Identifying Taylor rules in macro-finance models*

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Abstract

Identification problems arise naturally in forward-looking models when agents observe more than economists. We illustrate the problem in several New Keynesian and macro-finance models in which the Taylor rule includes a shock unseen by economists. We show that identification of the rule's parameters requires restrictions on the form of the shock. A state-space treatment verifies that this works when we observe the state of the economy and when we infer it from observable macroeconomic variables or asset prices.

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1 Introduction

The field of macro-finance has the potential to give us deeper insights into macroeconomics and macroeconomic policy by combining information about aggregate quantities with asset prices. The link between bond-pricing and monetary policy seems particularly promising if central banks implement monetary policy through short-term interest rates, as they do in models with Taylor rules.

If the combination of macroeconomics and finance holds promise, it also raises challenges. We address one of them here: the challenge of identifying monetary policy parameters in a modern forward-looking macroeconomic model. Identification problems arise in many economic models, but they play a particularly important role in assessments of monetary policy. If we see that the short-term interest rate rises with inflation, does that reflect the policy of the central bank, the valuation of private agents, or something else? Can we tell the difference?

Some prominent scholars argue that the answer is no. Cochrane (2011, page 606) puts it this way: "The crucial Taylor rule parameter is not identified in the new-Keynesian model." He devotes most of his paper to making the case. Joslin, Le, and Singleton (2013, page 597) make a related point about interpretations of estimated bond pricing models. In their words: "Several recent studies interpret the short-rate equation as a Taylor-style rule. ... However, without imposing additional economic structure, ... the parameters are not meaningfully interpretable as the reaction coefficients of a central bank." Canova and Sala (2009), Carrillo, Feve, and Matheron (2007), and Iskrev (2010) also question aspects of the identification of New Keynesian models.

We illustrate the identification problem and point to its possible resolution in several examples that combine elements of New Keynesian and macro-finance asset pricing models. The first example is adapted from Cochrane (2011) and consists of the Fisher equation and a Taylor rule. Later examples introduce exponential-affine pricing kernels and New Keynesian Phillips curves. The source of the identification problem in these models is that the agents populating the model observe the shock to monetary policy, but we economists do not. This approach dates back at least to Hansen and Sargent (1980, 1991) and has become standard practice in the New Keynesian literature. One consequence is that it can be difficult, or even impossible, for economists to disentangle the systematic aspects of monetary policy from unseen shocks to it.

We show, in these models, that even when we do not observe the shock, restrictions on the form of the shock allow us to identify the parameters of the Taylor rule. We do two things that we think make the logic clearer. First, we express shocks as linear functions of the state, a vector of dimension n. The state has linear dynamics, which are then inherited by the shocks. This is a modest but useful reframing of what we see in the New Keynesian literature, where shocks are typically independent ARMA processes.

Second, we deal separately with issues concerning the observation of shocks and those concerning observation of the state. We start by assuming that we observe the state but not the shock, by which we mean that we do not know the coefficients connecting the shock to the state. A clear conclusion emerges in this case: we need restrictions on the shock to the Taylor rule to identify the rule's parameters. In our examples, we need one restriction for each parameter to be estimated. If the state is not observed, we use a more general state-space framework in which the dynamics of the state are augmented with measurement equations connecting observable variables to the state. The state-space framework encompasses earlier examples as special cases. We find that the identification problem appears here in the relation between coefficients of measurement equations and structural parameters. The same reasoning applies: we need restrictions on the coefficients of the shocks to identify structural parameters such as those of the Taylor rule.

If this seems clear to us now, it was not when we started. We thought, at first, that identification required shocks in other equations, as in the Cowles Commission approach to simultaneous equation systems. We find, instead, that the conditions for identification don't change when we eliminate other shocks. They depend entirely on what we know about the shock in the equation of interest, the Taylor rule. We also thought that knowledge of the term structure of interest rates might help with identification. We find, instead, that knowing the term structure can be helpful in estimating the state, but knowing the state is not enough to identify the Taylor rule. What matters in all of these examples is whether we have restrictions on the shock to the Taylor rule that allow us to distinguish the shock from the systematic part of the rule.

One last thing before we start: We need to be clear about terminology. When we say that a parameter is *identified*, we mean that it is *locally point identified*. Local means here that we can distinguish a parameter value from local alternatives. Point means that we have identified a unique value, rather than a larger set.

2 The problem

Two examples illustrate the nature of identification problems in macro-finance models with Taylor rules. The first comes from Cochrane (2011). The second is an exponential-affine bond-pricing model. The critical ingredient in each is what we observe. We assume that economic agents observe everything, but we economists do not. In particular, we do not observe the shock to the Taylor rule. The question is how this affects our ability to infer the Taylor rule's parameters. We provide answers for these two examples and discuss some of the questions they raise about identification in similar settings.

2.1 Cochrane's example

Cochrane's example consists of two equations, an asset pricing relation (the Fisher equation) and a Taylor rule (which depends only on inflation):

$$i_t = E_t \pi_{t+1} \tag{1}$$

$$i_t = \tau \pi_t + s_t. \tag{2}$$

Here i is the (one-period) nominal interest rate, π is the inflation rate, and s is a monetary policy shock. The Taylor rule parameter $\tau > 1$ describes how aggressively the central bank responds to inflation. This model is extremely simple, but it's enough to illustrate the identification problem.

Let us say, to be specific, that the shock is a linear function of the state, $s = d^\top x$, and x is autoregressive,

$$x_{t+1} = A x_t + B w_{t+1}, \tag{3}$$

with A stable and disturbances $\{w_t\} \sim \mathrm{NID}(0, I)$. Although simple, this structure is helpful for clarifying the conditions that allow identification. It also allows easy comparison to models ranging from exponential-affine to vector autoregressions. For later use, we denote the covariance of innovations by $V_w = E[(Bw)(Bw)^\top] = BB^\top$ and the covariance matrix of the state by $V_x = E(xx^\top)$, the solution to $V_x = AV_xA^\top + BB^\top$.

We solve the model by standard methods; see Appendix A. Here and elsewhere, we assume agents know the model and observe all of its variables. Equations (1) and (2) imply the forward-looking difference equation or rational expectations model

$$E_t \pi_{t+1} = \tau \pi_t + s_t.$$

The solution for inflation has the form $\pi_t = b^\top x_t$ for some coefficient vector b to be determined. Then $E_t \pi_{t+1} = b^\top E_t x_{t+1} = b^\top A x_t$. Lining up terms, we see that b satisfies

$$b^\top A = \tau b^\top + d^\top \quad \Rightarrow \quad b^\top = -d^\top (\tau I - A)^{-1}. \tag{4}$$

This is the unique stationary solution if A is stable (eigenvalues less than one in absolute value) and $\tau > 1$ (the so-called Taylor principle). Equation (1) then gives us $i_t = a^\top x_t$ with $a^\top = b^\top A = -d^\top (\tau I - A)^{-1} A$.

Now consider estimation. Do we have enough information to estimate the Taylor rule parameter τ? If so, we can say it's identified. We might try to estimate equation (2) by running a regression of i on π, with the shock s as the residual. That won't work because s affects both i and π, and we need to distinguish its direct effect on i from its indirect effect through π. Least squares would deliver a coefficient of $\mathrm{Var}(\pi)^{-1}\mathrm{Cov}(\pi, i) = (b^\top V_x b)^{-1} b^\top V_x a$, which is not in general equal to τ.

How then can we estimate τ? The critical issue is observation of the shock s. Let us say for now that we — the economists — observe the state x, but may or may not observe the shock s. We return to the issue of state observability in Section 5. If we observe x, we can estimate A and V_x. We can also estimate the parameter vectors a and b connecting the interest rate and inflation to the state. If we observed the shock s, then we could estimate the parameter vector d. We now have all the components of (4) but τ, which we can infer. The Taylor rule parameter is not only identified, it's over-identified. If x has dimension n, we have n equations that each determine τ.

However, if we don't observe s, and therefore do not know d, we're in trouble. This is precisely the situation considered throughout the New Keynesian literature. In economic terms, we can't distinguish the effects on the interest rate of inflation (the parameter τ) and the shock (the coefficient vector d). This is a concrete example of the identification issue faced by economists using New Keynesian models.

2.2 An exponential-affine example

Another perspective on the identification problem is that we can't distinguish the pricing relation (1) from the Taylor rule (2). Sims and Zha (2006, page 57) put it this way: "The ... problem ... is that the Fisher relation is always lurking in the background. The Fisher relation connects current nominal rates to expected future inflation rates and to real interest rates[.] ... So one might easily find an equation that had the form of the ... Taylor rule, ... but was something other than a policy reaction function." Cochrane (2011, page 598) echoes the point: "If we regress interest rates on output and inflation, how do we know that we are recovering the Fed's policy response, and not the parameters of the consumer's first-order condition?" We'll see exactly this issue in the next example, in which we introduce an exponential-affine bond-pricing model into the problem.

Consider, then, an exponential-affine model of interest rates, a structure that's widely used in finance, in which bond yields are linear functions of the state. In the macro-finance branch of this literature, the state includes macroeconomic variables like inflation and output growth. Examples include Ang and Piazzesi (2003), Chernov and Mueller (2012), Jardet, Monfort, and Pegoraro (2012), Moench (2008), Rudebusch and Wu (2008), and Smith and Taylor (2009). In these models the short rate depends on, among other things, inflation.

An informative example starts with the log pricing kernel,

$$m_{t+1}^{\$} \quad = \quad -\lambda^\top\lambda/2 - \delta^\top x_t + \lambda^\top w_{t+1}, \tag{5}$$

and the linear transition equation (3). Here the nominal (log) pricing kernel $m_t^{\$}$ is connected to the real (log) pricing kernel m_t by $m_t^{\$} = m_t - \pi_t$. The one-period nominal interest rate is then

$$
\begin{aligned}
i_t &= -\log E_t \exp(m_{t+1} - \pi_{t+1}) && (6)\\
&= -\log E_t \exp(m^{\$}_{t+1}) \;\; = \;\; \delta^\top x_t. && (7)
\end{aligned}
$$

If we observe the state x, we can estimate δ by projecting the interest rate onto it.

If the first element of x is the inflation rate, it's tempting to interpret equation (7) as a Taylor rule, with the first element of δ the inflation coefficient τ. But is it? The logic of equation (7) is closer to the asset-pricing relation, equation (1), than to the Taylor rule, equation (2). But without more structure, we can't say whether it's one, the other, or something else altogether. This is, of course, the point made by Sims and Zha and echoed by Cochrane. Joslin, Le, and Singleton (2013, page 583) make a similar point in a model much like this one: "the parameters of a Taylor rule are not econometrically identified" in affine macro-finance models.

More formally, consider an interpretation of (7) as a Taylor rule (2). Since we observe inflation π_t and the state x_t, we can estimate the coefficient vector b connecting the two: $\pi_t = b^\top x_t$. Then the Taylor rule implies

$$
i_t \;\; = \;\; \tau\pi_t + s_t \;\; = \;\; \tau b^\top x_t + d^\top x_t.
$$

Equating our two interest rate relations gives us $\delta^\top = \tau b^\top + d^\top$. It's clear, now, that we have the same difficulty we had in the previous example: If we do not know the shock parameter d, we cannot infer τ from estimates of δ. If x_t has dimension n, we have n equations to solve for $n+1$ unknowns (d and τ).

If we interpret (7) as an asset pricing relation, then it's evident that we can't distinguish asset pricing (represented by δ) from monetary policy (represented by $\tau b + d$) without more information about the shock coefficients d. Generalizing the asset pricing relation from (2) to (7) has no effect on this conclusion.

2.3 Discussion

These examples illustrate the challenge we face in identifying the parameters of the Taylor rule, but they also suggest follow-up questions that might lead to a solution.

One such question is whether we can put shocks in other places and use them for identification. Gertler (private communication) suggests putting a shock in Cochrane's first equation, so that the example becomes

$$
\begin{aligned}
i_t &= E_t \pi_{t+1} + s_{1t}\\
i_t &= \tau\pi_t + s_{2t}.
\end{aligned}
$$

Can the additional shock identify the Taylor rule?

Suppose, as Gertler suggests, that s_1 and s_2 are independent. If s_1 is observed, we can use it as an instrument for π to estimate the Taylor rule equation, which gives us an estimate of τ. Given τ, we can then back out the shock s_2. We'll see in the next section that this example is misleading in one respect — we do not need a shock in the other equation — but there are two conclusions here of more general interest. One is that identification requires a restriction on the Taylor rule shock. Here the restriction is independence, but in later examples other restrictions serve the same purpose. The other is that identifying τ and backing out the unobserved shock are complementary activities. Generally if we can do one, we can do the other.

A second question is whether we can use long-term interest rates to help with identification. The answer is no if the idea is to use long rates to observe the state. In exponential-affine models, the state spans bond yields of all maturities. In many cases of interest, we can invert the mapping and express the state as a linear function of a subset of yields. In this sense, we can imagine using a vector of bond yields to observe the state. We have seen, though, that observing the state is not enough. We observe the state in both examples, yet cannot identify the Taylor rule. We explore the issue of state observability further in Section 5.

3 Macro-finance models with Taylor rules

Macro-finance models, which combine elements of macroeconomic and asset-pricing models, bring evidence from both macroeconomic and financial variables to bear on our understanding of monetary policy. It's not easy to reconcile the two, but if we do, we gain perspective that's missing from either approach on its own.

We show how Gertler's insight can be developed to identify the Taylor rule in such models. We use two examples, one based on a representative agent, the other on an exponential-affine model. We explore identification in these models when we observe the state, the short rate, and inflation, but not the shock to the Taylor rule. The identification issues are the same: we need one restriction on the shock to identify the (one) policy parameter.

3.1 A representative-agent model

One line of macro-finance research combines representative-agent asset pricing with a rule governing monetary policy. Gallmeyer, Hollifield, and Zin (2005) is a good example. We simplify their model, using power utility instead of recursive preferences and a simpler transition equation for the state.

The model consists of the bond-pricing relation, equation (6), plus

$$m_t = -\rho - \alpha g_t \tag{8}$$
$$g_t = g + s_{1t} \tag{9}$$
$$i_t = r + \tau \pi_t + s_{2t}. \tag{10}$$

Equations (6) and (10) mirror the two equations of Cochrane's example. The former is a more complex version of the Fisher equation — equation (1) — that represents the finance component of the model. The latter is a Taylor rule, representing monetary policy. Equations (8) and (9) characterize the real pricing kernel. The first is the logarithm of the marginal rate of substitution of a power utility agent with discount rate ρ, curvature parameter α, and log consumption growth g. The second connects fluctuations in log consumption growth to a shock s_{1t}. As in Section 2, the state x obeys the transition equation (3) and shocks are linear functions of it: $s_i = d_i^\top x$ for $i = 1, 2$. For simplicity, we choose r to reconcile the two interest rate equations, which makes mean inflation zero.

The solution now combines asset pricing with a forward-looking difference equation. We posit a solution of the form $\pi = b^\top x$. Solving (6) then gives us

$$i_t = \rho + \alpha g - V_m/2 + a^\top x_t, \tag{11}$$

with

$$\begin{aligned} a^\top &= (\alpha d_1^\top + b^\top)A \\ V_m &= a^\top B B^\top a. \end{aligned}$$

Note that the short rate equation (11) now has a shock, as Gertler suggests. Equating (10) and (11) gives us

$$(\rho + \alpha g - V_m/2) + (\alpha d_1^\top + b^\top)A x_t = r + (\tau b^\top + d_2^\top) x_t.$$

Lining up similar terms, we have $r = \rho + \alpha g - V_m/2$ and

$$(\alpha d_1^\top + b^\top)A = \tau b^\top + d_2^\top \quad \Rightarrow \quad b^\top = (\alpha d_1^\top A - d_2^\top)(\tau I - A)^{-1}.$$

As before, this gives us a unique stationary solution under the stated conditions: A stable and $\tau > 1$.

Now consider identification. Suppose we observe the state x, the interest rate i, the inflation rate π, and log consumption growth g, but not the shock s_2 to the Taylor rule. From observations of the state, we can estimate the autoregressive matrix A, and from observations of consumption growth we can estimate the shock coefficients d_1. We can also estimate a and b by projecting i and π on the state. With $a^\top = (\alpha d_1^\top + b^\top)A$ known, that leaves us to solve

$$a^\top = \tau b^\top + d_2^\top \tag{12}$$

for the Taylor rule's inflation parameter τ and shock coefficients d_2: n equations in the $n+1$ unknowns (τ, d_2). The identification problem is the same as in Cochrane's example: without further restrictions, the Taylor rule is not identified. This is Cochrane's conclusion in somewhat more general form.

We can, however, identify the monetary policy rule if we place one or more restrictions on its shock coefficients d_2. One such case was mentioned earlier: choose d_1 and d_2 so that the shocks s_1 and s_2 are independent. We'll return to this shortly. Another example is a zero in the vector d_2 — what is traditionally termed an exclusion restriction. Suppose the ith element of d_2 is zero. Then the ith element of (12) is

$$a_i \quad = \quad \tau b_i.$$

If $b_i \neq 0$ (a regularity condition we'll come back to), this determines τ. Given τ, and our estimates of a and b, we can now solve (12) for the remaining components of d_2. We can do the same thing with restrictions based on linear combinations. Suppose $d_2^\top e = 0$ for some known vector e. Then we find τ from $a^\top e = \tau b^\top e$. Any such linear restriction on the shock coefficient d_2 allows us to identify the Taylor rule.

Cochrane's example is a special case with shocks to consumption growth turned off: $d_1 = 0$. As a result, all the variation in inflation and the short rate comes from monetary policy shocks s_2. Special case or not, the conclusion is the same: we need one restriction on d_2 to identify the (one) Taylor rule parameter τ. Note well: The restriction applies to the Taylor rule shock, and does not require a shock in the other equation.

3.2 An exponential-affine model with a Taylor rule

We take a similar approach to an exponential-affine model, adding a Taylor rule to an otherwise standard bond-pricing model. The model consists of a real pricing kernel, a Taylor rule, and the transition (3) for the state. The first two are

$$\begin{aligned} m_{t+1} &= -\rho - s_{1t} + \lambda^\top w_{t+1} \\ i_t &= r + \tau \pi_t + s_{2t}. \end{aligned}$$

We refer to s_1 as the real interest rate shock and s_2 as the Taylor rule or monetary policy shock. As usual, the shocks are linear functions of the state: $s_i = d_i^\top x$. This model differs from the example in Section 2.2 in having a Taylor rule as well as a bond pricing relation. The question is what we need to tell them apart.

We solve the model by the usual method. Given a guess $\pi = b^\top x$ for inflation, the nominal pricing kernel is

$$m_{t+1}^{\$} \quad = \quad m_{t+1} - \pi_{t+1} \quad = \quad -\rho - (d_1^\top + b^\top A) x_t + (\lambda^\top - b^\top B) w_{t+1}.$$

The short rate follows from (7):

$$i_t \;=\; \rho - V_m/2 + (d_1^\top + b^\top A)^\top x_t,$$

where $V_m = (\lambda^\top - b^\top B)(\lambda - B^\top b)$. Equating this to the Taylor rule gives us $r = \rho - V_m/2$ and

$$d_1^\top + b^\top A \;=\; \tau b^\top + d_2^\top \quad\Rightarrow\quad b^\top \;=\; (d_1^\top - d_2^\top)(\tau I - A)^{-1}.$$

This is the unique stationary solution for b under the usual conditions.

Identification follows familiar logic. Let us say, again, that we observe the state x, the short rate i, and inflation π, which allows us to estimate A, a, and b. The interest rate expression $a^\top = d_1^\top + b^\top A$ therefore identifies d_1. The Taylor rule then implies $d_1^\top + b^\top A = \tau b^\top + d_2^\top$: n equations in the $n+1$ unknowns (τ, d_2). The model is identified only when we impose one or more restrictions on the coefficient vector d_2 of the monetary policy shock. If, for example, the ith element of d_2 is zero, then τ follows from $a_i = \tau b_i$ as long as $b_i \neq 0$.

This model is a generalization of the previous one in which we've given the real pricing kernel a more flexible structure. It's apparent, then, that the structure of the pricing kernel has little bearing on identification. We need instead more structure on the shock to the Taylor rule to compensate for not observing it.

3.3 Discussion

We have seen that we need one restriction on the shock coefficients to identify the Taylor rule in these examples. We gain some useful perspective into this result with the concept of *set identification*, which has been applied to far more complex environments by, among others, Chernozhukov, Hong, and Tamer (2007) and Manski (2008). We showed in Section 3.1 that a restriction of the form $d_2^\top e = 0$ suffices to (point) identify the Taylor rule parameter τ. In the absence of such a restriction, the linear combination can take on any real value: $d_2^\top e = \theta$ for any real θ. Equation (12) then implies

$$a^\top e \;=\; \tau b^\top e + d_2^\top e,$$

so that $\tau = (a^\top e - \theta)/(b^\top e)$, a function of the unknown θ. We might say that τ is set identified, with the set being the real line.

We can make the set smaller by limiting the range of θ. If we believe the shock is small, in the sense that $-\theta \leq d_2^\top e \leq \theta$ for some positive θ, we can restrict τ to an interval. As we drive θ to zero, the interval shrinks to a point. Other definitions of small can be used to generate other intervals.

Another source of perspective comes from comparison with identification in models of simultaneous equations. Many econometrics textbooks illustrate (point) identification with zero restrictions ("exclusions"). We typically need a variable in one equation that's missing (excluded) from the other. Consider a model of supply and demand. To identify the demand equation, we need a variable in the supply equation that's excluded from demand. That's not the case here. We can identify the Taylor rule even when there are no shocks in the other equation if we have a restriction on the Taylor rule shock. The issue is not whether we have the right configuration of shocks across equations, but whether we observe them. When we don't observe the shock to the Taylor rule, we need additional structure in the same equation to deduce its parameters.

The same logic applies to Gertler's example in Section 2.3, where we used independence of the two shocks to identify the Taylor rule. Doesn't that involve shocks in the second equation? Well, yes, but the critical feature of independence here is the restriction it places on the Taylor rule shock. The shocks are uncorrelated, hence independent, if $d_2^\top V_x d_1 = 0$. But that's a linear restriction $d_2^\top e = 0$ on the coefficient vector d_2 of the Taylor rule shock. In this case, $e = V_x d_1$. The same holds for restrictions on innovations to the shocks. They're independent and uncorrelated if $d_2^\top V_w d_1 = 0$. In the representative agent model of Section 3.1, such a restriction is easily implemented. If we observe consumption growth (9), then we also observe s_{1t} and can use it to estimate d_1 and compute the restriction on d_2, the coefficient vector of the Taylor rule shock. We give some illustrative numerical examples in Appendix B. These restrictions have no particular economic rationale in this case, but they illustrate how independence works. Similar "orthogonality conditions" for unobserved shocks appear throughout applied econometrics. In the New Keynesian literature, the shocks are typically low-order ARMA models, assumed to be independent of the rest of the model. Independence serves as a set of restrictions on the shocks that identify the model parameters, including the parameters of the monetary policy rule.

A similar question arises with restrictions on interest rate coefficients. Suppose we know that a linear combination of interest rate coefficients is zero: $a^\top e = 0$ for some known e. Then (12) gives us a restriction connecting the Taylor rule shock and its coefficient vector: $\tau b^\top e + d^\top e = 0$. One interpretation is that we've used a restriction from another part of the model for identification. We would say instead that any such restriction on interest rate behavior implies a restriction on the Taylor rule, namely $d^\top e = 0$, which identifies the policy rule for the usual reasons.

Another difference from traditional simultaneous equation methods is that single-equation estimation methods generally won't work. We need information about the whole model to deduce the Taylor rule. In the model of Section 3.1, for example, we need to estimate an interest rate equation to find a and an inflation equation to find b, before applying (12) to find τ. This reflects what Hansen and Sargent (1980, page 37) call the "hallmark" of rational expectations models: cross-equation restrictions connect the parameters in one equation to the parameters in the others.

4 A model with a Phillips curve

The next model has a stronger New Keynesian flavor. We add a Phillips curve to the representative agent model of Section 3.1 and an output gap to the Taylor rule. As a result, output growth g_t becomes endogenous. Models with similar features are described by Carrillo, Feve, and Matheron (2007), Canova and Sala (2009), Christiano, Eichenbaum, and Evans (2005), Clarida, Gali, and Gertler (1999), Cochrane (2011), Gali (2008), Iskrev (2010), King (2000), Shapiro (2008), Smets and Wouters (2007), Woodford (2003), and many others.

Despite the additional economic structure, the logic for identification is the same: we need restrictions on the shock coefficients to identify the Taylor rule. What changes is that we need two restrictions, one for each of the two parameters of the rule. We face similar issues in identifying the Phillips curve. If its shock isn't observed, we need restrictions on its coefficients to identify its parameters.

Our model consists of a pricing relation [equation (6)], a real pricing kernel [equation (8)], and

$$\begin{aligned} \pi_t &= \beta E_t \pi_{t+1} + \kappa g_t + s_{1t} \\ i_t &= r + \tau_1 \pi_t + \tau_2 g_t + s_{2t}. \end{aligned}$$

The first equation is a New Keynesian Phillips curve. The second is a Taylor rule, which now includes an output growth term. In addition, we have the transition equation (3) for the state and the shocks $s_i = d_i^\top x$ for $i = 1, 2$.

We now have a two-dimensional rational expectations model in the forward-looking variables π and g. The solution of such models is described in Appendix A. As others have noted, the conditions for a unique stationary solution are more stringent than before. We'll assume that they're satisfied.

We solve by guess and verify. We guess a solution that includes $\pi = b^\top x$ and $g = c^\top x$. Then the pricing relation gives us

$$i_t = \rho - V_m/2 + a^\top x_t$$

with $a^\top = (\alpha c^\top + b^\top) A$ and $V_m = a^\top B B^\top a$. If we equate this to the Taylor rule and collect terms, we have $r = \rho - V_m/2$ and

$$a^\top = \tau_1 b^\top + \tau_2 c^\top + d_2^\top. \tag{13}$$

Similarly, the Phillips curve implies

$$b^\top = \beta A b^\top + \kappa c^\top + d_1^\top. \tag{14}$$

We then solve equations (13) and (14) for the unknowns $(\tau_1, \tau_2, \beta, \kappa, d_1, d_2)$.

Suppose we, the economists, observe the state x, the interest rate i, the inflation rate π, and log consumption growth g, but not the shocks (s_1, s_2) to the Phillips curve and Taylor rule, respectively. From the observables, we can estimate the autoregressive matrix A and the coefficient vectors (a, b, c). In equation (13), representing the Taylor rule, the unknowns are the policy parameters (τ_1, τ_2) and the coefficient vector d_2 for the shock. If we do not observe the shock, we need two restrictions on its coefficient vector d_2 to identify (τ_1, τ_2). The logic is the same as before, but with two parameters to identify we need two restrictions on the vector of shock coefficients d_2.

The same logic applies to identifying the parameters of the Phillips curve. If we do not observe the shock s_{1t}, then two restrictions are needed to identify the parameters β and κ. The identification problem for the Phillips curve has the same structure as the Taylor rule, although in practice they've been treated separately. See the extensive discussions in Canova and Sala (2009), Gali and Gertler (1999), Iskrev (2010), Nason and Smith (2008), and Shapiro (2008).

Standard implementations of New Keynesian models typically use independent AR(1) or ARMA(1,1) shocks. See, for example, Gali (2008, ch 3) and Smets and Wouters (2007). In our framework, an independent AR(1) amounts to $n-1$ zero restrictions on the coefficient vectors d_i: none of the other state variables affect the shock. That's generally sufficient to identify the structural parameters of the model, including those of the Taylor rule. With respect to the Taylor rule, each element i for which $d_{2i} = 0$ leads, via equation (13), to an equation of the form $a_i = \tau_1 b_i + \tau_2 c_i$. As long as $(b_i, c_i) \neq (0, 0)$, any two such equations will identify the Taylor rule parameters (τ_1, τ_2). Similar logic applies to the Phillips curve.

5 Observing the state

Our approach so far is predicated on observing the state. But what happens if we observe the state indirectly? Or observe only a noisy signal of the state? These questions lead us to state-space models, in which we add to the transition equation for the state a so-called measurement or observation equation connecting an unseen state to a collection of observable variables. State-space models not only give us a way of estimating the state, they also give us a useful new perspective on the identification problem in forward-looking economies.

5.1 State-space models

The classic state-space framework consists of the transition equation (3) and a related measurement equation for observables,

$$y_t \quad = \quad C x_t + D v_t. \tag{15}$$

The measurement errors $v_t \sim \text{NID}(0, I)$ are independent of the w's.

A state-space model is a description of the distribution of observables y, but this distribution is invariant to linear transformations of the state x. Consider a model with state $\tilde{x} = Tx$, where T is an arbitrary square matrix of full rank. The transformed model is

$$\begin{aligned} \tilde{x}_{t+1} &= TAT^{-1}\tilde{x}_t + TBw_{t+1} &= \widetilde{A}\tilde{x}_t + \widetilde{B}w_{t+1} \\ y_t &= CT^{-1}\tilde{x}_t + Dw_t &= \widetilde{C}\tilde{x}_t + Dw_t, \end{aligned}$$

where $\widetilde{A} = TAT^{-1}$, $\widetilde{B} = TB$, and $\widetilde{C} = CT^{-1}$. The observational equivalence of models based on x and $\tilde{x}$ raises new identification issues that are not related to those we discussed earlier. These issues are generally managed by choosing a canonical form. See, for example, the extensive discussions in De Schutter (2000), Gevers and Wertz (1984), and Hinrichsen and Pratzel-Wolters (1989). Variants of this approach are used in dynamic factor models (Bai and Wang, 2012; Bernanke, Boivin, and Eliasz, 2005; Boivin and Giannoni, 2006; Stock and Watson, 2012) and exponential-affine term structure models (Joslin, Singleton, and Zhu, 2011). Given a canonical form — and the traditional controllability and observability conditions — we can generally estimate the matrices (A, B, C, D).

We give two examples of canonical forms in Appendix C for the case in which x, w, and y have the same dimension and B and C are nonsingular. In one, A has real Jordan form. In the other, $C = I$, so that the state and the observables are the same. Both feature lower triangular B. The two generate exactly the same distribution for y.

In some cases the measurements determine x exactly — for example, if C is square and nonsingular and $D = 0$ — but generally they do not. Does this affect our conclusions about identification? The answer is no. The Kalman filter is a recursive algorithm for computing the distribution of x from observations of y, and through them the distribution of y; see, among many others, Anderson and Moore (1979, Chapters 3-4), Boyd (2009, Lecture 8), and Hansen and Sargent (2013, Chapter 8). One of the intermediate outputs of the estimation process is a series of estimates (conditional means) of the state:

$$\hat{x}_{t|s} = E(x_t | y^s),$$

where $y^s = (y_s, y_{s-1}, ...)$ is a history of measurements. The Kalman filter produces, among other things, $\hat{x}_{t|t}$ and $\hat{x}_{t|t-1}$. They allow us to estimate the state-space parameters and explore their implications for identification exactly as before.

5.2 Structural interpretations of the measurement equation

The examples of Sections 2 to 4 fit neatly into state-space form. Since the state is exogenous, the economic structure shows up in the measurement equation (15).

Consider Cochrane's example from Section 2.1. If, as we've assumed throughout, we observe the interest rate i and inflation rate π, then two rows of the measurement equation are

$$\left[\begin{array}{c} i_t \\ \pi_t \end{array}\right] \;=\; \left[\begin{array}{c} a^\top \\ b^\top \end{array}\right] x_t \;=\; \left[\begin{array}{c} -d^\top(\tau I - A)^{-1}A \\ -d^\top(\tau I - A)^{-1} \end{array}\right] x_t. \tag{16}$$

The expressions on the right give us two rows of the matrix C. The identification question then takes this form: If we know C, represented here by a and b, can we back out values of the structural parameters (τ, d)?

More concretely, suppose we have estimates of A and C. Each row of C has n elements, one for each component of x. Identification consists of using these known values to determine values for the structural parameters (τ, d). Since we know A, the two rows of (16) contain essentially the same information. The estimate of the row corresponding to the inflation rate gives us values for the n elements of $-d^\top(\tau I - A)^{-1}$. Since we can estimate A separately, that leaves us $n+1$ structural parameters: the Taylor rule parameter τ and the monetary shock coefficients d. In the language we used earlier, these parameters are set-identified, with the set consisting of all the values of (τ, d) consistent with C. We can achieve point identification if we impose one or more restrictions on the shock coefficients.

Identification takes a particularly simple form when $C = I$. If $D = 0$ as well, the observables coincide with the state. The example then implies

$$\left[\begin{array}{c} a^\top \\ b^\top \end{array}\right] \;=\; \left[\begin{array}{c} e_1^\top \\ e_2^\top \end{array}\right] \;=\; \left[\begin{array}{c} -d^\top(\tau I - A)^{-1}A \\ -d^\top(\tau I - A)^{-1} \end{array}\right],$$

where e_i is a vector of zeros with one in the ith location. This restricts A to quasi-companion form: $e_2^\top A = e_1^\top$ (the interest rate is expected future inflation), so the first row of A is e_1. The first row then implies

$$e_2^\top(\tau I - A) \;=\; e_2^\top \tau - e_1^\top \;=\; d^\top.$$

Writing out the equations one by one, we have $-1 = d_1$, $\tau = d_2$, and $0 = d_j$ for $j \geq 3$. The form is different (the result of our choice of state), but the conclusion is the same: we need a restriction on d to (point) identify τ (namely, $d_2 = \tau$).

The other examples are similar. The exponential affine model in Section 3.2, for example, includes measurement equations (ignoring intercepts)

$$\left[\begin{array}{c} i_t \\ \pi_t \end{array}\right] \;=\; \left[\begin{array}{c} a^\top \\ b^\top \end{array}\right] x_t \;=\; \left[\begin{array}{c} d_1^\top + (d_1^\top - d_2^\top)(\tau I - A)^{-1}A \\ (d_1^\top - d_2^\top)(\tau I - A)^{-1} \end{array}\right] x_t.$$

The two equations connect estimated rows of C, labelled $a^\top$ and $b^\top$, to the structural parameters (d_1, d_2, τ). Here d_1 (the coefficient vector of the real interest rate shock s_1) is point-identified: $d_1 = a^\top - b^\top A$. The Taylor rule parameters (τ, d_2) are again set-identified, with point identification following from restrictions on d_2.

5.3 Term structures of measurements

Term structures of interest rates are a standard part of the collection of observables in bond-pricing models. Can they help us with identification? The answer is no, but let's work through it.

In models with an exponential-affine structure, including all of the models in this paper, forward rates are natural state variables. If q_t^h is the price at date t of a claim to one dollar at $t+h$, then continuously-compounded forward rates are defined by $f_t^h = \log(q_t^h/q_t^{h+1})$. The short rate is $i_t = f_t^0$. In our examples, the short rate takes the form (ignoring the intercept) $i_t = a^\top x_t$ and forward rates are $f_t^h = a^\top A^h x_t$. The vector f_t of the first n forward rates has the form

$$f_t \;=\; \begin{bmatrix} f_t^0 \\ f_t^1 \\ \vdots \\ f_t^{h-1} \end{bmatrix} \;=\; \begin{bmatrix} a^\top \\ a^\top A \\ \vdots \\ a^\top A^{n-1} \end{bmatrix} x_t \;=\; T x_t. \tag{17}$$

We can interpret this as a measurement equation with $y = f$, $C = T$, and $D = 0$. If $(A, a^\top)$ is observable (see Appendix C), then T is nonsingular and other measurement equations are redundant.

These measurements help us to estimate the state more precisely — that's how the measurement equation works. But they do not contribute anything new to the identification of the Taylor rule. We saw in Section 5.2 that rows of C may have structural interpretations that raise identification issues. The first row of (17) is a good example. We can estimate (identify) a, but we may not be able to point-identify the structural parameters on which it is based. The other rows, add nothing more. They include the same estimate multiplied by a power of the autoregressive matrix A, which is separately identified. Evidently, then, the term structure of interest rates is not a solution to the problem of identifying the Taylor rule.

Forecasts have a similar mathematical structure — and have similar consequences. Consider the forecast of variable z_t at a horizon of h periods. If a variable $z = a^\top x$ for some arbitrary coefficient vector a, then a forecast of future z might be denoted $F_t^h = E_t z_{t+h}$. The transition equation then implies $F_t^h = A^h a^\top x_t$. A collection of forecasts can be used the same way we used forward rates. Or we could add forecasts to our collection of observables. Chernov and Mueller (2012), Chun (2011), and Kim and Orphanides (2012) are examples that use survey forecasts in state-space frameworks. The forecasts add useful information in all of these applications, but they do not resolve the identification problem.

5.4 Transformations and restrictions

We've seen that linear transformations have no detectable impact on a state-space model, but they change the form of any restrictions we might place on the shocks. Consider the representative agent model in Section 3.1. The short rate is related to a transformed state $\tilde{x} = Tx$ by $i = r + a^\top x = r + \tilde{a}^\top \tilde{x}$ with $\tilde{a}^\top = a^\top T^{-1}$. Similarly, inflation is $\pi = b^\top x = b^\top T^{-1}\tilde{x} = \tilde{b}^\top \tilde{x}$ and the shocks become $s_i = d_i^\top x = d_i^\top T^{-1}\tilde{x} = \tilde{d}_i^\top \tilde{x}$. In this form, the Taylor rule implies

$$\tilde{a}^\top \quad = \quad \tau \tilde{b}^\top + \tilde{d}_2^\top,$$

the analog of equation (12) for the transformed state. The identification problem is the same as before: we need one restriction on $\tilde{d}_2$ to identify the single Taylor rule parameter τ.

The structure of the identification problem is the same, but suppose we observe the transformed state $\tilde{x}$ and not the original state x or the matrix T. Are the restrictions on d_2 intelligible when we translate them to $\tilde{d}_2$? Consider a general linear restriction of the form $d_2^\top e = 0$, where at least one element of e is non-zero. This restriction can be expressed $\tilde{d}_2^\top Te = 0$, so the restricting vector is $\tilde{e} = Te$ in the new coordinate system. If we don't know T, can we deduce $\tilde{e}$? There are at least two cases where the restrictions translate cleanly to the transformed state. These cases have nearly opposite economic interpretations, so they suggest a range of choices that can lead to identification.

In the first case, suppose the Taylor rule shock is uncorrelated with the other shock. Such "orthogonality conditions" are standard in the New Keynesian literature, where most shocks are assumed to be independent of the others. We gave an example in Section 3.3. In terms of the original state x, the restriction takes the form $d_2^\top V_x d_1 = 0$. In terms of the transformed state $\tilde{x}$, we have

$$\tilde{d}_2^\top E(\tilde{x}\tilde{x}^\top)\tilde{d}_1 \quad = \quad (d_2^\top T^{-1})(TV_xT^\top)(d_1^\top T^{-1})^\top \quad = \quad d_2^\top V_x d_1.$$

It's clear that this restriction is invariant to linear transformations of the state. We give a numerical example in Appendix B.

In the second case, optimal monetary policy dictates a connection between the two shocks. See, for example, Gali (2008, Section 3.4) and Woodford (2003). When a monetary authority minimizes an objective function, all variables of interest are affected by s_1. As a result, an optimal policy rule will make s_2 proportional to s_1. We express this by setting $s_2 = ks_1$ for some constant k, which implies the restriction $d_2^\top - kd_1^\top = 0$ in terms of the original state variable x. In terms of the transformed state $\tilde{x}$, the restriction is

$$\tilde{d}_2^\top - k\tilde{d}_1^\top \quad = \quad d_2^\top T^{-1} - kd_1^\top T^{-1} \quad = \quad (d_2^\top - kd_1^\top)T^{-1} \quad = \quad 0,$$

which is independent of the transformation T.

5.5 Vector autoregressions

There's an influential body of research in which vector autoregressions (VARs) are used to characterize the effects of monetary policy. Typically restrictions are imposed to identify policy innovations. See, for example, the many studies cited by Christiano, Eichenbaum, and Evans (1999, Sections 3 and 4) and Watson (1994, Section 4). Such models fit nicely into a state-space framework, which makes it easy to compare their approach to ours.

A generic VAR(q) might be expressed

$$A_0 y_t \;=\; A_1 y_{t-1} + A_2 y_{t-2} + \cdots + A_q y_{t-q} + u_t, \tag{18}$$

where y and u are vectors of the same dimension, $u_t \sim \mathcal{N}(0, \Sigma)$, and A_0 and Σ are nonsingular. Most VAR work starts with what is conventionally referred to recursive identification: A_0 is lower triangular and $\Sigma = I$. See Appendix D. Equation (18) is typically estimated in the form

$$y_t \;=\; (A_0)^{-1} A_1 y_{t-1} + (A_0)^{-1} A_2 y_{t-2} + \cdots + (A_0)^{-1} A_q y_{t-q} + (A_0)^{-1} u_t.$$

The matrix A_0 is implicit in $\mathrm{Var}[A_0^{-1} u_t] = A_0^{-1} (A_0^{-1})^\top$. We can compute a lower triangular $(A_0)^{-1}$ from its Choleski decomposition.

This approach, and other variations on the same theme, delivers a dynamic model in which each variable is associated with a specific disturbance. We can then use the rest of the model to compute impulse responses for each of them. As Watson (1994, page 2898) puts it: "[The] model provides answers to the 'impulse' and 'propagation' questions often asked by macroeconomists." It supports statements of the form: a contractionary shock to monetary policy is followed by a persistent increase in the federal funds rate, a U-shaped decrease in GDP, and a persistent decrease in commodity prices (adapted from Christiano, Eichenbaum, and Evans, 1999, page 87).

For comparison, consider a VAR interpretation of a state-space model. If x, y, and w all have dimension n, B and C are nonsingular, and $D = 0$, the state-space model can be expressed as a VAR(1). (We can do the same with higher-order VARs, but the notation is more cumbersome.) We choose a canonical form with B lower triangular and $C = I$; see Appendix C. The measurement equation then implies $y = x$ and the transition equation becomes

$$B^{-1} y_t \;=\; B^{-1} A y_{t-1} + w_t, \tag{19}$$

an example of equation (18) with $q = 1$, $A_0 = B^{-1}$, $A_1 = B^{-1} A$, and $u_t = w_t$. Since B is lower triangular, so is B^{-1}, and the model satisfies the conditions for recursive identification. We're missing, however, the connection between C and structural parameters that we described in Section 5.2, as well as the restrictions on this structure we used to identify the Taylor rule.

Christiano, Eichenbaum, and Evans (1999, Section 6) make a similar point: "Why did we not display or interpret the [relevant equation of a VAR as a monetary policy rule]? The answer is that these parameters are not easily interpretable." Why, you might ask? They continue: "In [two of our] examples the decision maker reacts to a variable that is not in the econometrician's data set. The policy parameters are a convolution of the parameters of the rule ... and the projection of the missing data onto the econometrician's data set." That's the essence of our problem: "the missing data" (the shock) is not in "the econometrician's data set." In short, recursive identification is not a solution to this particular problem.

6 Conclusion

Identification is always an issue in applied economic work, perhaps nowhere more so than in the study of monetary policy. That's still true. We have shown, however, that (i) the problem of identifying the systematic component of monetary policy (the Taylor rule parameters) in New Keynesian and macro-finance models stems from our inability to observe the nonsystematic component (the shock to the rule) and (ii) the solution is to impose restrictions on the shock. We are left where we often are in matters of identification: trying to decide which restrictions are plausible, and which are not.

A Solutions of forward-looking models

Consider the class of forward-looking linear rational expectations models,

$$\begin{aligned} z_t &= \Lambda E_t z_{t+1} + D x_t \\ x_{t+1} &= A x_t + B w_{t+1}. \end{aligned}$$

Here x_t is the state, Λ is stable (eigenvalues less than one in absolute value), A is also stable, and $w_t \sim \text{NID}(0, I)$. The goal is to solve the model and link z_t to the state x_t.

One-dimensional case. If z_t is a scalar and the shock is $s_t = d^\top x_t$, we have

$$z_t = \lambda E_t z_{t+1} + d^\top x_t. \tag{20}$$

Repeated substitution gives us

$$z_t = \sum_{j=0}^{\infty} \lambda^j d^\top E_t x_{t+j} = d^\top \sum_{j=0}^{\infty} \lambda^j A^j x_t = d^\top (I - \lambda A)^{-1} x_t.$$

The last step follows from the matrix geometric series if A is stable and $|\lambda| < 1$. Under these conditions, this is the unique stationary solution.

The same solution follows from the method of undetermined coefficients, but the rationale for stability is less obvious. We guess $z_t = h^\top x_t$ for some vector h. The difference equation tells us

$$h^\top x_t = h^\top \lambda A x_t + d^\top x_t.$$

Collecting coefficients of x_t gives us $h^\top = d^\top (I - \lambda A)^{-1}$.

This model is close enough to the examples of Sections 2 and 3 that we can illustrate their identification issues in a more abstract setting. Suppose we observe the state x_t and the endogenous variable z_t, but not the shock s_t. Then we can estimate A and h. Equation (20) then gives us

$$h^\top = \lambda b^\top A + d^\top.$$

If x has dimension n, we have n equations in the $n + 1$ unknowns (λ, d); we need one restriction on d to identify the parameter λ.

Multi-dimensional case. If z_t is a vector, as in Section 4, repeated substitution gives us

$$z_t = \sum_{j=0}^{\infty} \Lambda^j D A^j x_t.$$

That gives us the solution $z_t = Hx_t$ where

$$H \;=\; \sum_{j=0}^{\infty} \Lambda^j D A^j \;=\; D + \Lambda H A$$

or

$$\mathrm{vec}(H) \;=\; (I - A^\top \otimes \Lambda)^{-1}\mathrm{vec}(D).$$

See, for example, Anderson, Hansen, McGrattan, and Sargent (1996, Section 6) or Klein (2000, Appendix B). The same sources also explain how to solve rational expectations models with endogenous state variables.

B Numerical examples

We illustrate some of the issues raised in the paper with numerical examples of the model in Section 3.1: a representative agent with power utility and given consumption growth. We show how identification works when we observe the state and when we observe only a linear transformation of the state. In each case we use an orthogonality restriction on the shock to the Taylor rule.

We give the model a two-dimensional state and and choose parameter values $\tau = 1.5$, $\alpha = 5$, and

$$A \;=\; \begin{bmatrix} 0 & 1 \\ -0.05 & 0.9 \end{bmatrix}, \qquad B \;=\; \begin{bmatrix} 0.0078 & 0 \\ -0.0004 & 0.0003 \end{bmatrix}.$$

The consumption growth shock is governed by $d_1^\top = (1,0)$. The monetary policy shock is $d_2^\top = (\delta, 1)$, with δ chosen to make s_2 uncorrelated with s_1. These inputs imply

$$V_x \;=\; \begin{bmatrix} 0.6432 & -0.0069 \\ -0.0069 & 0.0258 \end{bmatrix} \cdot 10^{-4},$$

so $d_2^\top V_x d_1 = 0$ implies $\delta = 0.0108$.

These are the inputs, the source of data that we can use to estimate the Taylor rule. The question is whether we can do that under different assumptions about observability of the state.

State observed. Suppose, first, that we observe the state x_t. Then we can use observations of the interest rate i_t and inflation rate π_t to recover the coefficient vectors

$$a \;=\; \begin{bmatrix} -0.3152 \\ 10.4566 \end{bmatrix}, \qquad b \;=\; \begin{bmatrix} -0.2174 \\ 6.3044 \end{bmatrix}.$$

Similarly, observations of consumption growth allow us to recover d_1. We do not observe the Taylor rule shock s_{2t}, so its coefficient vector d_2 remains unknown. A least squares estimate of the Taylor rule here gives us $\tau = 1.6510$ which, of course, isn't the value that generates the data.

Identification requires a restriction on d_2. We know d_2 satisfies the orthogonality condition $d_2^\top e = 0$ with $e = V_x d_1$. With our numbers, $e^\top = (0.6364, -0.0069) \cdot 10^{-4}$. We post-multiply (12) by e to get $a^\top e = \tau b^\top e$. This implies $\tau = 1.5$, the value we started with. We can now recover d_2 from the same equation: $d_2^\top = a^\top - \tau b^\top = (0.0108, 1.0000)$.

This is simply the procedure we outlined in Section 3.1, but it gives us a concrete basis of comparison for situations in which we don't directly observe the state.

State observed indirectly. Now suppose we don't observe the state, but we observe enough variables to deduce a linear transformation of the state. We consider two examples.

In our first example, we observe the interest rate and inflation rate and use them as our transformed state: $\tilde{x} = (i, \pi)^\top$. Then $\tilde{x} = Tx$ with

$$T = \begin{bmatrix} a^\top \\ b^\top \end{bmatrix}.$$

This has something of the flavor of a vector autoregression, albeit a simple one. The transformed transition matrix is

$$\widetilde{A} = \begin{bmatrix} -4.6170 & 9.1006 \\ -2.8044 & 5.5170 \end{bmatrix},$$

which is easily estimated.

Now consider identification. If we regress i, π, and log consumption growth g on $\tilde{x}$, we get the coefficient vectors

$$\tilde{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \tilde{d}_1 = \begin{bmatrix} 22.07 \\ -36.60 \end{bmatrix}.$$

From observations of $\tilde{x}$, we can estimate its covariance matrix

$$V_{\tilde{x}} = \begin{bmatrix} 0.2932 & 0.1775 \\ 0.1775 & 0.1075 \end{bmatrix} \cdot 10^{-3}.$$

Finally, the orthogonality condition in these coordinates is $\tilde{d}_2^\top \tilde{e} = 0$ with $\tilde{e} = V_{\tilde{x}} \tilde{d}_1$. With our numbers, we have $\tilde{e}^\top = (-0.2718, -0.1812) \cdot 10^{-4}$. As before, we apply the orthogonality condition to equation (12), which gives us $\tau = \tilde{a}^\top \tilde{e} / \tilde{b}^\top \tilde{e} = 1.5$, the number we started with. It is clear from this that we are still able to recover the Taylor rule from this linear transformation of the state.

In our second example, we use the first two forward rates as the state: $\tilde{x}_t = (f_t^0 = i_t, f_t^1)^\top$. As we saw in Section 5.3, forward rates are connected to the original state x by $\tilde{x} = Tx$ with

$$T \;=\; \begin{bmatrix} a^\top \\ a^\top A \end{bmatrix}.$$

The same series of calculations gives us $\tau = 1.5$ in this case, too.

C State-space fundamentals

We outline some of the concepts of state-space modeling. Hansen and Sargent (2013) is a standard reference for economists. Anderson and Moore (1979) and Boyd (2009) are readable technical references.

The starting point is the state-space system (3,15). The state x has dimension n, the measurement y has dimension p. The matrices (A, B, C, D) are conformable.

We say (A, B) is *controllable* if

$$\mathcal{C} \;=\; \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

has rank n. The word controllable is misleading in this context; some say *reachable* instead. The idea is simply that w_t generates variation across all n dimensions of x. We say (A, C) is *observable* if

$$\mathcal{O} \;=\; \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has rank n. The idea here is that observing the history of y is enough to generate a full-rank estimate of x.

Controllability example. Here's one with x of dimension two and w of dimension one that fails:

$$A \;=\; \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}, \quad B \;=\; \begin{bmatrix} b_1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \mathcal{C} \;=\; \begin{bmatrix} b_1 & a_{11}b_1 \\ 0 & 0 \end{bmatrix},$$

which has rank $1 < n = 2$. Here the innovation w never generates variation in x_2, so we don't span the whole two-dimensional state. However, if a_{21} is nonzero we get controllability, because w affects x_2 with a one-period lag through its impact on x_1. A similar example is an AR(2) in companion form.

Observability example. The logic is similar. Suppose x is n-dimensional and the nth column of C consists of zeros. There's no direct impact of the nth state variable on the observations y. In the bond-pricing literature, this might be a case in which one of the state variables doesn't appear in bond yields of any maturity. Nevertheless, the nth state variable might be (indirectly) observable if it feeds into other state variables: if a_{jn} is nonzero for some $j \neq n$. Here's an example similar to our previous one:

$$A \;=\; \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix}, \;\; C \;=\; \begin{bmatrix} c_1 & 0 \end{bmatrix} \;\;\Rightarrow\;\; \mathcal{O} \;=\; \begin{bmatrix} c_1 & 0 \\ a_{11}c_1 & 0 \end{bmatrix}.$$

Since $a_{12} = 0$, matrix has rank one and the condition fails.

Canonical forms. We consider canonical forms for a state-space model in which x, w, and y all have dimension n and B and C are nonsingular conformable matrices. Since B and C have rank n, the model is controllable and observable. This is a (very) special case of the general state-space model, but illustrates how we might use canonical forms eliminate redundant parameters. The model as stated has $3n^2$ parameters, n^2 for each matrix.

These two canonical forms generate the same distribution of y:

- Joslin, Singleton, and Zhu (2011) suggest Jordan form for the transition matrix. In our version, A has real Jordan form (loosely speaking, diagonal), B is lower triangular, and C is unrestricted. The structure of B has no observational consequences: the transition equation has symmetric conditional variance matrix $V_w = BB^\top$, whose Choleski decomposition is lower triangular with the same number of distinct elements: $n(n+1)/2$. Since B is nonsingular, its diagonal elements are nonzero, and we can normalize x by setting them equal to one. Together this reduces the number of parameters in (A, B, C) to $n + n(n-1)/2 + n^2 = n(n+1)/2 + n^2$.
- An alternative is to set $C = I$, which defines x as y. This puts all the restrictions in C and leaves us with $n^2 + n(n+1)/2$ parameters, the same as the previous example.

D Recursive identification strategies

We review recursive approaches to identification in two common models.

Simultaneous equations. We follow Rothenberg's (1971, Section 6) classic presentation. The structure is

$$By_t + \Gamma x_t \;=\; u_t,$$

where endogenous variables y_t and disturbances u_t have dimension g, exogenous variables x_t have dimension k, $u_t \sim \mathcal{N}(0, \Sigma)$, Σ is symmetric, and both B and Σ are nonsingular. The reduced form is

$$y_t \;=\; \Pi x_t + v_t,$$

where $\Pi = -B^{-1}\Gamma$ $v_t = B^{-1}u_t \sim \mathcal{N}(0, \Omega)$, and $\Omega = B^{-1}\Sigma(B^{-1})^\top$.

The reduced form can obviously be estimated: it's identified. The identification question is whether we can recover the structural parameters (B, Γ, Σ) from the reduced form parameters (Π, Ω) and a collection of restrictions $\psi(B, \Gamma, \Sigma)$; that is, whether the conditions

$$\begin{aligned} B\Pi + \Gamma &= 0 \\ B\Omega B^\top - \Sigma &= 0 \\ \psi(B, \Gamma, \Sigma) &= 0 \end{aligned}$$

uniquely determine (B, Γ, Σ).

Rothenberg gives conditions for local point identification of the structural parameters. The example of interest is a recursive scheme with lower triangular B (elements above the diagonal are zero), diagonal Σ (off-diagonal elements are zero), and normalizations (traditionally the diagonal elements of B equal one). None of these restrictions involve Γ. Since Σ is diagonal, this scheme associates each disturbance with a specific equation and endogenous variable.

Vector autoregressions. The VAR (18) is a system of simultaneous equations. As above, a recursive scheme establishes identification: lower triangular A_0, diagonal Σ, and normalizations (here $\Sigma = I$). None of these restrictions involve $(A_1, A_2, \ldots, A_q)$. Again, each disturbance is associated with a specific equation and variable. Typically we estimate this in the form

$$y_t = (A_0)^{-1}A_1 y_{t-1} + (A_0)^{-1}A_2 y_{t-2} + \cdots + (A_0)^{-1}A_q y_{t-q} + (A_0)^{-1}u_t.$$

The matrix A_0 is implicit in $\mathrm{Var}[(A_0)^{-1}u_t] = \Omega$. Given estimates of the parameters, we can compute impulse responses to each of the disturbances. The VAR literature starts here, but goes on to consider a variety of other restrictions that serve the same purpose.

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