Journal of Mathematics Research; Vol. 6, No. 2; 2014
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Agmon–Kolmogorov Inequalities on $\ell^2(\mathbb{Z}^d)$

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Received: December 13, 2013 Accepted: February 13, 2014 Online Published: April 9, 2014

doi:10.5539/jmr.v6n2p38 URL: http://dx.doi.org/10.5539/jmr.v6n2p38

Abstract

Landau–Kolmogorov inequalities have been extensively studied on both continuous and discrete domains for an entire century. However, the research is limited to the study of functions and sequences on $\mathbb{R}$ and $\mathbb{Z}$, with no equivalent inequalities in higher-dimensional spaces. The aim of this paper is to obtain a new class of discrete Landau–Kolmogorov type inequalities of arbitrary dimension:

$$\|\varphi\|_{\ell^\infty(\mathbb{Z}^d)} \leq \mu_{p,d} \|\nabla_D \varphi\|^{p/2^d}_{\ell^2(\mathbb{Z}^d)} \|\varphi\|^{1-p/2^d}_{\ell^2(\mathbb{Z}^d)},$$

where the constant $\mu_{p,d}$ is explicitly specified. In fact, this also generalises the discrete Agmon inequality to higher dimension, which in the corresponding continuous case is not possible.

Keywords: Lieb–Thirring inequalities, Landau–Kolmogorov inequalities, discrete inequalities, discrete spaces, functional inequalities, sequence spaces

1. Introduction

In 1912, Hardy, Littlewood and Pólya proved the following inequalities for a function $f \in L^2(\mathbb{R})$:

$$\|f'\|_{L^2(-\infty,\infty)} \leq \|f\|^{1/2}_{L^2(-\infty,\infty)} \|f''\|^{1/2}_{L^2(-\infty,\infty)}, \tag{1}$$

$$\|f'\|_{L^2(0,\infty)} \leq \sqrt{2} \|f\|^{1/2}_{L^2(0,\infty)} \|f''\|^{1/2}_{L^2(0,\infty)}, \tag{2}$$

with the constants 1 and $\sqrt{2}$ being sharp. These results sparked interest in inequalities involving functions, their derivatives and integrals for a century to come. Specifically, in 1913, Landau proved the following inequality: For $\Omega \subseteq \mathbb{R}$, and $f \in L^\infty(\Omega)$:

$$\|f'\|_{L^\infty(\Omega)} \leq \sqrt{2} \|f''\|^{1/2}_{L^\infty(\Omega)} \|f\|^{1/2}_{L^\infty(\Omega)},$$

with the constant $\sqrt{2}$ being sharp. This result in turn was motivation for A. Kolmogorov, where in 1939 he found sharp constants for the more general case, using a simple, but very effective inductive argument to extend the case to higher order derivatives:

$$\|f^{(k)}\|_{L^\infty(\Omega)} \leq C(k,n) \|f^{(n)}\|^{k/n}_{L^\infty(\Omega)} \|f\|^{1-k/n}_{L^\infty(\Omega)},$$

where, for $k, n \in \mathbb{N}$ with $1 \leq k < n$, he determined the best constants $C(k,n) \in \mathbb{R}$ for $\Omega = \mathbb{R}$. Since then, there has been a great deal of work on what are nowadays known as the Landau–Kolmogorov inequalities, which are in their most general form:

$$\|f^{(k)}\|_{L^p} \leq K(k,n,p,q,r)\ \|f^{(n)}\|^{\alpha}_{L^q} \|f\|^{\beta}_{L^r},$$

with the minimal constant $K = K(k,n,p,q,r)$. The real numbers $p, q, r \geq 1$; $k, n \in \mathbb{N}$ with $(0 \leq k < n)$ and $\alpha, \beta \in \mathbb{R}$ take on values for which the constant K is finite (Gabushin, 1967).

However, literature on discrete equivalents of those inequalities remained very limited for a long time. In 1979, E. T. Copson was one of the first to find equivalent results for sequences, series and difference operators. Indeed, he found the discrete equivalent to (1) and (2). For a square summable sequence, $\{a(n)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ and a difference operator $(Da)(n) := a(n+1) - a(n)$, we have:

$$\|Da\|_{\ell^2(-\infty,\infty)} \leq \|a\|^{1/2}_{\ell^2(-\infty,\infty)} \|D^2 a\|^{1/2}_{\ell^2(-\infty,\infty)}, \tag{3}$$

$$\|Da\|_{\ell^2(0,\infty)} \leq \sqrt{2}\|a\|^{1/2}_{\ell^2(0,\infty)}\|D^2a\|^{1/2}_{\ell^2(0,\infty)}, \tag{4}$$

with the constants 1 and $\sqrt{2}$ yet again being sharp. Z. Ditzian (1982/83) then extended those results to establish best constants for a variety of Banach spaces, adding equivalent results for continuous shift operators $f(x+h) - f(x);\ x \in \mathbb{R}, f \in Ł^2(\mathbb{R})$.

Comparing inequalities such as (1) and (2), with (3) and (4) respectively, it was suspected that sharp constants were identical for equivalent discrete and continuous Landau–Kolmogorov inequalities for $1 \leq p = q = r \leq \infty$. Indeed, in the cases $p = 1, 2, \infty$, this was true for the whole and semi-axis. However, the general case has since been shown to be false, as for example demonstrated in 1988 by M. K. Kwong and A. Zettl, where they prove that for many values of p, the discrete constants are strictly greater than the continuous ones.

Another important special case of the Landau–Kolmogorov inequalities is the Agmon inequality, proven by Agmon (2010). Viewed as an interpolation inequality between $L^\infty(\mathbb{R})$ and $L^2(\mathbb{R})$, he states the following:

$$\|f\|_{L^\infty(\mathbb{R})} \leq \|f\|^{1/2}_{L^2(\mathbb{R})}\|f'\|^{1/2}_{L^2(\mathbb{R})}.$$

Thus, throughout this paper we shall call, for a domain Ω, a function $f \in L^2(\Omega)$, a sequence $\varphi \in \ell^2(\Omega)$, α, β being $\mathbb{Q}$-valued functions of the integers k, n with $k \leq n$ and constants $C(\Omega, k, n),\ D(\Omega, k, n) \in \mathbb{R}$:

$$\|f^{(k)}\|_{L^\infty(\Omega)} \leq C(\Omega, k, n)\ \|f\|^{\alpha(k,n)}_{L^2(\Omega)}\ \|f^{(n)}\|^{\beta(k,n)}_{L^2(\Omega)}, \tag{5}$$

$$\|D^k\varphi\|_{\ell^\infty(\Omega)} \leq D(\Omega, k, n)\ \|\varphi\|^{\alpha(k,n)}_{\ell^2(\Omega)}\ \|D^n\varphi\|^{\beta(k,n)}_{\ell^2(\Omega)}. \tag{6}$$

Agmon–Kolmogorov inequalities, where (6), for $\Omega := \mathbb{Z}^d$ will be the central concern of this paper. Specifically we only require the case where $k = 0$ and $n = 1$, whereas the other inequalities, i.e. those concerned with higher order, have been discussed in Sahovic (2013). These have a variety of applications in spectral theory for example. They can be used to obtain the Generalise Sobolev inequality and thus afterwards to obtain a variety of Lieb–Thirring class inequalities. These in turn have vast applications in the theory of quantum mechanics. Alternatively, Agmon–Kolmogorov inequalities on higher dimensional discrete spaces are a bit of a novelty in the field discrete inequalities, as usually only one-dimensional inequalities are studied. The induction method contains intrinsic ideas translateable to other classes of inequalities.

2. Agmon–Kolmogorov Inequalities Over $\mathbb{Z}^d$

We introduce our notations for the d-dimensional inner product space of square summable sequences. For a vector of integers $\zeta := (\zeta_1, \ldots, \zeta_d) \in \mathbb{Z}^d$, we say $\{\varphi(\zeta)\}_{\zeta\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$, if and only if the following norm is finite:

$$\|\varphi\|_{\ell^2(\mathbb{Z}^d)} := \Big(\sum_{\zeta\in\mathbb{Z}^d} |\varphi(\zeta)|^p\Big)^{1/2}.$$

Then, for φ, $\phi \in \ell^2(\mathbb{Z}^d)$, we let $< .,. >_d$ be the inner product on $\ell^2(\mathbb{Z}^d)$:

$$\langle\varphi, \phi\rangle_d := \sum_{\zeta\in\mathbb{Z}^d} \varphi(\zeta)\overline{\phi(\zeta)}.$$

We then let $D_1, \ldots, D_d$ be the partial difference operators defined by:

$$(D_i\varphi)(\zeta) := \varphi(\zeta_1, \ldots, \zeta_i + 1, \ldots, \zeta_d) - \varphi(\zeta_1, \ldots, \zeta_d).$$

The discrete gradient ∇_D shall thus take the following form:

$$\nabla_D\varphi(\zeta_1, \zeta_2, \ldots, \zeta_d) = \big(D_1\varphi(\zeta), D_2\varphi(\zeta), \ldots, D_d\varphi(\zeta)\big).$$

Thus, combining this definition with that of our norm above, we obtain:

$$\|\nabla_D\varphi\|^2_{\ell^2(\mathbb{Z}^d)} = \|D_1\varphi\|^2_{\ell^2(\mathbb{Z}^d)} + \ldots + \|D_d\varphi\|^2_{\ell^2(\mathbb{Z}^d)}.$$

Further, we require the following notation:

Definition 2.1 For a sequence $\varphi(\zeta) \in \ell^2(\mathbb{Z}^d)$ with $\zeta := (\zeta_1, ..., \zeta_d) \in \mathbb{Z}^d$, for $0 \le k \le d$ we define:

$$[\varphi]_k := \left(\sum_{\zeta_1 \in \mathbb{Z}} ... \sum_{\zeta_k \in \mathbb{Z}} |\varphi(\zeta)|^2 \right)^{1/2}.$$

Remark We identify that $[\varphi]_0 := |\varphi(\zeta)|$ and if we apply this operator for $k = d$, i.e. sum across all coordinates, we obtain the $\ell^2(\mathbb{Z}^d)$-norm:

$$[\varphi]_d = \|\varphi\|_{\ell^2(\mathbb{Z}^d)}.$$

We are interested in a higher-dimensional version of the discrete Agmon inequality (Sahovic, 2010), which estimates the sup-norm of a sequence $\phi \in \ell^2(\mathbb{Z})$ as follows:

$$\|\phi\|^2_{\ell^\infty(\mathbb{Z})} \le \|\phi\|_{\ell^2(\mathbb{Z})} \|D\phi\|_{\ell^2(\mathbb{Z})}.$$

Thus we commence by 'lifting' this estimate to encompass more variables:

Lemma 2.2 (Agmon–Cauchy Inequality) *For the operator D_{k+1}, acting on a sequence $\varphi(\zeta) \in \ell^2(\mathbb{Z}^d)$, we have:*

$$\sup_{\zeta_{k+1} \in \mathbb{Z}} [\varphi]_k \le [D_{k+1}\varphi]^{1/2}_{k+1} \, [\varphi]^{1/2}_{k+1}.$$

Proof. Using the discrete Agmon inequality on the $(k+1)^{th}$ coordinate, we find:

$$|\varphi(\zeta_1, \ldots, \zeta_d)|^2 \le \Big(\sum_{l \in \mathbb{Z}} |D_{k+1}\varphi(\zeta_1, \ldots, \zeta_k, l, \zeta_{k+2}, \ldots, \zeta_d)|^2 \Big)^{1/2} \Big(\sum_{l \in \mathbb{Z}} |\varphi(\zeta_1, \ldots, \zeta_k, l, \zeta_{k+2}, \ldots, \zeta_d)|^2 \Big)^{1/2}.$$

Now we sum with respect to the other coordinates:

$$\begin{aligned} \sum_{\zeta_1 \in \mathbb{Z}} ... \sum_{\zeta_k \in \mathbb{Z}} |\varphi(\zeta_1, \ldots, \zeta_d)|^2 \quad \le \quad & \sum_{\zeta_1 \in \mathbb{Z}} ... \sum_{\zeta_k \in \mathbb{Z}} \Bigg[\Big(\sum_{l \in \mathbb{Z}} |D_{k+1}\varphi(\zeta_1, \ldots, \zeta_k, l, \zeta_{k+2}, \ldots, \zeta_d)|^2 \Big)^{1/2} \cdot \\ & \Big(\sum_{l \in \mathbb{Z}} |\varphi(\zeta_1, \ldots, \zeta_k, l, \zeta_{k+2}, \ldots, \zeta_d)|^2 \Big)^{1/2} \Bigg], \end{aligned}$$

and use the Cauchy–Schwartz inequality on the k^{th} coordinate:

$$\begin{aligned} \sum_{\zeta_1 \in \mathbb{Z}} ... \sum_{\zeta_k \in \mathbb{Z}} |\varphi(\zeta_1, \ldots, \zeta_d)|^2 \quad \le \quad & \sum_{\zeta_1 \in \mathbb{Z}} ... \sum_{\zeta_{k-1} \in \mathbb{Z}} \Bigg[\Big(\sum_{\zeta_k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} |D_{k+1}\varphi(\zeta_1, \ldots, \zeta_k, l, \zeta_{k+2}, \ldots, \zeta_d,)|^2 \Big)^{1/2} \cdot \\ & \Big(\sum_{\zeta_k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} |\varphi(\zeta_1, \ldots, \zeta_k, l, \zeta_{k+2}, \ldots, \zeta_d)|^2 \Big)^{1/2} \Bigg]. \end{aligned}$$

We repeat this process to finally obtain:

$$\begin{aligned} \sum_{\zeta_1 \in \mathbb{Z}} ... \sum_{\zeta_k \in \mathbb{Z}} |\varphi(\zeta_1, \ldots, \zeta_d)|^2 \quad \le \quad & \Big(\sum_{\zeta_1 \in \mathbb{Z}} ... \sum_{\zeta_k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} |D_{k+1}\varphi(\zeta_1, \ldots, \zeta_k, l, \zeta_{k+2}, \ldots, \zeta_d)|^2 \Big)^{1/2} \cdot \\ & \Big(\sum_{\zeta_1 \in \mathbb{Z}} ... \sum_{\zeta_k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} |\varphi(\zeta_1, \ldots, \zeta_k, l, \zeta_{k+2}, \ldots, \zeta_d)|^2 \Big)^{1/2}. \end{aligned}$$

□

We estimate the $\ell^2(\mathbb{Z}^d)$-norm of a partial difference operator with the $\ell^2(\mathbb{Z}^d)$-norm of the sequence itself:

Lemma 2.3 *For a sequence $\varphi \in \ell^2(\mathbb{Z}^d)$ and for $i \in \{1, \ldots, d\}$, we have:*

$$\|D_i\varphi\|_{\ell^2(\mathbb{Z}^d)} \le 2\|\varphi\|_{\ell^2(\mathbb{Z}^d)}.$$

Proof. We show the argument for D_1 and note that due to symmetry the other cases follow immediately.

$$\begin{aligned} \|D_1\varphi\|^2_{\ell^2(\mathbb{Z}^d)} \quad &= \quad \sum_{\zeta \in \mathbb{Z}^d} |\varphi(\zeta_1 + 1, \ldots, \zeta_d) - (\zeta_1, \ldots, \zeta_d)|^2 \\ &\le \quad 2 \Big(\sum_{\zeta \in \mathbb{Z}^d} |\varphi(\zeta_1 + 1, \ldots, \zeta_d)|^2 + \sum_{\zeta \in \mathbb{Z}^d} |\varphi(\zeta_1, \ldots, \zeta_d)|^2 \Big) \\ &= \quad 4 \sum_{\zeta \in \mathbb{Z}^d} |\varphi(\zeta_1, \ldots, \zeta_d)|^2 = 4\|\varphi\|^2_{\ell^2(\mathbb{Z}^d)}. \end{aligned}$$

□

This implies that we can obtain an estimate for any mixed difference operator as follows:

$$\|D_1 \ldots D_k\varphi\|_{\ell^2(\mathbb{Z}^d)} \leq 2\|D_1 \ldots D_{l-1}D_{l+1} \ldots D_k\varphi\|_{\ell^2(\mathbb{Z}^d)}.$$

Therefore, by eliminating l difference operators, our inequality will contain the constant 2^l.

We arrive at our main result, the Agmon–Kolmogorov inequalities on $\ell^2(\mathbb{Z}^d)$.

Theorem 2.4 *For a sequence $\varphi \in \ell^2(\mathbb{Z}^d)$, and $p \in \{1, \ldots, 2^{d-1}\}$:*

$$\|\varphi\|_{\ell^\infty(\mathbb{Z}^d)} \leq \mu_{p,d}\|\nabla_D\varphi\|^{p/2^d}_{\ell^2(\mathbb{Z}^d)} \|\varphi\|^{1-p/2^d}_{\ell^2(\mathbb{Z}^d)},$$

where

$$\mu_{p,d} := \left(\frac{\kappa_{p,d}}{d^{p/2}}\right)^{1/2^d},$$

and $\kappa_{p,d}$ is a constant to be determined in the following section.

Proof. We use Lemma 2.2 and Lemma 2.3 repeatedly:

$$\begin{aligned}
\|\varphi\|_{\ell^\infty(\mathbb{Z}^d)} &\leq [D_1\varphi]_1^{1/2}\,[\varphi]_1^{1/2} \\
&\leq [D_2D_1\varphi]_2^{1/4}\,[D_1\varphi]_2^{1/4}\,[D_2\varphi]_2^{1/4}\,[\varphi]_2^{1/4} \\
&\;\;\vdots \\
&\leq [D_d \ldots D_1\varphi]_d^{1/2^d}\,\ldots\,\ldots\,[\varphi]_d^{1/2^d} \\
&= \|D_d \ldots D_1\varphi\|^{1/2^d}_{\ell^2(\mathbb{Z}^d)}\,\ldots\,\ldots\,\|\varphi\|^{1/2^d}_{\ell^2(\mathbb{Z}^d)} \\
\Rightarrow \|\varphi\|^{2^d}_{\ell^\infty(\mathbb{Z}^d)} &\leq \|D_d \ldots D_1\varphi\|_{\ell^2(\mathbb{Z}^d)}\,\ldots\,\ldots\,\|\varphi\|_{\ell^2(\mathbb{Z}^d)}.
\end{aligned}$$

We have generated an estimate by 2^d norms, with exactly 2^{d-1} norms originating from the term $[D_1\varphi]_1^{1/2}$. All those will thus involve the operator D_1, or more formally: $|\Xi_1| = 2^{d-1}$, where we let

$$\Xi_1 := \left\{\|D_{a_1} \ldots D_{a_k}D_1\varphi\|_{\ell^2(\mathbb{Z}^d)} \,\middle|\, a_i \neq a_j \;\forall\, i \neq j;\; \{a_1, \ldots, a_k\} \subset \{2, \ldots, d\}\right\}.$$

We note that we could also employ estimates by $\|D_i\varphi\|_{\ell^2(\mathbb{Z}^d)}$ for any $i \in \{1, \ldots, 2^d\}$, but our inequality will not change due to our symmetrising argument. Similarly, we have 2^{d-1} norms originating from the term $[\varphi]_1^{1/2}$, whose estimates will not involve the operator D_1. Hence $|\Xi_2| = 2^{d-1}$, where we let

$$\Xi_2 := \left\{\|D_{a_1} \ldots D_{a_k}\varphi\|_{\ell^2(\mathbb{Z}^d)} \,\middle|\, a_i \neq a_j \;\forall\, i \neq j;\; \{a_1, \ldots, a_k\} \subset \{2, \ldots, d\}\right\}.$$

We will now apply Lemma 2.3 repeatedly, to reduce the order of the operator inside the norms to either 0 or 1. We recognise that we have to estimate all ${}^1\xi \in \Xi_1$ by ${}^1\xi_1 := \|D_1\varphi\|_{\ell^2(\mathbb{Z}^d)}$ or alternatively by $\|\varphi\|_{\ell^2(\mathbb{Z}^d)}$.

Hence, we choose a $p \in \{0, \ldots, 2^{d-1}\}$ to estimate p elements in Ξ_1 by $\|D_1\varphi\|_{\ell^2(\mathbb{Z}^d)}$, leaving $2^{d-1} - p$ elements in Ξ_1 to be estimated by $\|\varphi\|_{\ell^2(\mathbb{Z}^d)}$. However, for all 2^{d-1} elements ${}^2\xi \in \Xi_2$, we have to provide an estimate by ${}^2\xi_1 := \|\varphi\|_{\ell^2(\mathbb{Z}^d)}$ only. This means we have $2^d - p$ elements in $\Xi := \Xi_1 \bigcup \Xi_2$ to be estimated by $\|\varphi\|_{\ell^2(\mathbb{Z}^d)}$:

$$\|\varphi\|^{2^d}_{\ell^\infty(\mathbb{Z}^d)} \leq \kappa_{p,d}\|D_1\varphi\|^p_{\ell^2(\mathbb{Z}^d)} \|\varphi\|^{2^d-p}_{\ell^2(\mathbb{Z}^d)},$$

where $\kappa_{p,d}$ remains a constant of the form 2^z with $z \in \mathbb{Q}$, which we leave to be identified in the next section. We thus obtain the following estimate:

$$\|\varphi\|^{2^{d+1}/p}_{\ell^\infty(\mathbb{Z}^d)} \leq \kappa^{2/p}_{p,d}\|D_1\varphi\|^2_{\ell^2(\mathbb{Z}^d)} \|\varphi\|^{(2^{d+1}-2p)/p}_{\ell^2(\mathbb{Z}^d)}.$$

We now exploit the symmetry of the argument:

$$\begin{aligned}
d\,\|\varphi\|^{2^{d+1}/p}_{\ell^\infty(\mathbb{Z}^d)} &\leq \kappa^{2/p}_{p,d}\left(\|D_1\varphi\|^2_{\ell^2(\mathbb{Z}^d)} + \ldots + \|D_d\varphi\|^2_{\ell^2(\mathbb{Z}^d)}\right)\|\varphi\|^{(2^{d+1}-2p)/p}_{\ell^2(\mathbb{Z}^d)} \\
&= \kappa^{2/p}_{p,d}\|\nabla_D\varphi\|^2_{\ell^2(\mathbb{Z}^d)} \|\varphi\|^{(2^{d+1}-2p)/p}_{\ell^2(\mathbb{Z}^d)},
\end{aligned}$$

and finally rearrange:

$$\|\varphi\|_{\ell^\infty(\mathbb{Z}^d)} \leq \left(\frac{\kappa_{p,d}}{d^{p/2}}\right)^{1/2^d} \|\nabla_D\varphi\|^{p/2^d}_{\ell^2(\mathbb{Z}^d)} \|\varphi\|^{1-p/2^d}_{\ell^2(\mathbb{Z}^d)}.$$

□

3. The Constant $\kappa_{p,d}$

It remains to identify the constant $\kappa_{p,d}$, we thus give:

Theorem 3.1 *We have, for arbitrary dimension d and $p \in \{1, \ldots, 2^{d-1}\}$:*

$$\kappa_{p,d} = 2^{d \cdot 2^{d-1} - p}.$$

We will break the proof down into several steps. The method for finding $\kappa_{p,d}$ will rely largely on the following observation:

Let $\tau(\xi)$ be the order of the operator contained in any given $\xi \in \Xi$. Then we let $\Omega_i := \{\xi \,|\, \tau(\xi) = i\}$, be the set of all terms in the estimate whose operator has a given order i. In Ξ_1 we have $1 \leq i \leq d$, and in Ξ_2, $0 \leq i \leq d-1$.

Lemma 3.2 *For the size of Ω_i, we have for $d \geq 2$:*

For Ξ_1:

$$|\Omega_i| = \binom{d-1}{i-1}, \qquad 1 \leq i \leq d,$$

and Ξ_2:

$$|\Omega_i| = \binom{d-1}{i}, \qquad 0 \leq i \leq d-1.$$

Proof. We follow by induction and prove the case of Ξ_2, noting that the argument for Ξ_1 is symmetrically identical. We have already seen that the formula is correct for $d = 2$, and now we assume it is true for $d = l$, i.e. for $0 \leq i \leq l-1$:

$$|\Omega_i| = \binom{l-1}{i},$$

and thus we have the following list:

$$\begin{array}{cccccccccccc} \Xi_2 & {}^2\xi_{2^{d-1}} & \ldots & \ldots & {}^2\xi_2 & {}^2\xi_1 & |\Omega_0| & |\Omega_1| & |\Omega_2| & \ldots & |\Omega_{l-1}| \\ \mathbb{Z}^l\colon & D_l \ldots D_2 & \ldots & \ldots & D_2 & 1 & \binom{l-1}{0} & \binom{l-1}{1} & \binom{l-1}{2} & \ldots & \binom{l-1}{l-1} \end{array}$$

Now each term of a given order τ will, by the Agmon–Cauchy inequality (Lemma 2.2), generate a term of order τ and one of order $\tau + 1$. Thus we have:

$$\begin{array}{cccccccccccc} \Xi_2 & {}^2\xi_{2^d} & \ldots & \ldots & {}^2\xi_2 & {}^2\xi_1 & |\Omega_0| & |\Omega_1| & |\Omega_2| & \ldots & |\Omega_l| \\ \mathbb{Z}^{l+1}\colon & D_{l+1} \ldots D_2 & \ldots & \ldots & D_2 & 1 & \binom{l-1}{0} & \binom{l-1}{0} + \binom{l-1}{1} & \binom{l-1}{1} + \binom{l-1}{2} & \ldots & \binom{l-1}{l-1} \end{array}$$

Now we apply the standard combinatorial identity ${}^aC_b + {}^aC_{b+1} = {}^{a+1}C_{b+1}$ and consider ${}^aC_0 = {}^aC_a = 1$, which immediately implies:

$$\begin{array}{cccccccccccc} \Xi_2 & {}^2\xi_{2^d} & \ldots & \ldots & {}^2\xi_2 & {}^2\xi_1 & |\Omega_0| & |\Omega_1| & |\Omega_2| & ... & |\Omega_{l+1}| \\ \mathbb{Z}^{l+1}\colon & D_{l+1} \ldots D_2 & \ldots & \ldots & D_2 & 1 & \binom{l}{0} & \binom{l}{1} & \binom{l}{2} & \ldots & \binom{l}{l} \end{array}$$

and hence for $d = l + 1$, we have:

$$|\Omega_i| = \binom{l}{i},$$

completing our inductive step. □

As discussed previously, if we consider to estimate a given $\xi \in \Xi$ using Lemma 2.3, we will, for example, obtain $\|D_1 \ldots D_k\varphi\|_{\ell^2(\mathbb{Z}^d)} \leq 2\|D_1 \ldots D_{l-1}D_{l+1} \ldots D_k\varphi\|_{\ell^2(\mathbb{Z}^d)}$. We can see that we generate a factor of 2 for every partial difference operator we eliminate, and thus have, for ${}^1\xi \in \Xi_1$ and ${}^2\xi \in \Xi_2$ with order $\tau({}^1\xi)$ and $\tau({}^2\xi)$ respectively:

$${}^1\xi \leq\; 2^{\tau({}^1\xi)-1} \|D_1\varphi\|_{\ell^2(\mathbb{Z}^d)}, \qquad \text{and} \qquad {}^2\xi \leq\; 2^{\tau({}^2\xi)} \|\varphi\|_{\ell^2(\mathbb{Z}^d)}.$$

We note here that $\kappa_{p,d}$ will not depend on which $\ell^2(\mathbb{Z}^d)$-norms in Ξ_1 are chosen to be estimated by ${}^2\xi_1 := \|\varphi\|_{\ell^2(\mathbb{Z}^d)}$. The reason for this is transparent when considering that the sum of all the orders $\sum_{i=1}^{2^{d-1}} \tau({}^1\xi_i)$ is a constant and needs to be reduced to the constant $p \cdot \tau({}^1\xi_1) = p$, generating a unique $\kappa_{p,d}$.

Lemma 3.3 *The* $\min_p \kappa_{p,d}$ *will be attained at* $p = 2^{d-1}$ *and takes on the following explicit form:*

$$\kappa_{2^{d-1},d} = \prod_{i=0}^{d-1} 2^{2i\binom{d-1}{i}}.$$

Proof. Our minimum constant for Ξ_1 in fact occurs if we choose all ${}^1\xi_1 \in \Xi_1$ to be estimated by $\|D_1\varphi\|_{\ell^2(\mathbb{Z}^d)}$, i.e. choose $p = 2^{d-1}$, the maximum p possible. Our minimum constant, denoted by ρ_d^1, for all terms in Ξ_1 will thus be:

$$\rho_d^1 = \prod_{k=1}^{2^{d-1}} 2^{\tau({}^1\xi_k)-1}.$$

Instead of examining each individual element ${}^1\xi$, we consider that all ${}^1\xi$ of equal order i generate the same constant, namely 2^{i-1}. Thus we collect all ${}^1\xi$ of the same order, and obtain:

$$\rho_d^1 = \prod_{i=1}^{d} 2^{(i-1)|\Omega_i|} = \prod_{i=1}^{d} 2^{(i-1)\binom{d-1}{i-1}}.$$

Then we need to estimate all ${}^2\xi \in \Xi_2$, and we proceed as for Ξ_1. All ${}^2\xi$ need to be estimated by $\|\varphi\|_{\ell^2(\mathbb{Z}^d)}$, each generating the constant 2^i, forming the equivalent pattern as that of Ξ_1. We thus obtain, for the minimal constant ρ_d^2:

$$\rho_d^2 = \prod_{i=0}^{d-1} 2^{i|\Omega_i|} = \prod_{i=0}^{d-1} 2^{i\binom{d-1}{i}}.$$

We now see that $\rho_d^2 = \rho_d^1$, and:

$$\kappa_{2^{d-1},d} = \rho_d^2\rho_d^1 = \prod_{i=0}^{d-1} 2^{2i\binom{d-1}{i}}.$$

□

We are now finally in a position to prove Theorem 3.1:

Proof. (Proof of Theorem 3.1) We are left to analyse the constant's dependence on our choice of p. First we note that in addition to the constant generated above, we will have chosen $2^{d-1} - p$ terms to be further reduced to $\|\varphi\|_{\ell^2(\mathbb{Z}^d)}$, each generating a power of 2. Hence we additionally need to multiply $\kappa_{2^{d-1},d}$ by $2^{2^{d-1}-p}$. Thus our final constant will be:

$$\kappa_{p,d} = 2^{2^{d-1}-p} \cdot \prod_{i=0}^{d-1} 2^{2i\binom{d-1}{i}} = 2^{2^{d-1}-p+2\sum_{i=0}^{d-1} i\binom{d-1}{i}}.$$

Then we can simplify this further by considering the binomial formula $(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$. We differentiate with respect to X and set $X = 1$:

$$n \cdot 2^{n-1} = \sum_{k=0}^{n} k \binom{n}{k}.$$

Thus we arrive at:

$$\kappa_{p,d} = 2^{d \cdot 2^{d-1}-p}.$$

□

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