AN INTRODUCTION TO THE STABILIZATION PROBLEM FOR PARAMETRIZED FAMILIES OF LINEAR SYSTEMS*

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Abstract

This talk provides an introduction to definitions and known facts relating to the stabilization of parametrized families of linear systems using static and dynamic controllers. New results are given in the rational and polynomial cases.

1. General discussion

We shall consider a set of problems which have appeared in algebraic system theory and whose solutions involve tools of various different types. These problems, in their simplest form, deal with parametrized families of pairs ("systems") $\{(A_\lambda,B_\lambda), \lambda\in\Lambda\}$, where A_λ is an $n\times n$ and B_λ is an $n\times m$ matrix for each λ (with n,m fixed integers). To be found is a new parametrized family $\{K_\lambda, \lambda\in\Lambda\}$ such that (1) a given design criterion is satisfied by the *closed-loop* matrix $A_\lambda+B_\lambda K_\lambda$ for all λ, and (2) the K_λ depend in a suitably 'nice' form on the parameter. (For the purposes of this talk, the entries of all matrices take real values, for each λ.)

For example, one design objective is that $A_\lambda+B_\lambda K_\lambda$ should (for each λ) have all its eigenvalues in the inside of the unit circle (discrete-time stabilization); one nice form of parameter dependence if, for instance, Λ is an Euclidean space $\Re^r$ (or an algebraic variety), is that K_λ be required to have entries which are polynomials in λ. Many other design objectives and types of parametrizations will be mentioned below. As a general rule the former will always deal with stability-related properties. Regarding parametrizations, we'll talk about the *continuous*, (real-)*analytic*, *rational*, or *polynomial* cases; when doing so, it will be implicit that the parameter set is respectively a topological space, a real-analytic manifold, or in the last two cases an Euclidean space, and that the given family $\{(A_\lambda,B_\lambda)\}$ is parametrized in this way. One may also consider of course other situations, for example the *smooth* ($=C^\infty$) case, or polynomial and rational families over algebraic varieties; for simplicity, we restrict attention to the above. For any given type of parametrization for the family $\{(A_\lambda,B_\lambda)\}$, we shall search for families of controllers parametrized in the same way. This insistence on a 'nice' parameter dependence for $\{K_\lambda\}$ (or its dynamic versions described later) is what distinguishes the area from the more classical study of single systems. Most

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***1980 Mathematics Subject Classification numbers 93B25,93D15**

****Research Supported in part by US Air Force Grant AFOSR-85-0196E**

results attempt in essence to establish local-global principles: does solvability for each individual λ imply the existence of a nicely parametrized solution?

There are various motivations for studying the general type of problems mentioned here. From a purely mathematical point of view, these are a natural next step after the solution of their non-parametrized versions, which constitute what a great deal of linear control theory is about. The system-theoretic interpretation of the present setup is as follows. The pairs (A_λ,B_λ) represent a discrete or continuous time system

$$x(t+1) \text{ [or } \dot{x}(t)] = A_\lambda x(t) + B_\lambda u(t) \tag{1}$$

whose general structure is known in advance but where certain parameters are *a priori* undetermined. (There may also be an output or *measurement* specified, of the type $y(t) = C_\lambda x(t)$, in which case the transposed family $\{(A_\lambda^T,C_\lambda^T)\}$ becomes also of interest, as in the example given below.)

Parametrized equations 1 arise for instance in the case where the (A_λ,B_λ) correspond to linearizations of a given nonlinear system at many different operating points, and one is interested in the design of controllers for all the linear systems so obtained. Such a situation appears frequently in aircraft control ("gain scheduling"), where controllers are precomputed for a large variety of operating conditions, with an on-board computer choosing the appropiate controller to be used at any given time based on environmental, geometric, flight-mode, and other factors, like pitch angle, air speed, angle of attack, and so forth. An alternative approach to this precomputation and storage would be to try to apply the tools of parametrized families to achieve the simultaneous design of these controllers, in the form of a parametrized controller which regulates once its parameters are properly tuned. Thus, only the functional form of the controllers needs to be stored. Together with an on-line identification procedure, this becomes in effect a method for *adaptive control*; recent work ([E2]) makes this application more precise. The resulting families will be typically analytic or rational, but other situations may appear too; for example polynomially parametrized families appear when dealing with systems with finite Volterra series (see [RU]).

The simplest control problem, that of stabilizing the above system by a static linear law $u(t) = K_\lambda x(t)$, consists mathematically of finding a (nicely parametrized) family $\{K_\lambda\}$ such that, say in continuous-time, all eigenvalues of $A_\lambda+B_\lambda K_\lambda$ have negative real parts, for each λ in the parameter set Λ. A more interesting problem is that of "pole-assignment", which consists of finding a family $\{K_\lambda\}$ such that $A_\lambda+B_\lambda K_\lambda$ has specified eigenvalues for each λ. (The terminology "poles" is due to the fact that the eigenstructure of this matrix gives rise to the poles of the transfer function of the closed loop system.)

A specific, though somewhat artificial, example of where a parametrized control problem appears is the following one. A cart can move horizontally, in one dimension, controlled by a motor. To its top is attached an inverted pendulum. The objective is to keep the pendulum in an upright position, using suitable controls (horizontal forces on the car). This is a standard textbook problem in control theory, and is analogous to a rocket stabilization problem. We shall assume that the mass M of the pendulum is concentrated at its top, and will disregard friction effects. The available observations will be the displacement of the cart and the angular velocity of the pendulum. Linearizing around the position corresponding to a static cart and a static vertical pendulum, the equations can be written as

$$\dot{x}(t) = Ax(t)+Bu(t),\ y(t) = Cx(t),$$

with A,B parametrized matrices as above, C a $p\times n$ matrix, m=1, n=4, p=2, and

$$A=\begin{matrix} 0 & 1 & 0 & 0\\ 0 & 0 & -\lambda & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & g+\lambda & 0\end{matrix} \qquad B=\begin{matrix}0\\1\\0\\-1\end{matrix} \qquad C=\begin{matrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{matrix} \tag{2}$$

(see [FH,p.47]), where the state coordinates are respectively the position and velocity of the cart, and the angle (with respect to the vertical) and the angular velocity of the pendulum. We are assuming that the pendulum has unit length, the cart has unit mass, g is the acceleration of gravity, and the parameter λ=Mg is scalar, $\Lambda=\Re$. The linearized model is controllable and observable for any λ (see below), so most design techniques can be easily applied, yielding in particular dynamic stabilizers for the above system. Here, however, we wish to consider the mass M of the pendulum, or equivalently λ, as a *parameter*, and we ask if it is possible to design a family of control systems $\{\Sigma_\lambda\}$ for 2, such that each Σ_λ results in a closed loop system with appropiate dynamic characteristics, and such that these controllers depend again nicely on λ.

The results to be reviewed later insure that pole assignment (in particular, stabilization with arbitrary degrees of convergence,) can be achieved, by regulators Σ_λ of fixed dimension and depending *polynomially* on λ, if and only if 2 is controllable and observable for all values of λ, real *and* complex. The controllability constraint means that the *controllability matrix* (B,AB,A^2B,A^3B) has full rank; since its determinant = g^2 = nonzero constant, this is clearly satisfied. The observability property deals with the analogous matrix for the dual pair (A^T,C^T). This *observability matrix* $(C^T,A^TC^T,(A^T)^2C^T,(A^T)^3C^T)^T$ has in particular minors $\Delta_1 = -\lambda^2$ (rows 1,3,5,7) and $\Delta_2 = g+\lambda$ (rows 1,2,3,4); since these two polynomials are relatively prime, the observability condition holds too. In fact, it is absolutely trivial (because m=1) to design a polynomially parametrized *state* feedback that achieves any desired pole location: if $k=k(\lambda)$ is desired so that $A+Bk^T$ has a characteristic polynomial $z^4+\alpha_3z^3+\alpha_2z^2+\alpha_1z+\alpha_0$, one may choose $k_1:=\alpha_0/g$, $k_2:=\alpha_1/g$, $k_4:=\alpha_3+\alpha_1/g$, and $k_3:=\alpha_2+g+(\alpha_0/g)+\lambda$, so that k is a polynomial of degree 1 in λ. In order to obtain a regulator that uses only the available measurements, we may proceed by constructing first a "Luenberger observer" for (A,C), which reduces to the solution of a pole-assignment problem for the transposed pair (A^T,C^T), and then combining this with the above feedback law. The usual methods for obtaining an observer, however, will *not* result in a polynomially parametrized observer. For example, since $A(\lambda)$ is cyclic for all λ, one may try to find a $v=v(\lambda)$ such that, for each λ, C^Tv is cyclic for A^T, reducing the construction to the single-output case ([WO]). But a simple calculation shows that no such v exists in this case, and the usual generalization, "Heymann's lemma," is not available over the polynomial ring $\Re[\lambda]$. Since $\Lambda=\Re$, $\Re[\lambda]$ is a *principal ideal domain*, however, and the construction of an observer *can* be completed, as discussed later. We omit the calculations here, but remark that they involve a certain amount of linear algebra over $\Re[\lambda]$, at the level of computations with Smith forms.

Another, more theoretical example, related to the "gain scheduling" idea mentioned earlier, of families of systems appearing as linearizations of nonlinear systems around different operating points (in the style of [BR]), is as follows. Consider the problem of obtaining observers for the linear systems that result when linearizing the 2-dimensional system $\dot{x}_1=u$, $\dot{x}_2=x_2+v$, $y_1=x_1x_2+x_1$, $y_2=x_1^2x_2-x_2$, at the natural equilibrium states $(\lambda,0)$, $\lambda\in\Re$. The duals of these systems have

$$A = \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} \qquad B = \begin{matrix} 1 & 0 \\ \lambda & \lambda^2-1 \end{matrix} \; . \tag{3}$$

It turns out that this is the example that we shall mention later in order to illustrate the fact that it is in general *impossible* to obtain arbitrary characteristic polynomials for A+BK, with polynomially parametrized K.

Finally, parametrized control problems are of interest in that they appear as one of the most interesting instances of *systems over rings*, or *generalized linear systems* (*g.l.s.*), and the results for these problems illustrate basically all the characteristics of that more general situation. The more general study of systems over rings deals with pairs of matrices (A,B) with entries on a commutative ring, not necessarily a ring of real functions. Since sheaf-theoretically rings can be represented in terms of functions on their prime spectra, the study of families of systems provides much intuition about the more general case. In other cases topological rings of various kinds are of interest, and systems over such may be seen as families induced by the possible representations of the ring; see for instance the references [B2], [GK], [BU].

It seems worth saying a few words about other applications of systems over rings. It is well known (and fairly obvious) that certain types of distributed linear systems can be written formally as 1 provided that one allows the matrices A, B to have operator entries. A typical example is that of delay-differential systems, for which the derivative $\dot{x}(t)$ may depend not just on present values of states and inputs but also on past values x(t-1), u(t-1), ..., etc. Introducing a shift operator $(\lambda v)(t) := v(t-1)$, the delay equation can also be modeled by an equation like 1 with the entries of A, B now *polynomials* in λ. (Polynomials in several variables appear if there are noncommensurate delay lenghts.) It is much less obvious that this formal procedure can be useful in solving actual control problems. An important contribution of E. Kamen (ca. 1973) was to realize that in this way one may apply results of systems over rings, whose study was initiated around that time ([RWK]). Other motivations for the study of g.l.s. are in the modeling of certain image processing algorithms and to model systems over the integers and residue rings. A large literature exists by now in the area; somewhat outdated surveys are [S1], [KN], and more up-to-date but much less complete is [S3]. A couple of textbooks are in preparation, and one is to appear soon ([BBV]). For application areas like delay systems, g.l.s. methods, in conjunction with certain techniques of analysis, offer an attractive alternative to methods based on functional analysis. We shall be here concerned only with families of systems, however.

2. Summary of the presentation

The organization of the talk is as follows. In the next section we review basic definitions and notations. After that, we shall cover various results, starting with those known under the most restrictive conditions on families, and progressing to those that hold under minimal conditions. The appendix contains proofs of new results.

The main concepts and results discussed are as follows (precise definitions to be given later). A *pointwise controllable* (resp., *asycontrollable*) family is one for which each of the systems (A_λ,B_λ) is controllable (resp., asycontrollable). "Controllability" of a given (A_λ,B_λ) corresponds to the standard system-theoretic notion of being able to control (in finite time) any state to any other state; it appears here mostly as a technical condition on the rank of a matrix associated to A_λ and B_λ (the condition is the same whether in discrete or continuous time). "Asycontrollablity", often called "stabilizability", corresponds to the possibility of finding, for each state x an infinite length control u(t), $0\le t<\infty$ which drives x asymptotically to the origin; its precise algebraic characterization depends on whether we are dealing with discrete or continuous time families.

In *pole assignment* problems we want to assign arbitrary eigenvalues to closed-loop dynamics, in *stabilization* problems the objective is just to stabilize the system. *Constant* solutions are in terms of static feedback $u(t)=K_\lambda x(t)$, *dynamic* solutions allow memory in the controller.

It is shown that, contrary to what could have been expected by analogy with the case of single systems, in general pointwise controllablity is *not* sufficient in order to insure *static* pole assignment. Somewhat surprisingly, it does suffice for concluding *dynamic* pole assignment. (In the case of polynomial families, a stronger property is needed for this later fact, however: controllability for complex parameter values as well.) In any case, if pointwise controllability holds, it is in general true that we can *stabilize* with static feedback, except in the case of discrete time polynomial families.

If the family is only pointwise asycontrollable, it is possible to stabilize using static feedback in the continuous and analytic cases, and to stabilize dynamically in all the others, except again for discrete time polynomial families.

3. Definitions and notations

There are various technical conditions that we may impose on the family $\{(A_\lambda,B_\lambda),\ \lambda\in\Lambda\}$. The first one, *pointwise controllability*, means that the controllability matrix

$$R(A_\lambda,B_\lambda):= [B_\lambda,A_\lambda B_\lambda,\cdots,A_\lambda^{n-1}B_\lambda] \tag{4}$$

has full rank n, i.e. that the system (A_λ,B_λ) is controllable, for each λ. (From now on, the statements "for each λ" or "for all λ" will mean always "for all $\lambda\in\Lambda$".) If (A_λ,B_λ) is only "stabilizable" for each λ, the family is *pointwise asy(mptotically)controllable*. Recall that (A_λ,B_λ) is "stabilizable" if the system 1 has the property that, for any initial state x(0) there is a control function u(·) such that the resulting trajectory x(t) converges to 0 as $t\to\infty$. Of course, this depends on the interpretation of (A_λ,B_λ) (discrete or continuous time), so to be more precise we should talk about "discrete" or "continuous" pointwise stabilizability; however, the meaning will be clear from the context. The "Hautus conditions" characterize these pointwise properties in the following way. Consider the matrix

$$S(A_\lambda,B_\lambda):= [zI-A_\lambda,B_\lambda] \tag{5}$$

for each fixed λ, with z a complex variable. Then controllability of any given (A_λ,B_λ) is equivalent to 5 having full rank for *every* $z\in\mathbf{C}$, and asycontrollability is equivalent to 5 having full rank for each $z\notin\mathbf{C}_s$, where $\mathbf{C}_s$ = the interior of the unit circle in the discrete time case, and $\mathbf{C}_s$ = the open left-hand plane in the continuous case.

Two other conditions are suggested from the theory of "systems over rings", and depend on the class of (real-valued) functions P allowed in parametrizations. For this exposition, P will be one of: $C^0(X)$, continuous functions on a topological space X, $C^\omega(X)$, real-analytic functions on a real-analytic manifold X, $\Re(X)$, rational functions on $X = \Re^r$ with no poles in $\Re^r$, or $\Re[X]$, polynomial functions on $X = \Re^r$. (We shall henceforth refer to the "cases C^o, C^ω, $\Re(X)$, $\Re[X]$" respectively.) All these P are seen as rings under pointwise multiplication. By a "family (of systems) over P" we mean a $\{(A_\lambda,B_\lambda),\ \lambda\in\Lambda\}$, such that, when thought of as a pair of matrices (A,B) whose entries are parametrized by λ, all such entries are functions of P. If P is not specified in a statement, the statement is meant to apply to all P as above. The conditions that we want to consider are then *ring controllability* and *ring asycontrollability*. Both can be defined in various ways, the most intuitive being as controllability and "stabilizability" of systems of difference equations over the appropiate rings; such definitions are given in [KS]. In particular, ring controllability means that 4 has a *right-inverse* over the ring P, i.e. that there exists a matrix $Q(\lambda)$ over P such that RQ = n by n identity.

Since here our interest in these conditions is purely technical, we prefer another type of definition. Consider the S in 5, now seen as a matrix over the polynomial ring P[z] obtained by adjoining an indeterminate to P. Then, ring controllability of $\{(A_\lambda,B_\lambda)\}$ (a family over P) means that S has a right inverse over P[z]. Similarly, ring asycontrollability means that S has a right inverse over the ring P_s of stable P-rational functions. This is the fraction ring of P[z] obtained by allowing denominators of the form $\theta(\lambda,z)$, where θ is monic in z and stable (i.e. all roots of $\theta(\lambda,\cdot)$ are in $\mathbf{C}_s$) for each fixed λ. A more elementary way of saying the same thing is that there must exist a (polynomial) matrix $Q(\lambda,z)$ over P[z] such that SQ = θI_n, with θ as above. Equivalences among various types of definitions are given for example in [HS1] and in section 3 of [KS]. Of course, in both cases the ring notion implies the corresponding pointwise concept.

It is easy to verify that in the cases P = C^o, C^ω, $\Re(X)$ pointwise controllability is equivalent to ring controllability. This is because the problem of finding a right inverse over P to the matrix R in 4 is equivalent to the problem of expressing a unit as a linear combination over P of the n-minors of R; since the pointwise condition insures that these have no common zeroes, adding the squares of these minors results in a nowhere vanishing function, which is a unit in the above cases. Of course, this doesn't work in the polynomial case; in the $\Re[X]$ case ring controllability is equivalent to R being right invertible for all λ *complex*. (A more elegant discussion of these points is in terms of [finitely generated] maximal ideals of P; see [S1].) The relations between the ring and pointwise *asycontrollability* notions are more delicate; we'll come back to this point at the end of the talk.

There are various different stabilization-related design objectives that we shall study. Each of these is defined with respect to a fixed ring P as above. The family (over P) $\{(A_\lambda,B_\lambda)\}$ is *coefficient assignable* (*c.a.*) (with static feedback) if for each polynomial $\theta(z)=\theta(\lambda,z)\in P[z]$ monic in z and of degree n, there exists a K in $P^{m\times n}$ (i.e., a family of m by n matrices $\{K_\lambda\}$ parametrized in the same way as the original family,) such that the characteristic polynomial of A+BK (calculated over P) is θ. Equivalently, the characteristic polynomial of each matrix $A_\lambda+B_\lambda K_\lambda$ is the corresponding $\theta(\lambda,\cdot)$. *Pole-assignability* (*p.a.*) means that such K can be found for all monic θ that factor linearly over P, $\theta = (z-\alpha_1)\cdots(z-\alpha_n)$, $\alpha_i\in P$. (Weaker notions of pole assignability are discussed later.) *(Discrete or continuous time) stabilizability* means that *there exists* a stable polynomial θ for which such a K can be found, where "stability" is understood as before. Finally, we shall talk about *dynamic* versions of these problems, meaning the following. A family $\{(A_\lambda,B_\lambda)\}$ is *dynamically c.a.* if there exists a nonnegative integer κ such that the new family

$$\tilde{A}_\lambda = \begin{matrix} A_\lambda & 0 \\ 0 & 0 \end{matrix} \qquad \tilde{B}_\lambda = \begin{matrix} B_\lambda & 0 \\ 0 & I_\kappa \end{matrix} \tag{6}$$

(having $\tilde{n}:= n+\kappa$ and $\tilde{m}:= m+\kappa$) is c.a.; similar definitions apply to p.a. and stabilization. The interpretation of this is: assume that $\tilde{K}$ is found so that $\tilde{A}+\tilde{B}\tilde{K}$ has desired dynamic behavior (characteristic polynomial). Writing $\tilde{K}$ in block form, the "closed loop system" with dynamics $\tilde{A}+\tilde{B}\tilde{K}$ corresponds (say in continuous time) to the system

$$\begin{aligned} \dot{x}_1 &= (A+BK_1)x_1 + BK_2x_2 \\ \dot{x}_2 &= K_3x_1 + K_4x_2, \end{aligned} \tag{7}$$

that is, there is a new system, with state variables x_2, whose inputs are the states x_1 of the original system, and which feeds a control $u = K_2x_2$ to the original system, such that the closed loop behavior is as desired. From an applied point of view, such a controller, easily implementable with digital technology, is as acceptable, for most applications, as a "static" feedback u = Kx. One of the surprising characteristics of parametrized families of systems is that, as we shall see, in many cases there are dynamic solutions but no static ones; this is in sharp contrast to the "classical" situation (single systems) where, for state-feedback problems such as those considered here, static controllers are sufficient.

Some notation will be useful to help organize the labeling of remarks and results. We will use a 4-tuple (a,b,c,d) where a = 'd' or 'Δ' indicates continuous or discrete time respectively, b = 'C' or 'D' indicates static or dynamic feedback respectively, c = 'S' or 'P' is used to indicate that the remarks concern just stabilization or pole-assignment (including c.a. and weaker versions of p.a.), and the last entry indicates the class P for which the results apply. A "*" in any position indicates that *all* options are appropiate. These notations will be used informally, for easy reference.

4. Pole assignment and related problems

The strongest assumption that we can make about the family $\{(A_\lambda,B_\lambda)\}$ is that of ring controllability. In fact, this condition is *necessary* for most objectives involving arbitrary modification of dynamics, like p.a. and c.a. (but not for just stabilization), even if dynamic feedback is used; for simplicity we give the p.a. version, but it is clear from the argument that many weaker notions of p.a. will still imply ring controllability.

Proposition 1: Assume that the dynamic p.a. problem is solvable for the family $\{(A_\lambda,B_\lambda), \lambda\in\Lambda\}$. Then the family is ring controllable.

Proof: We shall prove that each system (A_λ,B_λ) in the family is controllable, and in the case $P = \Re[X]$ that this is even true for complex values of the parameters λ. By the remarks in the previous section, this will imply the desired conclusion. So consider any such system $(A_{\lambda_o}, B_{\lambda_o})$. Let κ be as in the definition of dynamical p.a., and pick a set of real numbers $\{\alpha_1, \cdots, \alpha_{n+\kappa}\}$ disjoint from the eigenvalues of A_{λ_o}. By assumption, the polynomial (independent of λ) $\theta := (z-\alpha_1)\cdots(z-\alpha_{n+\kappa})$ can be assigned for the system 6, so in particular arguing pointwise it can be (dynamically) assigned for the given $(A_{\lambda_o}, B_{\lambda_o})$. (In the polynomial case $P = \Re[X]$ one may still formally evaluate the feedback matrix $\breve{K}$ at complex λ to obtain the same conclusion.) It follows by a classical argument that this system is controllable: otherwise there exists a basis in $\Re^n$ for which $(A_{\lambda_o}, B_{\lambda_o})$ has the block form

$$\begin{matrix} * & * & & * \\ 0 & F & & 0 \end{matrix} \qquad \textbf{(8)}$$

with F of size at least 1, and the eigenvalues of F appear among those of any closed loop system of the type 7, contradicting the pole placement result. QED

This remark out of the way, let's concentrate now on sufficiency results, assuming ring controllability. The first observation, which we will not explain in any detail, is that when m (number of controls) = 1 everything is as expected from the "classical" case. This is because the controllability matrix R in 4 is then a square matrix, and by assumption of ring controllability has a determinant invertible over P; thus global changes of basis are possible just as for single systems, resulting in the construction of various canonical forms, from which (static) c.a. is immediate. (See for instance [ZA] for some facts about the single-input case when somewhat less than ring controllability is available.) This kind of argument can be extended to the case where there is a square matrix, invertible over P, of the form

$$R^* := [b_1,b_2,\cdots,b_{m_1},Ab_1,\cdots,Ab_{m_2},\cdots,A^{r-1}b_1,\cdots,A^{r-1}b_{m_r}] \qquad (9)$$

such that $m_1\geq\cdots\geq m_r$, $\sum m_i=n$, and the b_i are columns of a matrix of the type BV, where V is a (square) invertible P-matrix. In the case of single systems (no parameters), such a matrix always exists, with V = identity, and finding it is often the first step in obtaining a canonical form which allows an easy solution of the c.a. problem (see for instance [KA]). In the parametrized case, *if* such an R^* exists, then the classical arguments can be generalized with no difficulty whatsoever, because all changes of coordinates used in obtaining canonical forms starting with R^* are given by unimodular matrices; thus the c.a. problem is solvable in that case.

In general, however, such a matrix will not exist (e.g. for the counterexamples given later), but the above remark is nonetheless of interest for two reasons. First, there is the result in [B1] that states that (for appropiate P) a good R^* will exist provided that the family $\{(A_\lambda,B_\lambda)\}$ has constant Kronecker indices, in other words, that the rank of the matrices $[B_\lambda,\cdots,A_\lambda{}^jB_\lambda]$ be independent of λ for each j = 0, $\cdots$, n-1. Though a highly restrictive condition, it does provide a nontrivial result, which depends on basic facts about vector bundles in the topological case, and on the solution of Serre's problem in the polynomial case. (To be more precise, [B1] deals with complex polynomial families, but the result can be proved for $\Re[X]$ and for families over $P = C^o(X)$, X contractible.) Another reason that 9 is of interest will be clear soon.

In relation to the remarks in the last paragraph, it is worth mentioning some recent results in [TK], where a much weaker assumption than constancy of Kronecker indices is used to obtain (still under the ring controllability hypothesis) a weak form of p.a.: the authors assume only that the rank of B_λ is constant, and conclude that any *constant* polynomial of degree n can be assigned. Their result applies in the case $C^o(X)$, X contractible (and for certain rings of complex analytic functions); its proof is based on algebro-geometric arguments concerning the existence of continuous sections of fibrations, applied to the system of algebraic equations on K that give the desired characteristic polynomial. Generalizations to other rings P are unknown.

The nonconstancy of the rank of B_λ is in fact at the root of most known counterexamples relating to p.a. and c.a. problems, via the argument used in [BSSV]. This argument, in its form given in [HS2], is based on the following observation. Here, a "unimodular" family of vectors $\{v_\lambda\}$ is a P-parametrized vector such that $v_\lambda \neq 0$ for all λ; a unimodular eigenvector of $\{A_\lambda\}$ is such a $\{v_\lambda\}$ with the property that, for some $\sigma \in P$, $A_\lambda v_\lambda = \sigma_\lambda v_\lambda$ for all λ; (unimodular *left* eigenvectors are defined similarly); a "unimodular column" for $\{B_\lambda\}$ is a unimodular family of vectors of the form $\{B_\lambda w_\lambda\}$, for some (necessarily itself unimodular!) $\{w_\lambda\}$.

Lemma 2: If the family $\{(A_\lambda,B_\lambda)\}$ is pointwise controllable and if $\{A_\lambda\}$ has a unimodular left eigenvector then $\{B_\lambda\}$ has a unimodular column.

The idea of the proof rests upon the "Hautus controllability condition" in terms of the matrix S introduced in 5. Indeed, the rank condition, applied pointwise, implies that a left eigenvector $v_\lambda{}^T$ of $\{A_\lambda\}$ must be necessarily a "unimodular row" of $\{B_\lambda\}$: $v_\lambda{}^T B_\lambda$ is nonzero for all λ. Thus $B_\lambda(B_\lambda{}^T v_\lambda)$ is a unimodular column of $\{B_\lambda\}$. QED

Now, if we produce a ring controllable polynomial family $\{(A_\lambda,B_\lambda)\}$ such that $\{B_\lambda\}$ has no unimodular column even for $P = C^o$, we will have established that, using the notational conventions introduced earlier, (*,C,P,*) *cannot be solved in general*:

Theorem A. There is a ring controllable polynomial family which cannot be (statically) pole-assigned, even if only *continuous* feedback is required.

Indeed, if for instance the constant polynomial $(z-1)^{n-1}z$ could be assigned over $C^o(\Re^r)$, i.e. there is a continuous $\{K_\lambda\}$ such that D:= A+BK has this characteristic polynomial, then $\ker(D_\lambda{}^T)$ (being of constant rank 1) is in a natural way a line bundle over $\Re^r$, hence is trivial. Thus there is a nowhere zero continuous section of this bundle, which is the same as a left eigenvector (with $P = C^o(\Re^r)$) of $D=\{D_\lambda\}$. Since the new family $\{(D_\lambda,B_\lambda)\}$ is *again* pointwise controllable (elementary linear system theory!), this contradicts via lemma 2 the above assumptions about $\{B_\lambda\}$. Thus there can exist no continuous (hence no C^ω, etc.,) feedback assigning this polynomial. Specific examples of such families are given in [BSSV] (see also [TA] and [SH] for a more methodical approach); for instance,

$$A_\lambda = \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \qquad\qquad B_\lambda = \begin{matrix} \lambda_2 & \lambda_2 \\ -\lambda_1 & z \end{matrix} \tag{10}$$

where $z = -\lambda_1-(1 - \lambda_1{}^2 - \lambda_2{}^2)$. A fixed point argument is used in [BSSV] to conclude that $\{B_\lambda\}$ has no unimodular columns.

Note that the above counterexample uses families over $\Re^2$. There is a good reason for this: In the case of polynomial (or even rational) families over $\Re$ (*scalar* parameters), the result in [MO] insures that the p.a. problem is *always* solvable. That proof can be easily extended to the real-analytic case over X = $\Re$ ([BSSV]) and, with some more difficulty, to $C^o(\Re)$ and some other cases ([HS2]). The basic method for assigning $\theta := (z-\alpha_1)\cdots(z-\alpha_n)$ is to inductively find first a $\{K_\lambda\}$ and a unimodular eigenvector of $\{A_\lambda+B_\lambda K_\lambda\}$ corresponding to α_1, and so on recursively for the rest of the α_i's. We illustrate only this first step. The controllability hypothesis insures that the preimage $(\alpha_1 I-A_\lambda)^{-1}(im B_\lambda)$ is nowhere zero; under suitable assumptions on P this is sufficient to conclude that there are families of vectors $\{w_\lambda\}$ and $\{v_\lambda\}$, the latter unimodular, such that $(\alpha_1 I-A_\lambda)v_\lambda = B_\lambda w_\lambda$. Hence we may define $\{K_\lambda\}$ so that $K_\lambda v_\lambda = -w_\lambda$. In the $\Re[X]$ and $\Re(X)$ cases, these families exist because P is a principal ideal domain, in the $C^\omega(\Re)$ case because P is an elementary divisor ring, and in the $C^o(\Re)$ case by some results on singular distributions. A generalization of this argument, using line bundles instead of unimodular vactors, is needed to obtain similar results for functions on certain noncontractible spaces, as done in [HS2] for functions on the unit circle.

The above positive results refer to the p.a. case; *for c.a. not even the case of "one-dimensional" P is well-behaved*. This was studied in some detail in [S1] and [BSSV]; the later reference shows (except for a coordinate change) that the family in 3 cannot be coefficient assigned over P = $\Re[\lambda]$. The polynomial $z^2-(2\lambda)z+1$ cannot be obtained by any feedback law, as established via a quadratic reciprocity argument.

Faced with these negative results, [HS1] began the study of the existence of *dynamic* solutions to the c.a. problem. Only partial results were given there, however. The proof that *the case* (*,D,*,*) *is always solvable* (under the assumed ring controllability condition) is due to [EK]:

Theorem B. Ring-controllability is equivalent to dynamic pole assignability.

A particularly nice (and very simple!) proof of this fact is due to P.P. Khargonekar (see a time-varying analogue in [KK2]), and is as follows. Choose κ (in the definition of dynamic controllers) as n^2. We claim that the extended system 6 is coefficient assignable. In order to prove this, it is enough to find a $K=\{K_\lambda\}$ and a family of matrices $\{L_\lambda\}$ such that the new family $\{(F_\lambda,G_\lambda)\} := \{(\bar{A}_\lambda+\bar{B}_\lambda K_\lambda,\bar{B}_\lambda L_\lambda)\}$ admits a global basis as in 9. For then one can assign $F+G\bar{K} = \bar{A}+\bar{B}K+\bar{B}L\bar{K} = \bar{A}+\bar{B}[K+L\bar{K}]$. Assume then that $C = \{C_\lambda\}$ is a right inverse (over P) of the matrix R in 4. Partition C (an nm by n matrix) in such a way that C_1 is the matrix obtained from its first m rows, C_2 from the next m, etc.,

$$C^T = [C_1^T, C_2^T, \cdots, C_n^T] .$$

Let the added coordinates be denoted by blocks (of dimension n each) $Z_1,\cdots,Z_n$. Now let K be in block form (K_{ij}) with $K_{11}:= 0$ (size m by n), K_{12} (m by n^2) be in terms of the above coordinates the combination $\sum C_i Z_i$, K_{21} again 0, and K_{22} the $n^2 \times n^2$ matrix having shift block structure

$$\begin{matrix}
0 & I & 0 & 0 & . & . & 0 \\
0 & 0 & I & 0 & . & . & 0 \\
. & . & . & . & & & . \\
. & . & . & . & & & . \\
0 & 0 & 0 & 0 & . & I & 0 \\
0 & 0 & 0 & 0 & . & 0 & I \\
0 & 0 & 0 & 0 & . & 0 & 0
\end{matrix}$$

(blocks of size $n \times n$). Finally, let L_λ be the (constant) matrix with $m+n^2$ rows and n columns which is zero except for an identity matrix in the last n rows. Thus, $\{(F_\lambda,G_\lambda)\}$ is the family

$$\begin{array}{cccccc} A & BC_1 & BC_2 & \cdot & \cdot & BC_n \\ 0 & I & 0 & \cdot & \cdot & 0 \\ 0 & 0 & I & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & I & 0 \\ 0 & 0 & 0 & \cdot & 0 & I \\ 0 & 0 & 0 & \cdot & 0 & 0 \end{array} \qquad \begin{array}{c} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ I \end{array}$$

(all blocks of size $n\times n$). The matrix $[G,FG,\cdots,F^nG]$ is then a square matrix with anti-diagonal blocks I_n, and thus is an R^* as in 9 (with all m_i=n, r=n+1, and V=I).

Note that the above solution results in a huge number κ; it is of course desirable to have this number be as small as possible. For the $C^o(X)$ case, X contractible, this can be achieved, at least for a weak form of p.a., with κ = m. This fact follows from the result already quoted from [TK], regarding constant rank families $\{B_\lambda\}$. Indeed, a $\{K_\lambda\}$ and $\{L_\lambda\}$ can be found to transform as before the extended system:

$$K = \begin{array}{cc} 0 & I \\ 0 & 0 \end{array} \qquad\qquad L = \begin{array}{c} 0 \\ I \end{array}$$

(both constant) resulting in a family $\{(F_\lambda,G_\lambda)\} := \{(\bar{A}_\lambda+\bar{B}_\lambda K_\lambda,\bar{B}_\lambda L_\lambda\}$ that is again ring controllable but for which the family $\{G_\lambda\}$ now has constant rank.

In general the question of how small a κ should be taken is interesting and relatively unexplored. Using facts from K-theory one may obtain negative results by methods analogous to those used above to show that p.a. is in general impossible with static feedback. Basically the idea is to find controllable systems for which the family $\{B_\lambda\}$ not only does not admit unimodular columns but such that no large-dimensional nonsingular sections exist for the column space of the corresponding extended $\{\bar{B}_\lambda\}$. We do not know of any definitive results in this direction.

5. Stabilization problems for controllable families

In this section we collect results obtained still under assumptions of controllability, but only providing stabilizability rather than pole or coefficient assignment. Since p.a. properties are not desired, it is not needed now to impose ring controllability; so we will only assume pointwise controllability. Of course we have the analogue of proposition 1; in fact this is proved in [KS]:

Proposition 3: Assume that the dynamic stabilization problem is solvable for the family $\{(A_\lambda,B_\lambda), \lambda\in\Lambda\}$. Then the family is ring asycontrollable.

When assuming controllability in order to establish a conclusion about stabilization (not pole assignment), we are in a sense requiring more than, on the basis of this proposition and the intuition provided by the classical case, should be necessary. Notice however that controllability of the family $\{(A_\lambda,B_\lambda)\}$ implies also (easy exercise!) controllability of any family of the type $\{(A_\lambda+\alpha I,B_\lambda)\}$, for any real α. If a stabilization result is applied to this *new* family, the conclusion for the original family is that eigenvalues are being placed in the translated set $\mathbf{C}_s-\alpha$. In the continuous-time case, this means that the original family can be *stabilized with arbitrary degree of convergence*. Conversely, if such a type of stabilization is possible (for any α, find a $\{K_\lambda\}$ such that all eigenvalues of A+BK are in $\mathbf{C}_s-\alpha$), the given family *must* be controllable; this is clear from the proof of proposition 1. (Analogous statements for dynamic stabilization.) Thus, if instead of stabilization we had introduced "stabilization with arbitrary degree of convergence", all results will still be true, and the assumption of pointwise controllability would be necessary as well as sufficient. In any case, as seen in the previous section, p.a. problems are not usually solvable if one requires static solutions, even under such stronger assumptions. On the other hand, *the cases* (*,*,S,*) *have positive solutions* (for pointwise controllable families), *with the exception of the cases* $(\Delta,*,S,\Re[X])$:

Theorem C. Except for the discrete-time polynomial case, pointwise controllable familes can be stabilized with static feedback.

An easy counterexample for the d.t. polynomial case is provided by the family (with m=n=r=1) $\{(1,\lambda^2+1)\}$. This cannot be stabilized by a static feedback, since for no possible polynomial $k(\lambda)$ is the polynomial $1+k(\lambda)(\lambda^2+1)$ less than 1 for all $\lambda\in\Re$. Note that in *continuous* time this *can* indeed be stabilized: we may use for example k:= -2, which insures that $1+k(\lambda)(\lambda^2+1)$ is always negative. And in either discrete or continuous time, for other P than polynomials we may divide by λ^2+1 so that c.a. is possible for this example.

In fact, the above example cannot be stabilized by dynamic feedback either, by the following argument. In general, if a family over $\Re[X]$ is dynamically discrete-time stabilizable then *the systems obtained by evaluation at complex* $\lambda\in\mathbf{C}^r$ *are all asycontrollable*. Indeed, let θ be the closed-loop characteristic polynomial (degree $n+\kappa$). If discrete-time stable for all real λ, it must be *constant as a function of* λ. This is because the possible coefficients of polynomials of any given degree whose roots are in the unit circle form a bounded set. Thus, when the parameters appearing in the K_λ used are evaluated over **C**, the same constant stable θ is obtained, proving the claim. We conclude that the example in the previous paragraph cannot be d.t. stabilizable, because the complex system obtained when λ=i is not asycontrollable.

The proof of Theorem C is due to many authors. Consider first P = C^0 and C^ω. There are basically three known ways to prove the positive results in these cases. The first, which will not generalize to the other cases, is based on the smooth dependence of solutions to quadratic optimal control problems, and was introduced in this context by [D1], [D2]. (See also [BT].) Consider for instance the continuous-time case, and the algebraic Riccati equation

$$QA + A^TQ = QBB^TQ - I,$$

to be solved for a family Q = $\{Q_\lambda\}$ each of whose members is a real symmetric matrix of size n. Since each (A_λ,B_λ) is controllable, and hence "stabilizable" in the classical sense, there is a *unique* solution Q_λ for each λ, and the feedback

$$K_\lambda := -B_\lambda{}^TQ_\lambda \tag{11}$$

stabilizes each such system. So if we prove that Q_λ depends smoothly on the data (A_λ,B_λ), we'll have the desired result over the above P. But smooth dependence can be concluded from an argument based on the implicit function theorem; see the above references for details. The argument in the discrete-time case is entirely analogous, but using the A.R.E. corresponding to discrete-time systems. Note that we then have the desired stabilization result, for P = C^0, C^ω, assuming *only* pointwise asycontrollability.

Another approach, which works for the above cases as well as in the discrete-time rational case $\Re(X)$, was taken by [KK1], based on a method due to [KL]. Consider first the discrete-time cases, and let W:= RR^T, where R is the matrix over P introduced in 4. Since the original family is pointwise controllable, this W is pointwise nonsingular, so in the cases P = C^0, C^ω, $\Re(X)$ we conclude that it is invertible over P. Then, the feedback law $K := -B^T(A^T)^nW^{-1}A^{n+1}$ will stabilize (see the above references). Note that K is indeed defined over P. In the continuous case, one uses a matrix analogous to R but involving an integral (the controllability Gramian of the system), and the proof is analogous, and in fact standard (see for instance the textbook [KA], exercise 9.2-11). However, the integration destroys the rational dependence on parameters, and hence doesn't apply to the $\Re(X)$ continuous time case.

To summarize, we are left with establishing the continuous-time $\Re(X)$ and $\Re[X]$ cases. This requires different techniques from the above. (Note however that, on compact subsets of parameter space, the Stone-Weirstrass theorem can be used to conclude polynomial stabilizability from the continuous case.) It will follow from the material in the next section and the appendix that one may stabilize systems in these cases provided that we allow for *dynamic* feedback. But static feedback will work too. This can be established as in [S4], which is in turn based on the method introduced in [BA]. The following fact is proved in the former:

Proposition 4: Let n,m be integers, and $A=(a_{ij})$, $B=(b_{ij})$ two matrices of distinct indeterminates, of sizes $n\times n$ and $n\times m$ respectively. There exist then:

- an $m\times n$ matrix $K(A,B,\gamma)$ of real polynomials in the a_{ij}, b_{ij}, and another variable γ, and
- (scalar) polynomials $p(A,B,\gamma)$ and $s(A,B,\gamma)$ in the variables a_{ij},b_{ij}, and γ,

such that: (a) when the variables a_{ij}, b_{ij} take values making (A,B) controllable, $p(A,B,\gamma)$ is nonzero, for every real γ, and (b) for any such values of the a_{ij}, b_{ij}, and for each γ, the matrix $A+B(qK)$ has all eigenvalues with real part less than $-\gamma$ whenever q is a (real) number such that $pq>s$. QED

Consider now the case $\Re(X)$ (or, for that matter, C^o, C^ω), and the family $\{(A_\lambda,B_\lambda)\}$. Pick γ:=0 and choose K, p, s, as in proposition 4. Substituting the expressions of the a_{ij}, b_{ij} as functions of λ into the entries of K, p, s, we may assume that these are also in P. By (a), $p(\lambda)$ has no real zeroes. Thus it is invertible over P, and q:= (s+1)/p is in P, so that qK is a stabilizing feedback as desired. The polynomial case is a bit more delicate, and depends on the fact that if $p(\lambda)$ is a polynomial in the variables $\lambda = (\lambda_1,\cdots,\lambda_r)$ which is never zero and if $s(\lambda)$ is also polynomial, then there is a *polynomial* function $q(\lambda)$ such that $pq>s$ for all λ. Thus qK is a polynomially parametrized stabilizing feedback. The existence of such a q follows from the "Real Nullstellensatz", and is discussed in [BS]. (See also the material in the Appendix.)

The proof of proposition 4 is based on the solution of the Lyapunov equation $QA + A^TQ = BB^T$, but using a suitable translate of the original matrix A, and the use of a K analogous to that in 11. We omit the details here. Reference [S4] includes the detailed computer results obtained, using a symbolic manipulation system, when one applies this method to the example described in 10.

In the case of continuous-time pointwise controllable polynomial families, somewhat more than stabilization is achievable. For the case of single-input (m=1) systems, one can "almost" assign (by static feedback) any n-1 poles, with the remaining pole being close to $-\infty$; if m>1 this is in general impossible, but a dynamic version is possible; a discussion is given in [S4]. While on the topic of dynamic stabilization, note that in the $\Re(X)$ case a pointwise controllable family is necessarily also ring controllable, and hence ring asycontrollable. It will then follow from the results in the next section that in the $\Re(X)$ case pointwise controllable families can be stabilized by dynamic feedback, even in the discrete-time case; as we'll see, however, a stronger result is possible, assuming only pointwise asycontrollability.

6. Asycontrollable families

Because of proposition 3 it is of interest to try to establish that asycontrollablity implies stabilizability, at least dynamically. For the cases C^o and C^ω there is no problem in this regard, since we had the results based on Riccati equations, which did *not* require controllability: $(*,*,S,\{C^o,C^\omega\})$ *is solvable* under the pointwise asycontrollability assumption:

Theorem D. For continuous or analytic families, pointwise asycontrollability is equivalent to static stabilizability.

Another positive result in this direction is that (*,D,S,*) *is solvable* for *ring*-asycontrollable families:

Theorem E. Ring-asycontrollability is equivalent to dynamic stabilizability.

This was proved in [E1] (a dual but weaker detectability result had been the main topic of [HS1]). Very briefly, the main point here is that, if one translates the meaning of feedback for an extended system into transfer-matrix terms, one is reduced to trying to solve an equation of the type

$$(zI-A)M(z) + BN(z) = I$$

over a ring P_s as in the definition of ring asycontrollability, but with the added constraint that M be in a suitable sense invertible and NM^{-1} be "causal" or "proper". The proof that one can indeed always satisfy the added constraint given just the ring asycontrollability assumption, is based essentially on a matrix division algorithm; see [E1], [E3], and the simplified version [RO] for details.

A natural question to ask is then, when does pointwise asycontrollability imply the ring notion? In [HS1] it was emphasized (in a more general systems over rings context) that one is really asking for a characterization of the maximal ideals of the rings P_s, and in particular whether these maximal ideals are in natural one-to-one correspondence to those ideals of P[z] that are kernels of evaluations at points $\lambda\in\Re^r$, $z\notin\mathbf{C}_s$. If such is the case, it will follow from local-global principles that the two notions of asycontrollability coincide. For instance, it is proved in [KS] that for r=1 (single-parameter families), $P=\Re[X]$, and *continuous time*, indeed these notions coincide. In discrete time, however, the example $\{(1,\lambda^2+1)\}$ given earlier is pointwise asycontrollable (since pointwise controllable) but not ring asycontrollable, since as remaked earlier it cannot be stabilized by dynamic feedback. In fact, we have now proved that these notions do indeed coincide in all other cases, that is, (*,D,S,*) have positive solutions (for pointwise asycontrollable families), *with the exception of the case* (Δ,D,S,$\Re$[X]), just as with the results in the previous section. It is only necessary to prove the continuous $\Re$(X) and $\Re$[X] cases, and the discrete $\Re$(X) case; these are treated in the appendix. In summary:

Theorem F. Except for the discrete-time polynomial case, pointwise asycontrollable familes can be stabilized with dynamic feedback. Thus, except for that case, pointwise asycontrollability, ring asycontrollability, and dynamic stabilizability are all equivalent notions.

Note the difference with Theorem C, which concludes *static* feedback but at the expense of *controllability* of the family. Further, for a particular system it is perfectly possible that no static stabilizing feedback exist but that a dynamic one be available. In fact, the following family (suggested to us by M.Hautus) provides an example for the discrete time $\Re$[X] case where ring asycontrollability holds but no static feedback stabilizes: (with n=m=r=1) take $\{(\lambda,\lambda^2),\ \lambda\in\Re\}$. Here $S = (z-\lambda,\lambda^2)$, and we can solve the equation $SQ = z^2$ = stable, taking for Q the transpose of $(z+\lambda,1)$. We don't know of any counterexamples showing that one cannot in fact find static stabilizers in the cases treated in the appendix, but it is likely that such a stronger result will be false.

7. Concluding remarks

Many other problems besides stabilization have been studied for systems over rings, and they could be specialized to the case of families; here I restricted attention to stabilization-related problems, but references on others can be found for instance in [S1], [RS], [S2], [E2], [JO], [DH], [CP]. Also, it is of interest to study the *genericity* of the various conditions obtained, like the results in [LO] for controllability in the polynomial ring case.

There is another, more general, definition of "family of systems", which arose originally when dealing with questions different from stabilization (see for example [HA] and [HP]). In this definition, a family over a topological space Λ is given by specifying a vector bundle E over Λ, a bundle homomorphism B: $\Lambda \times \Re^m \rightarrow E$ (the first seen as a trivial bundle over Λ) and a bundle endomorphism A: $E \rightarrow E$. Thus the families considered in this talk correspond to the case of trivial E. Algebraically, the more general definition deals with "systems over rings" with projective state spaces, while that used in this talk restricts to "free systems". In the recent work [HS2] the authors show how this more general notion of families is important when dealing with stabilization problems, even if ultimately interested in the free case. Specifically, when proving a pole assignability result for rings of functions on the unit circle, the natural induction proof proceeds by working with certain line bundles associated to the given family, and their complements, which give rise to families in this more general sense.

Because of the many possible assumptions on the original family $\{(A_\lambda,B_\lambda)\}$, and the possible design objectives, even the number of actual problems fitting the loosely stated goals mentioned in the introduction is larger than that treated here. I apologize in advance to any authors whose work was overlooked when preparing this talk. As a corollary to the above disclaimer, it is clear that omission of a problem does *not* qualify it as an "open problem" whose solution merits automatic publication. Further, the list of references that follows is far from being comprehensive, though we believe that those included provide pointers to most of the literature.

As a final remark, note that in all the problems considered, controllers (the feedback matrices K_λ) depend on the parameters λ; this is in contrast to the analogous, but technically completely different, problem appearing in *robust control* (see e.g. [GO]) where the obtained regulator must be independent of the parameters.

Acknowledgment. I'd like to thank Malo Hautus and Pramod Khargonekar for many useful suggestions on ways to improve this presentation.

I. Appendix: Proofs of new results

We establish here that pointwise asycontrollable continuous-time polynomial or rational families, and discrete time rational families, are in fact ring asycontrollable, and hence, by the results mentioned earlier, can be dynamically stabilized.

We first prove a result about matrices over rings, which generalizes a number of facts concerning pseudoinverses. For this paragraph and the next lemma, P is an arbitrary commutative ring, not necessarily one of the function rings considered elsewhere. Let $Q = (q_{ij})$ be an $n \times m$ matrix over P. For any positive $r \leq \min\{n,m\}$, we denote by $\mathrm{I}_r(Q)$ the ideal of P generated by all the $r \times r$ minors of Q. In general, we let $Q(\alpha,\beta)$, where α and β are ordered sets of indices for rows and columns respectively, denote the minor obtained from the rows/columns indexed by α, β. Thus $\mathrm{I}_r(Q)$ is the set of all linear combinations, with coefficients in P, of the $Q(\alpha,\beta)$ with α and β ordered index sets of cardinality r. If $\alpha = (\alpha_1,\cdots,\alpha_r)$ and ν is an integer, we write "$\nu \in \alpha$" to indicate that there is an index k such that $\alpha_k = \nu$; this index k is then denoted by $\alpha[\nu]$. If $\nu \in \alpha$, $\alpha \backslash \{\nu\}$ denotes the (r-1)-tuple obtained by deleting ν; if $\nu \notin \alpha$, $\alpha \cup \{\nu\}$ is the (r+1)-tuple obtained by inserting ν in the appropiate position of α. Finally, we also let $Q(\phi,\phi) := 1$ and $\mathrm{I}_s(Q) := \{0\}$ if s is larger than $\min\{n,m\}$.

Lemma 5: Let Q be as above, and let θ be an arbitrary element of $\mathrm{I}_r(Q)$. Then there exists a matrix H over P such that

$$QHQ = \theta Q + L$$

for some matrix L all whose entries are in $\mathrm{I}_{r+1}(Q)$.

Proof: This is an easy generalization of the argument given in [BH] as an explicit proof of the result in [S5] about weak generalized inverses of matrices. (The argument in [S5] could also be used, but it would need to be modified in less trivial ways.) Let $\theta = -\sum m_{\alpha,\beta}Q(\alpha,\beta)$ be an expresion in terms of the generators of $I_r(Q)$ (we will omit summation indices when clear from the context). Then, define $H := (h_{ij})$, where

$$h_{ji} := \sum (-1)^{\alpha[i]+\beta[j]+1}Q(\alpha\backslash\{i\},\beta\backslash\{j\})m_{\alpha,\beta} \tag{12}$$

with the sum over all ordered index sets, of cardinality r, α and β for which $i\in\alpha$ and $j\in\beta$. We must prove that, for each indices ν, μ, $(QHQ)_{\nu\mu} = \theta q_{\nu\mu} + L$, with L in $I_{r+1}(Q)$. This is done exactly as in [BH] (which deals essentially with the case $\theta = 1$). First note that, for any such ν, μ, and any fixed index sets as above α, β,

$$\sum (-1)^{\alpha[i]+\beta[j]+1}q_{\nu j}q_{i\mu}Q(\alpha\backslash\{i\},\beta\backslash\{j\}) + q_{\nu\mu}Q(\alpha,\beta) = L, \tag{13}$$

(sum over all $i\in\alpha$ and $j\in\beta$) with L in $I_{r+1}(Q)$. This is proved as follows. Let L := det(C), where C is obtained by adjoining row ν and column μ to the matrix corresponding to α and β. Thus either det(C) = 0 (if $\nu\in\alpha$ or $\mu\in\beta$) or $\det(C) = \pm Q(\alpha\cup\{\nu\},\beta\cup\{\mu\})$, so that L is as required. The formula now follows by expanding first in terms of the last row and then the last column. Now just calculate $(QHQ)_{\nu\mu} = \sum_{i,j}q_{\nu j}h_{ji}q_{i\mu}$. Substituting 12 into h_{ji}, and using property 13, this equals $\theta q_{\nu\mu} + l\sum m_{\alpha,\beta}$. QED

For any family $\{(A_\lambda,B_\lambda)\}$ and each $0 \le r \le \min\{n,m\}$ denote $\Lambda_r = \Lambda_r(A,B) := \{\lambda\in\Lambda$ such that $\text{rank}(B_\lambda)\le r\}$. Note that Λ_o is the set of parameters where B_λ vanishes, and $\Lambda_{\min\{n,m\}} = \Re^r$. Let N = N(B) be the determinantal rank of B, i.e. the largest r such that $I_r(B)$ is nonzero; this is also the smallest r such that $\Lambda_r = \Re^r$. Since parametrizations over P are (at least) continuous and the rank condition is given by the simultaneous vanishing of all $(r+1)\times(r+1)$ minors, Λ_r is always *closed*. When P = $\Re(X)$ or $\Re[X]$, Λ_r is in fact an algebraic subvariety of $\Lambda = \Re^r$.

We shall say that a fixed pair (F,G) is *full*(-control) if the column space of G is F-invariant, and that $\{(A_\lambda,B_\lambda)\}$ is a (pointwise) full family if for each λ the correponding (A_λ,B_λ) is full-control. Being full is of course an extremely restrictive condition: it says that whatever can be controlled can in fact be "instantaneously" controlled. The interesting point, to be made more precise later, is that *for dynamic feedback problems one may reduce problems to the full case*. (To some extent, this is the idea behind theorem (3.22) in [HS1].) For reasons that will become clear later, we prefer to use the notation $\{(A_\lambda,R_\lambda)\}$ for full families.

Lemma 6: Assume that $\{(A_\lambda,R_\lambda)\}$ is a continuous-time pointwise asycontrollable *full* family, with P = $\Re(X)$ or $\Re[X]$, and that A_λ is stable whenever λ is in Λ_{r-1}. Then, there exists a family $\{K_\lambda\}$ over P such that $A_\lambda+B_\lambda K_\lambda$ is stable whenever λ is in Λ_r.

Proof: We apply lemma 5 with $\theta := \sum(R(\alpha,\beta))^2$ (sum over all possible r-minors), and Q:= R. Let H be the matrix obtained there. The feedback to be obtained will have the form

$$K := -\gamma\theta H,$$

where γ is a polynomial to be chosen. Let E:= RH. Consider the function $\tau: \Re\to\Re$,

$$\tau(\rho) := \inf\{g\in\Re \text{ s.t. for } g'>g \text{ and } \lambda\in\Lambda_r \text{ with } \|\lambda\|^2\le\rho^2,\ A_\lambda - g'\theta(\lambda)E_\lambda \text{ is stable}\} \tag{14}$$

($\|\cdot\|$ indicates Euclidean norm on $\Re^r$). Assume for now that this function is well defined, i.e. that for each ρ the set in question is indeed nonempty. Then τ is symmetric and is nondecreasing for $\rho\ge0$. More interestingly, τ is a *semialgebraic function*, in the following sense. Consider the graph $GR(\tau) = \{(\rho,\tau(\rho)),\rho\in\Re\}$. Then this graph, as a subset of $\Re^2$, can be described by an equation in the first order theory of the real numbers with addition and multiplication (the first-order theory of real-closed fields; see for instance the survey in [RA] and references there). It is a straightforward (but very tedious) exercise in elementary logic to rewrite the definition of τ in terms of universal and existential quantification over real numbers, and we omit this. Note that Λ_r can be described by algebraic equalities, which is essential here, as well as that stability of a matrix can be described by a first-order statement over the reals, by writing complex numbers in terms of their real and imaginary parts (or, by application of the Routh-Hurwicz criterion). Further, in the rational case $P=\Re(X)$ it is necessary to take a common denominator of all rational functions appearing, since division is not explicitely allowed in the language, but this presents no difficulty.

The important fact is that, by the Tarski-Seidenberg theorem on quantifier elimination, it is also possible to express this graph in *quantifier-free* form. In particular, following the algorithm in [CO], one has that there exists a positive number ρ_o such that, for $\rho > \rho_o$, the graph of τ satisfies an algebraic equation $P(\rho,\tau(\rho))=0$, for some polynomial $P(x,y)$. Write P as a polynomial in y: $P(x,y) = a_\mu(x)y^\mu + \cdots + a_o(x)$. Taking if necessary a larger ρ_o, we may assume that $a_\mu(x)^2 > 1$ for $x > \rho_o$. For any fixed x, all roots of P(x,y) are bounded in magnitude by the rational function $= \mu + a_\mu(x)^{-2}[(a_o(x))^2 + \cdots + (a_{\mu-1}(x))^2]$, and hence by the polynomial $\phi(x) = \mu + (a_o(x))^2 + \cdots + (a_{\mu-1}(x))^2$. Thus, $\tau(\rho) < \phi(\rho)$ for $\rho>\rho_o$. Since τ increases for positive ρ, one may add a linear function to $\phi(\rho)$ to obtain a new polynomial ϕ' such that $\tau(\rho) < \phi'(\rho^2)$ for positive ρ. Let $\gamma(\lambda):= \phi'(\|\lambda\|^2)$. It then follows from the definition of τ that $A_\lambda-\gamma(\lambda)\theta(\lambda)E_\lambda$ is indeed stable for all λ in Λ_r.

We are left with proving that τ is well-defined. Since the intersection of Λ_r and any ball of radius ρ is compact, it is sufficient to prove that

(*) for each $\lambda\in\Lambda_r$ there exists a neighborhood U of λ and a $g\geq 0$ such that, for each $\mu \in U\cap\Lambda_r$ and each $g'>g$, $A_\mu-g'\theta(\mu)E_\mu$ is stable.

This is proved as follows. First note the following property:

(**) If $g\geq 0$ and $\lambda \in \Lambda_r\backslash\Lambda_{r-1}$, then each eigenvalue of $A_\lambda-gE_\lambda$ is either a stable eigenvalue of A_λ or is of the form $\sigma-\theta(\lambda)g$, with σ an eigenvalue of A_λ.

Indeed, $E^2 = \theta E + LH$, so when λ is in Λ_r the last matrix vanishes and E_λ is annihilated by the polynomial $z(z-\theta(\lambda))$. If, further, $\lambda\notin\Lambda_{r-1}$ then $\theta(\lambda)\neq 0$, so that there exists for each such λ an invertible real matrix T such that $F:= T^{-1}A_\lambda T$ and $G:= T^{-1}E_\lambda T$ have the block forms, respectively,

$$\begin{matrix} X & Y \\ 0 & Z \end{matrix} \qquad\qquad \begin{matrix} \theta(\lambda) & 0 \\ 0 & 0 \end{matrix}\quad , \tag{15}$$

with Z a stable matrix. The form for G is due to the fact that in this situation the minimal polynomial of E_λ is $z(z-\zeta)$, $\zeta=\theta(\lambda)\neq 0$. The form of F is established as follows. Since $E_\lambda R_\lambda = \zeta R_\lambda$, the column spaces of R_λ and of E_λ coincide for this λ. Thus, (A_λ,E_λ) is again full, and is also asycontrollable ("stabilizable" in the usual sense). So the same properties hold for the pair (F,G). Since ζ is nonzero, this means that the (2,1) block of F corresponding to the above form of G must indeed be zero. By the stabilizability property, Z must be already stable. Finally, since the eigenvalues of $A_\lambda-gE_\lambda$ are those of F-gG, we conclude that (**) is true.

We are now ready to prove (*), and hence complete the proof of the lemma. Consider first the case when $\lambda\in\Lambda_{r-1}$. Here let

$$U := \{\lambda \in \Re^r \text{ such that } A_\lambda \text{ is stable}\}. \tag{16}$$

Note that this is an open set, since the parametrization is continuous. Further, the hypotheses of the lemma imply that Λ_{r-1} is contained in U. Let g:= 0. Pick any $\mu \in U \cap \Lambda_r$ and any g'>g. If μ happens to be in Λ_{r-1} then $A_\mu - g'\theta(\mu)E_\mu = A_\mu$ (because θ vanishes on Λ_{r-1}), and is hence stable. If μ is in $U \cap (\Lambda_r \backslash \Lambda_{r-1})$ then by (**) the eigenvalues of $A_\mu - g'\theta(\mu)E_\mu$ are either those of A_μ or left translates of such; in either case, they are stable.

Now consider the case when $\lambda \in \Lambda_r \backslash \Lambda_{r-1}$. Note that $\theta(\lambda)>0$. Thus there exists a relatively compact neighborhood U of λ where θ is bounded away from zero, say such that $\theta^2 > c > 0$ there. Since U is relatively compact, there is an upper bound d>0 on the real parts of the eigenvalues of A_λ for λ in U. Pick g:= d/c. Then (**) implies that (*) holds around λ. QED

It is important to note that the compactness argument used above was introduced only to *prove the nonemptyness* of the set in question; the actual algorithm for elimination of quantifiers, when applied to the corresponding first order formula, will result in the same conclusion. So no approximations are used in implementing the above. (On the other hand, from a "practical" point of view the proof given is perhaps not too interesting, since the elimination of quantifiers for real-closed fields is a process of very high time complexity.)

There is an analogue in the *rational* discrete time case, but of course not in the polynomial case, as the previously seen example $\{(1,\lambda^2+1)\}$ showed.

Lemma 7: Assume that $\{(A_\lambda,R_\lambda)\}$ is a discrete-time pointwise asycontrollable *full* family, with P = $\Re(X)$, and that A_λ is stable whenever λ is in Λ_{r-1}. Then, there exists a family $\{K_\lambda\}$ over P such that $A_\lambda+R_\lambda K_\lambda$ is stable whenever λ is in Λ_r.

Proof: The proof is similar to that of the above lemma, but there are a few technical complications. The feedback K will now have the form

$$K := -T(RT+I)^{-1}A,$$

where T is a rational family to be chosen. Since RT commutes with $(RT+I)^{-1}$, the closed loop matrix A+RK when using this K will be $(RT+I)^{-1}A$. The problem is then to find a family T such that

$$\det(RT+I) \neq 0 \text{ on all of } \Re^r, \text{ and} \tag{17}$$

$$(RT+I)^{-1}A \text{ is stable (eigenvalues of magnitude < 1) on } \Lambda_r. \tag{18}$$

We shall proceed in two parts. First we show that

(i) there is a matrix T_o of rational functions on $\Re^r$ (possibly with singularities) such that all eigenvalues of $(RT_o)_\lambda$ have nonnegative real part when $\lambda \in \Lambda_r$ (so, in particular, $\det(RT_o+I)\neq 0$ on Λ_r) and $(RT_o+I)^{-1}A$ is stable (eigenvalues of magnitude < 1) on Λ_r,

and, if r<N (so that $\Lambda_r \neq \Re^r$,) we then establish that, for any fixed ε with $0<\varepsilon<1$, and for each k with $0 \leq k \leq N-r$,

(ii) there exists a matrix T_k of rational functions on $\Re^r$ (possibly with singularities) such that (a) all eigenvalues of $(RT_k)_\lambda$ have real part > $-\varepsilon$ whenever $\lambda \in \Lambda_{r+k}$, and (b) $(RT_k)_\lambda = (RT_o)_\lambda$ when $\lambda \in \Lambda_r$.

The case k=N-r of the above will then provide a T = T_k that satisfies the desired properties 17 and 18. The proof of (ii) will be by induction on k; note that the case k=0 follows from (i).

The form of T_o will be as follows. Let θ be as in the previous lemma, the sum of the squares of all r-minors of R, and pick H, E=RH as there. Then $T_o := \gamma\theta H$, where γ is a polynomial to be chosen. We introduce a function $\tau: \mathfrak{R}\to\mathfrak{R}$ analogous to that in 14:

$\tau(\rho) := \inf\{g\in\mathfrak{R}$ s.t., for each g'>g and each $\lambda\in\Lambda_r$ with $\|\lambda\|^2\le\rho^2$, all eigenvalues of $g'\theta(\lambda)E_\lambda$ have nonnegative real part and $(g'\theta(\lambda)E_\lambda+I)^{-1}A_\lambda$ is stable$\}$

The same general properties of τ hold as with 14. Note that one can rewrite the stability eigenvalue constraint for each λ without any divisions, using cofactor matrices. The proof is completed as before, provided we can prove that τ is well-defined: a polynomial function γ that bounds τ from above will then exist (and be constructible from the formula defining τ) and the resulting T_o will satisfy (i). We need to prove then the analogue of (*):

(2) for each $\lambda\in\Lambda_r$ there exists a neighborhood U of λ and a $g\ge 0$ such that, for each $\mu\in U\cap\Lambda_r$ and each g'>g, (a) all eigenvalues of $g'\theta(\mu)E_\mu$ have nonnegative real part and (b) $(g'\theta(\mu)E_\mu+I)^{-1}A_\mu$ is stable}

The proof is again analogous to that of the earlier statement. Note the following property:

(22) If $g\ge 0$ and $\lambda\in\Lambda_r\backslash\Lambda_{r-1}$, then (a) each eigenvalue of $(g'\theta(\lambda)E_\lambda+I)^{-1}A_\lambda$ is either a stable eigenvalue of A_λ or is of the form $(g'\theta(\lambda)^2+1)^{-1}\sigma$, with σ an eigenvalue of A_λ, and (b) each eigenvalue of $\theta(\lambda)E_\lambda$ is either 0 or $\theta(\lambda)^2$.

As before, $E^2 = \theta E + LH$, so when λ is in Λ_r the last matrix vanishes and E_λ is annihilated by the polynomial $z(z-\theta(\lambda))$. If, further, $\lambda\notin\Lambda_{r-1}$ then $\theta(\lambda)\neq 0$, so that there exists for each given such λ an invertible real matrix T such that $F:= T^{-1}A_\lambda T$ and $G:= T^{-1}E_\lambda T$ have the block forms, respectively, given in 15, with Z now stable in the discrete-time sense. Thus (b) is clear, and (a) follows from the fact that the eigenvalues in question are those of $(g'\theta(\lambda)G+I)^{-1}F$.

We use this to prove (2). Condition (a) on the eigenvalues of $g'\theta(\lambda)E_\lambda$ is clearly satisfied globally (on Λ_r) with g=0: on Λ_{r-1}, $\theta(\lambda) = 0$ so the eigenvalues are all zero; on $\Lambda_r\backslash\Lambda_{r-1}$, (22) insures that these are (real and) nonnegative. We now prove (b). Consider first the case when $\lambda\in\Lambda_{r-1}$. Here let U be exactly as with 16, except that stability is of course meant in the discrete time sense. This is again an open set containing Λ_{r-1}. As before, we choose g:= 0. Pick any $\mu\in U\cap\Lambda_r$ and any g'>g. On Λ_{r-1}, $(g'\theta(\mu)E_\mu+I)^{-1}A_\mu = A_\mu$ and is hence stable. If μ is in $U\cap(\Lambda_r\backslash\Lambda_{r-1})$ then by (22) the eigenvalues of $(g'\theta(\mu)E_\mu+I)^{-1}A_\mu$ are either those of A_μ or multiples of such by a number <1; in either case, they are stable. Now consider the case when $\lambda\in\Lambda_r\backslash\Lambda_{r-1}$. Note that $\theta(\lambda)>0$. Thus there exists a relatively compact neighborhood U of λ where θ is bounded away from zero, say such that $\theta^2 > c > 0$ there. Since U is relatively compact, there is an upper bound d>0 on the magnitudes of the eigenvalues of A_λ for λ in U. Pick now g:= (d-1)/c. Then (22) implies that (2) holds around such λ too. We have then completed the proof of (i).

There remains to establish (ii), by induction on k. As remarked earlier, the case k=0 is a consequence of (i). Assume then that (ii) is true for a given k. Apply lemma 5, with θ:= sum of squares of minors of size r+k+1. Let H be as there, and denote E:= RH. We shall find a T_{k+1} of the form

$$T_{k+1} = T_k + \gamma\theta H,$$

where γ is a polynomial to be chosen. Note that, since this θ vanishes on Λ_{r+k}, property (b) in (ii) is automatically satisfied with any γ, by induction. So we only need to guarantee invertibility. The critical observation now is that

On $\Lambda_{r+k+1} \backslash \Lambda_{r+k}$, each eigenvalue of $(RT_k+gE)_\lambda$ is either 0 or is of the form $\sigma + g\theta(\lambda)$, with σ an eigenvalue of RT_k.

Arguing as before, E can be brought into the block form of the second matrix in 15. Using the same similarity, RT_k becomes a matrix as the first one in 15 but with Z=0. This is because for each such λ the column space of E_λ coincides with the column space of R, and hence includes the column space of RT_k. This proves the above observation. Now consider $\tau: \Re \to \Re$ defined by

$\tau(\rho) := \inf\{g \in \Re$ s.t., for each $g' > g$ and each $\lambda \in \Lambda_{r+k+1}$ with $\|\lambda\|^2 \leq \rho^2$, all eigenvalues of $(RT_k)_\lambda + g'\theta(\lambda)E_\lambda$ have real part $> -\varepsilon\}$

Again this is definable in first-order terms, and admits a polynomial as an upper bound, as long as we can prove the appropiate local result. Around a λ in Λ_{r+k} we may chose the open neighborhood U:= $\{\lambda \in \Re^r$ such that all eigenvalues of RT_k have real part larger than $-\varepsilon\}$; by inductive hypothesis, this set indeed contains Λ_{r+k}. For μ in $U \cap \Lambda_{r+k+1}$, either the above matrix reduces to RT_k (when $\mu \in \Lambda_{r+k}$) or each of its eigenvalues is either zero or a nonnegative translate of an eigenvalue of RT_k, and so is $>-\varepsilon$. For λ in $\Lambda_{r+k+1} \backslash \Lambda_{r+k}$, pick a relatively compact neighborhood where θ is bounded away from zero; since the eigenvalues σ of RT_k are bounded on U, an appropiate g will exist such that $\sigma + g\theta(\lambda)^2$ is nonnegative there. This completes the proof of the lemma. QED

Corollary 8: Assume that $\{(A_\lambda, R_\lambda)\}$ is a pointwise asycontrollable *full* family, in continuous-time with P = $\Re(X)$ or $\Re[X]$ or in discrete-time with P = $\Re(X)$. Then, there exists a family $\{K_\lambda\}$ over P such that $A_\lambda + B_\lambda K_\lambda$ is stable for all λ. In particular, $\{(A_\lambda, R_\lambda)\}$ is *ring*-asycontrollable.

Proof: This is by induction on r on the lemmas. For r=1, the hypothesis on Λ_{r-1} holds, because if a pair (A,0) is asycontrollable, then A must be necessarily stable. So it is only necessary to remark that, if $\{(A_\lambda, R_\lambda)\}$ is pointwise asycontrollable and full, the same holds for any new family $\{(A_\lambda + R_\lambda K_\lambda, R_\lambda)\}$. But these are easily seen to be feedback invariant properties. QED

We now turn to the proof of the general case. The following lemma is valid for any of the rings P considered in this talk, and for both discrete and continuous time.

Lemma 9: Let $\{(A_\lambda, B_\lambda)\}$ be a family for which the associated family $\{(A_\lambda, R_\lambda)\}$ is ring asycontrollable, where $R_\lambda = R(A_\lambda, B_\lambda)$ is as in 4. Then $\{(A_\lambda, B_\lambda)\}$ is itself ring asycontrollable.

Proof: We argue spectrally as in [HS1]. Let M be a maximal ideal of the ring of stable rational functions P_s introduced earlier, and consider the reductions modulo M of the matrices $S(A_\lambda, B_\lambda)$ and $S(A_\lambda, R_\lambda)$, say S and S' respectively. Let k be the residue field P_s/M. Thus there are matrices F,G over k, and a $w \in k$, such that

$$S = [wI-F, G]$$

$$S' = [wI-F, G, FG, \cdots, F^{n-1}G],$$

with F of size n by n. Further, by assumption $S(A_\lambda, R_\lambda)$ is right invertible over P_s, so the reduced matrix S' is right invertible, i.e. full rank, over the *field* k. We want to prove that S is full rank over k; since M was arbitrary, this will imply the desired right invertibility (over P_s) of S(A,B). But this is easy to establish: a vector ν in the left nullspace of S is necessarily a left eigenvector of F and hence is also in the nullspace of S'. QED

Lemma 10: Let R_λ be as above. If $\{(A_\lambda,B_\lambda)\}$ is pointwise asycontrollable, then $\{(A_\lambda,R_\lambda)\}$ also is.

This is clear because the column space of R_λ includes that of B_λ.

Theorem 11: For pointwise asycontrollable families, the cases $(d,D,S,\{\mathfrak{R}[X],\mathfrak{R}(X)\})$ are solvable for continuous-time and $(\Delta,D,S,\mathfrak{R}(X))$ is solvable for discrete-time.

Proof: We must prove that pointwise asycontrollability implies the ring notion. By 9 and lemma 10, we may deal with $\{(A_\lambda,R_\lambda)\}$, which is full. The result then follows from corollary 8. QED

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