Algebraic semantics and model completeness for Intuitionistic Public Announcement Logic

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Abstract

In this paper, we start studying epistemic updates using the standard toolkit of duality theory. We focus on public announcements, which are the simplest epistemic actions, and hence on single-agent[1] Public Announcement Logic (PAL) without the common knowledge operator. As is well known, the epistemic action of publicly announcing a given proposition is semantically represented as a process of relativization of the model encoding the current epistemic setup of the given agents; from the given model to its submodel relativized to the announced proposition. We give the dual characterization of the corresponding submodel-injection map, as a certain pseudo-quotient map between the complex algebras respectively associated with the given model and with its relativized submodel. As is well known, these complex algebras are complete atomic BAOs (Boolean algebras with operators). The dual characterization we provide naturally generalizes to much wider classes of algebras, which include, but are not limited to, arbitrary BAOs and arbitrary modal expansions of Heyting algebras (HAOs). In this way, we access the benefits and the wider scope of applications given by a point-free, intuitionistic theory of epistemic updates. As an application of this dual characterization, we axiomatize the intuitionistic analogue of PAL, which we refer to as IPAL, and prove soundness and completeness of IPAL w.r.t. both algebraic and relational models. We also discuss and motivate the conjecture that the muddy children puzzle can be formalized using IPAL, leaving the actual formalization to future work.

1 Introduction

The Logic of Public Announcements (PAL) is the simplest logical framework within the family of Dynamic Epistemic Logics (DELs). It was introduced by Plaza in [14] and subsequently intensively studied, both specifically and as part of the DEL-family, viz. [1, 9, 5] and references therein. Public announcements are those epistemic actions whose corresponding epistemic updates can be semantically represented by means of an operation of *relativization*. Namely, the epistemic action of publicly announcing α corresponds to a shift in the models encoding the current epistemic setup of the given agents: the given model M shifts to its submodel M^α, based on $[\![\alpha]\!]_M = \{w \in M \mid M, w \Vdash \alpha\}$. It is also well known that PAL, as well as other logics in the DEL family, is difficult to treat by means of standard algebraic methods, due to the fact that its axiomatization is not schematic (i.e., closed under substitutions). However, studies which address these logics from an algebraic viewpoint are present in the literature, viz. [2].

In this paper, we present an approach which is yet different from the ones mentioned above, and which makes use of the standard duality toolkit. Namely, we give the dual characterization of the injection map from M^α into M. Unsurprisingly, this injection map can be dually characterized as a certain pseudo-quotient between the complex algebras of the underlying frames of M and of M^α (which, as is well known, are - up to isomorphism - complete atomic BAOs). This dual characterization naturally generalizes to much wider classes of algebras, which include, but are not limited to, arbitrary BAOs and arbitrary modal expansions of Heyting algebras (HAOs). In this way, we access the benefits and the wider scope of applications of a point-free, intuitionistic perspective on epistemic updates. As an application of this construction, we axiomatically introduced a logic, IPAL, which is intended to be the intuitionistic analogue of PAL. Just as the static fragment of PAL is given by classical S5, the static fragment of IPAL is given by the intuitionistic modal logic MIPC, which is considered with large consensus the intuitionistic counterpart of S5. We prove soundness and completeness of IPAL with respect to both algebraic and relational models. Just as the proof of completeness for the classical PAL is obtained by a reduction process of PAL formulas to S5-formulas, the completeness proof for IPAL is is obtained by reducing IPAL formulas to MIPC-formulas. The structure of the paper goes as follows: in Section 2, we collect the needed preliminaries on public announcement logic and intuitionistic modal logic. In Section 3 we provide the dual, algebraic characterization of public announcements. In Section 4, the intuitionistic public announcement logic IPAL is axiomatically introduced as well as its interpretation on models based on abstract Heyting algebras. We prove algebraic soundness and completeness for IPAL, both w.r.t. algebraic and relational models. We discuss and motivate the conjecture that the muddy children

[1]The results straightforwardly extend to the multi-agent setting.

puzzle can be formalized using IPAL in section 5, leaving the actual formalization to future work. Conclusions and open problems are listed in Section 6. Details of proofs are collected in Section 7, the appendix.

2 Preliminaries

2.1 The logic of public announcements

Let AtProp be a countable set of proposition letters. The formulas of (single-agent) public announcement logic PAL are built by the following inductive rule:

$$\varphi ::= p \in \mathsf{AtProp} \mid \neg\varphi \mid \varphi \vee \psi \mid \Diamond\varphi \mid \langle\alpha\rangle\varphi.$$

Models for PAL are Kripke models $M = (W, R, V)$ such that R is an equivalence relation. The evaluation of the static fragment of the language is standard. Formulas of form $\langle\alpha\rangle\varphi$ are evaluated as follows:

$$M, w \Vdash \langle\alpha\rangle\varphi \quad \text{iff} \quad M, w \Vdash \alpha \text{ and } M^\alpha, w \Vdash \varphi,$$

where $M^\alpha = (W^\alpha, R^\alpha, V^\alpha)$ is defined as follows: $W^\alpha = [\![\alpha]\!]_M$, $R^\alpha = R \cap (W^\alpha \times W^\alpha)$ and for every $p \in \mathsf{AtProp}$, $V^\alpha(p) = V(p) \cap W^\alpha$.

Proposition 1 ([4, Theorem 27])**.** *PAL is axiomatized completely by the axioms for the modal logic S5 plus the following axioms:*

1. $\langle\alpha\rangle p \leftrightarrow (\alpha \wedge p)$;
2. $\langle\alpha\rangle\neg\varphi \leftrightarrow (\alpha \wedge \neg\langle\alpha\rangle\varphi)$;
3. $\langle\alpha\rangle(\varphi \vee \psi) \leftrightarrow (\langle\alpha\rangle\varphi \vee \langle\alpha\rangle\psi)$;
4. $\langle\alpha\rangle\Diamond\varphi \leftrightarrow (\alpha \wedge \Diamond(\alpha \wedge \langle\alpha\rangle\varphi))$.

2.2 The intuitionistic modal logic MIPC

Introduced by Prior with the name MIPQ [15], the intuitionistic modal logic MIPC is largely considered the intuitionistic analogue of S5. The logic MIPC has been studied by many authors, viz. [6, 7] and the references therein. In this section we briefly review the notions and facts needed for the purposes of the present paper, and we refer to [6, 7] for their attribution. The formulas of MIPC are built by the following inductive rule:

$$\varphi ::= \bot \mid p \in \mathsf{AtProp} \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \to \psi \mid \Diamond\varphi \mid \Box\varphi.$$

Let $\top$ be defined as $\bot \to \bot$ and, for very formula φ, let $\neg\varphi$ be defined as $\varphi \to \bot$. The logic MIPC is the minimal set of formulas in this language which contains all the axioms of intuitionistic propositional logic, the following modal axioms:

$\Box p \to p,\ p \to \Diamond p,;$

$(\Box p \wedge \Box q) \to \Box(p \wedge q),\ \Diamond(p \vee q) \to (\Diamond p \vee \Diamond q),;$

$\Diamond p \to \Box\Diamond p,\ \Diamond\Box p \to \Box p,;$

$\Box(p \to q) \to (\Diamond p \to \Diamond q);$

and is closed under substitution, modus ponens and necessitation ($\varphi/\Box\varphi$).

The relational structures for MIPC, called *MIPC-frames*, are triples $\mathcal{F} = (W, \leq, R)$ such that $(W, \leq)$ is a nonempty poset and R is a binary equivalence relation such that $(R \circ \leq) \subseteq (\leq \circ R)$. *MIPC-models* are structures $M = (\mathcal{F}, V)$ such that $\mathcal{F}$ is a relational structure as specified above and $V : \mathsf{AtProp} \to \mathcal{P}^{\uparrow}(W)$ is a function mapping proposition letters to upward-closed subsets of W[2]. For any such model, its associated extension map $[\![\cdot]\!]_M \to \mathcal{P}^{\uparrow}(W)$ is defined recursively as follows:

$$
\begin{array}{rcl}
[\![p]\!]_M & = & V(p); \\
[\![\bot]\!]_M & = & \varnothing; \\
[\![\varphi \vee \psi]\!]_M & = & [\![\varphi]\!]_M \cup [\![\psi]\!]_M; \\
[\![\varphi \wedge \psi]\!]_M & = & [\![\varphi]\!]_M \cap [\![\psi]\!]_M; \\
[\![\varphi \to \psi]\!]_M & = & ([\![\varphi]\!]_M \cap [\![\psi]\!]^c_M){\downarrow}^c; \\
[\![\Diamond\varphi]\!]_M & = & R^{-1}[[\![\varphi]\!]_M]; \\
[\![\Box\varphi]\!]_M & = & ((\leq \circ R)^{-1}[[\![\varphi]\!]^c_M])^c;
\end{array}
$$

where $(.)^c$ is the complement operation. For any model M and any MIPC formula φ, we write:

$M, w \Vdash \varphi$ if $w \in [\![\varphi]\!]_M$;

$M \Vdash \varphi$ if $[\![\varphi]\!]_M = W$;

$\mathcal{F} \Vdash \varphi$ if $[\![\varphi]\!]_M = W$ for any model M based on $\mathcal{F}$.

Proposition 2. *MIPC is sound and complete with respect to the class of MIPC-frames.*

The algebraic semantics for MIPC is given by a variety of Heyting algebras with operators (HAOs) which are called monadic Heyting algebras:

Definition 3. The algebra $\mathbb{A} = (A, \wedge, \vee, \to, \bot, \Diamond, \Box)$ is a *monadic Heyting algebra* (MHA) if $(A, \wedge, \vee, \to, \bot)$ is a Heyting algebra and the following inequalities hold:

$\Box x \leq x$, $x \leq \Diamond x$;

$\Box x \to \Box y \leq \Box(x \to y)$; $\Diamond(x \vee y) \leq (\Diamond x \vee \Diamond y)$;

$\Diamond x \leq \Box\Diamond x$, $\Diamond\Box x \leq \Box x$;

$\Box(x \to y) \leq \Diamond x \to \Diamond y$.

The inequalities above can be equivalently written as equalities, thanks to the fact that, in any Heyting algebra, $x \leq y$ iff $x \to y = \top$. Clearly, any formula in the language $\mathcal{L}$ of MIPC can be regarded as a term in the algebraic language of MHAs. Therefore, given a monadic Heyting algebra $\mathbb{A}$ and an interpretation $V : \mathsf{AtProp} \to \mathbb{A}$, an MIPC formula φ is *true* in $\mathbb{A}$ under the interpretation V (notation: $(\mathbb{A}, V) \models \varphi$) if the unique homomorphic extension of V, which we denote $[\![\cdot]\!]_V : \mathcal{L} \to \mathbb{A}$, maps φ to $\top^{\mathbb{A}}$. An MIPC formula is *valid* in $\mathbb{A}$ (notation: $\mathbb{A} \models \varphi$), if $(\mathbb{A}, V) \models \varphi$ for every interpretation V.

MIPC-frames give rise to complex algebras, just as Kripke frames do: for any MIPC-frame $\mathcal{F}$, the *complex algebra* of $\mathcal{F}$ is

$$\mathcal{F}^+ = (\mathcal{P}^{\uparrow}(W), \cap, \cup, \Rightarrow, \varnothing, \langle R \rangle, [\leq \circ R]),$$

where for every $X, Y \in \mathcal{P}^{\uparrow}(W)$,

$$\langle R \rangle X = R^{-1}[X], \quad [\leq \circ R]X = ((\leq \circ R)^{-1}[X^c])^c, \quad X \Rightarrow Y = (X \cap Y^c){\downarrow}^c.$$

Clearly, given a model $M = (\mathcal{F}, V)$, the extension map $[\![\cdot]\!]_M : \mathcal{L} \to \mathcal{F}^+$ is the unique homomorphic extension of $V : \mathsf{AtProp} \to \mathcal{F}^+$.

Proposition 4. *For every MIPC-model $(\mathcal{F}, V)$ and every MIPC formula φ,*

1. $(\mathcal{F}, V) \Vdash \varphi$ *iff* $(\mathcal{F}^+, V) \models \varphi$.
2. $\mathcal{F}^+$ *is a monadic Heyting algebra.*

[2]For every poset $(W, \leq)$, a subset Y of W is *upward-closed* if for every $x, y \in W$, if $x \leq y$ and $x \in Y$ then $y \in Y$.

3 Epistemic updates on algebras

In this section, the operation of epistemic update on Kripke models is dualized: for every algebra $\mathbb{A}$ and every $a \in \mathbb{A}$, a quotient-like algebra $\mathbb{A}^a$ is defined in such a way that, whenever $\mathbb{A}$ is the algebraic dual of some frame $\mathcal{F}$, $\mathbb{A}^a$ is (isomorphic to) the algebraic dual of the subframe $\mathcal{F}^a$. We will proceed in two stages: first, for every $a \in \mathbb{A}$ we will introduce the equivalence relation which will be used to quotient out the algebra $\mathbb{A}$; this equivalence relation is a congruence on several classes of algebras which include Boolean and Heyting algebras, but it is not in general compatible with the modal operators, hence it is not in general a congruence on BAO's or on HAO's. This is of course unsurprising, since the corresponding operation of epistemic update on relational models produces submodels which are not in general generated submodels of the original models. The second stage focuses on the definition of the modal operators on the algebra $\mathbb{A}^a$, the proof of their normality and that indeed, whenever $\mathbb{A}$ is the algebraic dual of some frame $\mathcal{F}$, $\mathbb{A}^a$ is (isomorphic to) the algebraic dual of the subframe $\mathcal{F}^a$. This construction and facts hold uniformly in the Heyting algebra setting.

3.1 Updates as pseudo-quotients

Throughout this subsection, and unless specified otherwise, let $\mathbb{A}$ be a $\wedge$-semilattice and let $a \in \mathbb{A}$. Then, define the following equivalence relation $\equiv_a$ on $\mathbb{A}$: for every $b, c \in \mathbb{A}$,

$$b \equiv_a c \text{ iff } b \wedge a = c \wedge a.$$

Let $[b]_a$ be the equivalence class of $b \in \mathbb{A}$. Mostly we will drop the subscript when there is no risk of confusion. Let us denote the quotient set $\mathbb{A}/{\equiv_a}$ by $\mathbb{A}^a$. Clearly, $\mathbb{A}^a$ is an ordered set by putting $[b] \leq [c]$ iff $b' \leq_{\mathbb{A}} c'$ for some $b' \in [b]$ and some $c' \in [c]$. Let $\pi_a : \mathbb{A} \to \mathbb{A}^a$ be the canonical projection given by $b \mapsto [b]$.

The relation $\equiv_a$ and its properties are well known[3]. Particularly relevant for us is that $\equiv_a$ is a congruence if $\mathbb{A}$ is a Boolean algebra, a Heyting algebra, a bounded distributive lattice or a frame (as shown in Fact 7 below). Hence, $\mathbb{A}^a$ is canonically endowed with the same algebraic structure of $\mathbb{A}$ in each of these cases. Another property of $\equiv_a$ is as crucial as is straightforward:

Fact 5. *Let $\mathbb{A}$ be a $\wedge$-semilattice and let $a \in \mathbb{A}$. For every $b \in \mathbb{A}$, there exists a unique $c \in \mathbb{A}$ such that $c \in [b]_a$ and $c \leq a$.*

Proof. As to existence, since $\wedge$ is idempotent, $(b \wedge a) \wedge a = b \wedge a$; hence $b \wedge a \in [b]_a$. As to uniqueness, if $c \in [b]_a$ and $c \leq a$, then $c = c \wedge a = b \wedge a$. □

The fact above implies that each $\equiv_a$-equivalence class has a canonical representant, namely the only element in that class that is less than or equal to a. Hence, the map $i' : \mathbb{A}^a \to \mathbb{A}$ given by $[b] \mapsto b \wedge a$ is well defined. Clearly, $\pi \circ i'$ is the identity on $\mathbb{A}^a$. The map i' will be a critical ingredient for the definition of the interpretation of IPAL-formulas on algebraic models (cf. Definition 12), justified by the following:

Proposition 6. *If $\mathbb{A} = \mathcal{F}^+$, then for every $c \in \mathbb{A}^a \cong \mathcal{F}^{a+}$, $i'(c) = i[c]$.*

Proof. For every $c \in \mathcal{F}^{a+}$, $c \subseteq a$, hence $i'(c) = c \wedge a = c = i[c]$. □

The remainder of this subsection establishes the mentioned compatibility properties of $\equiv_a$, and shows that it is not in general compatible with the modal operators.

Fact 7. *For every $\wedge$-semilattice $\mathbb{A}$ and every $a \in \mathbb{A}$,*

1. *the relation $\equiv_a$ is a congruence of $\mathbb{A}$.*
2. *If $\mathbb{A}$ is a distributive lattice, then $\equiv_a$ is a congruence of $\mathbb{A}$.*
3. *If $\mathbb{A}$ is a frame, then $\equiv_a$ is a congruence of $\mathbb{A}$.*

[3] Cf. e.g. [12, Chapter 2], [13, Chapter 3, Section 4] pp 22-23, where it is used to define the local operators on locales which will be used to define the open sublocales (ibid. Chapter 5, Section 2).

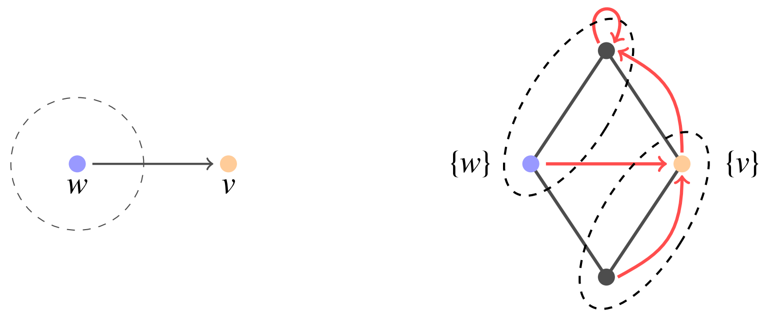

Figure 1: Frame and complex algebra of Example 8.

4. *If $\mathbb{A}$ is a Boolean algebra, then $\equiv_a$ is a congruence of $\mathbb{A}$.*

5. *If $\mathbb{A}$ is a Heyting algebra, then $\equiv_a$ is a congruence of $\mathbb{A}$.*

Proof. 1. Let $b_i \equiv_a c_i$, $i = 1, 2$. Now we have $(b_1 \wedge b_2) \wedge a = b_1 \wedge (b_2 \wedge a) = b_1 \wedge (c_2 \wedge a) = (b_1 \wedge a) \wedge c_2 = (c_1 \wedge a) \wedge c_2 = (c_1 \wedge c_2) \wedge a$.

2. This item is a special case of 3.

3. Let us show that for every $S \subseteq \mathbb{A}$, $\bigvee\{[s] \mid s \in S\} = [\bigvee S]$: clearly $s \leq \bigvee S$ for every $s \in S$ implies the '$\leq$' direction; as to the '$\geq$', it is enough to show that if $[c]$ is such that $[s] \leq [c]$ for every $s \in S$ then $[\bigvee S] \leq [c]$, i.e. $\bigvee S \leq c'$ for some c' such that $c' \wedge a = c \wedge a$. By assumption, for every $s \in S$ there exist some s' and some c_s such that $c \wedge a = c_s \wedge a$, $s' \wedge a = s \wedge a$ and $s' \leq c_s$. Hence, for every $s \in S$ there exists some $c_s \in [c]$ such that

$$s \wedge a = s' \wedge a \leq c_s \wedge a.$$

Let $c' := \bigvee\{c_s \mid s \in S\}$; then by frame distributivity $c' \wedge a = \bigvee\{c_s \wedge a \mid s \in S\} = \bigvee\{c \wedge a \mid s \in S\} = \bigvee\{c \mid s \in S\} \wedge a = c \wedge a$; moreover

$$\bigvee S \wedge a = \bigvee\{s \wedge a \mid s \in S\} \leq \bigvee\{c_s \wedge a \mid s \in S\} \leq c'.$$

4. Assume that $b \equiv_a c$; then $b \wedge a = c \wedge a$ hence, by distributivity and de Morgan law,

$$\neg b \wedge a = (\neg b \vee \neg a) \wedge a = (\neg c \vee \neg a) \wedge a = \neg c \wedge a.$$

5. Recall that for every $b, c \in \mathbb{A}$, $b \to c = \bigvee\{x \mid b \wedge x \leq c\}$. Define $[b] \to [c] = \bigvee\{[x] \mid b \wedge x \leq c\}$. By item 3, $[b] \to [c] = [b \to c]$. Let us verify that for every $b, c, x \in \mathbb{A}$,

$$[b] \wedge [x] \leq [c] \text{ iff } [x] \leq [b] \to [c]:$$

if $[b] \wedge [x] \leq [c]$ then $b' \wedge x' \leq c'$ for some $x' \equiv_a x$, $b' \equiv_a b$ and $c' \equiv_a c$. Hence $x' \leq b' \to c'$. So $[x] \leq [b \to c] = [b] \to [c]$. Conversely, if $[x] \leq [b] \to [c]$, then $b \wedge x \leq c$, hence $[b] \wedge [x] = [b \wedge x] \leq [c]$. $\Box$

Example 8. Let us consider the model $M = (\{w, v\}, \{(w, v)\}, V(p) = \{w\})$. The submodel $M^a = M^p = (\{w\}, \varnothing, V^p)$, represented in Figure 1 on the left by a dashed circle, is not a generated submodel of M. If $\mathcal{F}$ is the underlying Kripke frame of M, the complex algebra $\mathcal{F}^+$ is depicted in above figure, on the right side, where the red arrows represent the modal operator $\Box$. The dashed ellipses represent the equivalence classes of the relation $\equiv_a$, for $a = \{w\} \in \mathcal{F}^+$. Then clearly, $a \equiv_a \top$, but $\Box a = \{v\} \not\equiv_a \top = \Box\top$.

3.2 Modal operations on the pseudo-quotient algebra

Since $\equiv_a$ is not in general compatible with the modal operators, $\mathbb{A}^a$ does not canonically inherit the structure of modal expansion from $\mathbb{A}$. In the present subsection, the modalities will be defined on the algebra $\mathbb{A}^a$ in such a way that, when $\mathbb{A} = \mathcal{F}^+$ for some Kripke frame $\mathcal{F}$, we get $\mathbb{A}^a \cong_{BAO} \mathcal{F}^{a+}$. Throughout the present subsection, $\mathbb{A}$ will be a Heyting algebra.

3.2.1 The diamond operation

Let $(\mathbb{A}, \Diamond)$ be a HAO. Define for every $b \in \mathbb{A}$,

$$\Diamond^a[b] := [\Diamond(b \wedge a) \wedge a] = [\Diamond(b \wedge a)].$$

Fact 9. *For every HAO $(\mathbb{A}, \Diamond)$ and every $a \in \mathbb{A}$,*

1. *$\Diamond^a$ is a normal modal operator. Hence $(\mathbb{A}^a, \Diamond^a)$ is a HAO.*
2. *For every Kripke frame $\mathcal{F} = (W, R)$ and all $X, a \subseteq W$, $R^{a^{-1}}[X \cap a] = R^{-1}[X \cap a] \cap a$.*
3. *If $\mathbb{A} = \mathcal{F}^+$ for some Kripke frame $\mathcal{F}$, then $\mathbb{A}^a \cong_{BAO} \mathcal{F}^{a+}$.*

Proof. 1. $\Diamond^a[\bot] := [\Diamond(\bot \wedge a) \wedge a] = [\Diamond(\bot) \wedge a] = [\bot \wedge a] = [\bot]$. By distributivity, and the fact that π commutes with $\vee$,

$$\begin{array}{rcl}\Diamond^a[b \vee c] & := & [\Diamond((b \vee c) \wedge a) \wedge a] \\ & = & [\Diamond((b \wedge a) \vee (c \wedge a)) \wedge a] \\ & = & [(\Diamond(b \wedge a) \vee \Diamond(c \wedge a)) \wedge a] \\ & = & [(\Diamond(b \wedge a) \wedge a) \vee (\Diamond(c \wedge a) \wedge a)] \\ & = & [\Diamond(b \wedge a) \wedge a] \vee [\Diamond(c \wedge a) \wedge a] \\ & = & \Diamond^a[b] \vee \Diamond^a[c].\end{array}$$

2. Immediate.

3. By definition, $\mathcal{F}^{a+} = (\mathcal{P}(W^a), \langle R^a \rangle)$ and $\mathcal{F}^{+a} = \mathcal{F}^+/\!\equiv_a$. By Fact 5, for every $[X] \in \mathcal{F}^{+a}$ there exists a unique subset $Y = X \cap a$ of W that is a member of $[X]$ and is also a subset of a. So the assignment $[X] \mapsto X \cap a$ defines a map $\mu : \mathcal{F}^{+a} \longrightarrow \mathcal{F}^{a+}$. If $[X] \neq [X']$ then $X \cap a \neq X' \cap a$, which proves that μ is injective. If $X \in \mathcal{F}^{a+}$, then $X \subseteq a \subseteq W$, hence $X \in \mathcal{F}^+$; so $[X] \in \mathcal{F}^{+a}$ and moreover $\mu([X]) = X$, which shows that μ is surjective. Let us show that μ is a BAO homomorphism:

$$\mu(\neg^{\mathcal{F}^{+a}}[X]) = \neg^{\mathcal{F}^{a+}}(\mu([X]))$$

Since $\equiv_a$ is compatible with Boolean negation, $\neg^{\mathcal{F}^{+a}}[X] = [\neg^{\mathcal{F}^+}X] = [W \setminus X]$; hence $\mu(\neg^{\mathcal{F}^{+a}}[X]) = (W \setminus X) \cap a = a \setminus X$. On the other hand, $\neg^{\mathcal{F}^{a+}}(\mu([X])) = a \setminus \mu([X]) = a \setminus (X \cap a) = a \setminus X$. Let us show that

$$\mu(\Diamond^a[X]) = \langle R^a \rangle \mu([X]).$$

By definition, $\Diamond^a[X] = [\Diamond^{\mathcal{F}^+}(X \cap a)] = [\langle R \rangle(X \cap a)] = [R^{-1}[X \cap a]]$, hence $\mu(\Diamond^a[X]) = R^{-1}[X \cap a] \cap a$. On the other hand, $\langle R^a \rangle \mu([X]) = R^{a^{-1}}[\mu([X])] = R^{a^{-1}}[X \cap a]$. Then the claim immediately follows from the item 2 above. The remaining cases are left to the reader. $\square$

3.2.2 The box operation

Let $(\mathbb{A}, \Box)$ be a HAO. Define for every $b \in \mathbb{A}$,

$$\Box^a[b] := [a \to \Box(a \to b)] = [\Box(a \to b)].$$

The second equality holds since, by Fact 15.1, $a \wedge (a \to \Box(a \to b)) \leq \Box(a \to b)$, and by Fact 15.3, $a \wedge \Box(a \to b) \leq a \to \Box(a \to b)$.

Fact 10. *For every HAO $(\mathbb{A}, \Box)$ and every $a \in \mathbb{A}$,*

1. *$\Box^a$ is a normal modal operator.*
2. *If $(\mathbb{A}, \Box)$ is a BAO and $\Box = \neg\Diamond\neg$, then $\Box^a = \neg\Diamond^a\neg$.*
3. *If $\mathbb{A} = \mathcal{F}^+$ for some Kripke frame $\mathcal{F}$, then $\Box^a = [R^a]$, hence $\mathbb{A}^a \cong_{BAO} \mathcal{F}^{a+}$.*

Proof. 1. Since $\top \leq a \to \top$, $\Box^a[\top] := [a \to \Box(a \to \top)] = [a \to \Box(\top)] = [a \to \top] = [\top]$.

$$\begin{array}{rcll}
\Box^a[b \wedge c] & := & [a \to \Box(a \to (b \wedge c))] & \\
& = & [a \to (\Box((a \to b) \wedge (a \to c)))] & \text{(Fact 15.6)} \\
& = & [a \to (\Box(a \to b) \wedge \Box(a \to c))] & \\
& = & [(a \to \Box(a \to b)) \wedge (a \to \Box(a \to c))] & \text{(Fact 15.6)} \\
& = & [(a \to \Box(a \to b))] \wedge [(a \to \Box(a \to c))] & \\
& = & \Box^a[b] \wedge \Box^a[c]. &
\end{array}$$

2.

$$\begin{array}{rclcl}
\Box^a[b] & = & [a \to \Box(a \to b)] & = & [\neg a \vee \Box(\neg a \vee b)] \\
& = & [\neg\neg(\neg a \vee \Box(\neg a \vee b))] & = & \neg[\neg(\neg a \vee \Box(\neg a \vee b))] \\
& = & \neg[a \wedge \neg\Box(\neg a \vee b))] & = & \neg[a \wedge \neg\Box(\neg\neg(\neg a \vee b)))] \\
& = & \neg[a \wedge \neg\Box\neg(\neg(\neg a \vee b))] & = & \neg[a \wedge \Diamond(a \wedge \neg b)] \\
& = & \neg\Diamond^a[\neg b] & = & \neg\Diamond^a\neg[b].
\end{array}$$

3. By Fact 9.4, $(\mathbb{A}^a, \Diamond^a) \cong (\mathcal{P}(W^a), \langle R^a \rangle)$. So the statement immediately follows from this and item 2 above. $\square$

4 Intuitionistic PAL

4.1 Axiomatization

Let AtProp be a countable set of proposition letters. The formulas of the *intuitionistic public announcement logic* IPAL are built by the following syntax rule[4]:

$$\varphi ::= p \in \mathsf{AtProp} \mid \bot \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \varphi \to \psi \mid \Diamond\varphi \mid \Box\varphi \mid \langle\alpha\rangle\varphi \mid [\alpha]\varphi.$$

The same stipulations hold for the defined connectives $\top$, $\neg$ and $\leftrightarrow$ as introduced early on. IPAL is axiomatically defined by the axioms and rules of MIPC plus the following axioms:

Interaction with logical constants	**Preservation of facts**
$\langle\alpha\rangle\bot \leftrightarrow \bot$, $\langle\alpha\rangle\top \leftrightarrow \alpha$	$\langle\alpha\rangle p \leftrightarrow \alpha \wedge p$
$[\alpha]\top \leftrightarrow \top$, $[\alpha]\bot \leftrightarrow \neg\alpha$	$[\alpha]p \leftrightarrow \alpha \to p$
Interaction with disjunction	**Interaction with conjunction**
$\langle\alpha\rangle(\varphi \vee \psi) \leftrightarrow \langle\alpha\rangle\varphi \vee \langle\alpha\rangle\psi$	$\langle\alpha\rangle(\varphi \wedge \psi) \leftrightarrow \langle\alpha\rangle\varphi \wedge \langle\alpha\rangle\psi$
$[\alpha](\varphi \vee \psi) \leftrightarrow \alpha \to (\langle\alpha\rangle\varphi \vee \langle\alpha\rangle\psi)$	$[\alpha](\varphi \wedge \psi) \leftrightarrow [\alpha]\varphi \wedge [\alpha]\psi$
Interaction with implication	
$\langle\alpha\rangle(\varphi \to \psi) \leftrightarrow \alpha \wedge (\langle\alpha\rangle\varphi \to \langle\alpha\rangle\psi)$	
$[\alpha](\varphi \to \psi) \leftrightarrow \langle\alpha\rangle\varphi \to \langle\alpha\rangle\psi$	
Interaction with diamond	**Interaction with box**
$\langle\alpha\rangle\Diamond\varphi \leftrightarrow \alpha \wedge \Diamond\langle\alpha\rangle\varphi$	$\langle\alpha\rangle\Box\varphi \leftrightarrow \alpha \wedge \Box[\alpha]\varphi$
$[\alpha]\Diamond\varphi \leftrightarrow \alpha \to \Diamond\langle\alpha\rangle\varphi$	$[\alpha]\Box\varphi \leftrightarrow \alpha \to \Box[\alpha]\varphi$

4.2 Models

Definition 11. An *algebraic model* is a tuple $M = (\mathbb{A}, V)$ such that $\mathbb{A}$ is an MHA (cf. Definition 3) and $V : \mathsf{AtProp} \to \mathbb{A}$.

[4]The formulation of this section is for the single-agent case, in order to keep the axioms and proofs clearer. All the axioms also hold for the multi-agent case, obtained by just indexing the epistemic modalities. The multi-agent case is the version we shall use in the next section, where we discuss the muddy children puzzle.

Given such a model, we want to define its associated extension map $[\![\cdot]\!]_M : Fm \to \mathbb{A}$ so that, when $\mathbb{A} = \mathcal{F}^+$ for some Kripke frame $\mathcal{F}$, we recover the familiar extension map associated with the model $M = (\mathcal{F}, V)$. First off, we rewrite the satisfaction condition

$$M, w \Vdash \langle\alpha\rangle\varphi \quad \text{iff} \quad M, w \Vdash \alpha \text{ and } M^\alpha, w \Vdash \varphi :$$

this can be equivalently written as follows:

$$w \in [\![\langle\alpha\rangle\varphi]\!]_M \quad \text{iff} \quad \exists w' \in W^\alpha \text{ such that } i(w') = w \in [\![\alpha]\!]_M \text{ and } w' \in [\![\varphi]\!]_{M^\alpha}.$$

Because the map $i : M^\alpha \to M$ is injective, we get that $w' \in [\![\varphi]\!]_{M^\alpha}$ iff $w = i(w') \in i[[\![\varphi]\!]_{M^\alpha}]$. Hence, we obtain:

$$w \in [\![\langle\alpha\rangle\varphi]\!]_M \quad \text{iff} \quad w \in [\![\alpha]\!]_M \cap i[[\![\varphi]\!]_{M^\alpha}],$$

from which we get that

$$[\![\langle\alpha\rangle\varphi]\!]_M = [\![\alpha]\!]_M \cap i[[\![\varphi]\!]_{M^\alpha}]. \tag{1}$$

Next, we rewrite the satisfaction condition

$$M, w \Vdash [\alpha]\varphi \quad \text{iff} \quad M, w \Vdash \alpha \text{ implies } M^\alpha, w \Vdash \varphi :$$

this can be equivalently written as follows:

$$w \in [\![[\alpha]\varphi]\!]_M \quad \text{iff} \quad \forall w' \in W^\alpha \text{ if } i(w') = w \in [\![\alpha]\!]_M \text{ then } w' \in [\![\varphi]\!]_{M^\alpha}.$$

Because the map $i : M^\alpha \to M$ is injective, we get that $w' \in [\![\varphi]\!]_{M^\alpha}$ iff $w = i(w') \in i[[\![\varphi]\!]_{M^\alpha}]$. Hence, we get:

$$w \in [\![[\alpha]\varphi]\!]_M \quad \text{iff} \quad w \in [\![\alpha]\!]_M \Rightarrow i[[\![\varphi]\!]_{M^\alpha}],$$

from which we get that

$$[\![[\alpha]\varphi]\!]_M = [\![\alpha]\!]_M \Rightarrow i[[\![\varphi]\!]_{M^\alpha}], \tag{2}$$

where for every $X, Y \subseteq W$, $X \Rightarrow Y = (W \setminus X) \cup Y$.

Finally, we observe that because of Proposition 6, $i[[\![\varphi]\!]_{M^\alpha}] = i'([\![\varphi]\!]_{M^\alpha})$. So we can adopt equations (1) and (2), so modified, as the definitions of the extensions of $\langle\alpha\rangle\varphi$ and $[\alpha]\varphi$ respectively in any algebraic model $(\mathbb{A}, V)$:

Definition 12. For every algebraic model $M = (\mathbb{A}, V)$, the *extension map* $[\![\cdot]\!]_M : \mathcal{L} \to \mathbb{A}$ is defined recursively as follows:

$$\begin{array}{rcl}
[\![p]\!]_M & = & V(p); \\
[\![\bot]\!]_M & = & \bot^{\mathbb{A}}; \\
[\![\varphi \vee \psi]\!]_M & = & [\![\varphi]\!]_M \vee^{\mathbb{A}} [\![\psi]\!]_M; \\
[\![\varphi \wedge \psi]\!]_M & = & [\![\varphi]\!]_M \wedge^{\mathbb{A}} [\![\psi]\!]_M; \\
[\![\varphi \to \psi]\!]_M & = & [\![\varphi]\!]_M \to^{\mathbb{A}} [\![\psi]\!]_M; \\
[\![\Diamond\varphi]\!]_M & = & \Diamond^{\mathbb{A}}[\![\varphi]\!]_M; \\
[\![\Box\varphi]\!]_M & = & \Box^{\mathbb{A}}[\![\varphi]\!]_M; \\
[\![\langle\alpha\rangle\varphi]\!]_M & = & [\![\alpha]\!]_M \wedge^{\mathbb{A}} i'([\![\varphi]\!]_{M^\alpha}); \\
[\![[\alpha]\varphi]\!]_M & = & [\![\alpha]\!]_M \to^{\mathbb{A}} i'([\![\varphi]\!]_{M^\alpha}).
\end{array}$$

Here, $M^\alpha = (\mathbb{A}^\alpha, V^\alpha)$ such that $\mathbb{A}^\alpha = \mathbb{A}^{[\![\alpha]\!]_M}$ and $V^\alpha : \mathsf{AtProp} \to \mathbb{A}^\alpha$ is $\pi \circ V$, i.e. for every $p \in \mathsf{AtProp}$,

$$[\![p]\!]_{M^\alpha} = V^\alpha(p) = \pi(V(p)) = \pi([\![p]\!]_M).$$

Notice that, by Proposition 4, this definition specializes to those algebraic models $(\mathbb{A}, V)$ such that $\mathbb{A} = \mathcal{F}^+$ is the complex algebra of some MIPC-frame $\mathcal{F}$, and from those, to their relational counterparts $(\mathcal{F}, V)$. Hence, as a special case of the definition above we get an interpretation of IPAL on relational MIPC-models.

4.3 Soundness and completeness for IPAL

Proposition 13. *IPAL is sound with respect to algebraic MIPC-models, hence with respect to relational MIPC-models.*

Proof. The soundness of the preservation of facts and logical constants follows from Lemma 17. The soundness of the remaining axioms is proved in Lemmas 18, 19, 20, 21, 22 of the appendix. □

Theorem 14. *IPAL is complete with respect to relational MIPC-models.*

Proof. The proof is analogous to the proof of completeness of classical PAL, and follows from the reducibility of IPAL to MIPC via the reduction axioms [4, Theorem 27]. Let φ be a valid IPAL formula. Let us consider some innermost occurrence of any dynamic modality in a subformula of form $[\alpha]\psi, \langle\alpha\rangle\varphi$. The distribution axioms make it possible to equivalently transform this subformula by pushing the dynamic modality down the generation tree, through the static connectives, until it attaches to a proposition letter or to a constant symbol. Here dynamic modality disappears thanks to an application of the appropriate 'preservation of facts' or 'interaction with logical constant' axiom. This process is repeated for all the dynamic modalities of φ, so as to obtain an MIPC formula φ' which is provably equivalent to φ. Since φ is valid by assumption, and since the process preserves provable equivalence, by soundness we can conclude that φ' is valid. By the completeness of MIPC (Proposition 2), we can conclude that φ' is provable in MIPC, hence in IPAL. This concludes the proof, by the provable equivalence of φ and φ'. □

5 Preliminary application to reasoning about information update

Classical PAL and the logics in the DEL family have been famously used to reason about concrete scenarios in which information gets updated, for instance epistemic puzzles such as muddy children. Intuitionistic logic has also been used to interpret certain dynamic aspects of information [3]. Applying IPAL to these areas constitutes future work. Here we only provide an initial formalization and some preliminary insights on how to solve the muddy children puzzle; a full proof constitutes work in progress.

> After having played outside, $k \geq 1$ of n children have got mud on their foreheads. They can only see the others, so they do not know their own status. Now their Father comes along and says: "At least one of you is dirty". He then asks: "Do you know whether your own forehead is dirty?" Children answer truthfully, and this is repeated round by round. As questions and answers repeat, what will happen?
>
> There is a straightforward proof by induction that the first $k-1$ times he asks the question, they will all answer "No," but then, at the kth time, the children with muddy foreheads will all answer "Yes."

If $k = 1$, then the only dirty child knows that all the other children are clean, so his/her uncertainty is about whether the total number of dirty children is 0 or 1 (in the latter case, he/she will be dirty). *Learning* from Father that there is at least one dirty child among them takes away the uncertainty, and enables the conclusion that he/she is dirty[5]. As to the inductive step, suppose the statement is true for k dirty children, and let us show it for $k + 1$ dirty children. In this case, each dirty child sees k dirty children, so his/her uncertainty is about whether the total number of dirty children is k or $k + 1$ (in the latter case, he/she will be dirty). If there were only k dirty children, then, by induction hypothesis, each of them would know at round $k-1$. However, at round $k-1$, each dirty child *learns* that none of the others knows. Again, this takes away the uncertainty, and enables the conclusion that he/she is dirty[6].

Firstly, notice that the logical inferences in the above proof are intuitionistic: indeed, both in the base case and in the induction step, the inferences performed by the children are of the form

$$[(x \vee y) \wedge z] \wedge ((x \wedge z) \to \bot) \leq y;$$

[5] Notice that the difference with the clean children is that each clean child sees one dirty child, so each clean child's uncertainty is about whether the total number of dirty children is 1 or 2 (in the latter case, he/she will be dirty). Father's public announcement is uninformative for the clean children; the only point at which they learn what they need to conclude something about their own status is when the dirty child announces that he /she knows.

[6] Again, the same holds for clean children, but because they see more dirty children than each dirty child, the clean children are one step behind.

here, x and y represent the alternative possibilities about the total number of dirty children, and z represents what the k dirty children learn at round $k-1$ (z can also be taken to represent what the $n-k$ clean children learn at round k). This is a Heyting inequality (cf. Fact 15).

The aim of this section is to show that this proof by induction can be formalized in the language and by the entailment of IPAL. Of course, we will need the n-agent version of it, which we denote IPAL_n, whose language, if the set of agents is taken to be $\{1,\ldots,n\}$, is defined as one expects by considering indexed epistemic modalities $\Box_i$ and $\Diamond_i$ for $1 \leq i \leq n$, and whose axiomatization is given by correspondingly indexed copies of the IPAL axioms[7]. Derived modalities can be defined in the language of IPAL_n, which will act as finitary approximations of common knowledge: for every IPAL_n-formula φ, $E\varphi := \bigwedge_{i=1}^{n} \Box_i\varphi$. The intended meaning of E is 'Everybody knows'. It is easy to see that $E\top \dashv\vdash_{IK_n} \top$ and $E(\varphi \wedge \psi) \dashv\vdash_{IK_n} E\varphi \wedge E\psi$. So E is a box-type normal modality. Furthermore, for every $k \in \mathbb{N}$, the modality E^k is defined inductively by setting $E^0\varphi := \varphi$ and $E^{k+1}\varphi := EE^k\varphi$; likewise, $\odot^k$ can be defined for every k, for any other modal operation $\odot$ in the language. For the sake of this example, the set of atomic propositions can be restricted to $At = \{D_i, C_i \mid 1 \leq i \leq n\}$, where D_i is the proposition saying 'child i is dirty', and C_i is the proposition saying 'child i is clean'. Let us introduce the following abbreviations:

- $\mathsf{father} := \bigvee_{i=1}^{n} D_i$ expresses the proposition publicly announced by Father;
- $\mathsf{vision} := \bigwedge\{(D_i \to \Box_j D_i) \wedge (C_i \to \Box_j C_i) \mid 1 \leq i, j \leq n \text{ and } i \neq j\}$ expresses the fact that every child knows whether each other child is clean or dirty;
- $\mathsf{aut} := \bigwedge_{i=1}^{n}[(C_i \wedge D_i) \to \bot]$ expresses the fact that being clean or dirty are mutually incompatible conditions;
- $\mathsf{no} := \bigwedge_{i=1}^{n}(\Diamond_i D_i \wedge \Diamond_i C_i)$ expresses the ignorance of the children about their own status;
- $\mathsf{dirty}(J) := (\bigwedge_{j\in J} D_j) \wedge (\bigwedge_{h\notin J} C_h)$, for each $J \subseteq \{1,\ldots,n\}$, expresses that all and only the children in J have dirty foreheads.

The aim would be to prove something along side the following For every $\varnothing \neq J \subseteq \{1,\ldots,n\}$ such that $|J| = k$,

$$\mathsf{dirty}(J), \mathsf{aut}, \mathsf{vision}, E^k(\mathsf{aut} \wedge \mathsf{vision}) \vdash_{IPAL_n} [\mathsf{father}][\mathsf{no}]^{k-1}\Box_j D_j$$

for each $j \in J$. The full proof is left to future work.

6 Conclusions and open questions

The present paper is part of a line of investigation which aims at defining and studying dynamic epistemic logics based on weaker-than-classical propositional logic. The conceptual relevance of this investigation is providing a point-free, non-classical account on epistemic updates, which naturally generalizes the classical account. The approach we adopt is based on the dual characterization of the map which represents the update induced by an epistemic action. This dual characterization provides an equivalent description of the epistemic update, in an algebraic environment. The benefit brought in by this dual characterization is that it immediately generalizes to a much more general setting, in a modular way. In the present paper, we have treated the simplest epistemic actions, namely public announcements. We axiomatically defined IPAL, a logic of public announcements based on intuitionistic propositional logic. The static fragment of IPAL is the intuitionistic modal logic MIPC, which is widely considered the intuitionistic counterpart of S5. We have shown that IPAL is sound and complete w.r.t. MIPC-models, with an entirely analogous proof to the soundness and completeness proof for the classical PAL. Below we expand on some further remarks and directions.

The static fragment and epistemic operators. We have adopted the intuitionistic logic MIPC as our static fragment in this paper. However, different choices are possible: just to mention a few, we might wish to consider modal languages with just one modal operator (recall that box and diamond are not interdefinable in intuitionistic modal logic). As the reader might have noticed, the proofs of the relevant facts proceed modularly: for instance, the implication is not needed to describe the fundamental properties of the diamonds (either static or dynamic) and the same is true for box and discjunction.

[7]For the remainder of this section, if $\mathcal{L}$ is one of the logics introduced so far, $\mathcal{L}_n$ will denote its n-agent version.

This means that, in some cases, epistemic updates can be accounted for within strictly weaker settings than intuitionistic propositional logic. Also, more expressive languages can be considered, which admit modal operators expressing 'impossibility' or 'skepticism' as primitive[8] (viz. the distributive modal logic introduced in [11]). In an extended version of the present paper, we define axiomatically an expansion of IPAL with these operators taken as primitive, and prove analogous soundness and completeness results.

The static fragment and axiomatization. Another kind of variant involves the property we wish to impose on the relational structures: for instance, the debate on whether it is acceptable that the epistemic box satisfies the positive introspection axiom $\Box\varphi \rightarrow \Box\Box\varphi$ translates to the intuitionistic setting. Notice that none of our core results depend on the choice of the base logic. On the semantic side, none of our core results depend on the fact that the accessibility relation is an equivalence relation. Our treatment would have stood unchanged if instead of MIPC we had adopted e.g. Fischer-Servi's intuitionistic modal logic IK ([10]) as our static setting.

General epistemic actions and product updates. As discussed in [1], the relativization which semantically represents the epistemic update induced by public announcements can be regarded as a very special, almost degenerate case of product update. These too can be dualized, much in the same way as is done in the present treatment, by taking certain pseudo-quotients of coproducts of Heyting algebras. This direction is current work in progress.

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7 Appendix

7.1 Identities and inequalities valid on Heyting algebras

In a Heyting algebra $\wedge$ and $\to$ are *residuated*, namely, for all $x, y, z \in \mathbb{A}$,

$$x \wedge y \leq z \quad \text{iff} \quad x \leq y \to z. \tag{3}$$

Hence, by the general theory of residuation,

$$y \to z = \bigvee \{x \mid x \wedge y \leq z\}. \tag{4}$$

Using (3) and (4) above, it is not difficult to prove the following

Fact 15. *For every Heyting algebra $\mathbb{A}$ and all $x, y, z \in \mathbb{A}$,*

1. $x \wedge (x \to y) \leq y$;
2. $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$;
3. $x \wedge y \leq x \to y$;
4. $x \to y = x \to (x \wedge y)$;
5. $(x \wedge y) \to z = x \to (y \to z)$;
6. $x \wedge (y \to z) = x \wedge ((x \wedge y) \to z)$.

7.2 Properties of the map i'

Fact 16. *Let $\mathbb{A}$ be an MIPC-algebra, $a \in \mathbb{A}$, and let $i' : \mathbb{A}^a \to \mathbb{A}$ given by $[b] \mapsto b \wedge a$. Then, for every $b, c \in \mathbb{A}^a$,*

1. $i'(b \vee c) = i'(b) \vee i'(c)$;
2. $i'(b \wedge c) = i'(b) \wedge i'(c)$;
3. $i'(b \to c) = a \wedge (i'(b) \to i'(c))$;
4. $i'(\Diamond^a b) = \Diamond(i'(b) \wedge a) \wedge a$;
5. $i'(\Box^a b) = a \wedge \Box(a \to i'(b))$.

Proof. 1. Let $b', c' \in \mathbb{A}$ such that $b = [b']$ and $c = [c']$. Then $i'(b) = b' \wedge a$ and $i'(c) = c' \wedge a$, and because π commutes with $\vee$, we have $b \vee c = [b'] \vee [c'] = [b' \vee c']$. By distributivity, $i'(b \vee c) = i'([b' \vee c']) = (b' \vee c') \wedge a = (b' \wedge a) \vee (c' \wedge a) = i'(b) \vee i'(c)$.

2. Let $b', c' \in \mathbb{A}$ such that $b = [b']$ and $c = [c']$. Then $i'(b) = b' \wedge a$ and $i'(c) = c' \wedge a$, and because π commutes with $\wedge$, we have $b \wedge c = [b'] \wedge [c'] = [b' \wedge c']$. By idempotence and commutativity of $\wedge$, $i'(b \wedge c) = i'([b' \wedge c']) = (b' \wedge c') \wedge a = (b' \wedge a) \wedge (c' \wedge a) = i'(b) \wedge i'(c)$.

3. Let $b', c' \in \mathbb{A}$ such that $b = [b']$ and $c = [c']$. Then $i'(b) = b' \wedge a$ and $i'(c) = c' \wedge a$, and because π commutes with $\to$, we have

$$
\begin{array}{rclcl}
i'(b \to c) & = & i'([b'] \to [c']) & = & i'([b' \wedge a] \to [c' \wedge a]) \\
& = & i'([(b' \wedge a) \to (c' \wedge a)]) & = & a \wedge ((b' \wedge a) \to (c' \wedge a)) \\
& = & a \wedge (i'(b) \to i'(c)). & &
\end{array}
$$

4. Let $b' \in \mathbb{A}$ such that $b = [b']$. Then $i'(b) = b' \wedge a$, hence

$$b' \wedge a = (b' \wedge a) \wedge a = i'(b) \wedge a. \tag{5}$$

Moreover, $\Diamond^a b = [\Diamond(b' \wedge a)]$, hence $i'(\Diamond^a b) = \Diamond(b' \wedge a) \wedge a = \Diamond(i'(b) \wedge a) \wedge a$.

5. Let $b' \in \mathbb{A}$ with $b = [b']$. Then $i'(b) = a \wedge b'$ and $\Box^a b = [\Box(a \to b')]$. By Fact 15.4,

$$
\begin{array}{rclcl}
i'(\Box^a b) & = & i'([\Box(a \to b')]) & = & i'([\Box(a \to (a \wedge b'))]) \\
& = & i'([\Box(a \to i'(b))]) & = & a \wedge \Box(a \to i'(b)).
\end{array}
$$

□

7.3 Soundness Lemmas

In this subsection, the lemmas are collected which serve to prove Proposition 13.

Lemma 17. *Let $M = (\mathbb{A}, V)$ be an algebraic model. Let φ be a formula such that $[\![\varphi]\!]_{M^\alpha} = \pi([\![\varphi]\!]_M)$ for every formula α and model M. Then for every formula α,*

1. *$[\![\langle\alpha\rangle\varphi]\!]_M = [\![\alpha]\!]_M \wedge [\![\varphi]\!]_M$.*
2. *$[\![[\alpha]\varphi]\!]_M = [\![\alpha]\!]_M \to [\![\varphi]\!]_M$.*

Proof. 1.

$$
\begin{array}{rcl}
[\![\langle\alpha\rangle\varphi]\!]_M & = & [\![\alpha]\!]_M \wedge i'([\![\varphi]\!]_{M^\alpha}) \\
& = & [\![\alpha]\!]_M \wedge i'(\pi([\![\varphi]\!]_M)) \\
& = & [\![\alpha]\!]_M \wedge ([\![\varphi]\!]_M \wedge [\![\alpha]\!]_M) \\
& = & [\![\alpha]\!]_M \wedge [\![\varphi]\!]_M.
\end{array}
$$

2.

$$
\begin{array}{rcll}
[\![[\alpha]\varphi]\!]_M & = & [\![\alpha]\!]_M \to i'([\![\varphi]\!]_{M^\alpha}) & \\
& = & [\![\alpha]\!]_M \to i'(\pi([\![\varphi]\!]_M)) & \\
& = & [\![\alpha]\!]_M \to ([\![\varphi]\!]_M \wedge [\![\alpha]\!]_M) & \\
& = & [\![\alpha]\!]_M \to [\![\varphi]\!]_M. & \text{(Fact 15.4)}
\end{array}
$$

□

Lemma 18. *Let $M = (\mathbb{A}, V)$ be an algebraic model. For every formula α, φ and ψ,*

1. *$[\![\langle\alpha\rangle(\varphi \vee \psi)]\!]_M = [\![\langle\alpha\rangle\varphi]\!]_M \vee [\![\langle\alpha\rangle\psi]\!]_M$.*
2. *$[\![[\alpha](\varphi \vee \psi)]\!]_M = [\![\alpha]\!]_M \to ([\![\langle\alpha\rangle\varphi]\!]_M \vee [\![\langle\alpha\rangle\psi]\!]_M)$.*

Proof. 1.

$$
\begin{array}{rcll}
[\![\langle\alpha\rangle(\varphi \vee \psi)]\!]_M & = & [\![\alpha]\!]_M \wedge i'([\![\varphi \vee \psi]\!]_{M^\alpha}) & \\
& = & [\![\alpha]\!]_M \wedge i'([\![\varphi]\!]_{M^\alpha} \vee [\![\psi]\!]_{M^\alpha}) & \\
& = & [\![\alpha]\!]_M \wedge (i'([\![\varphi]\!]_{M^\alpha}) \vee i'([\![\psi]\!]_{M^\alpha})) & \text{(Fact 16.1)} \\
& = & ([\![\alpha]\!]_M \wedge i'([\![\varphi]\!]_{M^\alpha})) \vee ([\![\alpha]\!]_M \wedge i'([\![\psi]\!]_{M^\alpha}))) & \\
& = & [\![\langle\alpha\rangle\varphi]\!]_M \vee [\![\langle\alpha\rangle\psi]\!]_M. &
\end{array}
$$

2.

$$\begin{array}{rcll}
\llbracket [\alpha](\varphi \vee \psi) \rrbracket_M & = & \llbracket \alpha \rrbracket \to i'(\llbracket \varphi \vee \psi \rrbracket_{M^\alpha}) & \\
& = & \llbracket \alpha \rrbracket_M \to (i'(\llbracket \varphi \rrbracket_{M^\alpha}) \vee i'(\llbracket \psi \rrbracket_{M^\alpha})) & \text{(Fact 16.1)} \\
& = & \llbracket \alpha \rrbracket_M \to (\llbracket \alpha \rrbracket_M \wedge (i'(\llbracket \varphi \rrbracket_{M^\alpha}) \vee i'(\llbracket \psi \rrbracket_{M^\alpha}))) & \text{(Fact 15.4)} \\
& = & \llbracket \alpha \rrbracket_M \to ((\llbracket \alpha \rrbracket_M \wedge i'(\llbracket \varphi \rrbracket_{M^\alpha})) \vee (\llbracket \alpha \rrbracket_M \wedge i'(\llbracket \psi \rrbracket_{M^\alpha}))) & \\
& = & \llbracket \alpha \rrbracket_M \to (\llbracket \langle\alpha\rangle\varphi \rrbracket_M \vee \llbracket \langle\alpha\rangle\psi \rrbracket_M). &
\end{array}$$

□

Lemma 19. *Let $M = (\mathbb{A}, V)$ be an algebraic model. For every formula α, φ and ψ,*

1. $\llbracket \langle\alpha\rangle(\varphi \wedge \psi) \rrbracket_M = \llbracket \langle\alpha\rangle\varphi \rrbracket_M \wedge \llbracket \langle\alpha\rangle\psi \rrbracket_M$.
2. $\llbracket [\alpha](\varphi \wedge \psi) \rrbracket_M = \llbracket [\alpha]\varphi \rrbracket_M \wedge \llbracket [\alpha]\psi \rrbracket_M$.

Proof. 1.

$$\begin{array}{rcll}
\llbracket \langle\alpha\rangle(\varphi \wedge \psi) \rrbracket_M & = & \llbracket \alpha \rrbracket_M \wedge i'(\llbracket \varphi \wedge \psi \rrbracket_{M^\alpha}) & \\
& = & \llbracket \alpha \rrbracket_M \wedge (i'(\llbracket \varphi \rrbracket_{M^\alpha}) \wedge i'(\llbracket \psi \rrbracket_{M^\alpha})) & \text{(Fact 16.2)} \\
& = & (\llbracket \alpha \rrbracket_{M^\alpha} \wedge i'(\llbracket \varphi \rrbracket_{M^\alpha})) \wedge (\llbracket \alpha \rrbracket_M \wedge i'(\llbracket \psi \rrbracket_{M^\alpha})) & \\
& = & \llbracket \langle\alpha\rangle\varphi \rrbracket_M \wedge \llbracket \langle\alpha\rangle\psi \rrbracket_M. &
\end{array}$$

2.

$$\begin{array}{rcll}
\llbracket [\alpha](\varphi \wedge \psi) \rrbracket_M & = & \llbracket \alpha \rrbracket_M \to i'(\llbracket \varphi \wedge \psi \rrbracket_{M^\alpha}) & \\
& = & \llbracket \alpha \rrbracket_M \to i'(\llbracket \varphi \rrbracket_{M^\alpha} \wedge \llbracket \psi \rrbracket_{M^\alpha}) & \\
& = & \llbracket \alpha \rrbracket_M \to (i'(\llbracket \varphi \rrbracket_{M^\alpha}) \wedge i'(\llbracket \psi \rrbracket_{M^\alpha})) & \text{(Fact 16.2)} \\
& = & (\llbracket \alpha \rrbracket_M \to i'(\llbracket \varphi \rrbracket_{M^\alpha})) \wedge (\llbracket \alpha \rrbracket_M \to i'(\llbracket \psi \rrbracket_{M^\alpha})) & \text{(Fact 15.2)} \\
& = & \llbracket [\alpha]\varphi \rrbracket_M \wedge \llbracket [\alpha]\psi \rrbracket_M. &
\end{array}$$

□

Lemma 20. *Let $M = (\mathbb{A}, V)$ be an algebraic model. For every formula α, φ and ψ,*

1. $\llbracket [\alpha](\varphi \to \psi) \rrbracket_M = \llbracket \langle\alpha\rangle\varphi \rrbracket_M \to \llbracket \langle\alpha\rangle\psi \rrbracket_M$.
2. $\llbracket \langle\alpha\rangle(\varphi \to \psi) \rrbracket_M = \llbracket \alpha \rrbracket_M \wedge (\llbracket \langle\alpha\rangle\varphi \rrbracket_M \to \llbracket \langle\alpha\rangle\psi \rrbracket_M)$.

Proof. 1.

$$\begin{array}{rcll}
\llbracket [\alpha](\varphi \to \psi) \rrbracket_M & = & \llbracket \alpha \rrbracket_M \to i'(\llbracket \varphi \to \psi \rrbracket_{M^\alpha}) & \\
& = & \llbracket \alpha \rrbracket_M \to (\llbracket \alpha \rrbracket_M \wedge (i'(\llbracket \varphi \rrbracket_{M^\alpha}) \to i'(\llbracket \psi \rrbracket_{M^\alpha}))) & \text{(Fact 16.3)} \\
& = & \llbracket \alpha \rrbracket_M \to (i'(\llbracket \varphi \rrbracket_{M^\alpha}) \to i'(\llbracket \psi \rrbracket_{M^\alpha})) & \text{(Fact 15.4)} \\
& = & (\llbracket \alpha \rrbracket_M \wedge i'(\llbracket \varphi \rrbracket_{M^\alpha})) \to i'(\llbracket \psi \rrbracket_{M^\alpha}) & \text{(Fact 15.5)} \\
& = & (\llbracket \alpha \rrbracket_M \wedge i'(\llbracket \varphi \rrbracket_{M^\alpha})) \to ((\llbracket \alpha \rrbracket_M \wedge i'(\llbracket \varphi \rrbracket_{M^\alpha})) \wedge i'(\llbracket \psi \rrbracket_{M^\alpha})) & \text{(Fact 15.4)} \\
& = & \llbracket \langle\alpha\rangle\varphi \rrbracket_M \to (\llbracket \langle\alpha\rangle\varphi \rrbracket_M \wedge \llbracket \langle\alpha\rangle\psi \rrbracket_M) & \\
& = & \llbracket \langle\alpha\rangle\varphi \rrbracket_M \to \llbracket \langle\alpha\rangle\psi \rrbracket_M. & \text{(Fact 15.4)}
\end{array}$$

2.

$$\begin{array}{rcll}
\llbracket \langle\alpha\rangle(\varphi \to \psi) \rrbracket_M & = & \llbracket \alpha \rrbracket_M \wedge i'(\llbracket \varphi \to \psi \rrbracket_{M^\alpha}) & \\
& = & \llbracket \alpha \rrbracket_M \wedge (\llbracket \alpha \rrbracket_M \wedge (i'(\llbracket \varphi \rrbracket_{M^\alpha}) \to i'(\llbracket \psi \rrbracket_{M^\alpha}))) & \text{(Fact 16.3)} \\
& = & \llbracket \alpha \rrbracket_M \wedge (i'(\llbracket \varphi \rrbracket_{M^\alpha}) \to i'(\llbracket \psi \rrbracket_{M^\alpha})) & \text{(Fact 15.4)} \\
& = & \llbracket \alpha \rrbracket_M \wedge ((\llbracket \alpha \rrbracket_M \wedge i'(\llbracket \varphi \rrbracket_{M^\alpha})) \to i'(\llbracket \psi \rrbracket_{M^\alpha})) & \text{(Fact 15.6)} \\
& = & \llbracket \alpha \rrbracket_M \wedge (\llbracket \langle\alpha\rangle\varphi \rrbracket_M \to \llbracket \langle\alpha\rangle\psi \rrbracket_M). & \text{(proof above)}
\end{array}$$

$\square$

Lemma 21. *Let $M = (\mathbb{A}, V)$ be an algebraic model. For every formula α and φ,*

1. $[\![\langle\alpha\rangle\Diamond\varphi]\!]_M = [\![\alpha]\!]_M \wedge \Diamond^{\mathbb{A}}[\![\langle\alpha\rangle\varphi]\!]_M$.
2. $[\![[\alpha]\Diamond\varphi]\!]_M = [\![\alpha]\!]_M \to \Diamond^{\mathbb{A}}[\![\langle\alpha\rangle\varphi]\!]_M$.

Proof. We preliminarily observe that

$$\begin{array}{rcll} i'([\![\Diamond\varphi]\!]_{M^\alpha}) & = & [\![\alpha]\!]_M \wedge \Diamond^{\mathbb{A}}([\![\alpha]\!]_M \wedge i'([\![\varphi]\!]_{M^\alpha})) & \text{(Fact 16.4)} \\ & = & [\![\alpha]\!]_M \wedge \Diamond^{\mathbb{A}}[\![\langle\alpha\rangle\varphi]\!]_M. & \end{array}$$

Hence: 1.

$$\begin{array}{rcl} [\![\langle\alpha\rangle\Diamond\varphi]\!]_M & = & [\![\alpha]\!]_M \wedge i'([\![\Diamond\varphi]\!]_{M^\alpha}) \\ & = & [\![\alpha]\!]_M \wedge ([\![\alpha]\!]_M \wedge \Diamond^{\mathbb{A}}[\![\langle\alpha\rangle\varphi]\!]_M) \\ & = & [\![\alpha]\!]_M \wedge \Diamond^{\mathbb{A}}[\![\langle\alpha\rangle\varphi]\!]_M. \end{array}$$

2.

$$\begin{array}{rcll} [\![[\alpha]\Diamond\varphi]\!]_M & = & [\![\alpha]\!]_M \to i'([\![\Diamond\varphi]\!]_{M^\alpha}) & \\ & = & [\![\alpha]\!]_M \to ([\![\alpha]\!]_M \wedge \Diamond^{\mathbb{A}}[\![\langle\alpha\rangle\varphi]\!]_M) & \\ & = & [\![\alpha]\!]_M \to \Diamond^{\mathbb{A}}[\![\langle\alpha\rangle\varphi]\!]_M. & \text{(Fact 15.4)} \end{array}$$

$\square$

Lemma 22. *Let $M = (\mathbb{A}, V)$ be an algebraic model. For every formula α and φ,*

1. $[\![\langle\alpha\rangle\Box\varphi]\!]_M = [\![\alpha]\!]_M \wedge \Box^{\mathbb{A}}[\![[\alpha]\varphi]\!]_M$.
2. $[\![[\alpha]\Box\varphi]\!]_M = [\![\alpha]\!]_M \to \Box^{\mathbb{A}}[\![[\alpha]\varphi]\!]_M$.

Proof. We preliminarily observe that

$$\begin{array}{rcll} i'([\![\Box\varphi]\!]_{M^\alpha}) & = & [\![\alpha]\!]_M \wedge \Box^{\mathbb{A}}([\![\alpha]\!]_M \to i'([\![\varphi]\!]_{M^\alpha})) & \text{(Fact 16.5)} \\ & = & [\![\alpha]\!]_M \wedge \Box^{\mathbb{A}}[\![[\alpha]\varphi]\!]_M. & \end{array}$$

Hence: 1.

$$\begin{array}{rcl} [\![\langle\alpha\rangle\Box\varphi]\!]_M & = & [\![\alpha]\!]_M \wedge i'([\![\Box\varphi]\!]_{M^\alpha}) \\ & = & [\![\alpha]\!]_M \wedge ([\![\alpha]\!]_M \wedge \Box^{\mathbb{A}}[\![[\alpha]\varphi]\!]_M) \\ & = & [\![\alpha]\!]_M \wedge \Box^{\mathbb{A}}[\![[\alpha]\varphi]\!]_M. \end{array}$$

2.

$$\begin{array}{rcll} [\![[\alpha]\Box\varphi]\!]_M & = & [\![\alpha]\!]_M \to i'([\![\Box\varphi]\!]_{M^\alpha}) & \\ & = & [\![\alpha]\!]_M \to ([\![\alpha]\!]_M \wedge \Box^{\mathbb{A}}([\![[\alpha]\varphi]\!]_M)) & \\ & = & [\![\alpha]\!]_M \to \Box^{\mathbb{A}}[\![[\alpha]\varphi]\!]_M. & \text{(Fact 15.4)} \end{array}$$

$\square$