# Fractional Graph Theory

*A Rational Approach to the Theory of Graphs*

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*To Rachel, Sandy, Danny, Jessica, Naomi, Jonah, and Bobby*

# Contents

# Foreword

Graph theory is one of the branches of modern mathematics having experienced a most impressive development in recent years. In the beginning, Graph Theory was only a collection of recreational or challenging problems like Euler tours or the four coloring of a map, with no clear connection among them, or among techniques used to attach them. The aim was to get a "yes" or "no" answer to simple existence questions. Under the impulse of Game Theory, Management Sciences, and Transportation Network Theory, the main concern shifted to the maximum size of entities attached to a graph. For instance, instead of establishing the existence of a 1-factor, as did Petersen and also König (whose famous theorem on bipartite graphs was discovered 20 years earlier by Steinitz in his dissertation in Breslau), the main problem was now to study the maximum number of edges in a matching, even if not a 1-factor or "perfect matching". In this book, Scheinerman and Ullman present the next step of this evolution: Fractional Graph Theory. Fractional matchings, for instance, belong to this new facet of an old subject, a facet full of elegant results.

By developing the fractional idea, the purpose of the authors is multiple: first, to enlarge the scope of applications in Scheduling, in Operations Research, or in various kinds of assignment problems; second, to simplify. The fractional version of a theorem is frequently easier to prove than the classical one, and a bound for a "fractional" coefficient of the graph is also a bound for the classical coefficient (or suggests a conjecture). A striking example is the simple and famous theorem of Vizing about the edge-chromatic number of a graph; no similar result is known for the edge-chromatic number of a hypergraph, but the reader will find in this book an analogous statement for the "fractional" edge-chromatic number which is a theorem. The conjecture of Vizing and Behzad about the total chromatic number becomes in its fractional version an elegant theorem.

This book will draw the attention of the combinatorialists to a wealth of new problems and conjectures. The presentation is made accessible to students, who could find a clear exposition of the background material and a stimulating collection of exercises. And, above all, the pleasure afforded by these pages will contribute to making Combinatorics more appealing to many.

Claude Berge
Paris, France

# Preface

Graphs model a wide variety of natural phenomena, and it is the study of these phenomena that gives rise to many of the questions asked by pure graph theorists. For example, one motivation for the study of the chromatic number in graph theory is the well-known connection to scheduling problems. Suppose that an assortment of committees needs to be scheduled, each for a total of one hour. Certain pairs of committees, because they have a member in common, cannot meet at the same time. What is the length of the shortest time interval in which all the committees can be scheduled?

Let $G$ be the graph whose vertices are these committees, with an edge between two committees if they cannot meet at the same time. The standard answer to the scheduling question is that the length of the shortest time interval is the chromatic number of $G$. As an illustration, suppose that there are 5 committees, with scheduling conflicts given by the graph in Figure A. Since $G$ can be

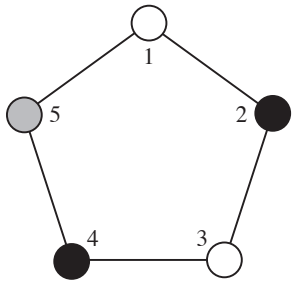


Figure A: The graph $C_5$, colored with three colors.

colored with 3 colors, the scheduling can be done in 3 hours, as is illustrated in Figure B.

It is a widely held misconception that, since the chromatic number of $G$ is 3, the schedule in Figure B cannot be improved. In fact, the 5 committees can be scheduled in two-and-a-half hours, as is illustrated in Figure C.

All that is required is a willingness to allow one committee to meet for half an hour, to interrupt their meeting for a time, and later to reconvene for the remaining half hour. All that is required is a willingness to break one whole hour into *fractions* of an hour—to break a discrete unit into *fractional* parts. The minimum length of time needed to schedule committees when interruptions are permitted is not the chromatic number of $G$ but the less well-known *fractional chromatic number* of $G$.

This example illustrates the theme of this book, which is to uncover the rational side of graph theory: How can integer-valued graph theory concepts be modified so they take on nonintegral values? This "fractionalization" bug has infected other parts of mathematics. Perhaps the best-known example is the fractionalization of the factorial function to give the gamma function. Fractal geometry recognizes objects whose dimension is not a whole number [126]. And analysts consider fractional derivatives [132]. Some even think about fractional partial derivatives!

We are not the first to coin the term *fractional graph theory*; indeed, this is not the first book

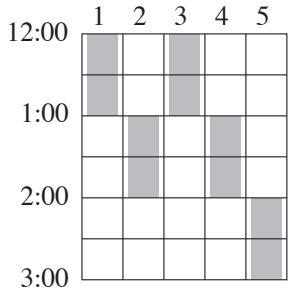


Figure B: A schedule for the five committees.

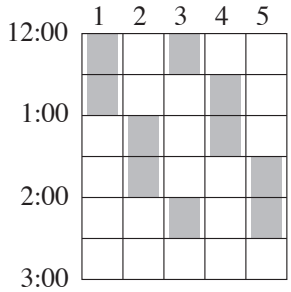


Figure C: An improved schedule for the five committees.

on this subject. In the course of writing this book we found that Claude Berge wrote a short monograph [16] on this very subject. Berge's *Fractional Graph Theory* is based on his lectures delivered at the Indian Statistical Institute twenty years ago. Berge includes a treatment of the fractional matching number and the fractional edge chromatic number.[1] Two decades have seen a great deal of development in the field of fractional graph theory and the time is ripe for a new overview.

## Rationalization

We have two principal methods to convert graph concepts from integer to fractional. The first is to formulate the concepts as integer programs and then to consider the linear programming relaxation (see §A.3). The second is to make use of the subadditivity lemma (Lemma A.4.1 on page 137). It is very pleasing that these two approaches typically yield the same results. Nearly every integer-valued invariant encountered in a first course in graph theory gives rise to a fractional analogue.

Most of the fractional definitions in this book can be obtained from their integer-valued counterparts by replacing the notion of a set with the more generous notion of a "fuzzy" set [190]. While membership in a set is governed by a $\{0,1\}$-valued indicator function, membership in a fuzzy set is governed by a $[0,1]$-valued indicator function. It is possible to devise fractional analogues to nearly

[1] More recently, Berge devotes a chapter of his monograph *Hypergraphs: Combinatorics of Finite Sets* [19] to fractional transversals of hypergraphs, which includes an exploration of fractional matchings of graphs.

any integer-valued graph-theoretic concept by phrasing the integer-valued definition in the right way and then inserting the word "fuzzy" in front of the word "set" in the appropriate place. Although this lends a unifying spirit to the book, we choose to avoid this language in favor of spelling out the definitions of fractional invariants directly in terms of $[0,1]$-valued labelings or functions. Nonetheless, the word "fractional" is meant to suggest the continuous unit interval $[0,1]$ in place of the "discrete unit interval" $\{0,1\}$. Further, "fractional" underscores the fact that the resulting real numbers are almost always *rational* values in $[0,1]$.

## Goals

The focus of this book is not on the theory of mathematical programming, although this theory is used and certain polytopes of interest to the programming theorist do make appearances here. Neither is the focus on algorithms and complexity, although these issues are discussed in places. Rather, the main goal is to prove theorems that are analogous to the main theorems of basic graph theory. For example, one might ask if there is a fractional analogue of the four-color theorem. One might hope to prove a fractional three-and-a-half-color result saying that the fractional chromatic number of any planar graph is no more than $7/2$ (in fact, this is obviously false) or one might hope to prove a fractional four-and-a-half-color result via an argument that does not rely on the four-color theorem itself (no such proof is known). The focus is on comparing and contrasting a fractional graph invariant to its integer-valued cousin and on discerning to what degree basic properties and theorems of the integer-valued invariant are retained by the fractional analogue.

Our goal has been to collect in one place the results about fractional graph invariants that one can find scattered throughout the literature and to give a unified treatment of these results. We have highlighted the theorems that seem to us to be attractive, without trying to be encyclopedic. There are open questions and unexplored areas here to attract the active researcher. At the same time, this book could be used as a text for a topics course in graph theory. Exercises are included in every chapter, and the text is meant to be approachable for graduate students in graph theory or combinatorial optimization.

## Chapter Overview

In Chapter 1 we develop a general fractional theory of hypergraphs to which we regularly refer in the subsequent chapters. In Chapters 2, 3, and 4 we study the fractional analogues of the matching number, the chromatic number, and the edge chromatic number. In Chapter 5 we study fractional arboricity via matroids. In Chapter 6 we discuss an equivalence relation on graphs that weakens isomorphism and is called fractional isomorphism. In Chapter 7 we touch on a number of other fractional notions. Finally, the Appendix contains background material on notation, graphs, hypergraphs, linear programming, and the subadditivity lemma.

Each chapter features exercises and a Notes section that provides references for and further information on the material presented.

## Acknowledgments

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And finally, thanks to our wives, Amy and Rachel, who endure our obsessions with patience and good humor.

## Feedback

We appreciate your comments and reactions. Feel free to reach us by e-mail at `ers@jhu.edu` (Scheinerman) or `dullman@math.gwu.edu` (Ullman).

## Finally

It is possible to go to a graph theory conference and to ask oneself, at the end of every talk, What is the fractional analogue? What is the right definition? Does the fractional version of the theorem still hold? If so, is there an easier proof and can a stronger conclusion be obtained? If the theorem fails, can one get a proof in the fractional world assuming a stronger hypothesis? We can personally attest that this can be an entertaining pastime. If, after reading this book, you too catch the fractional bug and begin to ask these questions at conferences, then we will consider ourselves a success. Enjoy!

—ES, Baltimore
—DU, Washington, D.C.

# Exercise

1. Let $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = x$. The half derivative of $f$ is $2\sqrt{x/\pi}$. Why?

# 1

# General Theory: Hypergraphs

Our purpose is to reveal the rational side of graph theory: We seek to convert integer-based definitions and invariants into their fractional analogues. When we do this, we want to be sure that we have formulated the "right" definitions—conversions from integer to real that are, in some sense, natural. Here are two ways we might judge if an integer-to-real conversion process is natural: First, when two seemingly disparate conversions yield the same concept, then we feel confident that the fractionalized concept is important (and not tied to *how* we made the conversion). Second, when the same conversion process works for a variety of concepts (e.g., we convert from matching number and chromatic number to their fractional analogues by the same methods), then we feel we have arrived at a reasonable way to do the integer-to-real transformation. Happily, we can often satisfy both of these tests for "naturalness". If a variety of attractive theorems arise from the new definition, then we can be certain that we are on the right track.

This chapter develops two general methods for fractionalizing graph invariants: linear relaxation of integer programs and applications of the subadditivity lemma.[1] We present both methods and prove that they yield the same results.

## 1.1 Hypergraph covering and packing

A hypergraph $\mathcal{H}$ is a pair $(S, \mathcal{X})$, where $S$ is a finite set and $\mathcal{X}$ is a family of subsets of $S$. The set $S$ is called the *ground set* or the *vertex set* of the hypergraph, and so we sometimes write $V(\mathcal{H})$ for $S$. The elements of $\mathcal{X}$ are called *hyperedges* or sometimes just *edges*.

A *covering* (alternatively, an *edge covering*) of $\mathcal{H}$ is a collection of hyperedges $X_1, X_2, \ldots, X_j$ so that $S \subseteq X_1 \cup \cdots \cup X_j$. The least $j$ for which this is possible (the smallest size of a covering) is called the *covering number* (or the *edge covering number*) of $\mathcal{H}$ and is denoted $k(\mathcal{H})$.

An element $s \in S$ is called *exposed* if it is in no hyperedges. If $\mathcal{H}$ has an exposed vertex, then no covering of $\mathcal{H}$ exists and $k(\mathcal{H}) = \infty$. Hypergraphs are sometimes known as *set systems*, although in that case one often forbids exposed vertices.

The covering problem can be formulated as an integer program (IP). To each set $X_i \in \mathcal{X}$ associate a 0,1-variable $x_i$. The vector $\mathbf{x}$ is an indicator of the sets we have selected for the cover. Let $M$ be the *vertex-hyperedge incidence matrix* of $\mathcal{H}$, i.e., the 0,1-matrix whose rows are indexed by $S$, whose columns are indexed by $\mathcal{X}$, and whose $i,j$-entry is 1 exactly when $s_i \in X_j$. The condition that the indicator vector $\mathbf{x}$ corresponds to a covering is simply $M\mathbf{x} \geq 1$ (that is, every coordinate of $M\mathbf{x}$ is at least 1). Thus $k(\mathcal{H})$ is the value of the integer program "minimize $\mathbf{1}^t\mathbf{x}$ subject to $M\mathbf{x} \geq 1$" where $\mathbf{1}$ represents a vector of all ones. Further, the variables in this and subsequent linear programs are tacitly assumed to be nonnegative.

[1] These central ideas (linear programming and subadditivity), as well as other basic material, are discussed in Appendix A. The reader is encouraged to peruse that material before proceeding.

The covering problem has a natural dual. Let $\mathcal{H} = (S, \mathcal{X})$ be a hypergraph as before. A *packing* (a *vertex packing*) in $\mathcal{H}$ is a subset $Y \subseteq S$ with the property that no two elements of $Y$ are together in the same member of $\mathcal{X}$. The *packing number* $p(\mathcal{H})$ is defined to be the largest size of a packing. The packing number of a *graph* is its independence number $\alpha$.

Note that $\emptyset$ is a packing, so $p(\mathcal{H})$ is well defined.

There is a corresponding IP formulation. Let $y_i$ be a 0,1-indicator variable that is 1 just when $s_i \in Y$. The condition that $Y$ is a packing is simply $M^t\mathbf{y} \leq 1$ where $M$ is, as above, the vertex-hyperedge incidence matrix of $\mathcal{H}$. Thus $p(\mathcal{H})$ is the value of the integer program "maximize $\mathbf{1}^t\mathbf{y}$ subject to $M^t\mathbf{y} \leq 1$." This is the dual IP to the covering problem and we have therefore proved the following result.

**Proposition 1.1.1** *For a hypergraph $\mathcal{H}$, we have $p(\mathcal{H}) \leq k(\mathcal{H})$.* □

Many graph theory concepts can be seen as hypergraph covering or packing problems. For instance, the chromatic number $\chi(G)$ is simply $k(\mathcal{H})$ where the hypergraph $\mathcal{H} = (S, \mathcal{X})$ has $S = V(G)$ and $\mathcal{X}$ is the set of all independent subsets of $V(G)$. Similarly, the matching number $\mu(G)$ is $p(\mathcal{H})$ where $\mathcal{H} = (S, \mathcal{X})$ has $S = E(G)$ and $\mathcal{X}$ contains those sets of edges of $G$ incident on a fixed vertex.

## 1.2 Fractional covering and packing

There are two ways for us to define the *fractional* covering and packing numbers of a hypergraph $\mathcal{H}$. The first is straightforward. Since $k(\mathcal{H})$ and $p(\mathcal{H})$ are values of integer programs, we define the *fractional covering number* and *fractional packing number* of $\mathcal{H}$, denoted $k^*(\mathcal{H})$ and $p^*(\mathcal{H})$ respectively, to be the values of the dual *linear* programs "minimize $\mathbf{1}^t\mathbf{x}$ s.t. $M\mathbf{x} \geq 1$" and "maximize $\mathbf{1}^t\mathbf{y}$ s.t. $M^t\mathbf{y} \leq 1$". By duality (Theorem A.3.1 on page 135) we have $k^*(\mathcal{H}) = p^*(\mathcal{H})$.

(Note that if $\mathcal{H}$ has an exposed vertex then the covering LP is infeasible and the packing LP is unbounded; let us adopt the natural convention that, in this case, we set both $k^*$ and $p^*$ equal to $\infty$.)

We will not use the $^*$ notation much more. There is a second way to define the fractional covering and packing numbers of $\mathcal{H}$ that is not *a priori* the same as $k^*$ and $p^*$; the $^*$ notation is temporary until we show they are the same as $k_f$ and $p_f$ defined below.

We begin by defining *$t$-fold coverings* and the *$t$-fold covering number* of $\mathcal{H}$. Let $\mathcal{H} = (S, \mathcal{X})$ be a hypergraph and let $t$ be a positive integer. A *$t$-fold covering* of $\mathcal{H}$ is a multiset $\{X_1, X_2, \ldots, X_j\}$ (where each $X_i \in \mathcal{X}$) with the property that each $s \in S$ is in at least $t$ of the $X_i$'s. The smallest cardinality (least $j$) of such a multiset is called the *$t$-fold covering number* of $\mathcal{H}$ and is denoted $k_t(\mathcal{H})$. Observe that $k_1(\mathcal{H}) = k(\mathcal{H})$.

Notice that $k_t$ is subadditive in its subscript, that is,

$$k_{s+t}(\mathcal{H}) \leq k_s(\mathcal{H}) + k_t(\mathcal{H})$$

since we can form a (perhaps not smallest) $(s+t)$-fold covering of $\mathcal{H}$ by combining a smallest $s$-fold covering and a smallest $t$-fold covering. By the lemma of Fekete (Lemma A.4.1 on page 137) we may therefore define the *fractional covering number* of $\mathcal{H}$ to be

$$k_f(\mathcal{H}) = \lim_{t\to\infty} \frac{k_t(\mathcal{H})}{t} = \inf_t \frac{k_t(\mathcal{H})}{t}.$$

Yes, we have called both $k^*$ and $k_f$ the *fractional covering numbers*—the punch line, of course, is that these two definitions yield the same result. However, it is clear that $k^*(\mathcal{H})$ must be a rational

number since it is the value of a linear program with integer coefficients; it is not clear that $k_f$ is necessarily rational—all we know so far is that the limit $k_t(\mathcal{H})/t$ exists. We also know that $k_t(\mathcal{H}) \leq tk_1(\mathcal{H}) = tk(\mathcal{H})$ and therefore $k_f(\mathcal{H}) \leq k(\mathcal{H})$.

In a similar way, we define a *t-fold packing* of $\mathcal{H} = (S, \mathcal{X})$ to be a multiset $Y$ (defined over the vertex set $S$) with the property that for every $X \in \mathcal{X}$ we have

$$\sum_{s \in X} m(s) \leq t$$

where $m$ is the multiplicity of $s \in S$ in $Y$. The *t-fold packing number* of $\mathcal{H}$, denoted $p_t(\mathcal{H})$, is the smallest cardinality of a $t$-fold packing. Observe that $p_1(\mathcal{H}) = p(\mathcal{H})$.

Notice that $p_t$ is superadditive in its subscript, i.e.,

$$p_{s+t}(\mathcal{H}) \geq p_s(\mathcal{H}) + p_t(\mathcal{H}).$$

This holds because we can form a (not necessarily largest) $(s+t)$-fold packing by combining optimal $s$- and $t$-fold packings. Using an analogue of the subadditivity lemma (exercise 1 on page 138), we define the *fractional packing number* of $\mathcal{H}$ to be

$$p_f(\mathcal{H}) = \lim_{t \to \infty} \frac{p_t(\mathcal{H})}{t}.$$

Note that since $p_t(\mathcal{H}) \geq tp_1(\mathcal{H}) = tp(\mathcal{H})$ we have that $p_f(\mathcal{H}) \geq p(\mathcal{H})$.

**Theorem 1.2.1** *The two notions of fractional covering and the two notions of fractional packing are all the same, i.e., for any hypergraph $\mathcal{H}$ we have*

$$k^*(\mathcal{H}) = k_f(\mathcal{H}) = p^*(\mathcal{H}) = p_f(\mathcal{H}).$$

**Proof**. If $\mathcal{H}$ has an exposed vertex, then all four invariants are infinite. So we restrict to the case that $\mathcal{H}$ has no exposed vertices.

We know that $k_f = p_f$ by LP duality. We show the equality $k^* = k_f$; the proof that $p^* = p_f$ is similar and is relegated to exercise 1 on page 12.

First we prove that $k^* \leq k_f$. Let $a$ be a positive integer. Let $\{X_1, \ldots, X_j\}$ be a smallest $a$-fold covering of $\mathcal{H}$ and let $x_i$ be the number of times $X_i$ appears in this covering. Then $M\mathbf{x} \geq a$, and therefore $M(\mathbf{x}/a) \geq 1$. Thus $\mathbf{x}/a$ is a feasible vector for the covering problem, hence $k^*(\mathcal{H}) \leq \mathbf{1}^t(\mathbf{x}/a) = k_a(\mathcal{H})/a$. Since this holds for all positive integers $a$, we have $k^*(\mathcal{H}) \leq k_f(\mathcal{H})$.

Next we prove that $k^* \geq k_f$. Let $\mathbf{x}$ be a vector that yields the value of the LP "minimize $\mathbf{1}^t \cdot \mathbf{x}$ s.t. $M\mathbf{x} \geq 1$". We may assume that $\mathbf{x}$ is rational (since the data in the LP are all integers). Let $n$ be the least common multiple of all the denominators appearing in the vector $\mathbf{x}$; thus the entries in $n\mathbf{x}$ are all integers. Further, we have $Mn\mathbf{x} \geq n$. We can form an $n$-fold cover of $\mathcal{H}$ by choosing hyperedge $X_i$ with multiplicity $nx_i$. The size of this $n$-fold cover is $\sum_i nx_i = \mathbf{1}^t \cdot n\mathbf{x}$. Thus, $nk^*(\mathcal{H}) = \mathbf{1}^t \cdot n\mathbf{x} \geq k_n(\mathcal{H})$. Furthermore, for any positive integer $a$, we also have $(an)k^*(\mathcal{H}) \geq k_{an}(\mathcal{H})$. Since $k^*(\mathcal{H}) \geq k_{an}(\mathcal{H})/(an)$ for all $a$, we have $k^*(\mathcal{H}) \geq k_f(\mathcal{H})$. □

## 1.3 Some consequences

Since the fractional covering/packing number of $\mathcal{H}$ is the value of a linear program with integer coefficients, we have the following.

**Corollary 1.3.1** *If $\mathcal{H}$ has no exposed vertices, then $k_f(\mathcal{H})$ is a rational number.* □

Not only is $k_f(\mathcal{H})$ rational, but we can choose the optimal weights $x_i$ of the $X_i$ to be rational numbers as well. Let $N$ be a common multiple of the denominators of the $x_i$'s and consider $k_N(\mathcal{H})$. One checks that we can form an $N$-fold covering of $\mathcal{H}$ if we choose $X_i$ with multiplicity $Nx_i$; moreover, this is best possible since $(\sum Nx_i)/N = \sum x_i = k_f(\mathcal{H})$.

**Corollary 1.3.2** *If $\mathcal{H}$ has no exposed vertices, then there exists a positive integer $s$ for which $k_f(\mathcal{H}) = k_s(\mathcal{H})/s$.* □

Thus we know not only that $\lim k_t(\mathcal{H})/t = \inf k_t(\mathcal{H})/t$ (Lemma A.4.1 on page 137) but also that both equal $\min k_t(\mathcal{H})/t$ and this minimum is achieved for infinitely many values of $t$. We use this fact next to obtain a strengthening of the subadditivity of $k_s(\mathcal{H})$.

**Proposition 1.3.3** *If $\mathcal{H}$ is any hypergraph, then there exist positive integers $s$ and $N$ such that, for all $t \geq N$, $k_s(\mathcal{H}) + k_t(\mathcal{H}) = k_{s+t}(\mathcal{H})$.*

**Proof**. Let $s$ be as in the previous corollary so that $k_f(\mathcal{H}) = k_s(\mathcal{H})/s$. Then, for any $n \geq 1$,

$$k_f(\mathcal{H}) \leq k_{ns}(\mathcal{H})/(ns) \leq k_s(\mathcal{H})/s = k_f(\mathcal{H}),$$

where the second inequality follows from the subadditivity of $k_s$. Hence $k_{ns}(\mathcal{H}) = nk_s(\mathcal{H})$. Now fix $j$ with $0 \leq j < s$ and for any positive integer $n$ let $a_n = k_{ns+j}(\mathcal{H}) - k_{ns}(\mathcal{H})$. Clearly $a_n$ is a nonnegative integer. Also,

$$\begin{aligned} a_{n+1} &= k_{(n+1)s+j}(\mathcal{H}) - k_{(n+1)s}(\mathcal{H}) \\ &= k_{(n+1)s+j}(\mathcal{H}) - (n+1)k_s(\mathcal{H}) \\ &\leq k_{ns+j}(\mathcal{H}) + k_s(\mathcal{H}) - (n+1)k_s(\mathcal{H}) \\ &= k_{ns+j}(\mathcal{H}) - nk_s(\mathcal{H}) \\ &= a_n, \end{aligned}$$

where again the inequality follows from subadditivity. A nonincreasing sequence of positive integers is eventually constant; we thus have an integer $M_j$ such that, if $m, n \geq M_j$, then $a_m = a_n$.

Let $M = \max\{M_0, M_1, \ldots, M_{s-1}\}$. Then, whenever $m, n \geq M$, we have

$$k_{ms+j}(\mathcal{H}) - k_{ms}(\mathcal{H}) = k_{ns+j}(\mathcal{H}) - k_{ns}(\mathcal{H})$$

for all $j$ with $1 \leq j < s$. In particular, with $m = n + 1$, we have

$$k_{(n+1)s+j}(\mathcal{H}) = k_{ns+j}(\mathcal{H}) + k_s(\mathcal{H})$$

for $1 \leq j < s$. Now let $N = Ms$. If $t \geq N$, then put $n = \lfloor t/s \rfloor$ and put $j = t - sn$. Then $t = sn + j$ with $n \geq M$ and $0 \leq j < s$, so

$$k_{s+t}(\mathcal{H}) = k_{(n+1)s+j}(\mathcal{H}) = k_{ns+j}(\mathcal{H}) + k_s(\mathcal{H}) = k_t(\mathcal{H}) + k_s(\mathcal{H}).$$

This completes the proof. □

In fact, the proof shows that any subadditive sequence $a_1, a_2, \ldots$ of nonnegative integers for which $\inf\{a_n/n\}$ equals $a_s/s$ satisfies the same conclusion: $a_{s+t} = a_s + a_t$ for sufficiently large $t$. This implies that the sequence of differences $d_n = a_{n+1} - a_n$ is eventually periodic, since $d_{t+s} = a_{t+s+1} - a_{t+s} = a_{t+1} + a_s - a_t - a_s = d_t$ for sufficiently large $t$.

In the presence of sufficient symmetry, there is a simple formula for $k_f(\mathcal{H})$. An *automorphism* of a hypergraph $\mathcal{H}$ is a bijection $\pi : V(\mathcal{H}) \to V(\mathcal{H})$ with the property that $X$ is a hyperedge of $\mathcal{H}$ iff $\pi(X)$ is a hyperedge as well. The set of all automorphisms of a hypergraph forms a group under the operation of composition; this group is called the *automorphism group* of the hypergraph. A hypergraph $\mathcal{H}$ is called *vertex-transitive* provided for every pair of vertices $u, v \in V(\mathcal{H})$ there exists an automorphism of $\mathcal{H}$ with $\pi(u) = v$.

**Proposition 1.3.4** *Let $\mathcal{H} = (S, \mathcal{X})$ be a vertex-transitive hypergraph and let $e = \max\{|X| : X \in \mathcal{X}\}$. Then $k_f(\mathcal{H}) = |S|/e$.*

**Proof**. If we assign each vertex of $\mathcal{H}$ weight $1/e$, then we have a feasible fractional packing. Thus $k_f(\mathcal{H}) = p_f(\mathcal{H}) \geq |S|/e$. Note that no larger uniform weighting of the vertices is feasible.

Let $\Gamma$ denote the automorphism group of $\mathcal{H}$. Let $w\colon S \to [0,1]$ be an optimal fractional packing of $\mathcal{H}$ and let $\pi \in \Gamma$. Then $w \circ \pi$ also is an optimal fractional packing of $\mathcal{H}$. Since any convex combination of optimal fractional packings is also an optimal fractional packing,

$$w^*(v) := \frac{1}{|\Gamma|} \sum_{\pi \in \Gamma} w[\pi(v)]$$

is an optimal fractional packing which, because $\mathcal{H}$ is vertex-transitive, assigns the same weight to every vertex of $\mathcal{H}$. Thus $w^*(v) = 1/e$ and the result follows. □

## 1.4 A game-theoretic approach

Consider the following game, which we call the *hypergraph incidence game*, played on a hypergraph $\mathcal{H} = (S, \mathcal{X})$. There are two players both of whom know $\mathcal{H}$. In private, the *vertex player* chooses an element $s$ of $S$. Also in private, the *hyperedge player* chooses an $X$ in $\mathcal{X}$. The players then reveal their choices. If $s \in X$ then the vertex player pays the hyperedge player \$1; otherwise there is no payoff. Assuming both players play optimally, what is the expected payoff to the hyperedge player? This is a classic, zero-sum matrix game whose matrix is $M$, the vertex-hyperedge incidence matrix. The optimal strategy for the players is typically a mixed strategy wherein the vertex player chooses an element $s_i \in S$ with probability $y_i$ and the hyperedge player chooses $X_j \in \mathcal{X}$ with probability $x_j$. The *value* of such a game is defined to be the expected payoff assuming both players adopt their optimal strategies.

**Theorem 1.4.1** *For a hypergraph $\mathcal{H}$ the value of the hypergraph incidence game played on $\mathcal{H}$ is $1/k_f(\mathcal{H})$.*

**Proof**. First note that if there is an element $s \in S$ that is in no $X \in \mathcal{X}$ (an exposed vertex) then by playing $s$ the vertex player can always avoid paying the hyperedge player; the value is $0 = 1/\infty$ as claimed. We therefore restrict ourselves to hypergraphs with no exposed vertices.

From game theory, we know that the value of this game $v^*$ is the largest number $v$ so that $M\mathbf{x} \geq v$ for any probability vector $\mathbf{x}$. Likewise, $v^*$ is the smallest number $v$ so that $M^t\mathbf{y} \leq v$ for all probability vectors $\mathbf{y}$. Our claim is that $v^* = 1/k_f(\mathcal{H}) = 1/p_f(\mathcal{H})$.

Let $v^*$ be the value of the game and let probability vector $\mathbf{x}^*$ be an optimal mixed strategy for the hyperedge player. We therefore have that $M\mathbf{x}^* \geq v^*$. Letting $\mathbf{x} = \mathbf{x}^*/v^*$, we note that $M\mathbf{x} \geq 1$ so $\mathbf{x}$ is a feasible covering and therefore $k_f(\mathcal{H}) \leq \mathbf{1}^t\mathbf{x} = \mathbf{1}^t\mathbf{x}^*/v^* = 1/v^*$ (since $\mathbf{x}^*$ is a probability vector).

Now let probability vector $\mathbf{y}^*$ be an optimal mixed strategy for the vertex player. Then $M^t\mathbf{y}^* \leq v^*$ and therefore $\mathbf{y} = \mathbf{y}^*/v^*$ is a feasible packing. We therefore have $p_f(\mathcal{H}) \geq \mathbf{1}^t\mathbf{y} = \mathbf{1}^t\mathbf{y}^*/v^* = 1/v^*$. Since $k_f = p_f$, the result follows. □

## 1.5 Duality and duality

The matching and covering problems are dual in the sense of linear/integer programming. If $\mathcal{H} = (S, \mathcal{X})$ is a hypergraph and $M$ is its vertex-hyperedge incidence matrix, then the problems of computing $k(\mathcal{H})$ and $p(\mathcal{H})$ are dual integer programs:

$$k(\mathcal{H}) = \min \mathbf{1}^t\mathbf{x} \text{ s.t. } M\mathbf{x} \geq 1, \text{ and}$$
$$p(\mathcal{H}) = \max \mathbf{1}^t\mathbf{y} \text{ s.t. } M^t\mathbf{y} \leq 1.$$

The problems of computing $k_f(\mathcal{H})$ and $p_f(\mathcal{H})$ are the dual linear programs obtained from these IPs by relaxing the condition that the variables be integers.

For example, let $G$ be a graph and let $\mathcal{H} = (V, \mathcal{X})$ where $V = V(G)$ and $\mathcal{X}$ is the set of all independent subsets of $V(G)$. Now $k(\mathcal{H})$ is the minimum number of independent sets that contain all vertices, i.e., $\chi(G)$. Similarly, $p(\mathcal{H})$ is the maximum number of vertices, no two of which are together in an independent set, i.e., the maximum number $\omega(G)$ of pairwise adjacent vertices. This is the sense in which the graph problems of computing the chromatic number and the size of the largest clique are dual.

Let us introduce a second notion of duality: *hypergraph duality*. Let $\mathcal{H} = (S, \mathcal{X})$ be a hypergraph and write

$$S = \{s_1, \ldots, s_n\} \qquad \text{and} \qquad \mathcal{X} = \{X_1, \ldots, X_m\}.$$

Then $M$ (the vertex-hyperedge incidence matrix) is $n \times m$ with $M_{ij} = 1$ if and only if $s_i \in X_j$. We define the *dual* hypergraph of $\mathcal{H}$, denoted $\mathcal{H}^*$, to be the hypergraph with vertex set $X$ and hyperedge set $\mathcal{S}$ with

$$X = \{x_1, \ldots, x_m\} \qquad \text{and} \qquad \mathcal{S} = \{S_1, \ldots, S_n\}$$

and we put $x_j \in S_i$ exactly when $s_i \in X_j$ (in $\mathcal{H}$). In matrix language, the vertex-hyperedge incidence matrices of $\mathcal{H}$ and $\mathcal{H}^*$ are transposes of one another.

We now consider the covering and packing problems in $\mathcal{H}^*$. A covering of $\mathcal{H}^*$ is the smallest number of $S_i$'s that include all the $x_j$'s. From the perspective of $\mathcal{H}$, we seek the minimum number of elements $s_i$ that intersect all the $X_j$'s. A subset of $S$ that is incident with all hyperedges is called a *transversal* of $\mathcal{H}$ and the smallest size of a transversal of $\mathcal{H}$ is called the *transversal number* of $\mathcal{H}$ and is denoted $\tau(\mathcal{H})$. Thus,

$$\tau(\mathcal{H}) = k(\mathcal{H}^*).$$

Next, consider $p(\mathcal{H}^*)$. The packing number of the dual of $\mathcal{H}$ is the largest number of $x_i$'s with no two in a common $S_j$. In terms of $\mathcal{H}$, we seek the maximum number of hyperedges $X_i$ that are

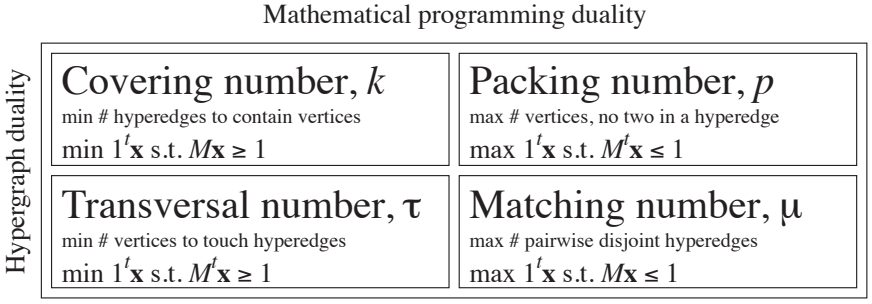


Figure 1.1. The dualities between the covering, packing, transversal, and matching numbers of a hypergraph.

pairwise disjoint. A *matching* in $\mathcal{H}$ is a collection of hyperedges that are pairwise disjoint. The *matching number* of $\mathcal{H}$, denoted $\mu(\mathcal{H})$, is the maximum size of a matching. Thus,

$$\mu(\mathcal{H}) = p(\mathcal{H}^*).$$

The relations between the four invariants are summarized in Figure 1.1.

Revisiting our example from before, if $G = (V, E)$ is a graph and $\mathcal{H}$ is the hypergraph whose vertex set $S$ is $V(G)$ and whose hyperedges are the independent subsets of $V$, then the transversal number $\tau(\mathcal{H})$ is the size of the smallest set of vertices that intersects every independent subset of $V$. Since any singleton vertex is independent, this is just $\nu(G) = |V(G)|$. The matching number is the maximum number of pairwise disjoint independent sets. Again, since singletons are independent, we have $\mu(\mathcal{H}) = \nu(G)$.

A less trivial example is afforded by taking $\mathcal{H} = G$. In this case $\tau(\mathcal{H})$ is the smallest size of a subset of vertices that intersects every edge.[2] The matching number $\mu(G)$ is the maximum number of pairwise disjoint edges, i.e., the maximum size of a matching in $G$.

Finally, we can consider the pair of *fractional* invariants $\tau_f$ and $\mu_f$ of a hypergraph: the *fractional transversal number* and the *fractional matching number*. They are simply

$$\tau_f(\mathcal{H}) = k_f(\mathcal{H}^*) \qquad \text{and} \qquad \mu_f(\mathcal{H}) = p_f(\mathcal{H}^*)$$

and therefore we have, by either sort of duality, $\tau_f = \mu_f$.

## 1.6 Asymptotic covering and packing

(This section may be safely skipped during a first reading of this book. This material is used again only in §3.5.)

The fractional covering number is one way to assign to a hypergraph a real number closely related to the covering number. Here is another way: Define the product $\mathcal{H} \times \mathcal{K}$ of two hypergraphs $\mathcal{H} = (S, \mathcal{X})$ and $\mathcal{K} = (T, \mathcal{Y})$ to be the hypergraph whose vertex set is $S \times T$ and whose hyperedges are all sets of the form $X \times Y$ where $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. See Figure 1.2 on the next page. Write $\mathcal{H}^n$

[2]Some authors call $\tau(G)$ the *covering* number of $G$. We believe it is more natural to call this the *transversal number* of $G$ and (for us) the covering number $k(G)$ is the minimum number of edges that contain all the vertices. These two notions are different. Consider the example $G = K_{1,n}$. In this case, $\tau(G) = 1$ (the non-leaf intersects all edges) but $k(G) = n$ (we must use all edges to touch all the leaves).

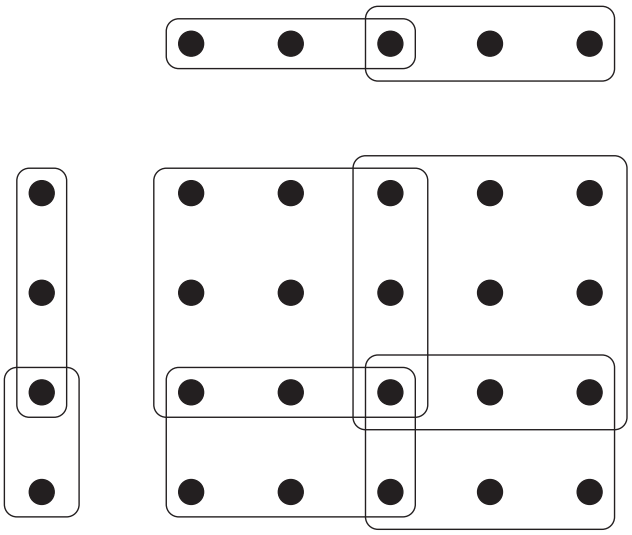

Figure 1.2. An example of hypergraph product. The 20-vertex hypergraph is the product of the 4-vertex hypergraph (left) and the 5-vertex hypergraph (top). The vertex set of the product is the Cartesian product of the vertex sets of the factors. Hyperedges in the product are the Cartesian products of hyperedges of the factors.

to be the $n$th power of $\mathcal{H}$ with respect to this product. We define the *asymptotic covering number* to be

$$k_\infty(\mathcal{H}) = \inf_n \sqrt[n]{k(\mathcal{H}^n)}.$$

If $\mathcal{A} \subseteq \mathcal{X}$ and $\mathcal{B} \subseteq \mathcal{Y}$ are coverings of $\mathcal{H}$ and $\mathcal{K}$ respectively, then clearly the sets of the forms $A \times B$, where $A \in \mathcal{A}$ and $B \in \mathcal{B}$, form a covering of $\mathcal{H} \times \mathcal{K}$. Hence $k(\mathcal{H} \times \mathcal{K}) \leq k(\mathcal{H})k(\mathcal{K})$. Thus if we define $g(n)$ to be $\log k(\mathcal{H}^n)$, we see that $g(m+n) \leq g(m) + g(n)$. An appeal to Lemma A.4.1 on page 137 then tells us that the infimum in the definition of $k_\infty$ is also a limit.

The dual invariant to the asymptotic covering number is naturally the *asymptotic packing number*, denoted $p_\infty(\mathcal{H})$ and defined as $\sup_n \sqrt[n]{p(\mathcal{H}^n)}$. If $A \subseteq S$ is a packing of $\mathcal{H}$ and $B \subseteq T$ is a packing of $\mathcal{K}$, then $A \times B$ is a packing of $\mathcal{H} \times \mathcal{K}$. Using exercise 1 on page 138 and the argument in the preceding paragraph, one sees that the supremum in this definition is also a limit.

We begin with the following result, which shows that the fractional covering number respects hypergraph products.

**Theorem 1.6.1** *If $\mathcal{H}$ and $\mathcal{K}$ are any hypergraphs, then*

$$k_f(\mathcal{H} \times \mathcal{K}) = k_f(\mathcal{H})k_f(\mathcal{K}) = p_f(\mathcal{H})p_f(\mathcal{K}) = p_f(\mathcal{H} \times \mathcal{K}).$$

**Proof**. As above, write $\mathcal{H} = (S, \mathcal{X})$ and $\mathcal{K} = (T, \mathcal{Y})$. If $\mathcal{A}$ is a smallest $s$-fold covering of $\mathcal{H}$ and $\mathcal{B}$ is a smallest $t$-fold covering of $\mathcal{K}$, then the family of sets of the form $A \times B$ where $A \in \mathcal{A}$ and $B \in \mathcal{B}$ is an $st$-fold covering of $\mathcal{H} \times \mathcal{K}$. Thus

$$\frac{k_{st}(\mathcal{H} \times \mathcal{K})}{st} \leq \frac{k_s(\mathcal{H})}{s} \cdot \frac{k_t(\mathcal{K})}{t}$$

and it follows that $k_f(\mathcal{H} \times \mathcal{K}) \leq k_f(\mathcal{H})k_f(\mathcal{K})$.

Similarly, if $A \subseteq S$ is a largest $s$-fold packing of $\mathcal{H}$ and $B \subseteq T$ is a largest $t$-fold packing of $\mathcal{K}$, then the set $A \times B$ is an $st$-fold packing of $\mathcal{H} \times \mathcal{K}$. It follows that $p_f(\mathcal{H} \times \mathcal{K}) \geq p_f(\mathcal{H})p_f(\mathcal{K})$. Three applications of the duality theorem (via Theorem 1.2.1 on page 3) yield

$$k_f(\mathcal{H} \times \mathcal{K}) \leq k_f(\mathcal{H})k_f(\mathcal{K}) = p_f(\mathcal{H})p_f(\mathcal{K}) \leq p_f(\mathcal{H} \times \mathcal{K}) = k_f(\mathcal{H} \times \mathcal{K}),$$

and we are done. □

There is a curious lack of symmetry between the two asymptotic invariants. It may come as a surprise that the asymptotic covering number of a hypergraph is always equal to the fractional covering number of the hypergraph. Perhaps a greater surprise is that the same is *not* true of the asymptotic packing number. This is discussed in §3.5.

**Theorem 1.6.2** *If $\mathcal{H}$ is any hypergraph, then $k_\infty(\mathcal{H}) = k_f(\mathcal{H})$.*

We provide two proofs: the first uses a random covering, and the second uses a greedy covering. We begin with a lemma.

**Lemma 1.6.3** *If $\mathcal{H}$ and $\mathcal{K}$ are any hypergraphs, then $k(\mathcal{H} \times \mathcal{K}) \geq k_f(\mathcal{H})k(\mathcal{K})$.*

**Proof**. Suppose that $\mathcal{H} = (S, \mathcal{X})$ and $\mathcal{K} = (T, \mathcal{Y})$. It is enough to show that there is a $k(\mathcal{K})$-fold covering of $\mathcal{H}$ of cardinality $k(\mathcal{H} \times \mathcal{K})$. Let $n = k(\mathcal{H} \times \mathcal{K})$. Consider a covering

$$\{X_1 \times Y_1, X_2 \times Y_2, \ldots, X_n \times Y_n\}$$

of $\mathcal{H} \times \mathcal{K}$. We claim that the multiset $\{X_1, X_2, \ldots, X_n\}$ is the desired covering. Clearly it has the right cardinality. To see that it is a $k(\mathcal{K})$-fold covering, we must show that every $s \in S$ is in $X_i$ for at least $k(\mathcal{K})$ of the indices $i$.

Fix $s \in S$. Let $I \subseteq [n]$ be the set of indices $i$ such that $s \in X_i$. We want to show that $|I| \geq k(\mathcal{K})$. Consider the multiset $M = \{Y_i\}_{i \in I}$. For every $t \in T$, we have $(s, t) \in X_i \times Y_i$ for some $i \in I$. Hence $t \in Y_i$. This shows that $M$ is a covering of $\mathcal{K}$ and so $I$ has cardinality at least $k(\mathcal{K})$. □

**Proof** (of Theorem 1.6.2). Suppose that $\mathcal{H} = (S, \mathcal{X})$. We first show that $k_f(\mathcal{H}) \leq k_\infty(\mathcal{H})$. Substituting $\mathcal{H}^n$ for $\mathcal{K}$ in Lemma 1.6.3 yields $k_f(\mathcal{H}) \leq k(\mathcal{H}^{n+1})/k(\mathcal{H}^n)$. Since this is true for all $n$ (including $n = 0$, if one understands $\mathcal{H}^0$ to be the hypergraph with one vertex and one nonempty hyperedge),

$$\begin{aligned}
k_f(\mathcal{H}) &\leq \inf_{n \geq 0} \frac{k(\mathcal{H}^{n+1})}{k(\mathcal{H}^n)} \\
&\leq \lim_{n \to \infty} \sqrt[n]{\frac{k(\mathcal{H}^1)}{k(\mathcal{H}^0)} \cdot \frac{k(\mathcal{H}^2)}{k(\mathcal{H}^1)} \cdots \frac{k(\mathcal{H}^n)}{k(\mathcal{H}^{n-1})}} \\
&= \lim_{n \to \infty} \sqrt[n]{\frac{k(\mathcal{H}^n)}{k(\mathcal{H}^0)}} \\
&= k_\infty(\mathcal{H}).
\end{aligned}$$

For the opposite inequality, suppose $k_f(\mathcal{H}) = k_t(\mathcal{H})/t$ and suppose $K$ is a $t$-fold covering of $\mathcal{H}$. Fix $\epsilon > 0$. Pick a multiset $M$ of hyperedges of $\mathcal{H}^n$ of cardinality

$$L = \left\lceil \left( \frac{k_t(\mathcal{H})}{t}(1 + \epsilon) \right)^n \right\rceil$$

at random, each choice made independently and uniformly among the hyperedges of the form

$$E = X_1 \times X_2 \times \cdots \times X_n$$

where each $X_i \in K$. We show that for large enough $n$ the probability that $M = \{E_1, E_2, \ldots, E_L\}$ is a covering of $\mathcal{H}^n$ is positive, and, in fact, arbitrarily close to unity.

Let $s = (s_1, s_2, \ldots, s_n)$ be an arbitrary element of the vertex set of $\mathcal{H}^n$. A random hyperedge $E = X_1 \times X_2 \times \cdots \times X_n$ contains $s$ if and only if $s_i \in X_i$ for all $1 \leq i \leq n$. For each $i$, this happens with probability at least $t/k_t(\mathcal{H}) = 1/k_f(\mathcal{H})$. Hence $s \in E$ with probability at least $(1/k_f(\mathcal{H}))^n$. Hence the probability that $s$ is not in any of the hyperedges in $M$ is less than

$$\begin{aligned}\left[1 - \left(\frac{1}{k_f(\mathcal{H})}\right)^n\right]^L &\leq \left[1 - \left(\frac{1}{k_f(\mathcal{H})}\right)^n\right]^{(k_f(\mathcal{H})(1+\epsilon))^n} \\ &= \left[1 - \left(\frac{1}{k_f(\mathcal{H})}\right)^n\right]^{(k_f(\mathcal{H}))^n(1+\epsilon)^n}.\end{aligned}$$

Since $\lim_{x\to 0}(1-x)^{1/x} = \frac{1}{e} < \frac{1}{2}$, the above probability is less than

$$\left(\frac{1}{2}\right)^{(1+\epsilon)^n}$$

for sufficiently large $n$. Since the cardinality of the vertex set of $\mathcal{H}^n$ is $|S|^n$, the probability that $M$ is not a covering of $H^n$ is less than

$$\frac{|S|^n}{2^{(1+\epsilon)^n}}.$$

The denominator grows superexponentially with $n$, but the numerator grows (merely) exponentially with $n$, so we can choose $n$ so that this fraction is less than 1. For such an $n$, the probability that $M$ is a covering of $\mathcal{H}^n$ is positive, hence at least one such $M$ exists. □

The following alternative proof of Theorem 1.6.2 produces a covering of $\mathcal{H}^n$ using a greedy approach. The proof is quite short once we establish the following lemma.

**Lemma 1.6.4** *If $\mathcal{H}$ is any hypergraph and $m$ is the maximum cardinality of a hyperedge in $\mathcal{H}$, then $k(\mathcal{H}) \leq (1 + \log m)k_f(\mathcal{H})$.*

**Proof**. Produce a covering of $\mathcal{H}$ in the following straightforward way. First pick a hyperedge $X_1$ of maximal cardinality $m$. Inductively, pick $X_i$ to be a hyperedge that contains the most elements of the vertex set not yet covered by $X_1, X_2, \ldots, X_{i-1}$. Continue picking hyperedges in this way until every element of the vertex set is in some $X_i$, and then stop.

The algorithm prescribes that $|X_i - \cup_{j=1}^{i-1} X_j|$ is a nonincreasing function of $i$. Let $a_k$ denote the number of indices $i$ such that $|X_i - \cup_{j=1}^{i-1} X_j| = k$. Let $a = \sum_{k=1}^{m} a_k$. We show that $a \leq (1 + \log m)k_f(\mathcal{H})$. Since we have produced a covering of cardinality $a$, the optimal covering can be no larger, and the result follows.

In fact, we show that $a \leq \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right) k_f(\mathcal{H})$. To see this, for $1 \leq j \leq m$ let $\mathcal{H}_j$ be the hypergraph whose ground set $S$ is obtained from $\mathcal{H}$ by deleting all vertices covered by the first $a_m + a_{m-1} + \cdots + a_{j+1}$ hyperedges from the algorithm, and whose edge set is obtained from $\mathcal{H}$ by intersecting every hyperedge with the set $S$. (Take $\mathcal{H}_m$ to be $\mathcal{H}$.) The hypergraph $\mathcal{H}_j$ has no edges

of cardinality greater than $j$, so no $j$-fold packing of $\mathcal{H}_j$ has fewer elements than the number of elements of the vertex set of $\mathcal{H}_j$, which is $a_1 + 2a_2 + \cdots + ja_j$. Hence, for $1 \leq j \leq m$,

$$p_j(\mathcal{H}) \geq p_j(\mathcal{H}_j) \geq a_1 + 2a_2 + \cdots + ja_j.$$

Now $k_f(\mathcal{H}) = p_f(\mathcal{H}) \geq (p_j(\mathcal{H})/j)$ for every $j$. It follows that, for $1 \leq j \leq m$,

$$jk_f(\mathcal{H}) \geq a_1 + 2a_2 + \cdots + ja_j. \qquad (*)$$

For $1 \leq j < m$, divide equation $(*)$ by $j(j+1)$ and for $j = m$, divide inequality $(*)$ by $m$ to obtain

$$\begin{aligned}
\frac{1}{2}k_f(\mathcal{H}) &\geq \frac{1}{2}a_1 \\
\frac{1}{3}k_f(\mathcal{H}) &\geq \frac{1}{2\cdot 3}a_1 + \frac{2}{2\cdot 3}a_2 \\
\frac{1}{4}k_f(\mathcal{H}) &\geq \frac{1}{3\cdot 4}a_1 + \frac{2}{3\cdot 4}a_2 + \frac{3}{3\cdot 4}a_3 \\
&\vdots \\
\frac{1}{m}k_f(\mathcal{H}) &\geq \frac{1}{(m-1)m}a_1 + \frac{2}{(m-1)m}a_2 + \cdots + \frac{m-1}{(m-1)m}a_{m-1} \\
k_f(\mathcal{H}) &\geq \frac{1}{m}a_1 + \frac{2}{m}a_2 + \cdots + \frac{m-1}{m}a_{m-1} + \frac{m}{m}a_m.
\end{aligned}$$

Adding these inequalities yields

$$\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{m} + 1\right) k_f(\mathcal{H}) \geq a_1 + a_2 + a_3 + \cdots + a_m. \qquad (**)$$

The left-hand side of inequality $(**)$ is less than $(1 + \log m)k_f(\mathcal{H})$ and the right-hand side of $(**)$ is no less than $k(\mathcal{H})$, since we have produced a covering of $\mathcal{H}$ of cardinality $a_1 + a_2 + a_3 + \cdots + a_m$. The result follows. □

**Proof** (of Theorem 1.6.2). We obtain the inequality $k_f(\mathcal{H}) \leq k_\infty(\mathcal{H})$ in the same way as the first proof of Theorem 1.6.2.

For the opposite inequality, apply Lemma 1.6.4 to the hypergraph $\mathcal{H}^n$ to obtain

$$k(\mathcal{H}^n) \leq (1 + \log(m^n))k_f(\mathcal{H}^n).$$

By Theorem 1.6.1, $k_f(\mathcal{H}^n) = (k_f(\mathcal{H}))^n$, and so

$$k(\mathcal{H}^n) \leq (1 + \log(m^n))\,(k_f(\mathcal{H}))^n = (1 + n\log m)\,(k_f(\mathcal{H}))^n\,.$$

Taking $n$th roots yields

$$\sqrt[n]{k(\mathcal{H}^n)} \leq \sqrt[n]{1 + n\log m}\, k_f(\mathcal{H}),$$

and letting $n$ approach infinity gives the desired inequality. □

Although the definition suggests otherwise, this theorem and Corollary 1.3.1 imply that $k_\infty$ is always rational.

In contrast to $k_\infty$, the invariant $p_\infty$ need not equal $p_f = k_f$ and in fact need not be rational. As an example, let $\mathcal{H}$ be the 2-uniform hypergraph (i.e., graph) $C_5$. Then $p(\mathcal{H}) = 2$, $k(\mathcal{H}) = 3$, and $p_f(\mathcal{H}) = k_f(\mathcal{H}) = k_\infty(\mathcal{H}) = 5/2$. The asymptotic packing number $p_\infty(\mathcal{H})$, however, turns out to equal what is called the Shannon capacity of $C_5$, which Lovasz [123] proved is equal to $\sqrt{5}$. (See §3.5.)

## 1.7 Exercises

1. Prove that $p^*(\mathcal{H}) = p_f(\mathcal{H})$. (This completes the proof of Theorem 1.2.1 on page 3.)

2. Find several graph invariants that can be expressed as either a hypergraph covering or a hypergraph packing problem.

   Find some graph invariants that *cannot* be so expressed.

3. Let $\mathcal{H} = (S, \mathcal{X})$ be a hypergraph. The automorphism group of $\mathcal{H}$ induces a partition of $S$ into subsets called *orbits*: two vertices $u, v$ are together in an orbit if and only if there is an automorphism $\pi$ so that $\pi(u) = v$.

   Prove that in an optimal fractional packing of $\mathcal{H}$ we may assume that all vertices in a given orbit are assigned the same weight.

   This is a generalization of Proposition 1.3.4 on page 5.

4. State and prove a result akin to Proposition 1.3.4 on page 5 but instead of assuming the hypergraph is vertex-transitive, assume that it is *edge-transitive*.

5. Let $n, r$ be positive integers and let $\mathcal{H}$ be the complete, $r$-uniform hypergraph, i.e., the vertex set of $\mathcal{H}$ is an arbitrary $n$-set and the hyperedges of $\mathcal{H}$ are all $\binom{n}{r}$ size $r$ subsets of the vertices.

   Compute $k(\mathcal{H})$, $p(\mathcal{H})$, $\mu(\mathcal{H})$, $\tau(H)$, $k_f(\mathcal{H})$, and $\mu_f(\mathcal{H})$.

6. A finite *projective plane* is a hypergraph $\mathcal{H}$ with the following properties:

   - For each pair of distinct vertices $u$ and $v$, there is exactly one hyperedge that contains both $u$ and $v$.
   - The intersection of any pair of distinct hyperedges contains exactly one vertex.
   - There exist four vertices, no three of which are contained in a single hyperedge.

   The vertices of $\mathcal{H}$ are called *points* and the hyperedges of $\mathcal{H}$ are called *lines* of the projective plane.

   If $\mathcal{H}$ is a finite projective plane, then there exists a positive integer $n$, called the *order* of the projective plane, with the following properties:

   - $\mathcal{H}$ contains exactly $n^2 + n + 1$ points and $n^2 + n + 1$ lines.
   - Every line of $\mathcal{H}$ contains exactly $n + 1$ points.
   - Every point of $\mathcal{H}$ is contained in exactly $n + 1$ lines.

   If $\mathcal{H}$ is a finite projective plane of order $n$, find $k(\mathcal{H})$, $p(\mathcal{H})$, $\tau(\mathcal{H})$, $\mu(\mathcal{H})$, $k_f(\mathcal{H})$, and $\tau_f(\mathcal{H})$.

7. Let $\mathcal{H} = (S, \mathcal{X})$ be a hypergraph. Define the *partitioning number* $\hat{k}(\mathcal{H})$ to be the smallest size of a *partition* of $S$ into parts chosen from $\mathcal{X}$. This is analogous to the covering number except that the parts must be pairwise disjoint.

   Give a definition for the *fractional* partitioning number $\hat{k}_f(\mathcal{H})$ of $\mathcal{H}$.

   Prove that if $\mathcal{H}$ is a simplicial complex, then $k(\mathcal{H}) = \hat{k}(\mathcal{H})$ and $k_f(\mathcal{H}) = \hat{k}_f(\mathcal{H})$.

   (A *simplicial complex* is a hypergraph in which every subset of an edge is also an edge. See §A.2 on page 134.)

8. In the fractional covering problem for a hypergraph, we assign weights to each hyperedge so that the sum of the weights of the hyperedges containing any particular vertex is at least one.

   Prove that in an optimal fractional covering of a *simplicial complex* we may assume (without loss of optimality) that the sum of the weights of the hyperedges containing a given vertex is exactly one, i.e.,

$$k(\mathcal{H}) = \min \mathbf{1}^t \mathbf{x} \text{ s.t. } M\mathbf{x} = 1.$$

## 1.8 Notes

For a general introduction to hypergraphs, see Berge [15] or [19]. For the general approach to fractional invariants of hypergraphs see Berge's *Fractional Graph Theory* [16] or Chapter 3 of *Hypergraphs* [19], as well as the survey article by Füredi [70]. See also Laskar, Majumdar, Domke, and Fricke's paper [111]. For information on the extension of the ideas in this chapter to infinite hypergraphs, see the work of Aharoni and Ziv, [1], [2].

The game theoretic approach to hypergraph covering and packing can be found in Fisher [63]. See also Brightwell and Scheinerman [30]. For background reading on linear programming and games, see the book by Gale, Kuhn, and Tucker [73].

The first appearance of a theorem like Proposition 1.3.3 seems to be in a paper of Stahl [171].

The first proof of Theorem 1.6.2 on page 9 can be traced back to an idea that appears, in a rather different context, in a paper of McEliece and Posner [129]. See also the paper by Berge and Simonovits [20]. The second proof is due to Lovász [121]. Propp, Pemantle, and Ullman [144] extend Theorem 1.6.2 to a more general class of integer programming problems.

Lemma 1.6.4 on page 10 is due to Lovász [121] and Stein [175].

# 2

# Fractional Matching

## 2.1 Introduction

A *matching* in a graph is a set of edges no two of which are adjacent. The *matching number* $\mu(G)$ is the size of a largest matching. A *fractional matching* is a function $f$ that assigns to each edge of a graph a number in $[0,1]$ so that, for each vertex $v$, we have $\sum f(e) \leq 1$ where the sum is taken over all edges incident to $v$. If $f(e) \in \{0,1\}$ for every edge $e$, then $f$ is just a matching, or more precisely, the indicator function of a matching. The *fractional matching number* $\mu_f(G)$ of a graph $G$ is the supremum of $\sum_{e \in E(G)} f(e)$ over all fractional matchings $f$.

These definitions coincide with the definitions of $\mu(G)$ and $\mu_f(G)$ from Chapter 1, where a graph $G$ is interpreted as a 2-uniform hypergraph. Alternatively, one can understand $\mu(G)$ as the packing number of a certain hypergraph. Given a graph $G$, construct a hypergraph $\mathcal{H}$ whose ground set is $E(G)$ and with a hyperedge $e_v$ for each vertex $v \in V(G)$ consisting of all edges in $E(G)$ incident to $v$. This is nothing more than the hypergraph dual of the 2-uniform hypergraph $G$. It is easy to see that $\mu(G) = p(\mathcal{H})$, the packing number of $\mathcal{H}$. The fractional analogue of $\mu(G)$ is naturally defined to be $\mu_f(G) = p_f(\mathcal{H})$.

Dual to the notion of matching is the notion of a transversal of a graph. A *transversal* of a graph $G$ is a set $S$ of vertices such that every edge is incident to some vertex in $S$. The following classical result of König-Egerváry (see [50], [109]) says that the duality gap for matching vanishes when the graph is bipartite.

**Theorem 2.1.1** *If $G$ is bipartite, then there is a transversal of $G$ of cardinality $\mu(G)$.* □

It is clear that $\mu_f(G) \geq \mu(G)$ for all graphs $G$. As an example, $\mu(C_5) = 2$, while the function that assigns the number $1/2$ to every edge of $C_5$ shows that $\mu_f(C_5) \geq 5/2$. In fact, $\mu_f(C_5) = 5/2$, as the following basic lemma guarantees.

**Lemma 2.1.2** $\mu_f(G) \leq \frac{1}{2}\nu(G)$.

**Proof**. For any vertex $v$ and fractional matching $f$, we have $\sum_{e \ni v} f(e) \leq 1$. Summing these inequalities over all vertices $v$ yields

$$\sum_{e \in E(G)} 2f(e) \leq \nu(G)$$

and the result follows. □

The example of $C_5$ also shows that $\mu_f(G)$ can exceed $\mu(G)$, although this does not happen for bipartite graphs.

**Theorem 2.1.3** *If $G$ is bipartite, then $\mu_f(G) = \mu(G)$.*

We offer two proofs, one combinatorial and one using the unimodularity theorem (Theorem A.3.3 on page 136).

**Proof**. In a bipartite graph, Theorem 2.1.1 provides us a transversal $K \subseteq V(G)$ whose cardinality is the same as that of a maximum matching. If $f$ is any fractional matching of $G$, then

$$\sum_{e \in E(G)} f(e) \leq \sum_{v \in K} \sum_{e \ni v} f(e) \leq \sum_{v \in K} 1 = |K|.$$

Hence $\mu_f(G) \leq |K| = \mu(G)$. However $\mu(G) \leq \mu_f(G)$ always, so we have equality here. □

Our second proof uses the total unimodularity (see page 136) of the vertex-edge incidence matrix $M(G)$ of $G$.

**Lemma 2.1.4** *A graph $G$ is bipartite if and only if $M(G)$ is totally unimodular.*

**Proof**. If $G$ contains an odd cycle, then select the submatrix of $M(G)$ whose rows and columns are indexed by the vertices and edges in the odd cycle. Since this submatrix has determinant $\pm 2$, $M(G)$ is not totally unimodular.

To prove the converse, suppose that $G$ is bipartite and pick any $m$-by-$m$ submatrix $B$ of $M$. If any column of $B$ is zero, then certainly $\det(B) = 0$. If any column of $B$ has a unique 1 in it, then the determinant of $B$ is (plus or minus) the determinant of the smaller square submatrix of $M$ obtained by removing the unique 1 and its row and column, and we proceed by induction on $m$. If every column of $B$ has two 1's in it, then the vertices and edges identified by $B$ form a 2-factor, a union of cycles, which according to our hypothesis must all be of even length. If in fact more than one cycle is identified by $B$, decompose $B$ into smaller square submatrices representing the cycles, and proceed by induction. Finally, if $B$ represents a single cycle, switch rows and columns of $B$ until the $i, j$ element is 1 just when $i = j$ or $i = j + 1$ or $i = 1, j = m$. (One way to do this is to first switch rows so that every row has a 1 immediately under another 1 from the row above, and then to switch columns.) This matrix has determinant 0, since the sum of the even columns is equal to the sum of the odd columns, which identifies a linear dependence among the columns. Hence in all cases the determinant of $B$ is 0, 1, or $-1$, and $M$ is totally unimodular. □

**Proof** (of Theorem 2.1.3). The fractional matching number is the value of the linear relaxation of the integer program that computes the matching number. The matrix of coefficients of this program is the incidence matrix $M(G)$ of $G$. Since this matrix is totally unimodular by Lemma 2.1.4, an appeal to Theorem A.3.3 on page 136 gives the result. □

The upper bound $\nu/2$ of Lemma 2.1.2 on the facing page is attained for the cycle $C_n$, showing that $\mu_f$ can take the value $n/2$ for any $n \geq 2$. It turns out that these are the only values that $\mu_f$ can take. (This is in stark contrast with the behavior of the fractional chromatic number or the fractional domination number, whose values can be any rational number greater than or equal to 2; see Proposition 3.2.2 on page 32 and Theorem 7.4.1 on page 117.)

**Theorem 2.1.5** *For any graph $G$, $2\mu_f(G)$ is an integer. Moreover, there is a fractional matching $f$ for which*

$$\sum_{e \in E(G)} f(e) = \mu_f(G)$$

*such that $f(e) \in \{0, 1/2, 1\}$ for every edge $e$.*

**Proof**. Among all the fractional matchings $f$ for which $\sum_{e \in E(G)} f(e) = \mu_f(G)$, choose one with the greatest number of edges $e$ with $f(e) = 0$. Let $H$ be the subgraph of $G$ induced on the set of edges with $f(e) > 0$. What does $H$ look like?

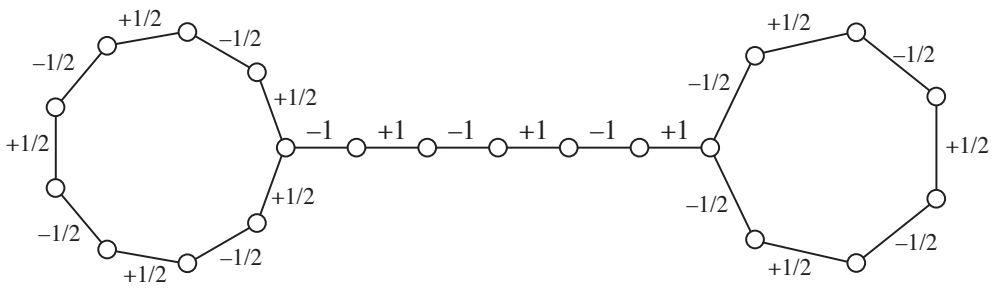


Figure 2.1. The nonzero values of the function $g$.

First, we show that $H$ can have no even cycles. Suppose that $C$ were such a cycle, and suppose that $\min_{e\in C} f(e) = m$ and that this minimum is achieved at edge $e_0$. Define $g : E(G) \to \{0, 1, -1\}$ to be the function that assigns 1 and $-1$ alternately to the edges of $C$ with $g(e_0) = -1$ and that assigns 0 to all other edges of $G$. Then $f + mg$ would be an optimal fractional matching of $G$ with at least one more edge, namely $e_0$, assigned a value of 0, contradicting the definition of $f$.

Second, we show that if $H$ has a pendant edge, then that edge is an entire component of $H$. For if $e = vw$ is an edge with $d(v) = 1$ and $d(w) > 1$ and if $f(e) < 1$, then increasing the value of $f(e)$ to 1 and decreasing the value of $f$ to zero on all other edges incident to $w$ produces a fractional matching of $G$ that attains the supremum and that has more edges assigned the value 0. This contradicts the definition of $f$.

Third, we show that if $H$ has an odd cycle $C$, that cycle must be an entire component of $H$. Suppose that $C$ were such a cycle. Suppose that some vertex $v \in C$ has $d_H(v) \geq 3$. Start at $v$, take a non-$C$ edge, and trace a longest path in $H$. Since this path can never return to $C$ (this would create an even cycle) or reach a vertex of degree 1 (there are no pendant edges), this path ends at a vertex where any further step revisits the path, forming another cycle. Thus $H$ contains a graph $K$ with 2 (necessarily odd) cycles connected by a path (possibly of length 0). Let $g : E(H) \to \{-1, -1/2, 0, 1/2, 1\}$ be the function that assigns 0 to edges not in $K$, $\pm 1$ alternately to the path connecting the two cycles of $K$, and $\pm 1/2$ alternately around the cycles of $K$ so that $\sum_{e\ni v} g(e) = 0$. An illustration of $g$ on $K$ appears in Figure 2.1. Note that $\sum_{e\in E(H)} g(e) = 0$.

Let $m$ be the real number of smallest absolute value such that $f + mg$ takes the value 0 on one of the edges of $K$. Then $f + mg$ is a fractional matching with more edges assigned the number 0 than has $f$, contradicting the definition of $f$.

Hence, $H$ consists only of components that are isomorphic to $K_2$ or odd cycles. The value of $f(e)$ is easily seen to be 1 if $e$ is an isolated edge and 1/2 if $e$ is in an odd cycle. Hence $f$ takes only the values 0, 1, and 1/2. □

A *fractional transversal* of a graph $G$ is a function $g : V(G) \to [0, 1]$ satisfying $\sum_{v\in e} g(v) \geq 1$ for every $e \in E(G)$. The *fractional transversal number* is the infimum of $\sum_{v\in V(G)} g(v)$ taken over all fractional transversals $g$ of $G$. By duality, the fractional transversal number of a graph $G$ is nothing more than the fractional matching number $\mu_f(G)$ of $G$.

**Theorem 2.1.6** *For any graph $G$, there is a fractional transversal $g$ for which*

$$\sum_{v\in V(G)} g(v) = \mu_f(G)$$

*such that* $g(v) \in \{0, 1/2, 1\}$ *for every vertex* $v$.

**Proof**. Suppose that $V(G) = \{v_1, v_2, \ldots, v_n\}$. Define $B(G)$ to be the bipartite graph whose vertex set is $V(B(G)) = \{x_1, x_2, \ldots, x_n\} \cup \{y_1, y_2, \ldots, y_n\}$ with an edge $x_i y_j \in E(B(G))$ if and only if $v_i v_j \in E(G)$. Another way of describing $B(G)$ is to say that it is the bipartite graph whose *bipartite* adjacency matrix is the adjacency matrix of $G$.

Given any fractional matching $f : E(G) \to \{0, 1/2, 1\}$, orient all the edges $e$ of $G$ for which $f(e) = 1/2$ so that no two directed edges have the same head or tail. This is easy to arrange, because the graph induced on the edges with $f(e) = 1/2$ has maximum degree 2; simply orient the cycles cyclically and orient the paths linearly. Now define a matching of $B(G)$ by selecting (1) edge $x_i y_j$ in case $f(v_i v_j) = 1/2$ and the edge $v_i v_j$ is oriented from $v_i$ to $v_j$, and (2) both edges $x_i y_j$ and $x_j y_i$ in case $f(v_i v_j) = 1$. The cardinality of this matching is equal to $2\sum_{e \in E(G)} f(e)$. Conversely, given any matching of $B(G)$, define a fractional matching $f : E(G) \to \{0, 1/2, 1\}$ by setting $f(v_i v_j) = 1$ if both $x_i y_j$ and $x_j y_i$ are in the matching, $1/2$ if only one is, and 0 if neither are. Owing to this correspondence, the fractional matching number of $G$ is half the matching number of $B(G)$.

Given an optimal fractional matching of $G$, we construct a maximum matching of $B(G)$ of cardinality $2\sum_{e \in E(G)} f(e)$. By Theorem 2.1.1, there is a transversal of $B(G)$ of the same cardinality. From this transversal, construct a fractional transversal $g : V(G) \to \{0, 1/2, 1\}$ of $G$ by setting $g(v_i) = 1$ if both $x_i$ and $y_i$ are in the transversal, $1/2$ if only one is, and 0 otherwise. The sum $\sum_{v \in V(G)} g(v)$ is clearly half the cardinality of the transversal of $B(G)$, namely, $\sum_{e \in E(G)} f(e)$, which is equal to $\mu_f(G)$. Hence $g$ is an optimal fractional transversal. □

## 2.2 Results on maximum fractional matchings

### Fractional Tutte's theorem

A *fractional perfect matching* is a fractional matching $f$ with $\sum f(e) = \nu(G)/2$, that is, a fractional matching achieving the upper bound in Lemma 2.1.2 on page 14. When a fractional perfect matching takes only the values 0 and 1, it is a perfect matching. In this section we present necessary and sufficient conditions for a graph to have a fractional perfect matching. The result is a lovely analogue of Tutte's [180] theorem on perfect matchings (see Theorem 2.2.3 below).

We begin with some simple propositions.

**Proposition 2.2.1** *Suppose that* $f$ *is a fractional matching. Then* $f$ *is a fractional perfect matching if and only if* $\sum_{e \ni v} f(e) = 1$ *for every* $v \in V$.

**Proof**. If the condition is satisfied at every vertex, then

$$2\sum_{e \in E} f(e) = \sum_{v \in V} \sum_{e \ni v} f(e) = \nu(G),$$

and we have a fractional perfect matching. Were there strict inequality in $\sum_{e \ni v} f(e) \leq 1$ for some $v \in V$, then there would be strict inequality when they are summed, and we would not have a fractional perfect matching. □

**Proposition 2.2.2** *The following are equivalent for a graph* $G$.

(1) $G$ *has a fractional perfect matching.*

(2) *There is a partition* $\{V_1, V_2, \ldots, V_n\}$ *of the vertex set* $V(G)$ *such that, for each* $i$*, the graph* $G[V_i]$ *is either* $K_2$ *or Hamiltonian.*

(3) *There is a partition* $\{V_1, V_2, \ldots, V_n\}$ *of the vertex set* $V(G)$ *such that, for each* $i$*, the graph* $G[V_i]$ *is either* $K_2$ *or a Hamiltonian graph on an odd number of vertices.*

**Proof**. (1) $\Longrightarrow$ (2): If $G$ has a fractional perfect matching, then by Theorem 2.1.5 on page 15 it has a fractional perfect matching $f$ that is restricted to the values 0, 1/2, and 1. Create the partition by putting two vertices in the same block if they are connected by an edge $e$ with $f(e) = 1$ or if they are connected by a path $P$ all of whose edges $e$ have $f(e) = 1/2$. The graphs induced on these blocks are all either isomorphic to $K_2$ or else have a spanning cycle.

(2) $\Longrightarrow$ (3): If any block $V_i$ of the partition induces a Hamiltonian graph on an even number of vertices, that block may be refined into blocks of size 2 each of which induces a $K_2$.

(3) $\Longrightarrow$ (1): If a block of the partition induces a $K_2$, let $f(e) = 1$ where $e$ is the included edge. If a block of the partition induces a Hamiltonian graph on an odd number of vertices, let $f(e) = 1/2$ for every edge $e$ in the Hamilton cycle. Let $f(e)$ be 0 for all other edges $e$. The function $f$ is clearly a fractional perfect matching. □

Let us write $o(G)$ to stand for the number of components of $G$ with an odd number of vertices and $i(G)$ to stand for the number of isolated vertices of $G$. The main theorem on perfect matchings is the following, due to Tutte [180].

**Theorem 2.2.3** *A graph* $G$ *has a perfect matching if and only if* $o(G - S) \leq |S|$ *for every set* $S \subseteq V(G)$*.* □

The analogous theorem for fractional perfect matchings is the following result.

**Theorem 2.2.4** *A graph* $G$ *has a fractional perfect matching if and only if* $i(G - S) \leq |S|$ *for every set* $S \subseteq V(G)$*.*

**Proof**. First assume that $G$ has a fractional perfect matching $f$ and, for some $S \subseteq V(G)$, the number of isolated vertices in $G - S$ is greater than $|S|$. Let $I$ be this set of isolated vertices and consider the graph $H = G[S \cup I]$. See Figure 2.2 on the facing page. Then $f$ restricted to $H$ is a fractional matching of $H$ with

$$\sum_{e \in E(H)} f(e) \geq |I| > \frac{1}{2}\nu(H),$$

which contradicts Lemma 2.1.2 on page 14.

To prove the converse, assume that $G$ has no fractional perfect matching. Then $G$ must have a fractional transversal $g : V(G) \to [0, 1]$ for which $\sum_{v \in V(G)} g(v) < \nu/2$. By Theorem 2.1.6 on page 16, this fractional transversal may be assumed to take only the values 0, 1/2, and 1. Let

$$\begin{aligned} S &= \{v \in V(G) : g(v) = 1\}, \\ I &= \{v \in V(G) : g(v) = 0\}, \quad \text{and} \\ C &= V(G) - S - I. \end{aligned}$$

Note that, because $g$ is a fractional transversal, no edge in $G$ joins a vertex in $I$ to another vertex in $I$, nor a vertex in $I$ to a vertex in $C$. Hence the vertices in $I$ are isolated vertices in $G - S$. So

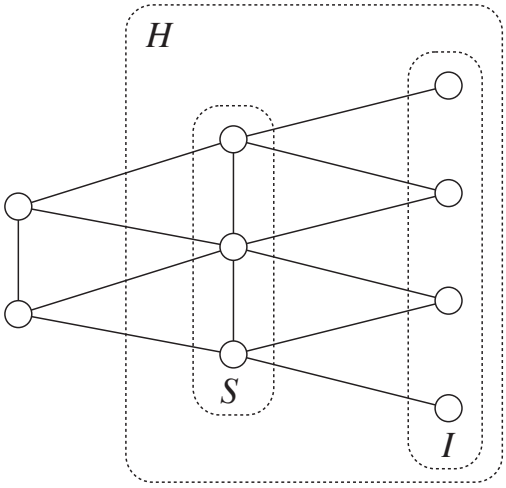


Figure 2.2. A graph $G$ that has no fractional perfect matching. Note that $|S| = 3$ and $i(G - S) = 4$. Subgraph $H$ is $G[S \cup I]$.

$i(G - S) \geq |I|$. But

$$\begin{aligned} |I| &= \nu(G) - |S| - |C| \\ &= \nu(G) - 2|S| - |C| + |S| \\ &= \left(\nu(G) - 2\sum_{v \in V(G)} g(v)\right) + |S| > |S|. \end{aligned}$$

Hence $i(G - S) > |S|$ and the condition is violated. □

A perfect matching is also called a 1-factor of a graph. It might seem sensible to find a generalization of Theorem 2.2.4 that provides a condition for the existence of a fractional $k$-factor. The only reasonable candidate for a definition of a fractional $k$-factor, however, is that it is $k$ times a fractional 1-factor. Hence the necessary and sufficient condition for the existence of a fractional perfect matching is also a necessary and sufficient condition for the existence of a fractional $k$-factor.

## Fractional Berge's theorem

The extent to which the condition in Tutte's theorem fails determines the matching number of the graph, in the sense of the following result due to Berge [14].

**Theorem 2.2.5** *For any graph $G$,*

$$\mu(G) = \frac{1}{2}\left(\nu(G) - \max\left\{o(G - S) - |S|\right\}\right),$$

*where the maximum is taken over all $S \subseteq V(G)$.* □

The analogous result works also for the fractional matching number.

**Theorem 2.2.6** *For any graph $G$,*

$$\mu_f(G) = \frac{1}{2}\left(\nu(G) - \max\left\{i(G-S) - |S|\right\}\right),$$

*where the maximum is taken over all $S \subseteq V(G)$.*

**Proof**. Suppose that the maximum is $m$, achieved at $S$. The case $m = 0$ is Theorem 2.2.3, so we may assume $m \geq 1$.

We first show that $\mu_f(G) \leq \frac{1}{2}(\nu(G) - m)$. Let $C \subseteq V(G)$ be the set of vertices that are neither in $S$ nor isolated in $G - S$. If $f$ is any fractional matching of $G$, then

$$\sum_{e \in E(G)} f(e) \leq \sum_{v \in S} \sum_{e \ni v} f(e) + \frac{1}{2} \sum_{v \in C} \sum_{e \ni v} f(e),$$

since no edge connects an isolated vertex in $G-S$ to another isolated vertex in $G-S$ or to a vertex in $C$. But then

$$\sum_{e \in E(G)} f(e) \leq |S| + \frac{1}{2}|C| = \frac{1}{2}\Big[i(G-S) + |S| + |C| - (i(G-S) - |S|)\Big]$$

$$= \frac{1}{2}(\nu(G) - m).$$

To obtain the opposite inequality, let $H$ be the join[1] of $G$ and $K_m$. We show that $H$ has a fractional perfect matching. Choose $T \subseteq V(H)$. If some vertex in the $K_m$ portion is not in $T$, then either $i(H-T) = 0$ and so $|T| \geq i(H-T)$, or else $H - T$ is just a single vertex in $K_m$ in which case $i(H-T) = 1 \leq |T|$. If $T$ contains all the vertices of the $K_m$ portion, let $T' = T \cap V(G)$. Then $i(H-T) = i(G-T')$, and so

$$|T| - i(H-T) = m + |T'| - i(G-T') \geq 0.$$

In all cases, $i(H-T) \leq |T|$, so by Theorem 2.2.4 on page 18, $H$ has a fractional perfect matching $f$. The restriction of $f$ to $E(G)$ is then a fractional matching on $G$ with

$$\sum_{e \in E(G)} f(e) = \sum_{e \in E(H)} f(e) \quad - \sum_{e \in E(H) - E(G)} f(e)$$

$$= \frac{\nu(H)}{2} - m = \frac{\nu(G) + m}{2} - m = \frac{\nu(G) - m}{2}$$

as required. □

## Fractional Gallai's theorem

Gallai's theorem relates the covering number $k$ of a graph to its matching number $\mu$.

**Theorem 2.2.7** *Let $G$ be a graph without isolated vertices. Then $k(G) + \mu(G) = \nu(G)$.* □

This result remains true in the fractional case.

[1]The *join* of graphs $G$ and $H$, denoted $G \vee H$, is formed by taking copies of $G$ and $H$ on disjoint sets of vertices and adding edges between every vertex of $G$ and every vertex of $H$.

**Theorem 2.2.8** *Let $G$ be a graph without isolated vertices. Then $k_f(G) + \mu_f(G) = \nu(G)$.*

**Proof**. By duality, it is enough to show that $p_f(G) + \tau_f(G) = \nu(G)$. Choose $f, g : V(G) \to [0, 1]$ with $g(v) = 1 - f(v)$. For any pair of vertices (but especially for $uv \in E(G)$) we have

$$f(u) + f(v) \geq 1 \iff g(u) + g(v) \leq 1.$$

If follows that $f$ is a fractional transversal if and only if $g$ is a fractional packing. If $f$ is a minimum fractional transversal, then $g$ is a feasible fractional packing and so

$$\tau_f(G) + p_f(G) \geq \sum_v f(v) + \sum_v g(v) = \nu(G).$$

Similarly, if $g$ is a maximum fractional packing, then $f$ is a feasible fractional transversal and so

$$\tau_f(G) + p_f(G) \leq \sum_v f(v) + \sum_v g(v) = \nu(G). \qquad \square$$

## 2.3 Fractionally Hamiltonian graphs

A *Hamiltonian cycle* in a graph is a cycle that contains every vertex, i.e., a spanning cycle. Graphs that have Hamiltonian cycles are called *Hamiltonian*.

Here is an alternative definition: Let $G$ be a graph on $n$ vertices. For a nonempty, proper subset $S \subset V$, write $[S, \overline{S}]$ for the set of all edges with exactly one end in $S$. A set of this form is called an *edge cut* of $G$ and is a set of edges whose removal disconnects $G$. Note that, if $G$ is disconnected, then the null set is an edge cut of $G$, obtained by selecting $S$ to be one component of $G$. A *Hamiltonian cycle* in $G$ is a subset $F \subseteq E(G)$ for which (1) $|F| = n$ and (2) every edge cut of $G$ contains at least two edges of $F$.

This definition has a natural fractional analogue. Consider the indicator function of the Hamilton cycle, and then relax! A graph $G$ is called *fractionally Hamiltonian* if there is a function $f : E(G) \to [0, 1]$ such that the following two conditions hold:

$$\sum_{e \in E(G)} f(e) = \nu(G)$$

and for all $\emptyset \subset S \subset V(G)$

$$\sum_{e \in [S, \overline{S}]} f(e) \geq 2.$$

We call such a function $f$ a *fractional Hamiltonian cycle*. Note that connectedness is a necessary condition for fractional Hamiltonicity.

For example, Petersen's graph (see Figure 3.1 on page 31) is fractionally Hamiltonian (let $f(e) = \frac{2}{3}$ for all edges $e$), but is *not* Hamiltonian; see also exercise 10 on page 28.

It is helpful to consider an LP-based definition of fractional Hamiltonicity. Let $G$ be a graph. We consider $f(e)$ to be the "weight" of edge $e$, which we also denote $w_e$. Consider the LP:

$$\begin{array}{lll} & \min \sum_{e \in E(G)} w_e & \\ \text{subject to} & \sum_{e \in [S, \overline{S}]} w_e \geq 2 & \forall S,\ \emptyset \subset S \subset V(G) \\ & w_e \geq 0, & \forall e \in E(G) \end{array}$$

This linear program is called the *fractional Hamiltonian LP*.

**Proposition 2.3.1** *Let $G$ be a graph on at least 3 vertices. Then $G$ is fractionally Hamiltonian if and only if the value of the fractional Hamiltonian LP is exactly $\nu(G)$.*

**Proof**. Exercise 7 on page 27. $\square$

Note that every edge cut in a fractional Hamiltonian cycle must have weight *at least* 2, but certain natural cuts must have weight *exactly* two.

**Proposition 2.3.2** *Let $f$ be a fractional Hamiltonian cycle for a graph $G$ and let $v$ be any vertex of $G$. Then*

$$\sum_{e \ni v} f(e) = 2.$$

**Proof**. Let $v$ be any vertex and let $S = \{v\}$. Then

$$\sum_{e \ni v} f(e) = \sum_{e \in [S, \overline{S}]} f(e) \geq 2.$$

To conclude, we compute

$$2\nu(G) \leq \sum_{v \in V(G)} \sum_{e \ni v} f(e) = 2 \sum_{e \in E(G)} f(e) = 2\nu(G)$$

and the result follows. □

If $G$ has an even number of vertices and a Hamiltonian cycle, then clearly $G$ has a perfect matching: simply take half the Hamiltonian cycle, i.e., every other edge. Likewise, we have the following simple result.

**Theorem 2.3.3** *If $G$ is fractionally Hamiltonian, then $G$ has a fractional perfect matching.*

**Proof**. If $f$ is a fractional Hamiltonian cycle, then, by Proposition 2.3.2, $\frac{1}{2}f$ is a fractional perfect matching. □

We say that $G$ is *tough* provided that $c(G-S) \geq |S|$ for every $S \subseteq V(G)$, where $c(G)$ stands for the number of connected components in $G$. (Compare the definition of toughness to the conditions in Theorems 2.2.3 and 2.2.4.) Perhaps the best-known necessary condition for a graph to be Hamiltonian is that the graph must be *tough*. This fact can be strengthened to the following.

**Theorem 2.3.4** *Let $G$ be a graph with $\nu(G) \geq 3$ that is fractionally Hamiltonian. Then $G$ is tough.*

**Proof**. Suppose, for the sake of contradiction, that $G$ is not tough. Let $S = \{u_1, \ldots, u_s\}$ be a set of vertices so that $c(G - S) > s = |S|$. Let the components of $G - S$ be $H_1, H_2, \ldots, H_c$.

Consider the dual to the fractional Hamiltonian LP:

$$\max \sum 2y_F \qquad \text{s.t.} \qquad \sum_{F \ni e} y_F \leq 1 \text{ for all } e \in E(G), \text{ and } \mathbf{y} \geq 0.$$

The variables $y_F$ are indexed by the edge cuts of $G$. To show that $G$ is not fractionally Hamiltonian, it is enough to present a feasible solution to the dual LP with objective value greater than $\nu(G)$.

Let $F_i = [V(H_i), V(G) - V(H_i)]$ and set $y_{F_i} = \frac{1}{2}$. For each vertex $u \in V(G) - S$ let $F_u = [\{u\}, V(G) - \{u\}]$ and put $y_{F_u} = \frac{1}{2}$. All other edge cuts are assigned weight 0.

First we check that this weighting is feasible for the dual LP. Let $e$ be any edge of $G$.

- If $e$ has both ends in $S$, then $\sum_{F \ni e} y_F = 0$.
- If $e$ has exactly one end in $S$, then $\sum_{F \ni e} y_F = \frac{1}{2} + \frac{1}{2} = 1$.

- If $e$ has no ends in $S$, then $e$ has both ends in a component $H_i$, and $\sum_{F \ni e} y_F = \frac{1}{2} + \frac{1}{2} = 1$.

Therefore the weighting $y_F$ is feasible.

Second we compute the value of the objective function $\sum_F 2y_F$. We have

$$\begin{aligned}\sum_F 2y_F &= \sum_{i=1}^{c} 2y_{F_i} + \sum_{u \in V(G)-S} 2y_{F_u} \\ &= c + 2\left[\frac{1}{2}(\nu(G) - s)\right] \\ &= \nu(G) + c - s \\ &> \nu(G)\end{aligned}$$

and therefore $G$ is not fractionally Hamiltonian. □

It is known that being tough is not a sufficient condition for a graph to be Hamiltonian; Petersen's graph is tough but not Hamiltonian. However, Petersen's graph is fractionally Hamiltonian, so one might ask, Is being tough sufficient for fractional Hamiltonicity? The answer is no. Indeed, we can say a bit more.

The *toughness* of a graph $G$ is defined to be

$$\sigma(G) = \min\left\{\frac{|S|}{c(G-S)} : S \subseteq V(G),\ c(G-S) > 1\right\}.$$

A graph is called *t-tough* provided $\sigma(G) \geq t$. A graph is tough if and only if it is 1-tough.

**Proposition 2.3.5** *Let $t < \frac{3}{2}$. There is a graph $G$ with $\sigma(G) \geq t$ that is not fractionally Hamiltonian.*

**Proof**. Let $n$ be a positive integer. Let $G$ be a graph with $V(G) = A \cup B \cup C$ where $A$, $B$, and $C$ are disjoint with $|A| = 2n+1$, $|B| = 2n+1$, and $|C| = n$. Thus $\nu(G) = 5n+2$. Both $A$ and $C$ are cliques in $G$, while $B$ is an independent set. The only other edges in $G$ are a matching between $A$ and $B$, and a complete bipartite graph between $B$ and $C$. See Figure 2.3 on the next page (where $n = 2$). One checks (exercise 11 on page 28) that the toughness of $G$ is $(3n+1)/(2n+1)$ (let $S = A \cup C$ in the definition), which can be made arbitrarily close to $\frac{3}{2}$.

To show that $G$ is not fractionally Hamiltonian we work with the dual to the fractional Hamiltonian LP. We assume that the vertices in $A, B, C$ are labeled so that $A = \{a_1, \ldots, a_{2n+1}\}$, $B = \{b_1, \ldots, b_{2n+1}\}$, and $C = \{c_1, \ldots, c_n\}$ and that $a_i$ is adjacent to $b_i$ for $1 \leq i \leq 2n+1$. We define a weighting $y_F$ of the edge cutsets $F$ of $G$ as follows:

- If $F = [\{a_i\}, V(G) - \{a_i\}]$, then $y_F = \frac{1}{4}$.
- If $F = [\{b_i\}, V(G) - \{b_i\}]$, then $y_F = \frac{3}{4}$.
- If $F = [\{a_i, b_i\}, V(G) - \{a_i, b_i\}]$, then $y_F = \frac{1}{4}$.
- For all other edge cuts, $y_F = 0$.

One now checks, patiently, that for every edge $e \in E(G)$ that $\sum_{F \ni e} y_F \leq 1$. (There are four cases depending on whether the endpoints of $e$ are (1) both in $A$, (2) both in $C$, (3) in $A$ and $B$, or (4) in $B$ and $C$.) We leave this verification to the reader.

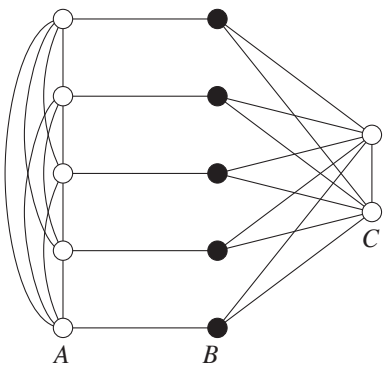


Figure 2.3. A tough graph that is *not* fractionally Hamiltonian.

The value of the objective function is

$$\begin{aligned}\sum_F 2y_F &= \frac{2}{4}|A| + \frac{6}{4}|B| + \frac{2}{4}|A| \\ &= \frac{2(2n+1) + 6(2n+1) + 2(2n+1)}{4} \\ &= 5n + \frac{5}{2} > 5n + 2 = \nu(G).\end{aligned}$$

Therefore $G$ is not fractionally Hamiltonian. □

It is conjectured that if a graph's toughness is sufficiently high, then it must be Hamiltonian; indeed, it is believed that $\sigma(G) \geq 2$ is enough. It therefore seems safe to make the analogous conjecture.

**Conjecture 2.3.6** *If $\sigma(G) \geq 2$, then $G$ is fractionally Hamiltonian.* □

## The middle levels problem: a fractional solution

The *middle levels problem* is a notorious problem in the study of Hamiltonian graphs. It concerns the following family of graphs. Let $n, a, b$ be integers with $0 \leq a < b \leq n$. Let $G(n; a, b)$ denote the bipartite graph whose vertices are all the $a$-element and $b$-element subsets of an $n$-set, say $[n] = \{1, 2, \ldots, n\}$. The $a$-sets are called the *bottom* vertices and the $b$-sets are called the *top* vertices. A bottom vertex $A$ and a top vertex $B$ are adjacent exactly when $A \subset B$.

The middle levels problem is as follows. For a positive integer $k$, is the graph $G(2k+1; k, k+1)$ Hamiltonian? The middle levels problem is so-named because it deals with the central levels of the Boolean algebra $2^{[2k+1]}$. More generally, the following is believed.

**Conjecture 2.3.7** *Let $k$ and $n$ be integers with $0 < k < n/2$. Then the graph $G(n; k, n-k)$ is Hamiltonian.*

We prove the fractional analogue.

**Theorem 2.3.8** *Let $k$ and $n$ be integers with $0 < k < n/2$. Then the graph $G(n;k,n-k)$ is fractionally Hamiltonian.*

**Proof**. Note that $G(n;k,n-k)$ is a regular bipartite graph in which every vertex has degree $\binom{n-k}{k}$. Let $w = 2/\binom{n-k}{k}$ and assign weight $w$ to every vertex. The sum of the weights of the edges is $w\binom{n}{k}\binom{n-k}{k} = 2\binom{n}{k} = \nu(G(n;k,n-k))$. To see that this weighting is a fractional Hamiltonian cycle we just need to show that there are at least $\binom{n-k}{k}$ edges in every edge cut.

We prove this below (Proposition 2.3.11 on the next page) and this completes the proof. □

Note that $G(n;k,n-k)$ is edge-transitive. It follows that $G(n;k,n-k)$ is fractionally Hamiltonian if and only if it has a fractional Hamiltonian cycle in which every edge has the same weight. This, in turn, is equivalent to showing that the edge connectivity $\kappa'$ of $G(n;k,n-k)$ equals its minimum degree. It follows that $\kappa' = \delta$ for $G(n;k,n-k)$ is a necessary condition for $G(n;k,n-k)$ to be Hamiltonian.

In the remainder of this section we analyze the edge connectivity of $G(n;k,n-k)$.

**Proposition 2.3.9** *Let $0 \le a < b \le n$ be integers. Then every pair of distinct bottom vertices of $G(n;a,b)$ is joined by $\binom{n-a}{b-a}$ edge-disjoint paths.*

**Proof**. Let $f(n;a,b)$ denote the maximum number of edge-disjoint paths between pairs of distinct bottom vertices. Since bottom vertices have degree $\binom{n-a}{b-a}$, we have $f(n;a,b) \le \binom{n-a}{b-a}$ and we work to show that equality holds.

The proof is by induction on $n$ and $a$. The basis cases $n = 1$ and/or $a = 0$ are easy to check.

Let $X \ne Y$ be bottom vertices. We claim that we can restrict to the case where $X$ and $Y$ are disjoint, for suppose $X$ and $Y$ have nonempty intersection, say $n \in X \cap Y$. There are $f(n-1;a-1,b-1)$ edge-disjoint paths joining $X-\{n\}$ to $Y-\{n\}$ in $G(n-1;a-1,b-1)$. By the induction hypothesis $f(n-1;a-1,b-1) = \binom{(n-1)-(a-1)}{(b-1)-(a-1)} = \binom{n-a}{b-a}$. If we insert element $n$ into the vertices (sets) on those paths we create $\binom{n-a}{b-a}$ edge-disjoint paths from $X$ to $Y$ in $G(n;a,b)$. Thus we may assume that $X$ and $Y$ are disjoint. This implies that $n \ge 2a$.

We claim that we can further restrict to the case that $X \cup Y = [n]$, for suppose that $X \cup Y \ne [n]$. Without loss of generality, $n \notin X \cup Y$.

In case $b > a+1$, there are $f(n-1;a,b)$ edge-disjoint $X$-$Y$ paths that avoid element $n$. There are also $f(n-1;a,b-1)$ edge-disjoint $X$-$Y$ paths all of whose top sets use element $n$. Thus

$$\begin{aligned} f(n;a,b) &\ge f(n-1;a,b) + f(n-1;a,b-1) \\ &= \binom{n-1-a}{b-a} + \binom{n-1-a}{b-a-1} = \binom{n-a}{b-a} \end{aligned}$$

as required.

In case $b = a+1$, there are $f(n-1;a,a+1) = \binom{(n-1)-a}{1} = n-a-1$ edge-disjoint paths that avoid element $n$. We can create one additional path that begins $X \to X \cup \{n\}$ and ends $Y \cup \{n\} \to Y$ all of whose internal vertices contain the element $n$. This gives a total of $n-a \ge a+1$ edge-disjoint $X$-$Y$ paths as required.

Thus we may assume $X \cup Y = [n]$ and $X \cap Y = \emptyset$. This means that $n = 2a \ge b > a$ and we work to construct $\binom{n-a}{b-a} = \binom{a}{b-a}$ edge-disjoint $X$-$Y$ paths.

Write $X = \{x_1, \ldots, x_a\}$ and $Y = \{y_1, \ldots, y_a\}$. For a subset $J \subset [a]$ we define $X[J]$ to be the set of elements in $X$ with subscripts in $J$, i.e., $X[J] = \{x_j : j \in J\}$. Define $Y[J]$ analogously. Choose

$J \subset [a]$ with $|J| = b - a$; there are $\binom{a}{b-a}$ choices for this set. We now construct a path from $X$ to $Y$ that begins $X \to X \cup Y[J]$ and ends $X[J] \cup Y \to Y$. The top vertices on this path satisfy the following conditions:

- They contain $X[J] \cup Y[J]$.
- For any index $k \notin J$, they contain exactly one of $x_k$ or $y_k$.

It is easy to find such paths; here is an explicit, if inefficient, construction. Let $T$ be a top vertex on this path, which contains $x_k$, but not $y_k$. From $T$ delete $x_k$ and any $b - a - 1$ other elements to reach a bottom vertex; next add to this bottom vertex $y_k$ and the $b - a - 1$ other elements previously deleted. We repeat these steps until we have transformed $X \cup Y[J]$ into $X[J] \cup Y$.

It is immediate that if $J \neq J'$, the corresponding paths are edge-disjoint because the top vertices on the $J$-path are disjoint from the top vertices on the $J'$-path.

Thus $f(n; a, b) = f(2a; a, b) \geq \binom{a}{b-a}$ as required. □

**Proposition 2.3.10** *Let $0 \leq a < b \leq n$ be integers. Then every pair of distinct top vertices of $G(n; a, b)$ is joined by $\binom{b}{a}$ edge-disjoint paths.*

**Proof**. The mapping $X \mapsto [n] - X$ is an isomorphism from the graph $G(n; a, b)$ to $G(n; n-b, n-a)$ that exchanges top and bottom vertices in the two graphs. The maximum number of edge-disjoint paths between top vertices of $G(n; a, b)$ equals the maximum number of edge-disjoint paths between bottoms in $G(n; n - b, n - a)$. By Proposition 2.3.9, this is precisely $\binom{n-(n-b)}{(n-a)-(n-b)} = \binom{b}{b-a} = \binom{b}{a}$. □

**Proposition 2.3.11** *Let $n$ and $k$ be integers with $0 \leq k < n/2$. Then the edge connectivity of $G(n; k, n - k)$ is $\binom{n-k}{k}$.*

**Proof**. Let $S$ be a nonempty, proper subset of $V = V(G(n; k, n - k))$. We must show that the number of edges between $S$ and $V - S$ is at least $\binom{n-k}{k}$. If $S$ and $V - S$ both contain a bottom vertex, or if they both contain a bottom vertex, then by Propositions 2.3.9 and 2.3.10 we have at least $\binom{n-k}{k}$ edges in $[S, V - S]$. Otherwise, the set of top vertices is entirely contained in (say) $S$ and the set of bottom vertices is entirely contained in $V - S$, and since the graph is $\binom{n-k}{k}$-regular, there are at least that many edges between $S$ and $V - S$. □

## 2.4 Computational complexity

The fractional matching number of a graph can be computed in polynomial time. Indeed, the constraint matrix in the fractional matching number problem has size $|V| \times |E|$ and therefore polynomial-time LP solutions for this problem exist (see the discussion on page 136 of §A.3). A more efficient approach, however, is to find the matching number of $B(G)$ using a standard bipartite matching algorithm. See also Bourjolly and Pulleyblank [29] for an efficient algorithm.

Fractional Hamiltonicity can also be tested in polynomial time. Here are two methods. First, we can formulate fractional Hamiltonicity as a multicommodity flow problem. There are two types of variables in this formulation: capacities and flows. To begin, arbitrarily assign a direction to every edge. To each edge $e$ we assign a "capacity" variable $c_e$ in the interval $[0, 1]$. For every pair of vertices $(u, v)$ and to every edge $e$ we assign a flow $f_{u,v,e}$. One should imagine that there are $n(n - 1)$ different commodities flowing through the graph and $f_{u,v,e}$ represents the amount of flow of commodity $(u, v)$ on edge $e$. The flow must satisfy $-c_e \leq f_{u,v,e} \leq c_e$. At every vertex other than $u$ or $v$ the net flow of commodity $(u, v)$ must be zero (flow in must equal flow out). The

objective is to minimize the sum of the capacities $\sum c_e$ subject to the additional constraint that the net flow of each commodity must be at least 2. One can show (exercise 12 on the following page) that the value of this multicommodity flow problem is $\nu(G)$ if and only if $G$ is fractionally Hamiltonian. We have introduced a number of new variables and many constraints, but note that the number of variables and constraints is polynomial in $\nu(G) + \varepsilon(G)$. Thus the value of this LP and the determination of $G$ being fractionally Hamiltonian can be found in polynomial time.

The fractional Hamiltonicity problem admits another polynomial-time solution, again using linear programming. This time we formulate the problem in the natural way. We assign a weight to each edge of $G$. We seek to minimize the sum of these weights subject to the constraints that the sum of the weights across any edge cut is at least 2. Now there are exponentially many cuts so this LP formulation is not polynomial in the size of $G$. However, the separation problem can be efficiently solved for this problem: Given a weight vector $\mathbf{w}$ we can, in polynomial time, identify an edge cut of minimum weight. This detects any edge cut with total weight less than 2. If this cut has total weight at least 2, then we know $\mathbf{w}$ is feasible. Otherwise, we know $\mathbf{w}$ is infeasible and we have identified a violated constraint. Therefore (see the discussion on page 137) we can find the value of the fractional Hamiltonian LP in polynomial time (polynomial in the size of $G$).

Since there exist polynomial-time algorithms to recognize fractionally Hamiltonian graphs, it would seem that there should be a "nice" theorem laying out necessary and sufficient conditions for fractional Hamiltonicity. Finding (and proving!) such a result is an open problem.

## 2.5 Exercises

1. Petersen's theorem [146] states that every 2-edge-connected 3-regular graph has a perfect matching. Prove this in the fractional case. Indeed, it is easy to prove that any $r$-regular graph has a fractional perfect matching.

2. Tutte's theorem (2.2.3) characterizes those graphs that have perfect matchings. Characterize those graphs that are "matching perfect", i.e., graphs $G$ such that $\mu(H) = \mu_f(H)$ for all induced subgraphs $H$ of $G$.

3. Use the previous two exercises to prove Hall's marriage theorem: *Every $r$-regular bipartite graph has a perfect matching.*

4. It is known that if $G$ is connected and has an even number of edges then $L(G)$, the line graph of $G$, has a perfect matching. Prove the easier result that asserts that every such line graph has a fractional perfect matching.

5. A graph is called *semi-bipartite* if any two vertex disjoint odd cycles are joined by an edge. Prove that a regular, connected, semi-bipartite graph has a matching with at most one unsaturated vertex.

6. Let $M$ be a matrix. The *term rank* of $M$ is the maximum number of nonzero entries of $M$ no two of which are in the same row or column of $M$.

   Let $G$ be a graph. Prove that the term rank of $A(G)$ is $2\mu_f(G)$.

7. Prove Proposition 2.3.1 on page 21. Why do we need the condition $\nu(G) \geq 3$?

   Note: The principal difference between the two definitions of fractionally Hamiltonian is whether or not we allow edge weights to exceed one.

8. Prove or disprove: Every fractionally Hamiltonian graph has a fractional Hamiltonian cycle with edge weights in $\{0, 1/2, 1\}$.

9. Define a *t-fold Hamiltonian cycle* to be a closed walk in a graph that visits every vertex exactly $t$ times (we do not double count the first/last vertex on the walk). A Hamiltonian cycle is just a 1-fold Hamiltonian cycle. An alternative definition for fractional Hamiltonicity might be: A graph is fractionally Hamiltonian provided there is a positive integer $t$ for which it has a $t$-fold Hamiltonian cycle.

   To what extent is this definition equivalent to the one presented in §2.3? Is the alternative notion implied by the one in §2.3? Does the alternative notion imply the one in §2.3?

10. Let $G$ be Petersen's graph with one edge deleted. Show that $G$ is fractionally Hamiltonian. What happens if we delete two edges from Petersen's graph?

11. Prove that $\sigma(G) = (3n+1)/(2n+1)$ for the graph in the proof of Proposition 2.3.5 on page 23.

12. Let $G$ be a graph and consider the multicommodity flow problem (see the discussion on page 27) on $G$. Prove that $G$ is fractionally Hamiltonian if and only if the value of the flow problem is $\nu(G)$.

13. An *Eulerian tour* is a walk in a graph that traverses every edge exactly once. Develop a sensible definition for a graph to have a *fractional Eulerian tour*.

    For your definition, prove or disprove: A graph has a fractional Eulerian tour if and only if it has an Eulerian tour.

14. In formulating fractional Hamiltonicity as a multicommodity flow problem, we have a commodity for every pair of vertices. Show that this is excessive and that one needs to consider only $n - 1$ commodities (flowing between a selected vertex and every other vertex).

## 2.6 Notes

The material on fractional perfect matchings is based on the work of Balas, Balinski, Bourjolly, Lovász, Plummer, Pulleyblank, and Uhry. See [8], [9], [28], [29], [124], [147], [148], and [181].

In [37], Chung *et al.* show that all denominators are possible for the fractional transversal number of a hypergraph of rank 3. They go on to explore, for each integer $r > 2$, the set of rational numbers that can be the fractional transversal number of a rank $r$ hypergraph. A complete characterization of these sets remains open.

The fractional Gallai theorem (Theorem 2.2.8) was first noticed by Grinstead and Slater; see [79], [166].

Aharoni and Ziv [1], [2] have studied matchings and fractional matchings in infinite graphs and hypergraphs. In particular, they show that Theorems 2.1.5 and 2.1.6 hold for infinite graphs as well.

The construction in the proof of Proposition 2.3.5 is based on an example due to Leslie Hall. The 2-tough conjecture is due to Chvátal [39]. See also Enomoto, Jackson, Katerinis, and Saito [51].

The middle levels problem has been attributed to various researchers. See, for example Dejter [42]. Unable to find Hamiltonian cycles in the middle levels graph, some researchers try to find long cycles; see the work of Savage [157], and Savage and Winkler [158]. An alternative approach is to consider levels that are widely separated. Hurlbert [97] shows that if $n$ is large enough (on the

order of $k^2$) then the graph $G(n; k, n-k)$ is Hamiltonian. See also Duffus, Sands, and Woodrow [47] as well as Kierstead and Trotter [106].

Exercise 5 on page 27 is based on Fulkerson, Hoffman, and McAndrew's paper [69]. See also Berge [16], page 27.

# 3

# Fractional Coloring

## 3.1 Definitions

The most celebrated invariant in all of graph theory is the *chromatic number*. Recall that an *$n$-coloring* of a graph $G$ is an assignment of one of $n$ colors to each vertex so that adjacent vertices receive different colors. The *chromatic number* of $G$, denoted $\chi(G)$, is the least $n$ for which $G$ has an $n$-coloring.

The *fractional chromatic number* is defined as follows. A *$b$-fold coloring* of a graph $G$ assigns to each vertex of $G$ a set of $b$ colors so that adjacent vertices receive disjoint sets of colors. We say that $G$ is *$a$:$b$-colorable* if it has a $b$-fold coloring in which the colors are drawn from a palette of $a$ colors. We sometimes refer to such a coloring as an $a$:$b$-coloring. The least $a$ for which $G$ has a $b$-fold coloring is the *$b$-fold chromatic number* of $G$, denoted $\chi_b(G)$. Note that $\chi_1(G) = \chi(G)$.

Since $\chi_{a+b}(G) \le \chi_a(G) + \chi_b(G)$, we define the *fractional chromatic number* to be

$$\chi_f(G) = \lim_{b\to\infty} \frac{\chi_b(G)}{b} = \inf_b \frac{\chi_b(G)}{b}.$$

(See §A.4 on page 137.)

We can also use the methods of Chapter 1 to describe the fractional chromatic number. Associate with a graph $G$ its *vertex-independent set hypergraph* $\mathcal{H}$ defined as follows. The vertices of $\mathcal{H}$ are just the vertices of $G$, while the hyperedges of $\mathcal{H}$ are the independent sets of $G$. Then $k(\mathcal{H})$ is the minimum number of independent sets of vertices needed to cover $V(G)$. Since a subset of an independent set is again independent, this is equivalent to the minimum number of independent sets needed to *partition* $V(G)$, i.e., the chromatic number. (See also exercise 7 on page 13.) Thus $k(\mathcal{H}) = \chi(G)$. Furthermore, the $b$-fold chromatic number of $G$ is just the $b$-fold covering number of $\mathcal{H}$ and so $\chi_f(G) = k_f(\mathcal{H})$.

We know from Corollary 1.3.1 on page 4 that $\chi_f(G)$ is a rational number and from Corollary 1.3.2 that there is a $b$ so that $\chi_f(G) = \chi_b(G)/b$. If $G$ has no edges, then $\chi_f(G) = 1$. Otherwise, $\chi_f(G) \ge 2$.

**Proposition 3.1.1** *For any graph $G$, $\chi_f(G) \ge \nu(G)/\alpha(G)$. Furthermore, if $G$ is vertex-transitive, then equality holds.*

**Proof**. Immediate from Proposition 1.3.4 on page 5. □

What is the dual notion? There is a natural interpretation of $p_f(\mathcal{H})$ (where $\mathcal{H}$ is the vertex-independent set incidence hypergraph of $G$). Note that $p(\mathcal{H})$ is the maximum number of vertices no two of which are together in an independent set. Stated another way, $p(\mathcal{H})$ is the maximum size

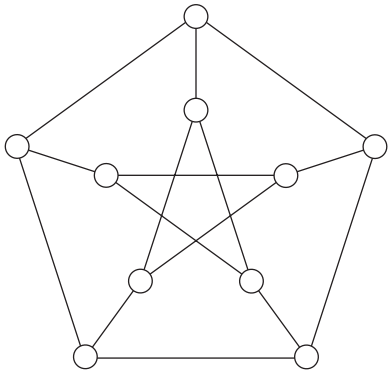

Figure 3.1. The Petersen graph $K_{5:2}$.

of a clique, so $p(\mathcal{H}) = \omega(G)$. Thus $p_f(\mathcal{H})$ is the *fractional clique number* of $G$, denoted $\omega_f(G)$. By duality, $\chi_f(G) = \omega_f(G)$.

The fractional clique number can also be defined as follows. For a graph $G$ and positive integers $a, b$, we call a multiset of vertices $K$ an $a{:}\,b$-*clique* if $|K| = a$ and if for every independent set of vertices $S$ the number of vertices of $K \cap (aS)$ (counting multiplicities) is at most $b$. The $b$-fold clique number of $G$, denoted $\omega_b(G)$, is the largest $a$ such that $G$ has an $a{:}\,b$-clique, and then

$$\omega_f(G) = \lim_{b\to\infty} \frac{\omega_b(G)}{b} = \sup_b \frac{\omega_b(G)}{b}.$$

A graph $G$ is called *perfect* if $\chi(H) = \omega(H)$ for all induced subgraphs of $H$. Perfection becomes trivial in fractional graph theory: for all graphs $G$, $\chi_f(H) = \omega_f(H)$ for all induced subgraphs $H$ of $G$! See also exercise 9 on page 54.

**Proposition 3.1.2** $\chi_f(C_{2m+1}) = 2 + (1/m)$

**Proof**. Since cycles are vertex-transitive and $\alpha(C_{2m+1}) = m$, the result follows from Proposition 3.1.1 on the facing page. □

## 3.2 Homomorphisms and the Kneser graphs

Given positive integers $a$ and $b$, define a graph $K_{a:b}$ as follows: the vertices are the $b$-element subsets of a fixed $a$-element set. There is an edge between two of these vertices if they are disjoint sets. The graphs $K_{a:b}$ are known as the *Kneser graphs* and they play a central role in the theory of the fractional chromatic number. As an illustration, $K_{5:2}$ is pictured in Figure 3.1.

We restrict attention to the case where $a \geq 2b$, since otherwise $K_{a:b}$ has no edges. Note that $K_{a:1} = K_a$, the complete graph on $a$ vertices. The graph $K_{a:2}$ is just $\overline{L(K_a)}$, the complement of the line graph of the complete graph on $a$ vertices.

Suppose that $G$ and $H$ are graphs. A *homomorphism* $\phi : G \to H$ is a mapping from $V(G)$ to $V(H)$ such that $\phi(v)\phi(w) \in E(H)$ whenever $vw \in E(G)$.

There is a simple connection between graph coloring and graph homomorphisms, namely, a graph $G$ is $n$-colorable if and only if there is a homomorphism $\phi{:}\, G \to K_n$.

This easily generalizes to $a{:}\,b$-colorings and motivates our study of Kneser graphs.

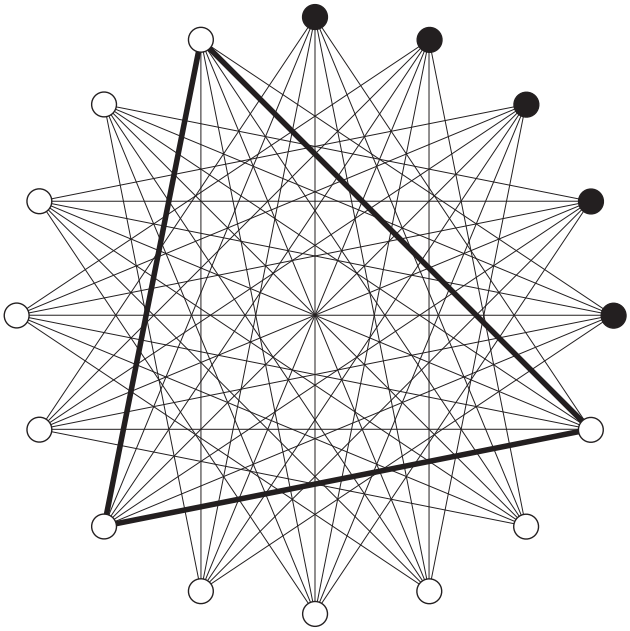

Figure 3.2. The graph $G_{16,5}$ from Proposition 3.2.2. Note that $\alpha(G_{16,5}) = 5$ (black vertices) and $\omega(G_{16,5}) = \lfloor 16/5 \rfloor = 3$ (thick edges). See also exercise 1 on page 54.

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**Proposition 3.2.1** *A graph $G$ has an $a{:}\,b$-coloring if and only if there is a graph homomorphism $\phi : G \to K_{a:b}$.* □

We know that $\chi_f$ can take only rational values and that $\chi_f(G) = 0$, $\chi_f(G) = 1$, or $\chi_f(G) \geq 2$ (see exercises 2 and 3 on page 54). In fact, all such values are actually achieved.

**Proposition 3.2.2** *Let $a$ and $b$ be positive integers with $a \geq 2b$. Let $G_{a,b}$ be the graph with vertex set $V(G) = \{0, 1, \ldots, a-1\}$. The neighbors of vertex $v$ are $\{v+b, v+b+1, \ldots, v+a-b\}$ with addition modulo $a$.*

*Then $\chi_f(G_{a,b}) = a/b$ and $\chi_b(G_{a,b}) = a$.*

Think of the vertices of $G_{a,b}$ as equally spaced points around a circle, with an edge between two vertices if they are not too near each other; see Figure 3.2.

**Proof**. Note that $G_{a,b}$ has $a$ vertices and is vertex-transitive. The maximum independent set of $G_{a,b}$ are all those of the form $\{i+1, i+2, \ldots, i+b\}$, so $\alpha(G_{a,b}) = b$ (exercise 1 on page 54). By Proposition 3.1.1 on page 30, $\chi_f(G_{a,b}) = a/b$. This implies $\chi_b(G) \geq a$ so it remains to exhibit an $a{:}\,b$-coloring of $G_{a,b}$. Let the colors be $\{0, 1, \ldots, a-1\}$ and assign to vertex $v$ the color set $\{v, v+1, \ldots, v+b-1\}$ with addition modulo $a$. It is easy to check that if $vw \in E(G_{a,b})$ then the color sets assigned to $v$ and $w$ are disjoint. □

Another graph whose fractional chromatic number is $a/b$ is the Kneser graph $K_{a:b}$. There is a close connection between this fact and the following well-known theorem [55] of extremal set theory, which has a simple phrasing in terms of the independence number of the Kneser graphs.

**Theorem 3.2.3 (Erdős-Ko-Rado)** *If $a$ and $b$ are positive integers with $a > 2b$, then*

$$\alpha(K_{a:b}) = \binom{a-1}{b-1}.$$

We prove this theorem below and, in fact, show that it is equivalent to the following proposition.

**Proposition 3.2.4** $\chi_f(K_{a:b}) = a/b$.

We provide two proofs, both very short. The first proof uses the Erdős-Ko-Rado theorem 3.2.3. The second uses composition of graph homomorphisms and is independent of Theorem 3.2.3. We may thus use Proposition 3.2.4 to prove the Erdős-Ko-Rado theorem without circularity.

**Proof 1**. The Kneser graph $K_{a:b}$ has $\binom{a}{b}$ vertices and is vertex-transitive. By Theorem 3.2.3, $\alpha(K_{a:b}) = \binom{a-1}{b-1}$. Hence, by Proposition 3.1.1

$$\chi_f(K_{a:b}) = \binom{a}{b} \Big/ \binom{a-1}{b-1} = a/b. \qquad \square$$

**Proof 2**. Let $G_{a,b}$ be the graph of Proposition 3.2.2 on the facing page. That result tells us that there is a homomorphism $\phi : G_{a,b} \to K_{a:b}$. Suppose that $K_{a:b}$ has a $c\colon d$-coloring; in other words, suppose that there is a homomorphism $\psi : K_{a:b} \to K_{c:d}$. Then $\psi \circ \phi$ would be a homomorphism from $G_{a,b}$ to $K_{c:d}$, so $G_{a,b}$ would have a $c\colon d$-coloring. Hence $\frac{c}{d} \geq \chi_f(G_{a,b}) = \frac{a}{b}$. We conclude that the natural $a\colon b$-coloring of $K_{a:b}$ (i.e., the identity coloring) is optimal. $\square$

As promised, we now use Proposition 3.2.4 to give a proof of the Erdős-Ko-Rado theorem.

**Proof** (of Theorem 3.2.3). Let $S \subseteq V(K_{a:b})$ be the set of vertices that (considered as sets) contain the number $a$. There are exactly $\binom{a-1}{b-1}$ elements in $S$, one for every subset of $[a-1]$ of size $b-1$. It is clear that $S$ is an independent set, since all elements of $S$ contain $a$ and so no two can be disjoint. Hence $\alpha(K_{a:b}) \geq \binom{a-1}{b-1}$.

To see that one can do no better than the set $S$, suppose that there is an independent set $T$ of vertices in $K_{a:b}$ of cardinality $c$. Let the full symmetric group on $[a]$ act on $T$, permuting the elements of $[a]$, to obtain $a!$ independent sets (counting multiplicities) of cardinality $c$. By symmetry, every vertex of $K_{a:b}$ is in the same number $t$ of these independent sets $\pi(T)$. If we count in two ways the number of pairs $(v, \pi)$ with $v \in \pi(T)$ we have $a! \cdot c = \binom{a}{b} \cdot t$. Thus, assigning one color to each of these independent sets gives a $t$-fold coloring of $K_{a:b}$ using $a!$ colors. By Proposition 3.2.4,

$$\frac{a!}{t} = \frac{\binom{a}{b}}{c} \geq \chi_f(K_{a:b}) = \frac{a}{b}.$$

Simplifying this expression yields $|T| = c \leq \binom{a-1}{b-1} = |S|$. Hence $S$ is maximal and the theorem is proved. $\square$

The fact that homomorphisms can be composed with one another is used again profitably in the following proposition.

**Proposition 3.2.5** *If $K_{a:b}$ has a $c\colon d$-coloring, then every graph that has an $a\colon b$-coloring also has a $c\colon d$-coloring.*

**Proof**. Suppose that $K_{a:b}$ has a $c\colon d$-coloring. Then there is a homomorphism $\phi : K_{a:b} \to K_{c:d}$. Suppose also that $G$ has an $a\colon b$-coloring. Then there is a homomorphism $\psi : G \to K_{a:b}$. But then $\phi \circ \psi$ is a homomorphism from $G$ to $K_{c:d}$, and the conclusion follows. $\square$

It would be nice to be able to characterize those integers $a, b, c, d$ that make the following assertion true: Whenever a graph $G$ has an $a{:}\,b$-coloring, it must have a $c{:}\,d$-coloring. Because of Proposition 3.2.5, the assertion is true exactly when $K_{a:b}$ has a $c{:}\,d$-coloring. This occurs exactly when $\chi_d(K_{a:b}) \geq c$. Unfortunately, the values of $\chi_d(K_{a:b})$ are not known for all $a, b, d$. Even the case $d = 1$ is difficult, Kneser having offered a conjecture in 1955 [108] that was not proved until 1977 by Lovász [122].

**Theorem 3.2.6** $\chi(K_{a:b}) = a - 2b + 2$. □

Lovász's proof is deep and we omit it. A short proof by Bárány appears in [10].

The following result of Stahl [171] allows us to extend Theorem 3.2.6.

**Lemma 3.2.7** *Given $c > 2d > 0$, there is a homomorphism mapping $K_{c:d}$ to $K_{(c-2):(d-1)}$.*

**Proof**. Let $S = \{s_1, s_2, \ldots, s_d\}$ be a vertex of $K_{c:d}$ with $s_1 < s_2 < \cdots < s_d$. We define $\phi(S)$ as follows. First, remove $s_d$ from $S$. If the resulting $d-1$ element set has all its entries less than $c-1$, we call that set $\phi(S)$. Otherwise, suppose that $s_{d-1} = c-1$ and suppose that $k$ is the largest integer such that $s_{d-k} = c-k$. Then subtract 1 from $s_{d-j}$ for each $j$ with $1 \leq j \leq k$ to give $\phi(S)$.

Clearly $\phi(S)$ is a $(d-1)$-element subset of $[c-2]$. The reader is left to verify (exercise 6 on page 54) that $\phi$ is a homomorphism. □

**Theorem 3.2.8** *If $1 \leq m \leq b$, then $\chi_m(K_{a:b}) = a - 2b + 2m$.*

**Proof**. First note that the result holds when $m = 1$ by Theorem 3.2.6 and when $m = b$ by Proposition 3.2.2. Define a sequence $s_2, s_3, \ldots, s_b$ by $s_d = \chi_d(K_{a:b}) - \chi_{d-1}(K_{a:b})$. We now show that $s_d \geq 2$ for every $d$ with $2 \leq d \leq b$. To see this, suppose that $\chi_d(K_{a:b}) = c$. Then there is a homomorphism $\psi : K_{a:b} \to K_{c:d}$. Compose $\psi$ with the homomorphism $\phi : K_{c:d} \to K_{(c-2):(d-1)}$ given by Lemma 3.2.7 to obtain a homomorphism from $K_{a:b}$ to $K_{(c-2):(d-1)}$, whose existence guarantees that $\chi_{d-1}(K_{a:b}) \leq c-2$. This simplifies to $s_d \geq 2$.

The sum of $s_d$ over all $d$ telescopes and yields

$$\sum_{d=2}^{b} s_d = \chi_b(K_{a:b}) - \chi(K_{a:b}) = a - (a - 2b + 2) = 2(b-1).$$

Were even one of the numbers $s_d$ greater than 2 for some $d$ with $2 \leq d \leq b$, this sum would be greater than $\sum_{d=2}^{b} 2 = 2(b-1)$, and we would obtain a contradiction. Hence $s_d = 2$ for every $d$ with $2 \leq d \leq b$ and the result follows. □

Stahl has conjectured that the formula given in the statement of Theorem 3.2.8 holds for all values of $m$.

## 3.3 The duality gap

While a linear program and its dual must have the same value, the same is not true for integer programs. Somewhere in this gap lies the common value of the linear relaxations of an IP and its dual. As an example, consider the fact that the clique number and the chromatic number of a graph need not be the same. Somewhere between these two numbers lies the fractional chromatic number. In this section, we explore examples that illustrate the various possibilities for the relative positions of these quantities.

When $\chi(G) = \omega(G)$, as is the case for perfect graphs, then obviously $\chi_f(G) = \chi(G)$. When $\chi(G) > \omega(G)$, many possibilities arise.

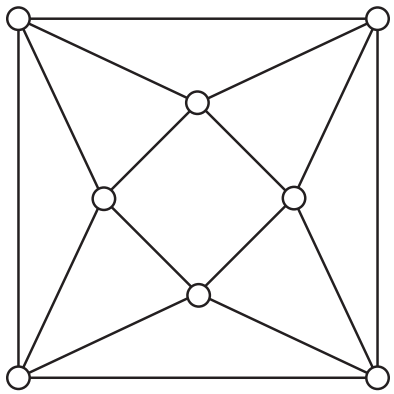

Figure 3.3. A graph with $[\omega, \chi] = [3, 4]$ and with $\chi_f = 4$.

**Example 3.3.1** Choose integers $n \geq 2$ and $b \geq 2$. Let $G$ be $K_{nb:b}$. Then $\omega(G) = \chi_f(G) = n$, but $\chi(G) = (n-2)b + 2$ by Theorem 3.2.6. This is an instance where the fractional chromatic number is at the left endpoint of the interval $[\omega(G), \chi(G)]$. By choosing $n$ large, one obtains an example where the gap between $\omega(G)$ and $\chi(G)$ is as large as one pleases. Indeed, for $n = 3$ this gives examples in which $\chi_f$ is bounded but $\chi$ is arbitrarily large.

**Example 3.3.2** Let $G$ be the graph pictured in Figure 3.3.

Note that $\omega(G) = 3$, but that $\alpha(G) = 2$, so $\chi_f(G) \geq 8/2 = 4$. It is just as easy to see that $\chi(G) = 4$ and so we have an example where $\chi_f(G)$ is at the right endpoint of the interval $[\omega(G), \chi(G)]$. Moreover, if we let $H = G[K_n]$, the lexicographic product[1] of $G$ and $K_n$, then $\omega(H) = 3n$, $\chi_f(H) = \chi(H) = 4n$. Hence we get such an example with a gap between $\omega(G)$ and $\chi(G)$ as large as we want.

One of the well-known ways of producing examples of graphs $G$ with a large gap between $\omega(G)$ and $\chi(G)$ is by iterating the following graph function—an idea due to Mycielski [136]. Given a graph $G$, define the graph $Y(G)$ as follows: $V(Y(G)) = (V(G) \times \{1, 2\}) \cup \{z\}$ and with an edge between two vertices of $Y(G)$ if and only if

(1) one of them is $z$ and the other is $(v, 2)$ for some $v \in V(G)$, or

(2) one of them is $(v, 1)$ and the other is $(w, 1)$ where $vw \in E(G)$, or

(3) one of them is $(v, 1)$ and the other is $(w, 2)$ where $vw \in E(G)$.

Figure 3.4 on the next page shows $Y(C_5)$, also known as the Grötzsch graph.

This function $Y$ preserves the clique number of a graph but increases the chromatic number. Mycielski [136] proved the following.

**Theorem 3.3.3** *If $G$ is a graph with at least one edge, then $\omega(Y(G)) = \omega(G)$ and $\chi(Y(G)) = \chi(G) + 1$.* □

[1]The lexicographic product of graphs $G$ and $H$, denoted $G[H]$, is defined to have vertex set $V(G) \times V(H)$ with an edge $(v_1, w_1)(v_2, w_2)$ if and only if $v_1v_2 \in E(G)$ or both $v_1 = v_2$ and $w_1w_2 \in E(H)$. Informally, we substitute a copy of $H$ for each vertex of $G$. If $vw \in E(G)$, then every vertex in $v$'s copy of $H$ is adjacent to every vertex in $w$'s copy.

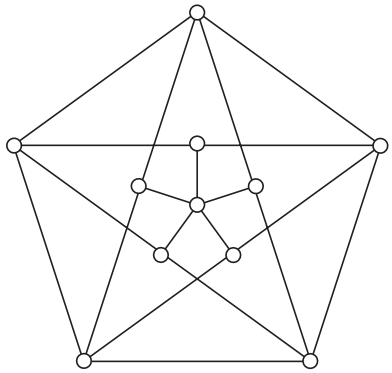

Figure 3.4. The Grötzsch graph $Y(C_5)$. Its fractional chromatic number is $\frac{5}{2} + \frac{2}{5} = \frac{29}{10}$.

It turns out that there is a simple but surprising formula relating $\chi_f(Y(G))$ to $\chi_f(G)$.

**Theorem 3.3.4** *If $G$ is any graph, then*

$$\chi_f(Y(G)) = \chi_f(G) + \frac{1}{\chi_f(G)}.$$

**Proof**. Suppose that $G$ has an $a{:}\,b$-coloring and that $\chi_f(G) = \frac{a}{b}$. To show that $\chi_f(Y(G)) \leq \chi_f(G)+1/\chi_f(G)$, it is enough to find an $(a^2+b^2){:}\,(ab)$-coloring of $Y(G)$. For the sake of definiteness, we use the numbers in $\{1, 2, \ldots, a\}$ to stand for the colors on $G$ and we write $C(v)$ for the set of colors assigned to $v$.

The vertices of $Y(G)$ are assigned sets of colors drawn from a palette of $a^2$ "ordered pair" colors and $b^2$ "new" colors. The former are all ordered pairs of the form $(i, j)$ with $1 \leq i, j \leq a$ and the latter are simply another $b^2$ additional colors, the set of which we denote $N$.

These $a^2 + b^2$ colors are assigned to vertices of $Y(G)$ as follows:

- $C'(v, 1) = \{(i, j) : i \in C(v),\ 1 \leq j \leq a\}$,
- $C'(v, 2) = \{(i, j) : i \in C(v),\ b < j \leq a\} \cup N$, and
- $C'(z) = \{(i, j) : 1 \leq i \leq a,\ 1 \leq j \leq b\}$.

Note that this assigns exactly $ab$ colors to each vertex, using $a^2$ ordered pair colors and $b^2$ new colors, a total of $a^2 + b^2$ colors. Since adjacent vertices of $Y(G)$ are assigned disjoint sets of colors by this scheme, we have our $(a^2 + b^2){:}\,(ab)$-coloring of $Y(G)$ and we have our upper bound.

To prove the opposite inequality, we consider fractional cliques. Suppose that $g : V(G) \to [0, 1]$ is a fractional clique on $G$ such that $\sum_{v \in V(G)} g(v) = \chi_f(G)$. We must manufacture a fractional clique $h$ on $H$ such that

$$\sum_{v \in V(Y(G))} h(v) = \frac{a^2 + b^2}{ab}.$$

Here is an $h$ that works: Let $h(v, 1) = \frac{a-b}{a} g(v)$, let $h(v, 2) = \frac{b}{a} g(v)$, and let $h(z) = \frac{b}{a}$. We now show that this is a fractional clique.

Let $I$ be an independent set of vertices in $H$. If $z \in I$, then clearly $I$ contains no vertices of the form $(v, 2)$, so

$$\sum_{x \in I} h(x) = h(z) + \sum_{(v,1) \in I} h(v, 1) = \frac{b}{a} + \frac{a-b}{a} \sum_{(v,1) \in I} g(v).$$

The sum in this last expression is no more than 1, since $g$ is a fractional clique on $G$. Hence

$$\sum_{x \in I} h(x) \leq \frac{b}{a} + \frac{a-b}{a} = 1.$$

If $z \notin I$, then $I = [J \times \{1\}] \cup [K \times \{2\}]$, where $J \subseteq V(G)$ is independent and $K \subseteq V(G)$ has no vertices adjacent to any vertices in $J$. We then compute

$$\sum_{x \in I} h(x) = \sum_{v \in J} h(v, 1) + \sum_{v \in K} h(v, 2) \tag{3.1}$$

$$= \sum_{v \in J} \frac{a-b}{a} g(v) + \sum_{v \in K} \frac{b}{a} g(v) \tag{3.2}$$

$$= \frac{a-b}{a} \sum_{v \in J} g(v) + \frac{b}{a} \sum_{v \in J \cap K} g(v) + \frac{b}{a} \sum_{v \in K - J} g(v) \tag{3.3}$$

$$\leq \sum_{v \in J} g(v) + \frac{b}{a} \sum_{v \in K - J} g(v) \tag{3.4}$$

Let $H = G[K - J]$. Suppose that $H$ has a $c$:$d$-coloring and that $\chi_f(H) = c/d$. Since $H$ is a subgraph of $G$,

$$\frac{c}{d} = \chi_f(H) \leq \chi_f(G) = \frac{a}{b}. \tag{3.5}$$

If $C_1, C_2, \ldots, C_c$ be the color classes of this coloring, then the sets of the form $J \cup C_i$ are independent sets of vertices of $G$. Hence

$$\sum_{v \in J \cup C_i} g(v) \leq 1$$

for $1 \leq i \leq c$. Summing this inequality over all $i$ yields

$$c \sum_{v \in J} g(v) + d \sum_{v \in K - J} g(v) \leq c.$$

Dividing through by $c$ gives

$$\sum_{v \in J} g(v) + \frac{d}{c} \sum_{v \in K - J} g(v) \leq 1,$$

which together with (3.1) through (3.4) and (3.5) yields

$$\sum_{x \in I} h(x) \leq 1.$$

Hence $h$ is a fractional clique and so $\chi_f(G)$ is at least as large as

$$\sum_{x \in V(H)} h(x) = \sum_{v \in V(G)} h(v, 1) + \sum_{v \in V(G)} h(v, 2) + h(z)$$

$$= \frac{a-b}{a} \sum_{v \in V(G)} g(v) + \frac{b}{a} \sum_{v \in V(G)} g(v) + \frac{b}{a}$$

$$= \sum_{v \in V(G)} g(v) + \frac{b}{a}$$

$$= \frac{a}{b} + \frac{b}{a} = \chi_f(G) + (1/\chi_f(G)).$$

This gives the reverse inequality and the theorem is proved. □

Theorem 3.3.4 tells us that the fractional chromatic number of the Grötzsch graph of Figure 3.4 is $\chi_f(C_5) + 1/\chi_f(C_5) = (5/2) + (2/5) = 29/10$, which is near the center of the interval $[\omega(Y(G)), \chi(Y(G))] = [2, 4]$. If $G_n$ is defined inductively by $G_1 = K_2$ and $G_n = Y(G_{n-1})$ for $n > 1$, then $\omega(G_n) = 2$ and $\chi(G_n) = n+1$ for all $n$, while $\chi_f(G_n)$ grows without bound and is asymptotic to $\sqrt{2n}$. (See exercise 11 on page 54.)

The graph $G_n$ also provides an example of a graph for which $\chi_f(G) = \chi_k(G)/k$ for no small $k$. In fact, $\nu(G_n) = 3 \cdot 2^{n-1} - 1$, while $\chi_f(G)$ is a fraction whose denominator, when written in smallest terms, is greater than $2^{2^{n-2}}$. Hence this denominator is greater than $2^{\nu(G)/6}$. (See exercise 12 on page 54.) This shows that there is no bound on this denominator that is a polynomial function of the number of vertices of $G$. This is an obstacle to finding a good algorithm for the computation of $\chi_f$. (See Theorem 3.9.2 on page 53.)

## 3.4 Graph products

Given two graphs $G$ and $H$, there are many natural ways to define a product graph on the vertex set $V(G) \times V(H)$. In this section, we first focus on the *disjunctive product* of $G$ and $H$, which we denote $G * H$ and define as follows: $V(G * H) = V(G) \times V(H)$ and $(v_1, w_1)(v_2, w_2) \in E(G * H)$ if and only if $v_1 v_2 \in E(G)$ or $w_1 w_2 \in E(H)$. The next proposition explains the special importance of this product.

**Proposition 3.4.1** *Given a graph $G$, we write $\mathcal{H}(G)$ for the hypergraph whose vertex set $S$ is $V(G)$ and whose hyperedges are the maximal independent sets of $G$. Given any two graphs $G_1$ and $G_2$,*

$$\mathcal{H}(G_1 * G_2) = \mathcal{H}(G_1) \times \mathcal{H}(G_2).$$

**Proof**. The vertex sets of the hypergraphs on the two sides of the equation are both $V(G_1) \times V(G_2)$. We must show that they have the same hyperedges. A hyperedge in $\mathcal{H}(G_1 * G_2)$ is a maximal independent set in $G_1 * G_2$. A hyperedge in $\mathcal{H}(G_1) \times \mathcal{H}(G_2)$ is a product $S_1 \times S_2$ of a maximal independent set $S_1$ from $G_1$ and a maximal independent set $S_2$ from $G_2$. We show that these are the same thing.

Let $T$ be a maximal independent subset of $G_1 * G_2$. For $i = 1, 2$, let $S_i$ be the projection of $T$ onto $G_i$; in other words,

$$S_1 = \{v_1 \in V(G_1) : v_1 v_2 \in T \text{ for some } v_2 \in V(G_2)\}$$

and similarly for $S_2$. Clearly $S_i$ is independent in $G_i$. Moreover $S_i$ must be maximal independent in $G_i$, since $T$ is maximal independent in the product graph. Also $S_1 \times S_2$ is independent in $G_1 * G_2$. But since $T \subseteq S_1 \times S_2$ is maximal independent by assumption, we must have $T = S_1 \times S_2$, and we are done. □

This result has the following corollaries.

**Corollary 3.4.2** *If $G_1$ and $G_2$ are any two graphs, then $\chi_f(G_1 * G_2) = \chi_f(G_1)\chi_f(G_2)$.*

**Proof**. We use Theorem 1.6.1 on page 8 and compute:

$$\begin{aligned}\chi_f(G_1 * G_2) &= k_f(\mathcal{H}(G_1 * G_2)) \\ &= k_f(\mathcal{H}(G_1) \times \mathcal{H}(G_2)) \\ &= k_f(\mathcal{H}(G_1))k_f(\mathcal{H}(G_2)) = \chi_f(G_1)\chi_f(G_2).\end{aligned}$$

□

**Corollary 3.4.3** *Let $G$ be a graph. For a positive integer $n$, write $G^n$ for the $n^{th}$ power of $G$ using the disjunctive product. Then*

$$\chi_f(G) = \lim_{n\to\infty} \sqrt[n]{\chi(G^n)} = \inf_n \sqrt[n]{\chi(G^n)}.$$

**Proof**. Immediate from Theorem 1.6.2 on page 9. □

Borrowing in a similar way from the theory of hypergraph coverings, we obtain the following result.

**Proposition 3.4.4** *If $G$ and $H$ are any two graphs, then*

$$\chi_f(G)\chi(H) \le \chi(G * H) \le \chi(G)\chi(H).$$

**Proof**. The first inequality follows from Lemma 1.6.3 on page 9. The second follows from the simple observation that colorings on $G$ and $H$ give rise to a product coloring on $G * H$. (An alternative argument is to apply the inequality $k(\mathcal{H} \times \mathcal{K}) \le k(\mathcal{H})k(\mathcal{K})$, which was presented on page 8.) □

There is a result analogous to Corollary 3.4.2 for the lexicographic product (for the definition, see the footnote on page 35).

**Corollary 3.4.5** *If $G_1$ and $G_2$ are any two graphs, then $\chi_f(G_1[G_2]) = \chi_f(G_1)\chi_f(G_2)$.*

**Proof**. Since $G_1[G_2]$ is a subgraph of $G_1 * G_2$, it is clear that $\chi_f(G_1[G_2]) \le \chi_f(G_1 * G_2)$, which is equal to $\chi_f(G_1)\chi_f(G_2)$ by Corollary 3.4.2. For the opposite inequality, suppose, for $i = 1, 2$, that $S_i$ is an $a_i{:}\, b_i$-clique on $G_i$ with $a_i/b_i$ as large as possible. We claim that the multiset $S = S_1 \times S_2$ is an $a_1a_2 : b_1b_2$-clique on $G_1[G_2]$. Clearly $|S| = a_1a_2$.

It remains to show that no independent set of vertices intersects $S$ in more than $b_1b_2$ points. What are the independent sets in $G_1[G_2]$? They are simply the sets $I$ such that $J = \{v : (v, w) \in I$ for some $w\}$ is independent in $G_1$ and, for each $v \in V(G_1)$, $L_v = \{w : (v, w) \in I\}$ is independent in $G_2$. Hence

$$\begin{aligned}|S \cap I| &= \left|(S_1 \times S_2) \cap \left(\bigcup_{v\in J} \{v\} \times L_v\right)\right| \\ &= \left|\bigcup_{v\in J} (S_1 \cap \{v\}) \times (S_2 \cap L_v)\right| \\ &\le |S_1 \cap J| \cdot b_2 \\ &\le b_1b_2,\end{aligned}$$

as was to be shown. □

Although the lexicographic product is not commutative, Corollary 3.4.5 shows that the graphs $G_1[G_2]$ and $G_2[G_1]$ have the same fractional chromatic number.

## A communication complexity story

Alice and Bob have a graph $G$ that they know well. Alice knows the name of a vertex $v$ in $G$ and Bob knows the name of an edge $e$. Alice does not know $e$ and Bob does not know $v$, but they do know that $v$ is an endpoint of $e$.

*Alice and Bob teach at a school with 250 students. Each day at recess exactly 200 of these students form two teams of size exactly 100 to play a game of Capture the Flag.*[2] *Bob is on duty for the first half of the recess, so he knows the students on each team. However, he goes to have his lunch before recess is over. Alice is the gym teacher and later that day she learns which team won. In other words, Bob knows an edge of the Kneser graph $K_{250:100}$ and Alice knows an endpoint of that edge.*

What is the shortest possible message (minimum number of bits) that Alice should send to Bob to tell him $v$? Certainly Alice may take the simple approach of communicating to Bob the name (number) of her vertex; this transmission requires $\lceil \lg |V(G)| \rceil$ bits. But is there a more efficient way?

*Later that evening, Alice calls Bob to tell him which team won, but wants to get off the phone with him as quickly as possible.*

The answer is that $\lceil \lg \chi(G) \rceil$ bits are necessary and sufficient. Alice and Bob can precolor the graph using as few colors as possible. Then, when Alice wants to inform Bob which vertex she knows, she need only tell him the *color* of her vertex; this is sufficient information for Bob (who knows an edge containing $v$) to deduce which vertex Alice knows. On the other hand, if fewer bits are communicated, then a pair of adjacent vertices must have the same label and if Bob should hold that edge, he will be unable to deduce which vertex Alice holds.

*By Theorem 3.2.6 on page 34, $\chi(K_{250:100}) = 250 - 200 + 2 = 52$ so 6 bits of information suffice.*

Now suppose that Alice knows a list of vertices $v_1, v_2, \ldots, v_n$ and Bob knows a list of edges $e_1, e_2, \ldots, e_n$ with $v_i \in e_i$. If Alice wishes to communicate her entire list of vertices to Bob she can do so in $n$ separate messages, for a total of $n\lceil \lg \chi(G) \rceil$ bits.

Let $G^n$ denote the $n^{\text{th}}$ power of $G$ using the disjunctive product. The list $(v_1, v_2, \ldots, v_n)$ of vertices held by Alice is a single vertex of $G^n$. The endpoints of the $n$ edges held by Bob correspond to the $2^n$ possible vertices of $G^n$ that Alice might hold. Note, however, if $(v_1, \ldots, v_n)$ and $(v'_1, \ldots, v'_n)$ are two different vertices that Alice might hold, then in some coordinate we have $v_i \sim v'_i$ in $G$. Therefore, the $2^n$ vertices that Alice might hold form a clique of $G^n$. For Alice to send Bob her list of vertices, it suffices for them to precolor $G^n$ and then for Alice to send the color of her vertex (list). By Corollary 3.4.3, $\chi(G^n) \approx \chi_f(G)^n$. Thus the number of bits to send the vertex list drops from $n \lg \chi(G)$ bits (using $n$ messages) to $n \lg \chi_f(G)$ bits (under this new protocol).

*To greatly reduce her phone time with Bob, Alice calls Bob at the end of the school year to report all the results. Instead of using 6 bits per game result, she can speed up to $\lg \chi_f(K_{250:100}) = \lg \frac{5}{2} \approx 1.32$ bits per game or an overall savings of over 75% in phone time with Bob!*

[2] In *Capture the Flag* large teams are desirable, but if the teams are too large the situation is completely unmanageable. Let us imagine that one hundred players per team is ideal.

## 3.5 The asymptotic chromatic and clique numbers

Given any graph $G$, we may define the *asymptotic chromatic number* of $G$ to be $\chi_\infty(G) = \inf \sqrt[n]{\chi(G^n)}$, where the power of $G$ is with respect to the disjunctive product (page 38) and the infimum is taken over all natural numbers $n$. By Corollary 3.4.3 on page 39 we know that $\chi_\infty(G) = \chi_f(G)$ for all graphs $G$ and that the infimum in the definition of $\chi_\infty$ is in fact a limit.

**Example 3.5.1** Let $G = C_5$. It is no easy matter to compute $\chi(G^n)$ even for small values of $n$. For $n = 1$, it is clearly 3. To calculate $\chi(G^2)$, we appeal to Proposition 3.4.4 with $H = G$ to obtain $\frac{15}{2} \leq \chi(G^2) \leq 9$. In fact, it is not hard to find a coloring of $G^2$ with 8 colors, and so $\chi(G^2) = 8$. A similar analysis for $n = 3$ yields $20 \leq \chi(G^3) \leq 24$, while ad hoc methods provide a coloring of $G^3$ with 21 colors. Hence the sequence $\{\sqrt[n]{\chi(G^n)}\}$ begins $3, \sqrt{8}, \sqrt[3]{20}$ or $\sqrt[3]{21}, \cdots$. There is no hint from these data that this sequence converges to $\chi_f(G) = 5/2$.

**Proposition 3.5.2** *A graph $G$ has the property that $\chi(G*H) = \chi(G)\chi(H)$ for all graphs $H$ if and only if $\chi(G) = \chi_f(G)$.*

**Proof**. If $G$ has the property, then putting $H = G^n$ leads to the conclusion that $\chi(G^n) = \chi(G)^n$ for all $n$. Hence $\sqrt[n]{\chi(G^n)} = \chi(G)$ for all $n$, and so $\chi_f(G) = \chi_\infty(G) = \chi(G)$. Conversely, if $\chi(G) = \chi_f(G)$, then Proposition 3.4.4 implies that $\chi(G)\chi(H)$ is both an upper and a lower bound for $\chi(G * H)$. □

The situation with the dual invariant is not so simple. Define the *asymptotic clique number* of a graph $G$ to be $\sup \sqrt[n]{\omega(G^n)}$. This equals the asymptotic packing number of the corresponding hypergraph, that is, $p_\infty(\mathcal{H}(G))$. The complementary invariant $\Theta(G) = \omega_\infty(\overline{G})$ has been studied by several authors (e.g., Lovász [123], Shannon [165]) and is known as the Shannon capacity of $G$. Given the parallelism between dual invariants, one might expect that $\omega_\infty$ would behave like $\chi_\infty$ and, in particular, that $\omega_\infty(G)$ would equal $\omega_f(G)$ for all $G$. In fact, this is not the case even for $C_5$, as Lovász discovered in 1979 [123].

**Theorem 3.5.3** *The asymptotic clique number of $C_5$ is $\sqrt{5}$.*

**Proof**. If we label the vertices of $C_5$ in cyclic order $v_1, v_2, v_3, v_4, v_5$, then a clique in $C_5^2$ is given by $(v_1, v_1), (v_2, v_3), (v_3, v_5), (v_4, v_2), (v_5, v_4)$. Hence $\omega(C_5^2) \geq 5$ (in fact, there is equality here), and so $\omega_\infty(C_5) = \sup \sqrt[n]{\omega(C_5^n)} \geq \sqrt{\omega(C_5^2)} = \sqrt{5}$.

To obtain the opposite inequality, we introduce a graph invariant $\eta(G)$ and show that $\omega_\infty(G) \leq \eta(G)$ for all $G$ and that $\eta(C_5) \leq \sqrt{5}$. An *orthonormal representation* of a graph $G$ is a mapping that associates to every $v \in V(G)$ a unit vector $\mathbf{v}$ in some Euclidean space so that $\mathbf{v}$ and $\mathbf{w}$ are orthogonal whenever $vw \in E(G)$. We define $\eta(G)$ to be the

$$\inf \left\{ \max_{v \in V(G)} (\mathbf{c} \cdot \mathbf{v})^{-2} \right\}$$

where the infimum is taken over all orthonormal representations of $G$ and all unit vectors $\mathbf{c}$. By compactness, it is clear that this infimum is actually achieved.

Our first observation about $\eta$ is that $\eta(G*H) \leq \eta(G)\eta(H)$. To see this, suppose that the infimum in the definition of $\eta(G)$ is achieved at the orthonormal representation $v \in V(G) \mapsto \mathbf{v}$ and the unit vector $\mathbf{c}$ and the infimum in the definition of $\eta(H)$ is achieved at $w \in V(H) \mapsto \mathbf{w}$ and $\mathbf{d}$. Then form

the tensor product of the two representations by considering the map $(v, w) \in V(G * H) \mapsto \mathbf{v} \otimes \mathbf{w}$, where $\mathbf{v} \otimes \mathbf{w}$ is the tensor product[3] of $\mathbf{v}$ and $\mathbf{w}$. Also form the tensor product of $\mathbf{c}$ and $\mathbf{d}$. Then

$$\begin{aligned}
\eta(G * H) &\leq \max_{(v,w)\in V(G*H)} ((\mathbf{c} \otimes \mathbf{d}) \cdot (\mathbf{v} \otimes \mathbf{w}))^{-2} \\
&= \max_{\substack{v\in V(G)\\ w\in V(H)}} \sum_{i,j} (c_i d_j v_i w_j)^{-2} \\
&= \max_{\substack{v\in V(G)\\ w\in V(H)}} \sum_{i} (c_i v_i)^{-2} \sum_{j} (d_j w_j)^{-2} \\
&= \max_{\substack{v\in V(G)\\ w\in V(H)}} (\mathbf{c} \cdot \mathbf{v})^{-2} (\mathbf{d} \cdot \mathbf{w})^{-2} \\
&= \eta(G)\eta(H).
\end{aligned}$$

The importance of $\eta$ is that it is at once submultiplicative and an upper bound for $\omega$. To show that $\omega(G) \leq \eta(G)$ for any $G$, suppose that $S \subseteq V(G)$ is a clique of vertices of size $\omega(G)$. Then for the orthonormal representation of $G$ that achieves the infimum in the definition of $\eta(G)$, we must have

$$\omega(G) = \frac{\omega(G)}{|\mathbf{c}|^2} \leq \frac{\omega(G)}{\sum_{v\in S} (\mathbf{c} \cdot \mathbf{v})^2} \leq \frac{1}{\min(\mathbf{c} \cdot \mathbf{v})^2} = \max(\mathbf{c} \cdot \mathbf{v})^{-2} = \eta(G).$$

Since $\omega(G^n) \leq \eta(G^n) \leq \eta(G)^n$, taking $n$th roots and limits yields the fact that $\omega_\infty(G) \leq \eta(G)$ for any graph $G$. All that remains is to compute $\eta(G)$ for the graph $G = C_5$. Let $\mathbf{c}$ be the unit vector in $\mathbf{R}^3$ that corresponds with the north pole $[0, 0, 1]^t$ and let $\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_5, \mathbf{v}_2, \mathbf{v}_4$ (in that order) be vectors whose heads are arranged on the upper unit hemisphere in $\mathbf{R}^3$ equally spaced around a latitude line, the latitude chosen so that the angle between $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$ is exactly $\pi/2$. (See Figure 3.5.) Since this is an orthonormal representation of $C_5$, we have that $\eta(C_5) \leq (\mathbf{c} \cdot \mathbf{v}_1)^{-2}$.

We are left with a problem in 3-dimensional geometry: calculate $\mathbf{c} \cdot \mathbf{v}_1$. The spherical law of cosines asserts that for any triangle on a sphere, with sides that sweep out angles $x, y, z$ from the center of the unit sphere and with angle $Z$ opposite the side with central angle $z$,

$$\cos z = \cos x \cos y + \sin x \sin y \cos Z.$$

Applying this to the triangle formed by $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{c}$ on the unit sphere, we substitute $4\pi/5$ for $Z$, $\pi/2$ for $z$, and $\mathbf{c} \cdot \mathbf{v}_1$ for both $\cos x$ and $\cos y$. This yields

$$0 = (\mathbf{c} \cdot \mathbf{v}_1)^2 + \left(1 - (\mathbf{c} \cdot \mathbf{v}_1)^2\right) \cos(4\pi/5).$$

Since $\cos(4\pi/5) = -(1 + \sqrt{5})/4$, we may solve for $(\mathbf{c} \cdot \mathbf{v}_1)^2$ obtaining

$$(\mathbf{c} \cdot \mathbf{v}_1)^2 = \frac{(1 + \sqrt{5})/4}{1 + (1 + \sqrt{5})/4} = \frac{1}{\sqrt{5}}.$$

Hence $(\mathbf{c} \cdot \mathbf{v}_1)^{-2} = \sqrt{5}$.

In summary, $\omega_\infty(C_5) \leq \eta(C_5) = \sqrt{5}$, and the theorem is proved. □

Thus, in the chain

$$\omega(G) \leq \omega_\infty(G) \leq \omega_f(G) = \chi_f(G) = \chi_\infty(G) \leq \chi(G),$$

all the inequalities can be strict.

[3]The *tensor product* of two vectors $(v_1, \ldots, v_m)$ and $(w_1, \ldots, w_n)$ is the vector in $\mathbf{R}^{mn}$ whose entries are indexed by ordered pairs $(i, j)$ with $1 \leq i \leq m$ and $1 \leq j \leq n$ and whose $(i, j)$th entry is simply $v_i w_j$.

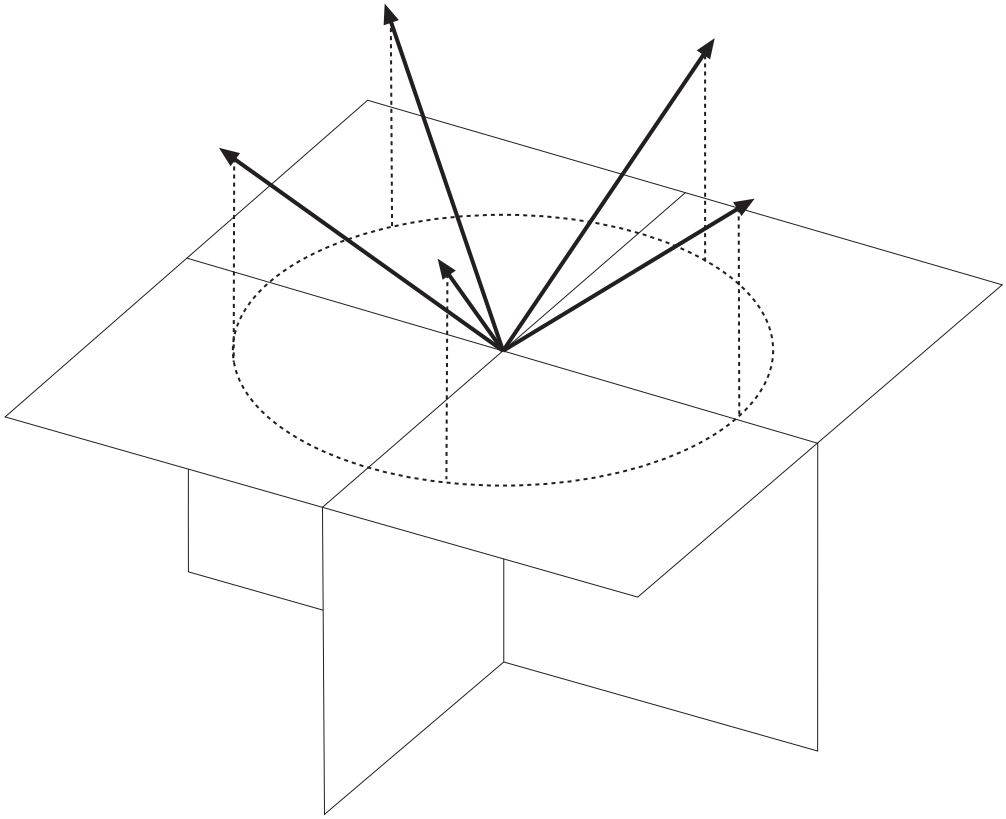

Figure 3.5. An umbrella of vectors.

## 3.6 The fractional chromatic number of the plane

In this section, we depart from the world of finite graphs to consider one infinite graph, the unit distance graph in the plane. This is the graph whose vertex set is the (uncountably infinite) set of points in the plane $\mathbf{R}^2$, with an edge between two points if and only if the points are exactly distance one from each other in the usual Euclidean metric. It is descriptive to abuse language and notation and call this graph "the plane" and denote it by $\mathbf{R}^2$. A *unit graph* is any subgraph of $\mathbf{R}^2$.

Much has been written about the chromatic number of $\mathbf{R}^2$ (see, e.g., Soifer [169]), a long-standing open problem attributed to Edward Nelson. The problem may be rephrased without the language of graph theory as follows: What is the smallest number of colors needed to color the plane if no two points of the same color may be exactly one unit apart? It can certainly be done with 7 colors owing to the coloring exhibited in Figure 3.6, while 3 colors is not enough even to color the 7 points which form the vertices of the unit graph in Figure 3.7, known as the spindle.

These examples show that $4 \leq \chi(\mathbf{R}^2) \leq 7$. What is the exact value? Despite the popularization of this question over the last 40 years, no further work has improved these bounds.

In this section, we consider the fractional chromatic number of the plane, and show that $3.555 \leq \chi_f(\mathbf{R}^2) \leq 4.36$.

**Theorem 3.6.1** $\chi_f(\mathbf{R}^2) \leq 4.36$.

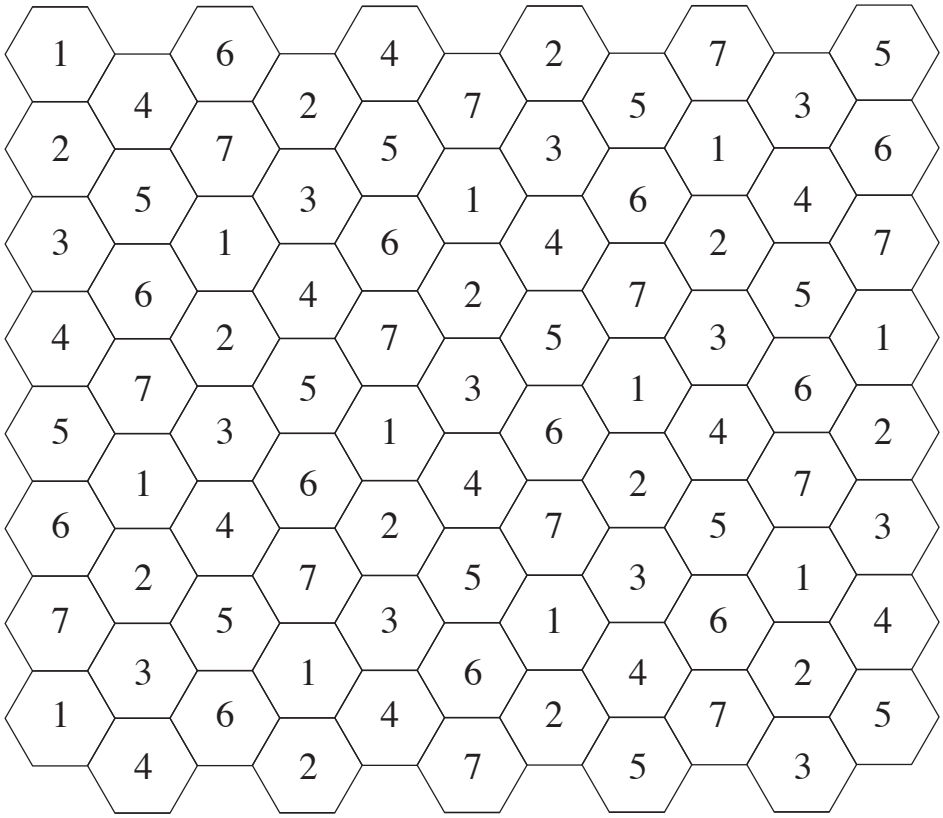


Figure 3.6. A 7-coloring of the plane.

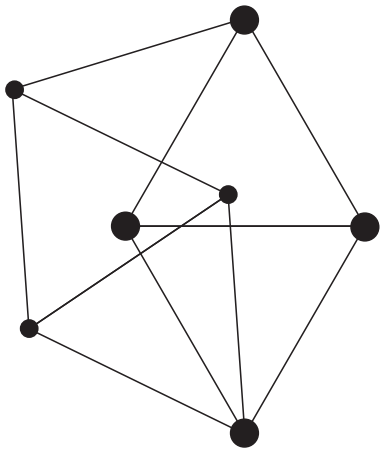

Figure 3.7. A unit graph with chromatic number 4.

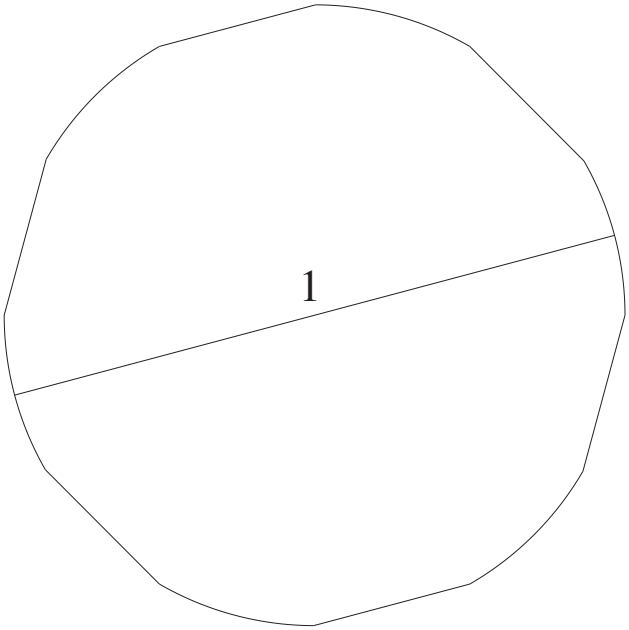


Figure 3.8. A set in the plane avoiding unit distance.

**Proof**. The key here is to find a set in the plane with density as large as possible while avoiding pairs of points at unit distance. The densest independent set in $\mathbf{R}^2$ that is known is the set $S$ which we now describe: Let $A$ be the open convex set pictured in Figure 3.8. The set $A$ may be obtained from an open disk of diameter one by removing 6 small circular segments, each of the same size and equally spaced around the circle, in such a way that the perimeter of $A$ is exactly 50% circular arcs and 50% segments. (The angle $\theta$ swept out by each circular arc of $A$ is the solution of $\theta = 2\sin\left(\frac{\pi}{6} - \frac{\theta}{2}\right)$.) Now frame $A$ in a hexagon whose sides are parallel to the segments of $A$ and whose boundary is exactly 1/2 unit from $A$. Then form $S$ by tiling $\mathbf{R}^2$ with these hexagons, letting $S$ be the union of the translates of $A$ that appear in this tiling. This is pictured in Figure 3.9 on the next page.

The density of $S$ in the plane may be computed numerically, and turns out to be $d = 0.229365$ to six decimal places. (See exercise 15 on page 54.)

To complete the proof, we illustrate why the set $S$ of density $d$ yields an upper bound for $\chi_f(\mathbf{R}^2)$ of $1/d$. For the sake of definiteness, coordinatize the plane by assuming that neighboring components of $S$ are centered at $(0,0)$ and $(s,0)$. (This defines $s$ and determines the center of all the components of $S$.) Fix a positive integer $n$ and, for $0 \le i, j < n$, let

$$S_{i,j} = S + (si/n)(1,0) + (sj/n)(1/2, \sqrt{3}/2),$$

a translate of $S$. Color the plane with $n^2$ colors, with one color class for each of the sets $S_{i,j}$.

How many colors does this assign to an arbitrary point $(x,y)$ in the plane? We have

$$(x,y) \in S_{i,j} \iff (x,y) - (si/n)(1,0) - (sj/n)(1/2, \sqrt{3}/2) \in S.$$

Let $H_{i,j}$ be the hexagon of width $s/n$ centered at

$$(si/n)(1,0) + (sj/n)(1/2, \sqrt{3}/2)$$

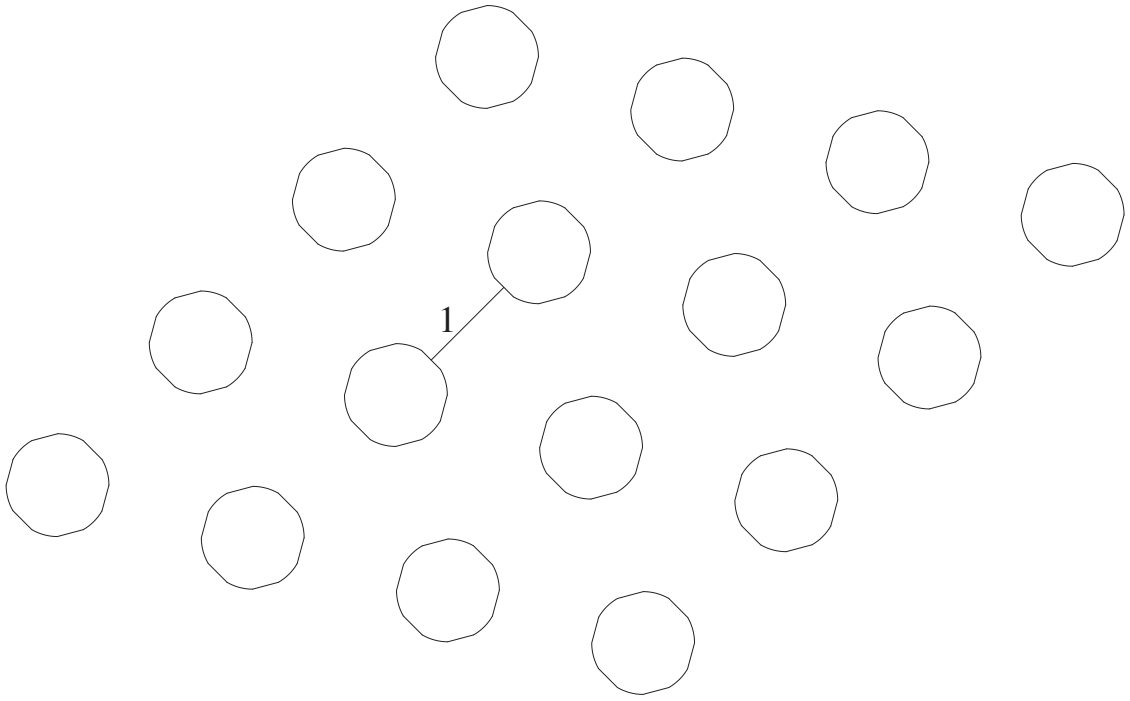


Figure 3.9. A periodic set in the plane avoiding unit distance.

with two sides parallel to the $y$-axis. Such hexagons tile the plane and have area $(\sqrt{3}/2)(s/n)^2$. Moreover, the number of ordered pairs $(i,j)$ satisfying $(x,y) \in S_{i,j}$ is at least the number of hexagons $H_{i,j}$ that lie entirely within $S-(x,y)$. Let $A_n$ be the set of points $p$ in the plane such that the disk centered at $p$ of radius $1/n$ lies entirely within $A$. Considerations of area show that the number of pairs $(i,j)$ with $(x,y) \in S_{i,j}$ is at least as large as the area of $A_n$ divided by the area $(\sqrt{3}/2)(s/n)^2$ of a hexagon $H_{i,j}$. Thus, this coloring assigns at least $2n^2\,\mathrm{Area}(A_n)/(\sqrt{3}s^2)$ colors to each point. Now the density of $S$ is given by $\mathrm{Area}(A)/((\sqrt{3}/2)s^2)$ and $\mathrm{Area}(A_n)$ clearly approaches $\mathrm{Area}(A)$ as $n$ gets large. Hence, when $n$ is large, this gives a coloring with $n^2$ colors and with each point in the plane assigned nearly $n^2 d$ colors. Hence the fractional chromatic number of the plane is at most $n^2/(n^2 d) = 1/d$.

Since we have an independent set of density 0.229365, we obtain an upper bound for $\chi_f(\mathbf{R}^2)$ of $1/0.229365 = 4.35987$ to six decimal places. □

To obtain lower bounds for $\chi_f(\mathbf{R}^2)$, one looks for finite unit graphs with as large a fractional chromatic number as possible. It is easy to show that the fractional chromatic number of the spindle of Figure 3.7 on page 44 is 7/2 and so $\chi_f(\mathbf{R}^2) \geq 7/2$. In the following theorem, the lower bound of 7/2 is slightly improved.

**Theorem 3.6.2** $\chi_f(\mathbf{R}^2) \geq 32/9$.

**Proof**. We describe a subgraph $G$ of $\mathbf{R}^2$ with 57 vertices and 198 edges that has fractional chromatic number equal to 32/9. Since a subgraph of $\mathbf{R}^2$ can only have smaller fractional chromatic number than does $\mathbf{R}^2$, this gives the desired result. In fact, to obtain the theorem it is sufficient to prove

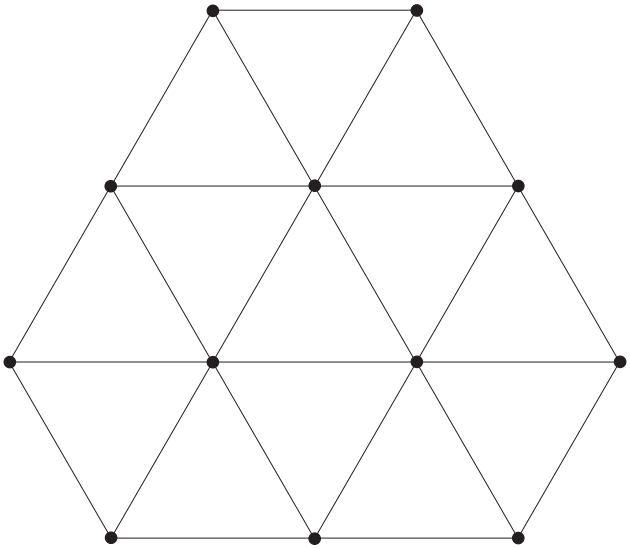

Figure 3.10. The core of $G$.

only that $\chi_f(G) \geq 32/9$. To do this, it is sufficient to provide a 96: 27-clique for $G$. This is what we do below.

Although the graph $G$ can be precisely described by giving the coordinates of the 57 vertices, the most readily absorbed description of $G$ is through pictures. We choose the latter approach so that we can provide a readable, self-contained proof that $\chi_f(G) \geq 32/9$. We build $G$ in stages. First give $G$ the 12 vertices and 24 edges pictured in Figure 3.10.

To this core we add 45 vertices, 3 vertices in each of 15 gadgets that we call spindles. A spindle is the graph with 7 vertices and 11 edges pictured in Figure 3.7 on page 44. Note that four of the vertices in Figure 3.7 are drawn slightly larger than the other three. These vertices are to be identified with vertices from the core as follows. Take five copies of the spindle as drawn in Figure 3.7. Without rotating them, translate them over to the core until the four larger vertices of the spindle lie over four vertices of the core. There are just five ways to do this. The three smaller vertices in each of the five spindles describe 15 vertices to be added to $G$ along with 30 new edges in the spindles and 21 new edges formed between new vertices in adjacent spindles. We have added 15 vertices and 51 edges to the core. Now rotate the core 120° and do the same thing, adding another 15 vertices and 51 edges. Then rotate 120° once again and do the same thing a third and last time. The resulting graph $G$ has $12 + 3 \cdot 15 = 57$ vertices. There are exactly 198 pairs of these vertices that are one unit apart, arising from the 24 edges in the core, the $51 \times 3 = 153$ edges described above in the process of adding spindles, and 21 other pairs of vertices that wind up exactly one unit apart in the construction. The resulting graph on 57 vertices and 198 edges is pictured in Figure 3.11.

To complete the proof, we define a 96: 27-clique on $G$. This multiset consists of all the non-core vertices of $G$, each taken with multiplicity 1, and all the core vertices of $G$, taken with multiplicity according to the labels in Figure 3.12 on page 49.

The cardinality of this multiset is $3 \cdot 7 + 3 \cdot 4 + 6 \cdot 3 + 45 \cdot 1 = 96$.

Refer to the multiplicity of a vertex in this multiset as its *weight* and let the weight of any set of vertices be the sum of the weights of the elements. We must show that no independent

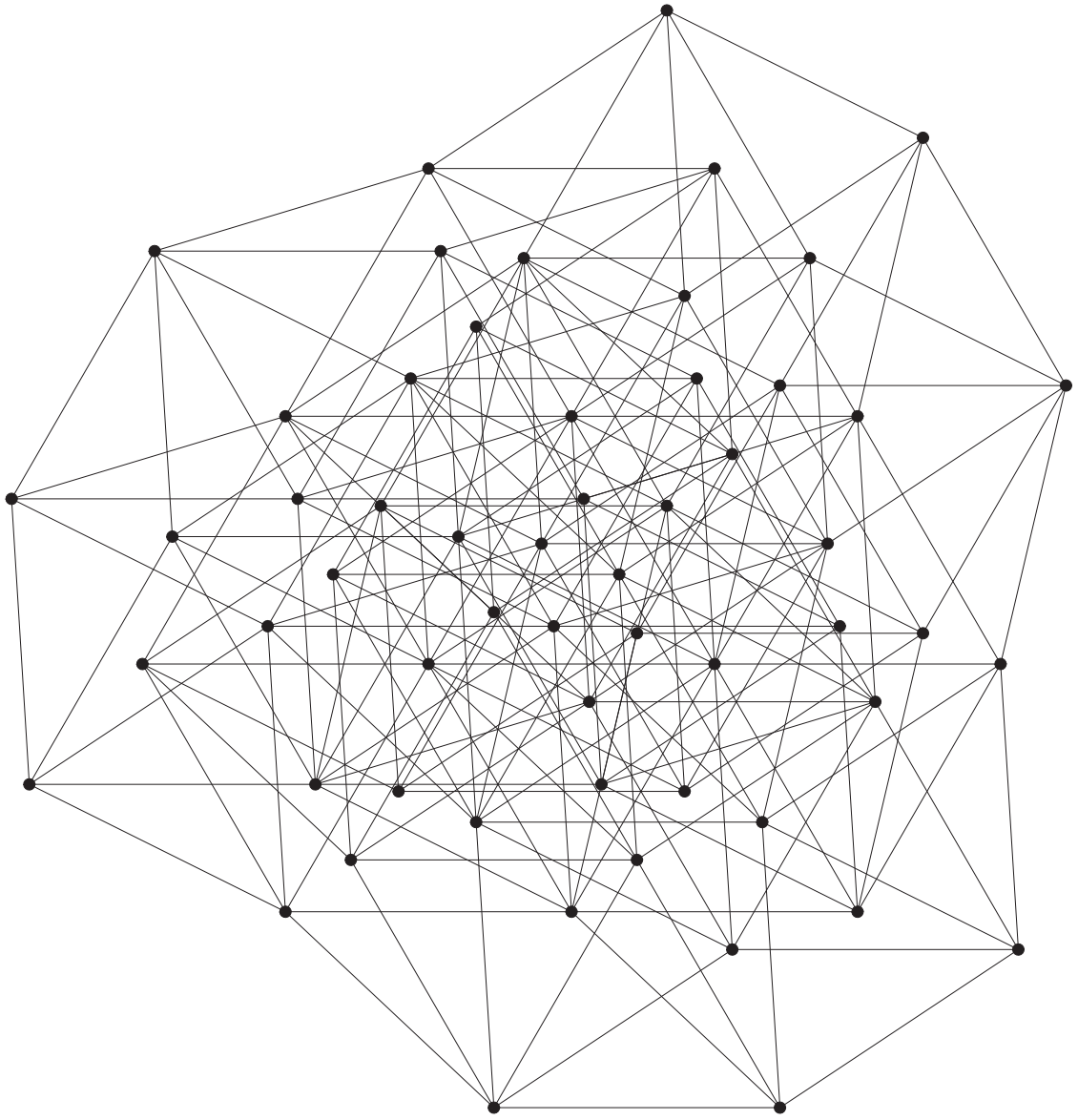

Figure 3.11. A unit graph with fractional chromatic number 32/9.

set of vertices in $G$ has weight greater than 27. To this end, note that no independent set has more than one vertex from the three new vertices in any one spindle. Therefore, no independent set in $G$ can pick up more than 15 spindle vertices. Thus we have something to prove only for independent sets that intersect the core in vertices with total weight at least 13. But there are only a few independent sets of core vertices that weigh that much. One consists of four vertices of weight 7, 4, 3, and 3, respectively. But then only 10 of the 15 spindles can contribute more vertices, leaving a total weight of $7 + 4 + 3 + 3 + 10 = 27$. Another has two vertices of weight 4 and two of weight 3. But then only 13 of the 15 spindles can contribute more vertices, leaving a total weight of $4 + 4 + 3 + 3 + 13 = 27$. The last possibility consists of a vertex of weight 7 and two of weight

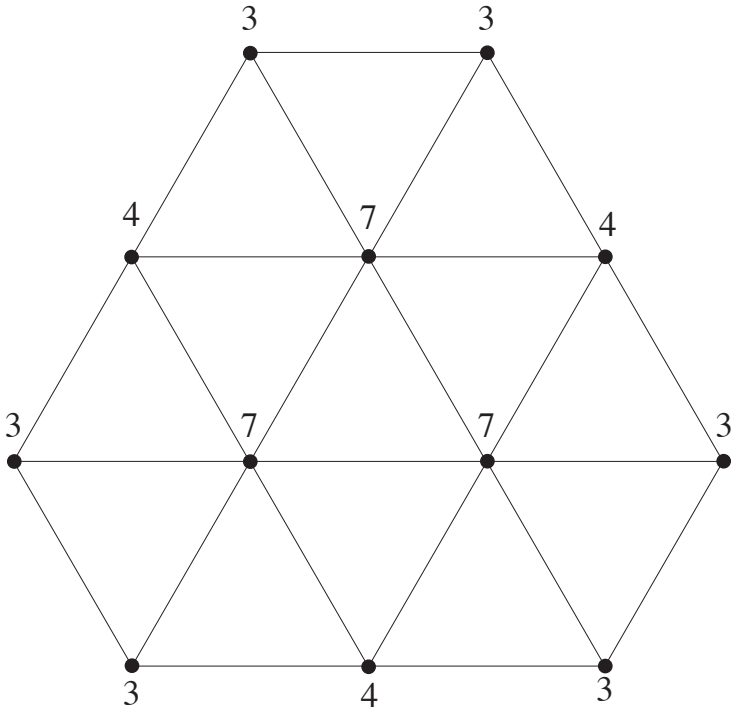


Figure 3.12. The 27-clique of $G$ restricted to the core.

3. No matter how these are chosen, only 14 spindle vertices can be added, leaving a total weight of $7 + 3 + 3 + 14 = 27$.

The fractional chromatic number of $G$ is thus at least $96/27 = 32/9$. This is enough to give the theorem. □

It can be shown that $\chi_f(G)$ does in fact equal $32/9$, so a different graph would be needed to improve the lower bound given here. However, the best lower bound might not be achieved by this method because the supremum of the fractional chromatic numbers of the finite subgraphs of $\mathbf{R}^2$ might not equal $\chi_f(\mathbf{R}^2)$; see exercises 23 and 24.

## 3.7 The Erdős-Faber-Lovász conjecture

In this section we prove the fractional analogue of a result whose integer version remains a notorious open problem.

In 1972, Erdős, Faber, and Lovász posed the following conjecture.

**Conjecture 3.7.1** *If a graph $G$ is the union of $n$ cliques of size $n$ no two of which share more than one vertex, then $\chi(G) = n$.* □

Erdős at first offered $50 for the proof of this innocent assertion, but later raised the amount to $500, when it became clear that the conjecture is deep. It remains open despite wide popularization for more than two decades. Here we prove the fractional version, due to Kahn and Seymour [105].

**Theorem 3.7.2** *If a graph $G$ is the union of $n$ cliques of size $n$ no two of which share more than one vertex, then $\chi_f(G) = n$.*

The proof depends on the following lemma, first mentioned by Motzkin [134].

**Lemma 3.7.3** *If $H$ is a bipartite graph with nonempty parts $X$ and $Y$ such that no vertex in $X$ is adjacent to every vertex in $Y$, then there are nonadjacent vertices $x \in X$ and $y \in Y$ satisfying $|X|d(x) \geq |Y|d(y)$.*

**Proof**. If any vertex $y \in Y$ is adjacent to every vertex in $X$, then we may throw $y$ away and apply an induction hypothesis to obtain

$$|X|\,(d(x)-1) \geq (|Y|-1)\,d(y).$$

The result follows by adding to this inequality the obvious inequality $|X| \geq d(y)$.

If no vertex in $Y$ is adjacent to every vertex in $X$, then

$$\begin{aligned}\sum_{\substack{x\in X, y\in Y:\\ xy\notin E(H)}} \frac{|X|d(x)-|Y|d(y)}{(|X|-d(y))\,(|Y|-d(x))} &= \sum_{\substack{x\in X, y\in Y:\\ xy\notin E(H)}} \frac{|Y|}{|Y|-d(x)} - \frac{|X|}{|X|-d(y)} \\ &= \sum_{x\in X} |Y| - \sum_{y\in Y} |X| = 0.\end{aligned}$$

It follows that the summand in the first expression must be nonnegative for some $xy \notin E(H)$. □

In fact, what we use below is a fractional version of Motzkin's lemma.

**Lemma 3.7.4** *Suppose that $H$ is a bipartite graph with nonempty parts $X$ and $Y$ such that no vertex in $X$ is adjacent to every vertex in $Y$. If $f : Y \to [0,\infty)$, then there are nonadjacent vertices $x_0 \in X$ and $y_0 \in Y$ satisfying*

$$|X| \sum_{\substack{y\in Y:\\ x_0y\in E(H)}} f(y) \geq d(y_0) \sum_{y\in Y} f(y).$$

**Proof**. If $f = 1$, this is just a restatement of Lemma 3.7.3. If $f$ is integral, it follows from Lemma 3.7.3 by replacing every $y \in Y$ with $f(y)$ copies of itself. If $f$ is rational, it follows by replacing $f$ by $cf$, where $c$ is a common denominator of the fractional values of $f$. Finally, for general $f$, we obtain the result by rational approximation. □

**Proof** (of Theorem 3.7.2). Since $G$ has cliques of size $n$, we know $\chi_f(G)$ is at least $n$. We show here that it is at most $n$ as well. Let $g : V(G) \to [0,1]$ be a fractional clique of maximum weight, so that $\chi_f(G) = \sum_{v\in V(G)} g(v)$. Let $G'$ be the subgraph of $G$ induced on vertices $v$ with $g(v) > 0$. (Note that $g$ is a fractional clique of $G'$ as well.) Let $\mathcal{I}$ be the collection of independent sets of vertices in $G'$ and let $f : \mathcal{I} \to [0,1]$ be a fractional coloring of $G'$ of minimum weight. Then

$$\chi_f(G) \geq \chi_f(G') = \sum_{I\in\mathcal{I}} f(I) \geq \sum_{v\in V(G')} g(v) = \chi_f(G),$$

so equality holds.

By the complementary slackness property (Theorem A.3.2 on page 135), we may assume, for every vertex $v \in V(G')$, that $\sum_{I \ni v} f(I) = 1$. By the same reasoning, for every $I \in \mathcal{I}$ with $f(I) > 0$ we may assume that $\sum_{v \in I} g(v) = 1$, which implies that every such $I$ is a maximal independent set in $G'$. Let $\mathcal{I}' = \{I \in \mathcal{I} : f(I) > 0\}$.

Let $\mathcal{C}$ be the $n$-set of $n$-cliques as described in the hypothesis of the Theorem. Let $\mathcal{C}'$ be the restriction of $\mathcal{C}$ to $V(G')$. If any clique $C \in \mathcal{C}'$ intersects every independent set $I \in \mathcal{I}'$, then we have

$$\chi_f(G) = \sum_{I \in \mathcal{I}'} f(I) = \sum_{v \in C} \sum_{I \ni v} f(I) = \sum_{v \in C} 1 \leq n,$$

which is what we want to prove. (The inequality follows from the fact that $\omega(G) = n$; see exercise 17 on page 55.)

Thus we may assume that no such clique $C$ exists. Build a bipartite graph $H$ with $V(H) = \mathcal{C}' \cup \mathcal{I}'$ and with an edge between clique $C$ and independent set $I$ if and only if $C \cap I \neq \emptyset$. By Lemma 3.7.4 on the facing page, there is a clique $C_0$ and an independent set $I_0$ such that $f(I_0) > 0$, $C_0 \cap I_0 = \emptyset$, and

$$|\{C \in \mathcal{C}' : C \cap I_0 \neq \emptyset\}| \sum_{I \in \mathcal{I}'} f(I) \leq |\mathcal{C}'| \sum_{\substack{I \in \mathcal{I}': \\ I \cap C_0 \neq \emptyset}} f(I) \tag{3.6}$$

$$\leq n \sum_{\substack{I \in \mathcal{I}': \\ I \cap C_0 \neq \emptyset}} f(I). \tag{3.7}$$

But

$$\sum_{\substack{I \in \mathcal{I}': \\ I \cap C_0 \neq \emptyset}} f(I) = \sum_{v \in C_0} \sum_{I \ni v} f(I) \tag{3.8}$$

$$= \sum_{v \in C_0} 1 \tag{3.9}$$

$$= |C_0| \tag{3.10}$$

$$\leq |\{C \in \mathcal{C}' : C \cap C_0 \neq \emptyset,\ C \neq C_0\} \tag{3.11}$$

$$\leq |\{C \in \mathcal{C}' : C \cap I_0 \neq \emptyset\}|. \tag{3.12}$$

The inequality in line (3.11) is derived from the maximality of $I_0$: Every $v$ in $C_0$ lies in (at least) one other $C \in \mathcal{C}'$, for otherwise $I_0 \cup \{v\}$ would be a larger independent set. The inequality at line (3.12) holds since any clique $C \in \mathcal{C}'$ that intersects $C_0$ must intersect $I_0$ (otherwise $I_0 \cup (C \cap C_0)$ would be a larger independent set of vertices). Inequalities (3.8)–(3.12) together with (3.6) and (3.7) give

$$\chi_f(G) = \sum_{I \in \mathcal{I}} f(I) \leq n. \qquad \square$$

## 3.8 List coloring

Recall that a graph is $a$-colorable provided we can assign a color to each vertex, from a palette of $a$ colors, so that adjacent vertices receive different colors. A *list coloring* of a graph is quite similar, only now each vertex may have its own private palette of colors. In other words, the list of acceptable colors for one vertex might be different from that of another.

More formally, let $a$ be a positive integer. An *$a$-palette* for $G$ is a mapping that assigns to each vertex $v$ a set $P(v)$ of cardinality $a$. Given $P$, we say $G$ is *$P$-choosable* provided we can properly color the graph $G$ such that each vertex $v$ is colored $C(v)$ where $C(v) \in P(v)$. A graph is *$a$-choosable* provided it is $P$-choosable for every $a$-palette $P$. Finally, the *list chromatic number* of $G$ is the least $a$ such that $G$ is $a$-choosable and is denoted $\chi^\ell(G)$. The list chromatic number is also known as the *choice number*.

Clearly if a graph $G$ is $a$-choosable it must be $a$-colorable. Thus $\chi^\ell(G) \geq \chi(G)$ and the inequality can be strict. For example, let $G = K_{3,3}$ with bipartition $\{x_1, x_2, x_3\} \cup \{y_1, y_2, y_3\}$. We claim that $K_{3,3}$ is not 2-choosable by setting $P(x_i) = P(y_i) = \{1, 2, 3\} - \{i\}$. Exercise 22 on page 55 asks the reader to check that $K_{3,3}$ is not $P$-choosable.

The fractional list chromatic number is analogous to the fractional chromatic number. In an $a{:}\,b$-coloring of a graph, we assign a set of $b$ colors to each vertex, with adjacent vertices receiving disjoint color sets; the colors are selected from a master palette of $a$ colors. Likewise, we say a graph $G$ is *$a{:}\,b$-choosable* if for every $a$-palette $P$ of $G$ we can assign to each vertex $v$ of $G$ a $b$-set of colors $C(v) \subseteq P(v)$ so that $v \sim w \Longrightarrow C(v) \cap C(w) = \emptyset$. The *$b$-fold list chromatic number* of $G$, denoted $\chi_b^\ell(G)$, is the least $a$ so that $G$ is $a{:}\,b$-choosable. It is easy to see that $\chi_b^\ell$ is subadditive in its subscript, so we define the *fractional list chromatic number* of $G$ to be

$$\chi_f^\ell(G) = \lim_{b\to\infty} \frac{\chi_b^\ell(G)}{b} = \inf_b \frac{\chi_b^\ell(G)}{b}.$$

Note that if a graph is $a{:}\,b$-choosable then it must be $a{:}\,b$-colorable. Thus

$$\chi_f(G) = \lim_{b\to\infty} \frac{\chi_b}{b} \leq \lim_{b\to\infty} \frac{\chi_b^\ell}{b} = \chi_f^\ell(G). \qquad (*)$$

However, in the fractional case there can be no gap!

**Theorem 3.8.1** *The fractional chromatic number of a graph equals its fractional list chromatic number, i.e., $\chi_f(G) = \chi_f^\ell(G)$ for all graphs $G$.*

**Proof**. In light of $(*)$ it is enough to show that $\chi_f(G) \geq \chi_f^\ell(G)$. Let $a, b$ be positive integers so that $\chi_f(G) = \frac{a}{b}$ and $G$ is $a{:}\,b$-colorable. Let $\varepsilon > 0$ be an arbitrary rational number. Let $n = \nu(G)$.

Let $m$ be a positive integer so that

$$A = (1+\varepsilon)am \qquad \text{and} \qquad B = bm$$

are integers. We claim that if $m$ is large enough then $G$ is $A{:}\,B$-choosable. Let $P$ be an $A$-palette of $G$ and let $X$ be the union of all the $P(v)$'s, i.e.,

$$X = \bigcup_{v\in V(G)} P(v).$$

We create a random partition of $X$ into $a$-parts

$$X = X_1 \cup X_2 \cup \cdots \cup X_a$$

placing a color $c \in X$ into $X_i$ with probability $1/a$, i.e., uniformly. The expected size of $P(v) \cap X_i$ is $\frac{1}{a}|P(v)| = (1+\varepsilon)m$. Thus if $m$ is sufficiently large we have (using well-known results for sums of independent Bernouli random variables, e.g., Chernoff bounds) that

$$\Pr\Big\{|P(v) \cap X_i| < m\Big\} < \frac{1}{na}.$$

It follows that, with positive probability, there is a partition of $X$ so that $|P(v) \cap X_i| \geq m$ for all $i$ and all $v$. Let $P_i(v)$ be an $m$-set contained in $P(v) \cap X_i$.

Now consider the $a:b$-coloring of $G$ and let $c(v) \subseteq \{1, 2, \ldots, a\}$ be the $b$-set of colors assigned to vertex $v$. Let

$$C(v) = \bigcup_{i \in c(v)} P_i(v).$$

Since the $X_i$ are pairwise disjoint and $c$ is a proper $b$-fold coloring we have that $C$ is a proper $B$-fold coloring. Thus

$$\chi_f^\ell(G) \leq \frac{\chi_B^\ell(G)}{B} \leq \frac{A}{B} = (1+\varepsilon)\frac{a}{b} = (1+\varepsilon)\chi_f(G).$$

As $\varepsilon$ was arbitrary, the result follows. □

## 3.9 Computational complexity

The decision problem of determining whether a graph is $k$-colorable is NP-complete for any fixed $k \geq 3$ [75],[76]. On the other hand, there is a simple, linear-time algorithm for checking 2-colorability. Where does checking $a:b$-colorability fit in?

The following theorem of Hell and Nešetřil [86] answers this question.

**Theorem 3.9.1** *If $H$ is a fixed graph that is not bipartite, then the problem of determining whether there is a homomorphism from a given graph $G$ to $H$ is NP-complete.* □

When $a > 2b$, the Kneser graph $K_{a:b}$ is not bipartite. Putting $K_{a:b}$ in place of $H$ in this theorem shows that checking $a:b$-colorability is NP-complete.

This does not immediately imply that determining whether a graph has fractional chromatic number less than or equal to $r \in \mathbf{Q}$ is NP-complete. There is no guarantee, after all, that in order to compute whether a graph has fractional chromatic number equal to $5/2$, say, one must check whether it is $5:2$-colorable. In fact, one way to compute the fractional chromatic number of a graph is to compute the value of the associated linear program. It is true that the problem of computing the value of an LP can be solved in polynomial time (via the ellipsoid algorithm) but this does not imply that computing $\chi_f$ can be done in polynomial time. The problem is that the associated LP potentially has exponentially many (in the number of vertices) variables, one for each maximal independent set in the graph.

In fact, the fractional chromatic number cannot be computed in polynomial time[4], as was first observed by Grötschel, Lovász, and Schrijver [81].

**Theorem 3.9.2** *For every real number $r > 2$, the problem of determining whether a graph $G$ has $\chi_f(G) \leq r$ is NP-complete.* □

Ironically, this negative result follows from the positive result that the ellipsoid algorithm for solving linear programs does run in polynomial time. It turns out that the ellipsoid algorithm gives a polynomial transformation between the fractional coloring problem and the problem of computing the independence number of a graph. The latter invariant is known to be NP-hard to compute.

[4]Unless P = NP.

## 3.10 Exercises

1. Show that the clique number is dual to the chromatic number by showing that the dual to the integer program that computes $\chi(G)$ computes $\omega(G)$.

2. Show that $\chi(G) = 2$ if and only if $\chi_f(G) = 2$.

3. Show that there is no graph $G$ with $0 < \chi_f(G) < 1$ or $1 < \chi_f(G) < 2$.

4. Show, without recourse to the fact that $\chi_\infty(G) = \chi_f(G)$, that $\chi(G) = 2$ if and only if $\chi_\infty(G) = 2$.

5. Let $G_{a,b}$ be the graph defined in Proposition 3.2.2 on page 32. Prove that $\alpha(G_{a,b}) = b$ and $\omega(G_{a,b}) = \lfloor a/b \rfloor$.

6. Show that the mapping $\phi$ described in the proof of Lemma 3.2.7 on page 34 is indeed a graph homomorphism, i.e., prove that if $S \cap T = \emptyset$, then $\phi(S) \cap \phi(T) = \emptyset$.

7. Prove the easy half of Theorem 3.2.6 on page 34 by showing that $a - 2b + 2$ is an upper bound for $\chi(K_{a:b})$.

8. The *star chromatic number* $\chi^*(G)$ of a graph $G$ is the infimum of all fractions $a/b$ such that there is a coloring $f : V(G) \to [a]$ satisfying $b \leq |f(u) - f(v)| \leq a - b$ for every $uv \in E(G)$. Prove that $\chi_f(G) \leq \chi^*(G)$ for every graph $G$.

9. Prove that a graph $G$ is perfect if and only if $\omega(H) = \omega_f(H)$ for all induced subgraphs $H$ of $G$.

10. For any graph $\chi(G)\chi(\overline{G}) \geq \nu(G)$. Prove the fractional analogue $\chi_f(G)\chi_f(\overline{G}) \geq \nu(G)$. Note that the fractional version is a stronger result than its integer original.

11. Show that the fractional chromatic number of the graph $Y^n(K_2)$ (see page 35) is asymptotic to $\sqrt{2n}$.

12. Let $a, b$ be positive integers. Prove that if $a$ and $b$ are relatively prime, then so are $a^2 + b^2$ and $ab$.

    Conclude that the denominator of $\chi_f(Y^n(K_2))$ is greater than $2^{2^{n-2}}$.

13. Show that $\chi_f(G + H) = \max\{\chi_f(G), \chi_f(H)\}$ and that $\chi_f(G \vee H) = \chi_f(G) + \chi_f(H)$.

    (For the definition of the join ($\vee$) of two graphs, see page 20.)

14. In this problem we generalize the notion of *chromatic polynomial*. Let $G$ be a graph and let $a, b$ be positive integers. Define $\chi(G; a, b)$ to be the number of $a$:$b$-colorings of $G$.

    (a) Find $\chi(\overline{K_n}; a, b)$.

    (b) Find $\chi(T; a, b)$ for a tree $T$ on $n$ vertices.

    (c) For any graph $G$ and fixed $b$, prove that $\chi(G; a, b)$ is a polynomial in $a$.

15. Imagine forming a periodic, independent set in the plane via the construction in the proof of 3.6.1 on page 43 but without the specification that the perimeter of the set $A$ is composed equally of circular and straight parts. There is a one-parameter family of such independent sets, ranging from periodically placed hexagons at one extreme to periodically placed disks

at the other. Show that the density of the resulting independent set is maximized when the perimeter of $A$ is composed in equal amounts of circular and straight parts. (This is a calculus exercise.) Show that the density of set $S$ appearing in Figure 3.9 on page 46 is approximately 0.229365. (This is a computational exercise.)

16. Show that the Erdős-Faber-Lovász conjecture follows from Vizing's theorem (Theorem 4.1.1 on page 57) in the special case that no three cliques have a common vertex.

    Conjecture a hypergraph version of Vizing's theorem that is strong enough to imply the Erdős-Faber-Lovász conjecture in the general case.

    Prove a fractional version of your hypergraph conjecture along the lines of the proof of Theorem 3.7.2 on page 50.

17. Let $G$ be a graph that satisfies the hypothesis of Theorem 3.7.2 on page 50. Prove that $\omega(G) = n$.

18. Show that a graph can have exponentially many maximal independent sets by constructing a sequence of graphs $G_n$ such that, for some constant $c > 1$, the number of maximal independent sets in $G_n$ is greater than $c^{\nu(G_n)}$ for all $n$.

19. Let $\mathbf{Q}^2$ be the subgraph of the unit distance graph $\mathbf{R}^2$ induced on the set of points both of whose coordinates are rational. Show that $\mathbf{Q}^2$ is bipartite.

20. In a fractional coloring of a graph, the sum of the weights of the independent sets containing a given vertex is at least one. Show that there is no loss of generality in assuming that the sum of the weights of the independent sets containing any given vertex is exactly one.

    In other words, prove that if $t \geq \chi_f(G)$, then there is a fractional coloring with total weight $t$ so that for every vertex $v$, the sum of the weights of the independent sets containing $v$ is exactly 1.

    [Hint: $\emptyset$ is independent.]

21. The chromatic number of a graph can be expressed as an integer program in the following curious way: Suppose that $V(G) = \{v_1, v_2, \ldots v_n\}$ and imagine a palette of $n$ colors (which is plenty to color a graph with $n$ vertices!). For $1 \leq j \leq n$, let $a_j$ be the indicator function that is 1 if color $j$ is used and 0 otherwise. For each $i$ and $j$ with $1 \leq i, j \leq n$, let $b_{i,j}$ be the indicator function that is 1 if vertex $v_i$ has color $j$ and 0 otherwise. The coloring problem may be expressed as the minimum value of $a_1 + a_2 + \cdots + a_n$ subject to the constraints that $\sum_{j=1}^{n} b_{i,j} \geq 1$ for every $i$ (every vertex needs a color), $b_{i,j} \leq a_j$ for every $i$ and $j$ (you can't paint a vertex with an unused color), and $b_{i,k} + b_{j,k} \leq 1$ for every $i, j, k$ with $v_i v_j \in E(G)$ (the coloring must be proper). One might try to define a new type of fractional chromatic number of $G$ to be the value of the real relaxation of this integer program. What is wrong with this definition?

22. Prove that $\chi^{\ell}(K_{3,3}) = 3$.

23. Let $G$ be an *infinite* graph and let $k$ be a positive integer. Then $G$ is $k$-colorable if and only if all finite induced subgraphs of $G$ are $k$-colorable. This result, which requires the Axiom of Choice when $G$ is uncountable, is due to Erdős and de Bruijn [53].

    Note that the $b$-fold chromatic number of an infinite graph can be defined in exactly the same manner as for finite graphs. One naturally defines $\chi_f(G)$ for infinite $G$ to be $\lim \chi_b(G)/b$. Show that there is no analogue of the Erdős-de Bruijn result for fractional chromatic number.

Hint: Let $G$ be the infinite graph formed by taking the disjoint union of the Kneser graphs $K_{3a:a}$ for $a = 1, 2, 3, \cdots$.

24. Let $G_n$ be the graph formed from $K_n$ by replacing each edge of $K_n$ with a length 3 path. Thus $G_n$ has $n + 2\binom{n}{2} = n^2$ vertices. Prove the following:

    (a) For fixed $b$ and $n$ sufficiently large, $\chi_b(G_n) = 3b$.

    (b) $\lim_{n\to\infty} \chi_f(G_n) = 1 + \sqrt{2}$.

    (c) Conclude that $\chi_f(G_\infty) = 3$ but for any finite subgraph $H$ of $G_\infty$ we have $\chi_f(H) < 1+\sqrt{2}$.

25. Let $G$ be an infinite graph. Prove that $\chi(G) = \infty$ if and only if $\chi_f(G) = \infty$.

## 3.11 Notes

The fractional analogue of the four-color theorem is the assertion that the maximum value of $\chi_f(G)$ over all planar graphs $G$ is 4. That this maximum is no more than 4 follows from the four-color theorem itself, while the example of $K_4$ shows that it is no less than 4. Given that the proof of the four-color theorem is so difficult, one might ask whether it is possible to prove an interesting upper bound for this maximum without appeal to the four-color theorem. Certainly $\chi_f(G) \leq 5$ for any planar $G$, because $\chi(G) \leq 5$, a result whose proof is elementary. But what about a simple proof of, say, $\chi_f(G) \leq 9/2$ for all planar $G$? The only result in this direction is in a 1973 paper of Hilton, Rado, and Scott [89] that predates the proof of the four-color theorem; they prove $\chi_f(G) < 5$ for any planar graph $G$, although they are not able to find any constant $c < 5$ with $\chi_f(G) < c$ for all planar graphs $G$. This may be the first appearance in print of the invariant $\chi_f$.

Girth and fractional chromatic number can be large simultaneously. Erdős [52] was the first to prove that girth and chromatic number can be simultaneously large. In fact, his argument shows that girth and $n/\alpha$ can be made simultaneously large. It follows from Proposition 3.1.1 on page 30 that girth and fractional chromatic number can also be made simultaneously large.

The communications complexity story is based on a paper by Alon and Orlitsky [4].

Corollary 3.4.3 on page 39 can be traced to the work of McEliece and Posner [129]. See also the paper by Berge and Simonovits [20]. For an alternative proof go to Hell and Roberts [87].

Theorem 3.3.4 is due to Larsen, Propp, and Ullman [114].

The graph in Figure 3.11 on page 48 was discovered by D. Fisher and D. Ullman. By considering larger graphs built in the same fashion as this one, it seems plausible that one can show that $\chi_f(\mathbf{R}^2) \geq 4$. The set illustrated in Figure 3.9 on page 46 was discovered by Croft [40] in 1967 and then rediscovered in 1993 by Hochberg and O'Donnell [92].

The material in Section 3.7 on page 49 is due to Kahn and Seymour [105].

The notion of list chromatic number is due to Vizing [185] and independently to Erdős, Rubin, and Taylor [56]. Noga Alon has written a nice survey article [3]. Theorem 3.8.1 on page 52 is due to Alon, Tusa, and Voigt [5].

Exercise 23 on fractional chromatic number of infinite graphs is based on a paper by Leader [115]. Leader asks if there is an infinite graph $G$ for which $\infty > \chi_f(G) > \sup_H \chi_f(H)$ where the supremum is over finite induced subgraphs of $G$, and exercise 24 answers that question. The example there is due to Levin [118, 119].

# 4

# Fractional Edge Coloring

## 4.1 Introduction

### The edge chromatic number

The *edge chromatic number* of a graph $G$, denoted $\chi'(G)$, is the smallest size of a partition of $E(G)$ into matchings of $G$. Since any subset of a matching is again a matching, we can formulate the edge chromatic number as the covering number of the hypergraph $\mathcal{H}$ whose vertices are the edges of $G$ and whose hyperedges are the matchings of $G$.

Perhaps the most important result on the edge chromatic number is the following celebrated theorem of Vizing [183] relating the edge chromatic number to the maximum degree $\Delta$.

**Theorem 4.1.1 (Vizing)** *For any graph $G$, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.* □

Thus it is easy to narrow down the value of $\chi'(G)$ to one of two possible values, but it turns out that it is NP-hard to determine which of $\Delta(G)$ or $\Delta(G)+1$ is the exact value. We call graphs for which $\chi'(G) = \Delta(G)$ *class one* and all other graphs *class two*. All bipartite graphs are class one (exercise 1 on page 70), while any regular graph on an odd number of vertices is class two.

### The fractional edge chromatic number

The *fractional edge chromatic number*, $\chi'_f(G)$, can be defined in the following equivalent ways:

- $\chi'_f(G) = k_f(\mathcal{H})$, where $\mathcal{H}$ is the edge-matching hypergraph described above.
- $\chi'_f(G) = \chi_f[L(G)]$, where $L(G)$ is the line graph of $G$.
- Let $\chi'_t(G)$ be the smallest size of a $t$-fold edge coloring of $G$, i.e., a coloring where we assign sets of $t$ colors to each edge so that no color is present twice at any vertex. (Equivalently, $\chi'_t(G) = \chi'(t \cdot G)$ where $t \cdot G$ is the multigraph formed from $G$ by replacing every edge with $t$ parallel edges.)

  Then $\chi'_f(G) = \lim_{t\to\infty} \chi'_t(G)/t$.
- A *fractional edge coloring* is an assignment of a nonnegative weight $w_M$ to each matching $M$ of $G$ so that for every edge $e$ we have

$$\sum_{M \ni e} w_M \geq 1.$$

  Then $\chi'_f(G)$ is the minimum value of $\sum_M w_M$ (where the sum is over all matchings $M$ and the minimum is over all fractional edge colorings $w$).

Dually, $\chi'_f(G)$ is the maximum sum of weights on edges in which no matching has total weight exceeding 1.

The notions of edge chromatic number and fractional edge chromatic number easily extend to multigraphs. However, if the multigraph has a loop, then no edge coloring or fractional edge coloring is possible. Thus we restrict our attention to loopless multigraphs.

## 4.2 An exact formula

Since no two edges incident at a vertex can be the same color in a proper edge coloring of a graph, we see that

$$\chi'(G) \geq \Delta(G).$$

(This is the easy portion of Vizing's theorem 4.1.1 on the preceding page.)

Here is another lower bound: Since each color class is a matching, it can contain at most $\lfloor \nu(G)/2 \rfloor$ edges. Thus,

$$\chi'(G) \geq \left\lceil \frac{\varepsilon(G)}{\lfloor \nu(G)/2 \rfloor} \right\rceil.$$

Furthermore, we can replace $G$ in the right-hand side of this inequality by any subgraph $H$ of $G$ since each color class uses at most $\lfloor \nu(H)/2 \rfloor$ of the $\varepsilon(H)$ edges in $H$. Thus,

$$\chi'(G) \geq \max_H \left\lceil \frac{\varepsilon(H)}{\lfloor \nu(H)/2 \rfloor} \right\rceil \qquad (*)$$

where the maximum is over all subgraphs $H$ of $G$ that have at least two vertices.

In searching for a subgraph $H$ that achieves the maximum in $(*)$, we can clearly restrict $H$ to be an *induced* subgraph of $G$. Also, we claim that we may assume that $H$ has an odd number of vertices. To see why, suppose that the maximum in $(*)$ is achieved for a graph $H$ with an even number of vertices. Let $v$ be a vertex of minimum degree in $H$. Since

$$\delta(H) \leq \frac{2\varepsilon(H)}{\nu(H)},$$

we obtain

$$\frac{\varepsilon(H) - \delta(H)}{\frac{1}{2}(\nu(H) - 2)} \geq \frac{2\varepsilon(H)}{\nu(H)},$$

which leads to

$$\frac{\varepsilon(H - v)}{\lfloor \nu(H - v)/2 \rfloor} \geq \frac{\varepsilon(H)}{\lfloor \nu(H)/2 \rfloor}.$$

Thus we can restrict the scope of the maximization in $(*)$ to induced subgraphs $H$ for which $\nu(H)$ is odd and at least 3.

We define

$$\Lambda(G) = \max_H \frac{2\varepsilon(H)}{\nu(H) - 1}$$

where the maximization is over all induced subgraphs $H$ of $G$ with $\nu(H) \geq 3$ and $\nu(H)$ odd. In case $G$ has fewer than three vertices, put $\Lambda(G) = 0$. Thus

$$\chi'(G) \geq \max \{\Delta(G), \lceil \Lambda(G) \rceil\}. \qquad (**)$$

Now (∗∗) is true not only for graphs $G$, but for loopless multigraphs as well. Furthermore, it is easy to check that

$$\Delta(t \cdot G) = t\Delta(G) \qquad \text{and} \qquad \Lambda(t \cdot G) = t\Lambda(G)$$

and therefore

$$\frac{\chi'_t(G)}{t} = \frac{\chi'(t \cdot G)}{t} \geq \max\{\Delta(G), \Lambda(G)\}$$

hence

$$\chi'_f(G) \geq \max\{\Delta(G), \Lambda(G)\}.$$

The lovely surprise is that this lower bound on $\chi'_f$ gives the exact value.

**Theorem 4.2.1** *For any loopless multigraph $G$,*

$$\chi'_f(G) = \max\{\Delta(G), \Lambda(G)\}.$$

## 4.3 The matching polytope

The key to proving Theorem 4.2.1 is a careful description of the *matching polytope*. Let $G$ be a loopless multigraph. To a subset $F$ of $E(G)$ we may associate a vector $\mathbf{i}_F$ in $\{0,1\}^{\varepsilon(G)}$, called the *incidence vector* of $F$, whose entry in position $e$ is a 1 just when $e \in F$.

The *matching polytope* of $G$, denoted $\mathbf{M}(G)$, is the convex hull of the incidence vectors of all the matchings of $G$. In symbols,

$$\mathbf{M}(G) = \Big\langle \{\mathbf{i}_M : M \text{ is a matching of } G\} \Big\rangle$$

where $\langle\cdot\rangle$ denotes convex hull. Note that since $\emptyset$ is a matching, $\mathbf{0} \in \mathbf{M}$. Also, any single edge of $G$ forms a matching so the $\varepsilon$ standard basis vectors are in $\mathbf{M}$. Thus $\mathbf{M}$ contains the simplex determined by these $\varepsilon + 1$ vectors and therefore has nonempty interior.

Because the matching polytope is defined as the convex hull of 0,1-vectors, the extreme points of $\mathbf{M}$ are exactly the incidence vectors of matchings. (That is, no incidence vector of a matching can be written as a convex combination of the others. See exercise 4 on page 70.)

We have described the matching polytope by naming its extreme points. A polytope can also be described as the intersection of a family of halfspaces. To present this alternative representation, we need some more notation.

Let $S$ be a subset of the vertex set of a graph $G$. Write $\partial S$ to stand for those edges that have exactly one end in $S$. For a vertex $v$, we write $\partial v$ for $\partial\{v\}$. We write $G[S]$ for the subgraph of $G$ induced on $S$, and we write $E[S]$ in place of $E(G[S])$, i.e., the set of edges with both ends in $S$.

**Theorem 4.3.1** *Let $\mathbf{M}$ be the matching polytope of a loopless multigraph $G$. Then a vector $\mathbf{x}$ is in $\mathbf{M}$ if and only if it satisfies all of the following:*

(i) $\mathbf{x} \geq 0$,

(ii) $\mathbf{x} \cdot \mathbf{i}_{\partial v} \leq 1$ *for all* $v \in V(G)$, *and*

(iii) $\mathbf{x} \cdot \mathbf{i}_{E[S]} \leq \lfloor |S|/2 \rfloor$ *for all* $S \subseteq V(G)$.

**Proof**. Let $\mathbf{M}$ be the matching polytope of $G$ and let $\mathbf{P}$ be the polyhedron defined by the inequalities (i), (ii), and (iii). Our claim is that $\mathbf{M} = \mathbf{P}$.

It is easy to check that $\mathbf{M} \subseteq \mathbf{P}$. It is enough to check that for any matching $M$ we have $\mathbf{i}_M \in \mathbf{P}$. Obviously, $\mathbf{i}_M \geq 0$. For any vertex $v$, at most one edge of $M$ is incident with $v$, so $\mathbf{i}_M \cdot \mathbf{i}_{\partial v} \leq 1$. Finally, let $S$ be any subset of $V$. Any matching $M$ can intersect $E[S]$ in at most $\lfloor |S|/2 \rfloor$ edges so, $\mathbf{i}_M \cdot \mathbf{i}_{E[S]} \leq \lfloor |S|/2 \rfloor$. Since the extreme points of $\mathbf{M}$ satisfy (i), (ii), and (iii), it follows that $\mathbf{M} \subseteq \mathbf{P}$.

Proving $\mathbf{P} \subseteq \mathbf{M}$ is more complicated. Suppose $\mathbf{P} \not\subseteq \mathbf{M}$. Let $\mathbf{z} \in \mathbf{P} - \mathbf{M}$, i.e., $\mathbf{z}$ satisfies (i), (ii), and (iii), but is not a convex combination of incidence vectors $\mathbf{i}_M$ of matchings. Let $\mathbf{\Pi}$ be a hyperplane that separates $\mathbf{z}$ from $\mathbf{M}$. We may take $\mathbf{\Pi}$ to be a *facet hyperplane* of $\mathbf{M}$, i.e., a hyperplane that contains a maximal face (facet) of $\mathbf{M}$. Thus $\mathbf{\Pi}$ is defined by the equation $\mathbf{a} \cdot \mathbf{x} = b$. The halfspace $\mathbf{H}$ defined by $\mathbf{a} \cdot \mathbf{x} \leq b$ contains $\mathbf{M}$ but not $\mathbf{z}$.

Note that since $\mathbf{M}$ has nonempty interior, each facet of $\mathbf{M}$ has at least $\varepsilon$ extreme points $\mathbf{i}_M$. We refer to the incidence vectors $\mathbf{i}_M$ that lie on $\mathbf{\Pi}$ as $\mathbf{\Pi}$-*extremal*.

Suppose $\mathbf{\Pi}'$ is another hyperplane that contains all the $\mathbf{\Pi}$-extremal incidence vectors. Then, since $\mathbf{\Pi}$ is a facet hyperplane, we must have $\mathbf{\Pi} = \mathbf{\Pi}'$. Moreover, if all the $\mathbf{\Pi}$-extremal incidence vectors satisfy some linear equation, then that linear equation must define $\mathbf{\Pi}$ and be the same as (up to nonzero multiple) the equation $\mathbf{a} \cdot \mathbf{x} = b$.

**Case I**: *Some entry in* $\mathbf{a}$ *is negative.*

Suppose that $e \in E(G)$ has $a_e < 0$. Let $\mathbf{i}_M$ be a $\mathbf{\Pi}$-extremal incidence vector. We claim that $e \notin M$. Suppose $e \in M$. Then

$$\mathbf{i}_{M-e} \cdot \mathbf{a} = \mathbf{i}_M \cdot \mathbf{a} - a_e = b - a_e > b,$$

implying that $\mathbf{i}_{M-e} \notin \mathbf{M}$, a contradiction. Thus every $\mathbf{\Pi}$-extremal incidence vector satisfies the equation $\mathbf{i}_M \cdot \mathbf{i}_{\{e\}} = 0$. Therefore $\mathbf{\Pi}$ is defined by the equation $x_e = 0$. So $\mathbf{H}$, which contains $\mathbf{M}$, is given by the inequality $x_e \geq 0$, but, since $\mathbf{z}$ is on the other side of $\mathbf{\Pi}$, we have $z_e < 0$, contradicting (i).

Thus we may assume $\mathbf{a} \geq 0$.

**Case II**: *Some vertex* $v$ *is saturated by every matching* $M$ *for which* $\mathbf{i}_M$ *is* $\mathbf{\Pi}$*-extremal.*

Let $\mathbf{i}_M$ be $\mathbf{\Pi}$-extremal. Since $v$ is $M$-saturated, $\mathbf{i}_M \cdot \mathbf{i}_{\partial v} = 1$. Since every $\mathbf{\Pi}$-extremal incidence vector satisfies the equation $\mathbf{x} \cdot \mathbf{i}_{\partial v} = 1$, this must be the equation for $\mathbf{\Pi}$ and $\mathbf{H}$ must be the halfspace defined by $\mathbf{x} \cdot \mathbf{i}_{\partial v} \leq 1$. Since $\mathbf{z}$ is on the other side of $\mathbf{\Pi}$, we have $\mathbf{z} \cdot \mathbf{i}_{\partial v} > 1$, a contradiction to (ii).

One case remains.

**Case III**: *For every vertex* $v$*, there is a* $\mathbf{\Pi}$*-extremal incidence vector* $\mathbf{i}_M$ *so that* $v$ *is not* $M$*-saturated.*

Let $H$ be the subgraph of $G$ given by those edges $e$ for which $a_e > 0$. Let $S$ be the vertex set of one component of $H$.

Claim A: *Some* $S$ *is not a singleton.* If every $S$ were a single vertex, this would imply that $\mathbf{a} = \mathbf{0}$, a contradiction since $\mathbf{a} \cdot \mathbf{x} = b$ is the equation of a hyperplane.

Thus we may choose $S$ containing at least two vertices.

Claim B: *For every* $\mathbf{\Pi}$*-extremal* $\mathbf{i}_M$*, at most one vertex in* $S$ *is* $M$*-unsaturated.* Suppose, for sake of contradiction, that two vertices, $u$ and $v$, are not saturated by $M$. Since $u$ and $v$ lie in the same component of $H$, we know there is a path in $H$ joining them.

We choose a matching $M$ and $u, v \in S$ so that $\mathbf{i}_M$ is $\mathbf{\Pi}$-extremal, $u$ and $v$ are $M$-unsaturated, and with the distance $d_H(u, v)$ as small as possible.

Note that the edge $uv$ (if it exists in $G$) is not in $H$ (i.e., $a_{uv} = 0$). For if $a_{uv} > 0$ we would have

$$\mathbf{i}_{M+uv} \cdot \mathbf{a} = \mathbf{i}_M \cdot \mathbf{a} + a_{uv} > b,$$

a contradiction. Thus $d_H(u, v) \geq 2$.

Let $P$ be any shortest path in $H$ from $u$ to $v$, and let $w$ be the vertex on $P$ immediately after $u$.

By our choice of $u$ and $v$, we know that $w$ is $M$-saturated. Also, there is *another* matching $M'$ for which $w$ is $M'$-unsaturated and $\mathbf{i}_{M'}$ is $\mathbf{\Pi}$-extremal. Starting from $w$, create a maximal path whose edges are alternately in $M$ and $M'$; call this path $Q$.

We claim that

$$\mathbf{i}_{M\cap Q}\cdot\mathbf{a}=\mathbf{i}_{M'\cap Q}\cdot\mathbf{a}. \qquad (*)$$

To see this, note that if these quantities were different, then the incidence vectors of the matchings

$$(M-Q)\cup(M'\cap Q) \qquad \text{and} \qquad (M'-Q)\cup(M\cap Q)$$

would be on opposite sides of $\mathbf{\Pi}$, a contradiction.

Finally, let $M''=(M-Q)\cup(M'\cap Q)$. By $(*)$ we have $\mathbf{i}_{M''}\cdot\mathbf{a}=b$ and therefore $\mathbf{i}_{M''}$ is $\mathbf{\Pi}$-extremal. However, $u$ and $w$ are $M''$ unsaturated and $d_H(u,w)=1<d(u,v)$, a contradiction to the choice of $u$, $v$, and $M$. This proves Claim B.

Claim C: *For every* $\mathbf{\Pi}$*-extremal* $\mathbf{i}_M$*, we have* $|M\cap E[S]|=\lfloor|S|/2\rfloor$. Note that Claim B says that $M$ fails to saturate at most 1 vertex in $S$. When $|S|$ is odd, this means that $M$ saturates all but one vertex of $S$, and when $|S|$ is even, $M$ must saturate all of $S$.

The question that remains is: Does every edge of $M$ that saturates a vertex of $S$ lie in $E[S]$? We claim the answer is yes. For suppose some edge of $e\in M$ has one end, say $v$, in $S$ and one end not in $S$. Then, by definition of $S$, we know $a_e=0$. Then $\mathbf{i}_{M-e}$ is again $\mathbf{\Pi}$-extremal, and therefore $M-e$ misses at most one vertex of $S$. If $|S|$ is odd, this is impossible. If $|S|$ is even, there is *another* edge $f$ in $M\cap\partial S$ that saturates *another* vertex $w\in S$. But then $\mathbf{i}_{M-e-f}$ is still $\mathbf{\Pi}$-extremal (because $a_e=a_f=0$) and $M-e-f$ fails to saturate two vertices of $S$, a contradiction. This proves Claim C.

We have shown that $\mathbf{i}_M\cdot\mathbf{i}_{E[S]}=\lfloor|S|/2\rfloor$ for every $\mathbf{\Pi}$-extremal $\mathbf{i}_M$ and every set $S\subseteq V(G)$. Thus the equation for $\mathbf{\Pi}$ must be $\mathbf{x}\cdot\mathbf{i}_{E[S]}=\lfloor|S|/2\rfloor$. We therefore have $\mathbf{z}\cdot\mathbf{i}_{E[S]}>\lfloor|S|/2\rfloor$, contradicting (iii) and proving the theorem. □

## 4.4 Proof and consequences

We now apply Theorem 4.3.1 on page 59 to prove Theorem 4.2.1 on page 59 that

$$\chi'_f(G)=\max\{\Delta(G),\Lambda(G)\}$$

where

$$\Lambda(G)=\max_H\frac{2\varepsilon(H)}{\nu(H)-1}$$

where the maximization is over all induced subgraphs $H$ with $\nu(H)\geq 3$ and odd.

**Proof** (of Theorem 4.2.1). We know that $\chi'_f(G)\geq\Delta(G)$ and $\chi'_f(G)\geq\Lambda(G)$.

The fractional edge chromatic number of $G$ can be expressed as a linear program:

$$\min\sum_M w_M \quad \text{s.t.} \quad \sum_M w_M\mathbf{i}_M\geq\mathbf{1}$$

where the sum is over all matchings $M$ of $G$, $w_M$ is a nonnegative weight for the matching $M$, and $\mathbf{1}$ is a vector of all 1's. Suppose that an optimal weighting assigns $w_M$ to matching $M$.

We may assume that $\sum_M w_M\mathbf{i}_M=\mathbf{1}$. To see this, let $e$ be an edge that receives total weight exceeding 1. Let $M$ be any matching that contains $e$ and has $w_M>0$. By decreasing $w_M$ and increasing $w_{M-e}$ by the same amount, we can reduce the excess weight on $e$ with no loss of optimality or feasibility. In this way, we may assume our optimal weighting satisfies $\sum_M w_M\mathbf{i}_M=\mathbf{1}$. (See also exercise 8 on page 13 and exercise 20 on page 55.)

Let $w^* = \sum_M w_M = \chi'_f(G)$. Then we may write

$$\sum_M \frac{w_M}{w^*}\mathbf{i}_M = \frac{1}{w^*}\mathbf{1}.$$

Thus, $(1/w^*)\mathbf{1} \in \mathbf{M}$, the matching polytope of $G$. We now apply Theorem 4.3.1 on page 59. We know that $w^*$ is the smallest number so that $(1/w^*)\mathbf{1}$ satisfies (i), (ii), and (iii). Now (ii) requires that $(1/w^*)\mathbf{1} \cdot \mathbf{i}_{\partial v} \leq 1$ for any $v \in V(G)$, which is equivalent to saying that $d(v) \leq w^*$ for all $v$, i.e., $\Delta(G) \leq w^*$. And (iii) requires that $(1/w^*)\mathbf{1} \cdot \mathbf{i}_{E[S]} \leq \lfloor |S|/2 \rfloor$ for all $S \subseteq V$. This is tantamount to requiring that

$$w^* \geq \max_S \frac{|E[S]|}{\lfloor |S|/2 \rfloor}$$

where the maximum is over all $S \subseteq V$ with $|S| > 1$. But we observed earlier that this maximum is simply $\Lambda(G)$. Thus the conditions of Theorem 4.3.1 tell us that $w^*$ must simply be the larger of $\Delta(G)$ and $\Lambda(G)$. This completes the proof. □

Let us consider the fractional edge chromatic number of a simple graph. We know, by Theorem 4.1.1 (Vizing) and the fact that $\chi'_f(G) \leq \chi'(G)$, that $\Delta(G) \leq \chi'_f(G) \leq \Delta(G) + 1$. We can use Theorem 4.2.1 to deduce the same result.

**Corollary 4.4.1 (Fractional Vizing)** *Let $G$ be a graph. Then*

$$\Delta(G) \leq \chi'_f(G) \leq \Delta(G) + 1.$$

**Proof**. The lower bound is trivial. For the upper bound, by Theorem 4.2.1 it is enough to show that $\Lambda(G) \leq \Delta(G) + 1$. Let $H$ be an induced subgraph of $G$ on $\nu(H) \geq 3$ vertices with $\nu(H)$ odd for which $\Lambda(G) = 2\varepsilon(H)/(\nu(H) - 1)$. Now, since $G$ is simple, $\Delta(H) \leq \nu(H) - 1$. Thus

$$2\varepsilon(H) \leq \Delta(H)\nu(H) \leq \Delta(H)\nu(H) - \Delta(H) + (\nu(H) - 1).$$

Dividing by $\nu(H) - 1$ we have

$$\Lambda(G) = \frac{2\varepsilon(H)}{\nu(H) - 1} \leq \Delta(H) + 1 \leq \Delta(G) + 1.$$ □

Now it is reasonable to ask: Which graphs have $\chi'_f(G) = \Delta(G)$ and which have $\chi'_f(G) = \Delta(G) + 1$? We answer the first for regular graphs and the second for connected graphs.

We say that a graph $G$ is an *$r$-graph* provided $G$ is $r$-regular and for any subset $X \subseteq V(G)$ with $|X|$ odd we have $|\partial X| \geq r$.

**Corollary 4.4.2** *Let $G$ be an $r$-regular graph. Then $\chi'_f(G) = r$ if and only if $G$ is an $r$-graph.*

**Proof**. Let $G$ be $r$-regular and let $H$ be an induced subgraph of $G$ with $V(H) = X$ and with $|X|$ odd. Clearly, if $|X| = 1$ we have $|\partial X| = r$ since $G$ is $r$-regular. So we may consider $|X| \geq 3$. Since

$$\varepsilon(H) = \frac{r|X| - |\partial X|}{2}$$

we have

$$\frac{2\varepsilon(H)}{\nu(H) - 1} \leq r \iff |\partial X| \geq r$$

and the result follows. □

**Corollary 4.4.3** *Let $G$ be a connected simple graph. Then $\chi'_f(G) = \Delta(G) + 1$ if and only if $G = K_{2n+1}$ (with integer $n \geq 1$).*

**Proof**. It is easy to check that $\Lambda(K_{2n+1}) = 2n + 1 = \Delta(K_{2n+1}) + 1$, so if $G = K_{2n+1}$ we have $\chi'_f(G) = \Delta(G) + 1$.

Suppose $G$ is not a complete graph. Let $H$ be any induced subgraph of $G$ with $\nu(H) \geq 3$ and $\nu(H)$ odd. We claim that $2\varepsilon(H)/(\nu(H) - 1) < \Delta(G) + 1$.

Now $2\varepsilon(H) \leq \nu(H)\Delta(H) = [\nu(H) - 1]\Delta(H) + \Delta(H)$ so,

$$\frac{2\varepsilon(H)}{\nu(H) - 1} \leq \Delta(H) + \frac{\Delta(H)}{\nu(H) - 1} \leq \Delta(H) + 1 \leq \Delta(G) + 1.$$

Now if $H$ is not regular or if $\Delta(H) < \nu(H)-1$ or if $\Delta(H) < \Delta(G)$, then one of the above inequalities is strict and so

$$\frac{2\varepsilon(H)}{\nu(H) - 1} < \Delta(G) + 1.$$

This implies that $\Lambda(G) < \Delta(G) + 1$ and, by Theorem 4.2.1 on page 59, $\chi'_f(G) < \Delta(G) + 1$.

The only case that remains is when $H$ satisfies (1) $H$ is regular, (2) $\Delta(H) = \nu(H) - 1$, and (3) $\Delta(H) = \Delta(G)$. Conditions (1) and (2) imply that $H$ is a complete graph. But since $G$ is connected but not complete, if follows that there is some vertex in $G$ that is not in $H$, but adjacent to a vertex $w$ of $H$. But then $\Delta(G) \geq d(w) > \Delta(H)$, a contradiction to condition (3).

Thus, if $\chi'_f(G) = \Delta(G) + 1$, $G$ must be complete. We know that $G \neq K_1$ (since $\chi'_f(K_1) = 0 < \Delta(K_1) + 1$). Further, $\Delta(K_{2n}) \leq \chi'_f(K_{2n}) \leq \chi'(K_{2n}) = 2n - 1 = \Delta(K_{2n})$ so it follows that $G$ must be a complete graph on an odd number (at least 3) of vertices. □

## 4.5 Computational complexity

The computation of the chromatic number and edge chromatic number are NP-hard [117]. As noted in the previous chapter, it is also NP-hard to compute the fractional chromatic number of a graph. Thus, at first glance, one might expect that the fractional edge chromatic number is just as intractable. However, Theorem 4.2.1 on page 59 expressed the fractional edge chromatic number (a minimization problem) in terms of $\Delta(G)$ and $\Lambda(G)$ (which are maximization problems). A consequence of this minimax theorem is that the problem "Given a graph $G$ and integers $a, b > 0$, decide if $\chi'_f(G) \leq a/b$" is in NP and co-NP. This suggests that a polynomial-time solution ought to exist, and indeed one does.

Consider the dual of the LP formulation of the fractional edge chromatic number. The variables are indexed by the edges of the graph and the constraints are indexed by its matchings. A given graph may have exponentially many matchings, so the LP formulation might not be of polynomial size. However, given a weighting of the edges (as a candidate dual solution), one can find a violated constraint (or verify feasibility) in polynomial time by solving a maximum weighted matching problem. Thus (see the discussion on page 137) the value of this LP can be determined in time polynomial in the size of the graph. Therefore one can compute in polynomial time the fractional edge chromatic number of a graph.

## 4.6 Fractional total chromatic number

Given any graph $G$, the *total graph* $T(G)$ is the graph whose vertex set is $V(G) \cup E(G)$ and with an edge between two elements if they are adjacent vertices of $G$, adjacent edges of $G$, or a vertex

and an incident edge of $G$. The *total chromatic number* $\chi''(G)$ of a graph $G$ is just the chromatic number of its total graph. In other words, $\chi''(G)$ is the smallest number of colors needed to color all the vertices and edges of $G$ in such a way that no two adjacent or incident objects are colored the same. We call a subset $S \subseteq V \cup E$ *totally independent* provided no two elements of $S$ are adjacent or incident.

It is easy to see that $\chi''(G) \geq \Delta(G) + 1$, since a vertex of highest degree and its incident edges must all receive different colors. For an upper bound, we can use disjoint palettes for coloring the vertices and the edges and by using the theorems of Brooks [31] and Vizing [183], to obtain $\chi''(G) \leq 2\Delta + 1$ for all graphs $G$. In fact, this upper bound can be substantially improved. In the 1960s, both Behzad [12, 13] and Vizing conjectured that the upper bound can be improved to $\Delta + 2$, a result that would be optimal, considering the case of $K_{2n}$.

**Conjecture 4.6.1 (Total Coloring)** *For any graph $G$, $\chi''(G) \leq \Delta(G) + 2$.*

The current best upper bound for $\chi''(G)$ is due to Hind, Molloy, and Reed [85], who prove that $\Delta + O(\log^8 \Delta)$ colors suffice. In this same paper, Molloy and Reed announce a proof that this can be improved to $\Delta + c$ for some large constant $c$.

The *fractional total chromatic number* $\chi''_f(G)$ is the fractional chromatic number of the total graph $T(G)$. The main result of this section, due to Kilakos and Reed [107], is to prove the total coloring conjecture in the fractional case.

**Theorem 4.6.2** *If $G$ is any graph, then $\chi''_f(G) \leq \Delta(G) + 2$.*

We begin by proving a slightly weaker version of Theorem 4.6.2. The basic ideas used in the proof of the weak version carry over to the proof of the full version.

**Proposition 4.6.3** *If $G$ is any graph, then $\chi''_f(G) \leq \Delta + 3$.*

**Proof**. Throughout this proof $\Delta$ refers to the maximum degree of $G$, i.e., $\Delta = \Delta(G)$. (When we want to speak of the maximum degree of some subgraph $H$ of $G$, we will always write $\Delta(H)$ without suppressing the argument.) Color the vertices of $G$ with $\Delta + 3$ colors. For each $i$ with $1 \leq i \leq \Delta + 3$, let $V_i$ be the set of vertices colored $i$. For each such $i$, apply Vizing's theorem (Theorem 4.1.1 on page 57) to the graph $G - V_i$ to obtain an edge coloring of $G - V_i$ using $\Delta + 1$ colors. For each such $i$ and each $j$ with $1 \leq j \leq \Delta + 1$, let $M_{i,j}$ be the set of edges in $G - V_i$ colored $j$. Note that $T_{i,j} = V_i \cup M_{i,j}$ is a totally independent set.

Assign weight $w_{i,j} = 1/(\Delta + 1)$ to each $T_{i,j}$. (The same weight is assigned to all of these sets.) To see that these weights create a fractional total coloring of $G$, note that, if $v$ is any vertex in $G$, then

$$\sum_{i,j:v \in T_{i,j}} w_{i,j} = \sum_{j=1}^{\Delta+1} \sum_{i:v \in V_i} \frac{1}{\Delta + 1} = 1,$$

since there is exactly one $i$ for which $v \in V_i$. Also, if $e$ is any edge in $G$ whose ends are colored $a$ and $b$, then

$$\sum_{i,j:e \in T_{i,j}} w_{i,j} = \sum_{i \neq a,b} \sum_{j:e \in M_{i,j}} \frac{1}{\Delta + 1} = 1,$$

since for each $i \neq a, b$ there is exactly one $j$ for which $e \in M_{i,j}$.

The total weight of this fractional total coloring is

$$\sum_{i=1}^{\Delta+3} \sum_{j=1}^{\Delta+1} w_{i,j} = \Delta + 3$$

and so $\chi''_f(G) \leq \Delta + 3$. □

We are ready to extend the ideas in the above proof to prove Theorem 4.6.2. The essential idea is to replace the appeal to $\chi'(H) \leq \Delta(H)+1$ (Vizing's theorem) with an appeal to $\chi'_f(H) \leq \Delta(H)$. Unfortunately, this inequality is not true in general; the correct result is given by Theorem 4.2.1 on page 59. A more detailed analysis is needed to circumvent this obstacle.

The basic approach imitates the proof of Proposition 4.6.3. We color the vertices of $G$ with $\Delta+2$ colors; the $i^{\text{th}}$ color class is denoted $V_i$. We then try to *fractionally* color the edges of $G-V_i$ with $\Delta$ colors and complete the proof in a manner similar to the above.

What might go wrong? The problem is that $G-V_i$ might not be fractionally $\Delta$-edge colorable. However, Theorem 4.2.1 tells us that

$$\chi'_f(G-V_i) = \max\{\Delta(G-V_i), \Lambda(G-V_i)\}.$$

Since $\Delta$ is certainly no smaller than $\Delta(G-V_i)$, the only way $G-V_i$ can fail to be $\Delta$-edge colorable is if $\Lambda(G-V_i) > \Delta$. Let $H$ be an induced subgraph of $G$. We call $H$ an *overfull* subgraph of $G$ provided $\nu(H)$ is odd and

$$\Lambda(H) = \frac{2\varepsilon(H)}{\nu(H)-1} > \Delta.$$

Note that if $H$ is a subgraph of $G$ for which $\chi'_f(H) > \Delta$, then, by Theorem 4.2.1, $H$ must contain an overfull subgraph of $G$. In the absence of overfull subgraphs of $G$, the proof proceeds without difficulty. Let us call an overfull subgraph $H$ of $G$ *minimal overfull* provided $H$ is overfull in $G$, but no proper induced subgraph of $H$ is overfull in $G$.

When $H$ is an induced subgraph of $G$ we write $\partial H$ for $\partial V(H)$, i.e., the set of edges of $G$ with exactly one end in $V(H)$.

**Lemma 4.6.4** *If $G$ is a graph of maximum degree $\Delta$ with overfull subgraph $H$, then the following hold:*

(1) $|\partial H| < \Delta$,

(2) $\nu(H) \geq \Delta + 1$, *and*

(3) *$H$ contains a vertex $w$ all of whose neighbors are in $H$.*

*Moreover,*

(4) *any two minimal overfull subgraphs are vertex disjoint, and*

(5) *if $K$ is an induced subgraph of $G$ that does not contain (as an induced subgraph) a minimal overfull subgraph of $G$, then $\chi'_f(K) \leq \Delta$.*

**Proof**. For (1), count the set

$$X = \{(v,e) : v \in V(H), e \in E(G), v \in e\}$$

in two ways. On the one hand, $|X| \leq \nu(H)\Delta$. On the other hand,

$$|X| = 2\varepsilon(H) + |\partial H| > (\nu(H)-1)\Delta + |\partial H|$$

and (1) easily follows.

For (2), note that the number of edges in $H$ is at most $\binom{\nu(H)}{2}$. But then

$$\frac{2\varepsilon(H)}{\nu(H)-1} \leq \nu(H) \leq \Delta,$$

so if $\nu(H) \leq \Delta$, then $H$ is not overfull.

Conclusion (3) follows easily from (1) and (2).

For (4), let $H, K$ be overfull subgraphs of $G$. Let $\varepsilon(A, B)$ denote the number of edges with one end in $A$ and one end in $B$. Without loss of generality, we may assume

$$\varepsilon(H \cap K, H - K) \leq \varepsilon(H \cap K, K - H).$$

It follows that

$$\begin{aligned}
&\varepsilon(H \cap K) + \varepsilon(H \cap K, H - K) \\
&\quad \leq \frac{\nu(H \cap K)\Delta - \varepsilon(H \cap K, G - (H \cap K))}{2} + \\
&\qquad + \varepsilon(H \cap K, H - K) \\
&\quad \leq \frac{\nu(H \cap K)\Delta}{2} - \frac{\varepsilon(H \cap K, K - H)}{2} - \frac{\varepsilon(H \cap K, H - K)}{2} + \\
&\qquad + \varepsilon(H \cap K, H - K) \\
&\quad \leq \frac{\nu(H \cap K)\Delta}{2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\varepsilon(H - K) &= \varepsilon(H) - \varepsilon(H \cap K, H - K) - \varepsilon(H \cap K) \\
&> \frac{(\nu(H) - 1)\Delta}{2} - \frac{\nu(H \cap K)\Delta}{2} \\
&= \frac{(\nu(H - K) - 1)\Delta}{2}.
\end{aligned}$$

Therefore $H - K$ is (or contains, if $\nu(H - K)$ is even) an overfull subgraph of $G$.

Thus, if $H$ and $K$ are distinct *minimal* overfull subgraphs, they must have empty intersection.

Finally, for (5), note that if $\chi'_f(K) > \Delta$, then, by Theorem 4.2.1, $\Lambda(K) > \Delta$. Therefore $K$ contains an overfull subgraph of $G$, and so $K$ must contain a minimal overfull subgraph of $G$. □

In the proof of Theorem 4.6.2 we color $V(G)$ with $\Delta + 2$ colors, but not in an entirely arbitrary manner. Our aim is to ensure that every overfull subgraph of $G$ has at least one vertex from every color class. Were we to be successful in this aim, then $G - V_i$ would be fractionally $\Delta$-edge colorable (where $V_i$ is color class $i$) and the proof would proceed smoothly. However, Lemma 4.6.4 only assures us that minimal overfull subgraphs have at least $\Delta + 1$ vertices, and clearly it is not possible to color the vertices of these with all $\Delta + 2$ colors. Further, it is not obvious that we can color the other minimal overfull subgraphs properly and use all $\Delta + 2$ colors. The next lemma is a partial solution to this problem. We define a minimal overfull subgraph $H$ of $G$ to be *small* if $\nu(H) = \Delta(G) + 1$.

**Lemma 4.6.5** *Let $G$ be a graph with maximum degree $\Delta$. There exists a proper vertex coloring of $G$ using $\Delta + 2$ colors, with classes $V_1, V_2, \ldots, V_{\Delta+2}$, that satisfies the following properties:*

(1) *If $H$ is a small minimal overfull subgraph, then $V(H) \cap V_i \neq \emptyset$ for all $i = 1, 2, \ldots, \Delta + 1$.*

(2) *Otherwise, if $H$ is a minimal overfull subgraph with at least $\Delta+2$ vertices, then $V(H) \cap V_i \neq \emptyset$ for all $i = 1, 2, \ldots, \Delta + 2$.*

**Proof**. We perform a "careful" greedy coloring of $G$ with $\Delta + 2$ colors as follows. Let $H_1, H_2, \ldots$ be the minimal overfull subgraphs of $G$. These subgraphs are pairwise disjoint by Lemma 4.6.4 (4). Having colored $H_1, \ldots, H_{j-1}$ in the manner described, color $H_j$ as follows. Sort the vertices in $H_j$ in decreasing order of their degree to non-$H_j$ neighbors, i.e., in decreasing order of $|\partial H_j \cap \partial v|$ for all $v \in V(H_j)$. From the palette $\{1, 2, \ldots, \Delta + 2\}$ assign to vertices (in the order described) the first proper (legal) color not already used in $H_j$. Since no vertex has degree greater than $\Delta$, there are more than enough colors to color $H_j$ properly. Since $H_j$ has at least $\Delta + 1$ vertices, but at most $\Delta - 1$ edges to $V(G) - V(H_j)$ (see Lemma 4.6.4 (1) and (2)), we know we can color $H_j$ properly using all $\Delta + 2$ colors, unless $H_j$ is small minimal overfull, in which case we have used colors 1 through $\Delta + 1$.

Finally, color all vertices not in a minimal overfull subgraph greedily. Again, since there are $\Delta + 2$ colors available, this gives a proper coloring of the entire graph $G$. □

**Proof** (of Theorem 4.6.2). We continue to write $\Delta$ for $\Delta(G)$.

Color $V(G)$ with $\Delta + 2$ colors as described in Lemma 4.6.5, i.e., so that every minimal overfull subgraph intersects all $\Delta+2$ color classes $V_i$ except for the small minimal overfull subgraphs, which intersect all color classes except $V_{\Delta+2}$.

Let $H_1, H_2, \ldots, H_r$ be the small minimal overfull subgraphs of $G$. By part (3) of Lemma 4.6.4 there is in each $H_k$ a vertex $w_k$ so that all neighbors of $w_k$ are contained in $H_k$. (Indeed, there are at least two such choices; let $w_k$ be any such vertex.) Set $W = \{w_1, w_2, \ldots, w_r\}$ and set $W_i = W \cap V_i$, i.e., $W_i$ is the set of $W$ vertices that have color $i$.

For $1 \leq i \leq \Delta + 1$, let $G_i$ be the multigraph formed from $G - (V_i - W_i)$ by doubling every edge except those incident with vertices in $W_i$. In other words, starting from $G$, delete all vertices of color $i$ except those in $W_i$. Then replace every edge by a double edge except for edges incident with vertices in $W_i$.

Similarly, let $G_{\Delta+2}$ be the multigraph formed from $G - V_{\Delta+2}$ by doubling every edge of $G$ except for those incident with $W$ (regardless of the color).

The key step is the following:

**Claim**. For $1 \leq i \leq \Delta + 2$, we have $\chi'_f(G_i) \leq 2\Delta$.

To prove the claim, we appeal to Theorem 4.2.1 on page 59. Since the maximum degree of $G_i$ is (at most) $2\Delta$, we must show that $\Lambda(G_i) \leq 2\Delta$. Suppose, for sake of contradiction, that $G_i$ contains a subgraph $L_i$ which is overfull (with respect to $G_i$). Without loss of generality, assume $L_i$ is minimal overfull in $G_i$.

Since $G_i$ is a (mostly) doubled version of an induced subgraph of $G$, any overfull subgraph of $G_i$ will also be overfull in $G$. To see this, let $L$ be the subgraph of $G$ induced on $V(L_i)$. Then

$$2\Delta < \frac{2\varepsilon(L_i)}{\nu(L_i) - 1} \leq \frac{4\varepsilon(L)}{\nu(L) - 1}$$

so $L$ is overfull in $G$.

Therefore, the only overfull subgraphs of $G$ that might "survive" in $G_i$ are some of the small minimal subgraphs $H_k$. Thus $V(L_i) = V(H_k)$ for some $k$. Now we compute

$$\frac{2\varepsilon(L_i)}{\nu(L_i) - 1} \leq \frac{2[2\binom{\Delta}{2} + \Delta]}{\Delta} = 2\Delta.$$

Thus $L_i$ is not overfull with respect to $G_i$ and therefore $\chi'_f(G_i) \leq 2\Delta$, proving the claim.

Now, for each $i$, let $f_i$ be an optimal fractional edge coloring of $G_i$, i.e., $f_i$ assigns a weight to each matching of $G_i$ so that

$$\sum_{M:e\in M} f_i(M) \geq 1 \qquad \text{and} \qquad \sum_{M} f_i(M) \leq 2\Delta.$$

The first sum holds for every $e \in E(G_i)$ and is over all matchings containing $e$. The second sum is over all matchings in $G_i$. However, using exercise 20 on page 55, we may assume that equality holds in both sums, i.e., the total weight of the matchings containing a given edge is exactly 1 and the total weight of all matchings is exactly $2\Delta$.

Each $f_i$ is a fractional edge coloring of $G_i$, a multigraph derived from $G$. A matching $M$ of $G_i$ can be considered to be a matching of $G$ as well. Formally, if $M$ is a matching of $G_i$, let $\pi(M)$ be the *projection* of $M$ into $G$ in which $e \in \pi(M)$ if and only if $e$ or its double is in $M$. We can extend $f_i$ to be defined on matchings of $G$ by $f_i(M) = \sum f_i(N)$ where the sum is over all matchings $N$ of $G_i$ for which $\pi(N) = M$.

Let $\mathbf{M}_i$ denote the set of all matchings we can form by projecting a matching of $G_i$ to $G$. Note that if $e$ is an edge of $G$ that is not incident to any vertex in $V_i \cup W$ (so $e$ is doubled in $G_i$), then

$$\sum_{M:e\in M\in\mathbf{M}_i} f_i(M) = 2$$

while if $e$ is incident with a vertex of $W_i$ (so $e$ is not doubled in $G_i$), then

$$\sum_{M:e\in M\in\mathbf{M}_i} f_i(M) = 1.$$

Thus if $w$ is a vertex in $W_i$, we have that

$$\sum_{\substack{M:M\in\mathbf{M}_i\\ \partial w\cap M\neq\emptyset}} f_i(M) \leq d(w) \leq \Delta$$

and

$$\sum_{\substack{M:M\in\mathbf{M}_{\Delta+2}\\ \partial w\cap M\neq\emptyset}} f_{\Delta+2}(M) \leq d(w) \leq \Delta,$$

hence

$$\sum_{\substack{M:M\in\mathbf{M}_i\\ \partial w\cap M=\emptyset}} f_i(M) \geq \Delta \qquad \text{and} \qquad \sum_{\substack{M:M\in\mathbf{M}_{\Delta+2}\\ \partial w\cap M=\emptyset}} f_{\Delta+2}(M) \geq \Delta. \qquad (*)$$

We are ready to present the fractional total coloring of $G$. For each $i$ with $1 \leq i \leq \Delta + 2$ and for each matching $M \in \mathbf{M}_i$ let

$$T_{i,M} = \begin{cases} M \cup (V_i - \{w : w \in W_i,\, \partial w \cap M \neq \emptyset\}) & \text{for } 1 \leq i \leq \Delta + 1 \\ M \cup V_{\Delta+2} \cup \{w : w \in W,\, \partial w \cap M = \emptyset\} & \text{for } i = \Delta + 2. \end{cases}$$

Roughly speaking, we form $T_{i,M}$ by including with matching $M$ all vertices of color $i$ we possibly can. Clearly, every $T_{i,M}$ is a totally independent set.

Next, we assign to $T_{i,M}$ the weight

$$w(T_{i,M}) = \frac{f_i(M)}{2\Delta}.$$

We claim that these weights give a total fractional coloring of $G$ with total weight $\Delta + 2$. To this end we show the following:

(1) the sum of the weights of all $T_{i,M}$ is exactly $\Delta + 2$;

(2) for every vertex, the sum of the weights of the $T_{i,M}$ containing that vertex is (at least) one; and

(3) for every edge, the sum of the weights of the $T_{i,M}$ containing that edge is (at least) one.

We take each of these tasks in turn.

For (1), the sum of the weights of all totally independent sets is

$$\sum_T w(T) = \sum_{i=1}^{\Delta+2} \sum_{M \in \mathbf{M}_i} \frac{f_i(M)}{2\Delta} = \sum_{i=1}^{\Delta+2} 1 = \Delta + 2.$$

For (2), we first consider a vertex $v$ *not* in $W$. Note that $v \in V_j$ for some $1 \leq j \leq \Delta + 2$ and the only totally independent sets that contain $v$ and to which we have assigned any weight are of the form $T_{j,M}$. Thus

$$\sum_{T \ni v} w(T) = \sum_{M \in \mathbf{M}_j} w(T_{j,M}) = \sum_{M \in \mathbf{M}_j} \frac{f_j(M)}{2\Delta} = \frac{2\Delta}{2\Delta} = 1$$

as required.

Next, let $w \in W$. We know that $w \in V_j$ for some $1 \leq j \leq \Delta + 1$. The total stable sets that contain $w$ are of two types: those of the form $T_{j,M}$ and those of the form $T_{\Delta+2,M}$. In both cases, we only consider those matchings $M$ for which $\partial w \cap M = \emptyset$. We apply $(*)$ to compute:

$$\sum_{T \ni w} w(T) = \sum_{\substack{M : M \in \mathbf{M}_j \\ \partial w \cap M = \emptyset}} \frac{f_j(M)}{2\Delta} + \sum_{\substack{M : M \in \mathbf{M}_{\Delta+2} \\ \partial w \cap M = \emptyset}} \frac{f_{\Delta+2}(M)}{2\Delta} \geq \frac{\Delta}{2\Delta} + \frac{\Delta}{2\Delta} = 1.$$

For (3), first consider an edge $e$ with $e \notin \partial W$. Suppose the ends of $e$ are in $V_a$ and $V_b$. Then we have

$$\sum_{T \ni e} w(T) = \sum_{\substack{j=1 \\ j \neq a,b}}^{\Delta+2} \sum_{M \in \mathbf{M}_j} \frac{f_j(M)}{2\Delta} = \sum_{\substack{j=1 \\ j \neq a,b}}^{\Delta+2} \frac{2}{2\Delta} = 1.$$

If on the other hand $e$ is an edge incident with a vertex of $W$, it has one end in $W \cap V_a$ and the other end in $V_b$. We compute

$$\sum_{T \ni e} w(T) = \sum_{\substack{j=1 \\ j \neq b}}^{\Delta+2} \sum_{M : e \in M \in \mathbf{M}_j} \frac{f_j(M)}{2\Delta}$$

$$= \sum_{\substack{j=1 \\ j \neq a,b}}^{\Delta+1} \sum_{M : e \in M \in \mathbf{M}_j} \frac{f_j(M)}{2\Delta} + \sum_{M : e \in M \in \mathbf{M}_a} \frac{f_a(M)}{2\Delta} + \sum_{M : e \in M \in \mathbf{M}_{\Delta+2}} \frac{f_{\Delta+2}(M)}{2\Delta}$$

$$= \frac{2(\Delta - 1)}{2\Delta} + \frac{1}{2\Delta} + \frac{1}{2\Delta} = 1.$$

This completes the proof. □

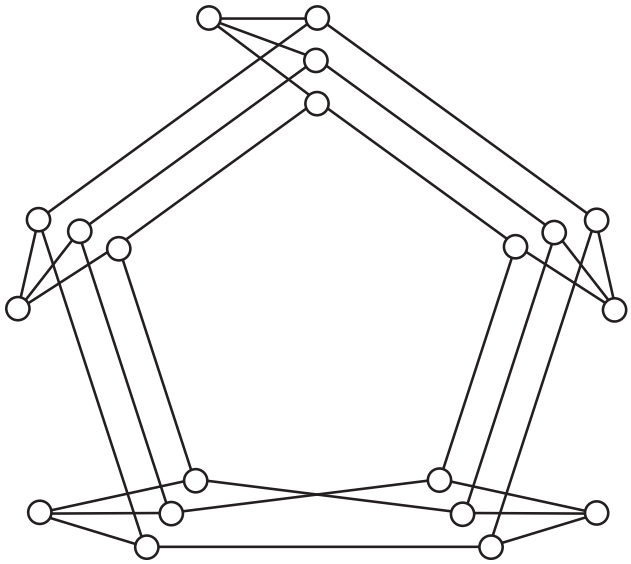

Figure 4.1. A graph $G$ with $\chi'(G) = 4$ but $\chi'_f(G) = 3$.

## 4.7 Exercises

1. Prove that all bipartite graphs are class one.

2. Here are two candidate definitions for "edge coloring perfect":

   - A graph $G$ is *edge coloring perfect* provided $\chi'(H) = \Delta(H)$ for all induced subgraphs $H$ of $G$.
   - A graph $G$ is *edge coloring perfect* provided $\chi'(H) = \chi'_f(H)$ for all induced subgraphs $H$ of $G$.

   Prove these definitions are equivalent and characterize this family of graphs.

3. Consider the following alternative approach to fractional edge coloring. Let $k$ be a positive integer. Call a graph $G$ *$k$-edge fractionally colorable* if there exist fractional matchings $f_1, f_2, \ldots, f_k$ of $G$ so that for every $e \in E(G)$ we have $\sum_j f_j(e) \geq 1$. Let $\tilde{\chi}'(G)$ be the least $k$ for which $G$ is $k$-edge fractionally colorable. Note that $\tilde{\chi}'$ is an integer.

   Why is $\tilde{\chi}'$ uninteresting?

   Hint: Prove a simple result about the value of $\tilde{\chi}'$.

4. Prove that the incidence vector of a matching cannot be written as the convex combination of incidence vectors of other matchings. Conclude that the extreme points of the matching polytope are exactly all the incidence vectors of the matchings.

5. Consider the graph $G$ in Figure 4.1. Show that $\chi'(G) = 4$ but $\chi'_f(G) = 3$.

6. Let $G$ be a connected graph other than $K_{2n+1}$ (for integer $n \geq 1$). Prove that

$$\chi'_f(G) \leq \begin{cases} \Delta + \frac{\Delta-1}{\Delta+1} & \text{if } \Delta \text{ is odd, and} \\ \Delta + \frac{\Delta}{\Delta+2} & \text{if } \Delta \text{ is even} \end{cases}$$

where $\Delta = \Delta(G)$.

7. By Tait's theorem, the four-color theorem is equivalent to the statement that all 3-regular, 2-edge-connected planar graphs have $\chi' = 3$. Prove that every 3-regular, 2-edge connected graph is a 3-graph. (See page 62 for the definition of a 3-graph.) Conclude that all such graphs have $\chi'_f = 3$. This gives a fractional analogue of the four-color theorem — though perhaps not the analogue one might have hoped for — whose proof does not depend on the four-color theorem itself.

## 4.8 Notes

Our exposition of fractional edge chromatic number is based on work by Edmonds, Lovász, Plummer, Seymour, and Stahl. The material on the total chromatic number is based on the work of Kilakos and Reed. See [48], [107], [124], [161], and [172].

The inequalities in Edmond's Theorem 4.3.1 on page 59 are somewhat redundant; all facet-defining inequalities are listed, but not all those listed are facet-defining. See Edmonds [48]. The proof we present is a simplified version of the one in Lovász and Plummer [124]; we save some effort by not attempting to sort out which of the inequalities are facet-defining.

Seymour [161] gives a non-polyhedral approach to proving Theorem 4.2.1 on page 59 by studying $r$-graphs (see Corollary 4.4.2 on page 62).

Corollary 4.4.3 and exercise 6 are due to Stahl [171].

The graph in Figure 4.1 (see exercise 5) is due to Isaacs [100].

The Kneser graphs show that the chromatic number and fractional chromatic number of a given graph can be quite different. However, for a graph $G$ we have $\Delta(G) \leq \chi'_f(G) \leq \chi'(G) \leq \Delta(G) + 1$, so the ordinary and fractional edge chromatic numbers of a graph are both close to its maximum degree. The situation is not so clear for multigraphs. Consider the multigraph formed from $K_3$ by replacing each edge with $t$ parallel edges. In this case $\Delta = 2t$ but $\chi' = \chi'_f = 3t$. Thus the gap between maximum degree and [fractional] edge chromatic number can be large. The best that can be said is that $\chi'(G) \leq \frac{3}{2}\Delta(G)$; this is a result of Shannon [164].

Instead, let us consider the gap between $\chi'(G)$ and $\Lambda(G)$ for a multigraph $G$. The following conjecture was proposed by Andersen [6], Goldberg [77], and Seymour [162].

**Conjecture 4.8.1** *Let $G$ be a multigraph with $\chi'(G) > \Delta(G) + 1$. Then $\chi'(G) = \lceil \Lambda(G) \rceil$.*

If correct, this conjecture would imply that $\chi'_f$ is between $\chi'$ and $\chi' + 1$. Thus it seems reasonable to suppose that $\chi'$ and $\chi'_f$ should always be close. Indeed, Kahn [103] proves just that.

**Theorem 4.8.2** *For every $\varepsilon > 0$ there exists an $N > 0$ so that for every multigraph $G$ with $\chi'_f(G) > N$ we have $\chi'(G) \leq (1 + \varepsilon)\chi'_f(G)$.* □

Stated succinctly, Kahn's result says that the edge chromatic number is asymptotic to the fractional edge chromatic number. In a subsequent paper, Kahn [104] shows that not only are the ordinary and fractional edge chromatic numbers asymptotic to one another, but they are also asymptotic to the list edge chromatic number. Of course, the *fractional* list edge chromatic number is the same as the ordinary fractional edge chromatic number by Theorem 3.8.1 on page 52.

# 5

# Fractional Arboricity and Matroid Methods

The material in this chapter is motivated by two notions of the density of a graph. The *arboricity* and the *maximum average degree* of a graph $G$ measure the concentration of edges in the "thickest" part of the graph.

## 5.1 Arboricity and maximum average degree

Suppose we wish to decompose the edges of a graph $G$ into acyclic subsets, i.e., if $G = (V, E)$ we want to find $E_1, E_2, \ldots, E_k \subseteq E$ so that (1) each of the subgraphs $(V, E_i)$ is acyclic and (2) $E = E_1 \cup E_2 \cup \cdots \cup E_k$. The smallest size of such a decomposition is called the *arboricity* (or *edge-arboricity*) of $G$ and is denoted $\Upsilon(G)$. If $G$ is connected, the arboricity is also the minimum number of spanning trees of $G$ that include all edges of $G$.

One can think of arboricity as being a variant of the edge chromatic number. We are asked to paint the edges of $G$ with as few colors as possible. In the case of edge chromatic number, we do not want to have two edges of the same color incident with a common vertex. In the case of arboricity, we do not want to have a monochromatic cycle.

There is an obvious lower bound on $\Upsilon(G)$. Since $G$ has $\varepsilon(G)$ edges and each spanning acyclic subgraph has at most $\nu(G) - 1$ edges we have $\Upsilon(G) \geq \varepsilon(G)/(\nu(G) - 1)$. Moreover, since $\Upsilon$ is an integer, we have $\Upsilon(G) \geq \left\lceil \frac{\varepsilon(G)}{\nu(G)-1} \right\rceil$.

This bound is not very accurate if the graph is highly "unbalanced"; for example, consider the graph $G$ consisting of a $K_9$ with a very long tail attached—say 100 additional vertices. We have $\nu(G) = 109$, $\varepsilon(G) = 136$, and therefore $\Upsilon(G) \geq \left\lceil \frac{136}{108} \right\rceil = 2$. The actual value of $\Upsilon(G)$ is larger since we clearly cannot cover the edges of $K_9$ with two trees; indeed, the arboricity of a graph is at least as large as the arboricity of any of its subgraphs. Thus we have

$$\Upsilon(G) \geq \max \left\lceil \frac{\varepsilon(H)}{\nu(H) - 1} \right\rceil$$

where the maximum is over all subgraphs of $H$ with at least 2 vertices. Indeed, this improved lower bound gives the correct value.

**Theorem 5.1.1**

$$\Upsilon(G) = \max \left\lceil \frac{\varepsilon(H)}{\nu(H) - 1} \right\rceil$$

*where the maximum is over all subgraphs of $H$ with at least 2 vertices.*

The proof of this theorem of Nash-Williams [137, 138] is presented in §5.4 below.

Notice that the arboricity of a graph can be expressed as a hypergraph covering problem. Given a graph $G = (V, E)$ we define a hypergraph $\mathcal{H} = (E, \mathcal{X})$ where $\mathcal{X}$ is the set of all acyclic subsets of edges of $G$. Then $\Upsilon(G) = k(\mathcal{H})$.

The *fractional arboricity of* $G$, denoted $\Upsilon_f(G)$, is simply $k_f(\mathcal{H})$. In other words, to compute $\Upsilon_f(G)$ we assign weights to the various acyclic subsets of edges so that every edge is contained in sets of total weight at least 1 and we seek to minimize the total of all the weights.

The arboricity of $G$ gives us some information about the "density" of $G$; graphs with higher arboricity are more tightly "packed" with edges. Another measure of the density of a graph is its *average degree*: let

$$\bar{d}(G) = \frac{\sum d(v)}{\nu(G)} = \frac{2\varepsilon(G)}{\nu(G)}.$$

Average degree gives us an overall measure of a graph's density, but does not indicate the density in the densest portion of the graph. Reconsider the graph $G$ consisting of a $K_9$ with a long tail. The average degree in such a graph is fairly low: $\bar{d}(G) = 2 \times 136/109 \approx 2.5$. However, the dense part of this graph—the $K_9$—has average degree 8. The value 8, in this case, is a better measure of the densest portion of the graph.

We define the *maximum average degree* of $G$, denoted $\text{mad}(G)$, to be the maximum of $\bar{d}(H)$ over all subgraphs $H$ of $G$.

The invariant $\text{mad}(G)$ arises naturally in the theory of random graphs; see, for example, [23, 141].

At first glance, $\text{mad}(G)$ does not seem to fit the paradigm of our other fractional graph-theoretic invariants. However, in §5.5, we see $\text{mad}(G)$ in the same light as $\Upsilon_f(G)$.

## 5.2 Matroid theoretic tools

The way to understand arboricity and maximum average degree is through matroid theory.

### Basic definitions

A *matroid* is a pair $\mathcal{M} = (S, \mathcal{I})$ where $S$ is a finite set and $\mathcal{I}$ is a subset of $2^S$ that satisfies the following three conditions:

- (nontriviality) $\emptyset \in \mathcal{I}$,
- (heredity) if $X \in \mathcal{I}$ and $Y \subseteq X$, then $Y \in \mathcal{I}$, and
- (augmentation) if $X, Y \in \mathcal{I}$ and $|X| > |Y|$, then there is an $x \in X - Y$ so that $Y \cup \{x\} \in \mathcal{I}$.

The set $S$ is called the *ground set* of the matroid and the sets in $\mathcal{I}$ are called the *independent sets* of $\mathcal{M}$. Note that the term *independent* is motivated by the concept of linear independence and is not related to graph-theoretic independence (what some graph theorists also call stability).

The following two basic examples provide the motivation for this abstraction.

**Example 5.2.1** *Columns of a matrix.* Let $A$ be a matrix and let $S$ be the set of its columns. We put a collection $I$ of columns of $A$ in $\mathcal{I}$ exactly when those columns are linearly independent. The resulting pair $\mathcal{M}(A) = (S, \mathcal{I})$ is a matroid. See exercise 3 on page 90.

**Example 5.2.2** *Acyclic sets in a graph.* Let $G = (V, E)$ be a graph. The *cycle matroid* of $G$, denoted $\mathcal{M}(G)$, has ground set $E$ and a subset $I \subseteq E$ is independent in $\mathcal{M}$ provided $(V, I)$ has no cycles. Proving that $\mathcal{M}(G)$ forms a matroid is relegated to exercise 4 on page 90. A matroid is called *graphic* if it is (isomorphic to) the cycle matroid of a graph.

## Bases

The maximal independent sets of $\mathcal{M}$ are called the *bases* of $\mathcal{M}$ and the set of all bases of $\mathcal{M}$ is denoted by $\mathcal{B}$ (or $\mathcal{B}(\mathcal{M})$). If $G$ is a connected graph, then the bases of $\mathcal{M}(G)$ are exactly the (edge sets of) spanning trees of $G$. If $A$ is a matrix, the bases of $\mathcal{M}(A)$ are those subsets of columns of $A$ that form bases, in the sense of linear algebra, of the column space of $A$.

**Theorem 5.2.3** *The set of bases $\mathcal{B}$ of a matroid $\mathcal{M}$ satisfies the following:*

- $\mathcal{B} \neq \emptyset$,
- *if $X, Y \in \mathcal{B}$, then $|X| = |Y|$, and*
- *if $X, Y \in \mathcal{B}$ and $x \in X$, then there is a $y \in Y$, so that $(X - \{x\}) \cup \{y\} \in \mathcal{B}$.*

*Moreover, if $\mathcal{B}$ is any collection of subsets of some set $S$ and $\mathcal{B}$ satisfies the above properties, then $\mathcal{B}$ forms the set of bases of a matroid on $S$.*

Note: The fact that any two spanning trees of a graph have the same number of edges, and the fact that any two bases of a vector space have the same cardinality, are instances of this theorem.

**Proof**. Since $\mathcal{I} \neq \emptyset$, we know that there are maximal independent sets, hence $\mathcal{B} \neq \emptyset$.

Suppose $X, Y \in \mathcal{B}$ with $|X| > |Y|$. Since $X$ and $Y$ are independent, there is an $x \in X - Y$ so that $Y \cup \{x\} \in \mathcal{I}$, contradicting the maximality of $Y$. Thus any two bases have the same size.

To show the third property, we note that $X - \{x\} \in \mathcal{I}$ and has one less element than $Y$, so there is a $y \in Y - (X - \{x\})$ so that $X' = (X - \{x\}) \cup \{y\}$ is independent. Were $X'$ not a basis, it would be contained in a strictly larger basis $X''$, but then $|X''| > |X'| = |X|$, which contradicts the second property.

Now suppose $\mathcal{B}$ satisfies the three stated properties. Define

$$\mathcal{I} = \{X \subseteq S : X \subseteq B \text{ for some } B \in \mathcal{B}\}.$$

Note that since $\mathcal{B} \neq \emptyset$, there is some $B \in \mathcal{B}$ and since $\emptyset \subseteq B$, we have $\emptyset \in \mathcal{I}$.

Clearly if $X \subseteq Y$ and $Y \in \mathcal{I}$, then $X \in \mathcal{I}$.

Finally, we show that $\mathcal{I}$ has the augmentation property. Suppose $X, Y \in \mathcal{I}$ and $|X| < |Y|$. Choose $A, B \in \mathcal{B}$ so that $X \subseteq A$ and $Y \subseteq B$. We know that $A$ and $B$ have the same cardinality, which we call $r$. We may also assume that we have chosen $A$ and $B$ so that $|A \cap B|$ is as large as possible.

Write

$$\begin{aligned} X &= \{x_1, x_2, \ldots, x_j\}, \\ A &= \{x_1, x_2, \ldots, x_j, a_1, a_2, \ldots, a_{r-j}\}, \\ Y &= \{y_1, y_2, \ldots, y_j, \ldots, y_{j+k}\}, \text{ and} \\ B &= \{y_1, y_2, \ldots, y_j, \ldots, y_{j+k}, b_1, \ldots, b_{r-j-k}\}. \end{aligned}$$

If some $a_i \in Y$ then $X \cup \{a_i\} \in \mathcal{I}$ and we're done. So we consider the case that no $a_i \in Y$. Without loss of generality, we may also assume $a_1 \notin B$ (since there are more $a$'s than $b$'s). Consider $A - \{a_1\}$. By hypothesis, we know there is a $z \in B$ so that $A' = (A - \{a_1\}) \cup \{z\} \in \mathcal{B}$. Since all members of $\mathcal{B}$ have the same cardinality, we know that $z \notin A$ and therefore $z \notin X$. If we are lucky, $z \in Y$ (and thus $z \in Y - X$) so then $X \cup \{z\} \subseteq A'$ so $X \cup \{z\} \in \mathcal{I}$ and we're done.

Otherwise ($z \notin Y$), we can replace $A$ by $A'$ and observe that $A'$ and $B$ have more elements in common than do $A$ and $B$. This contradicts the maximality of $|A \cap B|$. □

## Rank

We have seen that any two bases of a matroid have the same cardinality. Just as the maximum number of linearly independent columns in a matrix is called the rank of the matrix, so too do we define the *rank* of a matroid $\mathcal{M}$, denoted $\rho(\mathcal{M})$, to be the maximum size of an independent set in $\mathcal{M}$.

We can extend the notion of rank further. Let $X$ be any subset of the ground set of a matroid $\mathcal{M}$. One readily checks that any two maximal independent subsets of $X$ are necessarily of the same size. We therefore define the *rank* of $X$, denoted $\rho(X)$, to be the maximum cardinality of an independent subset of $X$.

When $A$ is a matrix and $X$ is a subset of its columns, then the rank of $X$ in the matroid $\mathcal{M}(A)$ is exactly the rank (in the usual linear algebra sense) of the matrix composed of those columns.

The rank function for a graphic matroid can be described as follows. Let $G = (V, E)$ be a graph and let $\mathcal{M}(G)$ be its cycle matroid. Let $F \subseteq E$. The rank $\rho(F)$ is the maximum size of an acyclic subset of $F$. If the graph $(V, F)$ has $c$ components (including isolated vertices, if any), then $\rho(F) = \nu(G) - c$.

Properties of the rank function of a general matroid are collected in the following result.

**Theorem 5.2.4** *Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid and let $\rho$ be its rank function. Then:*

- $\rho(\emptyset) = 0$,
- *if $X \subseteq Y \subseteq S$, then $\rho(X) \leq \rho(Y)$,*
- *if $X \subseteq S$, then $0 \leq \rho(X) \leq |X|$,*
- *if $X \subseteq S$ and $y \in S$, then $\rho(X) \leq \rho(X \cup \{y\}) \leq \rho(X) + 1$,*
- *if $X \subseteq S$ and $x, y \in S$ satisfy $\rho(X \cup \{x\}) = \rho(X \cup \{y\}) = \rho(X)$, then $\rho(X \cup \{x, y\}) = \rho(X)$, and*
- *if $X, Y \subseteq S$, then $\rho(X \cup Y) + \rho(X \cap Y) \leq \rho(X) + \rho(Y)$.*

**Proof**. The verification of most of these properties is routine; here we just prove the last property, which is known as the *submodular inequality*.

Pick $X, Y \subseteq S$. Let $I$ be a maximal independent subset of $X \cap Y$. Using the augmentation property, we can construct a $J \supseteq I$ that is maximal independent in $X \cup Y$. Furthermore, let $K = J - Y$ and $L = J - X$. See Figure 5.1. Note that $K \cup I$ is an independent subset of $X$ and

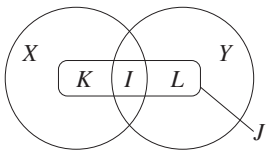


Figure 5.1. Understanding why the rank function of a matroid satisfies the submodular inequality.

$L \cup I$ is an independent subset of $Y$. Thus we may compute

$$\begin{aligned}\rho(X) + \rho(Y) &\geq |K \cup I| + |L \cup I| \\ &= 2|I| + |K| + |L| \\ &= |I| + |J| \\ &= \rho(X \cap Y) + \rho(X \cup Y).\end{aligned}$$

□

## Circuits

Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid. The subsets of $S$ not in $\mathcal{I}$ are called *dependent*. The minimal dependent subsets of $\mathcal{M}$ are called the *circuits* of $\mathcal{M}$. If $\mathcal{M}$ is the cycle matroid of a graph $G$, the circuits of $\mathcal{M}$ correspond exactly to the (edges sets of) cycles in $G$. The set of circuits of $\mathcal{M}$ is denoted $\mathcal{C}$ (or $\mathcal{C}(\mathcal{M})$).

If a singleton $\{x\}$ is dependent, it forms a circuit called a *loop* of the matroid. Note that a loop of a multigraph $G$ corresponds exactly to a loop of $\mathcal{M}(G)$. If $A$ is a matrix, a loop of $\mathcal{M}(A)$ corresponds to a column of all zeros.

If neither $x$ nor $y$ is a loop but $\{x, y\}$ is dependent (and therefore a circuit), then the elements $x$ and $y$ are called *parallel*. Edges $x, y \in E(G)$ are parallel edges (i.e., have the same endpoints) if and only they are parallel elements of $\mathcal{M}(G)$. Nonzero columns $x, y$ of a matrix $A$ are parallel elements of $\mathcal{M}(A)$ just when each is a scalar multiple of the other.

**Theorem 5.2.5** *If $\mathcal{M}$ is a matroid and $\mathcal{C}$ is the set of circuits of $\mathcal{M}$ then the following conditions hold.*

- *If $X, Y \in \mathcal{C}$ and $X \subseteq Y$, then $X = Y$, and*
- *if $X, Y \in \mathcal{C}$, $X \neq Y$, and $a \in X \cap Y$, then there exists a $Z \in \mathcal{C}$ with $Z \subseteq (X \cup Y) - \{a\}$.*

**Proof**. The first property is trivial. For the second, suppose $X, Y$ are distinct circuits with $a \in X \cap Y$. Then $X \cap Y$ is independent, and so

$$\begin{aligned}\rho(X) &= |X| - 1, \\ \rho(Y) &= |Y| - 1, \text{ and} \\ \rho(X \cap Y) &= |X \cap Y|.\end{aligned}$$

The submodularity inequality then gives

$$\rho(X \cup Y) \leq \rho(X) + \rho(Y) - \rho(X \cap Y)$$

$$= |X| - 1 + |Y| - 1 - |X \cap Y|$$
$$= |X \cup Y| - 2.$$

But then

$$\rho((X \cup Y) - \{a\}) \leq \rho(X \cup Y) < |(X \cup Y) - \{a\}|.$$

Thus $X \cup Y - \{a\}$ is dependent, and so contains a circuit. □

We know that if we add an edge between two vertices of a tree we create a unique cycle. A similar result holds for matroids.

**Theorem 5.2.6** *Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid, let $I$ be independent, and let $x \in S - I$. If $I \cup \{x\} \notin \mathcal{I}$, then there is a unique $C \in \mathcal{C}$ with $C \subseteq I \cup \{x\}$.*

**Proof**. Since $I \cup \{x\}$ is dependent, it contains a minimal dependent subset $C \in \mathcal{C}$. To show uniqueness, suppose $C_1, C_2 \in \mathcal{C}$ and $C_1, C_2 \subseteq I \cup \{x\}$, but $C_1 \neq C_2$. Note that $x \in C_1 \cap C_2$ (otherwise one of $C_1$ or $C_2$ would be contained in $I$, contradicting the independence of $I$) and therefore, by Theorem 5.2.5 on the facing page, there is a $C \in \mathcal{C}$ with $C \subseteq (C_1 \cup C_2) - \{x\} \subseteq I$, a contradiction. □

## 5.3 Matroid partitioning

Let us now revisit the arboricity of a graph in the language of matroid theory. Given a graph $G = (V, E)$, we wish to partition[1] $E$ into the smallest number of sets $E_1, E_2, \ldots, E_k$ so that each of the graphs $(V, E_i)$ is acyclic. In matroid language, we seek to partition $S$, the ground set of $\mathcal{M} = (S, \mathcal{I})$, into $S_1, S_2, \ldots, S_k$ so that each $S_i \in \mathcal{I}$.

Since a matroid $\mathcal{M} = (S, \mathcal{I})$ is a hypergraph, it makes sense to speak of the covering number of a matroid and to write $k(\mathcal{M}(G))$. Note that $\Upsilon(G) = k(\mathcal{M}(G))$.

It is perhaps useful to use the language of "coloring" to discuss matroid partitioning. We seek to color the elements of the ground set $S$ so that each color class (the set of elements of a given color) is independent. Indeed, we call a coloring of the elements *proper* in just this case.

Our goal is to find formulas and efficient (polynomial-time) algorithms for computing $k$ and $k_f$ of a matroid $\mathcal{M}$. For convenience we present the formulas here:

$$k(\mathcal{M}) = \max_{Y \subseteq S} \left\lceil \frac{|Y|}{\rho(Y)} \right\rceil \qquad \text{and} \qquad k_f(\mathcal{M}) = \max_{Y \subseteq S} \left( \frac{|Y|}{\rho(Y)} \right).$$

(See Corollary 5.3.3 on page 81 and Theorem 5.4.1 on page 83.)

We begin by developing an algorithm, known as the Matroid Partitioning Algorithm, that computes whether or not a given matroid $\mathcal{M}$ can be partitioned into a given number $k$ of independent sets. In this algorithm, we begin with all the elements of the ground set $S$ uncolored. As the algorithm progresses, colors are assigned to elements of $S$ and, at various times, the color assigned to an element can change. We first prove a lemma assuring ourselves that the recoloring step of the algorithm maintains the independence of the color classes. We then formulate the complete algorithm (on page 80) by iterating the lemma. We conclude with proofs of the formulas for $k(\mathcal{M})$ and $k_f(\mathcal{M})$.

[1] We may either *partition* or *cover* $E$ by acyclic subsets of edges—there is no difference since a subset of an acyclic set of edges is also acyclic. See exercise 7 on page 13.

Suppose $S_i$ (with $1 \leq i \leq k$) represents the set of elements that are currently color $i$ and let $x$ be any element. We can (re)assign color $i$ to $x$ (and still have a proper coloring) exactly when $S_i \cup \{x\} \in \mathcal{I}$. It is useful to have a notation for this. Let us write

$$x \leftarrow \langle i \rangle \quad \text{to mean} \quad x \notin S_i \text{ and } S_i \cup \{x\} \in \mathcal{I},$$

which we read as "$x$ may get color $i$".

At times, the algorithm transfers a color from one element to another. Let $x$ and $y$ be elements in $S$ with $y$ colored and $x$ either uncolored or colored differently from $y$. Let us write

$$x \leftarrow y \quad \text{to mean} \quad y \in S_i,\ x \notin S_i, \text{ and } (S_i - y) \cup \{x\} \in \mathcal{I} \text{ for some } i.$$

We read this as "$y$ may relinquish its color to $x$" or, more tersely, "$x$ may get $y$'s color".

Thus, given a partial proper coloring $S_1, \ldots, S_k$ of $\mathcal{M}$, the $\leftarrow$ relation defines a digraph on the set $S \cup \{\langle 1 \rangle, \langle 2 \rangle, \ldots, \langle k \rangle\}$.

The central idea in the Matroid Partitioning Algorithm is the following.

**Lemma 5.3.1** *Suppose $\mathcal{M} = (S, \mathcal{I})$ is a matroid and $S_1, S_2, \ldots, S_k$ forms a partial proper coloring of $\mathcal{M}$. Suppose there is a directed path of the form*

$$x_0 \leftarrow x_1 \leftarrow x_2 \leftarrow \cdots \leftarrow x_n \leftarrow \langle a \rangle$$

*that is minimal in the sense that $x_i \not\leftarrow x_j$ for $j > i+1$ and $x_i \not\leftarrow \langle a \rangle$ for $0 \leq i < n$. Furthermore suppose we recolor the elements of $S$ as follows: The new color of $x_i$ is the old color of $x_{i+1}$ (for $0 \leq i < n$) and the new color of $x_n$ is $a$. Then the recoloring described results in a proper partial coloring of $\mathcal{M}$.*

The notation $x_i \leftarrow x_{i+1}$ means that $x_i$ can acquire a new color from $x_{i+1}$. The assertion here is that it is permissible to make all these changes simultaneously. Because the $\leftarrow$ relation changes globally every time the partial coloring is amended, the lemma is not trivial.

**Proof**. We proceed by induction on $n$, the length of the path.

The basis case, $n = 0$, is trivial.

For $n \geq 1$, suppose

$$x_0 \leftarrow x_1 \leftarrow x_2 \leftarrow \cdots \leftarrow x_n \leftarrow \langle a \rangle$$

is a minimal path as described in the statement of the lemma and suppose the lemma has been proved for all smaller values of $n$. Suppose the current color of $x_n$ is $b$.

Recolor the matroid so that $x_n$ now has color $a$. That is, let $S'_1, S'_2, \ldots, S'_k$ be the new partial coloring with

$$S'_i = S_i \quad (i \neq a, b)$$
$$S'_a = S_a \cup \{x_n\}$$
$$S'_b = S_b - \{x_n\}$$

We know that all $S'_i$'s are independent because $x_n \leftarrow \langle a \rangle$ was true for the old coloring.

We claim that in this new partial coloring

$$x_0 \leftarrow x_1 \leftarrow x_2 \leftarrow \cdots \leftarrow x_{n-1} \leftarrow \langle b \rangle$$

is a valid minimal path. Note that once we have verified this claim, this proof is complete by induction.

To this end, we need to verify four claims.

1. *In the new coloring $x_{n-1} \leftarrow \langle b \rangle$.*

   This is correct because in the old coloring we had $x_{n-1} \leftarrow x_n$ and $x_n \in S_b$. This means that $S_b - \{x_n\} \cup \{x_{n-1}\}$ is independent. Since $x_n$ is not in $S'_b$, we may recolor $x_{n-1}$ to color $b$ as required.

2. *In the new coloring $x_i \leftarrow x_{i+1}$ for all $0 \leq i < n-1$.*

   If $x_{i+1}$ is any color *other* than $a$, then we clearly have $x_i \leftarrow x_{i+1}$ in the new coloring because $x_i \leftarrow x_{i+1}$ was true in the old coloring. Thus we may assume the color of $x_{i+1}$ is $a$.

   To show $x_i \leftarrow x_{i+1}$ (in the new coloring) we must show that the set $S''_a = S'_a - \{x_{i+1}\} \cup \{x_i\} = S_a - \{x_{i+1}\} \cup \{x_i, x_n\}$ is independent.

   Suppose $S''_a$ were dependent. Since $S''_a$ is the union of an independent set ($S'_a - \{x_{i+1}\}$) and a single element ($x_i$), by Theorem 5.2.6, $S''_a$ contains a unique cycle $C$. Note that $x_i \in C$. Also, $x_n \in C$ since $S''_a$ is the union of the independent set $S_a - \{x_{i+1}\} \cup \{x_i\}$ (independent because $x_i \leftarrow x_{i+1}$ in the old partial coloring) and the single element $x_n$.

   Now $x_i \not\leftarrow \langle a \rangle$ (by minimality) so $S_a \cup \{x_i\}$ is dependent and contains a unique circuit $C'$. Since $x_i \leftarrow x_{i+1}$ in the old coloring, $C'$ must contain both $x_i$ and $x_{i+1}$. Since $x_n \in S_b$, we know that $x_n \notin C'$. Thus $C$ and $C'$ are distinct circuits both containing $x_i$. By Theorem 5.2.5, there is a circuit $C'' \subseteq (C \cup C') - \{x_i\}$.

   Summarizing, we have

   $$\begin{aligned} C &\subseteq S''_a = S_a \cup \{x_i, x_n\} - \{x_{i+1}\}, \\ C' &\subseteq S_a \cup \{x_i\}, \text{ and} \\ C'' &\subseteq (C \cup C') - \{x_i\} \subseteq S_a \cup \{x_n\} \in \mathcal{I}, \end{aligned}$$

   which gives a contradiction.

3. *In the new coloring $x_i \not\leftarrow \langle b \rangle$ (for $i < n-1$).*

   If $x_i \leftarrow \langle b \rangle$ were true in the new coloring, then $x_i \leftarrow x_n$ would have been true in the old coloring—a contradiction to minimality.

4. *In the new coloring $x_i \not\leftarrow x_j$ (for $0 \leq i < j+1 \leq n$).*

   Suppose that, in the new coloring, we had $x_i \leftarrow x_j$ with $0 \leq i < j+1 \leq n$. If the color of $x_j$ is not $b$ (say it's $c$), then we have $S'_c \cup \{x_i\} - \{x_j\} \supseteq S_c \cup \{x_i\} - \{x_j\} \notin \mathcal{I}$ (with strict containment only if $c = a$), and therefore $S'_c \cup \{x_i\} - \{x_j\}$ is dependent, a contradiction. We may therefore restrict to the case that $x_j$ has color $b$.

   Since $x_i \leftarrow x_j$ in the new coloring, the set $I = S'_b - \{x_j\} \cup \{x_i\}$ is independent. Since $x_i \not\leftarrow x_j$ in the old coloring, the set $D = S_b - \{x_j\} \cup \{x_i\}$ is dependent. Now $D = I \cup \{x_n\}$ so $D$ contains a unique circuit $C$. Notice that $C$ necessarily contains $x_n$ and $x_i$.

   Now $x_i \not\leftarrow x_n$ in the old coloring, so $S_b - \{x_n\} \cup \{x_i\}$ must be dependent and contain a circuit $C'$. Note that $C'$ must contain $x_i$ but not $x_n$, and therefore $C' \neq C$ and both contain $x_i$. Thus there is a circuit $C'' \subseteq C \cup C' - \{x_i\}$. However $C \cup C' - \{x_i\} \subseteq S_b$, a contradiction.

This completes the proof. □

Lemma 5.3.1 gives us a method for extending a partial coloring of a matroid. Given a partial coloring $S_1, \ldots, S_k$ of $\mathcal{M}$, we construct the $\leftarrow$ digraph on $S \cup \{\langle 1\rangle, \ldots, \langle k\rangle\}$ and search for a minimal directed path to an unlabeled element of $S$ from one of $\langle 1\rangle$, $\langle 2\rangle$, $\ldots$, or $\langle k\rangle$.

If this is successful at each stage, we ultimately construct a partition of $\mathcal{M}$ into $k$ independent sets. However, at some point we might not be able to find such a path. In such a case, we would like to know that no partition of $\mathcal{M}$ into $k$ independent sets is possible.

There is a simple test we can apply to see if $\mathcal{M}$ is *not* partitionable into $k$ independent sets. Recall that $\rho(\mathcal{M})$, the rank of $\mathcal{M}$, is the maximum size of an independent set. If $\mathcal{M}$ had a partition $S = S_1 \cup S_2 \cup \cdots \cup S_k$ we would have $|S| \leq k\rho(\mathcal{M})$.

More generally, if $Y \subseteq S$, then $Y = (S_1 \cap Y) \cup (S_2 \cap Y) \cup \cdots \cup (S_k \cap Y)$ and therefore $|Y| \leq k\rho(Y)$. Thus if there is a $Y \subseteq S$ for which $|Y| > k\rho(Y)$ there can be no partition of $\mathcal{M}$ into $k$ independent sets.

**Theorem 5.3.2** *Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid and let $k$ be a positive integer. Then $\mathcal{M}$ has a partition into $k$ independent sets if and only if $|Y| \leq k\rho(Y)$ for all $Y \subseteq S$.*

The proof of this theorem is closely tied to the following algorithm.

---

**Matroid Partitioning Algorithm**

**Input:** A matroid $\mathcal{M} = (S, \mathcal{I})$ and a positive integer $k$.
**Output:** Either a partition $S = S_1 \cup \cdots \cup S_k$ with each $S_i \in \mathcal{I}$ or a $Y \subseteq S$ with $|Y| > k\rho(Y)$.

1. Let $S_1, S_2, \ldots, S_k$ all equal $\emptyset$.
2. If $S = S_1 \cup S_2 \cup \cdots \cup S_k$, output this partition and stop.
3. Otherwise (there are uncolored vertices), form the $\leftarrow$ directed graph based on the partial coloring $S_1, \ldots, S_k$.
4. If there is dipath to an uncolored vertex $x_i$ from a color class $\langle a\rangle$, recolor along a minimal such dipath, and go to step 2.
5. Otherwise (there is no such path), let $Y$ be the set of all vertices that can reach to uncolored vertices in the $\leftarrow$ digraph (i.e., $y \in Y$ if there is a $u \in S$ (with $u$ uncolored) and a dipath[2] $u \leftarrow \cdots \leftarrow y$). Output the set $Y$.

---

**Proof** (of Theorem 5.3.2 and the correctness of the Matroid Partitioning Algorithm).

It is enough to prove that if the algorithm reaches step 5, then $|Y| > k\rho(Y)$. Let $U$ denote the (nonempty) set of elements uncolored in the algorithm and let $V = Y - U$ (perhaps $V = \emptyset$). Since $U \neq \emptyset$, $|Y| > |V|$.

We claim that $\rho(Y) = |Y \cap S_i|$ for each $i$. Since $Y \cap S_i \in \mathcal{I}$ we know that $\rho(Y) \geq |Y \cap S_i|$. Suppose $\rho(Y) > |Y \cap S_i|$. This implies there is an $x \in Y - S_i$ with $(Y \cap S_i) \cup \{x\} \in \mathcal{I}$. If $S_i \cup \{x\} \in \mathcal{I}$ then we would have $x \leftarrow \langle i\rangle$, contradicting the fact that the algorithm passed step 4. Thus $S_i \cup \{x\}$ contains a unique circuit $C$. Now $C$ is not contained in $Y$ (since $(Y \cap S_i) \cup \{x\}$ is independent) so

[2]We allow dipaths of length zero. In other words, $Y$ contains all uncolored vertices.

there is an element $z \in C - Y$ and, since $z \neq x$, we have $z \in S_i - Y$. Therefore, $(S_i - \{z\}) \cup \{x\} \in \mathcal{I}$, so $x \leftarrow z$. However, this is a contradiction because $x \in Y$ but $z \notin Y$.

Finally, we compute:

$$\begin{aligned} k\rho(Y) &= \sum_{i=1}^{k} \rho(Y) \\ &= \sum_{i=1}^{k} |Y \cap S_i| \\ &= \left| \bigcup_{i=1}^{k} Y \cap S_i \right| \\ &= |V| < |Y| \end{aligned}$$

as required. □

If $\mathcal{M}$ has no loops (every singleton subset is independent), then there is a least $k$ so that $S = S_1 \cup \cdots \cup S_k$ with each $S_i \in \mathcal{I}$; indeed, this is just the covering number $k(\mathcal{M})$ of $\mathcal{M}$.

**Corollary 5.3.3** *If $\mathcal{M} = (S, \mathcal{I})$ is a matroid, then*

$$k(\mathcal{M}) = \max_{Y \subseteq S} \left\lceil \frac{|Y|}{\rho(Y)} \right\rceil .$$

**Proof**. Immediate from Theorem 5.3.2. □

A few words are in order about the worst-case run time of the Matroid Partitioning Algorithm. The "price" of a matroid-theoretic algorithm is often assessed by the number of times we "ask" the matroid if a given set is independent.

Suppose the ground set of a matroid $\mathcal{M}$ has $n$ elements. We run through steps 2, 3, and 4 at most $n$ times (we add one element to the partial coloring on each pass). To create the digraph in step 3, we check every pair of elements $x, y$ to see if $x \leftarrow y$. Each $x \leftarrow y$ and $x \leftarrow \langle a \rangle$ determination can be made by determining if a certain set is independent. We also compare each element $x$ to each color class $\langle a \rangle$ to check if $x \leftarrow \langle a \rangle$. These determinations take $O(n^2)$ tests of independence.

Thus, overall, we make $O(n^3)$ independence queries in the worst case.

To determine $k(\mathcal{M})$ we can run the Matroid Partitioning Algorithm for $k = 1, 2, 3, \ldots$ until we succeed in finding a partition. Even if this is done naively, we perform $O(n^4)$ independence queries. Thus $k(\mathcal{M})$ can be computed in polynomial time.

## 5.4 Arboricity again

Let $G$ be a graph. Recall that $\Upsilon(G)$ is the smallest size of a partition of $E(G)$ into acyclic subsets. In other words, $\Upsilon(G) = k(\mathcal{M}(G))$. The Matroid Partitioning Algorithm gives us a method to compute $k(\mathcal{M}(G))$ with a number of queries of independence that is a polynomial in the size of the matroid. Since we can check if a subset of $E(G)$ is acyclic in polynomial time, the overall time complexity of $\Upsilon$ is polynomial.

We may use Theorem 5.3.2 on the facing page to obtain a proof of Nash-Williams's Theorem 5.1.1 on page 72.

**Proof** (of Theorem 5.1.1). Let $G = (V, E)$ be a graph. We want to show

$$\Upsilon(G) = \max_H \left\lceil \frac{\varepsilon(H)}{\nu(H) - 1} \right\rceil \qquad (*)$$

where the maximum is over all subgraphs $H$ (with at least 2 vertices). We claim that we may restrict our attention to connected $H$. Suppose the maximum on the right-hand side of $(*)$ is achieved for a graph $H$ with more than one component. If one of those components is an isolated vertex $v$, then $\nu(H - v) < \nu(H)$, but $\varepsilon(H - v) = \varepsilon(H)$, a contradiction. Thus each component of $H$ has at least two vertices. If the components of $H$ are $H_1, H_2, \ldots, H_c$, then we have

$$\begin{aligned} \frac{\varepsilon(H)}{\nu(H) - 1} &= \frac{\sum_i \varepsilon(H_i)}{(\sum_i \nu(H_i)) - 1} \\ &\leq \frac{\sum_i \varepsilon(H_i)}{\sum_i (\nu(H_i) - 1)} \\ &\leq \max_i \left\{ \frac{\varepsilon(H_i)}{\nu(H_i) - 1} \right\}. \end{aligned}$$

(See exercise 10 on page 91.)

Now by Corollary 5.3.3 on the preceding page we know that

$$\Upsilon(G) = \max_{Y \subseteq E(G)} \left\lceil \frac{|Y|}{\rho(Y)} \right\rceil.$$

We claim that the maximum here is achieved at a set $Y \subseteq E(G)$ such that $(V, Y)$, the subgraph of $G$ induced on the set $Y$, has just one nontrivial[3] component. To see this, suppose $Y$ can be partitioned into $Y = Y_1 \cup Y_2 \cup \cdots \cup Y_c$ where each $(V, Y_i)$ is a nontrivial component of $(V, Y)$. Then

$$\begin{aligned} \frac{|Y|}{\rho(Y)} &= \frac{|Y_1| + \cdots + |Y_c|}{\rho(Y_1) + \cdots + \rho(Y_c)} \\ &\leq \max_i \left\{ \frac{|Y_i|}{\rho(Y_i)} \right\}. \end{aligned}$$

(Again, see exercise 10.) If $H = (V, Y)$ has a single, nontrivial component on $h$ vertices, then $\rho(V, Y) = h - 1$. Thus we have

$$\Upsilon(G) = \max_H \left\lceil \frac{\varepsilon(H)}{\nu(H) - 1} \right\rceil,$$

where the maximum is over all connected subgraphs $H$ of $G$ with at least 2 vertices. □

We now turn to a discussion of the fractional arboricity of $G$. We define $\Upsilon_f(G)$ to be $k_f(\mathcal{M}(G))$, the fractional covering number of the cycle matroid of $G$. Equivalently, we write $\Upsilon_t(G)$ to denote the $t$-fold arboricity of $G$: the minimum number of acyclic subsets of edges that include all edges of $G$ at least $t$ times. Alternatively, $\Upsilon_t(G) = \Upsilon(t \cdot G)$ where $t \cdot G$ is the multigraph formed from $G$ by replacing each edge of $G$ with $t$ parallel edges. Finally, $\Upsilon_f(G) = \lim_{t \to \infty} \Upsilon_t(G)/t$.

In a similar vein, the $t$-fold covering number $k_t(\mathcal{M})$ of a matroid $\mathcal{M}$ is the size of a smallest collection of independent sets of $\mathcal{M}$ that includes every element of $\mathcal{M}$ at least $t$ times. Alternatively, let $t \cdot \mathcal{M}$ denote the matroid formed by replacing each element of the ground set $S$ with $t$ parallel elements. (We leave it to the reader to make this precise; see exercise 11 on page 91.) Then $k_t(\mathcal{M}) = k(t \cdot \mathcal{M})$ and $k_f(\mathcal{M}) = \lim k_t(\mathcal{M})/t = \lim k(t \cdot \mathcal{M})/t$.

The following result is an analogue of Corollary 5.3.3 on the preceding page.

[3] A *trivial* component is an isolated vertex.

**Theorem 5.4.1** *Let $\mathcal{M}$ be a matroid. Then*

$$k_f(\mathcal{M}) = \max_{X \subseteq S} \frac{|X|}{\rho(X)}.$$

**Proof**. We know that $k_t(\mathcal{M}) = k(t \cdot \mathcal{M})$ so by Corollary 5.3.3 on page 81 we have

$$k_t(\mathcal{M}) = \max_{Y \subseteq t \cdot S} \left\lceil \frac{|Y|}{\rho(Y)} \right\rceil$$

where $t \cdot S$ is the ground (multi)set of $t \cdot \mathcal{M}$. Consider a $Y \subseteq t \cdot S$ that achieves the maximum. Notice that if $y \in Y$ then all elements of $t \cdot S$ parallel to $y$ are also in $Y$; otherwise we could add them to $Y$ without increasing $Y$'s rank. Thus we may restrict the scope of the maximization to those $Y$ of the form $t \cdot X$ where $X \subseteq S$. We have

$$k_t(\mathcal{M}) = \max_{X \subseteq S} \left\lceil \frac{t|X|}{\rho(X)} \right\rceil = \max_{X \subseteq S} \left\{ \frac{t|X|}{\rho(X)} \right\} + O(1) = t \max_{X \subseteq S} \left\{ \frac{|X|}{\rho(X)} \right\} + O(1)$$

and therefore

$$\frac{k_t(\mathcal{M})}{t} = \max_{X \subseteq S} \left\{ \frac{|X|}{\rho(X)} \right\} + O(1/t)$$

so as $t \to \infty$ we finally have

$$k_f(\mathcal{M}) = \max_{X \subseteq S} \left\{ \frac{|X|}{\rho(X)} \right\}.$$ □

We know (see Corollary 1.3.2 on page 4) that there is a finite positive integer $t$ for which $k_f(\mathcal{M}) = k_t(\mathcal{M})/t$. We claim that there is such a $t$ with $t \leq \rho(\mathcal{M}) \leq |S|$.

**Corollary 5.4.2** *Let $\mathcal{M}$ be a matroid. There exists a positive integer $t$ with $t \leq \rho(\mathcal{M})$ so that $k_f(\mathcal{M}) = k_t(\mathcal{M})/t$.*

**Proof**. We know that $k_f(\mathcal{M}) = \max_Y |Y|/\rho(Y)$. Let $Y$ be a subset of $S$ that achieves this maximum and let $t = \rho(Y)$. It follows that

$$k_t(\mathcal{M}) = \max_X \left\lceil \frac{t|X|}{\rho(X)} \right\rceil$$

and this maximum is certainly achieved by $Y$, so, since we chose $t = \rho(Y)$, we also have

$$k_t(\mathcal{M}) = \left\lceil \frac{t|Y|}{\rho(Y)} \right\rceil = |Y| = \frac{t|Y|}{\rho(Y)}$$

and therefore

$$\frac{k_t(\mathcal{M})}{t} = \frac{|Y|}{\rho(Y)} = k_f(\mathcal{M}).$$ □

It follows that there is a polynomial-time algorithm to compute $k_f(\mathcal{M})$: we simply compute $k(t \cdot \mathcal{M})/t$ for $1 \leq t \leq \rho(\mathcal{M})$ and take the smallest answer.

Let us apply what we have learned about $k_f(\mathcal{M})$ to give an analogue of Theorem 5.1.1 on page 72 for fractional arboricity.

**Corollary 5.4.3** *Let $G$ be a graph. Then*

$$\Upsilon_f(G) = \max_H \left\{ \frac{\varepsilon(H)}{\nu(H) - 1} \right\}$$

*where the maximum is over all subgraphs $H$ of $G$ with at least 2 vertices.* □

Thus the fractional arboricity of a graph can never be much less than its arboricity.

## 5.5 Maximum average degree again

Recall that $\mathrm{mad}(G)$ denotes the maximum average degree of $G$, i.e., $\mathrm{mad}(G) = \max_H \bar{d}(H)$ where $\bar{d}(H) = 2\varepsilon(H)/\nu(H)$ and the maximization is over all subgraphs $H$ of $G$.

Our goal is to show that $\mathrm{mad}(G)$ is actually an instance of the fractional covering problem for a matroid that is quite similar to $\mathcal{M}(G)$.

First note that if $G$ is a tree then $\mathrm{mad}(G) = \bar{d}(G) = 2(1 - \frac{1}{n})$ where $n = \nu(G)$. If $G$ is not connected, one checks (e.g., using exercise 10 on page 91) that $\mathrm{mad}(G)$ equals the maximum average degree of one of its components. Thus in the sequel we assume that $G$ is connected, but not a tree.

The first step is to define a new matroid on such graphs. Let $G = (V, E)$ be a connected graph that is not a tree. Let us call a subset $Y$ of the edges *nearly acyclic* if $(V, Y)$ has at most one cycle. Let $\mathcal{M}_1(G)$ be the pair $(E, \mathcal{I}_1)$ where $\mathcal{I}_1$ consists of all nearly acyclic subsets of edges. We claim that this forms a matroid. (In exercise 13 on page 91 this matroid is seen to be a special case of the *elongation* of a matroid.) The proof uses the following simple lemma.

**Lemma 5.5.1** *Let $G$ be a connected graph. Then $G$ has exactly one cycle if and only if $\nu(G) = \varepsilon(G)$.*

**Proof**. Exercise 12 on page 91. □

**Theorem 5.5.2** *Let $G = (V, E)$ be a connected graph. Then $\mathcal{M}_1(G)$ is a matroid.*

**Proof**. If $G$ is a tree then $\mathcal{M}(G) = \mathcal{M}_1(G)$ and the result is trivial, so we may assume that $G$ is not a tree.

Let the set of maximal members of $\mathcal{I}_1$ be denoted $\mathcal{B}_1$. In other words, $B \in \mathcal{B}_1$ just when $(V, B)$ is a spanning, connected, unicyclic subgraph of $G$. We use Theorem 5.2.3 on page 74 to show that $\mathcal{M}_1 = \mathcal{M}_1(G)$ is a matroid.

Note first that since $\emptyset \in \mathcal{I}_1$ we have that $\mathcal{B}_1 \neq \emptyset$.

Next, by Lemma 5.5.1, note that any two members of $\mathcal{B}_1$ have the same size $\nu(G)$.

Finally, choose $B, B' \in \mathcal{B}_1$ and $x \in B$. We must prove there is a $y \in B'$ so that $B - \{x\} \cup \{y\} \in \mathcal{B}_1$. If $x \in B'$ this is trivial, so we may suppose $x \notin B'$.

Now $(V, B)$ has a unique cycle $C$. Either $x \in C$ or $x \notin C$.

If $x \in C$, then $T = (V, B - \{x\})$ is a spanning tree of $G$. Now $|B'| = |B|$ and $B' \neq B$, so we select $y$ to be any member of $B' - B$. Observe that $T + y$ is therefore a spanning unicyclic subgraph of $G$, i.e., $B - \{x\} \cup \{y\} \in \mathcal{B}_1$ as required.

Otherwise $x \notin C$ and so $(V, B - \{x\})$ consists of two connected components: one containing $C$ and one that is acyclic. Now let $T'$ be a spanning tree of $(V, B')$ (we form $T'$ by deleting any edge in the unique cycle of $(V, B')$). There must be some edge $y \in B'$ that links the two components of $(V, B - \{x\})$. Thus $(V, B - \{x\} \cup \{y\})$ is a spanning unicyclic subgraph of $G$, i.e., $B - \{x\} \cup \{y\} \in \mathcal{B}_1$ as required. □

Now that we have verified that $\mathcal{M}_1(G)$ is a matroid, we can use Theorem 5.3.2 on page 80 to show that finding $\mathrm{mad}(G)$ is (essentially) an instance of the fractional covering problem for a matroid.

**Theorem 5.5.3** *Let $G$ be a connected graph that is not a tree. Then* $\mathrm{mad}(G) = 2k_f(\mathcal{M}_1(G))$.

**Proof**. Let $G = (V, E)$ be as in the theorem. Let $H$ be a subgraph of $G$ for which $\bar{d}(H) = \mathrm{mad}(G)$.

Now consider an optimal fractional covering of $\mathcal{M}_1(G)$, i.e., we have $J_1, J_2, \ldots, J_k \in \mathcal{I}_1$ and weights $w_1, w_2, \ldots, w_k \in [0,1]$ so that for any edge $e$ of $G$ we have

$$\sum_{i:e\in J_i} w_i \geq 1.$$

Since the fractional covering is optimal, $\sum w_i = k_f(\mathcal{M}_1(G))$.

Now we calculate:

$$\begin{aligned}\varepsilon(H) &\leq \sum_{e\in E(H)} \sum_{i:e\in J_i} w_i \\ &= w_1|J_1 \cap E(H)| + \cdots + w_k|J_k \cap E(H)| \\ &\leq w_1\nu(H) + \cdots + w_k\nu(H) \\ &= k_f(\mathcal{M}_1(G))\nu(H)\end{aligned}$$

and therefore

$$\mathrm{mad}(G) = \bar{d}(H) = \frac{2\varepsilon(H)}{\nu(H)} \leq 2k_f(\mathcal{M}_1(G)).$$

We now prove the opposite inequality. By Corollary 5.3.3 on page 81

$$k_f(\mathcal{M}_1(G)) = \max_{Y\subseteq E} \frac{|Y|}{\rho_1(Y)} \qquad (*)$$

where $\rho_1$ is the rank function for $\mathcal{M}_1(G)$. As in the proof of Theorem 5.1.1, it is enough to consider in $(*)$ just those $Y$ so that $(V, Y)$ has only one nontrivial connected component.

We know, by hypothesis, that $G$ has at least one cycle. If $C$ is the set of edges in that cycle, note that $|C|/\rho_1(C) = 1$. If $T$ is an acyclic set of edges, then $|T|/\rho_1(T) = 1$, so in computing $(*)$ we may further restrict our attention to those $Y$ for which $(V, Y)$ has one nontrivial component that is *not* a tree; let $H$ be the nontrivial component associated with an optimal $Y$. Note that $\rho_1(Y) = \nu(H)$, so we have

$$k_f(\mathcal{M}_1(G)) = \frac{|Y|}{\rho_1(Y)} = \frac{\varepsilon(H)}{\nu(H)} = \frac{1}{2}\bar{d}(H) \leq \frac{1}{2}\mathrm{mad}(G)$$

and we are done. □

Thus the Matroid Partitioning Algorithm can be used to compute, in polynomial time, the maximum average degree of a graph. Note that a "greedy" heuristic to compute $\mathrm{mad}(G)$ does not work (exercise 2 on page 90).

## 5.6 Duality, duality, duality, and edge toughness

We have expended considerable effort studying the covering number of a matroid $\mathcal{M}$. However, as we discussed in §1.5, there are three other related invariants closely allied to the covering number: the packing number, the transversal number, and the matching number. We consider each of these in turn.

We show that of the four invariants from Chapter 1 (covering, packing, transversal, and matching) only $k(\mathcal{M})$ is interesting.

We then introduce yet another notion of duality—matroid duality—and we consider the covering and fractional covering number of this dual. Finally, we show how the fractional covering number of the (matroid) dual of $\mathcal{M}(G)$ is closely connected to the edge toughness of $G$.

## Hypergraph duality and mathematical programming duality

We have examined $k$ and $k_f$ of a matroid in extensive detail. Here we consider $p$, $\tau$, and $\mu$ of a matroid (and their fractional counterparts) and find little of interest.

Let us begin with $p(\mathcal{M})$, the packing number of a matroid. Here we seek the maximum number of elements of $S$ no two of which are together in an independent set. Let $P$ be a maximal subset of $S$ with no two elements of $P$ forming an independent set. Certainly all loops of $\mathcal{M}$ are in $P$. Further, if $x, y \in P$ are not loops, then we must have $x$ and $y$ parallel. We therefore have the following.

**Theorem 5.6.1** *Let $\mathcal{M}$ be a matroid. Let $\ell$ denote the number of loops in $\mathcal{M}$ and let $m$ denote the maximum number of pairwise parallel elements of $\mathcal{M}$. Then $p(\mathcal{M}) = \ell + m$.* □

We know that $p(\mathcal{M}) \leq k(\mathcal{M})$, but Theorem 5.6.1 does not give an interesting lower bound for $k(\mathcal{M})$. If $\mathcal{M}$ has a loop, then $k(\mathcal{M}) = \infty$. If $\mathcal{M}$ is loopless, then $k(\mathcal{M})$ is finite but certainly we have $k(\mathcal{M}) \geq m$, the maximum number of pairwise parallel elements, since parallel elements must have different colors.

Next consider the transversal number of $\mathcal{M} = (S, \mathcal{I})$, the minimum number of elements of the ground set $S$ that intersect every member of $\mathcal{I}$. Since $\emptyset \in \mathcal{I}$, no transversal can exist, and therefore $\tau(\mathcal{M})$ is undefined (or $\infty$).

Finally, consider the matching number of $\mathcal{M}$. We want to compute the maximum number of pairwise disjoint independent sets of $\mathcal{M}$. It is clear that the best we can do is to take all the singleton sets in $\mathcal{I}$ (the non-loops) and $\emptyset$.

(See exercise 17 on page 91 for another approach to matching and packing.)

**Theorem 5.6.2** *Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid. Let $n = |S|$ and suppose $\mathcal{M}$ has $\ell$ loops. Then $\mu(\mathcal{M}) = n - \ell + 1$.* □

Now the fractional packing and fractional covering numbers of a matroid are the same, and these are covered by Theorem 5.4.1 on page 83. The common value of the fractional transversal and fractional matching number is $\infty$.

## Matroid duality

In addition to mathematical programming duality and hypergraph duality, matroids admit yet another form of duality. Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid and let $\mathcal{B}$ be the bases of $\mathcal{M}$. The *dual* of the matroid $\mathcal{M}$, denoted[4] $\tilde{\mathcal{M}} = (S, \tilde{\mathcal{I}})$, has the same ground set as $S$ and a subset $X$ of $\mathcal{S}$ is independent in $\tilde{\mathcal{M}}$ (i.e., is in $\tilde{\mathcal{I}}$) if and only if $X \subseteq S - B$ for some $B \in \mathcal{B}$. Indeed, it is easier to describe $\tilde{\mathcal{M}}$ in terms of its bases: The bases of $\tilde{\mathcal{M}}$ are the complements (relative to $S$) of the bases of $\mathcal{M}$. In symbols: $B \in \mathcal{B} \iff S - B \in \tilde{\mathcal{B}}$. We call $x \in S$ a *coloop* of $\mathcal{M}$ if $\{x\}$ is a dependent set in $\tilde{\mathcal{M}}$. In other words, a coloop of a matroid is an element contained in every basis.

The first order of business is to verify that $\tilde{\mathcal{M}}$ is a matroid.

**Theorem 5.6.3** *If $\mathcal{M}$ is a matroid, then so is its dual, $\tilde{\mathcal{M}}$.*

[4]The customary notation for a matroid dual is $\mathcal{M}^*$. We use the $*$ superscript to denote hypergraph duality, so we adopt this unconventional notation.

**Proof**. We check that $\tilde{\mathcal{B}}$ satisfies the conditions of Theorem 5.2.3 on page 74. It is immediate that $\tilde{\mathcal{B}}$ is nonempty and any two members of $\mathcal{B}$ have the same cardinality.

Suppose $\tilde{B}_1, \tilde{B}_2 \in \tilde{\mathcal{B}}$ (so that their complements $B_1, B_2$ are bases of $\mathcal{M}$) and let $x \in \tilde{B}_1$. We seek a $y \in \tilde{B}_2$ so that $\tilde{B}_1 \cup \{y\} - \{x\} \in \tilde{\mathcal{B}}$. If $x \in \tilde{B}_2$ we may take $y = x$, so suppose $x \notin \tilde{B}_2$. We now know that $x \in B_2 - B_1$. Now $B_1 \cup \{x\}$ must be dependent, and contain a unique circuit $C$. Note that $C$ is not entirely contained in $B_2$, so there must be a $y \in C - B_2$. Since $y \notin B_2$, we have $y \in \tilde{B}_2$. Finally $B_1 \cup \{x\} - \{y\}$ must be independent (since we have removed $y$ from the unique cycle $C$ of $B_1 \cup \{x\}$) and therefore a basis (since it is of full rank). Therefore its complement, $S - (B_1 \cup \{x\} - \{y\}) = \tilde{B}_1 \cup \{y\} - \{x\}$ is in $\tilde{\mathcal{B}}$ as desired. □

Let $G = (V, E)$ be a graph. The dual of the cycle matroid $\mathcal{M}(G)$ is denoted $\tilde{\mathcal{M}}(G)$ and is called the *cocycle matroid* of $G$. Let us describe the independent sets of $\tilde{\mathcal{M}}(G)$. Call a subset $F$ of $E$ *disconnecting* if $G - F$ has more components than $G$. Let $K$ be a maximal, nondisconnecting set of edges of $G$, and consider the graph $H = (V, E - K)$. Notice that $H$ has the same number of components as $G$ (since $K$ is nondisconnecting), but every edge of $H$ is a cut edge (since $K$ is maximal). Thus $H$ is a maximal spanning forest of $G$, i.e., $E - K$ is a basis of $\mathcal{M}$. We have shown that if $K$ is maximal, nondisconnecting, then $K$ is a basis of $\tilde{\mathcal{M}}$. One easily checks that the converse is true as well. Thus the independent sets of edges of $\tilde{\mathcal{M}}(G)$ are the subsets of the maximal, nondisconnecting subsets of edges of $G$. A less convoluted way of saying this is the following.

**Proposition 5.6.4** *Let $G = (V, E)$ be a graph and let $\tilde{\mathcal{M}}(G)$ be its cocycle matroid. A set of edges $F \subseteq E$ is independent in $\tilde{\mathcal{M}}$ if and only if $F$ is nondisconnecting in $G$.* □

It follows that the circuits of $\tilde{\mathcal{M}}(G)$ are the minimal disconnecting sets of edges of $G$; these are known as *bonds* or *cocycles* of $G$. A single edge of $G$ is a loop of $\tilde{\mathcal{M}}(G)$ if and only if it is a cut edge. A pair of edges of $G$ are parallel in $\tilde{\mathcal{M}}(G)$ if and only if they are not cut edges, but together they form a disconnecting set. Observe that in case $G$ is planar, then a loop of $G$ corresponds to a cut edge of $G$'s planar dual, and a pair of edges forming a minimal disconnecting set in $G$ corresponds to a pair of parallel edges in the dual. This is more than a curious observation. When $G = (V, E)$ is a planar graph, then $\tilde{\mathcal{M}}(G)$ has a particularly nice description.

**Theorem 5.6.5** *Let $G$ be a planar graph (embedded in the plane) and let $\tilde{G}$ be its dual. Then $\tilde{\mathcal{M}}(G) = \mathcal{M}(\tilde{G})$.* □

The equal sign in the conclusion makes sense since we can identify the edge set of an embedded planar graph with that of its dual. The proof of Theorem 5.6.5 is relegated to exercise 14 on page 91. A corollary of this result is that if $G$ is connected and planar, then the complement of a spanning tree in $G$ corresponds to a spanning tree in the dual of $G$.

## Dual rank

Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid and let $\tilde{\mathcal{M}}$ be its dual. Denote by $\rho$ and $\tilde{\rho}$ the rank functions of these matroids. Since bases of $\mathcal{M}$ and $\tilde{\mathcal{M}}$ are complements of one another, we clearly have $\rho(\mathcal{M}) + \tilde{\rho}(\tilde{\mathcal{M}}) = |S|$. The formula for $\tilde{\rho}(X)$ for any $X \subseteq S$ is presented next.

**Theorem 5.6.6** *Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid and let $\tilde{\mathcal{M}}$ be its dual. Denote by $\rho, \tilde{\rho}$ the rank functions of $\mathcal{M}, \tilde{\mathcal{M}}$ respectively. Then for any $X \subseteq S$ we have*

$$\tilde{\rho}(X) = |X| - \rho(S) + \rho(S - X).$$

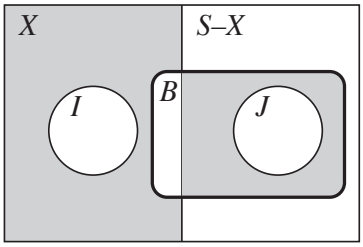


Figure 5.2. Deriving the formula for $\tilde{\rho}$.

**Proof**. Let $X \subseteq S$ and let $I \subseteq X$ be a maximal member of $\tilde{\mathcal{I}}$ contained in $X$. Let $J \subseteq S - X$ be a maximal member of $\mathcal{I}$ contained in the complement of $X$. Thus $\tilde{\rho}(X) = |I|$ and $\rho(S - X) = |J|$.

Since $I \in \tilde{\mathcal{I}}$, there exists a basis $B_0 \in \mathcal{B}$ for which $I \subseteq S - B_0$, or equivalently, $B_0 \subseteq S - I$. Observe, also, that $J \subseteq S - I$. Applying augmentation, we can add elements of $J$ to $B_0$ until we build a basis $B \in \mathcal{B}$ for which $J \subseteq B \subseteq S - I$. See Figure 5.2. Consider the shaded regions in the figure. By the maximality of $I$, the shaded portion of $X$ must be empty. Likewise, by the maximality of $J$, the shaded portion of $S - X$ is also empty. We may now calculate:

$$\begin{aligned}\tilde{\rho}(X) &= |I| \\ &= |X| - |X - I| \\ &= |X| - |B \cap X| \\ &= |X| - (|B| - |J|) \\ &= |X| - \rho(S) + \rho(S - X).\end{aligned}$$

□

## Covering number of the dual of a matroid

If $\mathcal{M} = (S, \mathcal{I})$ is a matroid and $\tilde{\mathcal{M}} = (S, \tilde{\mathcal{I}})$ is its dual, then $k(\tilde{\mathcal{M}})$ is the minimum size of a covering of $S$ by members of $\tilde{\mathcal{I}}$. Without loss of generality, we may assume the members of the cover are in $\tilde{\mathcal{B}}$. Thus we have

$$k(\tilde{\mathcal{M}}) \leq k \iff S = \tilde{B}_1 \cup \cdots \cup \tilde{B}_k \text{ with } \tilde{B}_i \in \tilde{\mathcal{B}}.$$

Taking complements with respect to $S$ we can rewrite this as

$$k(\tilde{\mathcal{M}}) \leq k \iff \emptyset = B_1 \cap \cdots \cap B_k \text{ with } B_i \in \mathcal{B}.$$

Thus we have the following.

**Theorem 5.6.7** *Let $\mathcal{M}$ be a matroid. Then $k(\tilde{\mathcal{M}})$ is the smallest number of bases of $\mathcal{M}$ whose intersection is empty. In case $\mathcal{M} = \mathcal{M}(G)$ for a connected graph $G$, then $k(\tilde{\mathcal{M}})$ is the minimum number of spanning trees of $G$ whose intersection is empty.* □

We can fractionalize this result. We weight the bases of $\mathcal{M} = (S, \mathcal{I})$ so that the total weight of bases not containing each $x \in S$ is at least 1. The minimum total weight is $k_f(\tilde{\mathcal{M}})$. When $G$ is a graph, there is a nice connection between $k_f(\tilde{\mathcal{M}}(G))$ and the *edge toughness* of $G$.

## Edge toughness

Let $G$ be a graph with at least one edge. The edge toughness of $G$ is an invariant that measures how resilient the graph is against edge deletion. Formally, the *edge toughness* of $G$, denoted $\sigma'(G)$, is defined to be

$$\sigma'(G) = \min_F \frac{|F|}{c(G-F) - c(G)}$$

where the minimization is over all disconnecting subsets of edges of $G$. For example, if $G$ is a connected graph with edge toughness $\sigma'$, then we must delete at least $\lceil \sigma' j \rceil$ edges to break $G$ into $j+1$ components. This invariant is also known as the *strength* of the graph.

Now the denominator $c(G-F)-c(G)$ is computable from the cycle matroid of $G$. If $F \subseteq E(G)$, then $\rho(F) = n - c(V,F)$ where $n = |V(G)|$ and $c(V,F)$ is, of course, the number of components of the graph $(V,F)$. Thus

$$c(G-F) - c(G) = [n - c(G)] - [n - c(G-F)] = \rho(E) - \rho(E-F).$$

Thus, letting $X = E - F$, we can rewrite the definition of edge toughness as

$$\sigma'(G) = \min_X \frac{|E-X|}{\rho(E) - \rho(X)}$$

where the minimization is over all subsets of edges $X$ for which $(V,X)$ has more components than $G$.

Notice that this new definition easily extends to matroids. If $\mathcal{M} = (S, \mathcal{I})$ is a matroid, define

$$\sigma'(\mathcal{M}) = \min_X \frac{|S-X|}{\rho(S) - \rho(X)}$$

where the minimization is over all $X \subseteq S$ for which $\rho(X) < \rho(S)$.

Note that if $G$ has at least one edge, then $\sigma'(G) = 1$ if and only if $G$ has a cut edge (exercise 15 on page 91). In a similar vein, $\Upsilon(G) = \infty$ if and only if $G$ has a loop.

More generally, for a matroid $\mathcal{M}$ (with rank at least 1) we have $\sigma'(\mathcal{M}) = 1$ if and only if $\mathcal{M}$ has a coloop (exercise 16), and $\Upsilon(\mathcal{M}) = \infty$ if and only if $\mathcal{M}$ has a loop.

If $\mathcal{M}$ has neither loops nor coloops, then there is a nice connection between $\sigma'(\mathcal{M})$ and $k_f(\tilde{\mathcal{M}})$.

**Theorem 5.6.8** *Let $\mathcal{M}$ be a matroid and suppose both $\mathcal{M}$ and $\tilde{\mathcal{M}}$ are loopless. Then*

$$\frac{1}{\sigma'(\mathcal{M})} + \frac{1}{k_f(\tilde{\mathcal{M}})} = 1.$$

**Proof**. Rewrite Theorem 5.6.6 on page 87 as $\rho(X) = |X| - \tilde{\rho}(S) + \tilde{\rho}(S-X)$ and use it to compute:

$$\begin{aligned}
1 - \frac{1}{\sigma'(\mathcal{M})} &= 1 - \max\left(\frac{\rho(S) - \rho(X)}{|S-X|}\right) \\
&= \min\left(1 - \frac{\rho(S) - \rho(X)}{|S-X|}\right) \\
&= \min\left(1 - \frac{[|S| - \tilde{\rho}(S)] - [|X| - \tilde{\rho}(S) + \tilde{\rho}(S-X)]}{|S-X|}\right) \\
&= \min\left(1 - \frac{|S| - |X| - \tilde{\rho}(S-X)}{|S-X|}\right)
\end{aligned}$$

$$= \min\left(\frac{\tilde{\rho}(S-X)}{|S-X|}\right)$$
$$= \frac{1}{k_f(\tilde{\mathcal{M}})}.$$ □

We can rewrite Theorem 5.6.8 in graph terms.

**Corollary 5.6.9** *Let $G$ be a graph. The edge toughness of $G$ is given by*

$$\sigma'(G) = \frac{k_f(\tilde{\mathcal{M}}(G))}{k_f(\tilde{\mathcal{M}}(G)) - 1}.$$

*If $G$ is planar with planar dual $\tilde{G}$, then*

$$\sigma'(G) = \frac{\Upsilon_f(\tilde{G})}{\Upsilon_f(\tilde{G}) - 1}.$$ □

In particular, the Matroid Partitioning Algorithm gives us a polynomial-time algorithm for computing the edge toughness of a graph.

## 5.7 Exercises

1. Prove that the arboricity of a planar graph is at most 3 and that this bound is the best possible.

2. Prove that $\text{mad}(G)$ cannot be computed in a greedy fashion. Specifically, find a graph $G$ for which the following algorithm fails.

   **Input**: a graph $G$.

   **Output**: (allegedly) $\text{mad}(G)$.

   (a) Let $X \leftarrow \emptyset$.
   (b) Let $X \leftarrow X \cup \{\bar{d}(G)\}$.
   (c) Let $v$ be a vertex of minimum degree in $G$.
   (d) Let $G \leftarrow G - v$.
   (e) If $G$ is not empty, go to step (b).
   (f) Output the largest element of $X$.

3. Prove that for any matrix $A$, $\mathcal{M}(A)$ is indeed a matroid.

4. Prove that for any graph $G$, $\mathcal{M}(G)$ is indeed a matroid.

5. Create a good definition of isomorphic matroids.

6. Prove or disprove: For all graphs $G$, $\mathcal{M}(G)$ is isomorphic to $\mathcal{M}(A(G))$ where $A(G)$ is the adjacency matrix of $G$.

7. Prove or disprove: For every graph $G$, there is a matrix $A$ so that $\mathcal{M}(G)$ is isomorphic to $\mathcal{M}(A)$.

8. Let $U_{n,k}$ denote the pair $(S, \mathcal{I})$ where $S$ is an $n$-set and $\mathcal{I}$ is the set of all subsets of $S$ with at most $k$ elements.

   Prove that $U_{n,k}$ is a matroid. We call $U_{n,k}$ a *uniform* matroid.

   Compute $k(U_{n,k})$ and $k_f(U_{n,k})$.

9. Prove that there is no graph $G$ for which $\mathcal{M}(G) \cong U_{4,2}$. (The preceding exercise defines $U_{n,k}$.)

10. Prove that if $a_i, b_i > 0$ (for $i = 1, \ldots, c$) then

$$\frac{a_1 + \cdots + a_c}{b_1 + \cdots + b_c} \leq \max_i \left\{ \frac{a_i}{b_i} \right\}.$$

    (We use this fact in the proof of Theorem 5.1.1.)

11. Write a careful definition for $t \cdot \mathcal{M}$: the matroid formed from $\mathcal{M}$ by replacing each element of the ground set of $\mathcal{M}$ with $t$ parallel elements.

12. Prove Lemma 5.5.1 on page 84: *Let $G$ be a connected graph. Then $G$ has exactly one cycle if and only if $\nu(G) = \varepsilon(G)$.*

13. Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid with rank function $\rho$. Let $t$ be a positive integer. Define

$$\mathcal{I}_t = \{X \subseteq S : \rho(I) \geq |I| - t\},$$

    i.e., all sets that can be formed from independent sets by the addition of at most $t$ additional elements.

    Prove that $\mathcal{M}_t = (S, \mathcal{I}_t)$ is a matroid, called an *elongation* of $\mathcal{M}$.

    Observe that this is a generalization of Theorem 5.5.2 on page 84 since if $\mathcal{M} = \mathcal{M}(G)$, then $\mathcal{M}_1 = \mathcal{M}_1(G)$.

14. Prove Theorem 5.6.5 on page 87: *Let $G$ be a planar graph (embedded in the plane) and let $\tilde{G}$ be its dual. Then $\tilde{\mathcal{M}}(G) = \mathcal{M}(\tilde{G})$.*

15. Let $G$ be a graph with at least one edge. Prove that $\sigma'(G) \geq 1$ and that $\sigma'(G) = 1$ if and only if $G$ has a cut edge.

16. (Generalizing the previous problem.) Let $\mathcal{M}$ be a matroid of rank at least 1. Prove that $\sigma'(\mathcal{M}) \geq 1$ and $\sigma'(\mathcal{M}) = 1$ if and only if $\mathcal{M}$ has a coloop.

17. Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid. Define a hypergraph $\mathcal{H} = (S, \mathcal{I} - \{\emptyset\})$, i.e., delete $\emptyset$ as a member of $\mathcal{I}$. Note that $k(\mathcal{M}) = k(\mathcal{H})$. State and prove simple formulas for $\mu(\mathcal{H})$, $p(\mathcal{H})$, and $\tau(\mathcal{H})$.

## 5.8 Notes

Matroids were invented by Hassler Whitney [189] in the 1930s. For background reading on matroids, see the monographs by Recski [152] or Welsh [186].

The results on arboricity (e.g., Theorem 5.1.1 on page 72) are due to Nash-Williams [137, 138]. See also Edmonds [49].

The Matroid Partitioning Algorithm is due to Edmonds [49], but he treats a more general situation. Given several matroids on the same ground set, e.g., $\mathcal{M}_i = (S, \mathcal{I}_i)$ for $1 \leq i \leq k$,

Edmonds seeks a partition $S = I_1 \cup I_2 \cup \cdots \cup I_k$ where each $I_i \in \mathcal{I}_i$. We only needed the case in which all $k$ matroids are the same.

The material on fractional arboricity is based on the work of Payan [143] and Catlin, Grossman, Hobbs, and Lai [33, 34]. See the latter papers for a treatment of edge toughness.

Random graph theorists (and others) are concerned with graphs $G$ whose densest portion is $G$ itself. In particular, a graph $G$ is called *balanced* provided $\text{mad}(G) = \bar{d}(G)$. Maximum average degree and balanced graphs arise in the theory of evolution of graphs. Let $G$ be a fixed graph and let $n$ be a very large positive integer. Form a graph by gradually adding edges at random between $n$ fixed vertices. As the density of the evolving random graph increases we eventually reach a stage where it is very likely that the random graph contains $G$ as a subgraph. The point at which this happens depends only on $\text{mad}(G)$. See Bollobás [23] or Palmer [141]. For more information on balanced graphs, see the papers by Lai and Lai [112] or Ruciński and Vince [155].

# 6

# Fractional Isomorphism

## 6.1 Relaxing isomorphism

What does it mean for two graphs to be "the same"? The fundamental equivalence relation defined between graphs is that of *isomorphism*: graphs $G$ and $H$ are *isomorphic*, denoted $G \cong H$, provided there is a bijection $\phi : V(G) \to V(H)$ so that $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(H)$. In other words, graphs $G$ and $H$ are isomorphic if they differ only in the names of their vertices.

In many branches of mathematics, it is useful to relax the notion of sameness. For example, in geometry, equality of figures is relaxed to congruence (isometry), which is further relaxed to similarity, or ultimately to the topological notion of homeomorphic.

We seek to relax the notion of graph isomorphism. To do so, let us recast graph isomorphism as an integer feasibility problem. Let $G$ and $H$ be graphs with adjacency matrices $A$ and $B$. Graphs $G$ and $H$ are isomorphic if and only if there is a permutation matrix[1] $P$ so that $A = PBP^{-1}$. In other words, we can form the adjacency matrix $A$ by applying the same permutation to both the rows and columns of $B$; this amounts to relabeling the vertices of $H$ to make $G$.

Now the relation $A = PBP^{-1}$ can be rewritten as $AP = PB$. The requirement that $P$ is a permutation matrix can be restated as (1) $P \cdot \mathbf{1} = \mathbf{1}$, (2) $P^t \cdot \mathbf{1} = \mathbf{1}$, and (3) the entries of $P$ are in $\{0, 1\}$. (Here and in the sequel, $\mathbf{1}$ stands for a vector of all 1's.)

Thus, isomorphism of graphs $G$ and $H$ (with adjacency matrices $A$ and $B$) is equivalent to the feasibility of the following system of linear equations (think of $A$ and $B$ as given and the $P_{ij}$'s as the unknowns):

$$AP = PB,$$

$$P \cdot \mathbf{1} = \mathbf{1},$$

$$P^t \cdot \mathbf{1} = \mathbf{1}, \text{ and}$$

$$\text{all } P_{ij} \in \{0, 1\}.$$

Our theme has been to replace integer programs by linear programs, thus it is natural in this setting to drop the requirement that $P$ be a 0,1-matrix and simply require the entries in $P$ to be nonnegative. A matrix $S$ whose entries are nonnegative, and whose rows and columns all sum to 1 [i.e., $S \cdot \mathbf{1} = \mathbf{1}$ and $S^t \cdot \mathbf{1} = \mathbf{1}$] is called a *doubly stochastic* matrix.

We say that graphs $G$ and $H$ are *fractionally isomorphic*, and we write $G \cong_f H$, provided there is a doubly stochastic matrix $S$ for which $AS = SB$ where $A, B$ are the adjacency matrices of the graphs $G, H$, respectively.

**Proposition 6.1.1** *The relation $\cong_f$ is an equivalence relation.*

**Proof**. Exercise 1 on page 108. □

[1] A *permutation matrix* $P$ is a square 0,1-matrix with exactly one 1 in every row and column.

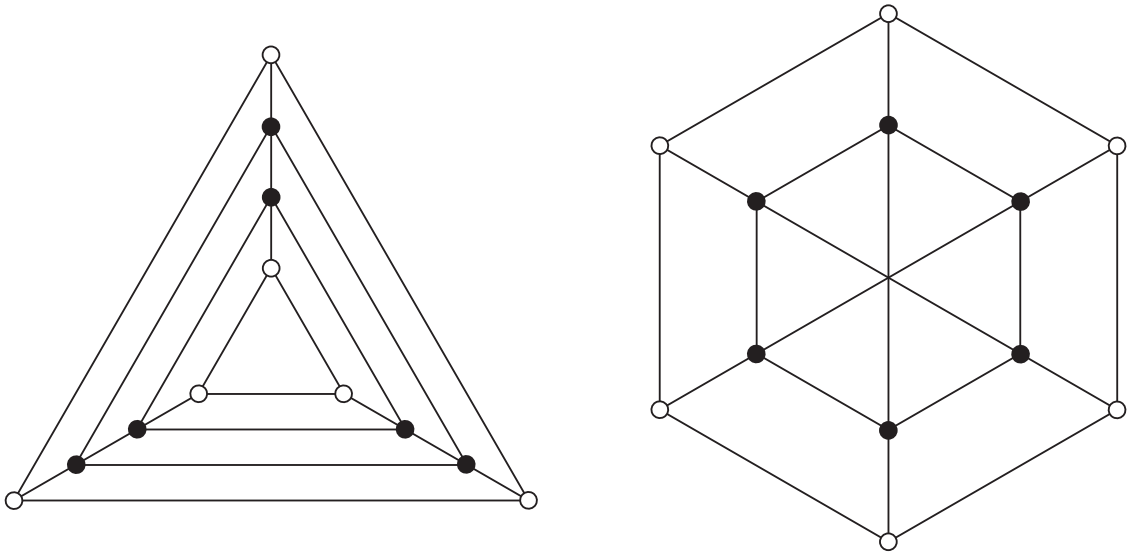

Figure 6.1. Two fractionally isomorphic graphs.

For example, let $G$ and $H$ be the two graphs in Figure 6.1. To verify that $G \cong_f H$, write their adjacency matrices as

$$A = \left[\begin{array}{c|c} 011100 & 100000 \\ 101010 & 010000 \\ 110001 & 001000 \\ 100011 & 000100 \\ 010101 & 000010 \\ 001110 & 000001 \\ \hline 100000 & 011000 \\ 010000 & 101000 \\ 001000 & 110000 \\ 000100 & 000011 \\ 000010 & 000101 \\ 000001 & 000110 \end{array}\right] \text{ and } B = \left[\begin{array}{c|c} 010101 & 100000 \\ 101010 & 010000 \\ 010101 & 001000 \\ 101010 & 000100 \\ 010101 & 000010 \\ 101010 & 000001 \\ \hline 100000 & 010001 \\ 010000 & 101000 \\ 001000 & 010100 \\ 000100 & 001010 \\ 000010 & 000101 \\ 000001 & 100010 \end{array}\right].$$

(To check this, note that we have listed the six vertices of degree 4 (black in the figure) first and the six vertices of degree 3 (white) last.) Let

$$S = \left(\frac{1}{6}\right)\left[\begin{array}{c|c} J & 0 \\ \hline 0 & J \end{array}\right]$$

where each $J$ is a $6 \times 6$ matrix of all ones and each 0 is a $6 \times 6$ matrix of all zeros. It is obvious that $S$ is doubly stochastic, and (with a little work) one checks that $AS = SB$. Thus the graphs $G$ and $H$ are fractionally isomorphic. They are not, however, isomorphic. Notice that $G$ is planar, but $H$ is not (its black vertices induce a $K_{3,3}$).

Although they are not isomorphic, these graphs have much in common. Let us note some of these common features, all of which hold in general for fractionally isomorphic graphs.

- $G$ and $H$ have the same number of vertices (12).
- $G$ and $H$ have the same number of edges (21).
- Both have the same degree sequence $(4, 4, 4, 4, 4, 4, 3, 3, 3, 3, 3, 3)$
- Both have the same maximum eigenvalue ($\frac{5+\sqrt{5}}{2} \approx 3.61803$).

These are not merely coincidences. Fractionally isomorphic graphs always share these common attributes.

**Proposition 6.1.2** *Fractionally isomorphic graphs have the same number of vertices.*

**Proof**. Check that doubly stochastic matrices must be square and the result follows easily. □

**Proposition 6.1.3** *Fractionally isomorphic graphs have the same number of edges.*

**Proof**. Let $A$ and $B$ be the adjacency matrices of fractionally isomorphic graphs $G$ and $H$, and let $S$ be doubly stochastic with $AS = SB$.

Notice that $\mathbf{1}^t \cdot A \cdot \mathbf{1} = 2\varepsilon(G)$ (we are summing the degrees of the vertices). Thus we compute

$$\begin{aligned} 2\varepsilon(G) &= \mathbf{1}^t \cdot A \cdot \mathbf{1} \\ &= \mathbf{1}^t \cdot A(S \cdot \mathbf{1}) \qquad \text{because } S \text{ is doubly stochastic} \\ &= \mathbf{1}^t \cdot (AS) \cdot \mathbf{1} \\ &= \mathbf{1}^t \cdot (SB) \cdot \mathbf{1} \\ &= (\mathbf{1}^t \cdot S)B \cdot \mathbf{1} \\ &= \mathbf{1}^t \cdot B \cdot \mathbf{1} \\ &= 2\varepsilon(H). \end{aligned}$$

□

We prove below that fractionally isomorphic graphs share the same degree sequence (Proposition 6.2.6 on page 98) and the same maximum eigenvalue (Proposition 6.5.2 on page 105).

In general, these conditions (same degree sequence, maximum eigenvalue, etc.) are not sufficient for fractional isomorphism (see exercise 5 on page 108). However, we do have the following.

**Proposition 6.1.4** *Let $G$ and $H$ be $r$-regular graphs on $n$ vertices. Then $G \cong_f H$.*

**Proof**. Let $A$ and $B$ be the adjacency matrices of $G$ and $H$. Let $S = \frac{1}{n}J$ where $J$ is an $n \times n$ matrix of all ones. Note that $AS = \frac{r}{n}J = SB$, hence $G \cong_f H$. □

## Another view

There is another way to view isomorphism and fractional isomorphism. Let $G$ and $H$ be isomorphic graphs on disjoint sets of vertices. Let us label the vertices of these graphs $V(G) = \{v_1, \ldots, v_n\}$ and $V(H) = \{w_1, \ldots, w_n\}$ so that $v_i \mapsto w_i$ is an isomorphism.

We can think of this isomorphism as a perfect matching between $V(G)$ and $V(H)$, i.e., the isomorphism is represented by a 1-regular graph $L$ on $V(G) \cup V(H)$. To distinguish the edge of $G$ and $H$ from the edges of $L$, we call the edges in the bipartite graph $L$ *links*. Let us now translate

the condition $v_iv_j \in E(G) \iff w_iw_j \in E(H)$ into a statement about edges (of $G$ and $H$) and links.

Let $v \in V(G)$ and $w \in V(H)$. We can find two sorts of 2-step paths from $v$ to $w$. We might first take an edge of $G$ followed by a link of $L$, or first take a link and then an edge (of $H$). These two kinds of paths are called *edge-link* or *link-edge*. The statement that $L$ represents an isomorphism between $G$ and $H$ can be re-expressed as follows: *For all $v \in V(G)$ and all $w \in V(H)$, there is a link-edge path from $v$ to $w$ if and only if there is an edge-link path from $v$ to $w$.*

We can describe fractional isomorphism in a similar way. Let $G$ and $H$ be graphs on $n$ vertices as before, with adjacency matrices $A$ and $B$ respectively. Let $S$ be doubly stochastic with $AS = SB$. The matrix $S$ can be represented as a weighted bipartite graph $L$ linking $V(G)$ and $V(H)$; there is a link from $v_i$ to $w_j$ just when $S_{ij}$ is positive. The weight of that link is, of course, $S_{ij}$.

For the graphs in Figure 6.1 on page 94, the linking graph consists of two disjoint copies of $K_{6,6}$ (one copy joining the black vertices and one copy joining the white vertices) in which every link has weight $\frac{1}{6}$. The condition $AS = SB$ can be re-expressed as follows: *For all vertices $v \in V(G)$ and $w \in V(H)$, the total weight of all links along edge-link paths from $v$ to $w$ equals the total weight of all links along link-edge paths from $v$ to $w$.*

## 6.2 Linear algebra tools

Our forthcoming analysis of fractional isomorphism relies on certain special properties of doubly stochastic matrices. First, they are positive matrices and therefore their eigenvalues and eigenvectors have special properties (see Theorem 6.2.2 on the facing page). Second, they can be written as convex combinations of permutation matrices (see Theorem 6.2.3 on the facing page).

In this section we collect matrix results that are not part of most undergraduate linear algebra curricula. Experts who are tempted to skip this section should read Theorem 6.2.4 on page 98 before proceeding.

### Direct sum and reducibility

Let $A$ and $B$ be square matrices. The *direct sum* of $A$ and $B$, denoted $A \oplus B$, is the matrix

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

where each "0" represents a rectangular block of 0's of the appropriate size. If $M = A \oplus B$ then we say $M$ is *decomposable*. More generally, we say $M$ is *decomposable* if there exist matrices $A$, $B$, $P$, and $Q$ such that $P$ and $Q$ are permutation matrices and $M = P(A \oplus B)Q$. Otherwise (if no such decomposition exists) we say that $M$ is *indecomposable*.

Let $M$ be an $n \times n$ matrix. Let $D(M)$ denote a digraph on $n$ vertices $v_1, \ldots, v_n$ with an arc from $v_i$ to $v_j$ just when $M_{ij} \neq 0$. We say that $M$ is *irreducible* provided $D(M)$ is strongly connected. Otherwise, we say that $M$ is *reducible*. Notice that $M$ is reducible just when there is a permutation matrix $P$ so that $PMP^t = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ where $A$ and $C$ are square matrices and 0 is a rectangular block of zeros.

Furthermore, we say that a matrix $M$ is *strongly irreducible* provided $PM$ is irreducible for any permutation matrix $P$. If $M$ is not strongly irreducible, then we can find permutation matrices $P$ and $Q$ so that $PMQ = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ where $A$ and $C$ are square matrices and 0 is a block of zeros.

For example, the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is irreducible, but neither strongly irreducible nor indecomposable.

**Proposition 6.2.1** *Let $S$ be a doubly stochastic, indecomposable matrix. Then $S$ is also strongly irreducible.*

**Proof**. Let $S$ be a doubly stochastic matrix. We prove the contrapositive of the Proposition. Suppose $S$ is not strongly irreducible, This means we can permute the rows and columns of $S$ so that $S = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}$ where $S_{11}$ and $S_{22}$ are square matrices and 0 represents a rectangular block of 0's. We claim that $S_{12}$ is 0. To see why, suppose $S_{11}$ has shape $a \times a$. Since $S$ is doubly stochastic, the sum of each column of $S_{11}$ is 1, so the sum of all the entries in $S_{11}$ is $a$. Now the sum of the entries in each row of $[S_{11}\ S_{12}]$ is also 1, so if any entry in $S_{12}$ were nonzero, there would be a row of $S_{11}$ whose sum would be less than 1, implying that the sum of all the entries in $S_{11}$ would be less than $a$, a contradiction. Thus $S_{12} = 0$ and it follows that $S = S_{11} \oplus S_{22}$, i.e., $S$ is decomposable. □

## Positive matrices: the Perron-Frobenius theorem

Let $M$ be a matrix. We write $M \geq 0$ to mean that every entry in $M$ is nonnegative and we say that $M$ is *nonnegative*. We write $M > 0$ if all of $M$'s entries are positive, and we call $M$ *positive*. Likewise, we may refer to vectors as being nonnegative or positive.

A *dominant eigenvalue* of a square matrix $M$ is an eigenvalue $\lambda$ of $M$ whose absolute value is at least that of any other eigenvalue, i.e., if $\lambda'$ is also an eigenvalue of $M$, then $|\lambda'| \leq |\lambda|$.

The principal theorem about nonnegative matrices is the celebrated Perron-Frobenius theorem.

**Theorem 6.2.2** *Let $M$ be an irreducible, nonnegative matrix. Then $M$ has a unique dominant eigenvalue $\lambda$ which is positive and has an associated eigenvector that is positive. Further, eigenvectors associated with other eigenvalues are not nonnegative.* □

## Doubly stochastic matrices: Birkhoff decomposition

Permutation matrices are doubly stochastic. Furthermore, doubly stochastic matrices can be decomposed into permutation matrices as follows.

**Theorem 6.2.3** *Let $S$ be a doubly stochastic matrix. There exist positive numbers $\alpha_1, \ldots, \alpha_s$ and permutation matrices $P_1, \ldots, P_s$ so that*

$$S = \sum_{i=1}^{s} \alpha_i P_i \quad \textit{and} \quad 1 = \sum_{i=1}^{s} \alpha_i. \qquad \square$$

This result can be expressed in words by saying *every doubly stochastic matrix is a convex combination of permutation matrices*. The key step in the proof is graph-theoretic (see exercise 2 on page 108). The decomposition $S = \sum \alpha_i P_i$ is known as a *Birkhoff decomposition* of the doubly stochastic matrix $S$.

The set of doubly stochastic matrices forms a compact, convex subset of the set of all real matrices; the extreme points of this convex set are exactly the permutation matrices.

Let $S$ be a doubly stochastic matrix and let $\mathbf{x}$ be a vector. The vector $S\mathbf{x}$ is a list of weighted averages of the entries in $\mathbf{x}$. If $R$ is another doubly stochastic matrix and $R(S\mathbf{x}) = \mathbf{x}$, then, in effect, no "loss of information" has taken place. Intuitively, this is only possible if $S\mathbf{x}$ and $\mathbf{x}$ have the same entries, i.e, $S\mathbf{x}$ is just a permutation of $\mathbf{x}$. Indeed, one can say more.

**Theorem 6.2.4** *Let $S$ and $R$ be $n \times n$ doubly stochastic matrices with Birkhoff decompositions $S = \sum \alpha_i P_i$ and $R = \sum \beta_j Q_j$. Let $\mathbf{x}, \mathbf{y}$ be $n$-vectors.*

(1) *If $\mathbf{y} = S\mathbf{x}$ and $\mathbf{x} = R\mathbf{y}$, then $\mathbf{y} = P_i\mathbf{x}$ and $\mathbf{x} = Q_j\mathbf{y}$ for all $i, j$.*

(2) *Let $\mathbf{x}$ and $\mathbf{y}$ be as in (1). If, in addition, either $S$ or $R$ is indecomposable, then $\mathbf{x} = \mathbf{y} = s \cdot \mathbf{1}$ for some scalar $s$.*

(3) *If $\mathbf{x}$ and $\mathbf{y}$ are 0,1-vectors and $\mathbf{y} = S\mathbf{x}$, then $\mathbf{y} = P_i\mathbf{x}$ for all $i$.*

The proof of this result makes use of the following convex set. Let $\mathbf{v}$ be a vector. Let $\Pi(\mathbf{v})$ denote the convex hull of all vectors formed by permuting the entries of $\mathbf{v}$. Note that the extreme points of $\Pi(\mathbf{v})$ are all the permutations of $\mathbf{v}$, i.e., the vectors of the form $P\mathbf{v}$ where $P$ is a permutation matrix (exercise 3 on page 108).

Now consider the statement $\mathbf{y} = S\mathbf{x}$ where $S$ is doubly stochastic. Since $S = \sum \alpha_i P_i$, we have $\mathbf{y} = S\mathbf{x} = \sum \alpha_i P_i\mathbf{x}$ and therefore $\mathbf{y}$ is a convex combination of permutations of $\mathbf{x}$, i.e., $\mathbf{y} \in \Pi(\mathbf{x})$. Furthermore, if $P'$ is any permutation matrix, then $P'\mathbf{y} = P'S\mathbf{x} = \sum \alpha_i (P'P_i)\mathbf{x}$, so $P'\mathbf{y} \in \Pi(\mathbf{x})$. In short, all the extreme points of $\Pi(\mathbf{y})$ are in $\Pi(\mathbf{x})$; hence we have proved the following.

**Lemma 6.2.5** *If $S$ is doubly stochastic and $\mathbf{y} = S\mathbf{x}$, then $\Pi(\mathbf{y}) \subseteq \Pi(\mathbf{x})$.* □

**Proof** (of Theorem 6.2.4). For (1), note that $\mathbf{y} = S\mathbf{x}$ and $\mathbf{x} = R\mathbf{y}$ imply, by Lemma 6.2.5, that $\Pi(\mathbf{x}) = \Pi(\mathbf{y})$. Thus $\mathbf{x}$ and $\mathbf{y}$ are extreme points of this polyhedron. As $\mathbf{y} = \sum \alpha_i P_i\mathbf{x}$ we must have that $\mathbf{y} = P_i\mathbf{x}$ for each $i$, and likewise $\mathbf{x} = Q_j\mathbf{y}$ for all $j$.

For (2), first note that if either $\mathbf{x}$ or $\mathbf{y}$ is $\mathbf{0}$, then both are $\mathbf{0}$ and the conclusion follows. So suppose that $\mathbf{x}, \mathbf{y}$ are nonzero and that $S$ is indecomposable. We know that $\mathbf{y} = S\mathbf{x}$ and that $\mathbf{x} = Q\mathbf{y}$ for some permutation matrix $Q$, thus $\mathbf{x} = QS\mathbf{x}$, i.e., $\mathbf{x}$ is an eigenvector of $QS$ with eigenvalue 1. Note that the vector $\mathbf{1}$ also is an eigenvector with eigenvalue 1. Now $S$ is nonnegative and indecomposable, and hence strongly irreducible (by Proposition 6.2.1 on the preceding page), so $QS$ is irreducible. Thus, by Theorem 6.2.2 on the preceding page, $QS$ has a *unique* dominant eigenvalue which is positive and has an associated eigenvector that is positive. Thus it follows that $\mathbf{x}$ is a scalar multiple of $\mathbf{1}$ as required.

For (3), we have only that $\mathbf{y} = S\mathbf{x}$, but we also know that $\mathbf{x}, \mathbf{y} \in \{0,1\}^n$. Now we know that $\mathbf{y} \in \Pi(\mathbf{x})$. Furthermore, $\mathbf{1}^t\mathbf{y} = \mathbf{1}^tS\mathbf{x} = \mathbf{1}^t\mathbf{x}$, so $\mathbf{x}$ and $\mathbf{y}$ have the same number of 1's, i.e., $\mathbf{x}$ is a permutation of $\mathbf{y}$. Thus $\mathbf{y}$ is an extreme point of $\Pi(\mathbf{x})$ and therefore, since $\mathbf{y} = \sum \alpha_i P_i\mathbf{x}$ we have $\mathbf{y} = P_i\mathbf{x}$ for all $i$. □

As an application of Theorem 6.2.4, we prove that fractionally isomorphic graphs have the same degree sequence. (In fact much more is true; see Theorem 6.5.1 on page 102.)

**Proposition 6.2.6** *If $G \cong_f H$, then $G$ and $H$ have the same degree sequence.*

**Proof**. Let $A, B$ be the adjacency matrices of $G, H$ and let $S$ be doubly stochastic with $AS = SB$. Let $\mathbf{d}_G = A \cdot \mathbf{1}$ and $\mathbf{d}_H = B \cdot \mathbf{1}$; these vectors record the degree sequences of $G$ and $H$ respectively.

Now

$$\mathbf{d}_G = A \cdot \mathbf{1} = AS \cdot \mathbf{1} = SB \cdot \mathbf{1} = S\mathbf{d}_H$$

and

$$\mathbf{d}_H = B \cdot \mathbf{1} = BS^t \cdot \mathbf{1} = S^t A \cdot \mathbf{1} = S^t \mathbf{d}_G$$

because $BS^t = S^t A$ follows from $AS = SB$ by transposition. Thus by Theorem 6.2.4 on the facing page we have that $\mathbf{d}_H$ and $\mathbf{d}_G$ are permutations of one another, i.e., $G$ and $H$ have the same degree sequence. □

## Kronecker product

Let $A$ and $B$ be $n \times n$ matrices. The *Kronecker product* of $A$ and $B$, denoted $A \otimes B$, is the $n^2 \times n^2$ matrix

$$\begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}B & A_{n2}B & \cdots & A_{nn}B \end{bmatrix}$$

where $A_{ij}B$ denotes an $n \times n$ block formed by multiplying each entry of $B$ by the scalar $A_{ij}$.

Matrices are convenient ways to represent linear transformations, so it is natural to seek a linear transformation that $A \otimes B$ represents.

For $n \times n$ matrices $A$, $B$, and $X$, let

$$F_{A,B}(X) = AXB^t.$$

Then $F_{A,B}$ is a linear transformation defined on the space $\mathbf{R}^{n \times n}$ of all $n \times n$ real matrices.

A natural basis for $\mathbf{R}^{n \times n}$ is the set $\{E(i,j) : 1 \le i, j \le n\}$ where $E(i,j)$ is a matrix all of whose entries are 0 except for a 1 in position $ij$. Note that $F_{A,B}(E(i,j)) = AE(i,j)B^t$ is a matrix whose $p, q$ entry is given by

$$[AE(i,j)B^t]_{pq} = \sum_{a,b} A_{pa} E(i,j)_{ab} B^t_{bq} = A_{pi} B_{qj}.$$

As $p$ and $q$ range over the numbers between 1 and $n$, we get the $n^2$ entries in the $j^{\text{th}}$ column in the $i^{\text{th}}$ block of columns in $A \otimes B$.

Further, for an $n \times n$ matrix $M$, let $\mathbf{c}(M)$ denote the $n^2$-column vector formed by stacking the columns of $M$ one atop the next. We have

$$\mathbf{c}\left(F_{A,B}(X)\right) = (A \otimes B)(\mathbf{c}(X)).$$

Thus $A \otimes B$ is the matrix for the linear transformation $F_{A,B}$ with respect to the basis $E(i,j)$.

When $A$ and $B$ are doubly stochastic, we see that $A \otimes B$ is as well. Furthermore, we have the following.

**Lemma 6.2.7** *Let $A$ and $B$ be doubly stochastic $n \times n$ matrices with Birkhoff decompositions $A = \sum_i \alpha_i P_i$ and $B = \sum_j \beta_j Q_j$. Then $A \otimes B$ is doubly stochastic with Birkhoff decomposition $A \otimes B = \sum_{i,j} \alpha_i \beta_j (P_i \otimes Q_j)$.*

**Proof**. Check that if $P$ and $Q$ are permutation matrices, then so is $P \otimes Q$. The lemma follows easily from there. □

## 6.3 Equitable partitions

The existence of a fractional isomorphism between two graphs is closely tied to the concept of an *equitable partition* of the vertices of a graph. To understand this concept, consider the black/white coloring of the graphs in Figure 6.1 on page 94. We mentioned earlier that the black vertices all have degree 4 and the white vertices all have degree 3. Actually, much more is true. Observe that every black vertex has exactly 3 black neighbors and exactly 1 white neighbor. Note further that every white vertex has 1 black and 2 white neighbors.

Let $S$ be a subset of the vertex set of a graph $G$. Let $d(v, S)$ denote the *degree of $v$ in $S$*, i.e., the number of neighbors of $v$ in $S$, i.e., $d(v, S) = |N(v) \cap S|$.

We say that a partition $V_1 \cup \cdots \cup V_s$ of $V(G)$ is *equitable* provided that for all $i, j$ and all $x, y \in V_i$ we have $d(x, V_j) = d(y, V_j)$. In other words, each of the induced subgraphs $G[V_i]$ must be regular and each of the bipartite graphs[2] $G[V_i, V_j]$ must be biregular.[3]

Every graph has an equitable partition: form the partition in which every part is a single vertex. If $G$ is a regular graph, then the partition with just one part containing all the vertices is equitable.

Now partitions of a set (including the equitable partitions of a graph) can be partially ordered by refinement.[4] Given two partitions of a set $\mathcal{P}$ and $\mathcal{Q}$, we can form their *join*, which is the finest partition that is coarser than both $\mathcal{P}$ and $\mathcal{Q}$. The join of $\mathcal{P}$ and $\mathcal{Q}$ is denoted $\mathcal{P} \vee \mathcal{Q}$.

Here is another way to describe the join of partitions $\mathcal{P}$ and $\mathcal{Q}$. Let $x \overset{\mathcal{P}}{\equiv} y$ mean that $x$ and $y$ are in the same part of the partition $\mathcal{P}$. It is easy to check that $\overset{\mathcal{P}}{\equiv}$ is an equivalence relation. Let $\mathcal{R} = \mathcal{P} \vee \mathcal{Q}$. Then $x \overset{\mathcal{R}}{\equiv} y$ if and only if we can find $z_1, z_2, \ldots, z_s$ with

$$x \equiv z_1 \equiv z_2 \equiv \cdots \equiv z_s \equiv y$$

where each $\equiv$ is either $\overset{\mathcal{P}}{\equiv}$ or $\overset{\mathcal{Q}}{\equiv}$ (exercise 6 on page 109). The relation $\overset{\mathcal{R}}{\equiv}$ is the transitive closure of the union of the relations $\overset{\mathcal{P}}{\equiv}$ and $\overset{\mathcal{Q}}{\equiv}$.

**Lemma 6.3.1** *Let $\mathcal{P}$ and $\mathcal{Q}$ be equitable partitions of a graph $G$. Then $\mathcal{P} \vee \mathcal{Q}$ is also an equitable partition.*

**Proof**. Let $\mathcal{R} = \{R_1, \ldots, R_r\}$. Note that each $R_i$ can be partitioned into sets from $\mathcal{P}$ or into sets from $\mathcal{Q}$ (because $\mathcal{P}$ and $\mathcal{Q}$ are refinements of $\mathcal{R}$).

Let $x, y \in R_i$. We must show that $d(x, R_j) = d(y, R_j)$ for all $j$. To do this, note that it is enough to consider the cases that $x \overset{\mathcal{P}}{\equiv} y$ and $x \overset{\mathcal{Q}}{\equiv} y$. Without loss of generality, suppose $x \overset{\mathcal{P}}{\equiv} y$.

Now $R_j$ can be partitioned into sets from $\mathcal{P}$, say $R_j = P_1 \cup P_2 \cup \cdots \cup P_p$ where each $P_a \in \mathcal{P}$. Thus

$$d(x, R_j) = d(x, P_1) + \cdots + d(x, P_p)$$
$$d(y, R_j) = d(y, P_1) + \cdots + d(y, P_p)$$

However, since $x$ and $y$ are in a common part of $\mathcal{P}$ it follows that $d(x, P_a) = d(y, P_a)$ for all $a$, and thus $d(x, R_j) = d(y, R_j)$ as required. □

[2]For disjoint $A, B \subseteq V(G)$, we write $G[A, B]$ to stand for the bipartite graph with vertex set $A \cup B$ in which there is an edge from an $A$ vertex to a $B$ vertex iff that edge exists in $G$.

[3]A bipartite graph $G = (X \cup Y, E)$ is *biregular* if any two vertices in $X$ have the same degree and if any two vertices in $Y$ have the same degree (not necessarily the same as the degree of an $X$ vertex).

[4]If $\mathcal{P}$ and $\mathcal{Q}$ are partitions of a common set $S$, we say that $\mathcal{P}$ is a *refinement* of $\mathcal{Q}$ provided every part of $\mathcal{P}$ is a subset of some part in $\mathcal{Q}$. When $\mathcal{P}$ is a refinement of $\mathcal{Q}$ we also say that $\mathcal{P}$ is *finer* than $\mathcal{Q}$ and the $\mathcal{Q}$ is *coarser* than $\mathcal{P}$.

**Theorem 6.3.2** *Let $G$ be a graph. Then $G$ has a unique coarsest equitable partition.*

**Proof**. Let $\mathcal{P}_1, \ldots, \mathcal{P}_k$ be a list of all of $G$'s equitable partitions; note that this list is nonempty since the "all singletons" partition of $V(G)$ is equitable. Let $\mathcal{Q} = \mathcal{P}_1 \vee \cdots \vee \mathcal{P}_k$. By Lemma 6.3.1, $\mathcal{Q}$ is the unique coarsest equitable partition of $G$. □

The set of equitable partitions of a graph is a join semilattice. Since this semilattice has a minimum element (the all singletons equitable partition), it is a lattice; see exercise 7 on page 109.

Consider again the graphs in Figure 6.1 on page 94. We would like to say that they have a *common* equitable partition. Let us define what we mean by this. Let $G$ be a graph and let $\mathcal{P} = \{P_1, \ldots, P_p\}$ be an equitable partition of $V(G)$. The *parameters* of $\mathcal{P}$ are a pair $(\mathbf{n}, D)$ where $\mathbf{n}$ is a $p$-vector whose $i^{\text{th}}$ entry is the size of $P_i$ and $D$ is a $p \times p$ matrix whose $ij$-entry is $d(x, P_j)$ for any $x \in P_i$. For the graphs in Figure 6.1, the partitions into black and white vertices both have parameters

$$\mathbf{n} = \begin{bmatrix} 6 \\ 6 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}.$$

We say that equitable partitions $\mathcal{P}$ and $\mathcal{Q}$ of graphs $G$ and $H$ have the *same parameters* if we can index the sets in $\mathcal{P}$ and $\mathcal{Q}$ so that their parameters $(\mathbf{n}, D)$ are identical. In such a case we say that $G$ and $H$ have a *common equitable partition*. If, in addition, $\mathcal{P}$ and $\mathcal{Q}$ are coarsest equitable partitions of $G$ and $H$, then we say that $G$ and $H$ have *common coarsest equitable partitions*.

The parameters $(\mathbf{n}, D)$ of an equitable partition $\mathcal{P} = \{P_1, \ldots, P_p\}$ of $V(G)$ obey a few simple relationships. First, since $\mathcal{P}$ is a partition of $V(G)$, we have $n_1 + \cdots + n_p = \nu(G)$. Second, since the number of edges from part $P_i$ to part $P_j$ equals the number edges from part $P_j$ to part $P_i$, we must have $n_i D_{ij} = n_j D_{ji}$.

Furthermore, suppose we order the vertices of $G$ so that vertices in $P_1$ are listed first, the vertices in $P_2$ second, etc. Then we can write the adjacency matrix $A(G)$ as

$$A(G) = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ A_{21} & A_{22} & \cdots & A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p1} & A_{p2} & \cdots & A_{pp} \end{bmatrix}$$

where block $A_{ij}$ is size $n_i \times n_j$ and the sum of each row of $A_{ij}$ is $D_{ij}$.

## 6.4 Iterated degree sequences

Another concept central to an understanding of fractional isomorphism is that of the *iterated degree sequence* of a graph. Recall that the *degree* of a vertex $v$ in $G$ is the cardinality of its neighbor set: $d(v) = |N(v)|$. The degree *sequence* of a graph $G$ is the multiset[5] of the degrees of the vertices,

$$d_1(G) = \{d(v) : v \in V(G)\}.$$

(The reason for the subscript '1' will become apparent shortly.)

[5]Usually, the degree sequence of a graph is a list of the degrees, often in descending numerical order. For our purposes here, however, the multiset formulation is more useful.

We can also define the degree sequence of a vertex to be the multiset of the degrees of its neighbors:

$$d_1(v) = \{d(w) : w \in N(v)\}.$$

For the graphs in Figure 6.1 on page 94, the white vertices have degree sequence $\{3, 3, 4\}$ and the black vertices have degree sequence $\{3, 4, 4, 4\}$.

The *degree sequence sequence* of $G$ is simply the multiset of degree sequences of the vertices of $G$:

$$d_2(G) = \{d_1(v) : v \in V(G)\}.$$

Likewise, the *degree sequence sequence* of a vertex is the multiset of degree sequences of its neighbors:

$$d_2(v) = \{d_1(w) : w \in N(v)\}.$$

For the graphs in Figure 6.1 the white vertices $w$ and the black vertices $b$ have

$$d_2(w) = \Big\{\{3, 3, 4\}, \{3, 3, 4\}, \{3, 4, 4, 4\}\Big\}, \quad \text{and}$$
$$d_2(b) = \Big\{\{3, 3, 4\}, \{3, 4, 4, 4\}, \{3, 4, 4, 4\}, \{3, 4, 4, 4\}\Big\}.$$

There is, of course, no reason to stop here. In general, define:

$$d_1(G) = \{d(v) : v \in V(G)\},$$
$$d_1(v) = \{d(w) : w \in N(v)\},$$
$$d_{k+1}(G) = \{d_k(v) : v \in V(G)\}, \quad \text{and}$$
$$d_{k+1}(v) = \{d_k(w) : w \in N(v)\}.$$

Finally, we may define the *ultimate degree sequence* of a vertex $v$ or a graph $G$ to be the infinite lists:

$$D(v) = [d_1(v), d_2(v), \ldots], \quad \text{and}$$
$$D(G) = [d_1(G), d_2(G), \ldots].$$

Notice that for the example graphs of Figure 6.1 on page 94 we have $D(H) = D(G)$. Indeed, the equivalence between having a fractional isomorphism, having a common coarsest equitable partition, and having the same ultimate degree sequence is the main theorem about fractional isomorphism.

## 6.5 The main theorem

**Theorem 6.5.1** *Let $G$ and $H$ be graphs. The following are equivalent.*

(1) $G \cong_f H$.

(2) *$G$ and $H$ have a common coarsest equitable partition.*

(3) *$G$ and $H$ have some common equitable partition.*

(4) $D(G) = D(H)$.

**Proof**. Let $G$ and $H$ be graphs on $n$ vertices with disjoint vertex sets $V(G) = \{v_1, \ldots, v_n\}$ and $V(H) = \{w_1, \ldots, w_n\}$. Let $A$ and $B$ denote their respective adjacency matrices.

We first show (1) $\Rightarrow$ (3). We use that to prove (1) $\Rightarrow$ (2), while (2) $\Rightarrow$ (3) is trivial. From there, we go on to show (3) $\Rightarrow$ (1), (3) $\Rightarrow$ (4), and (4) $\Rightarrow$ (3).

(1) $\Rightarrow$ (3): Suppose $G \cong_f H$ and let $S$ be doubly stochastic with $AS = SB$. By suitably renaming the vertices of $G$ and/or $H$, we may assume that $S$ has block structure $S = S_1 \oplus \cdots \oplus S_s$ where each $S_i$ is indecomposable (and therefore, by Proposition 6.2.1 on page 97, each $S_k$ is strongly irreducible). The matrix $S$ induces partitions on the vertices of $G$ and $H$ as follows. Let $\mathcal{P}_G = \{V_1, \ldots, V_s\}$ be the partition of $G$ in which $v_i$ and $v_j$ are in the same part just when indices $i$ and $j$ both fall inside the scope of the same $S_k$. Let $\mathcal{P}_H = \{W_1, \ldots, W_s\}$ be the partition of $H$ defined in the same way.

We show that $\mathcal{P}_G$ and $\mathcal{P}_H$ are equitable partitions with the same parameters.

We may partition the adjacency matrices $A$ and $B$ into blocks $A_{ij}$ and $B_{ij}$ following the partitions $\mathcal{P}_G$ and $\mathcal{P}_H$. The condition $AS = SB$ can be expanded to read

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ A_{21} & A_{22} & \cdots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s1} & A_{s2} & \cdots & A_{ss} \end{bmatrix} \begin{bmatrix} S_1 & 0 & \cdots & 0 \\ 0 & S_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_s \end{bmatrix} =$$

$$\begin{bmatrix} S_1 & 0 & \cdots & 0 \\ 0 & S_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_s \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1s} \\ B_{21} & B_{22} & \cdots & B_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ B_{s1} & B_{s2} & \cdots & B_{ss} \end{bmatrix}.$$

This can be rewritten more compactly as

$$A_{ij}S_j = S_iB_{ij}. \qquad (*)$$

Transposing $AS = SB$ yields $S^tA = BS^t$, which can be expressed

$$S_i^tA_{ij} = B_{ij}S_j^t. \qquad (**)$$

The vector $\mathbf{d}_{ij}(G) = A_{ij} \cdot \mathbf{1}$ gives the number of edges from each vertex in $V_i$ to all vertices in $V_j$ and the vector $\mathbf{d}_{ij}(H) = B_{ij} \cdot \mathbf{1}$ gives the number of edges from each vertex in $W_i$ to all vertices in $W_j$. We are done once we show that $\mathbf{d}_{ij}(G) = \mathbf{d}_{ij}(H) = c \cdot \mathbf{1}$ for some integer $c$.

To do this, multiply equations $(*)$ and $(**)$ on the right by the vector $\mathbf{1}$ to get

$$A_{ij}S_j \cdot \mathbf{1} = S_iB_{ij} \cdot \mathbf{1} \Rightarrow \mathbf{d}_{ij}(G) = S_i\mathbf{d}_{ij}(H), \quad \text{and}$$
$$S_i^tA_{ij} \cdot \mathbf{1} = B_{ij}S_j^t \cdot \mathbf{1} \Rightarrow S_i^t\mathbf{d}_{ij}(G) = \mathbf{d}_{ij}(H).$$

We may now apply part (2) of Theorem 6.2.4 on page 98 to vectors $\mathbf{d}_{ij}(G)$ and $\mathbf{d}_{ij}(H)$ to conclude that $\mathbf{d}_{ij}(G) = \mathbf{d}_{ij}(H) = c \cdot \mathbf{1}$ as required.

(1) $\Rightarrow$ (2): We have shown that (1) $\Rightarrow$ (3). Further, we have shown that the blocks of $S$ induce a common equitable partition in the graphs $G$ and $H$. We use this to show that (1) $\Rightarrow$ (2).

Suppose $G \cong_f H$ and $S$ is doubly stochastic with $AS = SB$. Let $\mathcal{P}_G$ be the coarsest equitable partition of $G$ with parameters $(\mathbf{n}, D)$. It is enough to show that $H$ has an equitable partition with the same parameters.

Let $\mathcal{P}_G = \{V_1, \ldots, V_r\}$ and assume that we have labeled the vertices of $G$ so that the first $n_1$ vertices of $G$ are in $V_1$, the next $n_2$ vertices of $G$ are in $V_2$, etc.

Let $R_i = (1/n_i)J_{n_i}$ where $n_i = |V_i|$ and $J_{n_i}$ is an $n_i \times n_i$ block of all ones. Put $R = R_1 \oplus R_2 \oplus \cdots \oplus R_r$. Note that $R$ is doubly stochastic. Furthermore, because $\mathcal{P}_G$ is an equitable partition, we have that $n_i D_{ij} = n_j D_{ji}$, which we rearrange to read $AR = RA$.

Since $AS = SB$, we compute that $ARS = RAS = RSB$.

Next we claim that $RS = R$. We know that the indecomposable blocks of $S$ correspond to a (perhaps finer) equitable partition of $G$; thus we may write $S = S_1 \oplus S_2 \oplus \cdots \oplus S_r$ where each $S_k$ is doubly stochastic (but not necessarily indecomposable) and has the same dimensions as $R_k$. Thus $RS = (R_1S_1) \oplus \cdots \oplus (R_rS_r)$ and $R_kS_k = R_k$ because $R_k$ is a scalar multiple of a matrix of all ones and $S_k$ is doubly stochastic.

Since we know that $ARS = RSB$, we now have $AR = RB$. We therefore know that $R$ induces an equitable partition in $H$ with the same parameters as $\mathcal{P}_G$, and we are done.

(2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (1): Suppose $G$ and $H$ have equitable partitions $\mathcal{P}_G$ and $\mathcal{P}_H$ with the same parameters $(\mathbf{n}, D)$. Let $\mathcal{P}_G = \{V_1, \ldots, V_s\}$, $\mathcal{P}_H = \{W_1, \ldots, W_s\}$ labeled so that $|V_i| = |W_i| = n_i$ and so that the number of edges from any vertex in $V_i$ to $V_j$ or from any vertex in $W_i$ to $W_j$ is $D_{ij}$. Furthermore, we may assume that the vertices of $G$ are labeled so that the first $n_1$ vertices are in $V_1$, the next $n_2$ vertices are in $V_2$, etc., and the same for $H$ and $W_1, W_2, \ldots$.

Let $S_i = (1/n_i)J_{n_i}$ and let $S = S_1 \oplus \cdots \oplus S_s$. Now just check that the doubly stochastic matrix $S$ satisfies $AS = SB$, hence $G \cong_f H$.

(3) $\Rightarrow$ (4): Let $\mathcal{P}_G = \{V_1, \ldots, V_s\}$ and $\mathcal{P}_H = \{W_1, \ldots, W_s\}$ be equitable partitions of $G$ and $H$ with the same parameters. We want to show that $D(G) = D(H)$. It is enough to show that for all $v, w \in V_i \cup W_i$ we have $D(v) = D(w)$, i.e., $d_k(v) = d_k(w)$ for all $k$. We do this by induction on $k$.

The basis case $k = 1$ follows from the fact that $\mathcal{P}_G$ and $\mathcal{P}_H$ are equitable partitions with the same parameters. Suppose we know that for all $v, w \in V_i \cup W_i$ we have $d_{k-1}(v) = d_{k-1}(w)$. Now $d_k(v) = \{d_{k-1}(x) : x \in N(v)\}$ and $d_k(w) = \{d_{k-1}(y) : y \in N(w)\}$. But we know that the number of neighbors of $v$ and $w$ in each part of the appropriate partition is the same, so by induction we have $d_k(v) = d_k(w)$.

(4) $\Rightarrow$ (3): Suppose $D(G) = D(H)$. Partition both $G$ and $H$ putting two vertices in the same part if they have the same iterated degree sequence; call these partitions $\mathcal{P}_G$ and $\mathcal{P}_H$. From the vector $D(v)$ we can read off how many neighbors of each type $v$ has. It follows that $\mathcal{P}_G$ and $\mathcal{P}_H$ are equitable with the same parameters. □

## Some consequences

Theorem 6.5.1 on page 102 has some interesting consequences. We list them briefly here.

Let $G$ and $H$ be graphs. They are fractionally isomorphic just when their coarsest equitable partitions have the same parameters. Thus we can test if $G$ and $H$ are fractionally isomorphic by finding their coarsest equitable partitions (see exercise 12 on page 109) and then comparing the parameters of the two partitions.

Let $\mathcal{P} = \{V_1, \ldots, V_p\}$ and $\mathcal{Q} = \{W_1, \ldots, W_p\}$ be coarsest equitable partitions of fractionally isomorphic graphs $G$ and $H$. We see from the proof of Theorem 6.5.1 that we may assume that the doubly stochastic matrix $S$ has the form $S = (1/n_1)J_{n_1} \oplus \cdots \oplus (1/n_p)J_{n_p}$. Thus we may assume that the linking graph is a disjoint union of complete bipartite graphs with uniform edge weights.

Further, if we partition the adjacency matrix $A$ of $G$ according to the coarsest equitable partition, i.e.,

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1p} \\ \vdots & \ddots & \vdots \\ A_{p1} & \cdots & A_{pp} \end{bmatrix}$$

then we know that $A_{ij}$ is a 0,1-matrix with constant row sum $d_{ij}$ (and constant column sum $d_{ji}$). We can form all graphs fractionally isomorphic to $G$ by the following operation: For every pair of indices $i, j$ with $1 \leq i \leq j \leq p$, replace $A_{ij}$ by any 0,1-matrix $C_{ij}$of the same shape with the same row and column sums (and replace $A_{ji} = A_{ij}^t$ with $C_{ij}^t$). This gives a complete description of the fractional isomorphism class of a graph.

Let $\mathcal{P} = \{V_1, \ldots, V_p\}$ be the coarsest equitable partition of $G$. For any two vertices $x, y$ in the same part of $\mathcal{P}$ the number of walks of length $k$ starting at $x$ must equal the number of walks of length $k$ starting at $y$. Further, if $G$ is fractionally isomorphic to $H$, then for any vertex $z$ linked to $x$, the number of walks starting at $z$ of length $k$ (in $H$) must equal the the number of length $k$ walks (in $G$) starting at $x$.

Let $\mathbf{w}(k)$ denote the vector of the number of walks of length $k$ starting anywhere in $G$. Note that $\mathbf{w}(k) = A^k \cdot \mathbf{1}$. If $G$ is connected, $A^k \cdot \mathbf{1}$ tends to the eigendirection corresponding to the largest eigenvalue $\lambda$ of $A$. Thus for any vertex $x$, the number of walks of length $k$ starting at $x$ is asymptotic to $c\lambda^k$ for some positive constant $c$. (If the graph is not connected, the number of walks of length $k$ out of a vertex grows like $\lambda^k$ where $\lambda$ is the maximum eigenvalue of the component of the graph containing the vertex.)

We have thus proved the following.

**Proposition 6.5.2** *If $G \cong_f H$ then the maximum eigenvalues of $G$ and $H$ are the same.* □

Note that fractionally isomorphic graphs need not be cospectral; see exercise 10 on page 109.

## 6.6 Other relaxations of isomorphism

The relation $G \cong H$ is equivalently expressed in matrix notation as $AP = PB$ or $A = PBP^t$ where $P$ is a permutation matrix; the equivalence follows from the fact that $P^t = P^{-1}$, i.e., permutation matrices are real orthogonal matrices. However, it is not necessarily the case that $S^t = S^{-1}$ for a doubly stochastic matrix. Thus the condition that $A = SBS^t$ for some $S$ is not equivalent to the condition that $AS = SB$ for some $S$.

Define a relation $\rightarrow$ on square matrices by $A \rightarrow B$ provided $A = SBS^t$ for some doubly stochastic matrix $S$. Furthermore, let us write $A \leftrightarrow B$ provided $A = SBS^t$ and $B = S^tAS$ for some doubly stochastic matrix $S$.

Surprisingly, the relations $\rightarrow$ and $\leftrightarrow$ on adjacency matrices are the same and are equivalent to isomorphism!

**Theorem 6.6.1** *Let $G$ and $H$ be graphs with adjacency matrices $A$ and $B$. The following are equivalent.*

(1) $G \cong H$.

(2) $A \leftrightarrow B$.

(3) $A \to B$ *and* $B \to A$.

(4) $A \to B$.

Note that the relations $\to$ and $\leftrightarrow$ are not the same for general square matrices and, in general, $\to$ is not an equivalence relation. For example, let $I$ be a $2 \times 2$ identity matrix and let $B = \frac{1}{2}J_2$, the $2 \times 2$ matrix all of whose entries are $1/2$. Then $A \to B$ but $B \not\to A$ and $A \not\to B$. In Theorem 6.6.1 the equivalences hold because the entries of the matrices are in $\{0, 1\}$.

**Proof**. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are all trivial. We show (4) $\Rightarrow$ (1).

Recall the material on Kronecker product (see page 99). The relation $A \to B$ can be restated as $A = F_{S,S}(B)$ for some doubly stochastic matrix $S$. Let $S = \sum_i \alpha_i P_i$ be a Birkhoff decomposition of $S$. Apply part (3) of Theorem 6.2.4 on page 98 and we have that $A = F_{P_i,P_j}(B)$ for all $i, j$. In particular, if we take $i = j$ we have $A = P_i B P_i^t$ and therefore $G \cong H$. □

Theorem 6.6.1 gives an interesting approach to checking if two graphs are isomorphic. One can check if the following system of quadratic equations, linear equations, and linear inequalities hold:

$$\begin{aligned} A &= SBS^t, \\ S \cdot \mathbf{1} &= \mathbf{1}, \\ S^t \cdot \mathbf{1} &= \mathbf{1}, \quad \text{and} \\ S &\geq 0. \end{aligned}$$

Of course, checking the feasibility of a general system of quadratic equations is a difficult problem.

## Semi-isomorphism

We relaxed the equation $A = PBP^t$ by allowing $P$ to stand for a doubly stochastic matrix instead of a permutation matrix. A further relaxation is to allow the two instances of $P$ to stand for *different* doubly stochastic matrices. Let us say that graphs $G$ and $H$ (with adjacency matrices $A$ and $B$) are *semi-isomorphic*, $G \cong' H$, provided there are doubly stochastic matrices $R$ and $S$ so that $A = SBR$.

It is not obvious that $\cong'$ is an equivalence relation. However, this fact is a consequence of the following.

**Lemma 6.6.2** *Let $G$ and $H$ be graphs with adjacency matrices $A$ and $B$. Then $G \cong' H$ if and only if there are permutation matrices $P$ and $Q$ so that $A = PBQ$.*

**Proof**. The ($\Leftarrow$) implication is trivial. For the ($\Rightarrow$) implication let $G$ and $H$ be semi-isomorphic with $A = SBR$. We can write this as $A = F_{S,R^t}(B)$. Applying part (3) of Theorem 6.2.4 on page 98 we have $A = F_{P,Q^t}(B)$ where $P$ and $Q$ are permutation matrices in Birkhoff representations of $S$ and $R$ respectively. Thus $A = PBQ$ as claimed. □

If $A = PBQ$ for permutation matrices $P$ and $Q$, then we have

$$QAQ^t = Q(PBQ)Q^t = (QP)B.$$

In other words, after suitably renaming the vertices of $G$, we see that the adjacency matrix of $G$ has the identical rows as the adjacency matrix of $H$, but perhaps in a different order. In other words, the neighborhoods of $G$ and $H$ are the same. Let's make this more precise.

Let $G$ be a graph. For $v \in V(G)$ its neighborhood is, of course, $N(v) = \{w \in V(G) : vw \in E(G)\}$. Define the *neighborhood multiset* of $G$ to be the multiset $N(G) = \{N(v) : v \in V(G)\}$.

Thus graphs $G$ and $H$ are semi-isomorphic if and only if we can relabel their vertices so that $N(G) = N(H)$.

Semi-isomorphism lives somewhere between isomorphism and fractional isomorphism.

**Theorem 6.6.3** *Let $G$ and $H$ be graphs. Then*

$$G \cong H \;\Rightarrow\; G \cong' H \;\Rightarrow\; G \cong_f H.$$

*Furthermore, neither implication may be reversed.*

**Proof**. The first implication is trivial. For the second, let $A$ and $B$ be the adjacency matrices of graphs $G$ and $H$, and (using Lemma 6.6.2) let $P$ and $Q$ be permutation matrices with $A = PBQ$. Taking transposes we have $A = Q^tBP^t$. These equations can be rewritten

$$AQ^t = PB$$
$$AP = Q^tB.$$

Adding these equations and dividing by two gives $AS = SB$, where $S = \frac{1}{2}(P + Q^t)$ is doubly stochastic. Hence $G \cong_f H$. The fact that neither implication can be reversed is left to the reader (exercise 11 on page 109). □

The semi-isomorphism of graphs can be related to the isomorphism of bipartite graphs as follows.

Let $G$ be a graph with $V(G) = \{v_1, \ldots, v_n\}$. Associate with $G$ a bipartite graph $B(G)$. The vertices of $B(G)$ are $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$. We have an edge $x_iy_j$ in $B(G)$ just when $v_iv_j \in E(G)$. If $A$ is the adjacency matrix of $G$, then the adjacency matrix of $B(G)$ is $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$. Notice that for a bipartite graph $G$, we have $B(G) \cong 2G$ (2 disjoint copies of $G$). See also the proof of Theorem 2.1.6 on page 16.

**Theorem 6.6.4** *Let $G$ and $H$ be graphs. Then $G \cong' H$ if and only if $B(G) \cong B(H)$.*

**Proof**. Let $G$ and $H$ be graphs with adjacency matrices $A$ and $B$.

If $G \cong' H$ then we can find permutation matrices $P$ and $Q$ so that $A = PBQ$. Check that

$$\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & Q^t \end{bmatrix} \cdot \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} \cdot \begin{bmatrix} P^t & 0 \\ 0 & Q \end{bmatrix}$$

and therefore $B(G) \cong B(H)$.

Now suppose $B(G) \cong B(H)$. Consider first the case that $B(G)$ and $B(H)$ are connected. Since $B(G)$ and $B(H)$ are isomorphic there is a permutation matrix $\begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ so that

$$\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix} = \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \cdot \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} \cdot \begin{bmatrix} P^t & R^t \\ Q^t & S^t \end{bmatrix}.$$

Now an isomorphism between connected bipartite graphs preserves (or reverses) the bipartition of the graph. Thus either $P = S = 0$ (and $Q$ and $R$ are permutation matrices) or $Q = R = 0$ (and $P$ and $S$ are permutation matrices). We know from the above matrix equation that $A = PBS^t + QBR^t$. In case $P = S = 0$ we have $A = QBR^t$ and in case $Q = R = 0$ we have $A = PBS^t$. In either case, we conclude that $G \cong' H$.

If $B(G)$ and $B(H)$ are not connected, we simply repeat the above analysis for their pairs of isomorphic components. □

The semi-isomorphism of graphs reduces to isomorphism of bipartite graphs. For bipartite graphs, isomorphism and semi-isomorphism are the same.

**Corollary 6.6.5** *If $G$ and $H$ are bipartite, then $G \cong H \iff G \cong' H$.*

**Proof**. $G \cong' H \iff B(G) \cong B(H) \iff 2G \cong 2H \iff G \cong H$. □

## 6.7 Exercises

1. Prove that $\cong_f$ is an equivalence relation. (Hint: First prove that if $S$ and $T$ are $n \times n$ doubly stochastic matrices, then so is $ST$.)

2. Let $S$ be a doubly stochastic matrix. Let $B(S)$ be the bipartite graph whose vertices correspond to the rows and columns of $S$, in other words, $V(B(S)) = \{r_1, r_2, \ldots, r_n, c_1, c_2, \ldots c_n\}$. Put an edge from $r_i$ to $c_j$ just when $S_{ij} \neq 0$.

   Prove that $B(S)$ has a perfect matching.

   Use this to prove that $S$ can be written as a convex combination of permutation matrices (Theorem 6.2.3 on page 97).

3. Let $\mathbf{v}$ be a vector and let

   $$X = \{P\mathbf{v} : P \text{ is a permutation matrix}\}.$$

   Thus $\Pi(\mathbf{v}) = \langle X \rangle$, i.e., the convex hull of the points in $X$. Prove that $X$ is the set of extreme points of $\Pi(\mathbf{v})$.

   Restated, prove that if $\mathbf{v} = \sum \alpha_i P_i \mathbf{v}$ where the $\alpha_i$'s are positive and sum to one, then $P_i\mathbf{v} = \mathbf{v}$ for all $i$.

4. Prove that if $G$ is a graph and $T$ is a tree and if $G \cong_f T$, then $G \cong T$.

5. Show that the eigenvalues and the degree sequences of the two graphs in Figure 6.2 on the facing page are the same, but the graphs are not fractionally isomorphic.

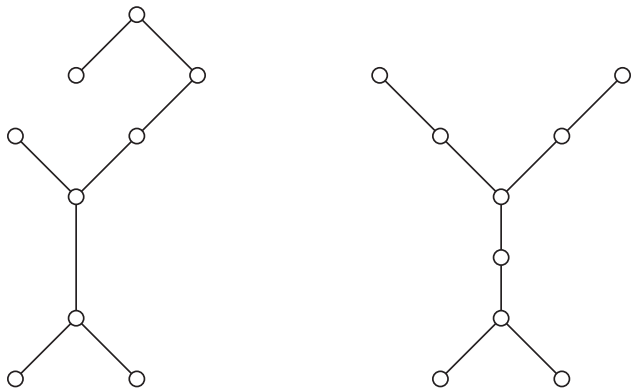

Figure 6.2. Two graphs with the same degree sequence and eigenvalues but that are not fractionally isomorphic.

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6. Let $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$ be partitions of a set $S$. Suppose $\mathcal{R} = \mathcal{P} \vee \mathcal{Q}$. Choose $x, y \in S$. Prove that $x \overset{\mathcal{R}}{\equiv} y$ if and only if there exist $z_1, \ldots, z_s \in S$ so that

$$x \equiv z_1 \equiv z_2 \equiv \cdots \equiv z_s \equiv y$$

where each $\equiv$ is either $\overset{\mathcal{P}}{\equiv}$ or $\overset{\mathcal{Q}}{\equiv}$. Indeed, show that we may assume that the $\equiv$ *alternate* between $\overset{\mathcal{P}}{\equiv}$ and $\overset{\mathcal{Q}}{\equiv}$.

7. Let $P = (X, \leq)$ be a finite partially ordered set in which the join of any two elements is defined. Suppose also that $P$ has a minimum element (an element that is below all other elements). Prove that the meet of any two elements of $P$ is also defined and therefore $P$ is a lattice.

   (The *meet* of $x$ and $y$, denoted $x \wedge y$, is the greatest element that is below both $x$ and $y$. A *lattice* is a partially ordered set in which the meet and join of any two elements are defined.)

8. Let $G$ be a graph and let $\mathcal{P}$ and $\mathcal{Q}$ be equitable partitions of $G$. Is it necessarily the case that the coarsest common refinement of $\mathcal{P}$ and $\mathcal{Q}$ is equitable?

9. What is the relationship between the tensor product of vectors $\mathbf{v} \otimes \mathbf{w}$ and the Kronecker product of matrices $A \otimes B$?

10. In Proposition 6.5.2 on page 105 we learn that maximum eigenvalue is a fractional isomorphism invariant. Find a pair of fractionally isomorphic graphs that are not cospectral (have the same eigenvalues).

11. Show that neither implication in Theorem 6.6.3 on page 107 can be reversed. That is (1) find a pair of graphs that are semi-isomorphic, but not isomorphic, and (2) find a pair of graphs that are fractionally isomorphic, but not semi-isomorphic.

12. Let $G$ be a graph and let $\mathcal{P} = \{V_1, \ldots, V_p\}$ be a partition of $V(G)$. Form a new partition $\mathcal{P}'$ as follows: two vertices $x, y \in V_i$ will be in a common part of $\mathcal{P}'$ just when $d(x, V_j) = d(y, V_j)$ for all $j$.

Consider the sequence $\mathcal{P} \to \mathcal{P}' \to \mathcal{P}'' \to \cdots$ beginning with the partition $\mathcal{P} = \{V(G)\}$, i.e., the one-part partition of $V(G)$.

Prove that the sequence leads to the coarsest equitable partition of $G$ at which point all further partitions are the same.

13. Is the problem of deciding whether two given graphs are fractionally isomorphic solvable in polynomial time, or is this decision problem NP-complete?

14. Let $G$ be a graph. Let $G'$ denote the bipartite graph formed from $G$ by replacing each edge of $G$ by a path length 2. (In other words, we form $G'$ from $G$ by subdividing each edge of $G$.)

    Prove that $G \cong H \iff G' \cong H'$.

    Conclude that the problem of recognizing a pair of isomorphic graphs is polynomially equivalent to the problem of recognizing a pair of semi-isomorphic graphs.

15. Let $G$ be a graph with adjacency matrix $A$. A *fractional automorphism* of $G$ is a doubly stochastic matrix $S$ so that $AS = SA$.

    Prove that the set of fractional automorphisms of $G$ forms a semigroup under matrix multiplication.

## 6.8 Notes

This chapter is based largely on the paper [151] by Ramana, Scheinerman, and Ullman. The notions of fractional isomorphism and equitable partitions can be traced back to the work of Brualdi [32], Leighton [116], McKay [130], Mowshowitz [135], and Tinhofer [177].

For background on Theorem 6.2.2 on page 97, commonly known as the Perron-Frobenius Theorem, see [95] and [96]. Theorem 6.2.4 on page 98 is due to Hardy, Littlewood, and Pólya [84]. See also Rado [150] as well as the book by Marshall and Olkin [128].

The graphs in Figure 6.2 on the preceding page are courtesy of Allen Schwenk.

Armed with the notion of fractional isomorphism, one can seek to prove a fractional version of the reconstruction problem; see the survey articles by Bondy [24] or Nash-Williams [139] for an overview. The fractional reconstruction problem is tantamount to the reconstruction of iterated degree sequences. This was considered by Manvel [127] and remains an open problem.

The computational complexity of graph isomorphism is unknown; clearly the problem is in NP, but it is not known if it is NP-hard or polynomial-time solvable. Booth and Colbourn [27] show that checking if bipartite graphs are isomorphic is just as hard as checking if general graphs are isomorphic. Thus the semi-isomorphism problem is again as hard as graph isomorphism. However, determining if two graphs are fractionally isomorphic is polynomial-time solvable.

# 7

# Fractional Odds and Ends

## 7.1 Fractional topological graph theory

Topological graph theory deals with finding "best" drawings of graphs on various surfaces. One may wish to minimize the genus of the surface, or the number of crossings. Or one may seek to decompose the graph into a minimum number of planar subgraphs. These ideas correspond to the graph invariants known as *genus*, *crossing number*, and *thickness*. We consider fractional analogues of each.

### Fractional genus

The *genus* $g(G)$ of a graph $G$ is the smallest genus of an oriented surface on which $G$ can be embedded. Planar graphs have genus 0, nonplanar graphs that are embeddable on the torus have genus 1, and so forth. In searching for an interesting fractional analogue of this notion, one is led naturally to the following: Write $tG$ for the graph composed of $t$ disjoint copies of $G$, and then define the fractional genus $g_f(G)$ of a graph $G$ by $\lim_{t\to\infty} g(tG)/t$. Unfortunately, this definition collapses completely, yielding $g_f = g$ for all graphs, owing to the following.

**Theorem 7.1.1** *The genus of a graph is the sum of the genera of its components.* □

Thus

$$g_f(G) = \lim_{t\to\infty} \frac{g(tG)}{t} = \lim_{t\to\infty} \frac{tg(G)}{t} = g(G).$$

The situation for nonorientable genus is more complicated. The nonorientable genus of a nonorientable surface is the number of crosscaps one must add to a sphere to achieve that surface. The surface with nonorientable genus 1 is called the *projective plane*. The *nonorientable genus* of a graph $G$, denoted $\tilde{g}(G)$, is the least nonorientable genus of a nonorientable surface upon which $G$ embeds. The *fractional nonorientable genus* of $G$ is, of course,

$$\tilde{g}_f(G) = \lim_{t\to\infty} \frac{\tilde{g}(tG)}{t}.$$

The nonorientable analogue of Theorem 7.1.1 fails. For example, $\tilde{g}(K_7) = 3$, but $\tilde{g}(2K_7) = 5 \neq 3 + 3$.

It is known that $\tilde{g}(G) \leq 2g(G) + 1$ for any graph $G$, but no lower bound can be given on $\tilde{g}(G)$ in terms of $g(G)$ as there are graphs of arbitrarily high genus that embed on the projective plane (so $g(G)$ is large with $\tilde{g}(G) = 1$). Of course, $g(G) = 0$ if and only if $\tilde{g}(G) = 0$ if and only if $G$ is planar.

Graphs $G$ for which $\tilde{g}(G) = 2g(G) + 1$ are called *orientably simple*. The relationship between $\tilde{g}(tG)$ and $\tilde{g}(G)$ depends on whether $G$ is orientably simple.

**Theorem 7.1.2** *Let $G$ be a graph and let $t$ be a positive integer. Then $\tilde{g}(tG) = t(\tilde{g}(G) - 1) + 1$ in case $G$ is orientably simple, and $\tilde{g}(tG) = t\tilde{g}(G)$ otherwise.* □

This gives the following corollary.

**Corollary 7.1.3** *Let $G$ be a graph. Then $\tilde{g}_f(G) = \tilde{g}(G) - 1$ in case $G$ is orientably simple, and $\tilde{g}_f(G) = \tilde{g}(G)$ otherwise.* □

## Fractional crossing number

Let $G$ be a graph. If $G$ is planar, then it can be drawn in the plane without edge crossings. Otherwise any drawing of $G$ must have crossings. The *crossing number* of $G$, denoted $x(G)$, is the minimum number of crossings in any drawing of $G$.

There is a natural way to fractionalize the crossing number.

Let $t \cdot G$ denote the multigraph formed from $G$ by replacing each edge by $t$ parallel copies.

Let $x_t(G) = x(t \cdot G)$. Note that $x_1(G) = x(G)$ and that $x_t(G) \leq t^2 x(G)$ by simply drawing $t \cdot G$ with each edge replaced by $t$ nearly identical curves. Each crossing in the old drawing becomes $t^2$ crossings in the new.

Let $x_f(G) = \lim_{t\to\infty} x_t(G)/t^2$. The existence of this limit is guaranteed by the following result, from which it follows also that $x_f(G)$ is nothing more than $x(G)$.

**Theorem 7.1.4** *For any graph $G$ and any positive integer $t$, $x_t(G) = t^2 x(G)$.*

**Proof**. Prepare a drawing of $t \cdot G$ with the fewest possible crossings, say $k = x_t(G) = x(t \cdot G)$. We know that $k \leq t^2 x(G)$.

Define a *selection* $S$ of $t \cdot G$ to be a copy of $G$ formed by taking one edge from each family of parallel edges. Clearly there are $t^m$ possible selections where $m = |E(G)|$. A *crossing* $X$ is a pair of edges of $t \cdot G$ that cross. A crossing $X$ is said to *belong* to a selection $S$ provided both edges in $X$ are in $S$. Finally, let $\Sigma$ denote the set of all ordered pairs $(X, S)$ where $X$ is a crossing that belongs to a selection $S$. We ask, What is the cardinality of $\Sigma$? We count in two different ways:

- $|\Sigma| \geq t^m x(G)$: For each selection $S$ there are at least $x(G)$ crossings belonging to that selection.
- $|\Sigma| = kt^{m-2}$. For each of the $k = x(t \cdot G)$ crossings $X$ in the drawing of $t \cdot G$ there are $t^{m-2}$ selections that include the edges in $X$ (since we have chosen the two edges in $X$ but the other $m - 2$ edges may be chosen freely).

Therefore

$$t^m x(G) \leq kt^{m-2}$$

and so $k \geq t^2 x(G)$ as required. □

## Fractional thickness

The *thickness* of a graph $G$, denoted $\theta(G)$, is the minimum number of planar graphs whose (edge) union is $G$. In other words, if $G = (V, E)$ we seek to decompose $E = E_1 \cup E_2 \cup \cdots \cup E_k$ so that each $(V, E_i)$ is a planar graph. The least $k$ for which such a decomposition is possible is the thickness.

In hypergraph language, given a graph $G = (V, E)$, form a hypergraph $\mathcal{H}$ whose vertices are the edges of $G$. A collection $F$ of edges of $G$ (vertices of $\mathcal{H}$) form a hyperedge just when $(V, F)$ is planar. Then the thickness of $G$ is just the covering number of $\mathcal{H}$, i.e., $\theta(G) = k(\mathcal{H})$.

We therefore define the *fractional thickness* of $G$ to be $\theta_f(G) = k_f(\mathcal{H})$.

Note that (fractional) thickness is much like (fractional) arboricity (see Chapter 5), but the hypergraph $\mathcal{H}$ defined above does not necessarily have a matroid structure; see exercise 1 on page 128. However, the following is clear.

**Lemma 7.1.5** *For any graph $G$ we have $\theta(G) \leq \Upsilon(G)$ and $\theta_f(G) \leq \Upsilon_f(G)$.*

**Proof**. Decomposing $G$ into forests certainly decomposes $G$ into planar subgraphs. □

For complete graphs $K_n$ the following result is known.

**Theorem 7.1.6** *For $n \neq 9, 10$ we have $\theta(K_n) = \lfloor (n+7)/6 \rfloor$, but $\theta(K_9) = \theta(K_{10}) = 3$.* □

For fractional thickness we know the following.

**Theorem 7.1.7** *For all $n \geq 3$ we have $\theta_f(K_n) = \binom{n}{2}/(3n-6)$.*

**Proof**. Note that $K_n$ is edge-transitive, hence its associated hypergraph is vertex-transitive. Therefore we may apply Proposition 1.3.4 on page 5. Since a maximum planar subgraph of $K_n$ has $3n-6$ edges, the result follows. □

The reader is invited to verify (exercise 2 on page 128) that for all $n \geq 3$ we have $\theta(K_n) = \lceil \theta_f(K_n) \rceil$ except for $n = 9, 10$.

## 7.2 Fractional cycle double covers

Let $G$ be a graph. A *cycle double cover* (*CDC*, for short) for $G$ is a family (i.e., a multiset) of cycles in $G$ with the property that every edge of $G$ is in exactly two members of the family. Clearly, if $G$ has a cut edge then no CDC of $G$ is possible. However, it is believed that all other graphs admit a CDC.

**Conjecture 7.2.1 (CDC)** *Every 2-edge-connected graph has a cycle double cover.* □

If every block of a graph has a cycle double cover, then these covers, taken together, give a CDC for the entire graph. Thus, to prove the CDC conjecture, it is enough to show that all 2-connected graphs admit CDCs.

It is easy to check (exercise 3 on page 128) that if a graph is Eulerian, then it has a CDC.

The CDC conjecture is related to graph embedding. Let $G$ be a 2-connected planar graph. Then every face of (an embedding of) $G$ is a simple cycle and every edge of $G$ is in exactly two faces. Thus the facial cycles of $G$ form a cycle double cover, and hence the CDC conjecture is true for planar graphs. For a general 2-connected graph $G$, if $G$ embeds on a surface $S$ so that every face is bounded by a simple cycle of $G$, then the same argument works.

Because the cycle double cover problem is a *covering* problem, it admits a natural fractionalization. Let $G$ be a graph and let $M$ be a matrix whose rows are indexed by the edges of $G$ and whose columns are indexed by the cycles of $G$. Let the $i$,$j$-entry of $M$ be a 1 in case the $i^{\text{th}}$ edge lies in the $j^{\text{th}}$ cycle of $G$, and 0 otherwise. Then a CDC corresponds to a nonnegative, integer vector $\mathbf{x}$ so that $M\mathbf{x} = \mathbf{2}$ (where $\mathbf{2}$ stands for a vector of all 2s). To fractionalize, we simply drop the integrality condition.

Stated another way, a *fractional cycle double cover* of $G$ assigns a (nonnegative, real) weight to every cycle of $G$ so that the sum of the weights of the cycles containing any given edge is exactly two. The natural fractional version of the CDC conjecture is true.

**Theorem 7.2.2** *Every 2-edge-connected graph has a fractional cycle double cover.*

**Proof**. By results of Fan [57] and Jaeger [101], every 2-edge-connected graph has a cycle $k$-fold cover for all even numbers $k \geq 4$. In particular, consider a cycle quadruple cover and assign each cycle to have weight exactly one-half its multiplicity in the quadruple cover. This gives a fractional cycle double cover of the graph. □

Indeed, this proof shows that every 2-edge-connected graph has a fractional cycle double cover using weights that are either integers or half integers. Because we can divide the cycle weights by (another) factor of 2, the fact that we are creating a *double* cover becomes somewhat irrelevant. We might just as well consider fractional cycle covers (no "double"): weightings of cycles so that the sum of the weights of the cycles containing any given edge must be exactly 1. Thus, every 2-edge-connected graph has a fractional cycle cover. (Fractional cycle covers are an appropriate generalization of fractional Euler tours; see exercise 13 on page 28.)

Bondy [25] poses a conjecture that is stronger than the cycle double cover conjecture.

**Conjecture 7.2.3 (Strong CDC)** *Every 2-edge-connected graph on $n$ vertices has a cycle double cover using at most $n-1$ cycles.* □

The appropriate fractionalization of this conjecture would be that every 2-edge-connected graph on $n$ vertices has a fractional cycle double cover in which the total weight of the cycles is at most $n-1$. This fractionalized version is an open problem.

## 7.3 Fractional Ramsey theory

The Ramsey number $r(k,l)$ is the smallest integer $n$ such that every graph $G$ on $n$ vertices has either $\alpha(G) \geq k$ or $\omega(G) \geq l$. Few exact values for $r(k,l)$ are known, although the values are known to grow exponentially with respect to $k+l$. See [78] for a thorough introduction to Ramsey theory.

We define the *fractional Ramsey number* $r_f(x,y)$ to be the smallest integer $n$ such that every graph $G$ on $n$ vertices has either $\alpha_f(G) \geq x$ or $\omega_f(G) \geq y$. (Remember that $\omega_f(G) = \chi_f(G)$ for all graphs $G$.) We allow $x$ and $y$ to take on any real values with $x, y \geq 2$.

Note that, in contrast to other fractional functions we have studied, fractional Ramsey numbers are integers.

Observe that $r_f(x,y)$ is symmetric in its two arguments, since $\alpha_f(\overline{G}) = \omega_f(G)$ and $\omega_f(\overline{G}) = \alpha_f(G)$.

It may come as a surprise that the fractional Ramsey number can be exactly computed for all $x$ and $y$, and moreover that it grows only polynomially with respect to $x$ and $y$.

**Theorem 7.3.1** *For all $x, y \geq 2$, we have*

$$r_f(x,y) = \min\left\{\left\lceil(\lceil x\rceil - 1)y\right\rceil, \left\lceil(\lceil y\rceil - 1)x\right\rceil\right\}.$$

**Proof**. The case when either $x$ or $y$ equals 2 is left to the reader (see exercise 5 on page 128). We proceed assuming that $x, y > 2$. Let

$$n = \min\left\{\left\lceil(\lceil x\rceil - 1)y\right\rceil, \left\lceil(\lceil y\rceil - 1)x\right\rceil\right\}.$$

We begin with the upper bound ($r_f(x,y) \le n$). Let $n_1 = \lceil (\lceil x \rceil - 1)y \rceil$ and let $G$ be any graph on $n_1$ (or more) vertices. If $\alpha(G) \ge \lceil x \rceil$ then clearly $\alpha_f(G) \ge \alpha(G) \ge \lceil x \rceil \ge x$. Thus we may suppose that $\alpha(G) \le \lceil x \rceil - 1$. But then it follows (by Proposition 3.1.1 on page 30) that $\omega_f(G) = \chi_f(G) \ge n_1/\alpha(G) \ge y$ as required. Likewise, if $n_2 = \lceil (\lceil y \rceil - 1)x \rceil$ then any graph $G$ on $n_2$ or more vertices has either $\alpha_f(G) \ge x$ or $\omega_f(G) \ge y$. Thus $r_f(x,y) \le \min\{n_1, n_2\} = n$ as required.

We now exhibit a graph with $n-1$ vertices for which $\alpha_f < x$ and $\chi_f < y$. The graph we use is of the form $G_{a,b}$ as defined in Proposition 3.2.2 on page 32. Recall that $\alpha(G_{a,b}) = b$ and $\omega(G_{a,b}) = \lfloor a/b \rfloor$ (exercise 5 on page 54).

Let $a = n-1$, let $b = \lceil x \rceil - 1$, and let $G = G_{a,b}$. By Proposition 3.2.2 we have

$$\begin{aligned}
\omega_f(G) = \chi_f(G) &= a/b \\
&= \frac{n-1}{\lceil x \rceil - 1} \\
&\le \frac{\left\lceil (\lceil x \rceil - 1)y \right\rceil - 1}{\lceil x \rceil - 1} \\
&< \frac{(\lceil x \rceil - 1)y}{\lceil x \rceil - 1} \\
&= y
\end{aligned}$$

so $\omega_f(G) < y$ as required.

We now consider $\alpha_f(G) = \omega_f(\overline{G})$. Since $\overline{G}$ is vertex-transitive, it follows (from Proposition 3.1.1) that $\alpha_f(G) = (n-1)/\omega(G)$. We claim that $\omega(G) = \lceil y \rceil - 1$. To show this, we first observe that

$$\omega(G) = \left\lfloor \frac{n-1}{\lceil x \rceil - 1} \right\rfloor \le \frac{n-1}{\lceil x \rceil - 1} = \frac{a}{b} < y,$$

and so $\omega(G) \le \lceil y \rceil - 1$. To prove the opposite inequality, we first recall that

$$\frac{n-1}{\lceil x \rceil - 1} = \frac{\lceil \min\{(\lceil x \rceil - 1)y, (\lceil y \rceil - 1)x\} - 1 \rceil}{\lceil x \rceil - 1}.$$

We check that

$$\begin{aligned}
\frac{\left\lceil (\lceil x \rceil - 1)y - 1 \right\rceil}{\lceil x \rceil - 1} &= \frac{\left\lceil (\lceil x \rceil - 1)\left(\lceil y \rceil - 1 + (y - \lceil y \rceil + 1)\right) - 1 \right\rceil}{\lceil x \rceil - 1} \\
&= \frac{(\lceil x \rceil - 1)(\lceil y \rceil - 1) + \left\lceil (\lceil x \rceil - 1)(y - \lceil y \rceil + 1) \right\rceil - 1}{\lceil x \rceil - 1} \\
&\ge \frac{(\lceil x \rceil - 1)(\lceil y \rceil - 1)}{\lceil x \rceil - 1} \\
&= \lceil y \rceil - 1.
\end{aligned}$$

Similarly, we also check that

$$\frac{\left\lceil (\lceil y \rceil - 1)x - 1 \right\rceil}{\lceil x \rceil - 1} = \frac{\left\lceil (\lceil y \rceil - 1)\left(\lceil x \rceil - 1 + (x - \lceil x \rceil + 1)\right) - 1 \right\rceil}{\lceil x \rceil - 1}$$

$$= \frac{(\lceil y \rceil - 1)(\lceil x \rceil - 1) + \left\lceil (\lceil y \rceil - 1)(x - \lceil x \rceil + 1) \right\rceil - 1}{\lceil x \rceil - 1}$$
$$\geq \frac{(\lceil y \rceil - 1)(\lceil x \rceil - 1)}{\lceil x \rceil - 1}$$
$$= \lceil y \rceil - 1.$$

It follows that

$$\frac{n-1}{\lceil x \rceil - 1} \geq \lceil y \rceil - 1,$$

and so

$$\omega(G) = \left\lfloor \frac{n-1}{\lceil x \rceil - 1} \right\rfloor \geq \lceil y \rceil - 1$$

as claimed.

We conclude that

$$\alpha_f(G) = \frac{n-1}{\omega(G)} = \frac{n-1}{\lceil y \rceil - 1} < \frac{(\lceil y \rceil - 1)x}{\lceil y \rceil - 1} = x.$$

Thus $\alpha_f(G) < x$ and $\omega_f(G) < y$, so $r_f(x, y) > n - 1$. Above we showed $r_f(x, y) \leq n$ and the result follows. □

It is known that the usual Ramsey number $r(k, k)$ grows exponentially with $k$ and lies between (roughly) $\sqrt{2}^k$ and $4^k$. By contrast, the fractional Ramsey number $r_f(k, k)$ grows at a merely polynomial rate with $k$.

## 7.4 Fractional domination

A *dominating set* in a graph $G$ is a set of vertices $S$ such that every vertex in $V(G)$ is either in or adjacent to a vertex in $S$. The *domination number* $\gamma(G)$ of a graph $G$ is the size of a smallest dominating set. (The problem of placing fire stations in an optimum way is a domination problem.) A *total dominating set* in a graph $G$ is a set of vertices $S$ such that every vertex in $V(G)$ is adjacent to a vertex in $S$. The *total domination number* $\Gamma(G)$ of a graph $G$ is the size of a smallest total dominating set.

These invariants can be viewed as the covering numbers of associated hypergraphs. Given a graph $G$, let $\mathcal{H}$ be the hypergraph whose vertex set is $V(G)$ and with, for each vertex $v \in V(G)$, a hyperedge consisting of the open neighborhood $N(v) = \{u \in V(G) | uv \in E(G)\}$. Then it is easy to see that $\Gamma(G) = k(\mathcal{H})$. Alternatively, if one creates a hyperedge for each *closed* neighborhood $N[v] = N(v) \cup \{v\}$, then $\gamma(G) = k(\mathcal{H})$.

Dual to the notion of domination number is the *closed neighborhood packing number* $\pi(G)$ of a graph. This is the maximum number of disjoint closed neighborhoods in the graph. Similarly, the dual to the total domination number is the maximum number $\Pi(G)$ of disjoint *open* neighborhoods in the graph.

The fractional analogues of these invariants may be obtained as the linear relaxation of the associated integer programs, or as the fractional covering number of the associated hypergraphs, or combinatorially as follows. Define a *dominating function* $f$ to be any $f : V(G) \to [0, 1]$ with $\sum_{u \in N(v)} f(u) \geq 1$ for every $v \in V(G)$. The *fractional domination number* $\gamma_f(G)$ is the minimum value of $\sum_{v \in V(G)} f(v)$ (the *weight* of $f$), where the minimum is taken over all dominating functions

$f$. Similarly one can define a *total dominating function* and the *fractional total* (!) *domination number* $\Gamma_f(G)$ by replacing closed neighborhoods with open ones.

Dual to the notion of a dominating function is the notion of a *closed neighborhood packing function*, defined to be a function $g$ that assigns a number to every vertex of $G$ (or, more formally, to every closed neighborhood of $G$, but this amounts to the same thing) in such a way that $\sum_{u\in N(v)} g(u) \leq 1$ for every $v \in V(G)$. By the duality theorem, the fractional domination number $\gamma_f(G)$ is the maximum of $\sum_{v\in V(G)} g(v)$ where the maximum is taken over all closed neighborhood packing functions on $G$.

It follows directly from these definitions that $\gamma_f(G) \leq \gamma(G) \leq \Gamma(G)$ and $\gamma_f(G) \leq \Gamma_f(G) \leq \Gamma(G)$ for any graph $G$. Considering the dual problem yields $\pi(G) \leq \gamma_f(G)$ and $\Pi(G) \leq \Gamma_f(G)$.

As a simple illustration of these ideas, we compute the various invariants on the complete graph $K_n$. Any set consisting of a single vertex is a dominating set in $K_n$, so $\gamma(K_n) = 1$. A singleton set is not a total dominating set, however. Any set consisting of two vertices is a total dominating set, so $\Gamma(K_n) = 2$. It is clear that $\pi(K_n) = \Pi(K_n) = 1$. No dominating function $f$ can have $\sum_{v\in V(K_n)} f(v) < 1$, so $\gamma_f(K_n) \geq 1$. But $\gamma_f(G) \leq \gamma(G)$ for any graph $G$, so $\gamma_f(K_n) = 1$. (An alternate proof that $\gamma_f(K_n) = 1$ is to use duality to see that $1 = \pi(K_n) \leq \gamma_f(K_n) \leq \gamma(K_n) = 1$.) Finally, we obtain the minimizing total dominating function $f$ by setting $f(v) = 1/(n-1)$ for every $v \in V(K_n)$. Clearly, $\sum_{v\in V(K_n)} f(v) = n/(n-1)$, so $\Gamma_f(K_n) \leq n/(n-1)$. One way to see that this $f$ is optimal is to consider any total dominating function $g$ and to observe that

$$n = \sum_{v\in V(G)} 1 \leq \sum_{v\in V(G)} \sum_{u\in N(v)} g(u) = (n-1) \sum_{v\in V(G)} g(v).$$

Dividing by $n-1$ shows that no total dominating function can have smaller total weight than $f$. Another way to see that $f$ is optimal is to use the following theorem.

**Theorem 7.4.1** *If $G$ has $n$ vertices and is $k$-regular, then $\gamma_f(G) = n/(k+1)$ and $\Gamma_f(G) = n/k$.*

**Proof**. The function $f$ that assigns $1/(k+1)$ to every vertex of $G$ is both a dominating function and a closed neighborhood packing function and has weight $n/(k+1)$. Thus this function is the minimum weight dominating function and simultaneously the maximum weight closed neighborhood packing function, and its weight $n/(k+1)$ is $\gamma_f(G)$.

An identical argument works for the fractional total dominating function, with $f$ assigning $1/k$ to every vertex of $G$. □

As an application of Theorem 7.4.1 with $k = 2$, we consider the cycles $C_n$. Note that $\gamma_f(C_n) = n/3$ and $\Gamma_f(C_n) = n/2$, while $\gamma(C_n) = \lceil n/3 \rceil$ and

$$\Gamma(C_n) = \begin{cases} \lceil n/2 \rceil & \text{if } n \text{ is odd, and} \\ 2\lceil n/4 \rceil & \text{if } n \text{ is even.} \end{cases}$$

There are $k$-regular graphs on $n$ vertices for every pair of numbers $(k, n)$ with $k \geq 1$, $n \geq k+1$, except when $k$ and $n$ are both odd. This gives an immediate way of constructing a graph $G$ that has fractional domination number $a/b$ for arbitrary integers $a \geq b$. If $a$ is even, construct a $(b-1)$-regular graph on $a$ vertices. If $a$ is odd, construct a $(2b-1)$-regular graph on $2a$ vertices instead. Similarly, the function $\Gamma$ takes on all rational numbers greater than 1.

## The duality gap

Suppose the vertices of a graph $G$ can be partitioned into blocks $V_1, V_2, \ldots, V_r$ in such a way that one vertex in each $V_i$ is adjacent to all the other vertices in $V_i$. Put another way, suppose $G[V_i]$

contains as a spanning subgraph a star (i.e., $K_{1,n}$ for some $n \geq 0$) for every $i$ with $1 \leq i \leq r$. In this case the indicator function of the centers of those stars is both a dominating function and a closed neighborhood packing function for $G$, and we are in the situation where the integer program and its dual have the same optimum value. In this case, there is no duality gap, and we have $\pi(G) = \gamma_f(G) = \gamma(G)$. An example of this phenomenon is when $G = C_{3k}$, where choosing every third vertex around the cycle gives an optimal dominating set and an optimum closed neighborhood packing simultaneously. Another example is when one vertex of $G$ is adjacent to all the others, in which case the partition of $V(G)$ into one block meets the condition.

Even when such a partition of $V(G)$ is not available, it may be that the duality gap vanishes. Such is the case for trees.

**Theorem 7.4.2** *If $T$ is a tree, then $\pi(T) = \gamma(T)$.*

**Proof**. Our proof is by induction on the number of vertices in $T$. If $T$ has only one vertex, in fact if $T$ has no path of length 3, then $\pi(T) = 1 = \gamma(T)$.

For the induction step, let $T$ be any tree and let $P$ be a longest path in $T$, with the vertices of $P$ in order $v_1, v_2, v_3, \ldots, v_r$. Note that $T - v_2$ can have only one component with more than one vertex, since otherwise we violate the maximality of $P$. We consider two cases, depending on whether $v_3$ has degree 2 (with neighbors $v_2$ and $v_4$) or degree greater than 2.

Suppose that $v_3$ has degree 2. Let $T'$ be the component of $T - \{v_2, v_3\}$ containing $v_4$. Apply the induction hypothesis to $T'$ to obtain a dominating set $S'$ and a collection $\mathcal{N}'$ of disjoint closed neighborhoods with $|\mathcal{N}'| = \pi(T') = \gamma(T') = |S'|$. Let $S = S' \cup \{v_2\}$ and $\mathcal{N} = \mathcal{N}' \cup \{N[v_2]\}$. It is clear that $S$ is a dominating set for $T$, that $\mathcal{N}$ is a neighborhood packing of $T$, and that $|S| = |S'| + 1 = |\mathcal{N}'| + 1 = |\mathcal{N}|$.

Now consider the alternative case, when $v_3$ has degree greater than 2 in $T$. Let $T'$ be the component of $T - v_2$ containing $v_3$. Applying the induction hypothesis to $T'$ yields as before a dominating set $S'$ and a collection $\mathcal{N}'$ of disjoint closed neighborhoods with $|\mathcal{N}'| = \pi(T') = \gamma(T') = |S'|$. The set $S = S' \cup \{v_2\}$ is clearly a dominating set for $T$. We now produce a neighborhood packing of $T$ of the same size. If $N[v_3] \notin \mathcal{N}'$, then put $\mathcal{N} = \mathcal{N}' \cup \{N[v_1]\}$. If $N[v_3] \in \mathcal{N}'$, let $u$ be a vertex adjacent to $v_3$ but not equal to $v_2$ or $v_4$. The maximality of $P$ implies that the neighborhood of $u$ contains, other than $v_3$, only leaves of $T$. Since $N[v_3] \in \mathcal{N}'$, neither $N[u]$ nor $N[v]$ for any of the leaves $v$ adjacent to $u$ are in $\mathcal{N}'$. Set $\mathcal{N} = \mathcal{N}' \cup \{N[v_1], N[u]\} - \{N[v_3]\}$. In either case, we have $|\mathcal{N}| = |S|$, and hence $\pi(T) = \gamma(T)$. □

It is possible to generalize this result in several directions. The class of graphs for which $\pi = \gamma$ includes not only the class of all trees but also the class of all strongly chordal graphs. A graph is *chordal* if it contains no induced cycle on more than 3 vertices. A graph is *strongly chordal* if it is chordal and in addition contains no induced "trampoline". A *trampoline* is the graph obtained from an even cycle by coloring the vertices alternately black and white and adding an edge between any two white vertices. (See Figure 7.1.)

Not every chordal graph has $\pi = \gamma$, and the trampoline $G$ on 6 vertices (see Figure 7.5 on page 124) is an example where $\pi(G) = 1$ but $\gamma(G) = 2$. Strongly chordal graphs, however, do have $\pi = \gamma$. This fact follows easily from the following theorem, discovered independently by Farber [58] and Iijima and Shibata [99]. The closed neighborhood matrix $C(G)$ of a graph $G$ is the matrix whose $ij$ entry is equal to 1 if $i = j$ or $v_i v_j \in E(G)$ and 0 otherwise. In other words, $C(G) = I + A(G)$.

**Theorem 7.4.3** *A graph is strongly chordal if and only if its neighborhood matrix $C(G)$ is totally balanced.* □

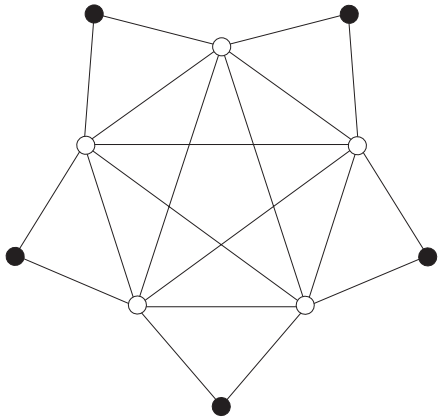

Figure 7.1. The trampoline on 10 vertices.

**Theorem 7.4.4** *If $G$ is strongly chordal, then $\pi(G) = \gamma(G)$.*

**Proof**. The integer program that computes $\gamma(G)$ is $\min \mathbf{x} \cdot \mathbf{1}$ subject to $C(G)\mathbf{x} \geq 1$. By Theorem 7.4.3, $C(G)$ is totally balanced. An appeal to Theorem A.3.4 on page 136 then gives the result. □

A tree is certainly strongly chordal, so Theorem 7.4.2 follows from Theorem 7.4.4.

The situation for the total domination number is similar. In this case we replace the notion of a strongly chordal graph by the notion of a chordal bipartite graph. A graph is *chordal bipartite* if it is bipartite and every cycle of length greater than 4 has a chord. Note that chordal bipartite graphs are not in general chordal.

The total domination number is computed by the integer program $\min \mathbf{x} \cdot \mathbf{1}$ subject to $A(G)\mathbf{x} \geq 1$, where $A(G)$ is the usual adjacency matrix of $G$. The following theorem is due to Hoffman, Kolen, and Sakarovitch [93].

**Theorem 7.4.5** *A graph is chordal bipartite if and only if $A(G)$ is totally balanced.* □

The next theorem follows in the same way as Theorem 7.4.4 above.

**Theorem 7.4.6** *If $G$ is chordal bipartite, then $\Pi(G) = \Gamma(G)$.* □

Trees are chordal bipartite, so we obtain the following analogue to Theorem 7.4.2.

**Corollary 7.4.7** *If $T$ is a tree, then $\Pi(T) = \Gamma(T)$.* □

The example of the cycles shows that the converses of most of the results in this section fail. Although $C_6$ is not strongly chordal, $\pi(C_6) = 2 = \gamma(C_6)$. Although $C_8$ is not chordal bipartite, $\Pi(C_8) = 4 = \Gamma(C_8)$. It seems to be a difficult problem to characterize those graphs for which $\pi = \gamma$ or $\Pi = \Gamma$.

Even when the duality gap does not vanish, i.e., when $\pi < \gamma$, it is possible for $\gamma_f$ to equal either $\gamma$ or $\pi$. The simplest example known where $\pi < \gamma_f = \gamma$ is the graph $G$ pictured in Figure 7.2, due to Fisher [64].

A simple example where $\pi = \gamma_f < \gamma$ is the graph pictured in Figure 7.3.

We leave it as an exercise to compute the closed neighborhood packing number, the domination number, and the fractional domination number of these examples to demonstrate that they are indeed examples illustrating what we claim. (See exercises 7–9 on page 128.)

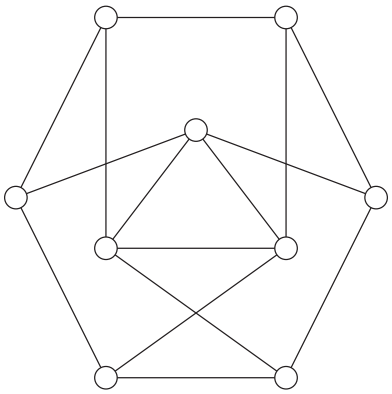

Figure 7.2. A graph with $\pi < \gamma_f = \gamma$.

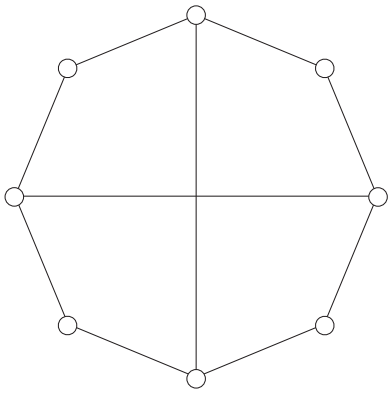

Figure 7.3. A graph with $\pi = \gamma_f < \gamma$.

## Fractional Vizing conjecture

In 1963, Vizing [182] made the following conjecture concerning the domination number of Cartesian products.

**Conjecture 7.4.8** *For all graphs $G$ and $H$, $\gamma(G \times H) = \gamma(G)\gamma(H)$.*

Here we show the fractional analogue holds. To this end, we need to introduce yet another graph product called the *strong direct product*, denoted $G \cdot H$. The graph $G \cdot H$ has vertex set $V(G) \times V(H)$ with an edge between $(v_1, w_1)$ and $(v_2, w_2)$ if any of the following hold:

(1) $v_1 = v_2$ and $w_1w_2 \in E(H)$,

(2) $v_1v_2 \in E(G)$ and $w_1 = w_2$, or

(3) $v_1v_2 \in E(G)$ and $w_1w_2 \in E(H)$.

The Cartesian product $G \times H$ is the spanning subgraph of the strong direct product $G \cdot H$ consisting of edges of type (1) and (2).

The strong direct product is a natural one in the study of domination, because dominating sets lift naturally to this product. To be precise, if $S \subseteq V(G)$ is a dominating set for $G$ and $T \subseteq V(H)$ is a dominating set for $H$, then $S \times T$ is a dominating set for $G \cdot H$. (Note that the same assertion cannot be made about the Cartesian product $G \times H$.) It follows that

$$\gamma(G \cdot H) \leq \gamma(G)\gamma(H) \tag{7.1}$$

Also

$$\gamma(G \cdot H) \leq \gamma(G \times H), \tag{7.2}$$

since removing edges from a graph cannot make it easier to dominate. Vizing's conjecture asserts that inequality (7.1) is stronger than inequality (7.2).

The fractional analogue of Vizing's conjecture follows from LP-duality.

**Theorem 7.4.9** *For all graphs $G$ and $H$, $\gamma_f(G \times H) \geq \gamma_f(G)\gamma_f(H)$.*

**Proof**. If $f$ is a dominating function for $G$ and $g$ is a dominating function for $H$, then the "product lifting" defined by $h(v, w) = f(v)g(w)$ is easily seen to be a dominating function for $G \cdot H$. If $f$ and $g$ are further assumed to be optimal, then the total weight of $h$ is

$$\sum_{(v,w) \in V(G \cdot H)} h(v, w) = \sum_{v \in V(G)} \sum_{w \in V(H)} f(v)g(w) = \gamma_f(G)\gamma_f(H).$$

We therefore obtain $\gamma_f(G \cdot H) \leq \gamma_f(G)\gamma_f(H)$, the fractional analogue of inequality (7.1).

The opposite inequality comes from duality. If $f$ is now taken to be a closed neighborhood packing function for $G$ and $g$ is taken to be a closed neighborhood packing function for $H$, then $h(v, w) = f(v)g(w)$ is a closed neighborhood packing function for $G \cdot H$. This allows us to obtain an inequality for the fractional closed neighborhood packing number of $G \cdot H$ exactly as in the previous paragraph, but (critically) with the inequality reversed, since the closed neighborhood packing number is a maximization parameter. By LP-duality, this gives $\gamma_f(G \cdot H) \geq \gamma_f(G)\gamma_f(H)$.

Hence $\gamma_f(G \cdot H) = \gamma_f(G)\gamma_f(H)$. Since the fractional analogue of inequality (7.2), namely that $\gamma_f(G \cdot H) \leq \gamma_f(G \times H)$, follows again from the simple fact that $G \times H$ is a subgraph of $G \cdot H$, we obtain our result. □

## 7.5 Fractional intersection number

An *intersection representation* of a graph $G = (V, E)$ is an assignment of a set $S_v$ to each vertex $v \in V$ so that $uv \in E$ if and only if $S_u \cap S_v \neq \emptyset$. It is not hard to see that every graph has an intersection representation: Let $S_v$ be the set of edges incident at $v$.

We have made no assumption on the structure of the sets $S_v$. If, however, we require that the sets $S_v$ be intervals on the real line, then not all graphs admit such *interval* representations. For example, $C_4$ cannot be represented as the intersection graph of real intervals. The intersection graphs of real intervals are known as *interval graphs*.

In this section, we consider intersection representations by *finite* sets. In this case, we define the *size* of an intersection representation $v \mapsto S_v$ to be the cardinality of the union of the sets assigned to vertices, i.e.,

$$\left| \bigcup_{v \in V} S_v \right|.$$

The *intersection number* of $G$, denoted $I(G)$, is the smallest size of an intersection representation of $G$. Clearly, $I(G) \leq \varepsilon(G)$.

Alternatively, we can express $I(G)$ as a covering number. Define a hypergraph $\mathcal{H}$ whose vertices are the edges of $G$ and whose hyperedges correspond to the cliques of $G$, that is, given a clique of $G$, the edges induced by that clique form a hyperedge of $\mathcal{H}$. One checks that $k(\mathcal{H}) = I(G)$ (exercise 16 on page 129).

**Proposition 7.5.1** *The intersection number of a graph is the size of a smallest covering of the edges of the graph by cliques.* □

Thus, it is natural to define the *fractional intersection number* $I_f(G)$ to be $k_f(\mathcal{H})$.

The fractional intersection number completes a "triangle" with the fractional chromatic and matching numbers. The fractional matching number can be cast as a covering problem involving vertices and edges. The fractional chromatic number can be cast as a covering problem involving vertices and cliques.[1] Thus we have studied vertex-edge and vertex-clique covering problems and the "triangle" is completed by considering edge-clique covering problems, i.e., intersection number.

In this section we explore the gap between $I(G)$ and $I_f(G)$ considering first instances where these invariants are the same, and then, instances where they are wildly different.

In case $G$ is a bipartite or, more generally, a triangle-free graph, the only cliques in $G$ are the edges of $G$. Thus $I(G) = I_f(G) = \varepsilon(G)$. We show that equality also holds for interval graphs and, more generally, chordal graphs. Recall that a graph is called *chordal* provided it does not contain $C_k$ with $k \geq 4$ as an induced subgraph. In a sense, chordal graphs are generalizations of trees: trees have no cycles, whereas chordal graphs have no chordless cycles. Analogous to leaves in trees, chordal graphs have *simplicial* vertices: vertices whose neighborhoods are cliques.

**Theorem 7.5.2** *Every chordal graph has a simplicial vertex.* □

We use the above result to prove the following.

**Theorem 7.5.3** *Let $G$ be a chordal graph. Then $I(G) = I_f(G)$.*

**Proof**. The packing problem dual to intersection number is to find the maximum number of edges in $G$ no two of which are in the same clique. Here we present an algorithm that simultaneously constructs a cover of $E(G)$ by cliques and a packing of edges into cliques, and the covering and packing have the same size. In other words, we show that $p(\mathcal{H}) = k(\mathcal{H})$ where $\mathcal{H}$ is the edge-clique incidence hypergraph of $G$.

Consider the following algorithm that takes as input a chordal graph $G$. During the course of this computation, we keep track of the "weight" of edges (either 0 or 1), the "weight" of various cliques (again, 0 or 1), and a label on each edge (either "covered" or "uncovered").

1. Initialize by setting all edge and clique weights to 0, and marking all edges "uncovered". Let $G_1 = G$.
2. FOR $i = 1$ to $|V(G)| - 1$ DO:
   (a) Let $v_i$ be a simplicial vertex of $G_i$, and let $S_i$ be the clique consisting of $v_i$ and all its neighbors in $G_i$.

[1] More properly, fractional chromatic number is a covering problem involving vertices and *independent sets* (not cliques), but the chromatic number of the complement of a graph deals with vertex-clique covering.

(b) IF some edge of $G_i$ incident with $v_i$ is marked "uncovered" THEN:

- Let clique $S_i$ have weight 1.
- Select an edge incident with $v_i$ that is marked "uncovered" and assign it weight 1.
- Mark all edges induced by $S_i$ "covered".

(c) Let $G_{i+1} = G_i - v_i$.

3. Output the sum of the weights of the edges of $G$.

To complete the proof, one notices that at every step of the FOR loop the total weight on the edges equals the total weight on the cliques. Further, by the end of the algorithm every edge is marked "covered" and is, indeed, covered by some clique $S_i$ that was given weight 1. Thus the set of cliques with weight 1 gives a covering of $E(G)$. Finally, one checks that no clique of $G$ can contain two edges of weight 1 (since covered edges are never selected) and therefore edges of weight 1 form a packing of edges into the cliques of $G$. Thus the sum of the edge weights $k(\mathcal{H})$ equals the sum of the vertex weights $p(\mathcal{H})$. □

Interval graphs form a subfamily of chordal graphs, so, naturally, it follows from Theorem 7.5.3 that $I = I_f$ for interval graphs. However, a bit more can be said.

**Theorem 7.5.4** *If $G$ is an interval graph, then $I_f(G) = I(G)$. Moreover, if $G$ has no isolated vertices, then $I(G)$ equals the number of maximal cliques in $G$.*

To prove Theorem 7.5.4 we use the following result due to Helly [88].

**Theorem 7.5.5** *If a collection of real intervals has nonempty pairwise intersection, then the intersection of all the intervals in the family is nonempty.*

**Proof**. Exercise 18 on page 129. □

**Proof** (of Theorem 7.5.4). Let $G$ be an interval graph with $M$ maximal cliques. Since every edge is in some maximal clique, we have $I_f(G) \leq I(G) \leq M$. Thus, it is enough to show $M \leq I_f(G)$. To this end, fix an interval representation of $G$ in which the endpoints of all intervals are distinct; let $I_v$ denote the interval assigned to $v \in V(G)$.

Let $S$ be a maximal clique of $G$. By the Helly property (Theorem 7.5.5), the intersection of all intervals assigned to vertices in $S$ is nonempty, and is therefore an interval, which we denote $J_S$. Note that $J_S$ is either (a) an interval assigned to a vertex $x$, i.e., $J_S = I_x$, or (b) the intersection of exactly two intervals assigned to vertices $x$ and $y$, i.e., $J_S = I_x \cap I_y$. See Figure 7.4 on the following page. We choose an edge $e_S$ depending on which of these cases, (a) or (b), occurs. In case (a) let $e_S$ be any edge incident with $x$, and in case (b) let $e_S = xy$.

The map $S \mapsto e_S$ selects exactly one edge in each maximal clique. Further, the only maximal clique that can contain $e_S$ is $S$ because if $e_S$ were in some other clique $T$, then $J_S$ and $J_T$ would intersect, implying that $S \cup T$ is also a clique, and thereby contradicting the maximality of $S$ and $T$. Thus the assignment $S \mapsto e_S$ is one-to-one and every clique of $G$ contains at most one selected edge. Thus the selected edges form a packing of edges into cliques, and therefore

$$M = p(\mathcal{H}) \geq p_f(\mathcal{H}) = I_f(G)$$

where $\mathcal{H}$ is the edge-clique hypergraph of $G$. □

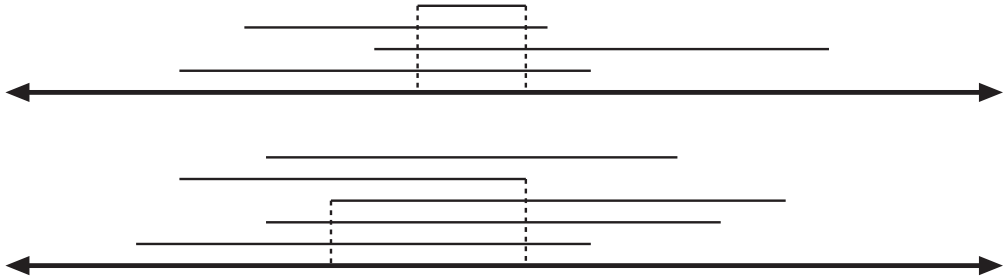

Figure 7.4. The intersection of pairwise intersecting intervals is either (a) one of the original intervals, or (b) the intersection of exactly two of the intervals.

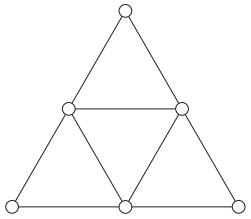

Figure 7.5. A chordal graph for which $I = I_f = 3$, but that has 4 maximal cliques.

Note that the stronger conclusion of Theorem 7.5.4 does not apply to chordal graphs: Figure 7.5 presents a chordal graph with $I = I_f = 3$ but that has 4 maximal cliques.

Theorems 7.5.3 and 7.5.4 describe situations in which intersection number and fractional intersection number are equal. One might ask: How different can these invariants be? The answer is: Very.

**Theorem 7.5.6** *Let $G = K(n_1, n_2, \ldots, n_p)$ be a complete multipartite graph with part sizes $n_1 \geq n_2 \geq \cdots \geq n_p$. Then $I_f(G) = n_1 n_2$.*

**Proof**. Let the $j^{\text{th}}$ part of $G$ be $A_j$ with $|A_j| = n_j$. The maximal cliques of $G$ are formed by selecting one vertex from each of the $p$ parts. We form a fractional intersection representation of $G$ by assigning weight $1/(n_3 n_4 \cdots n_p)$ to every maximal clique. Note that an edge between parts $i$ and $j$ is in exactly $(\prod n_k)/(n_i n_j)$ maximal cliques and receives total weight

$$\frac{n_1 n_2}{n_i n_j} \geq 1.$$

Thus this weighting is a feasible fractional edge cover by cliques. Furthermore, the sum of the weights on all cliques is precisely $n_1 n_2$ and therefore $I_f(G) \leq n_1 n_2$.

On the other hand, if we weight the edges between parts $A_1$ and $A_2$ with weight exactly 1 and all other edges with weight 0, then every clique contains total weight at most 1. This edge weighting is therefore feasible for the dual problem and uses total weight equal to $n_1 n_2$. Thus $I_f(G) \geq n_1 n_2$. □

Theorem 7.5.6 on the facing page implies that the fractional intersection number of the generalized octahedron graph $K(2,2,2,\ldots,2)$ is exactly 4. However, we show next that the intersection number of this graph goes to infinity as the number of parts increases.

**Theorem 7.5.7** *Let $n$ be a positive integer and let $G = K(2,2,\ldots,2)$ be the complete multipartite graph with $n$ parts of size 2. Then $I(G) \sim \lg n$.*

**Proof**. Label the vertices of $G$ as $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$ with $x_i \not\sim y_i$.

Let $U = \{1, 2, \ldots, 2t\}$ and construct an intersection representation of $G$ in which each vertex of $G$ is assigned a $t$-element subset of $U$. Note that there are $\binom{2t}{t}$ possible sets, so we take $t$ just large enough that $\binom{2t}{t} \geq 2n$. This gives $2t = (1 + o(1)) \lg n$. The assignment is now quite simple. Having chosen sets for $x_1, y_1, \ldots, x_i, y_i$, we assign to $x_{i+1}$ any as yet unassigned $t$-set of $U$ and we assign to $y_{i+1}$ its complement. With this assignment, any two sets intersect unless they correspond to a pair $x_i, y_i$. Thus we have $I(G) \leq 2t = (1 + o(1)) \lg n$.

For the lower bound, fix an intersection representation of $G$ of smallest size and let $I(G) = k$. We may assume that the representation chooses its subsets from $[k] = \{1, \ldots, k\}$. Note that no two vertices are assigned the same set (otherwise we would have a pair of adjacent vertices with identical neighbor sets). Thus $2^k \geq 2n$ because all $2n$ vertices must receive distinct subsets of $[k]$. Thus $I(G) = k \geq \lg n$. □

## 7.6 Fractional dimension of a poset

In this section, we leave graph theory to illustrate how the fractional paradigm can be applied in another part of combinatorics.

A *partially ordered set* (or *poset* for short) is a pair $P = (X, \leq)$ where $X$ is a finite set, called the *ground set* of $P$, and $\leq$ is a relation on $X$ that satisfies

- (reflexivity) $\forall x \in X,\ x \leq x$;
- (antisymmetry) $\forall x, y \in X,\ (x \leq y \text{ and } y \leq x) \Rightarrow x = y$; and
- (transitivity) $\forall x, y, z \in X,\ (x \leq y \text{ and } y \leq z) \Rightarrow x \leq z$.

We write $x < y$ if $x \leq y$ and $x \neq y$. Further, we write $x \geq y$ or $x > y$ provided $y \leq x$ or $y < x$ respectively. If either $x \leq y$ or $y \leq x$ we say $x$ and $y$ are *comparable*; otherwise, $x$ and $y$ are *incomparable*. A subset $C$ of $X$ is called a *chain* provided any two elements in $C$ are comparable. A subset $A$ of $X$ is called an *antichain* provided any two distinct elements in $A$ are incomparable.

A partially ordered set $P = (X, \leq)$ is called a *linear* or *total* order if any two elements in the ground set $X$ are comparable, in other words, if the entire ground set $X$ is a chain.

Let $P = (X, \leq)$ be a partially ordered set and let $L = (X, \leq')$ be a partially ordered set on the same ground set. We say that $L$ is a *linear extension* of $P$ provided (1) $L$ is a linear order, and (2) if $x \leq y$ (in $P$) then $x \leq' y$ (in $L$).

A *realizer* for a poset $P$ is a family $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ of linear extensions of $P$ with the property that, for all $x, y \in X$, we have $x \leq y$ in $P$ if and only if $x \leq y$ in every $L_i$.

The *dimension* of a poset $P$, denoted $\dim P$, is the smallest size of a realizer.

The dimension of a partially ordered set can be arbitrarily large. The following poset, denoted $S_n$, is a standard example. Let $n \geq 2$ be an integer. The ground set of $S_n$ is $\{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$. The order relation of $S_n$ can be described as follows. The $a$'s are pairwise incomparable, as are the $b$'s. The only strict relations are those of the form $a_i < b_j$ with $i \neq j$. See Figure 7.6. The dimension of $S_n$ is $n$; see exercise 14 on page 129.

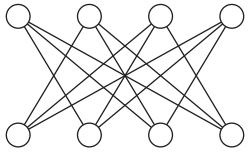

Figure 7.6. The poset $S_4$ has dimension 4 and fractional dimension 4.

The dimension of a poset can be described as the covering number of a hypergraph $\mathcal{H}$. Let $P = (X, \leq)$ be a poset, and let $\mathcal{L}$ be the set of all linear extensions of $P$. The vertices of $\mathcal{H}$ are all the ordered pairs $(x, y)$ with $x$ and $y$ incomparable. The hyperedges of $\mathcal{H}$ correspond to the linear extensions of $P$. For $L \in \mathcal{L}$, let $E_L$ consist of those ordered pairs $(x, y)$ for which $x < y$ in $L$. One checks that $\dim P = k(\mathcal{H})$.

The *fractional dimension* of $P$, denoted $\dim_f P$, can therefore be defined as $k_f(\mathcal{H})$. We can also define the fractional dimension as follows. Let $t$ be a positive integer. A *$t$-fold realizer* of $P = (X, \leq)$ is a multiset $\mathcal{R}_t = \{L_1, \ldots, L_m\}$ of linear extensions of $P$ with the property that, for all $x, y \in X$ with $x$ and $y$ incomparable, there are at least $t$ linear orders in $\mathcal{R}_t$ for which $x < y$ and at least $t$ for which $x > y$. We also call $t$-fold realizers *multirealizers*.

Let $\dim_t(P)$ denote the smallest size of a $t$-fold realizer. Then $\dim_f P = \lim_{t\to\infty}(\dim_t P)/t$.

For example, the fractional dimension of the standard example $S_n$ is $n$; see exercise 14.

Let $P = (X, \leq)$ and let $x \in X$. The *degree* of $x$ is the number of *other* elements in $X$ to which $x$ is comparable. We can bound the dimension of a poset in terms of its maximum degree.

**Theorem 7.6.1** *Let $P$ be a poset with maximum degree $d$. Then* $\dim P \leq 50d \log^2 d$. □

For fractional dimension, the following holds.

**Theorem 7.6.2** *Let $P$ be a poset with maximum degree $d$. If $P$ is not an antichain, then* $\dim_f P \leq d + 1$. □

Moreover, the standard example $S_n$ has maximum degree $n - 1$ and fractional dimension $n$, showing that the bound in Theorem 7.6.2 is the best possible.

Can dimension and fractional dimension be very different? The answer is yes. An *interval order* is a partially ordered set arising from the left-to-right order of intervals on the real line. More precisely, we say $P = (X, \leq)$ is an *interval order* provided we can assign to each $x \in X$ a real interval $I_x$ so that $x < y$ in $P$ if and only if $I_x$ is completely to the left of $I_y$. Interval orders can have arbitrarily high dimension.

**Theorem 7.6.3** *Let $k$ be a positive integer. Then there exists an interval order $P$ with* $\dim P = k$. □

By contrast, however, every interval order has small fractional dimension.

**Theorem 7.6.4** *Let $P$ be an interval order. Then* $\dim_f P < 4$.

To prove this, we need the following fact about interval orders.

**Theorem 7.6.5** *Let $P = (X, \leq)$ be an interval order. Then for every partition $X = A \cup B$ there exists a linear extension $L = (X, \leq')$ of $P$ with the property that for all $a \in A$ and $b \in B$ with $a \not> b$ we have $a <' b$.* □

Informally stated, Theorem 7.6.5 asserts that for any partition $A \cup B$ of the ground set of an interval order $P$ we can make a linear extension in which all the elements in $A$ are below all the elements in $B$, except when an element of $A$ is above an element of $B$ in $P$.

**Proof** (of Theorem 7.6.4). Let $P = (X, \leq)$ be an interval order and let $|X| = 2n$. (We may suppose $|X|$ is even. If not, we may add another element to $P$ that is below all the others. This augmented poset is also an interval order and its fractional dimension is at least that of $P$.) We build a multirealizer for $P$ as follows. Let $A \cup B$ be a partition of $X$ with $|A| = |B| = n$. Let $L_A$ be a linear extension of $P$ as described in Theorem 7.6.5. Let $\mathcal{R}$ be the multiset of all linear extensions $L_A$, as $A$ varies over the subsets of $X$ with $n$ elements. For every incomparable pair of elements $(x, y)$, there are at least $\binom{2n-2}{n-1}$ members of $\mathcal{R}$ in which $x < y$. Also, $|\mathcal{R}| = \binom{2n}{n}$, hence

$$\dim_f P \leq \frac{\binom{2n}{n}}{\binom{2n-2}{n-1}} = \frac{2n(2n-1)}{n^2} < 4.$$ □

Recently, Trotter and Winkler [179] have shown that the upper bound of 4 in Theorem 7.6.4 is tight.

## 7.7 Sperner's theorem: a fractional perspective

Let $P = (X, \leq)$ be a poset. A *chain* in $P$ is a subset $C \subseteq X$ in which any two elements are comparable. Similarly, an *antichain* is a subset $A \subseteq X$ in which any two distinct elements are incomparable. The *height* of $P$ is the maximum size of a chain in $P$ and the *width* of $P$ is the maximum size of an antichain.

Another way to express these ideas is to consider the *comparability graph* of $P$: Let $G(P)$ be the graph whose vertex set is $X$ with an edge joining $x$ and $y$ just when $x < y$ or $y < x$. Then a chain in $P$ is simply a clique in $G(P)$ and an antichain in $P$ is an independent set. The height and width of $P$ are just the clique and independence number of $G(P)$.

We may define the *fractional* height and width of a poset to be the fractional clique and independence numbers of the comparability graph of the poset. However, because comparability graphs are perfect, fractional height and width reduce to their integer counterparts. Indeed, the equality $\alpha(G(P)) = \chi(\overline{G(P)})$ is a restatement of the following result of Dilworth [43].

**Theorem 7.7.1 (Dilworth)** *The width of a poset equals the minimum size of a partition of the poset into chains.* □

Although the notions of fractional height and width yield nothing new, the fractional approach is helpful in considering the following classical problem.

Let $[n] = \{1, 2, \ldots, n\}$ and let $2^{[n]}$ denote the poset of all subsets of $[n]$ ordered by containment. The height of this poset is clearly $n+1$, but what of its width? Note that for each $k$ with $0 \leq k \leq n$, the subsets of $[n]$ of cardinality $k$ form an antichain of size $\binom{n}{k}$. The largest antichain of this sort has size $\binom{n}{\lfloor n/2 \rfloor}$. Can a larger antichain be found? The answer is no [170].

**Theorem 7.7.2 (Sperner)** *The width of $2^{[n]}$ is $\binom{n}{\lfloor n/2 \rfloor}$.*

**Proof**. Form a hypergraph $\mathcal{H}$ whose vertices are the $2^n$ subsets of $[n]$ and whose hyperedges are the maximal chains in $2^{[n]}$, i.e., the chains of size $n+1$. The width of $2^{[n]}$ is exactly the covering number $k(\mathcal{H})$, which by Dilworth's theorem equals $k_f(\mathcal{H})$. Note that $\mathcal{H}$ is edge-transitive (all maximal chains in $2^{[n]}$ look the same) so in a fractional covering of $\mathcal{H}$ we may assign all $n!$ of them the same weight $w$ without loss of optimality. (See Proposition 1.3.4 on page 5 and exercise 4 on page 12.) Choose $A \in 2^{[n]}$ with $|A| = a$. The total weight of all chains containing $A$ is $wa!(n-a)!$. The least $w$ that gives a feasible fractional cover satisfies

$$w \left\lfloor \frac{n}{2} \right\rfloor ! \left\lceil \frac{n}{2} \right\rceil ! = 1,$$

which gives

$$k_f(\mathcal{H}) = n!w = \frac{n!}{\lfloor \frac{n}{2} \rfloor ! \lceil \frac{n}{2} \rceil !} = \binom{n}{\lfloor n/2 \rfloor}. \qquad \square$$

## 7.8 Exercises

1. Let $\mathcal{H}$ be the hypergraph whose vertices are the edges of a graph $G$ and whose hyperedges correspond to the graph's planar subgraphs. Thus $\theta(G) = k(\mathcal{H})$. Show by example that $\mathcal{H}$ need not be a matroid.

2. Prove that for all $n \geq 3$ we have $\theta(K_n) = \lceil \theta_f(K_n) \rceil$ except for $n = 9, 10$.

3. Prove that if a graph has an Eulerian tour then it must have a cycle double cover.

4. Let $k$ be a positive integer and let $G$ be a graph. Call a family of cycles of a graph $G$ a *cycle $k$-fold cover* if every edge of $G$ is in exactly $k$ of the given cycles.

   Prove that if $k$ is odd, then $G$ has a cycle $k$-fold cover iff $G$ is Eulerian.

5. Show that for $x \geq 2$ we have $r_f(x, 2) = x$.

6. The fractional Ramsey number $r_f(x, y)$ as we have defined it is obviously integer-valued. Here is another way to fractionalize the Ramsey number.

   The notation $n \longrightarrow (a, b)$ means that for any partition $E(K_n) = A \cup B$ there must be either a $K_a$ on the $A$-edges or a $K_b$ on the $B$-edges. Now $r(a, b)$ is the least $n$ for which $n \longrightarrow (a, b)$ is true.

   Fractionalize this by defining $z \stackrel{*}{\longrightarrow} (x, y)$ to mean that if $G = (V, E)$ is any graph with $\omega_f(G) \geq z$, then for any partition $E = A \cup B$ we have either $\omega_f(V, A) \geq x$ or $\omega_f(V, B) \geq y$. Let $r^*(x, y)$ be the infimum of all $z$ for which $z \stackrel{*}{\longrightarrow} (x, y)$ is true.

   Prove that if $x, y > 2$ then $r^*(x, y) = xy$.

7. Compute $\pi$, $\gamma_f$, and $\gamma$ for the icosahedron and the dodecahedron.

8. Compute $\pi$, $\gamma_f$, and $\gamma$ for the graphs in Figures 7.2 and 7.3.

9. Show that no inequality holds between $\gamma(G)$ and $\Gamma_f(G)$ by finding two graphs $G$ and $H$ with $\gamma(G) < \Gamma_f(G)$ and with $\gamma(H) > \Gamma_f(H)$.

10. Show that $\Gamma_f(G) = 1$ if and only if $\Delta(G) = n - 1$.

11. Generalize Theorem 7.4.1 on page 117 by showing that, for any graph $G$, $n/(1+\Delta(G)) \leq \gamma_f(G) \leq n/(1+\delta(G))$ and $n/\Delta(G) \leq \Gamma_f(G) \leq n/\delta(G)$.

12. Find the domination number and the fractional domination number of the path $P_n$, the $2 \times n$ grid $P_n \times P_2$, and the $3 \times n$ grid $P_n \times P_3$.

13. A *block* in a graph is a maximal 2-connected subgraph. A *block graph* is a graph every one of whose blocks is complete. Show that $\pi(G) = \gamma(G)$ for every block graph $G$.

14. For $n \geq 2$ show that $\dim S_n = \dim_f S_n = n$, where $S_n$ is the standard example defined on page 125.

15. Prove Theorem 7.6.2 on page 126 in the special case that $P$ has height 2. (The *height* of a poset is the maximum cardinality of a chain.)

    Hint: Suppose there are $n$ minimal elements. Build a multirealizer by taking all $n!$ linear orders on the minimal elements and place the maximal elements into these linear orders as low as possible.

16. Prove that the intersection number of a graph equals the covering number of the edge-clique incidence hypergraph of the graph.

17. A graph $G$ is called *cobipartite* provided its complement, $\overline{G}$, is bipartite. It is called *locally cobipartite* if for every vertex $v$ the subgraph of $G$ induced by the neighbors of $v$, i.e., $G[N(v)]$, is cobipartite.

    It is conjectured that if a graph is locally cobipartite then $I(G) \leq \nu(G)$. Prove the fractional version of this conjecture, i.e., if $G$ is locally cobipartite, then $I_f(G) \leq \nu(G)$.

18. Prove Theorem 7.5.5 on page 123.

## 7.9 Notes

For an introduction to topological graph theory see Gross and Tucker [80], Mohar and Thomassen [133], Thomassen [176], or White and Beineke [188]. Theorem 7.1.1 on page 111 is a consequence of a result of Battle, Harary, Kodama, and Youngs [11]. Theorem 7.1.2 is from [173] by Stahl and Beineke. The history of Theorem 7.1.6 on page 113 is recounted in Thomassen [176], and in White and Beineke [188]. The result is based largely on the work of Beineke and Harary [21].

The Cycle Double Cover Conjecture 7.2.1 on page 113 was first posed by Szekeres in 1973 and independently by Seymour in 1979. Genghua Fan [57] has shown that every 2-edge-connected graph admits a cycle sextuple cover, i.e., a family of cycles that includes each edge of $G$ exactly 6 times. Together with Jaeger's [101] result, this means that every 2-edge-connected graph admits a cycle $k$-fold cover for all even $k \geq 4$. In case $k$ is odd, a graph has a cycle $k$-fold cover if and only if it is Eulerian. (See exercise 4.) Thus only the case $k = 2$ is open.

Paul Seymour [163] has proved a theorem on fractional directed cycle packing that is the fractional analogue of a conjecture of Younger. Recently, Reed, Robertson, Seymour, and Thomas [153] have proved this conjecture.

Theorem 7.3.1 and exercise 6 are from Jacobson, Levin, and Scheinerman [120], which also gives partial results on the more-than-two-color case. In particular, if $k \geq 2$ is an integer, one has

$$r_f(k, k, \ldots, k) = \frac{k^{p+1} - 2k^p + k}{k-1}$$

where $p$ is the number of $k$'s.

The fractional domination number was first encountered in a paper by Farber [58]. There is by now an extensive body of literature on domination, its relatives, and their fractional counterparts; see the paper [111] by Domke, Fricke, Laskar, and Majumdar for an overview of these results. Surveyed there are results not only on the domination number and the total domination number, but also on an assortment of other parameters known as the upper domination number, the independent domination number, the irredundance number, and the upper irredundance number, all of which can be fractionalized. Theorem 7.4.1 on page 117 was first noticed by Domke [44]. Theorem 7.4.2 on page 118 is due to Meir and Moon [131]. Theorem 7.4.9 on page 121 is due to Fisher, Ryan, Domke, and Majumdar [67].

It is an open problem to characterize those graphs $G$ every one of whose maximal independent sets contains $\alpha(G)$ elements. A fractional analogue is addressed in a paper by Currie and Nowakowski [41], where there appears a characterization of those graphs every one of whose minimal dominating functions attains the fractional domination number. For such graphs, the fractional domination number can be computed in a greedy way: Begin by assigning weight 1 to every vertex and then simply decrease the weights on vertices (taking the vertices in any order at all) making certain to maintain a weight of at least one on every closed neighborhood. When the last vertex is reached, the sum of all the weights will be $\gamma_f(G)$.

The material on fractional intersection number is due to Scheinerman and Trenk [159]. Exercise 17 on the preceding page, due to Chi Wang, is a weakened version of a conjecture of Opsut [140].

For an all-encompassing introduction to dimension theory for partially ordered sets, see Trotter's book [178]. Theorem 7.6.1 is due to Füredi and Kahn [72]. The log-squared term in this bound might not be best possible, but can't be replaced by anything less than a log term; see Erdős, Kierstead, and Trotter [54]. The results on fractional dimension are from Brightwell and Scheinerman [30], and Felsner and Trotter [61]. Theorem 7.6.2 can be improved by defining the *up-degree* and *down-degree* of $x$ to be the number of elements strictly greater than $x$ and less than $x$, respectively. Let $\Delta_U$ and $\Delta_D$ be the maximum up and down degrees of a poset. Then Felsner and Trotter [61] show that $\dim_f P \leq 1 + \min\{\Delta_U(P), \Delta_D(P)\}$.

For more information about interval orders and graphs, see Fishburn's book [62]. Theorem 7.6.3 on page 126 is due to Bogart, Rabinovitch, and Trotter [22]; see also Füredi, Hajnal, Rödl and Trotter [71]. Theorem 7.6.5 is from a paper by Rabinovitch [149].

The proof we present of Sperner's theorem 7.7.2 on page 127 is a fractionalized version of Lubell's proof in [125]. See also the monograph [7] by Anderson.

# Appendix A

# Background

This appendix has two purposes: to record basic notation (especially in graph theory) and to give a brief introduction to mathematical programming and subadditivity.

We assume our reader has a basic grounding in discrete mathematics, especially in graph theory. Because graph theory is a relatively young discipline, its notations are not 100% standardized. Here we simply list basic notation used in this book. Fundamental graph theory definitions are not always given; the reader should consult a text such as Berge [15], Bondy and Murty [26], Chartrand and Lesniak [35], Harary [83], or West [187].

We also give a brief introduction to linear (and integer) programming. Readers not familiar with basic LP theory (especially duality) should read this section before beginning to read the text. For a more thorough introduction, see a text such as Chvátal [38] or Schrijver [160].

The subadditivity lemma from analysis is developed in §A.4. Again, readers not conversant with this tool should read this section carefully before beginning the main portion of this book.

## A.1 Basic graph theory and notation

A *graph* is a pair $G = (V, E)$ where $V$ is a finite set and $E$ is a set of 2-element subsets of $V$. Thus, for us, a *graph* is what is commonly called a *simple graph*, i.e., a graph without loops or parallel edges. A *multigraph* may have loops and parallel edges.

We abbreviate $\{v, w\} \in E(G)$ to $vw \in E(G)$ or sometimes $v \sim w$. In this case we say that $v$ and $w$ are *adjacent*. When $e = vw$, we may write $v \in e$ and say that $v$ and $e$ are *incident*. We use $\nu(G)$ and $\varepsilon(G)$ to denote $|V(G)|$ and $|E(G)|$ respectively.

The following is a list of most of the notation used in this book. When a set of numbers follows an item, these numbers refer to the pages on which the relevant concepts are introduced or discussed.

If a symbol in this book appears with the subscript $f$, it is the fractional analogue of the unsubscripted symbol.

- $\alpha(G)$: independence number of $G$.
- $A(G)$: adjacency matrix of $G$.
- $B(G)$: bipartite split of $G$. {17, 107}
- $C_n$: cycle on $n$ vertices.
- $c(G)$: number of components of $G$. {22}
- $\chi(G)$: chromatic number of $G$. {30}
- $\chi'(G)$: edge chromatic number of $G$. {57}
- $\chi''(G)$: total chromatic number of $G$. {63}

- $d(v)$: degree of a vertex $v$.
- $d_k(v)$: iterated degree sequence of $v$. {102}
- $d(v, S)$: number of edges from $v$ to $S$. {100}
- $\bar{d}(G)$: average degree in $G$. {73}
- $\delta(G)$: minimum degree of a vertex in $G$.
- $\Delta(G)$: maximum degree of a vertex in graph $G$.
- $D(v)$, $D(G)$: ultimate degree sequence of a vertex $v$, of a graph $G$. {102}
- $\partial v$, $\partial S$, $\partial H$: set of edges with exactly one end at $v$, in $S$, in $V(H)$. {59}
- $\dim P$: dimension of poset $P$. {125}
- $E(G)$: edge set of $G$.
- $\varepsilon(G)$: number of edges in $G$, i.e., $|E(G)|$.
- $\eta$: orthonormal representation invariant. {41}
- $g(G)$, $\tilde{g}(G)$: genus, nonorientable genus of $G$. {111}
- $\gamma(G)$, $\Gamma(G)$: domination number, total domination number of $G$. {116}
- $i(G)$: number of isolated vertices in $G$. {18}
- $I(G)$: intersection number of $G$. {121}
- $J$: matrix of all ones. {94}
- $K_n$: complete graph on $n$ vertices.
- $K_{n,m}$: complete bipartite graph.
- $K(n_1, n_2, \ldots, n_p)$: complete multipartite graph.
- $k(\mathcal{H})$: covering number of hypergraph $\mathcal{H}$. {1}
- $\kappa'(G)$: edge connectivity of $G$. {25}
- $L(G)$: line graph of $G$.
- $\Lambda(G)$: lower bound for edge chromatic number. {58}
- $M(G)$, $M(\mathcal{H})$: vertex-edge incidence matrix of graph $G$, hypergraph $\mathcal{H}$. {1, 15}
- $\mathbf{M}(G)$: matching polytope of $G$. {59}
- $\mathcal{M}(G)$: cycle matroid of $G$. {74}
- $\mathcal{M}_1(G)$: near acyclic matroid of $G$. {84}
- $\mu(G)$, $\mu(\mathcal{H})$: matching number of graph $G$, of hypergraph $\mathcal{H}$. {7, 14}
- $\mathrm{mad}(G)$: maximum average degree of $G$. {73}
- $N(v)$, $N[v]$: open, closed neighborhood of vertex $v$. {116}
- $\nu(G)$: number of vertices in $G$, i.e., $|V(G)|$.
- $o(G)$: number of odd components in $G$. {18}
- $\omega(G)$: clique number of $G$.

- $P_n$: path on $n$ vertices.
- $p(\mathcal{H})$: packing number. {2}
- $\pi(G)$: closed neighborhood packing number of $G$. {116}
- $\Pi(G)$: open neighborhood packing number of $G$. {116}
- $\Pi(\mathbf{v})$: permutation polytope of vector $\mathbf{v}$. {98}
- $r(n, m)$: Ramsey number. {114}
- $\rho(\mathcal{M})$, $\rho(S)$: rank of a matroid $\mathcal{M}$, of a set $S$ in a matroid. {75}
- $\sigma(G)$, $\sigma'(G)$: toughness, edge toughness of $G$. {23, 89}
- $\tau(\mathcal{H})$: transversal number.
- $T(G)$: total graph of $G$. {63}
- $\theta(G)$: thickness of $G$. {112}
- $\Theta(G)$: Shannon capacity of $G$. {41}
- $\Upsilon(G)$: arboricity of $G$. {72}
- $V(G)$, $V(\mathcal{H})$: vertex set of graph $G$, of hypergraph $\mathcal{H}$.
- $x(G)$: crossing number of $G$. {112}
- $Y(G)$: Mycielski's construction on graph $G$. {35}
- $\overline{G}$: complement of $G$.
- $\mathcal{H}^*$: hypergraph dual of $\mathcal{H}$. {6}
- $\tilde{G}$, $\tilde{\mathcal{M}}$: dual of a planar graph $G$, of matroid $\mathcal{M}$. {86, 87}
- $G[A]$: induced subgraph of $G$ on vertex set $A$.
- $[A, B]$: set of edges with one end in $A$ and the other end in $B$.
- $G \times H$: Cartesian product of $G$ and $H$. {120}
- $G * H$: disjunctive product of $G$ and $H$. {38}
- $G[H]$: lexicographic product of $G$ with $H$. {35}
- $G \cdot H$: strong direct product of $G$ and $H$. {120}
- $\mathbf{v} \otimes \mathbf{w}$, $A \otimes B$: tensor product of vectors $\mathbf{v}$ and $\mathbf{w}$, Kronecker product of matrices $A$ and $B$. {42, 99}
- $G + H$: disjoint union of $G$ and $H$.
- $G \vee H$, $\mathcal{P} \vee \mathcal{Q}$: join of graphs $G$ and $H$, of partitions $\mathcal{P}$ and $\mathcal{Q}$. {54, 20}
- $A \oplus B$: direct sum of matrices $A$ and $B$. {96}
- $G \cong H$: $G$ is isomorphic to $H$.
- $G \cong' H$: $G$ is semi-isomorphic to $H$. {106}
- $G \leq H$: $G$ is an induced subgraph of $H$.
- $G \subseteq H$: $G$ is a subgraph of $H$.
- $\langle S \rangle$: convex hull of a set $S$.
- $\overset{\mathcal{P}}{\equiv}$: equivalence relation derived from partition $\mathcal{P}$. {100}
- $[n]$: the set $\{1, 2, \ldots, n\}$.

## A.2 Hypergraphs, multigraphs, multisets, fuzzy sets

A *hypergraph* $\mathcal{H}$ is a pair $(S, \mathcal{X})$ where $S$ is a finite set and $\mathcal{X} \subseteq 2^S$. The sets in the family $\mathcal{X}$ are called the *hyperedges* of $\mathcal{H}$. A hypergraph is also known simply as a *set system*. The *rank* of a hypergraph is the size of a largest hyperedge. If all members of $\mathcal{X}$ are of the same cardinality $r$, then $\mathcal{H}$ is called an *$r$-uniform* hypergraph. Graphs are 2-uniform hypergraphs.

A hypergraph $\mathcal{H} = (S, \mathcal{X})$ is called a *simplicial complex* in case $Y \in \mathcal{X}$ whenever $Y \subseteq X$ and $X \in \mathcal{X}$. If $\mathcal{H} = (S, \mathcal{X})$ is a simplicial complex with at least one edge, then we call $\mathcal{H}$ a *matroid* if it also satisfies the following condition: for all $X, Y \in \mathcal{X}$ with $|X| > |Y|$, there is an $x \in X - Y$ so that $Y \cup \{x\} \in \mathcal{X}$. (See §5.2.)

A *multiset* is a pair $M = (S, m)$ where $S$ is a set, called the *ground set* of $M$, and $m : S \to \mathbf{Z}^+$. For an element $s$ in the ground set $S$ the value $m(s)$ is called the *multiplicity* of $s$. We think of $m(s)$ as representing the number of times that $s$ is in $M$. If $x \notin S$ we may write $m(x) = 0$. The *cardinality* of $M$ is, naturally, $|M| = \sum_{s \in S} m(s)$.

If $M$ and $M'$ are multisets on ground sets $S$ and $S'$ respectively, their *multiset union* is $M \oplus M'$, the multiset on $S \cup S'$ in which the multiplicity of an element $s$ is $m(s) + m'(s)$.

When $t$ is a positive integer let $t \cdot M$ denote the $t$-fold multiset union $M \oplus M \oplus \cdots \oplus M$, i.e., the multiset in which the multiplicity of an element $s$ is $tm(s)$ where $m(s)$ is its multiplicity in $M$. We also write $t \cdot S$ where $S$ is a set to denote the multiset whose ground set is $S$ and all of whose elements have multiplicity equal to $t$.

A *multigraph* is a pair $G = (V, E)$ where $V$ is a finite set and $E$ is a multiset of 2-element multisets whose ground set is a subset of $V$. This definition allows for parallel edges (an edge $uv$ with multiplicity greater than 1) and loops (an edge of the form $vv$). Although some authors use the term "graph" to denote what we here call a "multigraph"; we use the term only in the more restrictive sense.

A *fuzzy set* is a pair $F = (S, w)$ where $S$ is a set and $w : S \to (0, 1]$. For an element $s \in S$, the value $w(s)$ is called the *weight* of $s$. When $x \notin S$ we may write $w(x) = 0$. The *weight* of $F$ is $w(F) = \sum_{s \in S} w(s)$. It is sometimes convenient to think of $w(s)$ as the "degree of membership" of the element $s$ in $F$.

Many of the fractional invariants in this book can be defined by taking a definition of a standard graph invariant verbatim and inserting the word "fuzzy" in an appropriate place. Thus it is often appropriate to understand the subscript "$f$", which we use throughout to denote a fractional analogue, to also stand for the word "fuzzy".

One can speculate what might be meant by a *fuzzy* or *fractional graph*. This could mean a pair $(V, E)$ in which $V$ is a finite set and $E$ is a fuzzy set of 2-element subsets of $V$. Alternatively, one might allow $V$ to be a fuzzy set as well. These objects are not considered in this book; we study graphs and, to the extent that they help us understand graphs, hypergraphs.

## A.3 Linear programming

We identify vectors in $\mathbf{R}^n$ ($n$-vectors) with $n$-by-1 matrices. We write $A^t$ for the transpose of the matrix $A$. If $A$ and $B$ are both $m$-by-$n$, we write $A \leq B$ to mean that each entry of $A$ is less than or equal to the corresponding entry of $B$. Further, when $\mathbf{v}$ is a vector and $s$ is a scalar, we write $\mathbf{v} \leq s$ to mean that every component of $\mathbf{v}$ is at most $s$.

A *linear program* (LP) is an optimization problem that can be expressed in the form "maximize $\mathbf{c}^t\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$", where $\mathbf{b}$ is an $m$-vector, $\mathbf{c}$ is an $n$-vector, $A$ is an $m$-by-$n$ matrix, and $\mathbf{x}$ varies over all $n$-vectors with nonnegative entries. The problem "minimize $\mathbf{c}^t\mathbf{x}$ subject to $A\mathbf{x} \geq \mathbf{b}$"

is also a linear program; again, we assume that $\mathbf{x} \geq 0$. It is easy to see that problems with equality constraints or with unconstrained variables can be put into the above form, so these variations may be considered. For our purposes, LPs always take the standard forms introduced here.

An *integer program* (IP) is an optimization problem of the same form as a linear program except that the vector $\mathbf{x}$ is subject to the additional constraint that all its entries must be integers.

In an LP or an IP, the expression $\mathbf{c}^t\mathbf{x}$ is called the *objective function*, a vector $\mathbf{x}$ satisfying the constraints $A\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq 0$ is called a *feasible solution*, and the optimum of the objective function over all feasible solutions is called the *value* of the program. It is natural to assign the value $-\infty$ to a maximization program with no feasible solutions and the value $+\infty$ if the objective function is unbounded on feasible solutions. The linear program obtained from an integer program $P$ by dropping the constraint that the entries of $\mathbf{x}$ be integers is called the *linear relaxation* of $P$. A main theme in this book is the connection between integer programs that compute graph-theoretic invariants and their linear relaxations, which compute the fractional analogues of those invariants.

If $P$ is the (linear or integer) program "maximize $\mathbf{c}^t\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq 0$", then the program "minimize $\mathbf{b}^t\mathbf{y}$ subject to $A^t\mathbf{y} \geq \mathbf{c}$, $\mathbf{y} \geq 0$" is called the *dual* of $P$. If $\mathbf{x}$ is a feasible solution for $P$ and $\mathbf{y}$ is a feasible solution for the dual of $P$, then, because $\mathbf{x}, \mathbf{y} \geq 0$, we have the *weak duality* inequality:

$$\mathbf{c}^t\mathbf{x} = \mathbf{x}^t\mathbf{c} \leq \mathbf{x}^t A^t\mathbf{y} = (A\mathbf{x})^t\,\mathbf{y} \leq \mathbf{b}^t\mathbf{y}. \tag{A.1}$$

This implies that the value of $P$ is less than or equal to the value of the dual of $P$. In fact, if $P$ is a linear program, more is true.

**Theorem A.3.1** *A linear program and its dual have the same value.* □

This is a central result in the theory of linear programming and is called the (strong) duality theorem. One way to prove it is to analyze an algorithm (the simplex algorithm) that gives a feasible solution $\mathbf{x}^*$ for $P$ and simultaneously a feasible solution $\mathbf{y}^*$ for the dual of $P$ such that $\mathbf{c}^t\mathbf{x}^* = \mathbf{b}^t\mathbf{y}^*$. This yields the opposite inequality from the one implied by (A.1) and the theorem follows. For details see any of the many standard books on linear programming.

Not only are the values of an LP and its dual equal, but the vectors achieving this value obey the *complementary slackness* condition described in the following result.

**Theorem A.3.2** *Let $\mathbf{x}^*$ be any optimal solution to the bounded, feasible linear program $\max \mathbf{c}^t\mathbf{x}$ subject to $A\mathbf{x} \leq b,\ \mathbf{x} \geq 0$, and let $\mathbf{y}^*$ be any optimal solution to the dual $\min \mathbf{b}^t\mathbf{y}$ subject to $A^t\mathbf{y} \geq \mathbf{c},\ \mathbf{y} \geq 0$. Then*

$$\mathbf{x}^* \cdot (A^t\mathbf{y}^* - \mathbf{c}) = \mathbf{y}^* \cdot (A\mathbf{x}^* - \mathbf{b}) = 0.$$

**Proof**. Equality holds in the weak duality condition (A.1) because $\mathbf{c}^t\mathbf{x}^* = \mathbf{b}^t\mathbf{y}^*$, i.e.,

$$\mathbf{c}^t\mathbf{x}^* = (\mathbf{y}^*)^t A\mathbf{x}^* = (\mathbf{y}^*)^t\mathbf{b},$$

from which it follows that $(\mathbf{c}^t - (\mathbf{y}^*)^t A)\,\mathbf{x}^* = 0$ and so $\mathbf{x}^* \cdot (A^t\mathbf{y}^* - \mathbf{c}) = 0$. We have $\mathbf{y}^* \cdot (A\mathbf{x}^* - \mathbf{b}) = 0$ by the same reasoning. □

Note that since $\mathbf{x}^*$ and $A^t\mathbf{y}^* - \mathbf{c}$ are nonnegative, Theorem A.3.2 implies that if some coordinate of either $\mathbf{x}^*$ or $A^t\mathbf{y}^* - \mathbf{c}$ is nonzero, then the corresponding coordinate of the other must be zero. In other words, complementary slackness holds in a term-by-term fashion. The same is true, of course, for $\mathbf{y}^*$ and $A\mathbf{x}^* - \mathbf{b}$.

The situation for integer programs is more complicated; the value of an integer program need not equal the value of its dual. The interval between these two values is called the *duality gap*. Any number in the interior of the duality gap is not achievable as a value of the objective function of either the primal program or the dual.

In some situations, the value of an integer program and its linear relaxation are equal. A $\{0,1\}$-matrix $M$ is *totally unimodular* if $\det S \in \{-1,0,1\}$ for all square submatrices $S$ (not necessarily formed from contiguous rows or columns). A $\{0,1\}$-matrix $M$ is (*totally*) *balanced* provided no odd-by-odd square submatrix has exactly two ones in every row and column. Note that total unimodularity implies total balance (exercise 3 on page 138).

The following theorem follows easily from the basic theory of linear programming and from Cramer's rule for solving linear systems.

**Theorem A.3.3** *If $A, \mathbf{b}, \mathbf{c}$ all have integer entries and $A$ is totally unimodular, then the value of the integer program "maximize $\mathbf{c}^t\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0$", and its linear relaxation are the same.* □

The same conclusion can be drawn from slightly different hypotheses.

**Theorem A.3.4** *If $A, \mathbf{b}, \mathbf{c}$ all have integer entries, at least one of $\mathbf{b}$ or $\mathbf{c}$ is a constant vector, and $A$ is totally balanced, then the value of the integer program "maximize $\mathbf{c}^t\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0$", and its linear relaxation are the same.* □

Under the hypotheses of these theorems, the duality gap associated with the IP reduces to a single point.

## Computational complexity

The classic method for finding an optimal solution to a linear program is the simplex algorithm. While this method works reasonably for most problems, it can, in the worst case, take an unreasonable amount of time to solve an LP. On the other hand, the ellipsoid algorithm always runs in at most polynomial time, but is not a practical method; on typical problems it is slower than the simplex method. Interior point algorithms appear to give us the best of both worlds: good performance on typical problems and a polynomial run time in the worst case.

More precisely, we measure the *size* of an LP as follows. Suppose the LP is

$$\max \mathbf{c}^t\mathbf{x} \quad \text{s.t.} \quad A\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq 0 \tag{A.2}$$

where $A$ is an $m \times n$ matrix. We suppose that the entries in $A$, $\mathbf{b}$, and $\mathbf{c}$ are rational numbers and we let $K$ be the maximum absolute value of the various numerators and denominators of these rational numbers when expressed in reduced form. Then the size of the LP is defined to be $mn \log K$. Were we to input an LP into a computer, the size of the LP is a good measure of how large (how many characters long) the input would be. Then when we assert that, say, the ellipsoid algorithm runs in polynomial time, we mean that it runs in time bounded by a polynomial in the size of the LP.

Sometimes the size of an LP is not a fair measure of the size of the problem it is formulated to solve. For example, consider the LP (Chapter 3) for the fractional chromatic number of a graph. A reasonable measure of the size of an instance of the fractional chromatic number problem of a graph $G = (V, E)$ is $|V| + |E|$. However, this LP formulation of fractional chromatic number features a matrix whose rows are indexed by $V$ but whose columns are indexed by the independent subsets of $V$, and there may be *exponentially many* (in $|V|$) independent subsets of $V$. Indeed, despite

the fact that linear programs can be solved in polynomial time, the fractional chromatic number problem is NP-hard.

Nevertheless, despite the combinatorial explosion inherent in formulating some fractional graph theory problems as linear programs, a polynomial-time solution (in the size of the *graph*) may still be possible. Consider again the LP in equation (A.2) in which the matrix $A$ has $m$ rows and $n$ columns and the entries in $A$, $\mathbf{b}$, and $\mathbf{c}$ are rational numbers. As before, let $K$ be the maximum absolute value of the various numerators and denominators of these rational numbers when expressed in reduced form. Suppose the number of constraints $m$ is exceedingly large compared to the number of variables $n$. It would be useful if we could still solve this LP efficiently, i.e., in time polynomial in $mn \log K$. Indeed, this is possible provided there is an efficient (polynomial-time) algorithm for solving the *separation problem*:

**Separation Problem**: Given a vector $\mathbf{x} \geq 0$, decide if $\mathbf{x}$ is feasible for the LP, and if not, find a constraint that is violated.

If we can solve the separation problem efficiently (in time polynomial in $n \log K$) then the ellipsoid algorithm can be used to solve the LP in polynomial time.

## A.4 The subadditivity lemma

A function $g : \mathbf{Z}^+ \to \mathbf{R}$ is called *subadditive* provided, for all $a, b$, we have

$$g(a+b) \leq g(a) + g(b).$$

**Lemma A.4.1 (Subadditivity)** *Suppose $g$ is subadditive and $g(n) \geq 0$ for all $n$. Then the limit*

$$\lim_{n\to\infty} \frac{g(n)}{n}$$

*exists and is equal to the infimum of $g(n)/n$ ($n \in \mathbf{Z}^+$).*

**Proof**. Let $x = \limsup g(k)/k$ and let $n$ be a fixed positive integer. Let $m$ be any (large) integer. Divide $m$ by $n$ and write $m = qn + r$ with $0 \leq r < n$. Note that by subadditivity, $g(m) \leq qg(n) + g(r)$. Dividing by $m$ we have

$$\begin{aligned} \frac{g(m)}{m} &\leq \frac{qg(n) + g(r)}{qn + r} \\ &\leq \frac{qg(n) + g(r)}{qn} \\ &\leq \frac{g(n)}{n} + \frac{g(r)}{qn}. \end{aligned}$$

Choose a sequence of $m$'s going to infinity so that $g(m)/m \to x$. Note that $\frac{g(r)}{qn} \to 0$ since $g(r)$ is bounded and $n$ is fixed but $q \to \infty$. Thus we have $x \leq g(n)/n$ for any $n$. This implies $\liminf g(n)/n \geq x = \limsup g(n)/n$, so $\lim g(n)/n = x$ and $\lim g(n)/n = \inf g(n)/n$. □

We often have use for a multiplicative version of this lemma. A function $h : \mathbf{Z}^+ \to \mathbf{R}^+$ is called *submultiplicative* provided, for all $a, b$, we have

$$h(a+b) \leq h(a)h(b).$$

**Lemma A.4.2** *Suppose that $h$ is a submultiplicative function with $h(n) \geq 1$ for all $n$. Then the limit*

$$\lim_{n\to\infty} \sqrt[n]{h(n)}$$

*exists and is equal to the infimum of $\sqrt[n]{h(n)}$.*

**Proof**. Let $g(n) = \log h(n)$. Then $g(a+b) = \log h(a+b) \leq \log h(a)h(b) = \log h(a) + \log h(b) = g(a) + g(b)$, so $g$ is subadditive. Lemma A.4.1 then implies that $\lim g(n)/n = \lim \log \sqrt[n]{h(n)}$ exists and equals its infimum. Thus the same is true for $\lim \sqrt[n]{h(n)}$. □

## A.5 Exercises

1. Formulate and prove a "superadditivity" lemma.

2. Let $g : \mathbf{Z}^+ \to \mathbf{R}$. Show that if $g(a+b+c) + g(c) \leq g(a+c) + g(b+c)$ for all $a, b, c \in \mathbf{Z}^+$, then $g(n)/n$ is a decreasing function of $n$ (hence approaches its limit monotonically).

3. Let $M$ be a 0,1-matrix. Prove that if $M$ is totally unimodular, then $M$ must be totally balanced.

4. Given a graph $G$, let $A$ be its adjacency matrix and let $\mathbf{b}$ and $\mathbf{c}$ be vectors of the appropriate size all of whose entries are 1. What graph theoretic parameter is given by the value of the LP "maximize $\mathbf{c}^t\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$"?

## A.6 Notes

Graph theory is a relatively young discipline, and as such there is not always a single standard for terminology and notation. We follow mostly the style of Bondy and Murty [26]. Other widely used introductions to graph theory include Chartrand and Lesniak [35] and Harary [83]. A more substantial introduction with a long, excellent list of exercises is West [187].

The theory of hypergraphs is explored in Berge [15] and [19].

Linear programming as a discipline goes back to 1947 when George Dantzig invented the simplex algorithm for solving certain military optimization problems. The proof of the duality theorem is due to Gale, Kuhn, and Tucker [73], appearing in 1951. A good mathematical treatment of linear programming theory is Schrijver [160]. In particular, our Theorem A.3.3 appears there on page 266 with proof, and a version of our Theorem A.3.4 appears there on page 305. A gentler introduction to the theory of linear programming is Chvátal [38]. See also Grötschel, Lovász, and Schrijver [82] or Papadimitriou and Steiglitz [142].

Lemmas A.4.1 and A.4.2 are widely known as folk theorems. They go back to the work of Fekete [60] in 1922.

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# Author Index

# Subject Index