Fault-Tolerant Secret Key Generation

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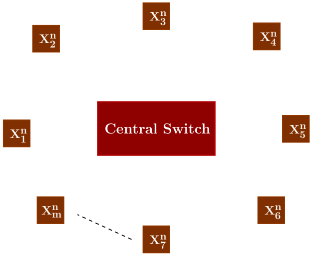
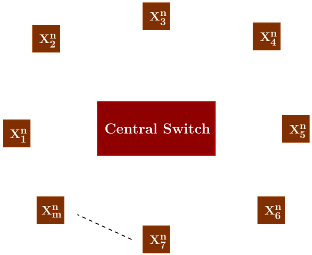
Multiterminal Source Model



Set of nodes: $\mathcal{M} = \{1, ..., m\}$

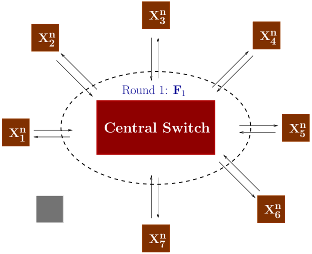
- Observations of the ith node: $X_i^n = (X_{i1}, ..., X_{in})$
- Denote by $X_{\mathcal{M}t}$ the correlated rvs $(X_{1t}, ..., X_{mt})$
- $X_{\mathcal{M}1}, ..., X_{\mathcal{M}n}$ are **finite, discrete valued, i.i.d. rvs**
  - with known probability distribution.

r-Rounds Adaptive Protocol



Available Nodes: $A_0 = \mathcal{M}$

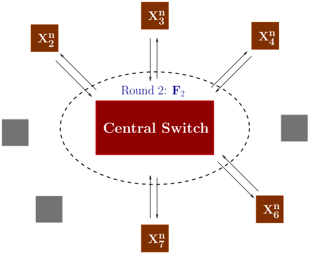
r-Rounds Adaptive Protocol



Nodes Remaining: $A_1 = \{1, 2, 3, 4, 5, 6, 7\}$

Communication in round j depends on:
local observations and the communication in the previous rounds.

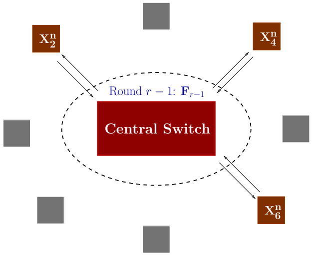
r-Rounds Adaptive Protocol



Nodes Remaining: $A_2 = \{2, 3, 4, 6, 7\}$

Communication in round j depends on:
local observations and the communication in the previous rounds.

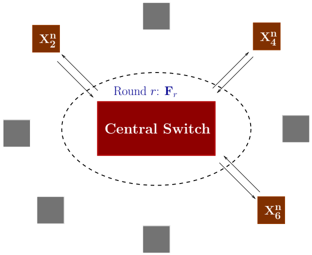
r-Rounds Adaptive Protocol



Nodes Remaining: $A_{r-1} = \{2, 4, 6\}$

Communication in round j depends on:
local observations and the communication in the previous rounds.

r-Rounds Adaptive Protocol

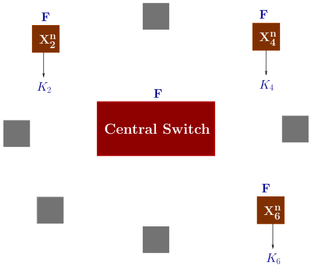


Nodes Remaining: $A_{r-1} = \{2, 4, 6\} = A_r$

Communication in round j depends on:
local observations and the communication in the previous rounds.

Assumption: $A_r = A_{r-1}$

r-Rounds Adaptive Protocol



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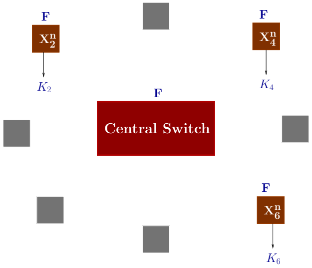
The overall communication depends on $A_r = A_{r-1} \subseteq ... \subseteq A_1$
- $\mathbf{F}$ denotes the overall communication.

r-Rounds Adaptive Protocol

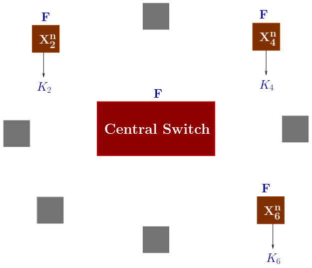


K constitutes a *secret key* if:

1. Recoverability: $\texttt{Pr}\left(K_i = K, i \in A_r\right) \approx 1$
2. Security: $I(K \wedge \mathbf{F}) \approx 0$

The rate of the SK: $\frac{1}{n} H(K)$

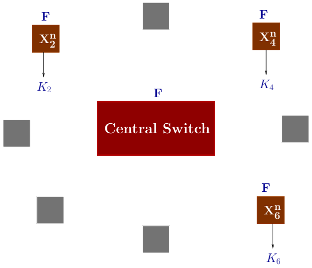
r-Rounds Adaptive Protocol



Definition (Achievable (r,t)-fault-tolerant SK rate)

$R \geq 0$ is an achievable (r,t)-fault-tolerant SK rate if there is an r-rounds adaptive protocol that generates an SK of rate greater than R whenever not more than t nodes drop out.

r-Rounds Adaptive Protocol

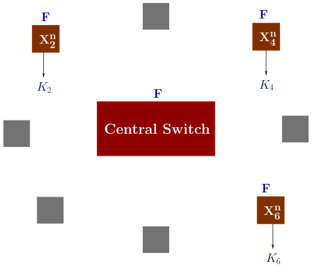


K constitutes a **perfect** *secret key* if:

1. **Perfect** Recoverability: $\mathtt{Pr}\left(K_i = K, i \in A_r\right) = 1$
2. **Perfect** Security: $I(K \wedge \mathbf{F}) = 0$

The rate of the SK: $\frac{1}{n}H(K)$

r-Rounds Adaptive Protocol



Definition (Achievable (r, t)-fault-tolerant perfect SK rate)

$R \geq 0$ is an achievable (r, t)-fault-tolerant **perfect** SK rate if there is an r-rounds adaptive protocol that generates a **perfect** SK of rate greater than R whenever not more than t nodes drop out.

Fault-Tolerant Secret Key Capacity

(r, t)-fault-tolerant SK capacity $C^{r,t}(\mathcal{M})$:

Supremum of all achievable (r, t)-fault-tolerant rates.

(r, t)-fault-tolerant perfect SK capacity $C_0^{r,t}(\mathcal{M})$:

Supremum of all achievable (r, t)-fault-tolerant perfect SK rates.

Lemma

For $r \geq 1$,

$$C_0^{1,t}(\mathcal{M}) \leq C^{r,t}(\mathcal{M}) \leq C^{r+1,t}(\mathcal{M}).$$

An Upper Bound on Fault-Tolerant SK Capacity

Theorem (Csiszár-Narayan 2004)

The secret key capacity (for t=0) is given by

$$C(\mathcal{M}) = H\left(X_{\mathcal{M}}\right) - \min\left(R_1 + R_2 + \ldots + R_m\right),$$

where the $\min$ *is taken over* $(R_1, \ldots, R_m)$ *that satisfy:*

$$\sum_{i \in B} R_i \geq H\left(X_B \mid X_{\mathcal{M} \setminus B}\right), \qquad B \subsetneq \mathcal{M}.$$

$\min$ value above is the minimum rate of communication for omniscience.

Lemma (Upper Bound on $C^{r,t}(\mathcal{M})$)

$$C_0^{1,t}(\mathcal{M}) \leq C^{r,t}(\mathcal{M}) \leq C^{r+1,t}(\mathcal{M}) \leq \min_{\substack{A \subseteq \mathcal{M} \\ |A| \geq m-t}} C(A), \qquad r \geq 1.$$

Proof Idea: Consider the sequence of sets $A_1 = \ldots = A_{r-1} = A_r = A$.

Monotonicity of SK Capacity

Theorem (Chan-Zheng 2010)

$$C(\mathcal{M}) = \min_{\mathcal{P}=\{C_1,\ldots,C_k\}} \frac{1}{k} D\left(X_{\mathcal{M}} \| X_{C_1}.X_{C_2}\ldots X_{C_k}\right),$$

where the minimization is over all partitions $\mathcal{P}$ of $\mathcal{M}$.

Lemma (Monotonicity of $C(\mathcal{M})$)

$$C(\mathcal{M}) \geq \min_{\substack{A\subseteq\mathcal{M}\\ |A|=m-1}} C(A).$$

Lemma (Upper Bound on $C^{r,t}(\mathcal{M})$)

$$C_0^{1,t}(\mathcal{M}) \leq C^{r,t}(\mathcal{M}) \leq C^{r+1,t}(\mathcal{M}) \leq \min_{\substack{A\subseteq\mathcal{M}\\ |A|=m-t}} C(A), \qquad r \geq 1.$$

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Yes.

When the observations of the nodes are symmetric

Exchangeable Random Variables

$\mathrm{P}_{X_1,\ldots,X_m} = \mathrm{P}_{X_{\sigma(1)},\ldots,X_{\sigma(m)}}$, for all permutations σ of $\{1, \ldots, m\}$

For disjoint sets B_1, B_2: $H\left(X_{B_1} | X_{B_2}\right)$ depends only on $|B_1|, |B_2|$

Define: $g(i|j) = H\left(X_1, \ldots, X_i | X_{i+1}, \ldots, X_{i+j}\right)$

Lemma (Minimum Rate of Communication for Omniscience)

For

$$\alpha_m = \frac{g(m-1|1)}{m-1},$$

$(\alpha_m, \ldots, \alpha_m)$ *is an optimal rate-vector for omniscience, i.e.,* $R_{CO} = m\alpha_m$.

Lemma

α_m *is nonincreasing in* m.

Proof: Uses properties $g(i|j)$ inherited from $H(\cdot)$.

Optimal Fault-Tolerant SK Generation Protocol

2-rounds adaptive protocol:

1. Each node communicates using random mapping of rate α_m.
   $A_1 =$ set of nodes that communicate in round 1, $|A_1| = k$
2. Nodes in A_1 send further communication of rate $\alpha_k - \alpha_m$
   - if $A_2 \neq A_1$ the protocol fails.

Observation: Two random mappings of rates R_1 and R_2 can serve as a single random mapping of rate $R_1 + R_2$ in (multiterminal) Slepian-Wolf coding.

Performance of the protocol:

- Nodes in $A_2 = A_1$ recover $X^n_{A_1}$
- Rate of communication $= k\alpha_k$
- Nodes in A_2 generate SK of rate $C(A_2)$

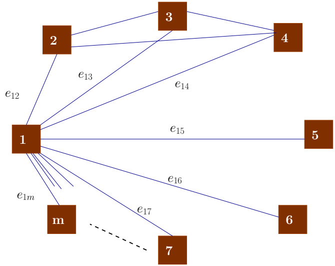
Theorem (Fault-Tolerant SK Capacity)

For exchangeable rvs, for $r \geq 2$,

$$C^{r,t}(\mathcal{M}) = \min_{\substack{A \subseteq \mathcal{M} \\ |A| = m-t}} C(A) = g(m-t|0) - \frac{(m-t)g(m-t-1|1)}{m-t-1}.$$

The Pairwise-Independent-Network Model

Graph $G = (\mathcal{V}, \mathcal{E})$



Ye-Reznik 2007, Nitinawarat et.al. 2010

B_{ij}: unbiased bit corresponding to the edge e_{ij}

Random Variables $\{B_{ij} : i, j \in \mathcal{M}\}$ are mutually independent.

- $X_i = \{B_{ij}$ corresponding to edges e_{ij} incident on $i\}$

The Pairwise-Independent-Network Model

$\boxed{\text{Assumption: The graph } G \text{ is complete}}$

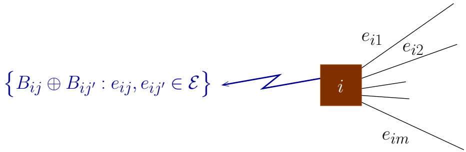
Symmetry: For $B_1 \cap B_2 = \emptyset$, $H\left(X_{B_1} | X_{B_2}\right)$ depends only on $|B_1|, |B_2|$.

$$C_0^{1,t}(\mathcal{M}) \leq C^{2,t}(\mathcal{M}) = g(m-t|0) - \frac{(m-t)g(m-t-1|1)}{m-t-1} = \frac{m-t}{2}$$

Generating 1-bit Fault-Tolerant SK

$\boxed{\text{Assume that } G \text{ is a } (t+1)\text{-}\textit{connected, spanning} \text{ graph.}}$

- Noninteractive protocol to generate 1-bit of fault-tolerant SK:



For $A \subseteq \mathcal{M}$ with $|A| \geq m - t$: let e_A be an edge between nodes in A.

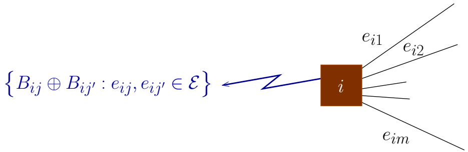
Claim: $H\left(B_{e_A} \middle| \left(F_A, X_i\right)\right) = 0$ *and* $I\left(B_{e_A} \wedge F_A\right) = 0$, $i \in A$.

B_{eA} constitutes a 1-bit SK for A

Generating 1-bit Fault-Tolerant SK

Assume that G is a $(t+1)$-*connected, spanning* graph.

- Noninteractive protocol to generate 1-bit of fault-tolerant SK:



This noninteractive protocol generates 1-bit SK for each spanning tree.

Nitinawarat et.al. use the interactive protocol of Csiszár-Narayan.

Optimal Fault-Tolerant SK Generation Protocol

Assumption: The graph G is complete

Noninteractive protocol above gives 1-bit of SK for each spanning tree

Find a "fault-tolerant" spanning tree packing

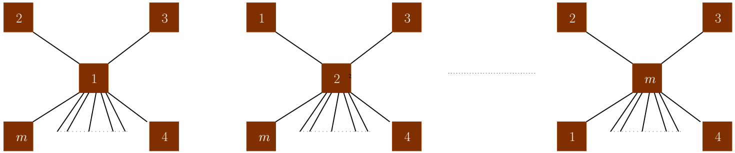
- sufficiently many spanning trees must remain when nodes drop out

- Consider $n = 2$: Any two nodes share 2 independent bits
- Can find a spanning tree packing such that:
  - any subset A contains $|A|$ spanning trees

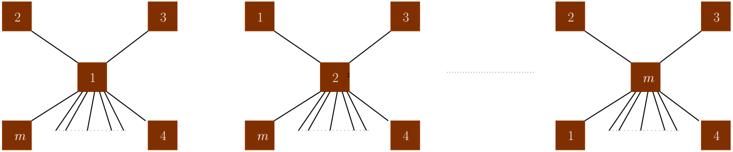
Thus, a subset of size $\geq m - t$ can pack $m - t$ spanning trees

Secret key rate attained: $\frac{m-t}{2}$

Optimal Fault-Tolerant SK Generation Protocol



Optimal Fault-Tolerant SK Generation Protocol



Theorem

For the PIN model corresponding to a complete graph,

$$C_0^{1,t}(\mathcal{M}) = C^{r,t}(\mathcal{M}) = \frac{m-t}{2}, \qquad r \geq 2.$$

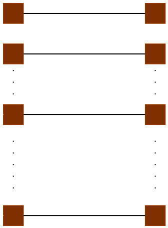
An Alternative Protocol

A protocol to generate $\lfloor \frac{m}{2} \rfloor - t$ bits of SK for $n = 1$:

First consider m even.

Tree remains connected if a leaf node drops out.

- Fix a matching in G.



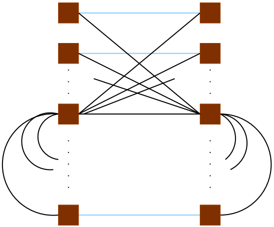
An Alternative Protocol

A protocol to generate $\lfloor \frac{m}{2} \rfloor - t$ bits of SK for $n = 1$:

First consider m even.

Tree remains connected if a leaf node drops out.

- Fix a matching in G.
- There is a spanning tree corresponding to each edge in the matching.



Future Directions

- This work is a first step towards the larger goal of information-theoretic SK agreement for dynamic groups.
- Incorporate rejoining of terminals that drop out.
- What if the central switch has additional side information?