so that $z \mapsto T(z,x)$ is invertible on $[t_m(x), t_{\max}]$, with inverse $Z(\cdot,x) : [t_m(x),t_{\max}] \to \mathbf{R}$. Relations between $v, t_{\max}$ and $x_{\max}$ which assure existance of a suitable t_m will be discussed in the next section.

Since T is smooth, invertibility is equivalent to the assumption that the logarithm of the *stretch factor*

$$s(t,x) = \frac{\partial Z}{\partial t}(t,x) = \left(\frac{\partial T}{\partial z}(Z(t,x),x)\right)^{-1} \tag{1}$$

is bounded on

$$R \equiv \{(t,x) : x\in[0,x_{\max}], t\in[t_m(x),t_{\max}]\} \tag{2}$$

Let $\phi_0 \in C_0^\infty((0,t_{\max})\times(0,x_{\max}))$ be a cutoff function or *mute*, with $\mathrm{supp}(\phi_0)\subset R$.

The convolutional model for the linearized forward map is:

$$D\mathcal{F}[v]r \simeq F[v]r(t,x) = \phi_0(t,x)w *_t [a(\cdot,x)r(Z(\cdot,x),x)](t,x) \tag{3}$$

where a, an *amplitude* derived from geometric acoustics, is also a functional of v.

The developments to follow will also assume that a process ("source signature deconvolution") has been applied to the data, so that $w \simeq \delta$. For realistic modeling, w must have Fourier transform of (essentially) compact support, so this assumption not really defensible. Nonetheless I shall assume it: the resulting errors do not seem to make the theory diverge too far from computational practice (Araya et al., 1996; Symes, 1998; Gockenbach and Symes, 1999). That is, replace (3) by

$$F[v]r(t,x) = \phi_0(t,x)a(t,x)r(Z(t,x),x) \tag{4}$$

Note that many of the assumptions introduced to arrive at the convolutional model - seismic waves as acoustic waves in a fluid, neglect of boundary conditions at the Earth's surface, isotropic point model for the energy source, source signature deconvolution, continuous sampling along a line to represent the data, linearization and high frequency asymptotics, and above all the layered medium assumption - are drastic approximations. Each and every one contradicts well-known features of field data. While such data can be preprocessed to remove inconsistencies with this model to some extent, the reader should bear in mind that the convolutional model is an oversimplification of seismic data formation.

The following computations will introduce yet more sources of asymptotic error - and only asymptotic error. In other words, operators will be replaced with others that differ by relatively smoothing perturbations. Rather than carry along a variety of expressions that accumulate this repeated dropping of a smoothing error, I will use the symbol "$O(\lambda)$" to suggest proportionality of the asymptotic error to a dominant wavelength in the data: $A = B + O(\lambda)$ means $A = B + K$ where K is smoothing relative to A and B. Of course statements like this are meaningful when A and B are operators defined by oscillatory integrals, and elliptic at least microlocally.

For example, equation (4) should really be regarded as an asymptotic approximation to a linearized fundamental solution of the wave equation:

$$F[v]r(t,x) = a(t,x)r(Z(t,x),x) = D\mathcal{F}[v]r + O(\lambda)$$

Admissible Models

The developments to follow require the restriction of v to *admissible sets* $\mathcal{A}$ of models, on which the convolutional model as defined above is reasonably well behaved. A simple way to define such sets is to impose simple bounds on the derivatives of v: for each $k \in \mathbf{Z}_+$, there exists $C_k \geq 0$ for which

$$\left|\frac{d^k}{dz^k}(\log v(z))\right| \leq C_k, z \in \mathbf{R}_+$$

That is, $\log v$ is restricted to a bounded subset of $C^\infty(\mathbf{R}_+)$.

For $k = 0$, I will be a bit more restrictive: impose smooth upper and lower "envelope" velocities as hard constraints: $v_{\rm min}(z) \le v(z) \le v_{\rm max}(z), z \in \mathbf{R}_+$. It is natural to assume that the velocity is known at and near the surface, so assume the existance of $z_{\rm min} > 0$ so that that $v_{\rm min}(z) = v_{\rm max}(z) \equiv v_0$ for $0 \le z \le z_{\rm min}$. These bounds derive from geophysical measurements and general knowledge about rock physics (for example $0 \le z \le z_{\rm min}$ could represent the water column in simulation of a marine survey), so should be regarded as distinct from the bounds implied by the first condition (membership in a bounded set in C^∞).

It will also be useful to have a constant velocity profile available, so assume also that $v(z) \equiv v_0 \in \mathcal{A}$, i.e. that $v_{\rm min}(z) \le v_0 \le v_{\rm max}(z)$, $z \in \mathbf{R}_+$.

Note that admissible sets as define here are convex.

Neither the traveltime $T(z,x)$ nor the other quantities introduced in the previous section are well-defined globally. For each choice of admissible set $\mathcal{A}$, it is possible to choose domains of definition uniformly for $v \in \mathcal{A}$, as follows.

Geometric optics implies the following formulae for the time $T(z,p)$ and offset $X(z,p)$ of a ray with ray parameter (slowness) p reflected from a horizontal surface at depth z:

$$X(z,p) = 2\int_0^z d\zeta\, \frac{v(\zeta)p}{\sqrt{1 - v^2(\zeta)p^2}},\ T(z,p) = 2\int_0^z d\zeta\, \frac{1}{v(\zeta)\sqrt{1-v^2(\zeta)p^2}} \tag{5}$$

(see (Aki and Richards, 1980), section 12.1). These formulae presume that $v(\zeta)p < 1$ for $0 \le \zeta \le z$. Choose $p_{\rm max}$ so that $v(z)p_{\rm max} < 1$ for $z \in \mathrm{R}_+, v \in \mathcal{A}$. Such a choice is possible in view of the bounds defining $\mathcal{A}$. Then (5) defines a C^∞ map $\Xi[v] : \mathbf{R}_+ \times [0, p_{\rm max}] \to \mathbf{R}^2$.

Claim: Set

$$x_{\rm max} = \frac{z_{\rm min} v_0 p_{\rm max}}{\sqrt{1 - v_0^2 p_{\rm max}^2}}$$

$$t_m(x) = \frac{\sqrt{z_{\rm min}^2 + x^2}}{v_0},\ 0 \le x \le x_{\rm max}$$

$$t_{\rm max} \ge t_m(x_{\rm max})$$

with $t_{\rm max}$ otherwise arbitrary. Let R be defined as in (2). Then $R \subset \cap_{v \in \mathcal{A}} (\mathrm{Range}\Xi[v])$, and $\Xi[v]$ is a diffeomorphism on the preimage of R. Moreover, decompose

$$\partial R = \Gamma_{\rm in} \cup \Gamma_0 \cup \Gamma_{\rm out}$$

with

$$\Gamma_{\rm in} = \{(t_m(x), x) : 0 \le x \le x_{\rm max}\}$$

$$\Gamma_0 = \{(t_m(x), x) : 0 \le x \le x_{\rm max}\}$$

$$\Gamma_{\rm out} = \{(t, x_{\rm max}) : t_m(x_{\rm max}) \le t \le t_{\rm max}\} \cup \{(t_{\rm max}, x) : 0 \le x \le x_{\rm max}\}$$

then $\Gamma_{\rm in}$ is an inflow boundary, and $\Gamma_{\rm out}$ is an outflow boundary, for the ray vector field

$$\frac{\partial}{\partial z} = \frac{\partial T}{\partial z}\frac{\partial}{\partial t} + \frac{\partial X}{\partial z}\frac{\partial}{\partial x} \tag{6}$$

uniformly for $v \in \mathcal{A}$.

Proof: It is straightforward to see that $R \subset \mathrm{Range}(\Xi[v_0])$. To see the same for arbitrary $v \in \mathcal{A}$, construct the homotopy in $\mathcal{A}$:

$$v_\sigma = (1-\sigma)v_0 + \sigma v$$

Pick $(t, x) \in R$; it is required to show that the equations

$$t = T[v_\sigma](z_\sigma, p_\sigma),\ \ x = X[v_\sigma](z_\sigma, p_\sigma)$$

have a unique solution (z_σ, p_σ) for all $\sigma \in [0, 1]$, this being clear for $\sigma = 0$.

Suppose (z_σ, p_σ) is a solution. Since

$$x = X(z_\sigma, p_\sigma) = \frac{z_{\rm min} v_0 p_\sigma}{\sqrt{1 - v_0^2 p_\sigma^2}} + \int_{z_{\rm min}}^{z_\sigma} dz \frac{z v_\sigma(z) p_\sigma}{\sqrt{1 - v_\sigma(z)^2 p_\sigma^2}}$$

$$\le x_{\rm max} = \frac{z_{\rm min} v_0 p_{\rm max}}{\sqrt{1 - v_0^2 p_{\rm max}^2}}$$

and

$$p \mapsto \frac{zvp}{\sqrt{1 - v^2 p^2}}$$

is increasing, it follows that $p_\sigma \le p_{\rm max}$, i.e. any path segment $\sigma \mapsto (z_\sigma, p_\sigma)$ stays within the strip $0 \le p \le p_{\rm max}$. The inverse Jacobian of Ξ, and all partial derivatives up to any finite order, are bounded uniformly in $v \in \mathcal{A}$ and over the half-strip, whence follows the existance of $\epsilon > 0$ for which the implicit function theorem guarantees a unique solution on $(\sigma - \epsilon, \sigma + \epsilon) \cap [0, 1]$ given a solution at σ. Since such a solution stays in the strip, it follows that the end is reached.

The inflow boundary property is clear, as the rays are the images of $p =$ const lines in the strip, and $\Gamma_{\rm in}$ is the image of $z = z_{\rm min}$. Similarly, since $dX/dz > 0$ and $dT/dz > 0$ throughout the strip, any ray which reaches either part of $\Gamma_{\rm out}$ makes a positive inner product with the outward pointing normal. **QED**

An important consequence is that the mute $\phi_0 \in C_0^\infty(\mathbf{R}^2)$ may be chosen uniform over $v \in \mathcal{A}$, and that uniform bounds exist for every value of the stretch factor $s(t, x)$.

All of these operators, suppressed smoothing errors included, depend parametrically on v. The estimates which establish the relative smoothing are uniform as v ranges over admissible classes of velocities $\mathcal{A}$. Therefore the errors are effectively small, not just smooth, in view of the assumed frequency dichotomy.

Asymptotic Approximation of Differential Semblance

According to the prescription given in (Symes, 2001), construction of the DS functional requires operators with canonical relation the same as, and inverse to, that of the linearized forward map, *restricted to each data bin.* In the case of CMP gathers considered here, a data bin is simply a trace, indexed by offset x. Thus the operator with canonical relation inverse to (respectively the same as) that of the linearized forward map is a prestack migration (respectively demigration) operator, yielding (respectively acting on) *x- dependent reflectivity functions* $r(z,x)$ (or image volumes).

In the development to follow, the asymptotic approximation (4) is used in place of the linearized forward map. This either reflects actual computation, if the asymptotic approximation is implemented numerically (as in (Araya et al., 1996; Symes, 1998; Gockenbach and Symes, 1999)) or is an approximation if some other technique is used (Symes, 1993; Symes and Versteeg, 1993; Kern and Symes, 1994).

To find an operator with the inverse canonical relation, it's merely necessary to use the inverse change of variables: set

$$G[v]d(z,x) = g(z,x)\phi_0(T(z,x),x)d(T(z,x),x) \tag{7}$$

where the amplitude $g(z,x)$ is at your disposal. Similarly

$$B[v]r(t,x) = \phi_0(t,x)b(t,x)r(Z(t,x),x) \tag{8}$$

with $b(t,x)$ equally arbitrary. The operator measuring semblance differentially is

$$\partial = \frac{\partial}{\partial x}$$

Select $\phi \in C_0^\infty(R)$ so that $\phi = 0$ on $\mathrm{supp}(1-\phi_0)$, and choose H to be a square root of the positive definite Helmholtz operator $I - \nabla^2_{t,x}$ with any convenient boundary conditions, say Dirichlet on $[0,t_{\max}] \times [0,x_{\max}]$. The *differential semblance operator* $W : \mathcal{A} \to OPS^0$ is defined as (Symes, 2001)

$$W[v] = \phi H \phi_0 B[v] \partial G[v]$$

and the basic differential semblance function $J_0[v,d]$ by

$$J_0[v,d] = \frac{1}{2}\|W[v]d\|^2_{L^2(R)}$$

The principal symbol of $W[v]$ will play a useful role in the sequel. To compute it, note that

$$\phi(t,x)H\phi_0 B[v]\partial G[v]d(t,x) = \phi(t,x)b(t,x)\left[\frac{\partial}{\partial x}[g(z,x)\phi_0(T(z,x),x)d(T(z,x),x)]\right]_{z=Z(t,x)}$$

$$= \phi(t,x)b(t,x)g(Z(t,x),x)\left(\frac{\partial T}{\partial x}(Z(t,x),x)\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)d(t,x)+O(\lambda)$$

$$= \phi_1(t,x)\left(p(t,x)\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)d(t,x)+O(\lambda)$$

where

$$P(t,x)=\frac{\partial T}{\partial x}(Z(t,x),x)$$

is the arrival (horizontal) slowness of the reflected ray passing offset x at time t,

$$\phi_1(t,x)=\phi(t,x)b(t,x)g(Z(t,x),x),$$

and the elided terms involve derivatives of b and g, but do *not* involve derivatives of the data d.

Note that $(t,x)\mapsto(Z(t,x),P(t,x))$ is simply the inverse diffeomorphism of the map defined in (5).

Applying the inverse square root of the Helmholtz operator produces

$$H\phi B[v]\partial G[v]d = H\phi_1\left(\frac{\partial d}{\partial x}+P\frac{\partial d}{\partial t}\right)+O(\lambda)$$

whence the symbol of $W[v]$ may easily be extracted.

The ray slowness $P(t,x)$ is locally a smooth function of the velocity v in any fixed open subset of the mute zone - here "smooth" means "when restricted to the intersection of $\mathcal{A}$ and any finite dimensional subspace of C^∞". Assume that the C^∞ multipliers b and g are smooth in their dependence on v also (for example, constant). Then $J_0[v,d]$ is a smooth function of $(v,d)\in\mathcal{A}\times L^2(R)$ as well.

Noise Free Data

Assume that the data d are model-consistent, that is

$$d(t,x)=a^*(t,x)r^*(Z^*(t,x))+O(\lambda)$$

for target offset independent reflectivity r^* and velocity v^*.

Note that

$$0=\frac{\partial}{\partial x}T(Z(t,x),x)=\frac{\partial T}{\partial z}(Z(t,x),x)\frac{\partial Z}{\partial x}(t,x)+\frac{\partial T}{\partial x}(Z(t,x),x)$$

so

$$\frac{\partial Z}{\partial x}(t,x)=-s(t,x)P(t,x)$$

(s being the stretch factor, defined above). Thus

$$\frac{\partial}{\partial x} r^*(Z^*(t,x)) = -s^*(t,x)P^*(t,x)(\frac{\partial r^*}{\partial z})(Z^*(t,x))$$

(s^* is the stretch factor belonging to v^*) whence

$$B[v]\partial G[v]d(t,x) = b(t,x)g(Z(t,x),x)\left(\frac{\partial}{\partial x} + P(t,x)\frac{\partial}{\partial t}\right)(a^*(t,x)r^*(Z^*(t,x)))$$

$$= b(t,x)g(Z(t,x),x)s^*(t,x)(P(t,x) - P^*(t,x))a^*(t,x)\frac{\partial r^*}{\partial z}(Z^*(t,x)) + O(\lambda)$$

According to the calculus of pseudodifferential operators,

$$H\phi B[v]\partial G[v]d =$$
$$(I - \nabla^2)^{-\frac{1}{2}}\phi_1\left(s^*(P - P^*)a^*\frac{\partial r^*}{\partial z}(Z^*)\right) + O(\lambda)$$
$$= (I - \nabla^2)^{-\frac{1}{2}}\phi_1\left(s^*(P - P^*)\frac{\nabla Z^* \cdot \nabla}{\nabla Z^* \cdot \nabla Z^*}d\right) + O(\lambda)$$
$$= W(t,x,\partial_t,\partial_x)d(t,x)$$

in which $W \in OPS^0$ has principal symbol

$$\phi_1 \frac{s^*(P - P^*)}{\sqrt{1 + s^{*,2}(1 + p^{*,2})}}$$

[Note that $H\phi B\partial G$ is equivalent to a ΨDO whose principal symbol is a multiplier only because noise free data $d(t,x) = a^*(t,x)r^*(Z^*(t,x))$ is conormal.] All of the factors this expression are *a priori* independent of v, except for ϕ_1. Accordingly assume from now on that $b(t,x)$ and $g(z,x)$, heretofore arbitrary, are chosen in such a way that

$$\phi_1(t,x) = \phi(t,x)b(t,x)g(Z(t,x),x)$$

is *independent of* v. This can be accomplished in a number of ways: since ϕ may be chosen independent of v, as noted above, one obvious way to achieve this property is to choose $b \equiv g \equiv 1$.

Granted this additional constraint,

$$J_0[v,d] = \int_R dt\,dx\, B^*(t,x)(P(t,x) - P^*(t,x))^2[r^*(Z^*(t,x))]^2 + O(\lambda) \tag{9}$$

where

$$B^*(t,x) = (a^*)^2\phi^2\frac{s^{*,2}}{1 + s^{*,2}(1 + p^{*,2})}$$

is independent of v, i.e. depends only on v^* and $\mathcal{A}$, and "$+O(\lambda)$" means, for this and other quadratic forms to follow, "differs by a quadratic form $< d, Kd >$ with $K \in OPS^{-1}$".

Analysis of Stationary Points

As $J_0[v,d]$ has just been revealed to be a weighted mean-square error in slowness $P(t,x)$, the first task is to establish a usable relationship between P and v. As it turns out, a nonlinear hyperbolic system links these functions.

Define

$$u(z) = \frac{1}{v^2(Z(t,x))},\ U_0(t) = U(t,0)$$

Then constancy of P along characteristics means that the vector field defined in (6) annihilates it. Up to a factor, this is

$$U\frac{\partial P}{\partial t} + P\frac{\partial P}{\partial x} = 0 \tag{10}$$

The fact that v is a function of z implies

$$P\frac{\partial U}{\partial t} + \frac{\partial U}{\partial x} = 0 \tag{11}$$

Cauchy data on inflow boundary components are

$$U(t,0) = U_0(t),\ P(t_m(x),x) = \frac{x}{v_0^2 t} \tag{12}$$

Denote from now on $\Delta P = P - P^*$, $\Delta U = U - U^*$. Then it follows from (10,11) that

$$U\frac{\partial \Delta P}{\partial t} + P\frac{\partial \Delta P}{\partial x} + \frac{\partial P^*}{\partial t}\Delta U + \frac{\partial P^*}{\partial x}\Delta P = 0 \tag{13}$$

$$P\frac{\partial \Delta U}{\partial t} + \frac{\partial \Delta U}{\partial x} + \frac{\partial U^*}{\partial t}\Delta P = 0 \tag{14}$$

with Cauchy data

$$\Delta U(t,0) = \Delta U_0(t) = U_0(t) - U_0^*(t),\ \Delta P(t_m(x),x) = 0 \tag{15}$$

This linear hyperbolic system for $\Delta U, \Delta P$ is most easily analysed in the (z,p) coordinate system (for v), as the coordinate lines are characteristic: set

$$\Delta u(z,p) = \Delta U(T(z,p),X(z,p)) = u(z) - u^*(Z^*(T(z,p),X(z,p)))$$

$$\Delta p(z,p) = \Delta P(T(z,p),X(z,p)) = p - P^*(T(z,p),X(z,p))$$

$$\alpha(z,p) = v(z)(1-v^2(z)p^2)^{-\frac{1}{2}}\frac{\partial P^*}{\partial t}(T(z,p),X(z,p))$$

$$\beta(z,p) = v(z)(1-v^2(z)p^2)^{-\frac{1}{2}}\frac{\partial P^*}{\partial x}(T(z,p),X(z,p))$$

$$\gamma(z,p) = \left(\int_0^z d\zeta v(\zeta)(1-v^2(\zeta)p^2)^{-\frac{3}{2}}\right)\frac{\partial U^*}{\partial t}(T(z,p),X(z,p))$$

In (z,p) coordinates, (13,14,15) become

$$\frac{\partial \Delta p}{\partial z} + \alpha \Delta u + \beta \Delta p = 0,\ \ \frac{\partial \Delta u}{\partial p} + \gamma \Delta p = 0 \tag{16}$$

$$\Delta p(z_{\min}, p) = 0,\ \ 0 \le p \le p_{\max} \tag{17}$$

The domain for the system (16,17) is the preimage S of R under the map $\Xi : (z,p) \mapsto (T(z,p), X(z,p))$. S is a subset of the strip $\mathbf{R}_+ \times [0, p_{\max}]$; the "top" part of its boundary consists of $\{(z_{\min}, p) : 0 \le p \le p_{\max}\}$.

Define $\psi_0 \in C_0^\infty(S)$ by $\psi_0 = \phi_0(\Xi)$. Then integrating the second equation in (16) and using the first gives

$$\psi_0 \alpha \Delta u(\cdot, 0) = -\psi_0 \left(\frac{\partial \Delta p}{\partial z} + \beta \Delta p + \alpha \int_0^p \gamma \Delta p \right)$$

Further integration in p yields

$$g(z)\Delta u(z,0) =$$

$$-\int_0^{p_{\max}} dp\, \psi_0(z,p) \left(\frac{\partial \Delta p}{\partial z}(z,p) + \beta(z,p)\Delta p(z,p) + \alpha(z,p) \int_0^p dp' \gamma(z,p')\Delta p(z,p') \right) \tag{18}$$

in which

$$g(z) = \int_0^{p_{\max}} dp\, \psi_0(z,p)\alpha(z,p)$$

Now set $\psi = \phi(\Xi)$; recall that $\operatorname{supp}\phi \subset \{(t,x) : \phi_0(t,x) = 1\}$, so ψ stands in the same relation to ψ_0. The functions $\log(g), \alpha, \beta$, and γ are bounded uniformly over $\operatorname{supp}\psi$ and over $v, v^* \in \mathcal{A}$. Thus

$$\mathcal{I}\Delta U_0(t) \equiv \int_0^t dt\, \phi(t,0)\Delta U_0(t) = \int_0^{Z(t,0)} dz\, \psi(z,0) \frac{\partial T}{\partial z}(z,0)\Delta u(z,0)$$

$$= -\int_0^{Z(t,0)} dz\, \frac{\psi(z,0)}{g(z)} \frac{\partial T}{\partial z}(z,0) \int_0^{p_{\max}} dp\, \psi_0(z,p)$$

$$\left(\frac{\partial \Delta p}{\partial z}(z,p) + \beta(z,p)\Delta p(z,p) + \alpha(z,p) \int_0^p dp' \gamma(z,p')\Delta p(z,p') \right)$$

$$= \int_0^{p_{\max}} dp\, (\lambda \Delta p)(Z(t,0), p) + \int_0^{Z(t,0)} dz \int_0^{p_{\max}} dp\, (\mu \Delta p)(z,p) \tag{19}$$

in which $\lambda, \mu \in C_0^\infty(S)$ also lie in a bounded set determined by $\mathcal{A}$.

Integration and repeated use of Cauchy-Schwarz inequality yields

$$\|\mathcal{I}\Delta U_0\|_{L^2([0,t_{\max})} \le C \|\phi_0 \Delta P\|_{L^2(R)} \tag{20}$$

Here and in the rest of this section, C stands for a constant dependending on on $\mathcal{A}$ but not otherwise on v, v^*.

The inequality (20) indicates that the L^2 norm of ΔP controls the discrepancy between v and v^*, in some sense, but $J_0[v,d]$ is a *weighted* L^2 mean square of ΔP. Therefore the next task is to understand the way in which the quantity $\sqrt{J_0[v,d]}$ dominates the unweighted norm. This would initially appear to be out of the question, as the weight contains as its essential factor a scaled version of the data, and there is no reason to expect the data to be uniformly non-vanishing - in fact, to model the behaviour of field data, r^* must be permitted to vanish or be very small over intervals of positive length. However, ΔP is not arbitrary: it satisfies a differential equation, and this fact comes to the rescue.

Lemma: Suppose that $f \in C^1[0,Z], g \in C^0[0,Z]$ and $w \in L^1[0,Z]$, are nonnegative functions related by

$$\left|\frac{df}{dz}\right| \le af + g$$

with $a \in \mathbf{R}_+$. Denote by $\mathcal{P}$ the set of all partitions $0 = z_0 < z_1 < ... < z_I = Z$ of $[0,Z]$ for which

$$\frac{1}{z_{i+1}-z_i}\int_{z_i}^{z_{i+1}} w = \frac{1}{Z}\int_0^Z w,\ i = 0,...,I-1$$

Set

$$E^* = \sup_{\mathcal{P}} \max_i e^{2a(z_{i+1}-z_i)}$$

$$\Delta^*[w] = \inf_{\mathcal{P}} \max_i (z_{i+1}-z_i)$$

$$\left(\frac{1}{Z}\int_0^Z w\right)\left(\frac{1}{E^*}\int_0^Z f - \Delta^*[w]\int_0^Z g\right) \le \int_0^Z fw \le \left(\frac{E^*}{Z}\int_0^Z w\right)\left(\int_0^Z f + Z\int_0^Z g\right) \tag{21}$$

Remark: The factor Δ^* encapsulates the degree of uniformity of the weight w. If $w = 0$ over an interval $[a,b] \subset [0,Z]$, then clearly $b - a \le \Delta^*$.

Proof: Follows from standard differential inequalities. **QED**

From the inflow/outflow structure of R, it follows that $S = \{(z,p) : z_{\min} \le z \le z_{\max}(p),\ 0 \le p \le p_{\max}\}$ for a piecewise smooth function $z_{\max} : [0,p_{\max}] \to \mathbf{R}_+$. So

$$J[v,d] = \int_0^{p_{\max}} dp \int_{z_{\min}}^{z_{\max}(p)} dz\, B_1(z,p)|\Delta p(z,p)|^2[r^*(Z^*(T(z,p),X(z,p)))]^2$$

in which B_1 depends on v as well as v^* but is uniformly bounded over S and $v, v^* \in \mathcal{A}$.

From (16) it follows that for $0 \le p \le p_{\max}$,

$$\left|\frac{\partial \Delta p}{\partial z}(z,p)\right| \le C(|\Delta u(z,p)|^2 + |\Delta p(z,p)|^2),\ z \in [z_{\min}, z_{\max}(p)]$$

with C as before uniform over $v, v^* \in \mathcal{A}$. Define

$$w(z,p) = B_1(z,p)[r^*(Z^*(T(z,p),X(z,p)))]^2$$

(21) implies that

$$\int_{z_{\min}}^{z_{\max}(p)} dz\, w(z,p)|\Delta p(z,p)|^2$$

$$\geq W_*(p)\left(\frac{1}{E^*}\int_{z_{\min}}^{z_{\max}(p)} dz\,|\Delta p(z,p)|^2 - C\Delta^*[w(\cdot,p)]\int_{z_{\min}}^{z_{\max}(p)} dz\,|\Delta u(z,p)|^2\right) \tag{22}$$

in which

$$W_*(p) = \frac{1}{z_{\max}(p)-z_{\min}}\int_{z_{\min}}^{z_{\max}(p)} dz\, w(z,p)^2$$

Using the bounds implicit in the definition of $\mathcal{A}$, it is easy to establish that

$$\Delta^*[w(\cdot,p)] \leq C\Delta^*[r^*]$$

with a constant C uniform over $v, v^* \in \mathcal{A}$ and $0 \leq p \leq p_{\max}$. Integrate (22) in p and use the second equation in (16) to get

$$J_0[v,d] \geq C\|\psi_1 r^*\|^2_{L^2(\mathbf{R}_+)}\left[\left(\frac{1}{E^*} - C_1\Delta^*[r^*]\right)\|\phi_0\Delta P\|^2_{L^2(R)} - C_2\Delta^*\int_{\mathbf{R}_+} dz\,|\psi_0(z,0)\Delta u(z,0)|^2\right] \tag{23}$$

The second integral on the RHS of (23) is a measure of the difference between v and v^*, or U_0 and U_0^*. The first term also bounds another such measure, namely $\|\mathcal{I}\Delta U_0\|^2$, according to (20). Without additional constraint, however, the L^2 norm of ΔU_0 can be arbitrarily larger than that of $\mathcal{I}U_0$. Assume now that $v, v^* \in \mathcal{A}_f \subset \mathcal{A}$, and that there exists K depending on $\mathcal{A}_f$ so that

$$v, v^* \in \mathcal{A}_f \;\Rightarrow\; \|\phi_0(\cdot,0)\Delta U_0)\|^2_{L^2(\mathbf{R}_+)} \leq K\|\phi_0(\cdot,0)\mathcal{I}U_0\|^2_{L^2(\mathbf{R}_+)}$$

Essentially this implies that $\mathcal{A}_f$ is the intersection of $\mathcal{A}$ with a finite dimensional submanifold of C^∞. It is guaranteed for example if $\mathcal{A}_f$ is chosen so that the reciprocal square velocities lie in a finite dimentional affine subspace of C^∞.

This inequality together with (20) and (23) imply that

$$J_0[v,d] \geq C\|\psi_1 r^*\|_{L^2(\mathbf{R}_+)}(C_3 - C_4K^2\Delta^*[r^*])\|\phi_0\mathcal{I}\Delta U_0\|^2_{L^2(\mathbf{R}_+)}$$

which establishes

Theorem 1: There exist constants C and δ, depending only on $\mathcal{A}$, so that if $K\Delta^*[r^*] \leq \delta$ then

$$\|\phi_0\Delta U_0\|_{L^2(\mathbf{R}_+)} \leq C\sqrt{KJ_0[v,d]}$$

Remark: The size of K roughly signifies the degree of oscillation permitted in the square slowness U_0. So the condition $K\Delta^*[r^*] \leq \delta$ can be interpreted to mean: as the degree of oscillation in U_0 (and U_0^*) increases, the density of reflectors required for unique velocity determination increases proportionally.

Denote by δv a perturbation in v, and by δP the corresponding perturbation in P: i.e. $\delta P = DP[v]\delta v$. The differentiability of P as a functional of v as v varies over a finite dimensional submanifold of C^∞ follows from Hamilton-Jacobi theory, or alternatively from the formulae presented above for P.

The analysis of stationary points hinges on showing that J_0 is *close to quadratic* in $v - v^*$, in the sense that if $\delta v = v - v^*$ then $DJ_0[v,d]\delta v \simeq \text{const.}\, J_0[v,d]$. To see this, note that (9) implies,

$$DJ_0[v,d]\delta v = \int_R dt\,dx\, B^*(t,x)\delta P(t,x)(P(t,x) - P^*(t,x))[r^*(Z^*(t,x))]^2 + O(\lambda)$$

$$= J_0[v,d] + \int_R dt\,dx\, B^*(t,x)(\delta P(t,x) - \Delta P(t,x))\Delta P(t,x)[r^*(Z^*(t,x))]^2 + O(\lambda) \quad (24)$$

which suggests that the key issue is the size of $\delta P(t,x) - \Delta P(t,x)$. As was the case for ΔP, δP is part of the solution of a first order hyperbolic system:

$$U\frac{\partial \delta P}{\partial t} + P\frac{\partial \delta P}{\partial x} + \frac{\partial P}{\partial t}\delta U + \frac{\partial P}{\partial x}\delta P = 0 \quad (25)$$

$$P\frac{\partial \delta U}{\partial t} + \frac{\partial \delta U}{\partial x} + \frac{\partial U}{\partial t}\Delta P = 0 \quad (26)$$

with the same Cauchy data as the system for $\Delta U, \Delta P$:

$$\delta U(t,0) = \Delta U_0(t) = U_0(t) - U_0^*(t),\ \delta P(t_m(x),x) = 0 \quad (27)$$

Subtracting (13), (14), and (15) from (25), (26), and (27) yields a system for $\delta P - \Delta P, \delta U - \Delta U$. As before, this system is best tackled in the characteristic (for v) (z,p) coordinates. Define

$$p'(z,p) = (\delta P - \Delta P)(T(z,p),X(z,p)),\ u'(z,p) = (\delta U - \Delta U)(T(z,p),X(z,p))$$

Then p', u' solve a hyperbolic system of the form

$$\frac{\partial p'}{\partial z} + \alpha' u' + \beta' p' = \alpha''\Delta u\frac{\partial \Delta p}{\partial z} + \beta''\Delta u\frac{\partial \Delta p}{\partial p} + \gamma''\Delta p\frac{\partial \Delta p}{\partial z} + +\delta''\Delta p\frac{\partial \Delta p}{\partial p} \quad (28)$$

$$\frac{\partial u'}{\partial z} + \gamma' p' = \lambda''\Delta p\frac{\partial \Delta p}{\partial z} + \mu''\Delta p\frac{\partial \Delta p}{\partial p} \quad (29)$$

$$u'(z,0) = 0,\ 0 \le z \le z_{\max};\ p'(z_{\min},p) = 0,\ 0 \le p \le p_{\max} \quad (30)$$

in which $\alpha', \beta', \gamma', \alpha'', \beta'', \gamma'', \delta'', \lambda'', \mu''$ and their derivatives satisfy uniform bounds for $(z,p) \in S$ and as v, v^* range over $\mathcal{A}$.

Standard energy estimates as in the preceding paragraphs lead to

$$\int_S \psi_0 (p')^2 \le$$

$$C\left(\int_S \psi_0 \Delta u^2 \left(\frac{\partial \Delta p}{\partial p}\right)^2 + \int_S \psi_0 \Delta u^2 \left(\frac{\partial \Delta p}{\partial z}\right)^2 + \int_S \psi_0 \Delta p^2 \left(\frac{\partial \Delta p}{\partial p}\right)^2 + \int_S \psi_0 \Delta p^2 \left(\frac{\partial \Delta p}{\partial z}\right)^2\right)$$

Claim: Each of the summands on the right is bounded by $\|\phi_0 \Delta P\|^4_{L^2(R)}$, with the bound uniform over $v, v^* \in \mathcal{A}_f$.

For example,

$$\int_S \psi_0 \Delta u^2 \left(\frac{\partial \Delta p}{\partial p}\right)^2 \le \left(\int_S \psi_0 \Delta u^2\right) \sup_{z_{\min}\le z \le z_{\max}} \int_0^{p_{\max}(z)} dp \left(\frac{\partial \Delta p}{\partial p}\right)^2$$

From (20) and the defining inequality of $\mathcal{A}_f$,

$$\int_S \psi_0 \Delta u^2 \le C\|\phi_0 \Delta P\|^2_{L^2(R)}$$

On the other hand, from (16),

$$\frac{\partial}{\partial z}\frac{\partial \Delta p}{\partial p} + \left(\alpha\gamma + \frac{\partial \beta}{\partial p}\right)\Delta p + \frac{\partial \alpha}{\partial p}\Delta u + \beta \frac{\partial \Delta p}{\partial p} = 0$$

whence follows by standard energy estimates that

$$\int_0^{p_{\max}(z)} dp\, \psi_0 \left(\frac{\partial \Delta p}{\partial p}(z,p)\right)^2 \le C\int_{z_{\min}}^{z} dz \int_0^{p_{\max}(z)} dp\, \psi_0(\Delta u^2 + \Delta p^2)$$

$$\le C\|\phi_0 \Delta P\|^2_{L^2(R)}$$

So the conclusion holds for the first summand. The others are handled similarly. **QED**

Changing variables again, it follows that

$$\|\phi_0(\delta P - \Delta P)\|_{L^2(R)} \le C\|\phi_0 \Delta P\|^2_{L^2(R)}$$

so

$$\left|\int_R dt\, dx\, B^*(t,x)(\delta P(t,x) - \Delta P(t,x))\Delta P(t,x)[r^*(Z^*(t,x))]^2\right|$$

$$\le \sqrt{J_0[v,d]}\left(\int_R dt\, dx\, B^*(t,x)(\delta P(t,x) - \Delta P(t,x))^2[r^*(Z^*(t,x))]^2\right)^{\frac{1}{2}}$$

$$\le C\sqrt{J_0[v,d]}\|r^*\|_{L^2(\mathbf{R}_+)}\|\phi_0(\delta P - \Delta P)\|_{L^2(R)}$$

$$leC\sqrt{J_0[v,d]}\|r^*\|_{L^2(\mathbf{R}_+)}\|\phi_0 \Delta P\|^2_{L^2(R)}$$

Theorem 1 and inequality (23) imply that this is in turn bounded by

$$\le CJ_0[v,d]^{\frac{3}{2}}$$

provided that $\Delta^*[r^*]$ is sufficiently small.

Together with the expression in (23) for the derivative of J_0, this bound implies that, for $\Delta^*[r^*]$ sufficiently small,

$$DJ_0[v,d]\delta v \leq J[v,d] - CJ_0[v,d]^{\frac{3}{2}} + O(\lambda)$$

from which follows

Theorem 2: Given $\mathcal{A}_f$ with the properties described above, there exists $\epsilon, \delta_1 > 0$ so that if $\Delta^*[r^*] \leq \delta_1$ and $J_0[v,d] \leq \epsilon$, then

$$\nabla J_0[v,d] = 0 \ \Rightarrow \ J_0[v,d] = O(\lambda)$$

Acknowledgements

The author performed part of the work reported here while a guest of the Stanford Exploration Project. The author thanks the Director of SEP, Prof. Jon Claerbout, and the other project members for their hospitality and for the stimulating intellectual atmosphere of SEP.

This work was supported in part by National Science Foundation grants DMS-9973423, DMS-9973308, and EAR-9977697, by the Los Alamos National Laboratory Computer Science Institute (LACSI) through LANL contract number 03891-99-23, by the Department of Energy EMSP grant DE-FG07-97 ER14827, and by The Rice Inversion Project (TRIP). TRIP sponsors for 2001 were Amerada Hess Corp., Conoco Inc., ExxonMobil Upstream Research Co., Landmark Graphics Corp., Shell International Research, and Western Geco.

REFERENCES

Aki, K. and Richards, P. (1980). *Quantitative Seismology: Theory and Methods*. Freeman, San Francisco.

Araya, K., Kim, S., Nolan, C. J., and Symes, W. W. (1996). Velocity inversion and high frequency asymptotics. In *Mathematical Methods in Geophysical Imaging*. Society of Photooptical Instrumentation Engineers.

Chauris, H. (2000). *Analyse de vitesse par migration pur l'imagerie des structures complexes en sismique réflexion*. PhD thesis, École des Mines de Paris.

Chauris, H. and Noble, M. (2001). Two-dimensional velocity macro model estimation from seismic reflection data by local differential semblance optimization: applications synthetic and real data sets. *Geophys. J. Int.*, 144:14–26.

Chauris, H., Noble, M., and Podvin, P. (1998). Testing the behaviour of differential semblance for velocity estimation. In *68th Annual International SEG Meeting*, New Orleans. Society of Exploration Geophysicists. Expanded Abstract.

Dobrin, M. (1976). *Introduction to Geophysical Prospecting*. McGraw-Hill, New York, third edition.

Gauthier, O., Tarantola, A., and Virieux, J. (1986). Two-dimensional nonlinear inversion of seismic waveforms. *Geophysics*, 51:1387–1403.

Gockenbach, M. and Symes, W. (1999). Coherent noise suppression in velocity inversion. In *Expanded Abstracts, Society of Exploration Geophysicists, 69th Annual International Meeting*, Tulsa. SEG.

Kern, M. and Symes, W. (1994). Inversion of reflection seismograms by differential semblance analysis: Algorithm structure and synthetic examples. *Geophysical Prospecting*, 99:565–614.

Lewis, M. and Symes, W. (1991). On the relation between the velocity coefficient and boundary values for solutions of the one-dimensional wave equation. *Inverse Problems*, 7:597–632.

Minkoff, S. E. and Symes, W. W. (1997). Full waveform inversion of marine reflection data in the plane-wave domain. *Geophysics*, 62(2):540–553.

Mulder, W. and ten Kroode, A. (2001). Automatic velocity analysis by differential semblance optimization. In *Expanded Abstracts, Society of Exploration Geophysicists, 71st Annual International Meeting*, Tulsa. SEG.

Robinson, E. and Treitel, S. (1980). *Geophysical Signal Analysis.* Prentice-Hall, Englewood Cliffs.

Scales, J., Smith, M., and Fischer, T. (1991). Global optimization methods for highly nonlinear inverse problems. In Cohen, G., Halpern, L., and Joly, P., editors, *Mathematical and Numerical Aspects of Wave Propagation Phenomena*, pages 434–444. SIAM, Philadelphia.

Sen, M. and Stoffa, P. (1991a). Nonlinear multiparameter optimization using genetic algorithms: Inversion of plane wave seismograms. *Geophysics*, 56:1794–1810.

Sen, M. and Stoffa, P. (1991b). Nonlinear one-dimensional seismic waveform inversion using simulated annealing. *Geophysics*, 56:1624–1636.

Symes, W. (1986). Stability and instability results for inverse problems in several-dimensional wave propagation. In Glowinski, R. and Lions, J., editors, *Proc. 7th International Conference on Computing Methods in Applied Science and Engineering.* North-Holland.

Symes, W. (1991). Layered velocity inversion: a model problem from reflection seismology. *SIAM J. Math. Anal.*, 22:680–716.

Symes, W. (1993). A differential semblance criterion for inversion of multioffset seismic reflection data. *J. Geoph. Res.*, 98:2061–2073.

Symes, W. (1998). High frequency asymptotics, differential semblance, and velocity analysis. In *Expanded Abstracts, Society of Exploration Geophysicists, 68th Annual International Meeting*, pages xx–xx, Tulsa. SEG.

Symes, W. and Carazzone, J. (1991a). Velocity inversion by differential semblance optimization. *Geophysics*, 56(5):654–663.

Symes, W. and Carazzone, J. (1991b). Velocity inversion by differential semblance optimization. *Geophysics*, 56(5):654–663.

Symes, W. and Versteeg, R. (1993). Velocity model determination using differential semblance optimization. In *Proc. 63rd Annual International Meeting*, pages 696–699, Washington, D.C. Society of Exploration Geophysicists. Expanded abstract.

Symes, W. W. (1997). High frequency asymptotics and velocity estimation. In *Depth Imaging Symposium*, Tulsa, Oklahoma, U.S.A. Geophysical Society of Tulsa.

Symes, W. W. (1999). All stationary points of differential semblance are asymptotic global minimizers: Layered acoustics. In *The Rice Inversion Project: Annual Report 1999*, Department of Computational and Applied Mathematics, Rice University, Houston, TX 77005-1892. also appeared in SEP-95.

Symes, W. W. (2001). Differential semblance: Formulation and implementation. In *The Rice Inversion Project: Annual Report 2001*, Department of Computational and Applied Mathematics, Rice University, Houston, TX 77005-1892.