ABSTRACT

Title of Document: MORE THAN JUST "PLUG-AND-CHUG": EXPLORING HOW PHYSICS STUDENTS MAKE SENSE WITH EQUATIONS

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Although a large part the Physics Education Research (PER) literature investigates students' conceptual understanding in physics, these investigations focus on qualitative, conceptual reasoning. Even in modeling expert problem solving, attention to conceptual understanding means a focus on initial qualitative analysis of the problem; the equations are typically conceived of as tools for "plug-and-chug" calculations. In this dissertation, I explore the ways that undergraduate physics students make conceptual sense *of physics equations* and the factors that support this type of reasoning through three separate studies.

In the first study, I investigate how students' can understand physics equations intuitively through use of a particular class of cognitive elements, *symbolic forms* (Sherin, 2001). Additionally, I show how students leverage this intuitive, conceptual meaning of equations in problem solving. By doing so, these students avoid algorithmic manipulations, instead using a heuristic approach that leverages the equation in a conceptual argument.

The second study asks the question why some students use symbolic forms and others don't. Although it is possible that students simply lack the knowledge required, I argue that this is not the only explanation. Rather, symbolic forms use is connected to particular *epistemological stances*, in-the-moment views on what kinds of knowledge and reasoning are appropriate in physics. Specifically, stances that value *coherence* between formal, mathematical knowledge and intuitive, conceptual knowledge are likely to support symbolic forms use. Through the case study of one student, I argue that both reasoning with equations and epistemological stances are

dynamic, and that shifts in epistemological stance can produce shifts in whether symbolic forms are used to reason with equations.

The third study expands the focus to what influences how students reason with equations across disciplinary problem contexts. In seeking to understand differences in how the same student reasons on two similar problems in calculus and physics, I show two factors, beyond the content or structure of the problems, that can help explain why reasoning on these two problems would be so different. This contributes to an understanding of what can support or impede transfer of content knowledge across disciplinary boundaries.

MORE THAN JUST “PLUG-AND-CHUG”: EXPLORING HOW PHYSICS STUDENTS MAKE SENSE WITH EQUATIONS

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I'm thankful to have been a part of weekly research meetings where the goal was to "play with new ideas" and find new ways to see the data. It's through participating in these meetings that I learned how to be an education researcher and how to be a member of this community.

I'm thankful to have been a part of a culture of exploration and innovation, where the freedom existed for a third year that mainly consisted of attending different research meetings and reading interesting papers. It was this year that totally resolved any misconceptions I possessed pertaining to linear notions of "progress in graduate school."

I'm thankful for being asked and for learning to ask interesting science questions. Having the chance to really explore questions deeply helped me relearn how to wonder again (and love it).

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Table of Contents

List of Tables

List of Figures

Chapter 1: Introduction and Literature Review

INTRODUCTION

A Tale of Two Problem-Solving Approaches in Physics

In the fall of 2005, I was taking an upper-division, undergraduate electricity and magnetism (E&M) course. As is standard in such courses, there were weekly problem sets assignments. I often met with a group of fellow undergraduates to work on the assignments for this class (and others). Here, I recount one memorable episode while working on a problem about the electromagnetic radiation from a rotating, charged ring:

> Griffiths 11.9: An insulating circular ring (radius b) lies in the $x\,y$ plane, centered at the origin. It carries a linear charge density $\lambda = \lambda_0 \sin \phi$, where λ_0 is constant and ϕ is the usual azimuthal angle. The ring is now set spinning at a constant angular velocity ω about the z axis. Calculate the power radiated (Griffiths, 1999).

The basic physics concept behind this phenomenon is that accelerating charges produce electromagnetic radiation. This problem can be solved by calculating the electric dipole moment of this ring and using that dipole moment with an equation for the power radiating from a rotating dipole (given in that section of the textbook) to compute the answer. But attending to more than just the solution to this problem, I want to contrast two different *approaches* to solving this problem that were present in our group's work.

I initially started this problem on my own while the other members in the group were working on a problem that I had already solved. I started this problem by looking over the previous section in the textbook, my eye particularly drawn to the "boxed" equations. I saw that the chapter the problem came from had at least three equations for calculating the radiated power, but I didn't immediately know from the form of the expressions in what situations they were applicable. I did see that all three equations contained terms that were not given in the problem statement: either $\ddot{\mathbf{p}}$, the second time derivative of the electric dipole moment, or m_0, the maximum magnetic dipole moment. It was not immediately obvious to me how to calculate either one of these values from the given values in the problem. In particular, I was confused about how to introduce time into a dipole moment, something for which we had never learned an equation.

At this point, the other members of the group finished the problem they had been working on. One of the other group members, Mark, came over to me and asked me what I was working on. I showed him the problem, and he read over it for a moment. He went up the chalkboard in our physics lounge area and said “ok, let’s draw a picture.” He sketched out the following picture (Fig. 1.1).

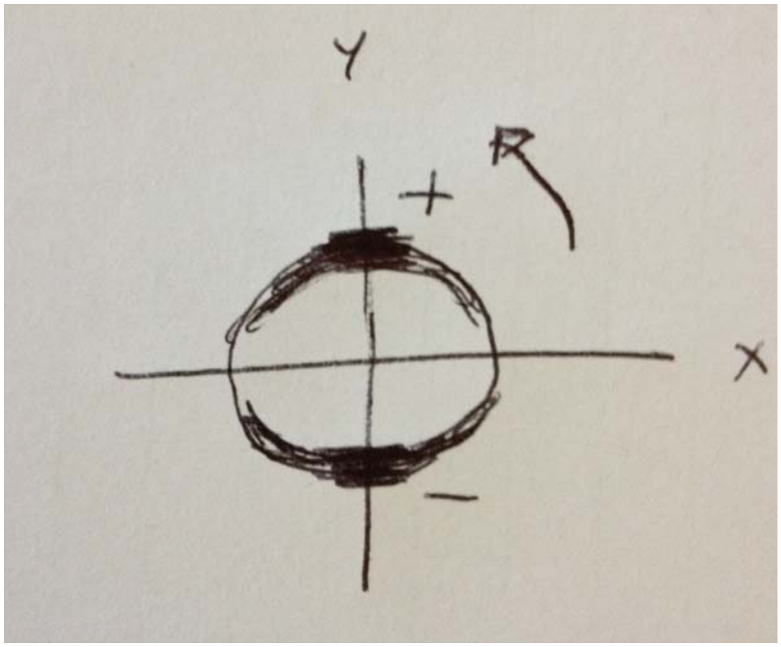

Figure 1.1: Mark’s diagram of the rotating charged ring in Griffiths problem 11.9

He explained some features of his diagram: the darker parts of the ring indicate a higher charge density, and half of the ring must be positive and half of the ring must be negative, which he got from the expression for the charge density. Next, he picked up a textbook and flipped through the pages. He spotted an equation for finding the radiated power from an arbitrary rotating charge distribution, one of the three equations I was deciding over. He remarked that we needed the dipole moment of our charged ring to solve this problem. He wrote an expression, $p = \int dQ\ \vec{r}$, on the board. I asked how we would get *dQ*. Mark drew a box around a small section of the ring in his diagram. He called this section of the ring *dQ* and said that the amount of charge in the box was equal to its length times its charge density, $(\lambda_0 sin\phi)\ r\ d\phi$. Combined with an expression for $\vec{r}$, this produced an integral that we could solve by integrating over the angle to get an expression for the dipole moment: $(\lambda_0 b^2 \pi)\ \hat{y}$. Mark liked this answer, because he saw that it agreed with his diagram, in which the dipole moment pointed in the $\hat{y}$ direction.

I brought up the fact that the equation actually needed $\ddot{\vec{p}}$, the second time derivative of the dipole moment, and this expression didn’t have time anywhere, so the answer would be zero. Mark agreed and said that this expression was for the dipole moment at time $t = 0$. Then, he wrote, with no explanation, the expression $(\lambda_0 b^2 \pi)\ (sin\omega t\ \hat{x} + cos\omega t\ \hat{y})$. After thinking for a second, he put a minus sign in front of the $\hat{x}$ term. At this point, Mark was satisfied that this problem was solved, except for some equation manipulation.

Blending Conceptual and Formal Mathematical Reasoning in Problem-Solving Expertise

My past self had trouble understanding how Mark had come to this solution. It seemed like he knew just the right equation for each part of the problem. How did he know the dipole moment integral? How did he know exactly how to transform an integral in dQ into one that can be evaluated? The time dependent rotation term he added into the dipole moment made sense as a way to get the rotation in there, but how did he know that he was allowed to just tack it on without some explicit formula for doing so?

There are many differences between Mark's solution and mine. I recount this (partially reconstructed) story to highlight two particular markers of problem-solving expertise evidenced by Mark's solution. First, Mark's solution includes many examples of *translating between a conceptual understanding of the situation and formal mathematical expressions*. Mark starts by translating the mathematical equation for the charge density of the ring to a picture that makes clear what the charge distribution on the ring actually looks like. This helps him to see features of the problem situation, such as the fact that the dipole moment is initially pointing in the y-direction. He understands dQ physically as an infinitesimally small piece of charge, not just as an indicator of which variable to take the anti-derivative with respect to. Mark also doesn't introduce rotation into the dipole moment expression through the application of some general formula or procedure. Instead, the fact that he immediately writes down an expression, modifying the sign afterwards, suggests that he is constructing on-the-spot the specific equation he needs to represent his understanding of the physical system. On the other hand, my approach dealt primarily with the formal mathematics without explicit consideration of the physical system. My approach could be labeled as "plug-and-chug:" I spent my time searching for relevant equations and considering whether I had been given enough information to determine the required quantities for using those equations.

Certainly it seems that Mark had stronger knowledge of E&M than I – he knew what the different radiated power formulae applied to, and he was able to calculate the dipole moment of our distribution from the definition – but beyond this, underlying our two approaches are two *different notions of what it means to learn and understand physics*. My approach to this problem was typical of my approach to learning physics at the time. I would sit in lectures and follow individual steps of derivations without seeing the big picture of what those derivations meant conceptually. I would approach problems by seeking out nearby equations and see if their application would help me solve the problem. It would not be until years later that I would start stably approaching problems more like Mark, using my conceptual understanding of situations to structure the mathematics.

I am embarrassed to say that I received an A+ as a final grade in this class. This grade represented my facility in finding and manipulating equations more than my understanding of the basic physics of E&M. Please note: I am not condemning the intentions of the instructor. I believe that the instructor's goal was to impart the fundamental ideas of E&M. For Mark, this course seemed to be successful in communicating those ideas. If I were to take this class again now, I believe that I would also be able to "see" those basic physics concepts and connect them to the equations I was manipulating. However, at the time, I was somehow not equipped to understand or take up that approach in physics class, just as I could not fully understand or take up Mark's approach when he presented it to me. As my views on what it means to learn physics have evolved, I can now understand what Mark was doing in his solution to the E&M problem, and I now see how his approach demonstrates problem-solving expertise. I also look back at my E&M course as a missed opportunity: I could have spent that time practicing this blending of physical understanding and formal mathematics rather than the manipulation of equations.

One goal of a physics classroom is to develop problem-solving expertise. As Mark's solution illustrates, this expertise is more than simply the ability to solve problems correctly; for Mark, it includes the interconnection of conceptual understanding and formal mathematical expressions. In this dissertation, I investigate aspects of this blending between conceptual and formal mathematical reasoning in working with physics equations. Specifically, I explore 1) how this type of reasoning is conceptualized in models of expert physics problem-solving practice, 2) how this reasoning is connected to *epistemological stances* towards what it means to learn and understand physics and physics equations, and 3) whether the use of this type of blended reasoning is affected by changing disciplinary contexts (i.e. will students similarly use this blended conceptual and formal mathematical reasoning in working across two different disciplines – for example, physics and math).

The three data chapters, each investigating one of these issues, are meant to stand alone as independent papers. They each contain more detailed literature reviews and descriptions of the methods specific to particular arguments in those chapters. In the rest of this chapter, I provide an overview of the literature point out how the work in this dissertation contributes to broader arguments in the field.

LITERATURE REVIEW

Physics Education Research Has Focused on Conceptual Reasoning as Separate from Reasoning with Equations

A resource letter in Physics Education Research (PER) was published in 1999 (McDermott & Redish, 1999), listing the systematic studies of student learning and of the efficacy of reformed instructional techniques. This resource letter lists 115 publications (of the 224 publications total) under the category of "conceptual understanding:" studies that attend to students' understandings of physics topics (often referred to as misconceptions, preconceptions, alternative conceptions, prior knowledge, difficulties, or understandings), instruction developed to improve those conceptual ideas, and assessments designed to probe for conceptual understanding. These papers span the physics topics of kinematics, dynamics, relativity, electric circuits, electricity & magnetism, light & optics, properties of matter, fluid mechanics, thermodynamics, waves, and modern physics.

Although this resource letter is not meant to be totally comprehensive, it is representative of a trend in PER that continue to be true: a primary area of focus in Physics Education Research is students' conceptual understanding of the physics topics. These studies tend to focus on students' qualitative understanding (or misunderstanding) of the physics content, separate from their conceptual understandings of the formal mathematical equations involved. Yet, this focus on conceptual understanding that is independent of the relevant physics equations is discontinuous with the importance of mathematics in physics courses and professional physics practice as tools for expressing and extracting conceptual meaning. How did it come to be that conceptual understanding in PER is studied separately from an understanding of physics equations?

The fruitfulness of research and instructional design centered on qualitative conceptual reasoning

A major contribution of PER is the growing body of qualitative questions that probe students' conceptual understanding of physics along with common student responses to these questions. The most coherent body of knowledge of this type has been produced by the University of Washington's Physics Education Group, who have systematically documented common conceptual difficulties with qualitative physics problems covering a wide range of physics content (e.g. Ambrose, Shaffer, Steinberg, & McDermott, 1999; Goldberg & McDermott, 1987; Kautz, Heron, Loverude, & McDermott, 2005; Scherr, Shaffer, & Vokos, 2001; Trowbridge & McDermott, 1980, 1981) and used these findings to design instructional interventions targeting these difficulties (McDermott, Shaffer, & the Physics Education Group at the University of Washington, 2002).

Another major area where PER has contributed in this way is in developing conceptual inventories: typically multiple-choice surveys consisting of qualitative questions designed to measure conceptual understanding on a certain subset of physics content (Chasteen & Pollock, 2009; Hestenes, Wells, & Swackhamer, 1992; Maloney, O'Kuma, Hieggelke, & Van Heuvelen, 2001; Thornton, Kuhl, Cummings, & Marx, 2009). These concept inventories have become commonly used tools for instructors in assessing student conceptual understanding, as well as tools for researchers to assess the success of reform-based instructional curricula in developing that conceptual understanding.

Success in quantitative problem solving does not necessarily indicate an understanding of physics concepts

In his guide to Peer Instruction, Mazur (1999) recounts a transformative experience in his teaching. Inspired by work in PER showing that, even after instruction, students have difficulty answering basic conceptual questions, Mazur tested his students with both a qualitative and quantitative electric circuit problem. He found that, although most physicists would judge the qualitative problem to be easier, students performed better on the quantitative problem. Mazur concluded that students could perform well on standard physics problems by memorizing algorithms without learning the underlying physics, and that success on these standard problems could mislead both teachers and students into thinking that they had mastered the topic. In response, Mazur developed his reformed physics curriculum, Peer Instruction, which includes qualitative discussion questions that target conceptual understanding.

Along these lines, other work has also explicitly argued against success with equation use as a marker of conceptual understanding. Lawson and McDermott (1987) found that students who successfully completed an introductory physics course could not make qualitative arguments with the work-energy or impulse-momentum theorems. These researchers argue that these students' success in their physics courses makes it likely that they were able to successfully use the work-energy theorem or impulse-momentum theorem to solve problems, even though they had not learned the underlying concepts well enough to make qualitative comparisons. Huffman (1997) showed that students who successfully adopted an explicit quantitative problem-solving strategy for solving dynamics problems did not show improvement over a control group in their conceptual understanding of dynamics – as measured by the Force Concept Inventory (Hestenes et al., 1992). Kim and Pak (2002) showed that even after solving thousands of physics problems, students still provided evidence of common conceptual difficulties in kinematics and dynamics.

The point that success in quantitative problem solving is not always an accurate measure of conceptual understanding, as well as the demonstrated fruitfulness in attending only to qualitative conceptual understanding has likely supported this divide between conceptual understanding and an understanding of physics equations. However, I propose that expertise in physics also involves blending the conceptual understanding of the physical situation with relevant mathematics, and that there is a dearth of research studying conceptual understanding of physics equations in PER. As I illustrated in the opening tale, Mark's problem-solving approach involves not just conceptual understanding or formal mathematical understanding, but the ability to translate between and blend the two in problem-solving practice.

All of this is not to say that PER has been devoid of research investigating physics students' mathematical understanding. However, swinging the pendulum the other way, these studies tend to focus on students' understanding of the formal mathematics used in physics, separate from a conceptual understanding of the underlying physics concepts. To be clear, students' understandings of the mathematics are often seen as connected to conceptual meaning at the level of the meanings of the individual symbols (e.g. m stands for mass, v stands for velocity, E stands for electric field) but independent from an understanding of the underlying physics that those mathematical formalisms represent (Breitenberger, 1992; Christensen & Thompson, 2012; Clement, Lochhead, & Monk, 1981; Cohen & Kanim, 2005; D. H. Nguyen & Rebello, 2011; N. L. Nguyen & Meltzer, 2003; Torigoe & Gladding, 2011). For example, Nguyen and Rebello (2011) studied student difficulties with integration in the context of electrostatics, concluding that physics students have difficulty understanding the infinitesimal term in the integral and have difficulty considering direction when integrating a vector quantity – difficulties that are not necessarily tied to an understanding of the underlying physics of the problems. This is not to say that no existing studies in the PER problem-solving literature investigate how a conceptual understanding of physics is blended with the use of mathematics. However, these few studies (Bing & Redish, 2009;

Redish & Smith, 2008; Sherin, 2001; Tuminaro & Redish, 2007) are the exception and not the rule.

One field of study that has explicitly proposed the importance of connecting conceptual reasoning to formal mathematics (in the form of physics equations) is the quantitative physics problem-solving literature. In the next section, I will describe how this connection between conceptual and formal mathematical reasoning tends to be described in the physics problem-solving literature, along with a possible directions for expanding this connection.

Expert Problem Solving Starts with Formal Physics Concepts to Select the Relevant Equations

Problem solving is central to physics. Because of this, one area of research has focused on describing, modeling, and teaching expert problem-solving practice (Hsu, Brewe, Foster, & Harper, 2004; Maloney, 1994, 2011). These problem-solving studies have attended to the importance of conceptual reasoning in problem solving, where the primary finding is that experts tend to begin problem-solving episodes with a qualitative, conceptual analysis that leads into the relevant mathematics and novices do not. Yet, there has not been attention to how that conceptual reasoning continues to be relevant to the mathematical equations even after they are selected. Most conceptions of expert problem solving treat the mathematical processing of equations, once selected, as algorithmic manipulations. In the following section, I will situate this argument in the literature, describing the findings of the expert problem-solving research, and follow up by describing the studies that have started to attend to how conceptual reasoning may be relevant in the mathematical processing of physics equations.

Research investigating how individuals solve quantitative problems in math and science spans many different research literatures so to attempt to discuss them all here is impossible. Because this dissertation pertains to quantitative physics problem solving, I limit the discussion of the literature to the problem-solving research published and commonly referred to in the PER literature. This will include discussion of some of the problem-solving research from cognitive science and psychology that formed the foundation on which the PER literature builds.

Experts start with what you know. Novices start with what you want to find.

One main finding in the problem-solving literature is that expert and novice approaches to quantitative problem solving differ. One characterization of this difference is that novices "backwards chain" and experts "forward chain:" novices initially select equations based on what quantity they wish to find, whereas experts initially select equations based on what quantities they already know (Larkin, McDermott, Simon, & Simon, 1980; Simon & Simon, 1978; Taasoobshirazi & Glynn, 2009).

Related studies show that experts tend to start by drawing a diagram of the problem situation, whereas novices tend to engage immediately with the mathematical equations. The common explanation is that experts tend to begin problem-solving with a qualitative analysis that leads into relevant calculations, whereas novices typically take a means-end approach to problem solving – attending to what quantity is desired and finding equations that can be used to calculate that quantity (Larkin & Reif, 1979; Walsh, Howard, & Bowe, 2007).

Experts draw on well-structured knowledge of physics principles to analyze physics problems

Building on this research investigating differences in expert and novice problem-solving approaches, some research focuses in on differences in the knowledge structures of those experts and novices. One well-known set of studies by Chi and colleagues (Chi, Feltovich, & Glaser, 1981; Chi, Glaser, & Rees, 1982) sought to understand how and why experts and novices differentially categorize physics problems according to how they would be solved. Experts tended to group these problems according to which general physics principles would be applied in the solutions – principles like conversation of energy, conversation of momentum, or Newton's second law, for example. Novices, on the other hand, tended to group these problems according to salient surface features – rotation, springs, inclined planes, etc. Although the presence of springs can indicate the relevance of particular physics concepts and equations related to springs – such as the energy stored in a spring being $\frac{1}{2}kx^2$ – these novices apparently tended to see these as primary instead of the overarching principle applied to solve the problem – such as conversation of energy.

To understand differences in expert and novice knowledge structures, interviews were conducted that prompted individuals to describe their problem-solving knowledge with respect to particular categories introduced by the interviewer. For the category of "inclined planes," the novice connected this to other relevant problem-solving features, such as the angle of the plane, whether there is a block on the plane, what the mass and height of the block are, whether there is friction, etc. Principles such as conversation of energy are mentioned but only at the end of the interview, after the relevant surface features of the problem are described. By contrast, the expert explanation starts with relevant physics principles, conservation of energy and Newton's laws and their applicability conditions. Chi et al. (1981) conclude that experts' knowledge is structured such that surface features can cue applicable physics principles – what are called "second-order features" – whereas novices do not directly read features beyond the relevant surface features. This could explain why novices often resort to means-end analysis while experts begin with a conceptual analysis: an expert's conceptual knowledge allows them to identify and start with the relevant physics principles.

Other research in this vein has specified theoretical models of effective problem-solving knowledge structures and the associated procedures for activating that knowledge in problem solving. Optimal problem-solving knowledge is described as hierarchically structured so that the search for applicable knowledge can proceed efficiently (Reif & Heller, 1982; Reif, 1995, 2008; Van Heuvelen, 1991a). These specified procedures define how the knowledge base is used to generate the initial problem description and the subsequent mathematics for solving the problem (J. I. Heller & Reif, 1984; Reif, 2008). These procedures inform the design of instructional expert problem-solving procedures.

Expert/novice differences inform the design of instructional problem-solving procedures

Although some research questions the validity of these characterization of expert/novice differences in problem solving – specifically, the finding that experts start with an initial conceptual analysis and novices do not (Chi et al., 1982) and the finding that experts forward chain and novices do not (Priest & Lindsay, 1992) – these expert/novice findings have been taken up to inform the design of instructional interventions aiming to develop students' problem-solving expertise in physics. Some interventions provide explicit procedures that direct students through a step-wise problem-solving approach (Gaigher, Rogan, & Braun, 2007; P. Heller, Keith, & Anderson, 1992; Huffman, 1997; Reif, Larkin, & Brackett, 1976; Reif, 2008; Van Heuvelen, 1991b). For example, Heller et al. (1992) direct students to generate a description of the problem, to create a representation of the problem incorporating a physical analysis, to select a relevant physics principle/equation, to apply this equation to the problem at hand to compute a symbolic or numerical solution, and then to check this answer. By following an expert procedure like this, the intention is that these students learn to internalize this procedure, and post-tests confirm that, unscaffolded, they continue to follow these expert-like problem-solving approaches (Gaigher et al., 2007; P. Heller et al., 1992; Huffman, 1997; Van Heuvelen, 1991b).

A primary goal of these step-by-step problem-solving prescriptions is to get novices to start the problem with a qualitative, conceptual analysis, mimicking expert approaches. Some instructional approaches focus only on developing this initial analysis, helping students to develop good initial problem descriptions and analyses and to search for the underlying concepts of physics problems (Ding, Reay, Lee, & Bao, 2008; Leonard, Dufresne, & Mestre, 1996; Mualem & Eylon, 2010). Related instructional interventions aim to help students develop expert categorization of problems according to physics concepts (Docktor, Mestre, & Ross, 2012; Dufresne, Gerace, Hardiman, & Mestre, 1992).

How Can Conceptual Understanding Inform the Mathematical Processing of the Relevant Equations?

Ultimately, the focus of the expert problem-solving research is the leveraging of an initial qualitative analysis to select relevant physics concepts and associated mathematical equations. But while the physics concepts related to Newton's 2^{nd} law would be used to point to the relevance of the equation $F = ma$, that equation would typically then be treated as a tool for algorithmic manipulations. Although experts do algorithmically manipulate equations in problem solving, this is not the only way they use mathematical equations. In this section, I revisit my and Mark's approaches to the E&M problem, to show how Mark's expertise went beyond using his conceptual understanding to pick out relevant generalized physics equations. He generated, evaluated, and modified novel mathematical expressions in ways that reflect his conceptual understanding of this specific problem.

Mark leverages conceptual reasoning to inform the use of equations in ways that cannot be described as "equation selection."

The expert/novice paradigm of initial qualitative analyses and well-structured knowledge describes some aspects of the differences in my and Mark's approaches to the radiation problem. In my "novice" approach, I started with a means-end or backwards-chaining approach, searching for equations that would give the desired final quantity, seeing what quantities in those equations were known or unknown, and looking for methods to solve those unknowns. Mark's "expert" approach started with a diagram of the problem situation, a brief planning episode for identifying relevant equations, and forward chaining to the final solution.

However, in other ways, this description misses aspects of Mark's expertise in relating mathematics to his conceptual understanding of the problem. For Mark, it wasn't just that the concepts pointed to which equations should be manipulated. Rather, his conceptual understanding influenced the creation and manipulation of those equations. In translating the formal definition of the electric dipole integral into one that can be evaluated, one could have relied on the application of completely generalized, formal definitions, possibly leading to an expression like: $dQ = \rho(\vec{r})dV = \left(\frac{\delta\left(\theta - \frac{\pi}{2}\right)}{r}\delta(r-b)\ \lambda_0 sin\phi\right) r^2 sin\theta\ dr\ d\phi\ d\theta$. Instead, Mark relies on his conceptual understanding of how to represent "a small piece of charge." He draws a box and says that the charge in that box is $(\lambda_0 sin\phi)\ b\ d\phi$. Mark's conceptual understanding of *dQ* as representing a tiny piece of charge helps him to evaluate the integral using expressions generated from his conceptual understanding of the physical situation, rather than starting from formal definitions and completely generalizable methods that would require explicit integration over r, θ, and ϕ

Similarly, Mark tacks on the dipole moment rotation term $(-sin\omega t\ \hat{x} + cos\omega t\ \hat{y})$ at the end of the problem, based on his conceptual understanding of the physical situation. In the expert problem-solving procedures, checking the answer against a conceptual understanding of the physical situation is an explicit step. However, Mark, finding an inconsistency, does not begin a search for where an error was made, redoing the mathematics to arrive at a different solution. Instead, he simply adds onto his final expression based on what he understands the answer must be. Again, Mark is generating, modifying, and manipulating mathematical expressions based on his conceptual understanding of the situation, rather than only using his conceptual understanding to pick out relevant physics concepts and the corresponding general equations.

The expert/novice research in general has not attended to the particular ways in which Mark flexibly blends his conceptual understanding with the mathematics. In the expert/novice paradigm, mathematical processing of the equations is typically conceived of only as the algorithmic manipulation of equations. Yet, this is not a sufficient descriptor of Mark's expertise.

Beyond the algorithmic manipulation of mathematics in problem solving

In elementary mathematics, Wertheimer (1959) showed that when given problems of the type (283+283+283+283+283)/5, students would find the sum of the numerator (in this case, 1415) and then divide by the denominator to reach the correct answer, 283. In these examples, students' mathematical processing relied on mastery of algorithmic computations, whereas a conceptual understanding of the operations involved could lead to a quick, heuristic solution that does not require explicit computation. In this way, a conceptual understanding of the mathematics involved can affect mathematical processing.

Similarly, Redish and Smith (2008) suggest that a conceptual understanding of the physical equations can alter the processing of those equations. In describing physical problem solving as the modeling of the physical system in a mathematical representation, the processing of those mathematics, and the translation back into the physical system, they warn that "…because of the fact that the equations are physical rather than purely mathematical, the processing can be affected by physical interpretations" (Redish & Smith, 2008, p. 302). While the problem-solving literature commonly treats mathematical processing as a purely formal mathematical enterprise (i.e. once the equations and variables are defined from the physical system, the resulting processing only depends on algorithmic machinery that doesn't depend on the previous conceptual analysis), Redish and Smith suggest that a conceptual understanding of the equations can affect the mathematical processing. But, so far, *how* the conceptual understanding of equations can affect the mathematical processing of physics equations has not been attended to in the literature.

One likely candidate for how a conceptual understanding of equations can affect the mathematical processing of equations is through symbolic forms (Sherin, 2001). As an example, Sherin found that students asked to come up with an equation for the acceleration of an object in free-fall experiencing air resistance do not use canonical force principles to generate the correct equation, $a = -g + f(v)/m$, where $f(v)$ is the drag force. A solution that aligns with common conceptions of expert problem-solving expertise would begin with a qualitative analysis of the forces through a free body diagram, selecting Newton's 2nd law ($F=ma$) as the applicable physics principle, filling in the specifics for this situation ($-mg + f(v) = ma$), and then dividing both sides by m to arrive at the final equation.

However, that is not what Sherin's interviewees do. Instead, they start by describing the total acceleration as being the net effect of two competing accelerations: the "upward" acceleration ($f(v)/m$) and the "downward" acceleration (g). They immediately write out their final equation, without going through the steps of Newton's 2nd law. Although it's possible that the students simply did some mental arithmetic, Sherin instead models the generation of this equation as an expression of students' intuitive idea of the upward and downward acceleration being in opposition. Rather than following the canonical physics principles, Sherin describes their reasoning as incorporating what he termed the *Opposition* symbolic form: a cognitive element that ties the intuitive idea of "two entities in opposition" to a mathematical structure.

Similarly to the rest of the physics problem-solving research, symbolic forms are used to model how conceptual reasoning can be used to model new equations and how conceptual reasoning can be read out from mathematical solutions (Sherin, 2001). However, there has thus far not been research indicating how symbolic forms could be used to influence the mathematical processing step in problem solving, although they serve as a plausible candidate.

Dissertation chapter 3 – How does the blending of conceptual and formal mathematical reasoning support expert problem-solving approaches?

I have pointed out the dearth of studies in PER that seek to incorporate investigations of students' qualitative conceptual reasoning with investigations of how they understand formal mathematics. Although the problem-solving literature more commonly speaks about both, conceptual reasoning is commonly related to the equations only in their selection, not in their use.

Building on the construct of symbolic forms, chapter 3 of this dissertation deals with one way in which the *blending* of conceptual and formal mathematical reasoning is a component of expert problem-solving practice. Through a case study of how two students take different approaches to explaining an equation and using that same equation in solving a physics problem, I illustrate how students can, using symbolic forms, interpret equations as expressing a conceptual meaning, and how they can leverage symbolic forms for heuristic shortcut solutions in physics problem solving that avoid algorithmic manipulation.

A Role for Epistemology in Studying the Blending of Conceptual and Formal Mathematical Reasoning in Physics

Thus far, questions of how conceptual and formal mathematical reasoning are blended have been discussed in terms of how the two can come together in problem solving for processing the mathematical equations. However, looking beyond just the relevant content knowledge and reasoning in problem solving, other studies have shown that how students reason about mathematical equations in physics is connected to their views towards knowledge and learning, or *epistemologies*. For this reason, understanding students' blending of conceptual and formal mathematical reasoning is naturally aligned with understanding students' epistemologies towards physics.

What is epistemology (in physics)?

Students' epistemologies in science have been studied along various dimensions - for instance, whether science is the accumulation of a static set of facts of whether it is the continued testing and reevaluation of ideas (Carey & Smith, 1993; Roth & Roychoudhury, 1994; Smith, Maclin, Houghton, & Hennessey, 2000; Songer & Linn, 1991). Specifically in physics, Hammer (1994) identifies three dimensions of students' epistemologies: whether physics consists of disconnected pieces or a coherent system; whether the content of physics is facts, formulae, and procedures or concepts; and whether physics is learned by authority or from independent thinking.

Redish, Saul, and Steinberg (1998) built on Hammer's work by constructing the Maryland Physics Expectations (MPEX) survey, probing students' epistemologies on Hammer's three dimensions as well as others. With the MPEX, they found that traditional physics courses typically have a negative effect on students' epistemologies (i.e. moved students towards less "expert" views). As such, reform efforts in physics have designed courses targeting the development of students' epistemologies in addition to conceptual knowledge (Elby, 2001; Hammer & Elby, 2003; Redish & Hammer, 2009). More recent work on surveying students' epistemologies and studying the effects of reformed instruction on those epistemologies has built on this original MPEX study (Adams et al., 2006; Gray, Adams, Wieman, & Perkins, 2008; Perkins, Adams, Pollock, Finkelstein, & Wieman, 2005).

How students reason with mathematics is connected to particular epistemological stances

Students' epistemologies have been shown to be connected to how they reason with mathematics in physics problem solving. Hammer (1994) presented a case study of two students who make the same mistake in solving a problem; both Roger and Tony noticed the same issue with their solution to a problem asking them to solve for the acceleration of two blocks connected by a taut string. They both found that the accelerations of the two blocks were different and both noticed that this didn't make physics sense – two blocks connected by a taut string should move together. Tony saw this inconsistency and went back to his work, finding his error and eventually correcting it. On the other hand, Roger noticed the error, but, in the end, chose to rely on his calculations. Hammer (1994) described this difference in approaches as reflecting an epistemological difference: Tony, who tended to view physics as consisting of concepts rather than facts and formulas, interpreted equations as representing common sense ideas, whereas Roger, who tended to view physics as consisting of formulas – taking up coherence between these formulas and conceptual reasoning when apparent, but not as a general expectation – viewed mathematics as both more trustworthy than and independent from his common sense understanding.

Similarly, Gupta & Elby (2011) showed that, at least initially, a student, Jim, working with a particular equation that doesn't seem to match with his everyday intuition, takes an epistemological stance that formal physics equations need not align with his intuitive ideas and that equations should be trusted over everyday common sense. However, later, when he resolves the inconsistency between the equation and his common sense, he takes a different epistemological stance that values this agreement, illustrating the tight coupling between reasoning with formal mathematics and the associated epistemological stances.

Lising and Elby (2005) presented an example of a student who did not resolve inconsistencies between formal mathematical reasoning and intuitive reasoning. They argued that a deficit in knowledge or skills alone couldn't explain this, because the student was skilled at both formal mathematical reasoning and intuitive reasoning separately. They therefore modeled the student as having an *epistemological barrier* separating formal physics reasoning and intuitive reasoning.

Bing and Redish (2009) identified four different ways that students epistemologically frame what kinds of knowledge and activity are appropriate in problem solving, including "calculation" and "physical mapping onto mathematics." These frames are also connected to particular aspects of students' epistemologies, such as "trusting in calculations" and "symbolic representation characterizes some features of a physical system," respectively. They show that the different stances on what kinds of knowledge or activity are appropriate can lead to different ways of using the equations. For example, a student activating a "physical mapping" frame may find taking a partial derivative with respect to Planck's constant ($\hbar$), a universal constant, problematic whereas a student activating an "invoking authority" frame may not if the problem explicitly requires it (Bing & Redish, 2012).

In these studies, students' epistemologies are connected to whether or not they reason with mathematical equations in ways that incorporate or value coherence with conceptual or intuitive meaning. Some studies show this coherence between equations and conceptual meaning in how equations are explained (Hammer, 1994; Lising & Elby, 2005), others in how quantitative solutions in problem solving are compared to conceptual understanding, and how that understanding might inform modifications to the solution (Bing & Redish, 2009; Gupta & Elby, 2011; Hammer, 1994). However, in none of these studies is epistemology seen as consequential for influencing different ways in which students can interpret equations as expressing conceptual meaning or leverage that conceptual meaning in the mathematical processing of equations. This is a specific connection I investigate in this dissertation.

Dissertation chapter 4: How is the blending of conceptual and formal mathematical reasoning connected to epistemologies?

Chapter 4 aims to connect symbolic forms use to epistemological stances that value *coherence* between formal physics knowledge and everyday reasoning, or between physics equations and physics concepts. In an intuitive way, this connection makes sense, because symbolic forms are nothing more than a way to combine formal physics equations with conceptual or everyday reasoning. I aim to investigate this connection broadly across 13 interviews, as well as within the interview of one individual, Devon, in order to explain the shifting presence and absence of symbolic forms use in his reasoning with equations throughout an interview. I will explain this shifting in symbolic forms use as aligning with corresponding shifts in his epistemological stance with respect to this *coherence*.

Does the Disciplinary Context Matter for How Conceptual and Formal Mathematical Reasoning Interact?

A large part of this dissertation focuses on the nature of mathematical sense making in physics – specifically, how conceptual understanding can influence the use of formal mathematics. One question this raises is whether the type of reasoning investigated is specific to the domain of physics, or whether this reasoning is more generally used across domains. How can different disciplinary contexts affect this reasoning?

Transfer of knowledge across the disciplines

This question about whether this reasoning is particular to physics problem contexts or whether it is used across problem contexts is fundamentally related to the studies of transfer of knowledge. One definition of the phenomenon of transfer is "the ability to extend what has been learned in one context to new contexts" (Bransford, Brown, & Cocking, 1999, p. 74). So what supports or impedes transfer from one context to the next?

One widespread finding is that specific contextual details of different problem contexts may impede transfer. One approach to fostering transfer is to provide the same underlying problem structure in many different contexts (Anderson, Reder, & Simon, 1996; Singley & Anderson, 1989). Gick and Holyoak (1980, 1983) aim to foster transfer by providing multiple problem contexts to help individuals develop a generalized “convergence” schema. In a military problem context, an army wants to conquer a fortress, but the roads are mined such that a large army cannot go on the roads. The solution to this problem is to split the army into smaller parts and to send each part down a different road to converge at the fortress. In a radiation problem context, radiation beams can destroy a tumor, but sufficiently strong beams will also destroy the incident healthy tissue. The solution in this problem also depends on the “convergence” schema: by aiming multiple weaker beams at the tumor from many different paths, only the tumor will experience the combined strength of the beams. Studies have shown that individuals instructed in only one example initially will have difficulty in spontaneously extending this “convergence” solution to an analogous problem situated in a different domain (Duncker, 1945; Gick & Holyoak, 1980; Novick, 1988; Reed, Ernst, & Banerji, 1974).

The aim of this classical view on transfer – the application of the same problem-solving knowledge and approaches across analogous problems situated in different domains – is naturally aligned with questions of transfer across disciplinary contexts. However, there have been a limited number of studies of transfer between physics and another disciplinary context. Bassok and Holyoak (1989) investigated transfer between isomorphic algebra and physics problems. Students trained on the algebra problems transferred their learned problem-solving approaches to the physics problems more often than students trained on the physics problems transferred to the algebra problems. They conclude that the original physics learning situation encodes the physics surface features as essential conditions for applicability (Dziembowski & Newcombe, 2005) rather than background details, whereas the algebra learning condition presents the problem solving approach in a more general manner that better supports abstraction. This aligns with work suggesting that learning in the most abstract context provides the best chance for transfer (Sloutsky, Kaminski, & Heckler, 2005). Bassok (1990) also showed that isomorphic problems in math and physics containing similar types of variables – both intensive or extensive – better support transfer than when isomorphic problems contain variables of different types.

In contrast to these classical views of transfer, there are more modern views that attend to more than structural similarities in the problems and the application of generalized abstract problem schemata (Bransford & Schwartz, 1999; Engle, Lam, Meyer, & Nix, 2012; Engle, 2006; Greeno, Moore, & Smith, 1993; Hammer, Elby, Scherr, & Redish, 2005; Lobato, Rhodehamel, & Hohensee, 2012; Lobato, 2003). However, these views have thus far not been applied to studying transfer across disciplinary contexts. In this dissertation, I argue for views of transfer across disciplines that look beyond similarities in content and structure, drawing on these more modern views of transfer for studying student reasoning across disciplinary problem contexts.

Dissertation chapter 5 – How is the interaction between conceptual and formal mathematical reasoning connected to disciplinary contexts?

Chapter 5 studies how student reasoning with mathematics may be affected by different disciplinary problem contexts in ways not accounted for by the classical transfer focus on content and structure of the problems. In this chapter, I aim to add to the studies of transfer across physics and math, in particular showing the productiveness of tools from more modern transfer perspectives. Specifically, I will show how a student takes two different approaches to similarly structured problems set across two disciplinary problem contexts and argue that an explanation of this student's reasoning that attends only to the content and structure in the problems (as from a classical transfer perspective) is insufficient for explaining the differences in his reasoning.

Dissertation Overview

The goal of this dissertation is to contribute to an understanding of how students make sense of the equations they encounter in physics. By "making sense of equations," I mean how students connect these formal mathematical equations to intuitive understandings or conceptual understandings in physics. Additionally, I aim to connect students' reasoning with mathematical equations to other factors beyond the content of their reasoning, such as their epistemological stances, in order to understand the dynamics of how students' reasoning may differ depending on the particular situation in which they are engaged.

Chapter 2 presents a theoretical framework for modeling the phenomena of students' expressed reasoning and their expressed epistemological stances in terms of cognitive resources.

Chapter 3 asks the question "how can the blending of conceptual and formal mathematical reasoning be productively leveraged for physics problem solving?" Sherin provides a starting point with the notion of symbolic forms: cognitive resources that blend an intuitive, conceptual schema with the symbol template of a mathematical equation. However, the research on symbolic forms has not shown how students leverage this type of reasoning in physics problem solving with canonical physics equations. We present a case study of how Alex and Pat explain and solve a problem with a physics equation, $v = v_0 + at$. Whereas Alex treats the equation as a computational tool, Pat uses the *Base + Change* symbolic form to interpret an intuitive meaning in the equation. In solving the problem, this symbolic forms-based interpretation of the equation affords him a heuristic approach to the problem that avoids explicit computation. We argue that the opportunistic use of a symbolic forms-based interpretation to leverage the equation in a more conceptual solution, what we call *blended processing,* is a component of problem-solving expertise in physics.

Chapter 4 asks the question "does the absence of symbolic forms use imply a knowledge deficit?" That is, how can we explain why some individuals use symbolic forms and others don't? Is it because some individuals simply don't "know" symbolic forms? I argue that the absence of symbolic forms use need not indicate a knowledge deficit. Rather, it could be the presence of an epistemological stance that does not support symbolic forms use. I code a series of interviews for both symbolic forms-based reasoning and epistemological stances that either value *coherence* – specifically, between formal reasoning in physics and everyday intuition or conceptual physics ideas – or see them as *disconnected*. The coding illustrates the plausibility of symbolic forms use being connected to epistemological stances valuing this *coherence*. I also show how one interviewee, Devon, does not use symbolic forms to interpret a physics equation initially but does so later on an isomorphic equation relating to money in one's bank account, indicating no deficit in symbolic forms knowledge. I argue that shifts in the presence or absence of symbolic forms use in Devon's reasoning correspond to shifts in Devon's epistemological stance.

Chapter 5 asks the question "what affects how individuals reason with mathematics across disciplinary contexts?" In this study, I present the interview of a student, Will, in which he was asked to reason about infinite series approximations on two isomorphic problems presented in the different disciplinary contexts of physics and calculus. Will's reasoning on these two problems takes different approaches. Although it is possible to explain some of these differences by attending to differences in content or structure in the problems that could support different approaches on the problems, I argue that attending to factors beyond the content and structure of the problems, such as Will's different epistemological stances towards the two problems and his different senses of accountability towards recalling relevant knowledge on the two problems, can help in explaining the differences in Will's reasoning approaches to those problems.

Chapter 6 summarizes the main points of this dissertation, discusses the implications of this work, and lists possible future directions.

Chapter 2: Theoretical Framework

INTRODUCTION

This dissertation deals extensively with students' reasoning around conceptual and formal mathematical ideas as well as students' in-the-moment epistemological stances. In this section, I will detail aspects of both the phenomenon of expressed reasoning and expressed epistemological stances, as well as the theoretical tools for modeling this reasoning through cognitive resources.

REASONING & CONCEPTUAL RESOURCES

Relevant Aspects of Expressed Reasoning for This Dissertation

In this dissertation, one major phenomenon of study is student reasoning on physics problems. There are many ways in which to understand and categorize students' espoused reasoning. For the purposes of this study, focused on the blending of conceptual and formal mathematical reasoning, I attend to the differences in the following kinds of reasoning:

> *Conceptual reasoning*: This is typically qualitative reasoning. Reasoning of this type may be related to physical concepts learned in a physics course. It may also be tied to common sense or everyday experience from living in the world.
> *Formal mathematical reasoning*: Students sometimes rely on the algorithmic machinery of mathematics. Equations can be used for algorithmic manipulations or as tools for transforming a desired numerical value from a set of known values (commonly referred to as "plug-and-chug" calculations). The use of the word "formal" means that this type of reasoning leverages the definitions and algorithmic procedures of mathematics. Of course there can also be a conceptual aspect of mathematical reasoning. However, I group this under "conceptual reasoning" in order to distinguish it from the "formal mathematical reasoning" I describe here. As a simple example, division by 3 can conceptually mean splitting up a set of objects into 3 equal groups. This conceptual understanding can live separately from the algorithmic rules for long division.

Of course, this split between conceptual and formal mathematical reasoning is not the only way to categorize student reasoning, nor are these the only two types of reasoning that might be espoused. However, in investigating how students may blend formal mathematical reasoning and conceptual reasoning, these are my definitions for identifying these two different types of reasoning.

Additionally I don't mean for formal mathematical reasoning and conceptual reasoning to be mutually exclusive. In fact, the search for the blending of these two types of reasoning assumes that individuals' expressed reasoning will sometimes incorporate reasoning of both types. The question in this dissertation is how these two can be blended together with a very particular goal: to make progress in physics problem solving.

One Possible Theoretical Model: Unitary Concepts

The relevant question here is how to model the reasoning a student expresses in a particular moment. In order to understand my theoretical framework, first I present another as a contrasting case. One approach, which I do not adopt in this dissertation, has been to identify an individuals' reasoning as resulting from a network of stable concepts (Carey, 1986; Clement, 1983; Driver, 1981; Halloun & Hestenes, 1985). These concepts are often seen as intact representations of the expressed reasoning. In studies of novices, these concepts are commonly seen as incorrect, often described as naïve theories or misconceptions. For example, McCloskey (1983) found that students' incorrect predictions of motion clustered around some common incorrect answers. These incorrect predictions are consistent with a theory of impetus: that objects set in motion possess an "impetus" that, unmaintained, gradually dies away. Because of the commonalities in students' incorrect predictions, and because of the parallels to the historical scientific theory of impetus, these students are often seen as having a stable, naïve theory. The presumption is that these stable misconceptions lead to consistent, though incorrect, reasoning.

These types of models have been referred to as *unitary* (Hammer et al., 2005), because they suggest that a student's expressed reasoning in a moment depends solely on what concepts the student has, suggesting stability and consistency in an individual's reasoning around particular topics and situations. Correct reasoning means that individuals possess the correct concepts; incorrect reasoning means that they possess incorrect concepts. Changing these unitary frameworks can be difficult, requiring the replacement of old core concepts with new ones. This process involves generating dissatisfaction in some way with existing concepts, presenting understandable and plausible alternative concepts, and coming to see these alternatives as ultimately more productive (Posner, Strike, Hewson, & Gertzog, 1982). The conceptual system is then restructured, accommodating the new concepts by restructuring or removing the old ones. This process has been described as "conceptual conflict" between competing concepts (Hewson & Hewson, 1984; Nussbaum & Novick, 1982). Underlying this model is an assumption that there exists an incorrect core concept to be confronted, deposed, and replaced by a canonical alternative.

Manifold Conceptual Resources

By contrast, an alternative view of cognition models an individual's expressed reasoning as consisting of the activation and coordination of multiple pieces of knowledge. In this view, an expressed reasoning doesn't arise from a static network of concepts. Rather, reasoning consists of the activation of some subset of an individuals' knowledge. Unlike the unitary models of knowledge, expressing incorrect reasoning does not imply having only incorrect conceptual knowledge and not having the canonically correct knowledge. Rather, individuals are seen as possessing *manifold* conceptual resources, many of which can be applied to a particular situation, leading to different possible lines of reasoning. Analogously, a novice construction worker may pick, out of all the tools on their tool belt, the wrong tool for a job. In this case, it would then be incorrect to necessarily infer that this construction worker only possessed this particular tool or lacked the appropriate tool for this job. In the same way, a student's knowledge is multifaceted, and the expressed reasoning represents just one of those facets.

For example, diSessa (1993) found that students made predictions about motion using different interpretations of forces: either force as a mover (forces can lead to translational motion), force as a spinner (forces can lead to rotation), or force as a deflector (forces can change the path of moving objects). Although a physicist views these as special cases of more fundamental physical laws, diSessa theorizes that these three intuitions about force are *phenomenological primitives* (or *p-prims*) that are abstracted from experiences from living in the physical world. Rather than appealing to a single, generalized notion of forces and motion across all the problems, diSessa sees novices' ideas of forces as more fragmented such that drawing on different fragments at different times can lead to different predictions of the resulting motion.

To address an issue of clarity, I point out that the "conceptual" in "conceptual reasoning" is not the same as the one in "conceptual resources." Conceptual reasoning refers to reasoning about qualitative physics concepts or everyday, intuitive ideas. It is defined to be separate from reasoning with equations (although this reasoning can be blended with formal mathematical reasoning. However, formal mathematical reasoning does not necessarily indicate the presence of conceptual reasoning, and vice versa). This comes from PER's focus on qualitative questions that do not include mathematics. "Conceptual resources" refer to pieces of knowledge that individuals possess that relate to the content of their reasoning. Therefore, "conceptual resources" are so named to be distinct from other cognitive resources that may relate to epistemological, affective, or motivational factors, for instance. This use of the word "conceptual" aligns more with the literature on conceptual change, which studies how naive content knowledge transforms into expert content knowledge. Conceptual knowledge resources can therefore be used to model both expressed conceptual reasoning and expressed formal mathematical reasoning.

Expanding on research on manifold frameworks, work has been done to show how students draw on different pieces of their knowledge, switching between different lines of reasoning quickly over short periods of time (e.g. Barth-Cohen, 2012; Bing & Redish, 2012; Sherin, Krakowski, & Lee, 2012), to identify the existence of other conceptual resources in addition to p-prims – e.g. symbolic forms (Sherin, 2001) – and to develop theoretical models of how reasoning emerges from assembly and coordination of these resources (diSessa & Sherin, 1998; diSessa, 1993; Wagner, 2006).

In manifold models, instruction proceeds from two underlying assumptions: 1) incorrect reasoning does not always stem from stable, alternative concepts and 2) there exist productive resources in an individual's cognitive ecology that can be drawn on for productive reasoning. From these assumptions, the strategy is not to confront and replace concepts but to help students draw on pieces of their own knowledge that will lead to productive reasoning (D. E. Brown, 1994; Clement, Brown, & Zietsman, 1989; Elby, 2001; Hammer, 2000). For example, Brown (1994) draws on "bridging analogies" for helping students to see that a book on a table experiences a normal force. He finds that students have a strong intuition that pushing their hand down on a spring means that the spring also pushes up against their hand. Through a series of analogies, students are shown the relevance of this intuition for understanding the situation of a book on a table. Rather than confronting and replacing incorrect force concepts, Brown seeks out existing student ideas that will lead to canonical physics interpretations. In a manifold framework, this process can be described as activating productive conceptual resources for reasoning about forces (Hammer, 2000).

In these manifold models of cognition, developing conceptual expertise means adopting stable patterns of activation in a wide variety of contexts. One would expect that expert physicists would reason from a coherent model of Newton's laws in a variety of situations. In manifold models, this development of a stable pattern of reasoning corresponds to the development of stable patterns of conceptual resource activation, rather than the highly-context sensitive activation of different p-prims for understanding forces seen in novice reasoning (diSessa & Sherin, 1998; Wagner, 2006). The manifold perspective challenges perspectives that exclude variability in an individual's reasoning but does not challenge the goal of stability in developing expertise.

One Notable Type of Conceptual Resource: Symbolic Forms

In general, the underlying philosophy of the multifaceted nature of an individual's knowledge is more important than identifying the specific conceptual resources that underlie an expressed reasoning. For this reason, in the data analysis, I will focus more on understanding the variability in an individual's expressed reasoning than on identifying all of the specific conceptual resources in play. For instance, I am more interested in identifying a student's reasoning either as "plug-and-chug" calculations or as intuitive or physical conceptual reasoning than in specifying what conceptual resources are activated in executing this reasoning.

However, one particular type of conceptual resource is central to this study of how students blend conceptual and formal mathematical reasoning: symbolic forms (Sherin, 2001). Symbolic forms are conceptual resources that contain a conceptual schema, an intuitive idea expressible in words, and a symbol template, the form and structure of a mathematical equation. This particular conceptual resource is important, because it reflects one particular way of blending conceptual and formal mathematical reasoning. In analyzing the data presented in this thesis, I will strongly attend to symbolic forms use or symbolic forms-based reasoning.

Considering symbolic forms as conceptual resources means that the absence of symbolic forms-based reasoning in a particular moment does not necessarily imply that that individual does not possess the relevant symbolic forms. Rather, in a particular moment, individuals may activate different sets of resources related to mathematics: resources for algorithmic manipulations or resources such as symbolic forms for blending conceptual reasoning with equations, to name two possible sets. Instead of seeking to identify which students do and which students do not "know" symbolic forms, I seek to understand why, in some situations, students do not draw on the knowledge of symbolic forms they possess.

EPISTEMOLOGICAL STANCES & EPISTEMOLOGICAL RESOURCES

Relevant Aspects of Expressed Epistemological Stances

Analogous to expressed reasoning, students also provide evidence of their in-the-moment views on what it means to learn and what kinds of reasoning are appropriate in a situation. I refer to this phenomenon as their *epistemological stances* (Hammer & Elby, 2002). Just as I take an individual's conceptual reasoning to be one in-the-moment facet of that individual's multifaceted knowledge, expressed epistemological stances are similarly one facet of that individual's multifaceted epistemology.

These epistemological stances can be identified through explicit statements. For example, Hammer (1994) documented students making explicit comments on how they make sense of equations. Tony said "...everything we, [the teacher] gave us, were the kinds of things we already knew but had never actually formalized, if that makes any sense...and that's what should happen, and it's just a matter of putting common sense into equations" (Hammer, 1994, p. 170). This statement indicates an epistemological stance that the formal mathematical equations reflect common sense. Roger, on the other hand, says that the only way he could teach the formula $v = v0 + at$ to another student is to work through the derivation. Hammer asks whether common sense applies to the formulas in Roger's physics course, to which Roger replies, "In the most simplest way" (Hammer, 1994, p. 171). This, and other evidence, supports Hammer's interpretation that Roger sees a limited connection between common sense and physics equations. For Roger, it seems that the equations are not as obviously expressing common sense as they are for Tony. These explicit statements towards the connection between common sense and equations indicate the epistemological stances that Roger and Tony are taking towards equations in that moment.

Roger and Tony reflect some of the dimensions of epistemology in which I am interested. Specifically, aligned with my focus on conceptual reasoning and formal mathematical reasoning, I attend to epistemological stances related to what kinds of reasoning are appropriate in learning and understanding physics – specifically, I investigate the roles of conceptual reasoning and formal mathematical reasoning in an individual's epistemological stances. This is related to Hammer's epistemological dimensions of "formulas vs. concepts" and "pieces vs. coherence"

(Hammer, 1994). This is also related to the additional dimensions surveyed in the MPEX: “reality link” and “math link.”

Manifold Epistemological Resources vs. Unitary Epistemological Beliefs

Similar to debates about the nature of conceptual knowledge, more recent theories of individuals’ epistemologies have proposed epistemologies as consisting of manifold epistemological resources rather than unitary epistemological beliefs (Hammer et al., 2005; Hammer & Elby, 2002). Although many studies have found consistency in individuals’ epistemologies (King & Kitchener, 1994; Perry, 1970; Schommer, 1990) and have modeled epistemological development as movement through stable stages (Carey & Smith, 1993; King & Kitchener, 1994; Smith et al., 2000), other studies have shown variability in what epistemological stances individuals can take (Bing & Redish, 2009; Gupta & Elby, 2011; Hammer, 1994; Louca, Elby, Hammer, & Kagey, 2004; Rosenberg, Hammer, & Phelan, 2006). These quick shifts cannot be explained by unitary epistemological beliefs. For example, Gupta and Elby (2011) present a student, Jim, who starts off expressing an epistemological stance that mathematics does not always cohere with his everyday thinking and that when mathematics and everyday intuition disagree, he would trust the mathematics. However, after working through a resolution for seeing an equation as coherent with his everyday intuitions, where there was initial disagreement, he expresses a different epistemological stance – one that values the coherence between the mathematical and everyday thinking. A unitary model of epistemological beliefs cannot describe this quick shift between these two different epistemological stances.

Connecting Conceptual and Epistemological Resources: Epistemological Framing

Other studies have shown that particular epistemological stances are connected to student learning and/or reasoning in science (Gupta & Elby, 2011; Lising & Elby, 2005; Perkins et al., 2005; Rosenberg et al., 2006; Songer & Linn, 1991; Windschitl & Andre, 1998) and math (Mason, 2003; Muis, 2004; Schoenfeld, 1985; Schommer, Crouse, & Rhodes, 1992). Therefore, developing students’ epistemologies also likely supports conceptual learning. In a common sense way, it makes sense that if an individual sees a certain kind of reasoning or approach as appropriate for a particular situation, then they’re likely to take on reasoning or approaches coherent with these views.

The process of epistemological framing (Hammer et al., 2005) models this connection at the level of conceptual and epistemological resources. The theory of epistemological framing posits that, in a situation, a coherent set of conceptual and epistemological resources are assembled and activated in response to the question: “what kinds of knowledge or reasoning is appropriate here?” The coherence between the conceptual and epistemological resources activated explains the coherence between an individual’s in-the-moment reasoning and epistemological stance. For example, Hammer et al. (2005) describe two students working on the following problem in physics class: if you are standing on a scale in an elevator moving at a constant speed, which forces would change magnitude if the elevator started accelerating downwards? Tracy starts by recording all the known quantities, apparently expecting to have to do a calculation. Sandy, on the other hand, questions this underlying premise, asking "do we even need to do all that calculation?" Sandy counters with a physical interpretation to the problem: the elevator pulls away from the person and so the person would have to catch up to it. The differences in Tracy's and Sandy's approaches relate to the differences in what kind of reasoning they believe the problem requires. In this way, we can understand Sandy's reasoning, for example, not just as a coordination of conceptual resources related to understanding forces and accelerating bodies, but also involving epistemological resources related to about what kinds of knowledge and approaches are appropriate here.

In a model of individuals' knowledge consisting of manifold cognitive resources - and therefore being multifaceted - the corresponding expressed reasoning and epistemological stances must also represent only one possible facet of the expressed lines of reasoning and epistemological stances that are available. Some work has shown that shifting between these different facets can be accomplished through shifts in either the conceptual or epistemological resources that are activated (Bing & Redish, 2012; Gupta & Elby, 2011; Louca et al., 2004; Rosenberg et al., 2006). As Gupta and Elby showed with Jim, a shift in Jim's reasoning with an equation corresponded to a shift in Jim's epistemological stance. At the level of the epistemological framing of specific cognitive resources, in order to maintain coherence between the conceptual and epistemological resources in play, a shift in the activated conceptual resources to ones connecting formal mathematical reasoning and conceptual reasoning must correspond to a similar shift to activate epistemological resources valuing coherence between formal mathematics and everyday intuition.

Conversely, Rosenberg et al. (2006) showed that epistemological shifts could lead to conceptual shifts. Sixth graders, creating a model for the rock cycle, start by trying to recall information learned in class. The teacher briefly intervenes, saying she wants students to start from their own ideas, rather than trying to recall ones from class. In what follows, the students enter a different mode of reasoning, using their everyday ideas to generate a story of how rocks transform over time. By simply shifting students' in-the-moment epistemological stances about what kinds of knowledge are productive in the rock cycle activity, a corresponding shift occurs in students' reasoning, reasoning that may not have otherwise been apparent. In this case, a shift in the active epistemological resources to ones supporting "storytelling" corresponds to a shift in the activated conceptual resources for generating parts of that story.

Jim in Gupta and Elby's study reveals important types of conceptual and epistemological resources that are relevant to the study of how conceptual and formal mathematical reasoning are blended together. As described previously, especially relevant to the analysis in this dissertation is the identification of conceptual reasoning and formal mathematical reasoning, as well as the identification of how the two are blended together or remain distinct. Along with the conceptual resources activated, epistemological framing requires coherence between these conceptual resources and the active epistemological resources.

THE RICHNESS OF INDIVIDUALS' REASONING

Briefly, I want to say a few words about relevant features of knowledge beyond conceptual and epistemological resources. My attention to these two specific aspects of knowledge and the related phenomena is not meant to imply that those are the *only* two aspects that matter in studying reasoning and learning. Other factors such as motivation (Pintrich, Marx, & Boyle, 1993), identity (Boaler & Greeno, 2000), and metacognition (Schoenfeld, 1987), to name three, have been shown to have an effect on the substance of individuals' reasoning as well. My focus on the connection between an individuals' reasoning and their epistemological stances is not meant to downplay or deny the importance of other factors. It is simply a pragmatic decision to focus on one particularly apparent aspect of the data.

One important theoretical and methodological issue is that this study focuses on how expressed reasoning corresponds to one particular facet of an individuals' knowledge rather than on how that expressed reasoning develops out of an interaction between individuals, physical artifacts, and other contextual features of the activity system (Carraher, Carraher, & Schliemann, 1985; Hutchins, 1995; Lave, 1988). In this matter, I align with Cobb (1994) in seeing these two types of perspectives as complimentary rather than dichotomous. In my theoretical framework, I acknowledge that the expressed reasoning in a situation develops out of a system of activity in which an individual is only one component.

In this dissertation, I foreground the cognitive resources and background the mechanics of how the activation of these resources is resultant from an interaction with people, artifacts, contexts, etc. One reason is that this dissertation is theoretically concerned with uncovering particular facets of students' reasoning related to blending conceptual and formal mathematical reasoning – specifically:

1) What does reasoning that blends conceptual and formal mathematical reasoning look like, and to what conceptual resources does this reasoning correspond?
2) Does the absence of such blended reasoning necessarily correspond to a knowledge deficit?
3) How is the activation of these conceptual resources connected to the activation of particular epistemological resources?

Although studying the situated nature of the activation of these conceptual resources is a valuable goal, I do not attend deeply to these issues here, in order to make this dissertation tractable. The identification of these resources and the conceptual and epistemological dynamics in which they are involved is a first step to understanding blended conceptual and formal mathematical reasoning.

Chapter 3: How Students Blend Conceptual and Formal Mathematical Reasoning in Solving Physics Problems[1]

INTRODUCTION

The science education literature on quantitative problem solving emphasizes the importance of incorporating conceptual reasoning in two phases of problem solving: (1) initial qualitative analysis of the problem situation to determine the relevant mathematical equations and (2) interpretation of the final mathematical answer, to check for physical meaning and plausibility (P. Heller et al., 1992; Redish & Smith, 2008; Reif, 2008). Without disputing the importance of these phases of problem solving, we note that almost no research has focused on the "mathematical processing" stage where the equations are used to obtain a solution. In this paper, we investigate different ways that students can process equations while problem solving. We argue that a feature of problem solving expertise — and a feasible instructional target in physics, chemistry, and engineering courses — is blended processing, the opportunistic blending of formal mathematical and conceptual reasoning (Fauconnier & Turner, 2003; Sherin, 2001) during the mathematical processing stage. In other words, we argue that expert problem solving involves exploiting opportunities to use conceptual reasoning in order to facilitate the manipulation of equations themselves.

To make our case, we first review how physics education researchers have conceptualized and taught quantitative problem solving. We then discuss research suggesting the importance of blending conceptual reasoning with symbolic manipulations in quantitative problem solving, and we propose *symbolic forms* (Sherin, 2001) as cognitive resources that facilitate such *blended processing*. Then we use contrasting case studies of two students solving a physics problem, to illustrate what we mean by blended processing. Alex solves the problem by representing the physical situation with a diagram, identifying the relevant physics equations, using those equations to compute a numerical answer, and reflecting upon that answer — in accord with problem-solving procedures taught in physics classrooms (e.g. Giancoli, 2008; Young & Freedman, 2003) and advocated in education research (P. Heller et al., 1992; Huffman, 1997; Reif, 2008; Van Heuvelen, 1991b). Pat, by contrast, blends symbolic equations with conceptual reasoning about physical processes to find a "shortcut" solution. After analyzing Alex's and Pat's responses in detail, we show that some other introductory physics students in our data corpus also do the type of blended processing done by Pat. In documenting what such blended processing can look like for undergraduate students in an introductory physics course, we make the case that (1) the opportunistic use of blended processing is part of quantitative problem-solving expertise; (2) a theoretical construct called *symbolic forms* (Sherin, 2001) contributes to a good cognitive account of Pat's (and the Pat-like students') blended processing; and (3) such blended processing is a feasible instructional target in science and engineering courses.

[1] This chapter is published in *Science Education* (Kuo, Hull, Gupta, & Elby, 2013).

LITERATURE REVIEW: CONCEPTUALIZATIONS OF EXPERT PROBLEM SOLVING

In this section, we present a common conceptualization of expertise in quantitative physics problem solving, as well as challenges to a particular aspect of that conceptualization. We limit our discussion to *quantitative* problem solving, because our argument specifically concerns the processing of equations in problem solving.

Research on Expert Problem Solving and Resulting Instructional Strategies Emphasize an Initial Conceptual Reasoning Phase

As a central feature of their professional practice, scientists apply domain-specific knowledge to solve quantitative problems (Redish & Smith, 2008; Reif & Heller, 1982; Reif, 2008). Partly for this reason, developing problem-solving expertise in students has become a central concern of science education researchers and practitioners (Hsu et al., 2004; Maloney, 1994, 2011).

Early research on physics problem solving suggests a difference between experts and novices. Experts tend to start with a conceptual analysis of the physical scenario, which then leads into the mathematics. By contrast, novices tend to start by selecting and manipulating equations that include relevant known and unknown quantities (Larkin et al., 1980; Simon & Simon, 1978). Specifically, on standard textbook physics problems, experts cue into relevant physics principles whereas novices cue into surface features and their related equations (Chi et al., 1981). Building on these findings, subsequent research has explored the benefits of helping students analyze the problem situation conceptually (Dufresne et al., 1992; Larkin & Reif, 1979) and has incorporated initial conceptual thinking into models of effective quantitative problem solving (J. I. Heller & Reif, 1984; Reif & Heller, 1982).

This research on expert-novice differences has also influenced researchers' formulation of multi-step problem-solving procedures intended for students to learn and apply (P. Heller et al., 1992; Huffman, 1997; Reif, 2008; Van Heuvelen, 1991a, 1991b). These procedures generally include versions of the following steps: (1) perform an initial conceptual analysis using relevant physics principles; (2) use this qualitative analysis to generate the relevant mathematical equations; (3) use equations to obtain a mathematical solution in a "mathematical processing" step; and (4) interpret that mathematical solution in terms of the physical scenario. These procedures incorporate the expert-novice findings by encouraging students to reason conceptually before jumping into mathematical manipulations.

In these strategies, the steps are meant to mirror behaviors exhibited by experts while also remaining accessible enough to be instructional targets. The explicit teaching and enforcement (through grading policies) of these problem-solving procedures has increased the quality and frequency of physical representations used in problem solving, as well as the correctness of students' answers, in comparison to traditional instruction (P. Heller et al., 1992; Huffman, 1997; Van Heuvelen, 1991b).

Studies of Quantitative Problem Solving Have Not Focused on How Equations Are Processed to Reach Solutions

The studies described above illustrate a common feature of research on students' quantitative problem solving: to the extent these studies focus on equations, they focus on how students *select* equations rather than on how students *use* those equations after their selection. While this focus has produced important findings and implications for instruction (for example, emphasizing initial conceptual reasoning for selecting relevant equations), it has also limited attention to how students process mathematical equations to obtain numerical or symbolic solutions.

In some research, the equations are treated (either explicitly or implicitly) as computational tools, devices to find unknown values from known values through symbolic and numeric manipulation. This is true of the problem-solving procedures described above (P. Heller et al., 1992; Huffman, 1997; Van Heuvelen, 1991b) and of studies about how successful problem solvers use mathematics (e.g. Dhillon, 1998; Taasoobshirazi & Glynn, 2009).

Other more recent studies have not attended to any aspect of how equations are processed. Walsh, Howard & Bowe (2007) focused mainly on how students *selected* relevant equations rather than on how those equations are subsequently used. Some studies have deemphasized the use of math completely and focused only on students' qualitative analysis, both in instructional interventions (e.g. Mualem & Eylon, 2010) and in finding predictors of problem-solving expertise (e.g. Shin, Jonassen, & McGee, 2003).

The first author (Kuo) did a search through *Science Education, Journal of Research in Science Teaching*, *Research in Science Education, International Journal of Science Education, American Journal of Physics*, *The Journal of Engineering Education*, *Cognition & Instruction*, and *Journal of the Learning Sciences* from January 2000 to March 2012 and *Physical Review Special Topics – Physics Education Research* from July 2005 (its inception) to March 2012. He looked for articles focusing explicitly on problem solving in which the analysis attended to the possibility of processing equations in multiple ways. First, the titles of all articles were scanned, and abstracts of articles with titles containing terms such as "problem solving" or "equations" were read. If the abstract described investigations of components of problem-solving expertise, the article itself was read. This search found no studies that focused upon the mathematical processing step in quantitative problem solving or described alternatives to using equations as computational tools.

Other Studies Suggest the Importance of Blending Conceptual Reasoning with Symbolic Manipulations

We have shown that research on quantitative problem solving has not attended to different ways that mathematical equations can be used to obtain numerical or symbolic solutions. This paper presents two different ways that such equations may be used: (i) as computational tools, manipulated to solve for unknown quantities, or

(ii) blended with conceptual meaning to produce solutions (or progress toward solutions). But why is this difference significant?

Other pockets of research suggest that using equations without looking to their conceptual meaning *during the processing* can, in certain situations, reflect a *lack* of expertise. In mathematics education research, for example, Wertheimer (1959) asked students to solve problems of the following type: (815+815+815+815+815)/5 = ?. Students who solved the problem by computing the sum in the numerator and then dividing by 5 missed a possible shortcut around explicit computation: using the underlying conceptual meanings of addition and division to realize that the solution is 815, without doing any computations. Students who missed the shortcut had demonstrated proficiency with the mathematical procedures, but not understanding of the underlying conceptual meaning. Additionally, Arcavi (1994) suggested the importance of *symbol sense*: an ability to reason conceptually about symbols. This *symbol sense* includes the ability to interpret the conceptual meaning behind symbolic relationships, generate expressions from intuitive and conceptual understanding, and decide when and how best to exploit one's conceptual understanding of symbols.

Redish and Smith (2008), writing about expert problem solving in science and engineering, also challenged the view that symbolic manipulation should be *a priori* divorced from conceptual reasoning, saying "…because of the fact that the equations are physical rather than purely mathematical, the processing can be affected by physical interpretations" (Redish & Smith, 2008, p. 302). Just as Wertheimer showed that students' conceptual understanding of *mathematical* operations influenced how they carry out calculations in an arithmetic problem, Redish and Smith suggest that students' interpretations of equations in terms of the physical scenario can influence how they use the equations in solving physics/engineering problems.

Again, we do not dispute the instructional value of prior research on problem solving procedures that emphasize conceptual reasoning at the start and the end of problem solving. As noted above, an instructional emphasis on such procedures has helped students to produce more and better representations and to produce correct solutions more frequently. However, Wertheimer, Arcavi, and Redish and Smith suggest the importance of focusing on *how* students process equations in their quantitative problem solving, a focus not present in the quantitative problem solving literature. Specifically, these researchers argue that blending conceptual reasoning with mathematical formalism in the processing of equations — what we refer to as *blended processing* — may be productive and reflect greater expertise in some situations than using an equation simply as a computational tool. In the next section, we discuss the use of *symbolic forms,* which we argue is one specific way that blended processing can occur in physics problem solving.

Symbolic Forms: A Blend of Conceptual Reasoning and Mathematical Formalism

In arguing that problem solving does not necessarily proceed from direct application of canonical physics principles, Sherin (2001) proposed the existence of knowledge structures called *symbolic forms,* which link mathematical equations to intuitive conceptual ideas. Specifically, in a *symbolic form*, a *symbol template* is blended with a *conceptual schema*.

A *symbol template* represents the general structure of a mathematical expression without specifying the values or variables. For example, $\square = \square$ is the symbol template for Newton's 2nd law ($F = ma$), while the symbol template for the first law of thermodynamics, $\Delta E = Q + W$, is $\square = \square + \square$. Each symbol template is not unique to a single equation. For instance, the symbol template $\square + \square + \square$ can describe both the expression $x_0 + v_0t + 1/2\, at^2$ and the expression $P_0 + 1/2\rho v^2 + \rho gh$.

A *conceptual schema* is an intuitive idea or meaning that *can* be (but does not *have* to be) represented in a mathematical equation or expression. By "intuitive" ideas, we mean ideas that are informal and drawn from everyday (non-academic) knowledge - ideas that make quick and immediate sense and that do not seem to require further explanation. One example of such a conceptual schema is the idea that *a whole consists of many parts*. For example, an automobile can be seen as an assembled whole of many parts such as the engine, the transmission, and the chassis; a wedding guest list can be conceptualized as consisting of the close relatives, the close friends, business contacts, and others; an essay might be viewed as the compilation of the introduction, the main body of argument, and the conclusion. Similarly, a physics student's conceptual understanding of the total mechanical energy of a system may be grounded in the idea that it is comprised of many different types of energy: kinetic, gravitational potential, spring potential, and so on.

Another conceptual schema, applicable to reasoning about a game of tug-of-war or about a marriage between a spendthrift and a miser, is the idea of *opposing influences*. In physics, this conceptual schema may apply to a student's conceptual understanding of a falling object, where air resistance opposes the influence of gravity (Sherin, 2001).

As these examples illustrate, a conceptual schema in Sherin's framework is an *intuitive* idea used in everyday, nonscientific reasoning, not a formal scientific concept. A student's understanding of a formal scientific concept (such as mechanical energy) can draw upon these intuitive conceptual schemata (such as *whole consists of many parts*); but the conceptual schema also plays a role in students' reasoning about other subjects, such as wedding guest lists.

A *symbolic form* is a cognitive element that blends a symbol template with a conceptual schema, such that the equation is interpreted as expressing meaning corresponding to the conceptual schema. For example, the *parts-of-a-whole* symbolic form blends the symbol template "$\square = \square + \square + \square + \ldots$" with the conceptual schema of *a whole consisting of many parts*. The box on the left side of the equation takes on the meaning of "whole" and the boxes on the right side take on the meaning of "parts." A student who uses the *parts-of-a-whole symbolic form* to interpret the equation $E = ½mv^2 + mgh + ½kx^2$ would say that the overall (whole) energy of the system consists of the sum of three separate parts (kinetic, gravitational potential, and spring potential). So, when a symbolic form is used, the reasoning is neither purely formal mathematical nor purely conceptual; it is blended into a unified way of thinking that leverages *both* intuitive conceptual reasoning and mathematical formalism. By contrast, a writer thinking about her essay might think of the parts of her essay (Introduction, etc.) using the conceptual schema corresponding to *parts-of-*

a-whole, but is unlikely to think of adding those parts in an equation. Only when a conceptual schema is blended with an equation's symbol template is a symbolic form present in the person's reasoning.

Sherin (2001) observed students productively using symbolic forms in two ways. One was to produce novel equations from an intuitive conceptual understanding of a physical situation. For example, figuring out how much rain would hit them in a rainstorm, a pair of students wrote the equation [*total rain*] = [*#raindrops/s*] + *C*. Their explanation of this equation reflected the use of *parts-of-a-whole.* Specifically, they said the total rain would come from two sources: the amount falling on top of the person, indicated by [*#raindrops/s*], and the amount striking the front of the person as they walk forward, indicated by *C*. Other research has also supported the explanatory power of symbolic forms in models of how students translate physical understandings into mathematical equations (Hestenes, 2010; Izsák, 2004; Tuminaro & Redish, 2007).

The other way in which Sherin's subjects used symbolic forms was to interpret mathematical equations in terms of a physical scenario, using functional relations expressed by the equation. For example, after deriving the terminal velocity of a falling object, $v = mg/k$, several students noticed that the mass, m, was in the numerator. Students interpreted this as meaning that a heavier object reaches a greater terminal velocity. Sherin modeled these students as using the *prop+* (positive proportionality) symbolic form — a blend of the symbol template […x… / …....] with the conceptual schema that one quantity increases as another one increases— to read out a physical dependence from the mathematical equation. Later, Sherin (2006) hypothesized that *prop+* was tied to physical notions of effort and agency, what we see as "cause-and-effect." Other researchers have also used symbolic forms to model how students translate from mathematical solutions into physical understanding (Hestenes, 2010; Tuminaro & Redish, 2007; VanLehn & van de Sande, 2009).

These two ways in which Sherin (2001) saw students use symbolic forms correspond roughly to the two "steps" involving conceptual reasoning, as described by the problem solving literature: (i) translating conceptual understanding of a physical scenario into mathematical equation(s) at the start of problem solving and (ii) giving a physical interpretation of a mathematical solution at the end[2]. We have seen no studies that look at how symbolic forms-based reasoning — or blended conceptual and formal mathematical reasoning more generally — might enter into the "mathematical processing" step in quantitative problem-solving.

Building on the Literature to Explore the Mathematical Processing Step

We contribute to the literature on quantitative problem solving in three ways. First, we focus on different ways in which mathematical equations can be used to

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[2] We do not mean to imply, however, that completing these steps necessarily or even commonly involves symbolic forms-based reasoning. Using an *informal* conceptual schema to *invent* an equation differs from common instantiations of the "conceptual reasoning" step, in which students generally use *formal* concepts to *select* equations. And giving a physical interpretation of a mathematical answer can involve attaching physical significance to a number rather than attaching meaning to a functional relation between variables.

reach a solution. This is a relatively unexplored topic, as discussed above. We find that equations can be used as tools for symbolic or numerical manipulations or as tools in blended processing. As we argue, the opportunistic use of such blended processing can reflect greater expertise than symbolic or numerical computation (Arcavi, 1994; Redish & Smith, 2008; Wertheimer, 1959).

Second, we argue that symbolic forms help to explain the patterns we document below in students' problem solving and are therefore productive analytical tools for researchers trying to understand the nature of blended processing.

Third, we argue that symbolic forms are also plausible targets for instruction in introductory physics, partly because they rely on intuitive rather than formal (or discipline-specific) reasoning. Students can therefore productively use symbolic forms *while* they are still learning difficult physics concepts.

METHODS AND DATA COLLECTION

Overview and Research Questions

We began with a broad, ill-defined question, "*how do students make sense (or not) of the mathematics they use in solving physics problems?*", accompanied by our own intuitions about how blended processing might play a role (as discussed below). This initial question, along with our intuitions, led to the design of an interview protocol. Looking at early interview data, we felt that students' responses to the first two prompts (described below) fell into two broad categories: ones that blended intuitive reasoning with the formal mathematical equations in a particular way and ones that took an algorithmic approach to using the equations. This sharpened our focus to specific research questions around how and when students blend intuitive and formal mathematical ideas: (1) *On the Two Balls Problem (described below), how can we characterize the differences between a solution that uses blended processing and a solution that does not*, and (2) *How can we describe the formal and informal knowledge that students bring to bear when using blended processing to address this problem?* We use Alex's and Pat's responses to explore these research questions.

Interview Context

Our data set consists of videotaped interviews with 13 students enrolled in a first-semester, calculus-based, introductory physics course at a large, public university in the United States. The course, geared toward engineering majors, covers mechanics. These students were interviewed between fall 2008 and spring 2011, and our recruitment did not target any particular demographic. The interviews lasted about one hour.

The two subjects on whom we focused our analysis, "Alex" and "Pat" (pseudonyms), were interviewed one and a half months into the course in fall 2008. By that time, the course had covered kinematics, including objects falling under the influence of gravity, which is the topic of the interview prompts discussed below. We chose Alex and Pat for fine-grained analysis because they were among the first students we interviewed and because the strong differences between their responses motivated us to seek an explanation for those differences.

Interview Protocols

We designed the semi-structured interviews to probe engineering majors' approaches to using equations while solving quantitative physics problems. Specifically, we wanted to explore what formal mathematical and conceptual tools they bring to bear and which epistemological stances they take toward the knowledge they use. Moreover, we designed some prompts specifically to probe whether and how students use blended processing when the opportunity arises. To that end, we had students think aloud while solving specific problems. We also asked them to explain the meaning of both familiar and unfamiliar equations and to discuss more generally how they know when they "understand" an equation. The complete protocol is online (http://hdl.handle.net/1903/12947). Our analysis in this paper focuses on the first two prompts in the interviews.

> *Prompt 1: Explain the velocity equation*
> The interviewer shows the student the equation $v = v_0 + at$ and asks, *Here's an equation you've probably seen in physics class. How would you explain this equation to a friend from class?*
>
> *Prompt 2: Two Balls Problem*
> *(a) Suppose you are standing with two tennis balls on the balcony of a fourth floor apartment. You throw one ball down with an initial speed of 2 meters per second; at the same moment, you just let go of the other ball, i.e., just let it fall. I would like you to think aloud while figuring out what is the difference in the speed of the two balls after 5 seconds – is it less than, more than, or equal to 2 meters per second? (Acceleration due to gravity is 10 m/s^2.)* [If the student brings it up, the interviewer says to neglect air resistance][3]
> (b) [Only if student solved part (a) by doing numerical calculations] *Could you have solved that without explicitly calculating the final values?*

In designing the Two Balls Problem, we anticipated that some students would find something like the following shortcut: according to the velocity equation, $v = v_0 + at$, since both balls *gain* the same amount of speed over 5 seconds (with the gain given by the mathematical term at), the final difference in speeds equals the initial difference in speeds, in this case 2 meters per second. Note that the shortcut uses blended processing, interpreting the equation conceptually as "the final velocity is the initial velocity plus the change in velocity." We wanted to see if students would notice this shortcut solution or something like it, either on their own or in response to

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[3] Through doing the interviews, we realized that for any reasonable fourth floor balcony, the two balls would hit the ground before 5 seconds. However, this did not present a significant problem as few students mentioned this issue in their solution, and the ones that did proceeded to solve the problem as if the balls would not hit the ground.

the follow-up prompt about whether the problem could have been solved without numerical calculations.

Although this problem is not as complex or difficult as some textbook introductory physics problems, it is still a problem that we would want students to be able to solve; reform-oriented physics classes often ask questions such as this (e.g. Redish & Hammer, 2009). Moreover, we believe that this problem's conceptual shortcut afforded a way to investigate differences in how students process mathematical equations in problem solving; we suspected that blended processing would contribute to finding the shortcut. Although blended processing is not explicitly sought in most textbook physics problems, the *opportunistic* use of such reasoning when it is possible is an aspect of problem-solving expertise, as argued above. In asking the Two Balls Problem, we hoped to elicit data on whether and how students perform blended processing.

Analysis Phase 1: Alex and Pat

We began with fine-grained, qualitative analysis of Alex's and Pat's approaches to the Two Balls Problem and their explanations of the velocity equation, $v = v_0 + at$. Our goal was to characterize how the two subjects were conceptualizing the equation and its role in problem solving.

Phase 1a: Two Balls Problem (prompt 2)

To start, we looked at video and corresponding transcript of the two subjects' responses to the Two Balls Problem (prompt 2). In trying to characterize how they were thinking about and using the velocity equation, we analyzed the solutions they were speaking and writing while thinking aloud. However, other markers of their thinking, such as word choice, pauses, and speech rhythm also informed our interpretations. Although these markers are discussed by the discourse and framing analysis literature, (Gee, 1999; Tannen, 1993), we do not claim to be doing discourse or framing analysis.

After formulating possible characterizations, we performed a line-by-line analysis through this particular section of the transcript (response to prompt 2) for confirmatory and/or disconfirmatory evidence. In this way we refined and narrowed down the plausible characterizations of how Alex and Pat were thinking about the equation (Miles & Huberman, 1994).

Phase 1b: the velocity equation (prompt 1)

With Alex and Pat, we had reached tentative consensus in phase 1a about how they were thinking about and using the velocity equation in solving the Two Balls Problem (though with Pat, we lacked evidence to decide between two subtly different interpretations). We then analyzed their responses to prompt 1, which asks how they would explain the velocity equation to a friend, in order to look for confirmatory or disconfirmatory evidence for our phase 1a interpretation. Using the same analytical tools as in phase 1a, we tried to characterize how they were thinking about the equation in the context of explaining it. We then compared our characterizations to what we had found in phase 1a. As discussed below, the alignment was strong. By providing detailed analysis below, and also the complete transcript (at http://hdl.handle.net/1903/12947), we give readers the opportunity to check if we matched up our characterizations in phase 1a and 1b when it was not warranted.

Analysis Phase 2: Looking at Other Students

Although the literature has not documented Pat-like blended processing of equations, our experiences as physics instructors and our subsequent interviews led us to think his reasoning might not be idiosyncratic. Furthermore, while the presence of blended processing distinguishes Pat's reasoning from Alex's, other students might not fall so cleanly on one side of this distinction or the other.

To explore these issues, we analyzed the responses of the 11 other students we interviewed (besides Alex and Pat). Specifically, the first and second author independently coded whether each student (1) used blended processing to find a shortcut solution to the Two Balls Problem, either initially or in response to our follow-up prompt asking if the problem could be solved without plugging in numbers; and (2) gave an explanation of the velocity equation that combined the symbol template with a conceptual schema — i.e., a symbolic forms-based explanation. The two independent coders initially agreed on 9 of 11 codes for code 1 and on all 11 codes for code 2.

Because the number of students in this study is small, we are not aiming to make statistically significant claims about patterns of reasoning. We use these data instead to bolster our arguments that blended processing is a useful analytical lens for understanding students' reasoning and a feasible instructional target.

RESULTS OF ANALYSIS PHASE 1a: THE TWO BALLS PROBLEM

How Alex Used the Velocity Equation While Solving the Two Balls Problem

Alex solved the problem with a numerical calculation

Alex started by drawing a diagram of the two balls and labeling their speeds (figure 3.1). After deciding to use the velocity equation to solve this problem, Alex paused and remarked that she did not have a value for *a* for the equation, $v = v_0 + at$. She realized that *a* should be 9.8 and wrote this value in her diagram. The interviewer interjected and said that she could use 10 if she wanted, and Alex responded that using 10 is probably easier. She then explicitly solved for the velocities of the thrown and dropped balls after five seconds and wrote down the difference. Figure 3.1 shows all of her work[4].

[4] We note that Alex incorrectly labels the acceleration of the dropped ball with units of m/s instead of m/s2. However, we do not focus on this mistake because it does not propagate into her manipulation of the equations, which is the focus of our analysis.

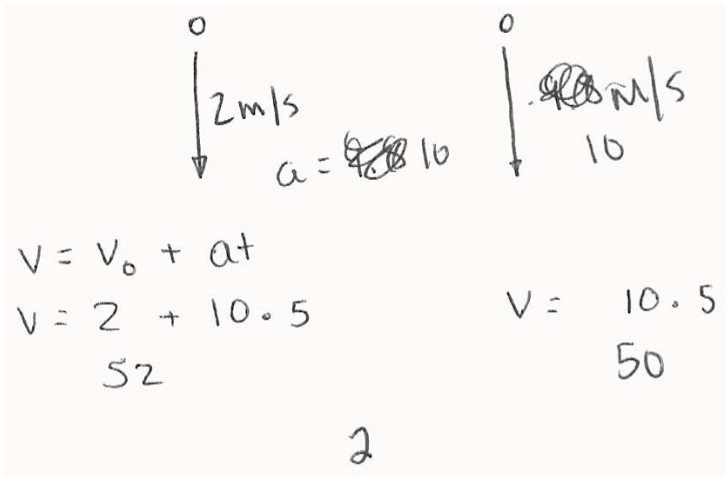

Figure 3.1: Alex's written work on the Two Balls Problem

After working out the speeds of the two balls to be 50 m/s and 52 m/s, Alex explained her thought process:

> A31 Alex: …Ok, so after I plug this into the velocity equation, I use the acceleration and the initial velocity that's given, multiply the acceleration by the time that we're looking at, five seconds, and then once I know the velocities after five seconds of each of them, I subtract one from the other and get two. So the question asks "is it more than, less than, or equal to two?" so I would say equal to two.

("A31" refers to the 31st conversational turn in the interview with Alex.)

In this segment, Alex followed a set of steps similar to the ones advocated in research-based problem-solving strategies: draw a picture of the situation (which can include labeling known values), choose the equation relevant to the physical situation, and calculate the desired unknowns to answer the question. Alex executed this procedure smoothly, with the only pause coming when she considered the value of the acceleration. We tentatively conclude that, during this segment, Alex used the velocity equation as a tool for numerical computation.

Alex exhibited hints of conceptual reasoning but not stably integrated with the velocity equation

After Alex gave her solution to the Two Balls Problem, the interviewer asked if someone could have answered the question without explicitly solving for the velocities of both balls. We designed this follow-up prompt to get at whether students (i) might have implicitly used blended processing in a way that their think-aloud and written solution did not reveal, or (ii) might get cued into blended processing by this prompt.

Alex's first answer was yes, but when asked to elaborate, she seemed unsure:

> A35 Alex: Well, I'd have to think about it, since you're dropping one and throwing one. If you're, I mean I guess if you think you're throwing one

2 m/s and the other has 0 velocity since you're just dropping it, its only accelerating due to gravity, you can just say that since you know one is going at 2 m/s, it's going to get there 2 m/s faster, so 5 seconds faster, it would get there 2 seconds, er, it's going 2 m/s faster, I guess.

A36 Interviewer: OK. So, they would say that you threw one, so this was getting 2 m/s faster. So what happens 5 seconds later?

A37 Alex: Uh, it's going, uh, I don't know. (laughs)

As the interviewer followed up, Alex continued to sound less and less sure about her answer. Finally, she changed her mind:

A51 Interviewer: So you're saying that they need not have actually plugged in the numbers? Is that what I'm hearing?

A52 Alex: No, I think you'd have to plug in the numbers because, uh, I mean you just would to be sure. I guess you, I don't think you can just guess about it.

In line A35, Alex attempted to show how the Two Balls Problem can be answered without an explicit calculation. One interpretation of this exchange is that she never came up with a firm conceptual explanation for how to solve the problem without calculations, as evidenced by her mixing up the units ("...so 5 seconds faster, it would get there 2 seconds, err, it's going 2 meters per second faster..."). A different interpretation is that she was trying to express the following conceptual argument: since both balls are accelerating due to gravity only, both balls will gain the same amount of speed, so the thrown ball will be traveling "2 meters per second faster" (line A35). Either way, she backed off this line of reasoning in lines A37 and A51, possibly because she felt on the spot trying to answer the interviewer's questions (evidenced by her greater hesitancy than when she presented her original solution). Our point is that, either way, any conceptual reasoning in line A35 was not *stably* integrated with Alex's mathematical, symbolic reasoning. Evidence for this lack of stable integration comes from (*i*) the lack of explicit mention of the equation or implicit reliance on its structure in line A35, and (*ii*) her view in line A51 of the calculation as a way to "be sure" of non-calculation-based reasoning, which is more of a "guess" than something reliably connected to the calculation in some way.

In summary, the procedural way in which Alex solved the problem, along with the lack of a stable connection between the equation and conceptual reasoning evidenced by her follow-up comments, points us toward the conclusion that Alex, in this context, was viewing the velocity equation as a computational tool for calculating a final velocity given an initial velocity, an acceleration, and a time. We will put this initial interpretation to the test below when we analyze how Alex explained the velocity equation. But first, to emphasize the contrast between Alex and Pat, we present Pat's solution to the Two Balls Problem, which he solved without calculating the final velocity of either ball.

How Pat Used the Velocity Equation While Solving the Two Balls Problem

<u>Pat solved the problem without plugging in numbers</u>

Like Alex, Pat also turned to the velocity equation. However, he used it very differently from the way Alex did:

> P41 Pat: …Well, the first thing I would think of is the equations. The velocity, I suppose, is the same equation as that other one [the velocity equation he had just explained in prompt 1], and I'm trying to think of calculus as well and what the differences do. So the acceleration is a constant and that means that velocity is linearly related to time and they're both at the same, so the first difference is the same. I *think* it's equal to two meters per second.

Later, when asked by the interviewer how he got this answer, Pat elaborated on his solution a little more:

> P45 Pat: So the first differences are the same.
> P46 Interviewer: Mhm.
> P47 Pat: And if the first differences are the same then the initial difference between the two speeds should not change.

When asked, Pat explained that the term "first differences" comes from his high school algebra class, where sets of data points would be analyzed by taking "delta y over delta x," which is called the "first difference." So, "first difference" connects at least roughly to the notion of slope.

A few moments later, Pat stated that "there's a couple of methods of attacking" the problem if he gets stuck. Pat then further discussed different ways to solve the Two Balls Problem:

> P61 Pat: So if I started from thinking about the equations and I'm not quite sure whether the velocities are changing at the same rate, then like sometimes I'll use several [solution methods] and see if they're consistent. Then I could switch to thinking about the derivatives of the velocity and I'll think, ok, so the initial conditions are off by 2 and then the velocities are changing at the same rate so that should mean they stay at 2…

Pat did not follow a set of steps similar Alex's. Instead, his solution is a shortcut around an explicit calculation: since the velocities of the two balls change at the same rate, the difference between those two velocities stays the same. Notably, the velocity equation ($v = v_0 + at$) plays a role in his shortcut, but his reasoning is not purely symbolic. Pat started his solution (line P41) by referring to the velocity equation, but he used it to point out that "velocity is linearly related to time" which led him to say that the "first difference" is the same. Since "first difference" is similar to the idea of slope, this aligns with his reasoning in line P61, where he offered a similar argument in terms of derivatives and explicitly stated that the velocities are changing at the same rate.

We use these case studies of Alex's and Pat's reasoning to make two points. First, although both Alex and Pat reasoned productively and correctly solved the Two Balls Problem, Pat's style of reasoning is not currently described in the problem solving literature. Illustrating this productive reasoning can help expand our understanding of how students approach similar problems.

Second, we view Pat's reasoning as aligning better with expert problem solving than Alex's more procedural approach does. Pat saw multiple solution paths, which he related to one another, while Alex saw just one. Pat flexibly used the available information, which is a component of what Hatano & Inagaki (1986) call "adaptive expertise," while Alex's approach appeared more step-by-step. Also, Pat connected *conceptual meaning* to mathematical formalism: the idea that if two things change at the same rate, then the difference between them stays the same. As Wertheimer (1959), Arcavi (1994), and Redish & Smith (2008) argue, such blended processing indicates a deeper, more expert understanding than simply using the formalism. Other researchers also emphasize the deeply connected nature of expert knowledge (Chi et al., 1981, 1982; Reif & Heller, 1982; Reif, 2008), though they do not explicitly discuss connections between informal conceptual knowledge and mathematical formalism. Linking conceptual reasoning to mathematical formalism as Pat does — using blended processing of the velocity equation — is arguably an example of forging or exploiting such connections. These connections can support quick and robust solutions through the flexible coordination of multiple strands of reasoning, as Pat illustrates.

Although we have hypothesized that Pat was reasoning by connecting a mathematical equation to an intuitive conceptual schema (i.e., "if two things change by the same amount, the difference between them stays the same"), we see at least one plausible alternative account. It is possible that Pat's reasoning was driven by a formal rule of mathematical operations and objects, such as "if the derivative/first difference of two quantities is equal, then the difference between them doesn't change." Our phase 1b analysis of Pat's explanation of the velocity equation will help distinguish between these possibilities, in the end favoring our initial interpretation that Pat was blending conceptual and symbolic reasoning.

RESULTS OF ANALYSIS PHASE 1b: EXPLAINING THE VELOCITY EQUATION

So far, we have tentatively concluded that in the context of the Two Balls Problem, Alex viewed the equation as a computational tool while Pat was more flexible, reasoning with the equation to find a conceptual shortcut. To find confirmatory or disconfirmatory evidence, we now look to Alex's and Pat's responses to prompt 1, which asked how they would explain the velocity equation to a friend. We show how the absence of symbolic forms-based reasoning in Alex's explanation and the presence of such reasoning in Pat's explanation help us understand the differences in their approaches to the Two Balls Problem.

Alex Explains the Velocity Equation as a Computational Tool

Alex initially seemed puzzled by this question but eventually answered:

> A10 Alex: Umm, Ok, well, umm, I guess, first of all, well, it's the equation for velocity. Umm, well, I would, I would tell them that it's uh, I mean, it's the integral of acceleration, the derivative of {*furrows brow*} position, right? So, that's how they could figure it out, I don't know. I don't really {*laughs*}, I'm not too sure what else I would say about it. You can find the velocity. Like, I guess it's interesting because you can find the velocity at any time if you have the initial velocity, the acceleration, and time…

Alex's explanation here has two main parts. First, the velocity equation is defined through its relation to other kinematic equations; it is the integral of acceleration and the derivative of the position equation. Second, the equation can be used as a computational tool: to calculate the velocity at some time if you know the other values in the equation.

The interviewer then asked if that is what she would have said on an exam. She said "no" and elaborated:

> A14 Alex: Um, well, it depends on what it was asking, 'cause I feel like your question's kind of vague, but, I mean, I would probably just say 'it's the velocity equation' {*nods and laughs*}. I mean, if it was a more specific question, I could probably like, elaborate, I guess.

Finally, Alex was asked to explain the equation to a 12-year old who knows math but does not really know physics.

> A16 Alex: Well, these two sums will tell you how fast something is going. If you know how fast it's going when it first starts and after it first starts moving and you know its speed when it first starts moving, and you know a certain point in time. You're looking at a certain point in time at which the object is moving, and you know how fast it's changing its speed, you can find how fast it's moving at that time, or you can find out the acceleration from it if you know how fast it's going at that time.

Unlike her response to a friend from class or on a test, Alex now explicitly described the mathematical variables in terms of physical ideas: "if you know *how fast it's going when it first starts* [v_0]...and you know a *certain point in time* [t]... *and you know how fast it's changing its speed* [a]…*you can find out... how fast it's moving at that time* [v]."

Yet, even in light of this conceptual interpretation of the variables, there is evidence that Alex's explanation of the equation as a whole is still as a computational tool, as in line A10: if you know any three variables, you can solve for the fourth. In line A16, Alex explained that you can solve for *v* if you know v_0, *a*, and *t*. Alternatively, she stated, you can solve for *a* if you know *v*. This interpretation is coherent with how she actually used the velocity equation to compute the final velocities in the Two Balls Problem.

Although Alex attached physical meaning to the individual variables in the velocity equation, Alex's explanation does not include a symbolic forms-based interpretation of the velocity equation *as a whole*. There is no evidence of an intuitive conceptual schema deeply associated with the symbol template reflecting the structure of the velocity equation (☐ = ☐ + Δ). The following analysis of Pat's explanation of the equation will provide a contrasting case to clarify what evidence we use to make claims about the presence of a symbolic form in a student's reasoning.

Pat Connected the Equation to a Physical Process

When asked to explain the equation to a friend from class, Pat started by looking at the units and meaning of the variables:

> P2 Pat: Well, I think the first thing you'd need to go over would be the definitions of each variable and what each one means, and I guess to get the intuition part, I'm not quite sure if I would start with dimensional analysis or try to explain each term before that. Because I mean if you look at it from the unit side, it's clear that acceleration times time is a velocity, but it might be easier if you think about, you start from an initial velocity and then the acceleration for a certain period of time increases that or decreases that velocity.

Pat started with the definitions of each variable, as Alex described in line A16; however, he then provided some preliminary evidence of interpreting the equation in terms of a symbolic form we now introduce, called *Base + Change* (Sherin, 2001, p.514). In *Base + Change*, the symbol template ☐ = ☐ + Δ is linked to the intuitive conceptual schema that the *final amount is the initial amount plus the change in that amount.* For example, a careful consumer might have this symbolic form active in her reasoning while balancing her checkbook.

In line P2, Pat provided glimmers of evidence that he was relying upon a *Base + Change* interpretation of the equation as a whole rather than just interpreting the individual variables. First, he signaled a shift away from discussing the meaning of the individual variables when he referred to getting to "the intuition part." Then, by doing dimensional analysis of the term *at*, he indicated that he is starting to think of *at* not only as a product of two different quantities, but as a single term with the same units as v_0, which hints at reasoning at the level of the symbol template ☐ = ☐ + Δ. Finally, he transitioned from talking about the individual terms in the equation into an overall "story" of a physical process that the equation represents: "*...you start from* an initial velocity *and then* the acceleration for a certain period of time *increases or decreases* that velocity" (emphasis ours). Part of the evidence for this transition is a shift in Pat's narrative perspective. Up until this point, Pat had been speaking from the perspective of a person working with the equation: how “you” or “I” would use or explain the equation. At the transition, the “you” shifted to an object or person that starts with an initial velocity and then undergoes a change in that velocity. This shift in perspective suggests a shift in meaning, from his previous ideas about the definition of variables and dimensional analysis to something else.

This “something else” for Pat relied upon the conceptual schema associated with the *Base + Change* symbolic form: *the final amount* (in this case, final velocity) *is the initial amount* (initial velocity) *plus the change to that initial amount* (due to acceleration).” Pat’s reliance on this conceptual schema would be solid evidence of symbolic forms-based reasoning except that it is not yet absolutely clear whether Pat was connecting this conceptual schema to the symbol template of the equation.

After Pat discussed how v_0 or a can take on positive and negative values, the interviewer asked Pat what he meant by “the intuition part” in line P2:

P9 Interviewer: So right when you started you said something about “well, then from the intuitive side.”
P10 Pat: Yeah, the problem is dimensions are just numbers really, or units, and it doesn’t really explain what’s going on in the motion.
…
P15 Interviewer: Ok, so how would you explain it intuitively?
P16 Pat: I would say that an acceleration is the change in velocity, so you start from the velocity you have in the beginning and you find out how the acceleration affects that velocity. Then that would be the significance of each term.

The first sentence in line P16 reiterates the “story” in line P2. It is the second sentence, “[t]hen that would be the significance of each term,” that provides solid evidence for the presence of the *Base + Change* symbolic form in Pat’s reasoning. Whereas before we could not be sure the idea of “final equals initial plus change” was connected to the symbol template, here we interpret Pat’s explanation as saying that “the velocity you have in the beginning” and “how the acceleration affects that velocity” correspond to the terms v_0 and *at*, respectively. This corresponds to a *Base + Change* symbolic form interpretation, where v_0 is the *base* velocity and *at* is the c*hange* to that base velocity.

This explanation contrasts with Alex’s. She attached conceptual meaning only to the individual variables in the equation. Here, Pat interpreted the *whole equation* as representing a process of starting with a base amount and changing that base by some value to obtain the final amount.

This analysis supports and refines our analysis from phase 1a, where we interpreted Pat’s reasoning as blending conceptual reasoning with formal mathematics. In phase 1a, we highlighted an alternative possibility: that Pat’s reasoning was driven by a *formal* rule rather than an *intuitive* schema rooted in a conceptual understanding of the physical process. Here, however, we see that Pat viewed the velocity equation as expressing an intuitive conceptual schema connected to his conceptual understanding of the physical process of speeding up, and specifically that his reasoning is rooted in the *Base + Change* symbolic form. Pat’s initial solution to the Two Balls Problem, as discussed above, came from thinking of the formal equation (lines P41 and P61) and relies on the conceptual reasoning that since the *initial* difference in velocities is 2 meters per second, and because both balls undergo the same *change* in velocity, the *final* difference in velocities is still 2 meters per second. Although it is unclear whether he was applying *Base + Change* to the individual velocities of each ball or directly to the difference in those velocities, he was connecting the equation — its general structure and its linearity — to the intuitive schema of "final equals initial plus (linear) change." Pat's explicit explanation of the velocity equation with *Base + Change* provides confirmatory evidence of blended processing in his solution to the Two Balls Problem.[5]

Summary of Differences Between Pat's and Alex's Views of the Velocity Equation

Looking across the first two prompts in the interviews ("explain the velocity equation" and "solve the Two Balls Problem"), we see a key difference in how Alex and Pat connected their conceptual understanding of a physical situation to an equation. Alex's connection was at the level of individual variables, while Pat additionally saw the equation as-a-whole expressing an intuitive conceptual idea about the physical process: the velocity you start with plus the velocity you gain (or lose) due to acceleration over a certain period of time is the velocity you end up with. We have also argued that this difference between Alex's and Pat's conceptualizations of the equation offers explanatory power for many of the differences in their responses to those two prompts, including Alex's explicit calculation and Pat's blended processing.

We are not claiming that Alex does not "have" the *Base + Change* symbolic form, or symbolic forms in general, in her repertoire of cognitive resources. Indeed, she showed evidence of blending symbolic and conceptual reasoning later in the interview. We are only claiming that, for whatever reasons, the *Base + Change* symbolic form was not tied to the velocity equation in Alex's responses to the first two prompts.

Additionally, our point is not only the specific form of Pat's blended processing, but also that Pat productively blended conceptual reasoning with mathematical formalism *at all* to inform his use of equations during problem solving. Throughout the interview, Pat blended conceptual and formal mathematical reasoning in ways that we sometimes, but not always, identify as reflecting the use of a symbolic form.

[5] It is possible that Pat's symbolic forms-based interpretation of the velocity equation was not active in his reasoning when he solved the Two Balls Problem. Indeed, several of our interviewees did not reason consistently across the two prompts. However, the coherence of Pat's reasoning across the two prompts suggests, though does not prove, that he was interpreting the velocity in the same way in both segments of the interview.

RESULTS OF ANALYSIS PHASE 2: LOOKING AT OTHER STUDENTS

To see if Pat represented an idiosyncratic case and if the presence or absence of blending conceptual and formal mathematical reasoning is a useful distinction more broadly, we coded the remaining 11 interviews (leaving out Alex and Pat). Specifically, we coded whether each student (1) used blended processing to find a shortcut solution to the Two Balls Problem, either initially or in response to our follow-up prompt asking if the problem could be solved without plugging in numbers; and (2) gave an explanation of the velocity equation that combined the symbol template with a conceptual schema — i.e., a symbolic forms-based explanation.

Other Examples of Blended Processing

Through this coding, we showed that Pat's blended reasoning is not completely idiosyncratic. Six of the eleven other students showed evidence of either a symbolic forms-based explanation of the velocity equation or a blended processing shortcut on the Two Balls Problem. Since we find few examples of such reasoning in the literature, we now present examples from those six students. First, we present two examples of blended processing on the Two Balls Problem, starting with Meg:

> [After Meg answers the Two Balls Problem simply using the velocity equation to compute the solution through symbol manipulation, the interviewer asks her if she was surprised by that answer.]
> Meg: I expected it to be two, because I just I remember something, you know, that if the acceleration will be the same, gravitational acceleration is the same, so what's, in my mind I just reason out that, you know, if one has more of the speed than the other, because the change is the same. So then it's still going to be, the difference is still going to be the same. If you're changing both by the same amount, then in the end one is going to have the same amount more [that it does initially] than the other.

Here, Meg used the idea that because the balls are both "changing [speed]...by the same amount," the "difference [in speeds] is still going to be the same" as it was initially.

Next, we present Sam's reasoning:

> Sam: There's no force acting on them after they have gone, they, one just has initial speed, so we know that, you know, the acceleration, um, you know, multiplied by, you know, they're both in there for, both um subjected to gravity [laughs] for five seconds, so they will both have the, you know, same accelera-, they're exposed to the same acceleration for the same amount of time, which would give, you know, an additional velocity, um, of the same. Um, however, you know, we could have this equation, you know. v_0, well one, we have an initial velocity of two, and one, we have an initial velocity of zero, so we know that one already is going to be experiencing, you know, faster, er well it will be going faster for two seconds, just before we even start.

Even though at the end Sam stated that one ball will be "going faster for two seconds," his previous utterances suggest that he meant that the thrown ball will be traveling two meters per second faster at five seconds. His argument, though perhaps harder to follow than Meg's, is substantively similar: Because both balls are "exposed to the same acceleration for the same amount of time," they both get "an additional velocity" that is "the same." Therefore, since for "one [ball], we have an initial velocity of two, and [for the other ball], we have an initial velocity of zero," the difference in speeds remains two. Supporting this blended-processing interpretation of Sam's explanation is his follow-up remark after plugging through the equations:

> Sam: You know, even before doing any math, like I said before, we already know that this is going to be, well it's going to be, not faster, but a quantity of two meters per second more regardless of what the acceleration is or time.

So, in both Meg's and Sam's reasoning, we see similarities to the blended processing described in Pat's solution: since both change by the same amount, the final difference equals the initial difference.

Next, we present two students' explanations of the velocity equation that we view as using the *Base + Change* symbolic form. We start with Meg:

> [Meg uses the example of a falling object to explain the velocity equation.]
> Meg: So the final velocity, the velocity that it hits the ground with is related to the initial velocity because the object has an initial velocity and if you think about it, if the object is moving and it's constantly changing velocity…You can start off with the initial velocity and then you multiply the change in velocity with time, how much time it took and that should theoretically give you how much the velocity has changed, correct? So if you, so then if you have the initial velocity and you have how much the velocity changed, um, and you add those two together that should, in theory, give you the final velocity.
>
> Stan: …Acceleration is a rate of change in speed and *t* is the time. *at*, like the whole thing, is what you changed in a period, so I'll say that *v initial* is the speed that you already had, plus the speed that you changed is the speed that you have right now.

In these two representative cases, the students interpreted the *at* term in the equation as the change in the initial velocity, which when added to the initial velocity gives the final velocity – which is typical of how the *Base + Change* symbolic form is applied to the velocity equation (Sherin, 2001, p.514)

Is Blended Processing a More Broadly Applicable Distinction?

On the Two Balls Problem, three students used blended processing, seven did not, and one was ambiguous. In explaining the velocity equation: six students used symbolic forms-based reasoning and five did not[6].

As mentioned above, each of two researchers independently produced 22 codes (two codes each for 11 students). They agreed on 20 of the codes. One of these disagreements was quickly resolved through discussion. The other disagreement, on one student's response to the Two Balls Problem, was deemed "ambiguous," because the evidence could be used to both support and refute the presence of blended processing. This agreement between independent coders and the small number of uncodable responses (one out of 22) suggest that the presence/absence of blended conceptual and formal mathematical reasoning is a meaningful distinction for other students in our study and plausibly, for larger populations as well.

DISCUSSION: REVISITING WHAT COUNTS AS PROBLEM-SOLVING EXPERTISE

We now argue that Pat's (and other students') productive use of blended processing gives us reason to amend the common view of what constitutes good problem solving in physics, and hence, of what strategies instructors should nurture in their students.

Most Research-based Problem-Solving Strategies Do Not Include Blended Processing

As discussed in the literature review, research on expert problem solving and instructional problem-solving procedures for fostering such expertise have focused on the finding that novices jump right into manipulating equations without a physical understanding of the problem. Without disputing the importance of research building on this finding, we note that the problem-solving procedures studied and advocated by researchers and instructors do not consider different ways that equations can be used to solve a problem. These procedures typically conceive of the mathematical processing step as manipulating symbols until you obtain an unknown (Giancoli, 2008; P. Heller et al., 1992; Huffman, 1997; Van Heuvelen, 1991a, 1991b; Young & Freedman, 2003). However, blending symbolic manipulations with conceptual reasoning when possible is a part of problem-solving expertise, because such

[6] Although our sample size was not large enough to do a correlational study, readers may be interested to know how many students, like Pat, both answered the Two Balls Problem with blended processing and explained the velocity equation with symbolic forms-based reasoning; and likewise, how many students answered both questions as Alex did. Of the 10 other unambiguously coded students, three students solved the Two Balls Problem with blended processing. All three of these students also gave a symbolic forms-based explanation of the velocity equation to a friend. Of the seven students who did not solve the Two Balls Problem through blended processing, five did not give a symbolic forms-based explanation of the velocity equation, while two did. Future work might sample larger populations to generalize a connection between a Base + Change symbolic forms-based explanation of the velocity equation and blended processing on the Two Balls Problem.

reasoning is adaptive and flexible and because it leads to quicker, more generalizable solutions.

Our contribution to this line of research is to illustrate in some detail this alternative way of using equations in problem solving. We contrasted students who did and did not incorporate conceptual reasoning into their mathematical processing. Through these qualitative case studies, we made the case that blended processing was demonstrably productive for the students who used it in solving the Two Balls Problem: they reached the solution quickly, without having to do extensive calculations.

Moreover, for the physics education research community, which has emphasized the importance of conceptual understanding (Elby et al., 2007; Hestenes et al., 1992; McDermott, 1991), the blended processing in students' solutions to the Two Balls Problem highlights an additional way in which conceptual reasoning can enter quantitative problem solving: *between* deciding which equations to use and evaluating the final answer. For these reasons, we urge both researchers and instructors to revisit the standard problem-solving procedures discussed above. Perhaps, instead of encouraging students (implicitly or explicitly) to treat the processing of equations as symbol manipulation, we should help students learn to spot and exploit opportunities where blended processing can help them find shortcuts and gain a deeper understanding of the physical meaning of their solutions.

Two Examples of Symbolic Forms as an Instructional Target

Our analysis also helps us refine our vague instructional suggestion to "teach students to blend conceptual and symbolic reasoning" into something more concrete. Symbolic forms-based reasoning, we argue, is a productive and feasible target of instruction, and techniques for helping students engage in such reasoning exist. Some researchers (such as Redish and Hammer (2009)) discuss instructional strategies intended to help students see the conceptual ideas in equations, yet they provide no specific examples. To build on this literature, we provide two concrete examples from our teaching.

The first example comes from a large lecture, undergraduate, introductory physics course. The instructors (the third author, Gupta, and another colleague[7]) use a modified treatment of a classic question in order to emphasize the symbolic forms-based reasoning implicit in the usual explanation. The question is this: A ranger aims a tranquilizer gun directly at a monkey, who is hanging from a branch. At the moment the gun is fired, the monkey drops from the branch. Will the tranquilizer dart hit the monkey?

The answer is that the dart hits the monkey, even though the monkey drops. Here is the usual reasoning. First, imagine the same scenario but with no gravitational force: The dart travels in a straight line and hits the motionless monkey. Now, consider how "turning on" gravitational force modifies this scenario. While the dart travels to the monkey, the monkey falls a certain distance; but during this time, the dart also falls by the *same* distance below that straight line — below where it

[7] This instructional modification is courtesy of David Hammer, the other instructor for the course.

would have been if there was no gravitational force. Usually, instructors use this scenario to illustrate the independence of the horizontal and vertical components of motion; the vertical distance the monkey falls equals the vertical distance the dart "falls" below where it would have been in the absence of gravity, because the dart's horizontal motion does not affect the vertical displacement to its motion caused by the gravitational force.

Building on this conceptual insight, the instructor highlights the *Base + Change* symbolic form that is implicit in this reasoning. Specifically, the instructor emphasizes that the height of either the monkey or the dart at time *T* (when the dart reaches the monkey) can be written as the equation in Fig. 3.2. The instructor discusses how this equation "says" that the *final* height equals the *base* height (in the absence of gravitational force) minus the *change* in height due to gravitational force. Since the *base* heights are the same for the monkey and dart (i.e., the dart would hit the monkey in the absence of gravitational force), and because the height of each object *changes* by the same amount due to the gravitational force, $-(1/2)gT^2$, the final heights are the same for each. So, the dart hits the monkey. In this way, the instructor illustrates the *Base + Change* symbolic form and shows how it affords a calculation-free shortcut highlighting the connection to the underlying physical processes in this scenario.

$$\begin{aligned} Height &= \begin{bmatrix} \text{Height where the object would} \\ \text{have been with no gravity} \end{bmatrix} - [\text{Distance it falls due to gravity}] \\ &= \begin{bmatrix} \text{Height where the object would} \\ \text{have been with no gravity} \end{bmatrix} - \frac{1}{2}gT^2 \end{aligned}$$

Figure 3.2: The equation describing the height of the dart and of the monkey.

Our second instructional example comes from the algebra-based introductory physics course described by Redish & Hammer (2009). In the discussion sections of this course, students engage in small-group collaborative learning using "tutorials," which are guided-inquiry worksheets (Elby et al., 2007). The Momentum Tutorial leads students to figure out the formula for "oomph," which turns out to be the physics concept of momentum, using their intuitive ideas of motion. First, students consider a rock and pebble thrown at the same speed. The tutorial asks, "Which one has more oomph?" It goes on to ask if the rock is twice as massive as the pebble, "*intuitively*, how does the rock's oomph compare to the pebble's oomph?" Students generally have no trouble formulating the idea that the more massive the rock is compared to the pebble, the greater its "oomph" as compared to the pebble's, sometimes expressing their ideas qualitatively, sometimes in more mathematical terms (e.g., "proportional").

Next, the tutorial asks students to consider two identical bowling balls rolling at different speeds. If the faster ball is exactly 7 times as fast as the slower one, "intuitively, how does the faster ball's oomph compare to the slower ball's oomph?" Again, most students find it obvious that the faster ball has more oomph and that 7 times faster probably corresponds to 7 times as much oomph.

Finally, this section of the tutorial asks,

> The physics concept corresponding to oomph is momentum. Building on your above answers, figure out a formula for momentum (oomph) in terms of mass and velocity. Explain how the formula expresses your intuitions from... above.

We view this tutorial as helping students construct and/or use the *Prop+* (positive proportionality) symbolic form (Sherin, 2001, p. 533), which combines the "direct proportionality" symbol template with the intuitive conceptual schema of one quantity increasing if another one does through a cause-and-effect relationship.

These instructional examples echo prior research suggesting that symbolic forms-based reasoning is something that can be scaffolded in instruction. For instance, Izsák (2004) documented eighth graders using *Base + Change* to construct a mathematical equation after interacting in a rich learning environment for several hours. Almost all of Sherin's undergraduate subjects displayed evidence of using symbolic forms (Sherin, 2001, 2006). Tuminaro and Redish (2007), studying students in a physics course for undergraduate life science and non-science majors, documented the use of symbolic forms-based reasoning during problem solving. For these reasons, we believe that some students' failure to blend conceptual and symbolic reasoning during mathematical processing reflects not a lack of ability, but a lack of scaffolding. As discussed above, the standard problem-solving procedures taught in textbooks and endorsed by researchers fail to support such scaffolding.[8] Indeed, such procedures may hinder students' development and use of symbolic forms, by implicitly encouraging students to view equations merely as tools for computation and symbol manipulation.

Nonetheless, we see a potential downside in recommending that symbolic forms become an instructional target. As an anonymous reviewer noted, constraining students to expert behavior may not be the road to developing expertise. We do not want the use of symbolic forms to become simply another step in problem-solving strategies, instantiated by making students explain the "symbolic form-based meaning" of each equation they use in a problem, because one consequence could be that symbolic forms become required elements of problem solving at the expense of being a tools that help students make sense of equations. We queasily imagine students flipping through "symbolic forms sheets" along with the standard formula sheets sometimes allowed in exams.

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[8] While research has shown that students exhibit symbolic forms-based reasoning in ways consistent with existing problem solving schemes, such schemes do not explicitly support symbolic forms-based reasoning. In translating a physical scenario into equations, students are encouraged to use formal physics concepts to select from equations they already know, not to use intuitive schema to invent or interpret equations. While productive, this kind of translation sidesteps the blending of conceptual and formal mathematical reasoning. And in checking their mathematical answers, students are encouraged to check if a numerical answer is physically plausible, which taps into intuitive knowledge but not into intuitive conceptual schemata. To be fair, however, students are also sometimes encouraged to check the functional relations in their final symbolic expression for plausibility, which can definitely involve the use of symbolic forms corresponding to direct and inverse proportionality. We advocate engaging students in this kind of answer checking more often.

We want students to use symbolic forms not as a required "step" but as an organic part of authentic sense-making in which they seek coherence between intuitive ideas and formal representations. For example, the momentum tutorial discussed above does not meet our goals if its scaffolding practically forces students to arrive at the correct expression for momentum, $p=mv$. We would rather see some students generate incorrect equations, such as $p = m + v$ or $p = mv^2$, both of which capture the idea that increasing an object's mass or speed increases its momentum. The presence of these incorrect guesses provides an opportunity for productive small-group discussions about which mathematical expressions best capture the students' intuitive ideas about "oomph." This kind of authentic sense-making provides students the opportunity to use symbolic forms as a tool in seeking understanding, not as part of a rote procedure.

Beyond Base + Change and Symbolic Forms: Blending Intuitive Ideas and Formal Representations

Because *Base +Change* played a central role in our analysis of Pat's reasoning and in one of our instructional examples above, a reader could conclude that we advocate *Base + Change*, in particular, as an element of problem-solving expertise and as an instructional target. Admittedly, one limitation of this research is that it investigates a small number of students' use of one particular symbolic form. We want to emphasize, however, that our argument is not specific to the *Base + Change* symbolic form. We believe that symbolic forms more generally—blends of conceptual schemata and algebraic symbol templates—can contribute to quantitative problem solving expertise. Sherin (2001) documented about 20 symbolic forms relevant to physics, including *Base + Change*, *Parts-of-a-Whole*, and *Prop+*. Although the Two Balls problem happened to afford blended processing using *Base + Change*, we expect that other problems afford blended processing using other symbolic forms. Future, research may investigate whether and how larger populations of students productively use these other symbolic forms in blended processing.

More broadly, just as *Base + Change* is an example of a broader category called symbolic forms, reasoning that employs symbolic forms is an example of a broader class of reasoning that combines formal representations (for example, in science: equations, graphs, free body diagrams, Lewis structures, etc.) with informal knowledge derived from everyday experiences, in order to give additional, intuitive meaning to the formal representations.

For example, in interpreting the force diagram in Figure 3.3, a student can blend her formal knowledge of force diagrams with the conceptual schema of *two influences that precisely cancel so that there is no net outcome*—the conceptual schema for the *Canceling* symbolic form (Sherin, 2001) — to quickly determine that the net force on the object is zero. That intuitive conceptual schema (i) allows the student to take a shortcut around the formal rule of summing separately over the horizontal and vertical forces ($F_{x,\text{ net}} = \sum F_x$, $F_{y,\text{ net}} = \sum F_y$), and (ii) contributes to an intuitive explanation of *why* the net force is zero, in terms of pairs of canceling forces. These examples illustrate our broader point that blending informal, everyday knowledge with knowledge of formal representations is part of problem-solving expertise and is a feasible instructional target.

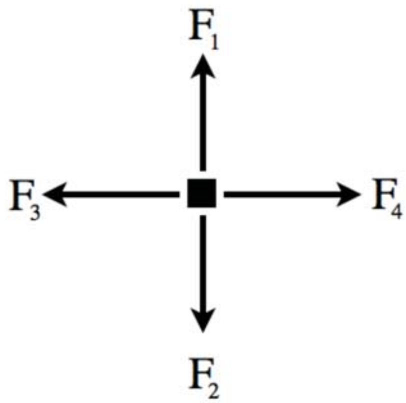


Figure 3.3: Free body diagram showing forces on an object. In interpreting the diagram a student could draw on the intuitive idea that *two opposing influences could precisely cancel so that there is no net outcome* to arrive at the conclusion that the net force on the object is zero.

CONCLUSION

A gap exists in the literature: research on quantitative problem solving has focused on how experts and novices *select* equations but not on how they *use* the selected equations to solve problems. This paper attempts to address this gap with an illustrative case study showing how two students process the same physics equation differently.

Analyzing how Alex and Pat explained and used a standard kinematic equation, $v = v_0 + at$, we attributed part of the difference in their reasoning patterns to the use or lack of use of a knowledge element called a *symbolic form*. A symbolic form is a blend of symbolic and conceptual knowledge, a "marriage" of a symbol template to an intuitive conceptual schema. Pat's use of a symbolic form enabled him to give an intuitive explanation of the velocity equation and to quickly find a non-computational shortcut to the Two Balls Problem. Alex's explanation and problem solving, although productive and correct, were more procedural, and her processing of the velocity equation was more computational. So, as we argued, Pat's solution to the Two Balls Problem shows more expertise. However, it is Alex's solution that aligns more closely with the standard problem-solving procedures advocated by researchers and taught to students. We have used this result, along with our sample instructional techniques, to argue that blending conceptual and symbolic reasoning can be a desirable and feasible instructional target.

Given our arguments, a researcher might take away the message that we advocate replacing the "mathematical processing" step in a problem solving procedure with a "symbolic forms-based reasoning" step. This is not what we are suggesting. Not all quantitative physics questions have a non-computational shortcut as the Two Balls Problem does. However, as we argued, a valuable piece of Pat's solution to the Two Balls Problem is the adaptive expertise displayed by his evaluation of multiple solution paths. Good problem solving involves making decisions, not just following a set procedure (Reif, 2008). Aligning with this argument, we support a model of expert problem solving that does not always require either symbol manipulation or symbolic forms-based reasoning. Instead, good problem solvers have these and other tools in their toolbox, and they select which tools to use based on the details of the problem (Reif & Heller, 1982; Reif, 2008; Schoenfeld, 1985, 1992).

Chapter 4: Connecting Epistemology to Symbolic Forms Use

INTRODUCTION

In a previous study, introductory physics students were asked in interview settings how they would explain a familiar physics equation for velocity, $v = v_0 + at$, to a friend from their class (Kuo, Hull, Gupta, & Elby, 2013). Alex explained the equation as a computational tool: if you know v_0, a, and t, then you can calculate v. Pat, on the other hand, read a conceptual meaning from the structure of the equation: v_0 is the starting velocity, at is the change in velocity from the start, and adding the two gives the final velocity v.

Alex and Pat similarly differed on how they solved a problem with that same equation. After explaining the equation, they were presented with the Two Balls Problem:

> Suppose you are standing with two tennis balls on the balcony of a fourth floor apartment. You throw one ball down with an initial speed of 2 meters per second; at the same moment, you just let go of the other ball, i.e., just let it fall. What is the difference in the speeds of the two balls after 5 seconds – is it less than, more than, or equal to 2 meters per second?

Alex approaches this problem with explicit computations, calculating the speeds of the balls after 5 seconds to be 50 and 52 meters per second and then subtracting one from the other to find that the difference remains two meters per second. By contrast, Pat's solution leverages an intuitive conceptual idea connected to the velocity equation. From the equation, he determines that the velocities of the balls change at the same rate. Through the intuitive idea "if two things change by the same amount, then the difference between them stays the same," Pat concludes that the final difference in speeds is still 2 meters per second.

In this previous work, the *Base + Change* symbolic form (Sherin, 2001) is modeled as playing a central role in Pat's reasoning with the velocity equation. A symbolic form is a cognitive element that blends the structure of an equation with an intuitive meaning. For example, in *Base + Change*, the conceptual schema "the final amount is the initial amount plus the change in that amount" is mapped onto the structure of the equation, represented by the symbol template $\square = \square + \Delta$. Reasoning about terms in the equation corresponding to "final amount," "initial amount," and "change" (as Pat does) provides evidence of the *Base + Change* symbolic form playing a role in that reasoning. Reasoning with these symbolic forms can lead to quick, heuristic problem solving approaches through *blended processing* (Kuo et al., 2013), combining conceptual meaning with formal mathematics to circumvent explicit computations. Because of this, symbolic forms-based reasoning is argued to be a form of problem-solving expertise.

In interviews, 6 of 13 students in this study did not use *Base + Change* in either explaining the velocity equation or solving the Two Balls Problem. In aiming to understand and foster problem-solving expertise, one important question to answer is "why do these students not demonstrate this type of reasoning?"

The problem solving literature has shown many examples of novices demonstrating different quantitative problem-solving approaches than experts (Chi et al., 1981, 1982; Larkin et al., 1980; Simon & Simon, 1978; Taasoobshirazi & Glynn, 2009; Walsh et al., 2007). When individuals fail to display such problem-solving expertise, this is often attributed to a deficit in the content or structure of their knowledge (Chi et al., 1982; P. Heller et al., 1992; Larkin et al., 1980; Reif et al., 1976). Applying this to Alex's and Pat's reasoning with the velocity equation, the absence of symbolic forms in Alex's reasoning could illustrate a lack of knowledge about symbolic forms. This type of interpretation commonly leads to instructional interventions that involve explicitly teaching students the skills or problem-solving procedures they ostensibly lack (P. Heller et al., 1992; Huffman, 1997; Larkin & Reif, 1979; Leonard et al., 1996; Mualem & Eylon, 2010; Van Heuvelen, 1991b). In the case of Alex, this approach would suggest the need for explicit instruction on how to interpret particular equations with symbolic forms such as *Base + Change* and how to use that interpretation in physics problem solving.

However, it also seems reasonable to believe that the 6 students who did not initially use *Base + Change* – and who were enrolled in an undergraduate physics course for engineering majors – have learned about and used symbolic forms such as *Base + Change* in working with algebraic equations like the money equation. Symbolic forms are typically tied to algebraic equations of the types I expect students to have seen frequently throughout their schooling careers in math and science. For example, in learning about linear equations, I expect symbolic forms such as *Base + Change* to be either explicitly or implicitly relevant. Additionally, there is evidence that 8^{th} graders (Izsák, 2000) as well as undergraduate physics majors (Sherin, 2001; Kuo et al., 2013) use *Base + Change* and other symbolic forms.

Furthermore, one interviewed student, Devon, provides an explicit counterexample to a deficit-based explanation. Devon, who is not counted in the original interview set, initially does not use *Base + Change* to interpret the velocity equation but does when later given a (non-physics) equation for how much money one has after working for a certain period of time, $m = m_0 + rd$ (where m is the ending amount of money, m_0 is the starting amount of money, r is the salary rate, and d is the number of working days). Between asking Devon about the velocity equation and "money" equation, the interviewer provides no instruction about symbolic forms; simply asking the money equation was enough to activate *Base + Change* in Devon's reasoning. So, if students such as Devon do not lack knowledge of symbolic forms, why do they not always bring that knowledge to bear in understanding physics equations?

In this chapter, I argue that, in addition to knowledge deficits, one possible explanation for the absence of symbolic forms-based reasoning when applicable is that such reasoning is not viewed as relevant or appropriate. I make the case that *epistemological stances* towards learning physics and understanding physics equations (i.e. in-the-moment views on what it means to learn and understand physics) are connected to whether students draw on symbolic forms-based reasoning.

First, using a set of interviews conducted to investigate how introductory physics students understand physics equations, I code individual interviews for both students' symbolic forms-based reasoning and expressed epistemological stances reflecting how they view understanding and learning physics and physics equations in those moments. I argue that students who tend to use symbolic forms in the interviews also tend to express epistemological stances that value *coherence* between two different types of reasoning, either (1) between formal physics and everyday intuition or (2) between equations and intuitive or conceptual meaning. Although student epistemology has been connected to reasoning in many different ways, I aim to show that, in particular, symbolic forms use is tied to expressed *coherence* epistemological stances in these interviews.

Next, I will apply this connection between symbolic forms use and epistemological stances to understand Devon's different responses to the physics and non-physics prompts (such as the "money" prompt). To investigate these different patterns of responses, I present a case study of Devon's interview, arguing that Devon's reasoning shifts between two modes, consisting of different reasoning with and epistemological stances towards equations. This provides an alternative to a knowledge deficit explanation of Devon's initial reasoning, which would model the initial absence of symbolic forms-based reasoning as resulting from a lack of knowledge about *Base + Change*. Rather, I model Devon's initial lack of symbolic forms-based reasoning as aligning with his epistemological stance towards physics equations, and his subsequent shift to symbolic forms use as aligning with a shift in that epistemological stance.

LITERATURE REVIEW

Research Supporting a Connection Between Student Reasoning and Epistemologies

Student reasoning or problem-solving approaches and students' *epistemologies*, or their views on what it means to learn and understand, are connected. For example, a student who views physics equations as distinct from qualitative physics concepts is less likely to use those physics concepts while reasoning with equations, even if they show expertise in such qualitative reasoning in other contexts (Lising & Elby, 2005). This is not to say that the amount of or structure of students' conceptual knowledge in physics does not affect their problem-solving approaches. Rather, it is to say that models of student thinking that only consider conceptual knowledge can explain additional reasoning phenomena by incorporating a connection between reasoning and epistemologies.

Two methods: case study analysis and large-N survey

The connection between student reasoning and epistemology has been investigated using two approaches. One approach is to construct case studies of individual or group reasoning (Bing & Redish, 2009; Gupta & Elby, 2011; Hammer, 1994; Lising & Elby, 2005; Rosenberg et al., 2006; Schoenfeld, 1988). For example, Lising and Elby (2005) showed that, across a series of classroom episodes and interviews, Jan demonstrated facility with resolving inconsistencies within either her formal reasoning or intuitive reasoning in order to argue that her tendency to not resolve inconsistencies across the two cannot be attributed to deficiencies in formal knowledge or intuitive knowledge or to deficiencies in resolving inconsistencies. Rather, they infer the existence of an epistemological stance separating formal and intuitive reasoning in physics that precludes resolutions that incorporate both. The approach of analyzing problem-solving episodes of student reasoning can provide plausible alternatives to knowledge deficit-based explanations as the cause of novice reasoning and behavior. Using this approach, Lising and Elby (2005) and others (Gupta & Elby, 2011; Hammer, 1994) have made the case that absence of particular kinds of reasoning in physics problem solving can reflect epistemological stances that do not support those particular reasoning approaches, rather than knowledge deficits.

A complementary approach, the large-N survey study, aims to show that particular epistemologies are correlated with higher post test scores after some intervention – a course, a curriculum, or an experimental learning treatment (Mason, 2003; Perkins et al., 2005; Schommer et al., 1992; Schommer, 1990; Songer & Linn, 1991; Windschitl & Andre, 1998). For example, Schommer et al. (1992) show the effect of epistemological beliefs in a short learning intervention. First, using a Likert-scale epistemological survey, subjects' epistemologies are measured along particular dimensions. Then, subjects are given short reading passages on statistics, followed by a multiple-choice survey designed to measure how much they learned. Schommer et al. show that, although prior knowledge is a predictor of success on the statistics post-test, so too is (what they label as) an epistemological belief in *simple knowledge* (i.e. "knowledge consists of disconnected, unambiguous facts"). They infer that students whose survey responses are consistent with *simple knowledge* perform worse on the post-test, because they approached learning the statistics passage as accumulating simple facts rather than connecting the ideas together to form a coherent set of knowledge. Common to these types of studies, statistical analysis is used to argue that novice reasoning (in this case, naïve approaches to learning, measured by correctness on a content post-test) is correlated with certain epistemologies.

Two types of inferences made in connecting student reasoning to student epistemologies

Both of these methodologies, case study analysis and large-N survey work, have contributed to knowledge about the connection between student reasoning and student epistemologies by, at times, making particular inferences from the data. Here, for each of the two methodologies above, I point out an inference that is commonly made.

One inference commonly made by case study analysis is to infer student epistemologies from student reasoning (Bing & Redish, 2009; Hammer, 1994; Lising & Elby, 2005; Rosenberg et al., 2006; Schoenfeld, 1988). In seeking to understand patterns in student reasoning or problem-solving approaches, researchers infer the underlying epistemologies that plausibly support those approaches, not necessarily requiring explicit confirmation. For example, if a student consistently uses physics equations for algebraic manipulations, that reasoning is used as evidence to infer an underlying epistemology that physics consists of facts and formulae rather than concepts (Hammer, 1994). In another example, Schoenfeld (1988) observed that high school geometry students could quickly solve mathematical proofs but then spend the majority of their time writing out the proof in a canonical, two-column format. Schoenfeld inferred that these students view correctness in mathematical proofs as depending on the formal notation as much as on the mathematical substance, even though students never make any explicit statements regarding that inferred view.

This is not to say that these studies necessarily exclude explicit evidence of students' epistemologies. However, these more explicit statements reflecting expressed epistemological stances are not treated as necessary evidence for claims about students' epistemologies. Instead, they are seen as additional evidence, supporting the inferred epistemologies already derived from student reasoning data. For example, Hammer (1994) explicitly asks Roger whether common sense applies to his physics course and to equations in physics. However, Roger's response is taken not as a crucial, independent piece of evidence of his epistemology, but rather as one triangulating piece of evidence, along with Roger's reasoning with physics equations, for understanding the nature of Roger's epistemology and its impact on his reasoning in physics.

One benefit of the case study approach is its ability to investigate student reasoning and problem-solving approaches. Looking only at post-test scores and final grades are not sufficient for determining the reasoning and approaches taken to achieve those scores and grades. A student who develops expertise in rote, algorithmic calculations and a student who seeks to connect those calculations to the conceptual meaning behind them may both be successful in solving problems correctly.

By contrast, large-N survey studies often seek to empirically establish, rather than assume, a correlation between reasoning and epistemology (Mason, 2003; Schommer et al., 1992; Schommer, 1990; Songer & Linn, 1991; Windschitl & Andre, 1998). In service of this goal, the inference sometimes made by these large-N survey studies is to use the *final products* of student reasoning to infer the *process* of student reasoning, the very inference the case studies seek to avoid. Schommer et al. (1992) used correct answers on the statistics post-test survey and responses on the epistemological survey to infer a plausible mechanism of how an epistemological belief in *simple knowledge* influences student reasoning: viewing knowledge as simple could lead to study strategies that emphasize memorizing disconnected facts, rather than learning to interrelate these facts into a coherent web of knowledge. However, the data collected cannot directly support hypotheses about how students approached learning from the text passages; it only measures their success in applying that knowledge as demonstrated on the post-test. The authors themselves state that research investigating student learning approaches on the reading passage is required to verify this plausible mechanism for connecting student reasoning to student epistemologies.

The Nature of the Connection Between Student Reasoning and Epistemologies: Unitary Vs. Manifold Cognitive Structures

An example: "unitary" vs. "manifold" interpretations of large-N epistemological surveys

One important issue in the large-N survey studies described previously is how students' responses on epistemological surveys are interpreted. One analytical approach treats students' responses on epistemological surveys as evidence of consistently held epistemological *beliefs* (Schommer, 1990; Songer & Linn, 1991). For example, Songer and Linn (1991) label students who tend to give survey responses aligned with a particular epistemology as *stably having that epistemology*. Hammer and Elby (2002) see a pattern in the literature of claiming students possess stable and consistent epistemological theories or beliefs that develop in stages, which they describe as modeling epistemology with *unitary* cognitive elements. By unitary, they mean that "each belief corresponds to a unit of cognitive structure, which an individual either does or does not possess" (Hammer & Elby, 2002, p. 169). They, on the other hand, argue that students' epistemologies consist of *manifold* epistemological resources that are contextually activated. In this manifold view, individuals' epistemologies are multifaceted, and individuals can take many possible epistemological stances depending on the situational context.

This theoretical difference in how epistemology is modeled leads to practical differences in how inconsistent survey results are interpreted and acted on. Where these unitary views might recommend discarding survey items that yield inconsistent responses as bad probes of students' stable and consistent epistemological beliefs, a manifold view seeks to understand how the different questions might cue different facets of students' epistemologies (Yerdelen-Damar, Elby, & Eryilmaz, 2012).

The manifold perspective is more aligned with my purpose in studying Devon: I aim to understand why Devon would respond differently on the physics and non-physics prompts, rather than to simply discard the interview prompts as "bad questions." In the next section, I describe how this debate between unitary and manifold views of cognition extends to the modeling of student reasoning.

Manifold views of conceptual knowledge

This idea of manifold models of students' epistemologies is situated in a broader argument between models of cognition that treat student reasoning as unitary and stable and models of reasoning as more manifold and dynamic (diSessa, Gillespie, & Esterly, 2004; diSessa & Sherin, 1998; diSessa, 1993; Hammer et al., 2005; Hammer, 2000; Ioannides & Vosniadou, 2002; Sherin et al., 2012). For example, in response to studies attributing incorrect student reasoning to naïve, but stable, theories of force and motion, diSessa (1993) argued that student reasoning about force and motion is multifaceted. He modeled student reasoning about conceptual physics problems as involving, in part, *phenomenological primitives (*or *p-prims)* – intuitive notions abstracted from experiences in the world. In contrast to a consistent theory of force and motion being applied across multiple situations, different p-prims can be activated in different moments, leading to different reasoning in and conclusions about the same underlying physical scenario. For example, in considering how a pull on a yo-yo string will cause it to move, an individual's reasoning could initially stem from the p-prim of *force as a mover*, predicting that the yo-yo will move in the direction of the pull. However, in other moments, that same individual might think of *force as a spinner* and predict that the yo-yo will spin away from the original pull.

Of course, in canonical physical theories, a force can cause either linear and rotational motion. The case that diSessa makes is that novice reasoning here does not follow from one stable and consistent theory of force and motion. Instead, an individual's knowledge can consist of manifold cognitive elements (here, different p-prims related to force and motion), and their reasoning is driven by the activation of a subset of these cognitive resources. Importantly, each of these p-prims can be either productive or unproductive in different situations, and each one can lead to different predictions about the motion of an object. Studying these different pieces of knowledge and the contexts in which they are activated can help us understand the variation in an individual's reasoning around the topic of force, variation that a unitary theory of motion does not adequately explain. In the same way that Hammer and Elby argue that an individual's epistemology is multifaceted, so then is that individual's conceptual knowledge.

Epistemological framing: a manifold model of the connection between student reasoning and epistemology

Both case-study analysis and large-N survey work has established the connection between student reasoning and epistemologies. *Epistemological framing* (Hammer et al., 2005) models this connection while maintaining the multifaceted nature of conceptual knowledge and epistemologies.

To illustrate this point, consider the following question: "which vehicle feels more force in a head-on collision between a large truck and a small car?" One possible answer to this question involves formal physics knowledge, such as Newton's 3rd law. Another possible answer draws on physical intuition and experiences, such as imagining what you would feel if you were in the truck versus if you were in the car. How one answers the question depends on "what kinds of knowledge or approaches are appropriate here?" *Epistemological framing* is the drawing on a coherent set of conceptual and epistemological resources in response to a situation or problem (Hammer et al., 2005).

For example, one epistemological framing of this question draws on conceptual resources related to Newton's 3rd law and epistemological resources supporting formal, classroom knowledge as appropriate; another epistemological framing of this question draws on conceptual resources related to physical experience and epistemological resources supporting informal, everyday reasoning as appropriate. Although these two epistemological framings of the problem are very different, they are both self-consistent and locally coherent (i.e. the type of conceptual resources drawn on are consistent with the types of knowledge supported by the active epistemological resources). Moreover, we might expect that some individuals – introductory physics students, for example – could take either approach to answering this problem. In this way, an individual possesses manifold ways of approaching this problem, and one espoused line of reasoning does not preclude knowledge of the other. Said differently, the absence of one kind of reasoning in a situation or problem does not necessarily imply a knowledge deficit.

Usefulness of epistemological framing in studying dynamic shifts

A benefit of manifold models (e.g. epistemological framing) over unitary models of cognition is that manifold models provide tools for investigating in-the-moment shifts in an individual's reasoning. In the unitary conceptual change literature, individuals are depicted as reasoning in ways consistent with a stable and consistent conceptual network. Reasoning is not multifaceted in the way that diSessa described novice reasoning about forces. Conceptual change is a non-trivial process, involving the introduction of cognitive conflict through the presentation of new concepts and evaluation of their usefulness over currently held concepts for describing and understanding phenomena (Hewson & Hewson, 1984; Posner et al., 1982).

On the other hand, epistemological framing leads to interpretations of student reasoning and epistemology as multifaceted. This means that the epistemological frame activated in response to a situation is only one of many possible frames, and that there is the possibility to shift between frames, even during a short reasoning episode. Rosenberg, Hammer, and Phelan (2006) showed how a group of students can shift from one locally coherent epistemological frame to another. They present a group of 8th grade students, who were asked by their teacher to come up with a model of how the three types of rocks they had learned about could be connected in a rock cycle. This group of students begins the activity by seeking out a relevant worksheet, accumulating facts, and ordering them to chronologically. This is modeled as reflecting a "cut-and-paste" epistemological framing. The teacher then intervenes, urging the students to start "from their own ideas" rather than from the worksheets. This intervention shifts how students frame the activity to "storytelling," activating a different set of mutually coherent conceptual and epistemological resources for creating a story from their own ideas. So, the teacher's intervention caused an epistemological shift in how students understand "what kinds of knowledge or approaches are appropriate here." In other words, the shifting of students' activated epistemological resources for understanding this activity as "storytelling" (and the demand of coherence between the active epistemological and conceptual resources) also shifts the activation of conceptual resources to ones for generating a plausible story, revealing that the lack of an initial "story" for the rock cycle was not due to a conceptual knowledge deficit.

Conversely, Gupta and Elby (2011) give an example of how a shift in the conceptual resources in play subsequently shifts which epistemological resources are active. In an interview, Jim articulated the epistemological stance that physics equations are disconnected from everyday common sense and that equations are more trustworthy than that common sense. In working on a problem that asked him to compare the pressure at two depths underwater, Jim's activated epistemological resources supported formal mathematical knowledge over conceptual reasoning. But in a moment where Jim finds his everyday ideas about pressure as useful for understanding the pressure equation, the epistemological resources, at least momentarily, shift to ones valuing that connection between intuitive reasoning and the formal physics equations. The result is Jim saying that it "makes more sense" and it feels "more comfortable" now that he sees this alignment between intuitive and formal mathematical reasoning.

Overall, these case studies show that epistemological framing can shift from one internally coherent set of conceptual and epistemological resources to another. A shift in either the conceptual or epistemological resources activation must lead to an accompanying shift in the other, to maintain coherence between the two.

DATA COLLECTION AND METHODOLOGY

Research Questions

This study started with a broad, ill-defined research question: *how do students make sense (or not make sense) of the mathematics they use in solving physics problems?* Following our intuitions, the research team designed a series of interview prompts that asked students to engage in two different kinds of tasks: (1) explaining and reasoning with physics equations and (2) explaining how they approach learning physics equations and physics more broadly in their courses. Previous work sharpened our understanding of how symbolic forms-based reasoning was used on these prompts (Kuo et al., 2013). In watching the interview videos, we also noticed that students who tended to use symbolic forms also tended to express epistemological stances that valued *coherence*, between either (1) formal ideas in physics and everyday reasoning or (2) physics equations and physics concepts. This led to the development of a specific research question, investigated in the phase 1 analysis: *Across the interviews, do students who tend to use symbolic forms in reasoning with physics equations also tend to express* coherence *epistemological stances towards learning physics and physics equations? (And also, do students who tend not to use symbolic forms-based reasoning also tend to express epistemological stances seeing formal physics as* disconnected *from conceptual reasoning or everyday intuition?)* To study this question, I coded the interviews for both symbolic forms use and expressed *coherence* or *disconnected* epistemological stances.

After an initial set of interviews revealed that some students do not use symbolic forms-based reasoning to explain equations in the interview, the research team designed a set of isomorphic non-physics questions (discussed in detail later) which we expected would cue symbolic forms-based reasoning. For one student, Devon, separate coding of the physics and non-physics prompts revealed that he tended to use symbolic forms-based reasoning more on the non-physics prompts. This led to a research question, investigated in the phase 2 analysis: *Why does Devon tend to use symbolic forms-based reasoning more often with non-physics equations than physics equations in his interview?* To investigate this question, I perform a more in-depth case study of Devon's responses to the two types of prompts in his interview.

Data Collection

Thirteen students who were enrolled in or had completed a first-semester, introductory physics course were interviewed. These semi-structured interviews consisted of prompts to get at both how students reasoned about and with physics equations (e.g. “explain the velocity equation” and the Two Balls Problem discussed in the introduction) and also at their epistemological stances towards learning and understanding equations in physics and math. Although there was a set of these preplanned prompts, the interviewer was free to make in-the-moment decisions about what questions to pursue based on emergent topics in the interview (Ginsberg, 1997). The interviewer was free to change the order of the prompts, omit prompts, and modify prompts. Additionally, the interviewer was free to ask follow-up questions, revoicing student ideas and selectively zooming in on and probing deeper into particularly interesting topics (Lee, Russ, & Sherin, 2012). In this way, the interviewer was not beholden to a set structure but was free to investigate interesting points and explore in-the-moment hypotheses. Consequently, not all interviewees received the same prompts or experienced them in the same order.

Phase 1 Analysis

In the phase 1 analysis, I code the interviews for both symbolic forms use and espoused epistemological stances. If there is a connection between the two, then the coding will reveal a pattern: that interviews that tend to contain symbolic forms-based reasoning also tend to contain *coherence* epistemological stances (and that interviews that tend not to contain symbolic forms-based reasoning tend to contain *disconnected* epistemological stances). Although the connection between student reasoning and epistemologies have been established, there has been no work that studies this connection at the grain size of particular cognitive elements (in this case, symbolic forms).

<u>Avoiding two common inferences in connecting student reasoning to epistemology</u>

In the phase 1 analysis, my purpose is to argue that epistemological stances that value *coherence* between physics and everyday life or between equations and intuitive or conceptual meaning are connected to symbolic forms use. To make this argument, this study aligns, in part, with the purpose of the large-N survey studies: I do not *a priori* assume a connection between epistemologies and symbolic forms-based reasoning. Rather, two independent types of evidence are collected: (1) data on whether a student uses symbolic forms in reasoning with equations in physics and (2) data on the expressed epistemological stances towards equations in physics, to test empirically whether the two are connected.

At the same time, I align with the case studies approach in how reasoning with equations in physics is studied. Although Alex’s and Pat’s reasoning with the velocity equation differed, they both gave valid explanations and solutions on the Two Balls Problem. Because correct use of either symbolic forms or algorithmic manipulations will lead to the same (correct) final answer, simply measuring their success on such prompts would be insufficient. For this reason, student reasoning with equations is investigated in this study by examining students’ approaches through video analysis of the interviews, to see if symbolic forms-based reasoning is present.

In sum, for these purposes, I aim to avoid the two inferences made in similar studies: inferring underlying epistemologies from student reasoning or inferring how a student reasoned on a problem through their success or failure on that problem. Although some more recent studies are also explicitly careful to avoid these inferences (e.g. Gupta & Elby, 2011), this study adds to these by using a novel coding scheme to expand the analysis beyond a single interview in order to investigate possible patterns across different interviews.

Interview segments to code for symbolic forms and epistemological stances

In order to investigate whether symbolic forms is connected to particular epistemological stances valuing *coherence*, these interviews are coded for

> (1) whether symbolic forms-based reasoning is used on the prompts that ask students to reason with or explain particular physics equations and
> (2) whether *coherence* or *disconnected* epistemological stances are expressed on the explicit epistemological prompts about learning and understanding physics and physics equations.

The interview is broken up into different segments, along the points where the interviewer introduces a new prompt or topic. These segments are then coded according to the symbolic forms or epistemology coding schemes, described in the following sections. Then, for each interview, I compare the percentage of "symbolic forms segments" that received a "used symbolic forms" code, as well as the percentage of "epistemology segments" that received a "*coherence* epistemology" code. Here, I list abridged versions of interviews prompts that make up the "symbolic forms segments" and "epistemology segments:"

Symbolic Forms Coding Segments

- *Explain the velocity equation:* "Here's an equation you have probably seen in physics class, $v = v_0 + at$. How would you explain this equation to a friend from class?"
- *Two Balls Problem:* Suppose you are standing with two tennis balls on the balcony of a 4th floor apartment. You throw one ball down with an initial speed of 2 m/s; at the same moment, you just let go of the other ball. Is the difference in the speeds of the two balls after 5 seconds less than, more than, or equal to 2 m/s?
- *Explain the pressure equation:* "Here's an equation you are probably unfamiliar with, $P = P_{at\ top} + \rho gh$ (variables defined in the problem). How would you explain that equation to yourself?
- *5/7 Meter Problem:* Is the pressure at h = 5 meters underwater greater than, less than, or equal to the pressure at h =7 meters underwater?
- *Mars Lake Problem:* Gravity on Mars is weaker than gravity on Earth (i.e. g is less on Mars). How does the pressure at a certain depth in a

lake on Mars compare to the pressure at the same depth in a lake on Earth?

- *Explain Coulomb's Law:* The force between two electric charges, q_1 and q_2, that are separated by a distance r is $F_{12} = \frac{kq_1q_2}{r^2}$, where k is a constant. How would you explain this equation to a friend in your physics class?

Epistemology Coding Segments

- "How do you know when you understand a physics equation?"
- "How do you approach learning equations in physics?"
- "What is difficult about understanding the math in your physics course?"
- "If you had a photographic memory for equations, would that improve your performance in your class?"
- *"Diana" Prompt*: "Imagine there is a student, Diana, who is not taking the course for credit. She wants to understand physics more deeply. She's not interested in learning to solve the quantitative problems, but she's willing to study outside of class to learn the concepts better. What role, if any, should equations play in her studying?"
- "How do you approach solving problems in physics?
- "What do you do when you get stuck in problem solving?"

In the analysis, I am not using an interviewee's reasoning on these prompts to suggest how they might behave or respond in other situations. Since the interview is an atypical, interactional event between interviewer and interviewee with particular materials in a particular environment, the reasoning and epistemological stances that emerge in the interview cannot be said to unaffected by the interview context (diSessa, 2007; Russ, Lee, & Sherin, 2012), and it is entirely possible that interviewees may exhibit totally different behavior in other situations.

Rather, I am looking for broad patterns in symbolic forms use and expressed epistemological stances *within the interview*. For this reason, my claims are ultimately about a connection between symbolic forms use and espoused epistemological stances, rather than claiming that the particular reasoning with equations and epistemological statements reflect how students would respond to similar prompts in other contexts.

Another issue is the lack of standardization across interviews. Because of the freedom the interviewer has to modify and adapt the interview protocol, no two interviews look exactly the same in terms of which prompts are posed, the order in which prompts are posed, and the amounts of time spent on each prompt. I do not claim that these differences have no effect on the outcomes of the interview. Rather, given an interviewee's statements in the interview, likely influenced by the particular configuration of interview prompts, I am looking for patterns between symbolic forms use and epistemological stances within that interview.

A coding scheme for symbolic forms

On prompts asking students to explain an equation or to solve a quantitative problem, I code students' responses for the presence or absence of symbolic forms-based reasoning, as illustrated for these same prompts in previous work (Kuo et al, 2013). I use the original list of symbolic forms as a guide to identify symbolic forms-based reasoning (Sherin, 2001, pp. 532-537). From this list, I code for symbolic forms from three clusters: the competing terms cluster, the proportionality cluster, and the terms as amounts cluster (which includes *Base + Change*). The symbolic forms ignored by the coding, such as *Dependence* or *Scaling*, were either widely used and considered too basic or were not applicable to our prompts.

Here, I describe the symbolic forms most relevant to the interview prompts and most prevalent in the coding:

- *Base + Change:* The symbol template ☐ = ☐ + Δ is connected to the conceptual schema "the final amount is the initial amount plus the change in that amount." Applied to the equation $v = v_0 + at$, for example, v_0 takes on a meaning of a starting velocity and at takes on a meaning of a change in velocity added onto that starting value.
- *Parts-of-a-Whole:* The symbol template ☐ = ☐ + ☐ + ☐ + … is connected to the conceptual schema that "the whole is made up of several parts." For example, in the mechanical energy equation $E = ½mv^2 + mgh + ½kx^2$, the total energy is the sum of three different kinds of energy: kinetic, gravitational potential, and spring potential energy.
- *Prop+:* Essentially meaning direct proportionality, the symbol template [… X …/ …] is connected to the conceptual schema that as X increases, the total value also increases. For example, for the underwater pressure term ρgh, as h (the depth under the surface of the water) increases, then this pressure term increases.
- *Opposition:* The symbol template ☐ - ☐ is connected to a conceptual schema of two influences opposing each other. For example, the net force on a space shuttle as it's launching into orbit, $F_{net} = F_{rocket} – GmM/r^2$, can be interpreted as the force of the rocket opposing the Earth's gravity.

These symbolic forms interview segments are coded as "used symbolic forms" if there is evidence of the structure of an equation being tied to a conceptual schema, through an explicit description of the equation or a *blended processing* solution, as described in the introduction. Description or use of an equation as a computation tool or only relating conceptual meaning to the individual variables (e.g. identifying m as the mass of an object) is coded as "did not use symbolic forms." At times there was weak evidence hinting that the interviewee could be using symbolic forms in their reasoning with equations. I take a strict approach to coding these segments based on Sherin's taxonomy of symbolic forms, so this weak evidence of symbolic forms use is coded as "did not use symbolic forms."

Some segments could not be coded according to this coding scheme, because of the absence of reasoning with equations. The interviewee may not have articulated their reasoning on the problem or may have taken a purely conceptual approach to the problem, avoiding mathematical equations or other formalism. Since I am interested in how students use equations, these segments are not counted in our final coding. The non-physics prompts that some interviewees were asked are excluded from the coding results, because, in phase 1, I am interested in how students use canonical physics equations. What remains are the coded segments: prompts where interviewees explained, used, or reasoned with physics equations.

Each interview prompt asking students to explain, use, or reason with physics equations counts as one "interview segment." The same prompt may have been asked in several different ways, to see if changes to the original problem could influence symbolic forms use by slightly altering the prompt. For example, Alex was asked to (1) explain the velocity equation to a friend from class, (2) explain the velocity equation to a 12-year old, and (3) explain the velocity equation on an exam. In the coding, these questions all relate to the same prompt ("explain the velocity equation") and are collapsed into one segment, such that if an interviewee uses symbolic forms-based reasoning on one of the three parts, this segment is coded as "used symbolic forms." The reason for this is because the follow-up prompts are only given if the initial form of the question does not elicit symbolic forms use. Additionally, I am interested in whether students use symbolic forms at all on these prompts, not in how many follow-up questions it took before students used symbolic forms-based reasoning. This collapsing of codes is used to keep the total number of codes for students who don't use symbolic forms from being artificially increased, artificially lowering their symbolic forms use percentage. Also, throughout all of the interviews, it was discovered that some prompts are better for eliciting symbolic forms reasoning than others. In the results, I exclude prompts on which no interviewees used symbolic forms-based reasoning.

<u>*A coding scheme for* coherence *and* disconnected *epistemological stances*</u>

In order to code for student epistemologies in the interview, I draw on work done on surveying student epistemologies in physics. Specifically, I draw on the Colorado Learning Attitudes about Science Survey (CLASS) (Adams et al., 2006).

The CLASS is a survey designed to probe a range of student attitudes towards learning physics, including epistemologies. Furthermore, the CLASS uses empirically validated categories. For the interview coding scheme, a team of researchers analyzed the categories of the CLASS and selected those believed to be relevant to symbolic forms use. In general, symbolic forms are supported by valuing coherence between two types of reasoning: connecting formal physics equations with either (1) everyday intuition or (2) conceptual meaning in physics. Below I describe the four CLASS categories used as indicators of a *coherence* epistemological stance and why those categories are believed to be relevant to symbolic forms use, along with statements that are coded into these categories. Following this, I describe the CLASS categories that are not included in the coding scheme along with what kinds of statements cannot be coded by this scheme.

The "epistemology segments" in the interview are broken up according to when the topic changes. For example, the interviewer may ask a prompt such as "how do you know when you understand an equation?" Follow-up questions to a student's response that produce an elaboration on the current topic – such as "what did you mean by X?" – do not signal a topic change and are counted as a continuation of the current interview segment. However, questions that potentially probe a different aspect of students' epistemologies – such as changing the topic to "what is difficult about the math in your physics class?" – do signal a topic change and the start of a new "epistemology segment."

For the coding, students' statements in the interview were used as evidence for how they might answer the CLASS survey items. These survey items were used as guides for coding students as expressing either a *coherence* or a *disconnected* epistemological stance in the epistemology segments. Although I describe these CLASS categories separately, items from all four categories are used indistinguishably in order to code along the one broad epistemological dimension relevant for this study: *coherence* versus *disconnected*. Also, since the specific categories and items aren't distinguished in the final coding, I do not attempt to be exhaustive in identifying the relevant CLASS items for coding a segment. Often, many CLASS items can be used to code a segment, ultimately leading to the same code. In cases where segments might be coded as *coherence* according to some items, and *disconnected* according to others, this segment is coded as "ambiguous" and is omitted from the final coding. In the results, only definitive evidence of *coherence* or *disconnected* epistemological stances, as defined by this coding scheme, are counted.

Four CLASS categories for coding coherence *and* disconnected *epistemological stances*

Here, I list the CLASS survey items from the four categories used in the interview coding scheme.

Real World Connection

R1) Learning physics changes my ideas about how the world works.
R2) Reasoning skills used to understand physics can be helpful to me in my everyday life.
R3) The subject of physics has little relation to what I experience in the real world.

Overall, these items probe the relation that students see between physics and everyday life. Although seeing coherence between physics and everyday reasoning might not necessarily involve symbolic forms reasoning, symbolic forms use is one way we can see this connection being instantiated. More broadly, this category also represents the seeking of connections between physics knowledge and everyday ideas and experiences.

Evidence of a *coherence* epistemological stance includes statements that emphasize connecting the formal ideas in physics with everyday examples, like using the example of a baseball player hitting a ball with a bat to understand Newton's laws. Evidence of a *disconnected* epistemological stance would include statements that physics is distinct from real life. However, no student in the interviews makes this type of explicit statement.

Sense Making/Effort

S1) I'm not satisfied until I understand why something works the way it does.
S2) In physics, it is important for me to make sense out of formulas before I can use them correctly.
S3) Spending a lot of time understanding where formulas come from is a waste of time.
S4) There are times I solve a physics problem more than one way to help my understanding.
S5) When I solve a physics problem, I explicitly think about which physics ideas apply to the problem.
S6) In doing a physics problem, if my calculation gives a result very different from what I'd expect, I'd trust the calculation rather than going back through the problem.
S7) When studying physics, I relate the important information to what I already know rather than just memorizing the way it is presented.

Items S1 through S4 have to do with seeking out the reasons why physics equations work. Symbolic forms are one type of tool students use to make sense of equations in this way. Statements related to wanting to understand the proof or derivation of an equation or wanting to know "where an equation comes from" and why it works would be coded as indicating a *coherence* epistemological stance.

Items S5 through S7 emphasize connections between quantitative problem solving and conceptual ideas, as well as connections between formal physics knowledge and "what I already know." Seeking out and finding these connections are naturally aligned with symbolic forms, as they are used to interpret formal equations from physics class in conceptual and intuitive ways. If a student mentions thinking physically about a problem situation, thinks about relevant physics concepts (such as conservation of momentum) when dealing with an equation, mentions a relevant conceptual analysis at the start of problem solving, or checks a mathematical answer against qualitative expectations, that would be coded as indicating *coherence*.

Conceptual Connections & Applied Conceptual Understanding

C1) Knowledge in physics consists of many disconnected topics.
C2) A significant problem in learning physics is being able to memorize all the information I need to know.
C3) If I don't remember a particular equation needed to solve a problem on an exam, there's nothing much I can do (legally!) to come up with it.
C4) If I get stuck on a physics problem, there is no chance I'll figure it out on my own.
C5) If I want to apply a method used for solving one physics problem to another problem, the problems must involve very similar situations.
C6) When I solve a physics problem, I locate an equation that uses the variables given in the problem and plug in the values.
C7) I do not expect physics equations to help my understanding of the ideas; they are just for doing calculations.

C8) After I study a topic in physics and feel that I understand it, I have difficulty solving problems on the same topic.

Although *conceptual connections* and *applied conceptual understanding* are two separate categories on the CLASS, 4 of the 9 items that make up these two categories are included in both. For that reason, it was practically useful to consider both categories together in the coding. One of the 9 items overlaps with and was already discussed in the *sense making/effort* category. Because I care only about the items and not the CLASS categories for our purpose, I describe only the remaining 8 items here.

The first five items (C1 through C5) all roughly have to do with the idea that physics consists of a large set of disconnected ideas that require memorization. In problem solving, this view implies that if you don't know the right equation or approach, then you won't be able to figure it out, and in order for a known approach to work, it has to be from a similar type of problem. Interviewees may indicate a *disconnected* epistemological stance by stating that "there's just so much to know" or that studying involves only memorizing the equations or other ideas. Alternatively, interviewees may disagree with one or more of the first five items and espouse a *coherence* epistemological stance, saying that you can derive all the various equations from a small set of basic relations or that topics are coherent and well-structured.

Another theme in this CLASS category is a "plug-and-chug" approach to problem solving, especially in items C6 and C7. While many PER-based, problem-solving approaches emphasize an initial conceptual analysis and tying that analysis to physics equations (P. Heller et al., 1992; Huffman, 1997; Leonard et al., 1996), a plug-and-chug approach might rely on equations as simply tools for turning one set of values into other values through algorithmic computation. Agreeing with a plug-and-chug approach to using equations is evidence of a *disconnected* epistemological stance, because a more computational approach to using equations does not necessarily involve coherence with conceptual or intuitive ideas. Specific evidence for this coding might be an interviewee stating that understanding an equation is equivalent to being able to use it in problem solving or stating that the difficult parts of problem solving are knowing which equations to use or how to use them. Furthermore, one might state that equations are for detailed calculations rather than for supporting conceptual understanding and that, in using an equation, the hard part is the algebraic manipulations.

In contrast, interviewees who would disagree with these statements, evidencing a *coherence* epistemological stance, might say that equations are not just computational tools, but also express a conceptual meaning. They may also say that equations are a compact way to express meaning that might otherwise be contained in words or diagrams, perhaps even giving examples of the kind of conceptual meaning conveyed (e.g. $F = ma$ expresses the common sense dependence that if the force doubles, then the acceleration should double).

CLASS categories and epistemology interview segments not included in the coding

There are four categories not included in the coding. The first is *personal interest*. This category consists of items probing whether the student enjoys physics and whether they relate the physics they learn to everyday life (which is redundant with the *real world connection* category). Since I don't expect interest in physics to necessarily be tied to *coherence* or *disconnected* epistemological stances, I don't include it in the coding.

The second is *problem solving general.* The items in this category that are not also included in categories discussed so far only have to do with self-efficacy (e.g. "I can usually figure out a way to solve physics problems."), which is also not necessarily related to *coherence* or overlaps in substance with other categories that are included. Therefore, this category is excluded to avoid redundancy and simplify the coding scheme. All of the items in the third category, *problem solving confidence*, overlap with *problem solving general*.

The final category not included in our coding scheme is *problem solving sophistication*, which only contains items that are also contained in another category.

In coding the epistemology segments, only statements about how the interviewee approaches learning physics, understands physics equations, and solves physics problems are included. Test-taking strategies or other classroom strategies distinct from interviewees' in-the-moment epistemological stances of what knowledge in physics is and how to learn it are not included (e.g. a segment where an interviewee describes a test-taking strategy of writing down equations and attempting to plug-and-chug to try and get partial credit even though they describe this strategy as conflicting with a deep understanding of equations as representing conceptual relationships would be coded as *coherence*.). Other segments cannot be coded, because the segments do not contain sufficient evidence to make a judgment for how a student would respond on the CLASS survey. Finally, some of the prompts ask students how they view equations in their math classes, in contrast to physics. These segments are omitted from the results, because the research question in the phase 1 analysis applies specifically to physics equations and epistemological stances towards physics.

Phase 2 Analysis

During the data collection process, non-physics interview prompts were designed to test the plausibility of knowledge deficit explanations over epistemology-based explanations of the initial absence of symbolic forms-based reasoning in the interviews. These non-physics prompts were designed to include equations isomorphic to the velocity equation, and therefore also embodying *Base + Change*, but situated in more everyday kinds of reasoning, related to money in a bank account or the speed of a car. An example of one prompt about money is given here:

- *Explain the money equation (with symbols):* You start out with \$$m_0$, and you make \$$r$ per day. How many dollars (\$$m$) would you have at the end of d working days? Could you express the number of dollars (\$$m$) in an equation?

In these modified interviews, these non-physics prompts were asked only if an interviewee *did not use* symbolic forms-based reasoning on the initial prompts related to velocity: "explain the velocity equation" and the Two Balls Problem. If the interviewees then did use *Base + Change* to explain a "money equation" or a "speed equation," that would provide evidence that they do possess knowledge about and facility in using the *Base + Change* symbolic form, precluding a pure knowledge deficit explanation of the initial absence of symbolic forms-based reasoning. As alluded to earlier, in the case of Devon, this is exactly what happens.

In Devon's coding in the previous analysis, only prompts dealing with physics equations and epistemological stances towards physics were included: symbolic forms are used on 50% of the possible segments (1 out of 2) and an espoused *coherence* epistemology is given in 20% of the possible segments (1 out of 5). However, in truth, Devon's interview also included prompts requiring reasoning with non-physics equations as well as epistemological prompts probing his epistemology of learning and understanding equations outside of physics. On these previously ignored prompts, Devon uses symbolic forms 80% of the time (4/5 segments) and expressed a *coherence* epistemological stance 100% of the time (2/2 segments)[9]. Devon's reasoning with and epistemological stances towards equations in and outside of physics is remarkably different.

Including these non-physics segments raises the question: How can Devon's shifts between the absence and presence of symbolic forms use on the physics and non-physics prompts be understood? In phase 2, through a case study of Devon's interview, I argue that Devon's reasoning within the interview shifts between two distinct modes, consisting of different reasoning with and epistemological stances towards equations. These two different modes and the shifts between them suggest he usefulness of epistemological framing, which demands local coherence between expressed reasoning and expressed epistemological stances, in understanding Devon's reasoning in the interview.

The phase 2 analysis consists of a case study of Devon's responses to both physics and non-physics interview prompts. A finer-grained look at the details of the interview may reveal how, what in unitary models might be interpreted as inconsistency, can be understood as shifts between different epistemological frames. The goals of this case study are two-fold: (1) to argue that Devon's initial lack of symbolic forms-based reasoning does not indicate a lack of symbolic forms knowledge and (2) to illustrate how Devon's changing reasoning throughout the interview can be understood through shifts between coherent sets of conceptual and epistemological resources. This suggests that the results of phase 1 should be interpreted through a manifold rather than unitary framework of cognition.

<u>Interview setting and prompts</u>

At the time of the interview, Devon is one month into a second semester, calculus-based physics course for engineering majors, having taken the first semester course in the previous semester. The first semester of this course covers mechanics

[9] The coding of Devon's epistemological stances expressed on non-physics prompts is done by naively applying the CLASS categories to learning math and equations more broadly. Although problematic, this coding provides a coarse comparison between Devon's expressed epistemological stances towards learning equations in and out of physics, which motivates more in-depth qualitative analysis in phase 2.

and the second semester covers topics such as oscillations, thermodynamics, electrostatics, and circuits.

After an initial set of interviews, the predesigned prompts were augmented with non-physics symbolic forms prompts based on two criteria: 1) the new prompts were to be similar to the two velocity prompts: "explain the velocity equation" and the Two Balls Problem, and 2) the new prompts were expected to elicit symbolic forms use from students who do not initially use symbolic forms on the velocity prompts. These new, non-physics prompts would be asked if initial reasoning on the velocity prompts did not include symbolic forms-based reasoning. Two sets of such prompts were designed:

Money Prompts

- *Explain the money equation (with numbers)*: Say you have $50 to start with. Working in a bookstore, you can make $40 per day. If you work a regular work week (Monday through Friday), then how much money would you have at the end of the week?
- *Explain the money equation (with symbols):* You start out with $\$m_0$, and you make $\$r$ per day. How many dollars ($\$m$) would you have at the end of d working days? Could you express the number of dollars ($\$m$) in an equation? [After the interviewee answers] How would you explain this to a 12-year old?
- *Two Accounts Problem:* You have a friend, Lisa. To start with, you have $50, but Lisa starts with $0. Both of you work at the same bookstore and make $40 per day. At the end of the work week (M-F), how much more would you have compared to your friend?

Speed Prompts

- *Explain the speed equation:* Say a car was moving with a constant speed, s_0 mph, before it started speeding up onto a highway at the rate of r mph each second. After t seconds, what is the speed of the car? [After the interviewee answers] How would you explain this to a 12-year old?
- *Two Cars Problem:* One car is moving at the constant speed of 5 mph and another at a constant speed of 7 mph. At the same moment, both cars start speeding up at the rate of 10 mph each second. After 5 seconds, what is the difference in the speeds of the two cars?

The two types of problems, "explain the [*X*] equation" and the Two [*X*] Problem, are designed to be similar across the three relevant problem contexts: velocity, money, and speed. All three problem contexts deal with mathematically isomorphic equations that take the form *[final amount] = [initial amount] + [rate of change][time]* and can be understood through *Base + Change*. The "explain the [*X*] equation" prompts ask interviewees to explain the equation and the Two [*X*] Problems contain the following isomorphic structure: "[Item 1] and [item 2] start at different amounts and increase at the same rate for the same amount of time. After that amount of time, what is the difference in amount between [item 1] and [item 2]?"

This is the relevant structure that has been shown to afford a *blended processing* shortcut on the Two Balls Problem.

Some differences exist for the purpose of eliciting symbolic forms use. The money and speed prompts are designed to be more everyday situations than considering velocity, a physics concept. Specifically for the velocity equation, studies have shown that isomorphic equations around more everyday content can cue both *Change* as rate-times-time (Sherin, 2001) and epistemological stances supporting the making sense of mathematics (Hammer, 1994). Also, the "explain the money equation" and "explain the speed equation" prompts require the construction of an equation rather than just the interpretation of a given equation, drawing on research showing that active production can help reveal deep structure (Schwartz & Martin, 2004). Finally, the "explain the money equation" prompt has a numerical and symbolic version. We expected that concrete values rather than symbolic variables might help in eliciting intuitive understanding of the equation.

If the money prompts are successful at eliciting symbolic forms use, the speed prompts aimed to bring this symbolic forms-based reasoning back closer to the initial velocity problem context. Although speed and velocity are very similar content topics, we expected that the speed of a car would be closer to everyday kinds of reasoning than the velocity equation, an equation from physics class.

Case study analysis of Devon's reasoning

Just as in the phase 1 analysis, in phase 2 I consider the symbolic forms segments and epistemology segments separately. I start by contrasting Devon's reasoning on the velocity and money prompts. Devon's different epistemological stances towards physics and math are consistent with how he reasons on those two prompts, respectively. I then argue that shifts in Devon's epistemological stances contribute to his shifting reasoning within the speed prompts.

In arguing that the shift in Devon's reasoning with equations can be explained, in part, by shifts in epistemological stances, there arise many plausible alternative explanations for Devon's shifting reasoning that don't require attention to epistemological stances. Throughout the analysis, I will pose these alternatives and argue that, for some parts of the interview, they alone cannot explain Devon's reasoning. Showing that these non-epistemological alternative explanations cannot completely explain the shifts in Devon's reasoning supports the explanatory power in attending to Devon's shifting epistemological stances in the interview.

PHASE 1 RESULTS

Examples of Coding Scheme Application

Previous work (Kuo et al., 2013) has presented and coded Alex's and Pat's work on two prompts, "explain the velocity equation" and the Two Balls Problem. Here, I use Alex's and Pat's responses on additional prompts from the interviews to provide examples of symbolic forms and epistemology segments and of how the coding schemes are applied.

Symbolic forms segment: explain the pressure equation

As summarized in the introduction, Alex's responses to the "explain the velocity equation" and the Two Balls Problem interview segments did not incorporate symbolic forms-based reasoning, whereas Pat's did include symbolic forms-based reasoning. Here, I present the coding of Alex's and Pat's reasoning on an additional symbolic forms interview segment, "explain the pressure equation."

> Explain the pressure equation
>
> Here's an equation you perhaps haven't yet learned. It's a formula for the pressure at a given depth under the surface of a lake, ocean, or whatever: $P = P_{at\ top} + \rho g h$, where $P_{at\ top}$ is the pressure at the surface of the water, ρ is the density of water, and h is the distance below the surface. How would you explain that equation to yourself?

Alex initially treats the equation as a computational tool, similar to how she does with the velocity equation. Later on, she describes the equation as a tool that can be used for plotting a graph. These two explanations treat the equation as an algorithmic formalism for computing either values or points on a graph, rather than connected to an intuitive meaning through symbolic forms. Later, Alex points out a particular similarity in structure between the velocity equation and the pressure equation:

> 126 A: The structure where, that, if you have an initial value that's not, that's only dependent on itself plus a set of other values that are being multiplied together, I guess in this case, and those two variables are changing, they're usually changing, which gives you a value that's at any given, a different time or a different height or depth, can change the whole equation. But then that first one, it's still going to stay the same, it's still going to be constant. So, I guess it's an equation that's mostly dependent on the second, these two, these two sets of variables at the end.

Here, Alex provides no evidence that she is treating the second term as a single quantity with a conceptual meaning. Instead, she talks about the $\rho g h$ term in the equations as a collection of symbols: "a set of other values that are being multiplied together." Overall, this segment is coded as "did not use symbolic forms."

In Pat's reasoning on this prompt, he makes a different kind of connection between the pressure equation and "more familiar ones:"

> 112 I: So you mentioned something about that this, you know you try to understand it. It looks analogous to some of the equations you've seen before. Uh, could you tell a little more about that? Which equations it is?
>
> 113 P: Well pretty much any of the kinematic equations that start with an initial condition, well a lot of equations start with an initial condition. So I think of $P_{at\ top}$, and I see the other one and think of change and, the other one, like in my mind I'm kind of thinking about the area that this equation is describing, so you have a point underwater, and you have a single line

shooting up to it, and I guess that's probably where I fall short on the equation, because when I think of pressures, I think of areas and there's not really any area involved.

114 I: Ok.

115 P: But it reminds me of potential energy problems or any problem where you have a certain condition and then something else happens to it, say a force or energy transfer or something else and you, that's expressed as a change and an initial and final condition.

116 I: Ok, so you're seeing the $P_{at\ top}$ as, as the initial condition and then the ρgh as the change.

117 P: Yes.

Pat's reasoning here is coded as using the *Base + Change* symbolic form. Like Alex, he draws a comparison to other ideas from physics he knows, but rather than computational or graphical similarities, he points out a similarity in the conceptual interpretation. Much like his interpretation of the velocity equation, Pat connects this equation to a conceptual process: you start with a certain condition and "something happens to it" that causes a change. In the equation this is expressed as a change, an initial condition, and a final condition, through the three terms in the equation.

Epistemology segment: how do you know when you understand an equation?

After the problems using the pressure equation, the interviewer asks Alex when she knows she understands an equation. Alex replies:

122 A: I mean, I still don't even, I mean, I can know this. V equals initial velocity plus acceleration, time, like, I can know that equation, but sometimes it's hard, even when you think you know an equation, sometimes it's hard to know when to use it or how to use it or, um, what it really means.

...

128 A: Um, because just sometimes when you're doing a physics problem, uh, you know something can seem really complicated, but really it would be really simple and you wouldn't know, you would, there's just so much to know, I guess, that you might not think to realize that maybe I can use this basic kinematic equation rather than some really complicated equation. And it's just knowing when to use which equation, I guess. That can make it hard sometimes.

Alex points out a difference between knowing the form of the equation and knowing when to use the equation. For example, you can know the velocity equation, but you might have trouble knowing when you can use it. She emphasizes that it could be simple, but that "there's just so much to know." This aligns with a sense that physics involves a lot of disconnected ideas or that there are a lot of ideas to memorize.

This segment is coded as indicating a *disconnected* epistemological stance, because of the emphasis that there are so many ideas to know, agreeing with items C1 (Knowledge in physics consists of many disconnected topics) and C2 (A significant problem in learning physics is being able to memorize all the information I need to know). Also, her description of knowing how to use each equation relies on knowing when to use an equation in problem solving rather than knowing the conceptual meaning behind that equation, agreeing with item C7 (I do not expect physics equations to help my understanding of the ideas; they are just for doing calculations).

In response to this same question, Pat, on the other hand, emphasizes understanding what the equation means in relation to the real world:

> 203 P: I know I really understand an equation when I can tell where each of the values, where each of the terms is coming from. I mean, the first thing you need to know is what each of the values represents in a real world application of the motion or phenomenon that it's describing, but I usually know when I really understand an equation when I understand what each term means and can conceptualize in my head where all of the values are, what each term is doing I suppose. In the sense of, say you have a function or a variety of values over a graph or something like that. Let's say like, if this value is higher, what does that mean for the motion and what does it, err, what does that mean for the, what you're given? And what does it mean for what you get? And how do they interact with each other? So if I have all of those interactions going together in my head, say for the F equals M A problems.
> 204 I: Ok.
> 205 P: Then I would know that, it makes sense to me that, uh acceleration would be proportional to the force and inversely proportional to the mass just from my own experiences, and I would understand what each of those values meant in relation to the others and why they are in the positions they're in.

To understand a physics equation, Pat thinks about a "real world application of the motion or phenomenon," which disagrees with item R3 (The subject of physics has little relation to what I experience in the real world.). He does so in order to understand the terms and values in the equation, agreeing with S2 (In physics, it is important for me to make sense out of formulas before I can use them correctly.). Pat gives the example of the equation $F = ma$, with which he considers proportional relationships between the three terms. For him, these mathematical relationships make sense in relation to what he knows from his own experiences, agreeing with S7 (When studying physics, I relate the important information to what I already know rather than just memorizing the way it is presented.). Pat seeks to understand the underlying meaning and structure of this equation with respect to his own experiences, so this segment is coded as indicating a *coherence* epistemological stance.

Epistemology segment: what's difficult about math in your physics course?

The interviewer asks what's difficult about math in physics class? Alex replied:

> 133 A: Um…I'm not, I don't think I really have trouble understanding the math. It's usually just, I mean, I can understand the basic equation and you

know, like algebra-wise what's going on, you know, like how they're moving everything around or whatever they're doing, but it's usually just that, it's, it's easy to see the teacher do it and watch him use all the equations, and like, you know, whatever order he uses them in, but then when you have to do it on your own, it's sometimes harder to, you know, remember, because there's just so many different types of problems. So you might think you understand it, but then you actually try to do it, like, in a different type of problem, and it might not work [laughs].

Alex's answer emphasizes the notion that there are different types of problems, and that you have to learn a particular methods to solve each type, agreeing with item C5 (If I want to apply a method used for solving one physics problem to another problem, the problems must involve very similar situations). This, along with a sense that it is difficult to remember how to solve each type of problem, is evidence that Alex, in this moment, is not treating physics as connected: coming from a small set of fundamental relations that can be used to solve many problems that seem different. Rather, physics is fragmented, and therefore it is hard to remember all the constituent parts, agreeing with items C1 (Knowledge in physics consists of many disconnected topics) and C2 (A significant problem in learning physics is being able to memorize all the information I need to know). This ultimately leads to a *disconnected* epistemology code for this segment.

The interviewer asks Pat what he finds difficult about the math used in his physics course. In his response, Pat explains the interconnectedness of physics and calculus:

277 P: Well I've already, I'm a freshman and when I was in high school, we had non-calculus based physics, but taking calculus then as well, meshing the two is very intuitive, since, I don't know for sure, but I assume that Newton's contributions to both fields weren't an accident, and they're very interconnected.

…

280 I: Can you tell me what are you thinking about the connection? You said that the two fields are connected?

281 P: Yeah, in that physical phenomena doesn't really follow linear equations very often. When they do, you're lucky, but even if it does, there are, you think about rates a lot and integration is also very helpful when you have a rate and you'd like to get a position, so tools like that have obvious applications like the first things you learn, or at least I learned in calculus when talking about derivatives and integrals were position, velocity, and acceleration graphs and seeing how they relate, how the graphs relate to each other, and how the equations relate to each other, the idea of the area under the curve or tangential slope.

282 I: Ok.

283 P: Just, that's the most, I think people use that example so often, because it's very intuitive and because you do have a little bit of background in that, in that when you're explaining it obviously velocity is the change in

distance over time and acceleration is the change of velocity over time, so relating that concept of slope is pretty easy. The integral is a little bit more of a pain, but it also starts to make a little more sense like when you do it over time, well suppose, I guess there's an inverse operation to that as well.
284 I: Mhm.
285 P: So the connection between calculus and physics is very clear to me.

Rather than just memorizing facts from calculus and physics, Pat here is describing how he relates the ideas of position, velocity, and acceleration to the graphs and equations to make sense of the material. This indicates that he sees these pieces of knowledge as connected, disagreeing with C1 (Knowledge in physics consists of many disconnected topics). It also indicates that in learning calculus and physics, Pat sees the connections between the two, agreeing with S7 (When studying physics, I relate the important information to what I already know rather than just memorizing the way it is presented.). Through this, Pat's statements are coded as indicating a *coherence* epistemological stance.

Results of the Coding

Illustrating the coding scheme by summarizing Alex and Pat

To illustrate how the interview coding is aggregated, the results of Alex's and Pat's interviews are summarized in tables 4.1 and 4.2 respectively. Alex used symbolic forms on 0 out of 5 of the coded symbolic forms segments and espoused a *coherence* epistemology stance on 0 out of 4 of the coded epistemology segments (omitting the ambiguous segment). Pat used symbolic forms on 4 out of 5 of the coded symbolic forms segments and espoused a *coherence* epistemological stance on 3 out of 4 of the coded epistemology segments. This qualitatively aligns with the prediction of the original research question: interviews that tend to include symbolic forms-based reasoning also tend to include expressed *coherence* epistemological stances.

Alex's Interview			
Symbolic Forms Segments	**SF Code**	**Epistemology Segments**	**Epistemology Code**
Explain the velocity equation	Did not use	When do you understand an equation?	Disconnected
Two Balls Problem	Did not use	Difference between equations in physics and math?	Disconnected
Explain the pressure equation	Did not use	What's difficult about math in physics class?	Disconnected
5/7 Meter Problem	Did not use	Photographic memory prompt	Disconnected
Mars Lake Problem	Did not use	"Diana" prompt	Ambiguous
Total SF used fraction	0/5	**Coherence Epistemology fraction**	0/4

Table 4.1. Results of coding the symbolic forms and epistemology segments in Alex's interview

Pat's Interview			
Symbolic Forms Segments	**SF Code**	**Epistemology Segments**	**Epistemology Code**
Explain the velocity equation	Used	When do you understand an equation?	Coherence
Two Balls Problem	Used	What's difficult about math in physics class?	Coherence
Explain the pressure equation	Used	Photographic memory prompt	Disconnected
5/7 Meter Problem	Did not use	"Diana" prompt	Coherence
Mars Lake Problem	Used		
Total SF used fraction	4/5	**Coherence Epistemology fraction**	3/4

Table 4.2. Results of coding the symbolic forms and epistemology segments in Pat's interview

Results of coding the 13 interviews

The 13 interviews in our data set were coded using the coding scheme. For each interview, I compared the fraction of symbolic forms segments where symbolic forms-based reasoning was used to the fraction of epistemology segments that provided evidence of a *coherence* epistemological stance. The results, shown below in table 4.3, are also given as decimal values.

	SF Fraction	Coh. E Fraction	SF %	Coh. E %
Colin	5/5	2/4	1.00	0.50
Meg	4/5	5/6	0.80	0.83
Pat	4/5	3/4	0.80	0.75
Sam	3/4	2/2	0.75	1.00
Vanessa	2/3	2/4	0.67	0.50
Sarah	3/5	0/3	0.60	0.00
Devon	1/2	1/5	0.50	0.20
Stan	2/5	3/6	0.40	0.50
Scott	1/5	3/4	0.20	0.75
Alex	0/5	0/4	0.00	0.00
Fred	0/5	4/6	0.00	0.67
Jim	0/6	0/4	0.00	0.00
Peter	0/5	2/5	0.00	0.40

Table 4.3: Fractions and decimal values of symbolic forms use and expressed *coherence* epistemological stances in the 13 interviews

Plausibility over Statistical Significance

Note that this is small-N data in two respects: the small number of total interviews (13) and the small number of symbolic form or epistemology segments within an interview, ranging from 2 to 6. Additionally, as discussed earlier, there is a lack of standardization across these interviews. Because of these two issues, the goal is not to arrive at generalizable statistical claims. Instead, I consider these results to be a rough, overall summary of each interview.

Rather than providing strong statistical support, I treat the coding as qualitative summaries of the 13 interviews. Looking at the coding results in this way suggests some broad trends. Grouping interviews according to whether they a) tend to include symbolic forms use or not and b) tend to espouse a *coherence* or a *disconnected* epistemological stance, leads to results shown in table 4.4. The 4 interviews that are split evenly (fraction = 50%) in either symbolic forms use or epistemology coding are omitted from this table, because it's unclear which way to group them.

		Symbolic Forms Segments	
		Used SF	Did not use SF
Epistemology Segments	Coh. E.	3	2
	Dis. E	1	3

Table 4.4: Grouping interviews according to tendency to use symbolic forms and tendency towards *coherence/disconnected* epistemological stances.

6 out of 9 interviews in this table align with the initial prediction that symbolic forms use tends to co-occur with *coherence* epistemological stances (3 interviews), and absence of symbolic forms use tends to co-occur with *disconnected* epistemological stances (3 interviews). Although we can perform some quantitative analyses on this table[10], this is not the goal. Rather, the coding of an interview broadly summarizes the results of a qualitative case study, focusing in particular on symbolic forms use and epistemological stances towards learning and understanding physics and equations in physics. In this way, 6 interview case studies support a connection between tendencies for symbolic forms use and tendencies to express *coherence* epistemological stances.

Qualitative investigation of why 3 interviews go against the prediction

Looking at table 4.4, how can we understand the interviews that disagree with the initial prediction? These 3 interviews, those of Fred, Sarah, and Scott, are also the only ones where the symbolic forms use and *coherence* epistemological stance percentages differ by more than 50%. What are the implications for the proposed connection between epistemology and symbolic forms use? One possibility is that this connection simply does not hold true for these interviews.

[10] Fisher's exact test reveals that the probability of this distribution arising (from the constraints that out of 9 interviews total, 4 interviews tended to include symbolic forms-based reasoning and 5 tended to included "coherence" epistemology statements) given the null hypothesis is p=.32.

Another possibility is that the coding scheme could be improved to better capture aspects of epistemologies that might plausibly contribute to symbolic forms use. Looking at the specific interview responses supports this latter interpretation. The benefit of the interview data over an epistemological survey in this case is that this interview coding also allows for exploration and reinterpretation of the coding scheme's definition of a *coherence* epistemology.

For example, Sarah tended to use symbolic forms but also tended to espouse a *disconnected* epistemological stance. One reason why this might make sense is if the *coherence/disconnected* categories of the coding scheme do not fit exactly with epistemological stances that support symbolic forms use. For example, Sarah, in describing a disagreement between her intuition and a demonstration in her physics class, stated that she trusts physics equations over her intuition, which was coded as espousing a *disconnected* epistemological stance:

> I: Ok, your intuition will tell you that it would end up going that way, but what ended up happening is that it sort of went that way.
> S: Yeah, and that's what physics requires. Like, your intuition can be wrong, yes, so you really can't use intuition. You just have to really rely on your equations, and then, you just have to, I guess, know how they have to go. Like not from your experience or, yeah, that what I'm saying.

It's possible that this epistemological stance, coded as *disconnected*, does not dampen symbolic forms-based reasoning. Although she is stating that she expects the results of physical intuition to at times disagree with calculation, this does not necessarily imply that Sarah does not seek a conceptual interpretation of the calculations or the equations that are used in those calculations. It is possible that one can take a stance that everyday intuition is unreliable for making predictions while simultaneously seeing conceptual ways of understanding the more reliable formal calculations.

On the other hand, Fred tended not to use symbolic forms but did espouse a *coherence* epistemological stance. One way this could align with the prediction in the original research question is if he seeks coherence between the math and the concepts in ways other than symbolic forms use. For example, while problem solving Fred evaluates his final numerical solutions to see if they make sense:

> F: Well, um, if, when I get the final answer on the exam, I always think about the number I get and think if it's logical or not. Like, for example if I'm solving a problem with a gun shooting a bullet, and for the speed of the bullet I get like 5 m/s, that's not logical. It should be like a couple of hundred, maybe even a thousand.
> I: Right.
> F: And if I get a really, number I know is not correct, I'll go back and see what I did wrong.

Although Fred is interpreting his numerical solution with his common sense ideas of how the world works, he is connecting numerical values, rather than the structure of equations, to conceptual meaning. Symbolic forms-based reasoning with equations is a separate type of mathematical sense-making from checking numerical solutions against common sense. It is possible that individuals espousing *coherence* epistemological stances seek connections between formal mathematics and intuitive reasoning in ways other than through symbolic forms use.

These examples point to the fact that the CLASS categories and the coding scheme for *coherence/disconnected* epistemological stances might not match 1-to-1 with symbolic forms use. Symbolic forms use is one particular aspect of mathematical sense-making, whereas the CLASS categories are meant to capture broad information on students' views towards learning physics. Further work needs to be done to see if the coding scheme derived from the CLASS categories can be refined to better target epistemological stances that plausibly contribute to symbolic forms use.

PHASE 2 RESULTS

The phase 1 analysis sought to establish a connection between symbolic forms use and particular epistemological stances. But should this connection be understood as a reflection of a unitary and stable cognitive structure or as a locally coherent activation of a subset of manifold cognitive resources? In the phase 2 analysis, I present a case study of how Devon shifts between two modes of reasoning with and epistemological stances towards equations in order to support the usefulness of the manifold perspective here.

Contrasting Devon's Reasoning on the Velocity and Money Prompts

Devon's interview starts with the same prompts as Alex's and Pat's interviews: "explain the velocity equation" and the Two Balls Problem. On these prompts, Devon does not use symbolic forms. After an 11 minute conversation about Devon's experience in his courses, his personal history, and how he approaches learning in physics and math courses, the interviewer (Ayush Gupta) asks him the money prompts: "explain the money equation (with numbers)," "explain the money equation (with symbols)," and the Two Accounts Problem. On these money problems, Devon does display symbolic forms-based reasoning. Here, I present Devon's reasoning on these prompts in more detail.

Devon explains the velocity equation as a computational tool

The interview starts with Ayush asking Devon to explain the velocity equation to a friend from class. Devon starts by checking to make sure the units of the terms are all the same. This type of dimensional analysis is a procedure common to physics courses. From there, Ayush asks Devon how he would explain this equation to a 12-year old middle school student. Devon replies:

> [00:04:39]
> D: I don't think little kids, middle school kids should know this stuff? I mean, I know I didn't. My first exposure wasn't until senior year of high school, so but if I had to, I'd probably put it in simpler terms. Like, you

> know velocity, it's a vector, but, you know, probably younger kids, middle school kids are more familiar with the concept of speed and not needing the direction, not needing the magnitude, so if I just keep it in terms like that, you know, they get the final speed of an object, you know, you need to know the initial, you need to know the rate it accelerates, you need to know the time, probably simpler terms, if that makes sense.

His description of the equation, like Alex's, focuses on its computational use ("…they get the final speed of an object, you know, you need to know the initial, you need to know the rate it accelerates, you need to know the time..."). Devon has not provided evidence that he is using *Base + Change* or any other symbolic form in his reasoning here.

<u>*Devon shies away from the Two Balls Problem*</u>

Next, when Ayush asks Devon how he would solve the Two Balls Problem, he seems apprehensive, checking with Ayush about what exactly the problem is asking. Ayush reassures Devon that he is only interested in how Devon thinks about this problem, not whether or not Devon can solve it correctly. Devon eventually says that he wasn't expecting to have to answer physics problems:

> [00:07:00]
> D: I feel so dumb, I feel like I didn't get anything out of, to be honest, I just forgot, OK. I forgot a lot of stuff over the winter holidays, because that's the way my mindset works. And I feel on the spot here, under pressure. I didn't expect to have to answer questions. I thought it was "what did you think of the class?"
> A: Actually, that's a good thing, what did you think of the class?

At this point, Ayush changes the topic here to talk about Devon's experiences in physics, talk about his personal history, and ask epistemological prompts about learning equations in math and physics. Devon describes himself as a "math guy," which Ayush leverages to introduce the following isomorphic, non-physics problems as "math problems."

In Devon's reasoning on the two velocity prompts, there is no evidence of symbolic forms-based reasoning.

<u>*Explain the money equation*</u>

On the "explain the money equations (with numbers)" prompt, Devon quickly and unproblematically generates the correct equation. However, there is no evidence yet that the *Base + Change* symbolic form is involved. Ayush quickly moves on to ask Devon the "explain the money equation (with symbols)" prompt.

Devon immediately writes out the correct equation ($\$m = rd + \$m_0$), and starts to talk about the similarities he sees between this equation and the velocity equation:

[00:20:45]
D: Err, yeah, so, your final, how much money you make is equal to your salary per day times the number of days you work plus the initial, yeah... OK, I kind of see where you're going now, OK...
A: Tell me a little bit about, tell me a little bit more about what you were saying about where I'm going.
D: I think you're trying to relate this back to the other velocity equation somehow, I don't know.   I mean it's in that form, like your total, I mean your final dollars is just like your final velocity and your initial is just like your v_0 and then *a* would be your rate per day, your dollars per day, I guess and then *t* would be like however many days you work, so, yeah, I mean it's a variable of time so…
A: So, you were seeing these two equations as kind of similar.
D: Yeah.

Here, Devon sees the connection between the money equation and the velocity equation as having the same sort of structure with the same variables (initial, rate, and time), but not yet seeing similarity as an underlying *Base + Change* meaning. Ayush follows up by asking Devon how he would explain the equation to a 12 year old, and Devon responds with an answer that incorporates the *Base + Change* symbolic form:

[00:22:10]
D: I mean, I'd just I'd probably just read out the problem in its entirety and just say "so, now count how many days is between Monday and Friday, you know Monday, Tuesday, Wednesday, Thursday, Friday, and that's something they can do on their fingers, and so they'd say five.   So, if you work five days and you get so much money per each day, what do you do to calculate your total earnings for those five days?   And I think by twelve they would know, oh just multiply by how much you get per day, OK, and that's going to take care of this, the r times d, and then you know to get the total, they already know what you start off with so they would know to add it to that.
A: Why add?
D: Because you want to have the total, like for, you start off with a certain amount, and you want to know how much you have after the week, so your initial amount plus how much you earned that week is equal to the total amount of money that you have.

Devon explains why *rd* represents the total earnings for a certain amount of time. Upon further questioning, Devon explains that this quantity ("how much you earned that week") is added to the initial amount to equal the total amount of money. This explanation reflects the use of *Base + Change*, since the conceptual schema of a change added onto a base value is tied to the form of the equation.

A blended processing solution on the Two Accounts Problem

Devon's solution on the Two Accounts Problem, which is immediate, leverages the *Base + Change* symbolic form:

> [00:23:46]
> D: You'd have $50 more. You work the same number of days and you get the same amount, so the only difference is the starting, how much you start with, so if you had $50 and she had none, then you'd have $50 more than her. Because you both get the same amount throughout the week, you both earn the same amount through working.
> …
> [00:26:00]
> D: …Basically, for Lisa, you just say your final money is just equal to the rate per day times the number of days and just not add your initial to it.

Devon's solution here uses a conceptual schema applicable to all of the Two [*X*] Problems: "if two things change by the same amount, then the difference between them stays the same." At the end of his reasoning, he explicitly connects this conceptual schema to the form of the money equation. This reasoning is supported by the *Base + Change* symbolic form, which brings attention to the conceptual entities *base*, *change*, and *final* in the equation (Kuo et al., 2013).

Differences between the velocity and money prompts that support different modes of reasoning with equations

Devon's use of *Base + Change* on the money prompts counters any unitary, deficit models of Devon's initial reasoning with the velocity equation. Although he doesn't use *Base + Change* in using or explaining the velocity equation, he does so with the money equation, so it cannot be that Devon has a "symbolic forms knowledge deficit." So why does Devon draw on symbolic forms-based reasoning for one equation and not the other? There are several differences in the content and substance of these two sets of prompts that may have contributed to the presence or absence of symbolic forms-based reasoning.

First, the content of the equations is different. We expect that students have more experience reasoning about tangible quantities such as money than with velocity and acceleration of objects. Research has shown that college physics students have difficulties in understanding velocity and acceleration, even after instruction (Trowbridge & McDermott, 1981). Sherin (2001) hypothesized that students may have difficulty in interpreting acceleration as a rate, because it is a rate of change of another rate, velocity. Although we see no evidence of this here, it is one example of a conceptual difficulty that students may have with the topic of velocity that is not present with the topic of money. These conceptual difficulties may suppress interpreting terms like *at* as changes in velocity added onto the initial velocity.

Second, the tasks ask students to engage with the equations differently. The velocity equation prompt required explaining a familiar equation from physics to a student. The money equation prompts were designed to require the construction of an equation from a given set of values. Constructing the equation may have helped Devon see the underlying connection between the structure of the equation and an intuitive interpretation of that structure. Perhaps prompting Devon to construct the velocity equation would have elicited *Base + Change* reasoning as well.

Without arguing against the possible influence of these two differences, I also aim to show how Devon's epistemological stances towards understanding physics and math in this interview are plausibly connected with his different reasoning with equations on the velocity and money prompts. In the next section, Devon's epistemological stances towards equations in physics and math are compared.

Devon's Expressed Epistemological Stances Towards Math and Physics

Here, I seek to identify Devon's epistemological stances towards understanding equations in physics and math. The word *stance* emphasizes the fact that Devon's epistemologies are not modeled as stable beliefs but rather in-the-moment positions on knowing and understanding, resulting from the in-the-moment activation of a subset of Devon's epistemological resources. Broadly, Devon's epistemological stance towards physics, as expressed in this interview, is that physics consists of a disconnected set of concepts and related equations, whereas he talks about math as coherent and well defined.

Immediately after Devon voices his apprehension towards the Two Balls Problem, Ayush takes Devon's suggestion and segues the conversation to talk about what Devon thought of the class. Subsequently, Devon articulates an epistemological stance towards physics as consisting of idiosyncratic concepts:

> [00:08:08]
> D: … I can do all these math, like number theory, linear algebra, you know, differential equations, but when it comes to physics, there's just so many other concepts, like you need to know what happens if there's gravity's involved, what happens if there's no gravity involved, and I just find it just too complex for my mind…

Whereas one could view having or not having gravity as special cases in a general reasoning framework of kinematics and dynamics, Devon hints at seeing these as different lines of reasoning ("what happens if there's gravity involved, what happens if there's no gravity involved"). Managing this large set of disconnected concepts introduces complexity and a sense of difficulty for Devon.

Devon goes on to talk about how he thinks about equations differently in math and physics:

> [00:08:42]
> A: So, when you're looking at equations in physics or equations in math, is there a difference in the way you look at them?
> D: Yes, there is. In the equations in math, it's just so structured, you know, in my linear algebra class right now, we're learning about matrices and how to find the reduced echelon form of a matrix, and it's just so structured. There's just one fixed rule of how to solve it. In physics, there's just so many different cases, you know, like I remember in [my 1st semester introductory physics course] there were all those different kinematic equations. You had to use one for one case and it's just... too much. Yeah.

> A: Is there a difference in the way you understand math equations versus physics equations?
> D: Yeah, I think so. I understand math a lot easier, because you don't have to think of all these different situations. There's not gravity involved in one case, or whatever, you know.
> A: It's all the special cases that makes the physics much more difficult.
> D: Yes, and it's in the math too, it's just, you don't need to know all these concepts like torque, and, you know, spring constants, and if friction is involved and, math is just, pure math is just so much easier. I'm enjoying my number theory class right now where we're just learning proofs and diophantine equations and, you know, stuff like that, I enjoy that.

Devon describes math as being more structured, and problems having one well-defined approach. However, in solving a kinematics problem in physics, the different conceptual factors that can be present or absent from a problem lead to a number of different considerations. For Devon, it's the consideration of these special cases that are difficult in physics, which contrasts with his perception of the straightforwardness of math.

This split in epistemological stances towards math and physics is also present when Devon describes his different approaches towards learning equations in math and physics. Later on in the interview, after the 3 sets of isomorphic problems, Devon is asked when he feels he understands an equation in both physics and math:

> [00:36:02]
> A: When do you feel really comfortable with an equation, when do you feel that you really understand an equation?
> D: In physics or math?
> A: Tell me about both.
> D: Well, in physics, I feel comfortable when I memorize the thing, and I know all the units that are attached to it. 'Cuz as I said, I like, I'm a concrete sequential kind of guy in the math, so if I know, if I could see that the units make sense, then I know what I'm doing must be right, I don't, I just don't like thinking of the concepts behind it, I don't like thinking of gravity, you know, Other people think because of this, because of gravity, it's going to do such and such, I'm not, you know, I like just focusing on units and just if it makes sense, and I just memorize the equation, I mean, other people can derive the equations by, I don't know, Newton's second law or doing the free body diagrams and they can derive an equation or a certain kinematic-, but I don't do that, I just think of what makes sense, unit wise, I guess.

Coherent with his stance that what makes physics difficult is all the disconnected concepts, Devon's approaches to learning physics equations explicitly avoids these concepts. Instead of seeking to sort out all of the conceptual issues, Devon approaches learning physics equations through memorization. Although he points out that other people might be able to derive equations through more fundamental principles and think about physics concepts – what Hammer would take as evidence of *weak coherence* or *weak concepts* (Hammer, 1994) – Devon points out that his own approach is to memorize the final equation.

Next, Devon describes how he approaches learning equations in math.

> A: And what about math? When do you feel that you really understand an equation in math?
> D: Well, in math, well like, there's so many proofs, and it just makes sense in my mind, I don't know, like derivatives and integrals and Jacobian transformations. It just all makes sense to me, because there's a reason it works, and it's just one reason. It's not like in physics really where there's so many different cases like I said before. In math, like if I understand the proofs of why it's that way, and then I'm comfortable using that equation.

Devon speaks about equations in math making sense and understanding why those equations make sense leads to comfort with using that equation. Although it's still unclear how or why math equations make sense to him, Devon contrasts his understanding of math to physics by saying that math seems more structured and that he feels he can make more sense of math. Importantly, Devon seeks out an understanding of the underlying reasons why equations in math work or make sense, whereas he just memorizes physics equations to explicitly avoid dealing with the underlying physics concepts behind the equations.

These different epistemological stances towards equations in physics and math align with his symbolic forms use on physics and money prompts. Devon's epistemological stance towards physics concepts as a set of disconnected ideas and rules likely suppresses symbolic forms use. Because these concepts overly complicate physics for Devon, he does not seek the underlying meaning of equations. Since symbolic forms, such as *Base + Change*, represent one underlying conceptual meaning of equations, avoiding the underlying conceptual meaning of physics equations likely suppresses symbolic forms-based reasoning with those equations. On the other hand, Devon's epistemological stance towards math is that it is well-structured and "makes sense," leading to learning approaches that seek the underlying reasons why equations make sense. Seeking the underlying meaning of why equations make sense likely supports the use of symbolic forms, intuitive ways to understand the underlying structure of equations.

Furthermore, there is explicit evidence that the velocity prompts are seen as "physics" in the interview and that the money prompts are seen as "math" in the interview. The velocity equation is introduced as an equation that he might have seen in his physics course, and in deferring solving the Two Balls Problem, Devon says that he forgot a lot over the winter break, suggesting that he views the velocity prompts as relating to last semester's physics course. For the money prompts, the interviewer introduces them as "math" questions, building off Devon's discussion on the distinction between math and physics. At the end of the interview, Devon himself indicates that he viewed the Two Accounts Problem as a "math" problem.

In the language of epistemological framing as the assembly of a set of coherent conceptual and epistemological resources, Devon epistemologically framed the velocity prompts by drawing on conceptual resources supporting "plug-and-chug" and epistemological resources supporting physics as idiosyncratic and made complicated by the various concepts. By contrast, he framed the money prompts by drawing on symbolic forms for reasoning with the equation and epistemological resources supporting math as "making sense." These two different modes of epistemological framing can help us understand the connections between Devon's expressed epistemological stances in the interview towards physics and math with his reasoning with equations on the velocity and money prompts, respectively.

Again, although I describe Devon's epistemological stances towards "physics" and "math," the manifold perspective I adopt here suggests that these stances towards the disciplines of physics and math do not necessarily represent stable epistemological beliefs about the two disciplines. Based on the stability and consistency of Devon's epistemological stance towards math in the interview, one might suspect that Devon, in general, tends to view math equations as making sense and seeks to understand the proofs behind why those equations work. Although this tendency may be true, it is similarly possible that, in a particular situation, Devon may enact an epistemological stance supporting the memorization of an equation when dealing with particular difficult topics in math.

Using this case study, connections can be drawn between Devon's reasoning with the velocity and money equations and Devon's epistemological stances towards physics and math, respectively. Going further into the interview, I argue that this alignment can help explain finer-grained dynamics in Devon's reasoning that are not apparent from the coarse coding of Devon's reasoning and epistemological stances on the physics and non-physics prompts alone.

Shifts in Reasoning with Symbolic Forms on the Speed Prompts

The third and final set of prompts relate to the speed of cars. These prompts were designed to bring the problem content closer to velocity while still maintaining a sense of everyday reasoning.

Devon explains the speed equation as a computational tool, at first

Devon again makes an explicit connection to the past problems. He constructs a similar equation ($s=s_0+rt$) and points out that it's similar to previous equations in the structure of *[final value] = [initial value] + [rate of increase][time]*. To try to get Devon to articulate the intuitive meaning behind the equation, Ayush prompts Devon to explain the equation to an 8th grader:

> [00:27:47]
> D: Um... Well, I mean, it's, it's a little tougher conceptually than doing money, because when you're, you can't really, I mean with money, it's so much easier, 'cause it's just like basic numbers. It's a physical amount, and here you're dealing with all these concepts, like speed, you know, it's not like something you can hold, you know, so I guess I'd say, I'd actually assign numbers to these. OK. Probably that would be the best way to do it…

Devon generates hypothetical values to plug into the equation in order to explain it more easily. In the end, he explains the equation as a computational tool, saying that he would also show the 8th grader how the units of the equation work out.

Here, Devon himself identifies a difference between this prompt and the velocity and money prompts: speed is not a tangible object, like money is, making the explanation of the equation more difficult. Granted, this comes in response to needing to explain to an 8th grader rather than in developing his own understanding of the equation. But even still, that Devon is conscious of this difference here provides some evidence that this difference is salient in his own reasoning. His explanation treats the equation as a computational tool and involves checking the units, two similar pieces to his explanation of the velocity equation.

Ayush then continues this line of questioning, leading to a different explanation of the equation:

> [00:29:36]
> A: OK, so I'm wondering, OK, I'm the 8th grader now and I'm wondering, I'm going fifty miles per hour, and I speed up 2 miles per hour every second and I'm wondering, at the end of ten seconds, how much is my speed? So, I'm still wondering how I get to the final speed. How would you explain that to me…as an 8th grader?
> D: I messed up in my last reasoning I guess.
> A: No, meaning, so far everything is good, I'm just trying to, you know, I'm being an annoying 8th grader who's…

Devon interrupts, noticing that he made the previous unit cancellation too complicated, because he tried to convert all the "hours" into "seconds," which is unnecessary to show that the units work out. He then initiates a *Base + Change* explanation of the equation, using some of the specific values he invented earlier.

> [00:30:33]
> D: …I don't know, I could just say, look at it this way. You're going ten seconds, and you know you're going two miles per hour faster each second, so two times ten. What is the total miles per hour you increase in that ten seconds then, so two times ten. That's twenty, so twenty plus your initial fifty miles per hour, so the final would be 70 miles per hour. There you go.
> A: 8th grader is happy.

In this exchange, Ayush restates his question, trying to get Devon to articulate a deeper conceptual meaning. Devon interprets this as a signal that he has given an incorrect explanation. Ayush assures Devon that his reasoning so far is appropriate, but that he is playing the role of an "annoying 8th grader." Devon then plunges into an explanation incorporating *Base + Change*: the rate times the time tells you "the total miles per hour you increase in that ten seconds" and so adding that to the initial speed will give the final speed. Ayush comments that this satisfies the annoying 8th grader.

Devon here begins his explanation with one similar to his explanation of the velocity equation, a computational tool, and moves to one similar to his explanation of the money equation, incorporating the *Base + Change* symbolic form. In this shift, the effect of Ayush's continued questioning is apparent. Ayush has expressed some dissatisfaction with Devon's first answer, so he gives another. In this way, Ayush signals that he is seeking a deeper explanation, which Devon initially interprets as signaling an error, and he takes up the question as requiring a different type of explanation. Had Ayush expressed such dissatisfaction with Devon's reasoning on the velocity problem, it's possible that Devon may have given a symbolic forms-based interpretation of the velocity equation.

Possible hypotheses of what Devon will do on the Two Cars Problem

Based on the interview so far, what reasoning should we expect from Devon on the Two Cars Problem? There is evidence to suggest two different possibilities.

Devon may use symbolic forms-based reasoning here. By this time in the interview, Devon has shown facility in using *Base + Change* to explain the money and speed equations and on the Two Accounts Problem. On the money prompts, he showed facility in using *Base + Change* to explain the equation he constructed, and he used a blended processing solution on the Two Accounts Problem. Devon has been building connections between the problems, explicitly pointing out the similar structure of the three equations. All of this supports the use of *Base + Change* on the Two Cars Problem for blended processing.

However, there is also reason to suspect that that Devon will not use symbolic forms-based reasoning here. Devon has pointed out here that thinking about speed is a little more difficult than thinking about tangible quantities like money. His initial explanation is more computational ("just plug it in here"), and it is only from the further prompting from Ayush that Devon starts to articulate symbolic forms-based reasoning. So it is reasonable to project that Devon's solution to the Two Cars Problem will be more computational, at least before further prompting from the interviewer.

Devon's reasoning on the Two Cars Problem: an initial computation use of equations, followed by a shift to symbolic forms use

What actually happens is neither of these two predictions: Devon starts down a computational path, and then spontaneously sees the blended processing solution on his own.

Devon starts out by referencing the previous numerical computation he performed for his initial explanation to the 8^{th} grader in the previous prompt, but then realizes the applicability of the blended processing shortcut from the previous problems:

> [00:31:43]
> D: I'd do it the same way I did it in the last problem, but just do it twice, and then find the difference. So I can just say this is car one and this is, s_0 would be five miles per hour plus um ten is your rate, ten miles per hour is your rate, times for each second, times the five seconds that it's asking for and so those

> cross out, and you're left with five plus fifty equals fifty five miles per hour. So that would do it for car two, except make the s_0 be 7... oh, man, I'm so stupid. [laughs] Duh. It's just the 2 miles per hour difference. I didn't even have to do all that. [laughs] I'm realizing it, like, that that was extraneous math.

Car 1: $S = 5\,mph + 10\,\frac{mph}{\cancel{s}} \cdot 5\cancel{s}$

$S = 5\,mph + 50\,mph = 55\,mph$

Car 2: $S = 7\,mph + 50\,mph = 57\,mph$

Figure 4.1. Devon's written work on the Two Cars Problem

Figure 4.1 shows Devon's work up until the point that he makes his realization. The use of "oh" in the moment when he makes his realization signals the beginning of a repair activity, where the reasoning that comes after is meant to replace what came before (Schiffrin, 1999). Devon explicitly says that the previous reasoning was extraneous, and he goes on to articulate another solution to the problem: the blended processing solution.

> A: Why do you say that?
> D: Because the rate at which they speed up is the same so, and the time allotted for the problem, five seconds for each car so, the rate is the same, so you know that it's going to be the same difference at the end, after you add the initial speed. It's just like the other problem with the, with Lisa, it's just like the money problem, yeah, in that I didn't even have to do the math because I knew fifty minus zero is fifty, and here seven minus five is two, so yeah.
> I: So you could have, so somebody could have answered this question without doing the calculations?
> D: Yes. Because the added miles per hour added to each car is the same. How much you add to the initial, that's why, so it doesn't matter if it's after the five seconds or not. Since they're speeding up at the same rate, you just take whatever the initial speed is to find the difference.

Devon explicitly connects his solution to the Two Accounts Problem, where he didn't have to do the math. His solution uses the same blended processing schema in the Two Accounts Problem. He refers explicitly to a *Change* being added onto a *Base* ("…the added miles per hour added to each car is the same. How much you add to the initial…").

What explains these shifts between computational and symbolic forms-based interpretations of the speed equation? Of the three differences I described earlier between the prompts – (1) differences in the specific content (velocity vs. money vs. speed), (2) differences in explanation versus construction of an equation in the prompts, and (3) differences in the persistence of interviewer questioning – the second and the third are easily excluded. Although the “explain the money equation” and “explain the speed equation” prompts are different from the “explain the velocity problem” in that they require construction of an equation in addition to explanation, none of the Two [*X*] Problems differ in this way. If he could immediately see the blended processing solution on the Two Accounts Problem, there wasn’t any obvious difference in what the Two Cars Problem was asking Devon to do that would obviously lead to added difficulty in seeing this same. Similarly, although Ayush asks Devon for an elaboration, it is after Devon himself realizes that the problem can be solved through blended processing, so the shift cannot be explained simply with further interviewer questioning.

One relevant difference is the difference in content, from money to the speed of the car. The topic of speed brings the interview prompts back to kinematics, although slightly less formally and more related to everyday experience and language. Devon explicitly says that more abstract concepts such as speed are harder to explain than more tangible objects, like money.

In addition to differing levels of abstractness, I argue that the differences between velocity, money, and speed are also connected to different epistemological stances. Money is both more “concrete” and also connected to, in this interview, epistemological stances that better support symbolic forms use than velocity is. That Devon’s reasoning with equations on the speed prompts shift from more computational or “plug-and-chug” approaches to symbolic forms use suggests that there is some feature of the prompts that is, in some sense, “in-between” the two epistemological stances he espoused earlier.

Supporting this interpretation, at the end of the interview, Devon provides evidence that he viewed the Two Cars Problem as “in-between” in terms of his epistemological stances. After the three sets of prompts, the interview again moves to the topics of Devon’s future plans and his experiences with math and physics. Ayush then asks Devon whether he viewed the Two Cars Problem as a “math” or “physics” problem.

> [00:45:23]
> A: So I'm wondering, when you saw the Lisa problem, or when you were looking at the whole "how much do you earn" kind of thing, did you think of it as kind of a math thing or a physics thing?
> D: A math thing.
> A: But when we came to the car problem, did you look upon that as a…?
> D: Not really. I still looked at it like math. I like problems like these, but when it gets to kinematics and advanced concepts, that's when it starts.

Although, ultimately, Devon viewed the Two Cars Problem as “math,” he also states that this problem is close to kinematics concepts in physics. When he refers to “when ‘it’ starts,” he is plausibly referring to his epistemological stance that physics consists of many disconnected concepts and equations. That Devon views the Two Cars Problem as both “math” and as starting to get into physics concepts helps to explain the shifts in Devon’s approaches on “explain the speed equation” and the Two Cars Problem. One way to explain why Devon doesn’t immediately leverage the isomorphism between the money prompts and the speed prompts to immediately assert a solution that avoids explicit computation or the *Base + Change* interpretation of the equation is that he initially views this problem as starting to deal with physics concepts, and therefore requiring his usual approach with physics problems and equations. This plausibly explains why Devon’s initial explanation of the speed equation similarly doesn’t leverage symbolic forms.

By the end of his work on the speed prompts, Devon has used symbolic forms on the two speed prompts. It could be that after some initial work, Devon was able to see the similarity between the speed prompts and the money prompts. What is consequential for my explanation is that Devon finally saw the Two Cars Problem as “math”. This supports the idea that Devon’s shift towards symbolic forms-based reasoning reflects an overall shift in his epistemological framing, which then also includes a shift to an epistemological stance that supports making sense of equations.

DISCUSSION

Establishing a Connection Between Symbolic Forms-based Reasoning and Coherence Epistemological Stances

The coding of Alex’s interview, Pat’s interview, and 4 others for symbolic forms use and *coherence/disconnected* epistemological stances support the initial prediction in the research question: symbolic forms use tends to co-occur with *coherence* epistemological stances. Although this fits with previous research on the connection between student reasoning and student epistemologies, previous studies have not investigated this connection between epistemology and specific cognitive elements, such as symbolic forms, which correspond to specific ways of reasoning with mathematical equations in physics. Although 3 interviews break this pattern, closer analysis of the specific responses in the interview hint at the coarseness of this coding scheme for epistemology and suggests possible revisions could increase alignment between the *coherence* epistemology coding scheme with epistemologies that plausibly relate to and support symbolic forms use. This work can inform the design of future studies that increase standardization and sample size, to study the broad generalizability of this connection that we illustrate across 6 interview case studies.

Although not generalizable, these 6 case studies that illustrate the connection I seek to show, pointing towards epistemology being consequential for student reasoning and away from descriptions of reasoning that only consider conceptual knowledge. Again, it must be that knowledge of and facility in using symbolic forms-based reasoning is important, but that models of student reasoning that only consider conceptual knowledge miss this connection to epistemology, providing a plausible alternative, in some cases, to pure knowledge deficit explanations of "naïve" student reasoning.

Manifold Cognitive Frameworks as a Theoretical Lens for Seeing Where Students Are and Where They Can Go

In Devon's case study, one factor connected to his reasoning on the different problems is his epistemology - specifically, his epistemological stances towards math and physics in this interview. However, the way in which this connection is modeled, either as instantiations of a unitary or of a manifold cognitive network, is consequential for diagnosing where students are and for plotting a trajectory of how to help these students develop problem-solving expertise.

Beyond just Devon, unitary and manifold frameworks for cognition lead to different interpretations of the coding results in phase 1. The coding broadly categorizes interviews for the tendencies to use symbolic forms or espouse a *coherence* epistemological stance. These tendencies can be interpreted in two ways. Through a unitary lens, interviews that tend not to include symbolic forms-based reasoning or coherence epistemologies suggest deficits in those two areas. These unitary interpretations would prescribe explicit instruction on symbolic forms – what they are and how to interpret particular equations with them in explanations and problem solving – or explicit instruction to develop more sophisticated epistemologies.

However, for Devon, a different intervention was successful in eliciting symbolic forms-based reasoning. The interviewer asks Devon to reason about other equations, on which it is expected that he will more likely use symbolic forms such as *Base + Change*, and this leads to symbolic forms-based reasoning in the interview. The success of this weaker intervention precludes a knowledge deficit interpretation of Devon's initial "plug-and-chug" explanation of the velocity equation. The interview built towards helping Devon see a facet of his own existing knowledge as relevant for reasoning with the velocity equation. It must be that Devon's knowledge about equations is manifold: multifaceted and contextually activated.

Devon's case study also suggests that his changing reasoning with equations in the interview is tied to changing epistemological stances towards understanding equations. The non-physics prompts were designed from the research team's intuitive expectations that they would elicit more everyday types of reasoning. Devon's responses in the interview suggest that the non-physics prompts accomplish this by, in part, supporting different epistemological stances towards knowing equations. Just as with Devon's reasoning with equations, the interview did not include explicit epistemological instruction. It must be that Devon's epistemology is also manifold: multifaceted and contextually activated.

In the education literature, conceptual change is a hotly debated process. How can we model individuals' knowledge from their expressed reasoning, and how can we understand how that system of knowledge evolves in the progression from novice to expert? One approach is to investigate and model microgenetic change in short, detailed episodes (e.g. Barth-Cohen, 2012; Gupta & Elby, 2011; Levrini & diSessa, 2008; Rosenberg et al., 2006; Schoenfeld, Smith, & Arcavi, 1993). These studies differ from pre-post survey studies by aiming to study how student reasoning changes over a brief episode, in order to understand processes of change. Devon's case study is one such example: through a sequence of interview prompts, Devon's reasoning with equations changes. In this case, Devon's reasoning throughout the interview illustrates one possible trajectory for learning to reason intuitively about physics equations by building on productive pieces of prior knowledge for interpreting algebraic equations more generally rather than requiring the acquisition of completely foreign knowledge and ways of thinking. Additionally, the success of the non-physics prompt sequence in tapping into Devon's productive knowledge for interpreting equations in intuitive ways suggests a possible design for instructional sequences.

Instructional Implications

Although this interview is not explicitly a teaching experiment, Devon's reasoning throughout the interview provides a possible instructional sequence for helping students see the intuitive meaning embedded in physics equations. By posing isomorphic, but more everyday and familiar, problems to Devon, the interviewer seeks to help Devon tap into the symbolic forms-based reasoning that he himself, in other contexts, uses. Although Devon's shifting reasoning on the speed prompts points to possible disruptions in the connection between *Base + Change* and the velocity equation for Devon by the end of the interview, this sequence could provide a starting point for helping students to see the symbolic forms in physics equations. Similarly, instructors could devise other sets of mathematically isomorphic equations and problems that might help students see how other ways in which students already understand equations are useful for understanding physics equations.

To evaluate the efficacy of Devon's interview as a possible instructional sequence, I present here the end of Devon's interview, where he is asked to reconsider the Two Balls Problem, the problem that Devon originally does not want to attempt to answer. At the end of the interview, Devon sees the blended processing solution to the Two Balls Problem.

> [00:46:33]
> D: I'm thinking, so you throw one ball down with an initial speed. When you throw it down, does that mean there's a force that you add to it to make it go at that initial speed?
> A: Yeah, One ball I throw down. I throw one at 2 m/s and the other one I just let go. So one I let go and the other, I'm throwing it down at the same instant.
> D: Uh... alright, well, alright, I'll just take a guess. I guess it's going to be equal to 2 m/s because, treating this, see, I don't even understand if it works this way, but treating this like the other ones, you have your initial speed and it's 2, whereas the other ball is zero and it goes down at the same rate and the same amount of time, so, well, 'cause the rate. The acceleration due to gravity is always the same for any object in free fall, 9.8 m/s, 10 m/s, so I guess just

like that, times the five seconds to get your meters per second and then you add that to initial to get your final.

A: OK

D: But like, the reason I'm wary is I don't, I mean, I know that if it's in free fall, I'm just so confused. I know if it's in free fall, then acceleration due to gravity is 10 m/s^2, but I don't know if it changes if there's a force added to it. Yeah, so I'm just confused about stuff like that, you know?

Devon's reasoning with the Two Balls Problem at the end of the interview indicates two aspects of the success of the interview sequence. First, Devon sees the blended processing solution that he used on the Two Accounts Problem and (eventually) on the Two Cars Problem as possibly fitting the Two Balls Problem, which he did not express initially in the interview.

Second, Devon also explains a conceptual confusion that keeps him from answering with certainty: the acceleration changes if the applied force changes, and Devon expresses his confusion about whether the thrown ball experiences an extra force from the throw that changes the acceleration during its fall. This is consistent with documented student ideas that applied forces persist with objects (McCloskey, 1983).

In previous work (Kuo et al., 2013), we argued that because symbolic forms directly leverage intuitive conceptual ideas rather than formal physics concepts, students can learn to use symbolic forms in understanding physics equations in parallel with, rather than after, learning difficult physics concepts. Here, Devon shows that the blended processing solution may even provide an opportunity for articulating and grappling with a difficult physics concept.

It's important to note that this conceptual difficulty cannot be responsible for the lack of symbolic forms-based reasoning in explaining the velocity equation, because the "explain the velocity equation" prompt doesn't contain any references to any applied forces, nor does Devon reference a difficulty in determining the acceleration from the force. For these reasons, this conceptual difficulty alone can't explain Devon's initial "plug-and-chug" explanation of the velocity equation.

Along with symbolic forms-based reasoning, *coherence* epistemological stances are also a target of physics instruction for two reasons: (1) such stances are connected to success in learning physics (Hammer et al., 2005; Perkins et al., 2005) – and specifically, in this study, symbolic forms use – and (2) *coherence* epistemological stances themselves are an intrinsic goal of science education and a marker of scientific expertise (Bing & Redish, 2012; Redish et al., 1998). Using isomorphic equations that get closer and closer to physics concepts, the goal is to scaffold Devon into drawing on both symbolic forms and epistemological stances that support seeking an underlying meaning and understanding of why equations make sense. In fact, the success of the money and speed prompts in helping Devon activate symbolic forms-based reasoning in the interview depends on these prompts tapping into certain epistemological stances that support symbolic forms use. The success of instructional sequences for developing symbolic forms-based reasoning with physics equations depends crucially on whether this sequence can tap into these epistemological stances.

How the Case Study Supports (and Goes Beyond) the Coding

This case study of Devon illustrates how such analyses can support our original coding scheme. By studying Devon's work on the velocity prompts and the money prompts, along with his explicit epistemological stances towards math and physics equations, the differences between Devon's symbolic forms and epistemological coding on the physics and non-physics prompts can be understood. This case study builds on the coding for understanding Devon's shifting reasoning in the interview.

Additionally, this case study goes beyond the original coding by showing shifts that are finer-grained than those captured in the coding. In the coding scheme, Devon's reasoning on the "explain the speed equation" and Two Cars Problem segments are both simply coded as "used symbolic forms." Because the coding scheme attends only to the presence or absence of symbolic forms-based reasoning, the shifts present in Devon's reasoning were not even initially apparent. Yet, in order to fully understand Devon's reasoning, these shifts add on another layer of moment-to-moment understanding of Devon's reasoning and his epistemological stances.

Another aspect lost by the coding is time ordering. By summing over the symbolic forms and epistemological interview segments for the physics and non-physics prompts, the sense of Devon's trajectory through the interview is lost. The non-physics prompts were designed as a sequence that would cue symbolic forms use, eventually bringing the problems closer to the original kinematics problem content. The coding loses this sense of a trajectory, totally ignoring a sense that each prompt is connected to and building on what has come before. As an extreme example, Devon's reasoning on the Two Balls Problem is simply coded as "used symbolic forms," neither taking into account his initial apprehension towards the problem nor how his final answer depends on the long sequence of prompts in the interview that came before.

Beyond Just Conceptual Knowledge and Epistemology

To make the case that it is productive to look beyond conceptual knowledge to understand the presence or absence of symbolic forms-based reasoning in undergraduate physics students' reasoning with equations, I, in this paper, point out that certain epistemological stances are aligned with symbolic forms use. However, this is not to say that conceptual knowledge and epistemology are the only factors relevant for understanding student reasoning.

For example, for Devon, it is clear that his affect plays a role in his reasoning in the interview. When the Two Balls Problem is initially posed, Devon's uncertainty towards physics and nervousness in the interview support his not wanting to even attempt the problem. One possible influence of the interview could be that it puts him at ease and makes him feel that it's safe to articulate his thinking on the Two Balls Problem when posed at the end of the interview.

In other parts of the interview not presented in this paper, Devon talks about his personal histories with physics and math, and how negative experiences in physics and positive ones related to math have led him to shift his major from engineering to secondary math education. The disciplines of physics and math in this interview carry different affective responses in addition to the different epistemological stances Devon takes towards understanding equations. Future work can enrich the understanding of cases such as Devon's by layering on attention to other factors, such as affect, in addition to attention to conceptual knowledge and epistemology to understand the dynamics of student reasoning.

CONCLUSION

In this paper, I have made several arguments.

1) The absence of symbolic forms-based reasoning does not necessarily imply a knowledge deficit. Symbolic forms are a cognitive element that are connected to epistemological stances that value *coherence* between physics equations and underlying conceptual meaning as well as coherence between the formal ideas of physics and everyday life. Therefore, the absence of symbolic forms from reasoning with equations could be due alignment with a *disconnected* epistemological stance.
2) Manifold models of cognition that treat reasoning as the dynamic activation of a subset of an individual's cognitive resources are more productive for understanding Devon's reasoning and shifts between his different modes of reasoning with equations.
3) Although the coding scheme is beneficial for illustrating broad patterns, it misses these fine-grained shifts in reasoning and ignores the consequentiality of the time ordering in the interview, which can be recovered by case study analysis.
4) The case study helps illustrate the two coherent modes around Devon's reasoning, as well as finer-grained shifts between the modes of reasoning. But more than classifying Devon's reasoning or epistemology, this illustrates one way that students can tap into their prior knowledge for viewing symbolic forms as useful in understanding and using physics equations and suggests one type of instructional sequence for doing so.

Chapter 5: Looking Beyond Content and Structure to Understand Transfer of Mathematical Approaches Across Disciplinary Contexts

INTRODUCTION

The transfer of content knowledge across courses in different disciplines is a common goal in a multidisciplinary education system. This is especially true in, although not unique to, physics, which relies heavily on mathematics– a reliance that is reflected in the prerequisite mathematics courses required for enrollment in most physics courses. Physics instructors hope that students will use the mathematics they learn in math class in order to make sense of and solve problems in physics. Just as introductory physics courses build on ideas from calculus, electrical engineering courses build on ideas from differential equations, and introductory biology courses build on ideas from introductory chemistry and physics.

However, sequencing disciplinary courses hierarchically to build on one another appears to be an inefficient or ineffective method for aiding students in learning that content and bringing it into another disciplinary course. Many efforts in aligning content across courses in different disciplines have been documented (e.g. Al-Holou et al., 1999; Dunn & Barbanel, 2000; Elliott, Oty, McArthur, & Clark, 2001; Loverude, Kautz, & Heron, 2002; Plomer, Jessen, Rangelov, & Meyer, 2010; Watkins, Coffey, Redish, & Cooke, 2012). These courses typically reorganize the topics covered in two or more disciplinary courses such that related topics are covered at the same time in an organized manner. The common goals of these multidisciplinary reforms are to avoid haphazard coverage and content misalignment while also decreasing the compartmentalization of content knowledge in order to encourage transfer (Al-Holou et al., 1999).

These goals align with views of transfer that focus on the content as explanatory in understanding why transfer succeeds or fails. Yet, more modern perspectives on transfer look beyond this attention to content and structure. I argue that interdisciplinary efforts can similarly benefit by shifting their attention beyond the alignment of content knowledge. In this paper, I show that attending to both a student's epistemological stances and his feelings of being accountable towards recalling particular pieces of knowledge can help explain why he takes different approaches to solve similarly structured problems that are situated in different disciplinary surface features. In other words, attending to these features, and other features of this type beyond the content the problems, can aid in understanding what impedes the transfer of knowledge across disciplines.

LITERATURE REVIEW: WHAT KINDS OF FACTORS CAN EXPLAIN SUCCESS OR FAILURE IN TRANSFER?

The phenomenon of transfer has been defined as "the ability to extend what has been learned in one context to new contexts" (Bransford et al., 1999, p. 74). While this phenomenon is clearly central to the interests of education research, there is much theoretical disagreement about what is learned, what is transferred, and what counts as a valid or useful investigation of transfer (Beach, 1999; Greeno et al., 1993; Lave, 1988; Lobato, 2006; Schwartz, Bransford, & Sears, 2005). Yet, even in light of these theoretical disagreements, having students extend, in some way, what is learned in one disciplinary context to another remains an important goal.

Attempts to support transfer across disciplines typically focus on the relevant content knowledge or similarities and differences in the structures of the disciplinary problem contexts. However, there are cases where this attention to content and structure is insufficient for fully understanding the transfer of relevant content knowledge. In this paper, I argue that newer transfer perspectives that expand their focus beyond content and structure can provide useful tools for understanding the transfer of knowledge across disciplines. In what follows, I elaborate on how this argument is instantiated in the literature.

Views of Transfer that Focus on Content and Structure

Most research into transfer attends primarily to the content knowledge being transferred or how this content is tied to the structure of the different problems across which knowledge is to be transferred (Barnett & Ceci, 2002; Schwartz & Nasir, 2003). Classical notions of transfer depend on constructions of mental representations of an initial learning and transfer situation. Transfer of knowledge occurs when these two representations are similar enough that knowledge or approaches applicable to the initial situation can be seen as also productive in the transfer situation (Anderson et al., 1996; Singley & Anderson, 1989). In a series of seminal studies, Gick and Holyoak (1980, 1983) performed a series of experiments to see whether individuals can sufficiently abstract the structure of a problem in one domain in order to apply its solution on a similarly structured problem in another domain. They argue that successful transfer depends on a learner's ability to induce an abstract schema from the concrete problems. This abstract schema helps learners see the similarities in the deep, conceptual structure across problem contexts, facilitating transfer. Based on their experimental findings, they argue that increased alignment between the deep structures of the problems increases transfer (Gick & Holyoak, 1980) and that providing multiple analogous cases in the learning condition helps subjects develop an abstract schema focused on the deep structure rather than idiosyncratic surface features of any one problem (Gick & Holyoak, 1983). The classical transfer literature has strongly attended to seeing the commonalities in deep structure between learning and transfer problems, which has been a continued focus of the transfer literature (Bassok, 1990; A. L. Brown, Kane, & Echols, 1986; Chi & VanLehn, 2012; Gentner, Loewenstein, & Thompson, 2003; Novick, 1988; Reed, Dempster, & Ettinger, 1985; Reed et al., 1974; Spencer & Weisberg, 1986).

Similarly, typical efforts to align courses across two disciplines focus on aligning content and highlighting the common deep structure between different disciplinary problems. For example, Plomer et al. (2010) developed a tighter coupling between a physics course and a neurophysiology course. Designing physics laboratories to connect physics concepts to neuroscience contexts improved student facility in relating physical ideas about electric circuits and optics to neuroscience problems on a transfer post-test. Similarly, Dunn and Barbanel (2000) co-taught an integrated physics/calculus sequence that aligned the content of the disciplines such that relevant calculus and physics topics were taught together. This produced enhanced opportunities for connections between the disciplines – for example, developing a physical interpretation of the divergence theorem in terms of sources and sinks and connecting that mathematical theorem to Gauss' law – but also led to some difficulties in helping students negotiate different notation and disciplinary approaches. Al-Holou et al. (1999) reviewed a series of programs that aligned content across engineering, physics, mathematics, and other disciplinary courses in order to reduce the compartmentalization of knowledge and improve connections across disciplines. These interdisciplinary course reforms align with classical transfer perspectives that similarly attend to the presentation of content and structure across problems and situations.

More Modern Perspectives on Transfer Attend to Content and Structure in New Ways

Some more recent perspectives in transfer move beyond abstracted schemata or experimenter-defined structural similarities as the crucial mediators of transfer, while maintaining the focus on relevant conceptual knowledge and similar problem structures. I elaborate on several of these more modern perspectives to show how they disagree with classical notions of schema acquisition and analogical transfer while maintaining a similar focus on content and structure.

Rather than the application of intact, generalized mental schema to different situations, Wagner (2006) describes transfer as the construction and coordination of different pieces of knowledge as applied to new situations. Transfer of knowledge is not the washing out of contextual details and application of a previously learned generalized concept or schema. Rather, it is the activation and coordination of knowledge elements in conjunction with specific contextual features that helps learners see how prior knowledge is relevant in new situations. In this way, consistent success in transferring knowledge comes not from an abstracted schema that ignores contextual features but facility in activating and building up the same underlying knowledge in ways that incorporate contextual features.

Greeno, Moore, and Smith (1993) exchange purely cognitive, mental constructs for situated activity in their studies of transfer. Their definition of transfer is the transformation of a current activity based on past activity. In one experiment, subjects throw darts at an underwater target. Because the apparent position of the underwater target is distorted by refraction, only darts aimed below where the target appears to be will hit the target. Of the two experimental groups, one that receives explicit instruction on the physics of refraction and one that does not, the one that received the refraction lesson performs better when the depth of the water changes, increasing the distortion of the apparent position of the target. In the original study, Scholckow and Judd (described in Judd, 1908) explain this finding by positing that instructed students had sufficient conceptual knowledge of the physics of refraction to guide adjustment in a new situation. Greeno et al. on the other hand argue that it's not conceptual physics knowledge that leads to better performance in the transfer situation but rather attention to particular affordances of the activity. For example, instruction on refraction could attune subjects to certain perceptual affordances of the situations, such as the apparent bending of the dart's path after it enters the water. In a situated theory of transfer, adjusting to the transfer situation requires tapping into this affordance to transform the activity in appropriate ways (i.e. aiming farther from the apparent position of the target to compensate for the increased deviation from the "apparent path" of the darts). In this situated approach to understanding transfer, it is the affordances and constraints of activities that are attuned to in transfer across problem situations, not an application of abstract conceptual knowledge.

An actor-oriented transfer perspective (Lobato et al., 2012; Lobato, 2003) focuses on what problem features students attend to. By not testing whether or not students are able to transfer particular pieces of knowledge that are pre-defined by the researcher, this attention to what students notice assumes that connections are always being made and asks how what features are noticed by learners influences what knowledge is transferred-in. For example, in working on algebra problems related to slope, students from two different classes attended to different features of a linear graph (Lobato et al., 2012). Students who attended to the x-y coordinates of points on the line were successful in calculating the slope, whereas students who attempted to use the gridlines on the graph to find "rise over run" were unsuccessful, because the irregular scaling of the gridlines made "counting boxes" more difficult. For these students, although success in their courses indicated that they had "learned" how to calculate slopes, differences in what students had been trained to notice in such problems led to differential success on a transfer problem. The actor-oriented perspective makes the case that students do not attend only to general mathematical similarities in the structure of these slope problems, but also to differences in the representational details that cue different approaches.

Yet, for all of their differences from classical transfer perspectives, these studies maintain an attention to the content and structure of the problems or activities. Wagner's "transfer-in-pieces" perspective maintains that the factors to consider in transfer are conceptual knowledge and how seeing similarity between problem contexts relies on that conceptual knowledge. Greeno et al.'s attention to the affordance of the apparent bending of the dart is a reinterpretation of what is similar in the structure and activity of two problem situations. Lobato et al. redefine the grain-size of what structural similarities are attended to, maintaining the importance of structural similarities associated with the relevant conceptual knowledge. Although these perspectives expand beyond the development of an abstracted general schema, the similarities in the structural features of the problems remain central to understanding transfer.

Transfer Mechanisms Beyond Content and Structure

There are many critiques of the classical perspective on transfer, suggesting factors beyond content and structure. In this study, I investigate how a student reasons similarly and differently on two problems designed to share the same deep structure. Therefore, I limit the discussion of transfer mechanisms to those potentially relevant to a "two-problem transfer paradigm," (Lave, 1988) where the relevant phenomenon is transfer of content knowledge from one problem to another.

Pugh and Bergin (2006) review the literature on how transfer depends on three aspects of motivation: achievement goals, interest, and self-efficacy. Although there are limited transfer studies in these areas, they argue that these motivational factors are correlated with successful transfer. Pugh and Bergin suggest that these factors can support a "motivation to transfer" and support persistence in transfer tasks. In this view, transfer does not depend only on an individual's ability to see similarity in different problems but also depends on motivation to seek out and persist in searching for similarity supporting transfer.

Engle and colleagues suggest that how individuals frame a learning situation as connecting to future situations affects transfer (Engle et al., 2012; Engle, 2006). *Expansive framing* of learning situations connects these initial learning contexts to future times and broader spaces, whereas bounded framing limits the learning situation to being relevant only to the present time and room in which the activity is occurring. Additionally, expansive framing promotes ownership and agency over the content in the learning situation, whereas bounded framing limits the student's role as understanding and regurgitating a source text. Expansive framing promotes transfer, because individuals expect that what was learned before is relevant to future situations and they take agency and authority to use and transform ideas from the initial learning situation for use in the transfer situation.

Importantly, the framework of expansive framing attends to factors involved in transfer that are broader than just the particular structure or knowledge content involved in the learning or transfer task, such as expectations that this learning activity will be relevant for future activities and authorship over the relevant content. Although Engle and colleagues do not ignore the importance of structural or content similarities between the learning and transfer situations, the notion of expansive framing suggests that *structure and content are not the only relevant factors* in understanding why transfer does or does not occur. A bounded framing could impede transfer even in the face of structural or conceptual similarity between two problems.

Other models of "transfer" describe the framing as *epistemological*, responding to "what kinds of knowledge or approaches are appropriate here?" (Hammer et al., 2005). In one example, Louis describes his approach to learning in physics and chemistry as starting from equations and memorizing facts. The instructor suggests that Louis should attempt to make a commonsense analogy to understand electric circuits, rather than starting with the relevant equation. In response, Louis comes up with an everyday analogy between electric circuits and dump trucks driving down the highway. This use of such analogies in learning physics (what some would describe as transfer-in of everyday knowledge) improves Louis' performance in the course. Importantly, the "learning condition" didn't aim to teach Louis about the structural similarities between dump trucks carrying cargo and electric circuits. Instead, asking Louis to come up with an analogy shifted his *epistemological stance* towards what kinds of knowledge are appropriate for understanding circuits, then leading to the construction of a productive analogy that builds on Louis' prior everyday knowledge. This is not to say that recognition of the structural similarities between cargo trucks and electric circuits are not crucial for constructing a valid analogy. However, this attention to structure alone does not explain the success of the instructor's intervention in leading to the construction of this analogy.

These perspectives on transfer all attend to features other than substantive content knowledge to be transferred. In the next section, I describe the purpose of this study and how looking beyond content knowledge can prove productive for studies of transfer as well as instructional efforts aiming to connect knowledge across disciplinary contexts.

"IT'S NOT A TRANSFER TEST, BUT CAN IT TELL US ABOUT TRANSFER?"

In this study, I show how one student, Will, reasons on two isomorphic problems in a clinical interview. The problems, situated in either physics or calculus content, are isomorphic in that they both ask Will to compare a mathematical expression to the first few terms of an infinite series, asking when one is a good approximation for the other. Although involving two isomorphic problems, this does not fall into the typical "two-problem transfer paradigm" for two reasons.

The first is that neither problem is the learning situation or the transfer situation. We pose both problems to Will with no explicit instruction, to see how he would reason on both. While Will's prior knowledge of physics and calculus are likely candidates to be "transferred-in," there is not a clear pre-defined delineation of learning and transfer situations.

Second, the criterion for successful transfer across these problems is not well defined. That is, there is no canonical solution for this type of problem that researchers or instructors would necessarily agree should be transferred to indicate expertise or success; we would expect both novices and experts to provide a range of reasoning and justification in their solutions. However, even though not a formal transfer test, I propose that Will's reasoning on these two problems has implications for studies of transfer.

The Purpose of This Study

A common result of transfer studies is the identification of factors that support or suppress transfer. The classical transfer question is whether the same pieces of content knowledge can be and are applied to different problems that share the same deep structure. Common approaches to this question attend to the content and structure of the problems as explanatory. In this paper, I aim to show how a focus on content and structure alone is insufficient to capture the factors relevant to Will's different approaches.

Although the two problems in this study are designed to be similar, Will uses different knowledge and approaches on these two problems. Even though successful transfer between these two problems is not well defined, Will illustrates that, for him, knowledge and approaches on one problem are not necessarily relevant for the other. In this case, Will's interview aligns with typical transfer experiments in that both provide the possibility for identifying differences that impede transfer of knowledge across isomorphic situations. This study contributes by illustrating what kinds of differences students may perceive in these isomorphic physics/calculus problems that may support different approaches and modes of reasoning on these two problems.

Specifically, drawing from factors that explain success or failure in transfer, I seek to identify factors that contributed to Will's different approaches to the two problems. I show that some of these factors are related to content and structure, but that Will's different *epistemological stances* towards and his differential *senses of accountability* to the disciplinary content of the two problems also contribute. Although factors related to content and structure are important, they cannot totally explain differences in Will's reasoning on these two problems. These additional factors have instructional implications for both disciplinary courses that rely on content from other disciplines and interdisciplinary efforts that explicitly aim to bridge content across disciplinary contexts.

METHODOLOGY

Setting

The data presented in this study was collected as part of a cross-institution, interdisciplinary research project studying how students reason about approximations in physics and calculus. The study was driven by two broad research questions: 1) "how do students approach approximation similarly/differently in calculus and physics?" and 2) "what factors support this similar/different reasoning?"

Students were interviewed at two universities, a large west coast public university and a large east coast public university. These students were either enrolled in or had completed within the past year a second-semester calculus course covering integration and Taylor series and a physics course covering oscillations. Interviewed students were asked two or three questions, asking them to make or consider approximations in the contexts of physics or calculus. At the end of the interviews, students were asked to reflect on the differences between the problems and how they approached them.

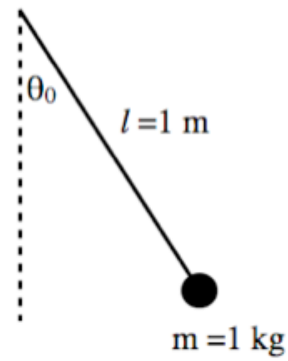
The interviews were semi-structured in that the interviewer (Eric Kuo) was free to ask follow-up questions, revoice student ideas and selectively zoom in on and probe deeper into particularly interesting topics (Lee et al., 2012). In this way, the interviewer was not beholden to a set structure but was free to investigate interesting points and explore in-the-moment hypotheses. This allowed for the interviewer to seek out additional depth in an interviewee's reasoning to help distinguish between different interpretations of what they were saying, as well as explore emergent facets of an interviewee's reasoning not originally anticipated by the interviewer. Because I aim to understand each interview on its own terms rather than to seek patterns across the interviews, standardization across interviews is unnecessary.

Because the interview is an interaction between interviewee and interviewer, it is likely that the interviewee's reasoning is affected by this interaction. As such, I do not claim ecological validity: that the responses in these interviews represent how these interviewees would reason in other, more naturalistic situations.

Two Problems for Student Reasoning Across Calculus and Physics

In order to investigate how students reason with infinite series expressions and approximations in physics and calculus, two problems were designed to contain similarly structured mathematics. In the physics problem, the infinite series is an equation for the period of a pendulum. In the calculus problem, the infinite series is the Taylor series expansion about $x = 0$ for the arctangent function. These two problems are shown in Figures 5.1 and 5.2, respectively.

PENDULUM



You have a pendulum made of a metal ball on a string. The string is 1 meter long and the metal ball has a mass of 1 kg. You might know that the approximation for the period of a pendulum for small oscillations is:

$$T = 2\pi\sqrt{\frac{l}{g}}$$

where T is the period of the pendulum, l is the length of the pendulum, and g is acceleration due to gravity (9.81 m/s^2)

This equation only holds for small angle oscillations of the pendulum. For larger angles, the period of a pendulum can be found with the following equation:

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{16}\theta_0^{\,2} + \frac{11}{3072}\theta_0^{\,4} + \ldots\right)$$

where θ_0 is the angle of displacement of the pendulum from vertical in <u>radians</u>.

You want to calculate the period of oscillation for this pendulum. How big can the angle of displacement of the pendulum be before the equation for small oscillations isn't a good approximation of the period?

Figure 5.1: The Pendulum problem

TAYLOR SERIES

The Taylor series about $x = 0$ for arctan(x) is:

$$\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots$$

How big a value can x be before stopping at the second term is a bad approximation?

Figure 5.2: The Arctan problem

These problems were designed to share a common structure in the following way: both problems asked students to make judgments about when the series expansion parameter, x or θ, gets so large that the exact expression is no longer close to what a truncated series would give. In a classical transfer perspective, the same abstract schema could be applied onto both problems: a schema related to "judging whether two expressions, an exact expression and a truncated series, are 'close enough." These problems were paired to see whether students bring similar or different approaches to both problems and what factors might contribute to those similar or different approaches.

Analysis of the Interview

Based in our research questions, the initial analysis focused on characterizing the content of students' reasoning, identifying similarities and differences in how the same student approaches the Pendulum and Arctan problems. This involved both understanding the substance of a student's final answer, as well as the knowledge and approaches that a student takes to arrive at that answer.

Once the differences in a student's reasoning on these two problems were characterized, the analysis moved beyond the content of and approaches in the student's reasoning to investigate factors that plausibly contribute to the development of that reasoning. The analysis focused in particular to the kinds of factors identified in the transfer literature that have been shown to influence students' reasoning, such as, but not limited to:

> <u>Content and structural factors of the problems</u>: *Does the particular content of the problems (pendulum or Taylor series) and how the problems are structured support the reasoning or approaches students' bring to the problem?*
> <u>Epistemological stances</u>: *What are students' notions of what kinds of knowledge are appropriate for these problems and do they cohere with and support students' reasoning on these problems?*
> <u>Expansive framing</u>: *How do students' view these problems as connected to previous times and places? Do students have a sense of authorship and/or*

agency with respect to the relevant content? Does this contribute to how students' reasoning on these problems develops?
Motivational factors: *How do achievement goals, interest, or self-efficacy influence students' reasoning on these problems?*

In this study, I present the case study of one student, Will, working on the Pendulum and Arctan problems. Again, this single case study is not meant to illustrate any generalizable pattern. Here, Will's interview is used as a proof-of-concept of the usefulness in attending to factors beyond content and structure in answering questions of transfer across disciplines. Will's interview stood out because of the unusual amount of depth and articulateness with which he spoke about more than just his reasoning on the problem, providing data that points to three factors that cohere with and support the differences in his reasoning: differences in content and structure, differences in his epistemological stances towards what kinds of knowledge are useful for solving these problems, and differences in his sense of accountability towards the content of the problems.

Will's interview is unique, because at the time of the interview in July, he had completed the second-semester calculus course but had not yet started the second-semester physics course in which he was enrolled in for the fall. Unlike the other interviewees, he had not yet seen the relevant pendulum equation in his college physics course, although he expresses familiarity with the equation $T = 2\pi\sqrt{\frac{l}{g}}$, from his high school physics course.

SUMMARY OF WILL'S REASONING ON THE PENDULUM AND ARCTAN PROBLEMS

Will's final answers to the Pendulum and Arctan problems are similar in some ways: he defines a range within which the approximation will be good from bounds derived in the problem situations. However, his approaches to these problems are very different. I start by summarizing key features of how he approaches the problems differently on the Pendulum and Arctan problems and then pointing to three factors that may have supported these different approaches on the two problems.

Will's Approach on the Pendulum Problem: Making Sense and Exploring

Making sense of an unfamiliar equation

Will starts the problem by noting that he hasn't seen this problem before. In reading through the problem, Will starts to make sense of what the problem is asking, noting that a good approximation has the two expressions for the period being equal to each other. He also notices that the two expressions are the same except for the additional $\left(\frac{1}{16}\theta_0{}^2 + \frac{11}{3072}\theta_0{}^4\right)$ term in the series expression. Moreover, he views this extra term as added to "make up for the error that occurs…when the angle gets too big." That is, when the angle is small, this extra term is small, so the two expressions are approximately equal. But when the angle gets large, then the extra term becomes large, so the approximation is no longer good.

Will moves to speak more broadly about the form of the series equation. From the decreasing coefficients and increasing powers of theta, he concludes that the terms in the series must get progressively smaller. He connects this to ideas in "calc 2" dealing with power series, but he also connects this idea to everyday life, saying "I think that that's how we do, like, everything." He goes on to explain how this connects to learning to shoot a basketball: when you start learning, you improve quickly, because you're learning the basics. As you continue, you make small adjustments to perfect your shot that you don't notice, because the changes are so small.

Bounded guess-and-check, based in a physical understanding of pendulum motion

At the beginning of the problem, Will also considers that one possible method is to plug in angles into the expression $\left(1+\frac{1}{16}\theta_0{}^2+\frac{11}{3072}\theta_0{}^4\right)$ and see when it becomes larger than 1, but he rejects this idea, stating that an analytic solution would be preferable even though he doesn't know exactly how to go about finding one.

A few minutes later in the interview, Will sets an upper bound on the angle of the pendulum, which he uses as a starting point for plugging in angles to see how different the two expressions for period will be. He sets this upper bound from his understanding of the physical motion of the pendulum.

> [00:13:53]
> W: Alright, so since I'm, like, not really sure how to do it, um, what I would do is look at the picture, which I know is dumb, but I would do that.
> E: Ok.
> W: I would just get an idea, and I would know that this angle, um, it's in radians, right? So my answer cannot be above pi over 2, is what I would say, or it wouldn't be, yeah, I would keep it in this quadrant. I don't know why, but I feel like that's a better chance.
> E: Just 'cause the picture?
> W: Just because of the way it looks. I doubt it would fly all the way up here. [motions with his hand swinging up past 90 degrees] Maybe it would, and I'm being wrong, but once you get up to a certain point up here, I feel like it wouldn't. That wouldn't be the way the problem works. And it's possible it does, but if I don't know how to do a problem, I'd rather just sort of guess at an answer than leave it blank. So I would try to confine my answer to, um, zero and pi over 2, and then, I guess I would start, if I had a calculator, by plugging in pi over 2 and seeing what answer it gave me. And then I'd try to find bounds for it and guess somewhere in the middle.

The interviewer then points to the calculator on the table, and Will evaluates the truncated series at $\theta = \pi/2$ and gets 1.17. He finds that the $\frac{11}{3072}\theta_0{}^4$ term comes out as .02, so he concludes that the additional higher-order terms will be even smaller and therefore negligible. He concludes that 1.17 is too different from 1. In his physics course experience, being within 0.1 is close enough, so, in a test situation, he would try angles less than *π/2* to try to bring the series expression closer to 1.

<u>*Exploring and evaluating algebraic manipulations for an analytic solution*</u>

After his "bounded guess-and-check," I, the interviewer, ask Will to consider how he would approach this problem if there weren't the time pressure of a test situation. Will declares that, given that extra time, he would try to manipulate the equations to find an analytic solution. He then sets the two equations for period equal to each other, yielding this equation: $\frac{1}{16}{\theta_0}^2 + \frac{11}{3072}{\theta_0}^4 = 0$. Will seems not to like this expression. I point out that $\theta = 0$ would work mathematically, to which Will counters that that would be impossible, for physical reasons:

> [00:27:33]
> W: Well, yeah. But that wouldn't be, that would be impossible.
> E: Ok.
> W: Theta equaling zero means that it would just be sitting there, right?
> …
> [00:28:31]
> W: 'Cause they're asking what the, ok, what's the angle for this, uh, for the equation for the period that doesn't work for small oscillations? When the angle equals zero? Well, then it's not even moving. And then there's like no per-, ok, I mean it's true, there's like no period, because it's just sitting there. Um, although technically if theta were equal, yeah, if theta were equal to zero, the two equations would equal each other, but it wouldn't be doing anything. It would just be hanging straight down. You can't really have a period when it's just sitting there not moving.
> E: Mhm.
> W: So that's why I wouldn't, I just, anytime I get an answer like that, I make absolutely sure it makes sense, and then, and then, 'cause a lot of times if you make a mistake, that's what you get. You get, like, a ridiculous answer.

In the end, Will decides that there must be a way to perform algebraic manipulations that lead to an analytic answer, but he can't figure it out. He goes back and reiterates his bounded guess-and-check as the final answer he would have to give in lieu of a more analytic solution.

Will's Approach on the Arctan Problem: Recalling Formal Knowledge

After Will's work on the Pendulum problem described above, followed by a brief reflection on what score he would expect to receive if the Pendulum problem was on a test in his physics class, Will begins work on the next task: the Arctan problem.

<u>*Connecting this problem to ideas from calculus class*</u>

Will starts by summarizing the problem and pointing out that he doesn't remember the general method for finding the Taylor series of a function, but he recognizes this problem as the kind he has done before in calculus class. He notes that arctangent has asymptotes but can't remember what they are. He focuses in on the "about $x = 0$" in the definition of the Taylor series and tries to remember what it means and how it is relevant for this problem.

Later on, after trying out some values for *x*, he tries to remember the Taylor series equations for sine and cosine. He also tries to recall the general formula for Taylor series:

> [00:50:40]
> W: Um, 'cause I know the, I'm trying to remember the stuff for, like, sine and cosine and sort of do it from there. Because you know, you don't really memorize stuff for arctangent, but like, you do for *sin*(*x*). Um, and I know that that we, like, it's frustrating, because we did learn, like, this exact thing. It's where it goes from, uh, the co-, uh, the power's always, uh, odd, and it's, alternates subtraction and, uh, addition. It's like one over, how do you do a [inaudible]. I'm trying to remember the very, just the very basic one, for a Taylor series.

Will struggles with the meaning of "bad approximation," feeling that there must be a formal definition to remember. He continues this line of reasoning by trying to remember equations "from that chapter of the book." He continues to try to figure out the relevance of "about $x = 0$." He also tries to remember the bounds of arctangent and the general formula for Taylor series expansions.

Bounded guess-and-check, based on recalling ideas from calculus class

After his initial recognition of the problem, Will uses the calculator provided in the interview to start to define bounds for a good approximation. He chooses $x = 2$ as a value to try and finds that it would be a bad approximation, because it makes the series evaluate to a negative number. However, he notes that he has no principled reason for not wanting the approximation to be negative.

In trying to remember ideas from his calculus course, he vaguely remembers conclusions of the various convergence tests for infinite series, recalling that there are ones where "*n* has to be less than 1, otherwise…it explodes." He then sets the bounds for $\left(x - \frac{1}{3}x^3\right)$ to be between zero to one, although he recognizes that this arbitrary and is not the "right way to do it."

In response to Will's statement that he can't remember much from that section in calculus, I ask him what information he would look up to help him do the problem. He asks for two things, 1) the asymptotic bounds of arctangent and 2) the general formula for Taylor series about $x = a$, which were given to him. After he continues struggling with the problem for a short time, I provide him with these two facts. He uses the range of arctangent ($-\pi/2$ to $\pi/2$) to set the bounds $-\frac{\pi}{2} < \left(x - \frac{1}{3}x^3\right) < \frac{\pi}{2}$, since the approximation should not go "beyond what [the] function is defined as." Will attempts to use the general formula for Taylor series to derive the Taylor series for arctangent about $x = 0$ but, realizing that he has to take derivatives of arctangent, eventually decides to drop that line of inquiry. In the end, he concludes that the general formula for Taylor series is not helpful for him here.

Attempting to recall an equation that can solve this problem

Throughout his work on the Arctan problem, Will seeks to recall the method from his calculus class that he believes can be used to solve this problem. As such, he seeks to recall ideas from his math course: the general formula for Taylor series, the Taylor series for sine and cosine, the asymptotes of arctangent, and the various convergence tests for infinite series. Throughout the problem, Will's approach is driven by an underlying assumption that there exists a relevant method from calculus class that can be used to solve this problem quickly.

At the end of the interview, when Will is comparing his different approaches on the two problems, I ask him if he could have pursued a solution that involved algebraic manipulations, like his work on the Pendulum problem. Will responds that he could not, because problems like this require specific methods and approaches:

> [01:22:33]
> W: No, you couldn't really do algebraically there. You have to prove it by, you know, using, you can't really, you know, you can't really, with Taylor series you don't, like, break it apart and say "x equals this." You use, like, all the different methods of approximation they have, or what do they have again? They have, like depending on what you need, they have different methods. Um, I just don't remember them. Uh, and if you don't know them, then you can't do the problem.
> E: Ok. So there's somethin-, there's some method they have that you just have to remember.
> W: Yeah. There's a way to do this, that problem. Takes, like, 2 minutes or even less, but if you don't know how to do it like that, you're screwed. Good luck.

DIFFERENCES IN CONTENT AND STRUCTURE THAT SUPPORT DIFFERENCES IN WILL'S REASONING

Attending to the content of Will's reasoning on the two problems, there are some similarities in how Will approaches the problem. In both the Pendulum and Arctan problems, he sets bounds for what counts as a good approximation, determined by the specifics of the problem situation, and uses these bounds to constrain his guess-and-check attempts. This application of "bounded guess-and-check," though instantiated in two slightly different ways, is a similarity in how he approaches these two problems. This similarity in approach between the problems is likely supported by the similar deep structures of the problems, requiring judgment as to how good an approximation a truncated series is for the whole series.

Yet, there are also differences in Will's reasoning across these two problems. I point out two such differences, along with the structural or content features of the problems that could have plausibly contributed to these in a classical transfer perspective. Although the target knowledge to be transferred is not pre-defined, the classical transfer lens can be used to investigate differences in the particular content or structural details of the problems that may guide students' reasoning in different directions.

A Difference in How the Bounds Are Set: Physical Intuition Versus Formal Mathematical Knowledge

In both problems, Will uses the problem features in order to set bounds on what a good approximation is. However, there is a difference in what knowledge Will draws on to set these bounds. On the Pendulum problem, the bounds are set according to Will's physical understanding of the motion of the pendulum, whereas the bounds in the Arctan problem are set according to the range of a mathematical function. This is not a surprise, because one problem is set against a physical backdrop and the other is not. Therefore, on the Pendulum problem, an everyday understanding of physical motion can set constraints upon the mathematics that define that motion. Furthermore, the presence of a diagram in the Pendulum problem, which Will explicitly points his attention to, likely cues or supports reasoning about the physical motion of the pendulum. If there had been an analogous diagram in the Arctan problem (i.e. a graph of $y = arctan(x)$), it seems reasonable that Will could similarly have defined bounds for arctangent without needing to ask me for the asymptotes of the function.

Different Approaches for Seeking an Analytic Solution

Will's "bounded guess-and-check" is not his preferred approach to either problem. For both, he would prefer more analytic solutions: formal manipulations that will deterministically lead to a well-defined boundary for a "good approximation." Yet, he seeks out different kinds of analytic approaches on these two problems. On the Pendulum problem, his approach is to set the two expressions equal to each other and attempt algebraic manipulations to reach a solution. In attempting these algebraic approaches, he checks his approaches against common sense and points out why his solution is not physically reasonable. On the Arctan problem, his approach is to try to recall relevant knowledge that may point to a correct method or approach rather than attempting algebraic manipulations.

There are two structural differences between the problems that may support these different lines of reasoning. The first is that the Arctan problem is explicitly labeled as being a Taylor series problem. As such, Will recognizes that this problem is explicitly connected with content from his calculus course. Although Will recognizes the "power series" in the Pendulum problem, neither the problem nor he explicitly references "Taylor series," a phrase which likely cues Will's knowledge from calculus class on the Arctan problem.

The second difference is in the complexity of the mathematics involved. In the Pendulum problem, the two expressions are algebraic, the only difference between the two being the additional polynomial terms added in the series expression. The Arctan problem contains a transcendental function, $arctan(x)$, which makes solving for x analytically impossible. Because of the relative simplicity of the pendulum expressions, algebraic manipulations can be used to solve for an exact expression for θ.

Implications of a Classical Transfer Perspective Focusing on Content and Structure

Taking a classical transfer perspective, differences in the content and structure of the two problems can help explain differences in Will's reasoning. The next step, according to the perspective, would be to test the consequentiality of these differences in problem structure by redesigning the problems to align along some of these structural features. For example, the Arctan problem could be changed to ask about the Taylor series of $\left(\frac{1}{1-x}\right)$, so that algebraic manipulation of the equations could be used to solve for x. Students' approaches to this problem could be used as evidence to argue for or against the relevance of this structural feature in how students approach this problem. Additionally, an instructional intervention could help draw students' attention away from surface differences and towards the underlying structural similarity of these two problems in order to support the application of similar reasoning on the two problems.

Looking Beyond Content and Structure To Seek Out Explanations for Other Differences

I have argued that the classical transfer lens here provides some insight into Will's different reasoning on these two problems. However, there are some issues that are not addressed from this focus on the content and structure of the problems.

More than differences in the content of his two approaches, his approaches to seeking out those solutions differ. On the Pendulum problem, Will takes an approach I label as *making sense and exploring*: he reads the unfamiliar problem to make sense of what the question is asking and tries out different ways to manipulate the equations to reach a solution. He tries out different manipulations to see if they'll produce an answer, evaluating his approaches to see why they fail. I label his approach on the Arctan problem as *recalling formal knowledge*: Will spends much of the problem attempting to remember Taylor series ideas from his calculus class, believing that there exists a particular equation that can solve this problem.

Given that Will cannot remember the relevant ideas from his calculus course, why does he not switch to an alternative approach? Why does he spend so much time failing to recall, rather than attempting to make sense of and explore different approaches as he does on the Pendulum problem?

In the following sections, I elaborate on these differences in how Will seeks a solution on these two problems and propose that these differences are supported by his epistemological stances towards each of these problems, as well as his feelings of accountability for the content of the problems.

<u>HOW WILL'S EPISTEMOLOGICAL STANCES SUPPORT DIFFERENT APPROACHES</u>

In this section, I argue that Will's epistemological stances of what kinds of knowledge and reasoning are appropriate for the two problems contributes to his *making sense and exploring* approach on Pendulum and his *recalling formal knowledge* approach on Arctan. Specifically, Will views "logical reasoning" and making connections as appropriate on the Pendulum problem, whereas the reasoning appropriate for the Arctan problem is "not natural" and needs to be learned in class.

Logical Reasoning Is Appropriate on the Pendulum Problem

In between his reasoning on the Pendulum problem and the Arctan problem, Will recognizes that his solution to the Pendulum problem not "correct" and starts to explain his strategy for how he would attempt to get partial credit on problems such as this:

> [00:42:26]
> W: A lot of teachers, if you put something down that sort of makes sense, you won't get, if you just make it up, you'll get a zero, but if you put something down and you show logical thoughts and sort of show how you got to a semi-close answer, they'll give you like a point or two and be like, "alright, nice try, but not even close." Um, and it's a big difference again between this stuff and like, history. This kind of stuff you can, even if you don't know it, you can use logic and you can, uh, make connections and rationalize certain things and know that they're true just by looking at what you are given. You don't need to know, I mean you've learned it over the years, but you don't need to know a specific date or a specific event to answer a question. Like if they ask you, you know, "what's the Battle of Hastings?" and you don't know anything about the Battle of Hastings, you're not, you can't just be like a, "there were swordsmen. They fight." You can't say that. But this one you could say, alright I don't know what this equation really means, but I'm told, and I've never seen it before, but I do know that T should equal T when the number of, when the, um, degree of displacement of the oscillation is correct. And you can show stuff like that. And I think if you show that you're willing to do that, they'll give you a little bit.
> E: So you're saying like, here you can sort of use logic to figure it out.
> W: Yeah. With these two things, told what L and G are, obviously. They won't just give you a variable and not tell you what it is. Um, and you show that this has to be between a certain number to be correct, just if you just think about it and show that you were, that you rationalized your answer, that's what they want. They want you to give an answer and then, if they think that you thought about it and used viable methods of reasoning to get there, that's what they are more likely to give credit for than just writing down numbers.
> E: I see. Versus in, like, history, you can't logic things out.

> W: Yeah, you can't be like, "it's logical that in this battle, this happened." I mean, if you knew the date you could maybe say what weapons they used, something like that, but yeah, you can't just look at it and be like, "oh, clearly from this I can reason this happened." I don't know. You just can't do that. So you know it or you don't. This kind of stuff, eh, you may not know it, but you might know part of it or you might understand something and from there you, extrapolate information.

Here, Will expresses an epistemological stance that "logical reasoning" is appropriate and valued on the Pendulum problem. Even though he hasn't seen the problem before, he can understand the problem enough to make connections and reason through the problem to produce solutions that will yield partial credit. The fact that logical reasoning is appropriate and is valued by teachers is discipline-dependent. Will stresses that the logical reasoning that he believes is appropriate on the Pendulum problem is not possible in history, where prior knowledge of particular facts are required.

Importantly, Will's statements are epistemological in that he is reflecting not only on his approach to the Pendulum problem, but also on *what kind of approach is appropriate here*. This is reflected especially in contrast to the types of reasoning that are appropriate for historical questions, such as "what was the Battle of Hastings?"

Will's epistemological stance towards the Pendulum problem coheres with and supports Will's approach on this problem. Given his stance that logical reasoning is appropriate, then it makes sense that Will would try to make sense of this unfamiliar problem rather than arguing that he doesn't possess enough knowledge from physics to solve a pendulum oscillations problem. As a specific example, in rejecting $\theta = 0$ as an answer to the problem, Will argues that that answer must be wrong, not because his method was not canonical, but because it is physically ridiculous result: it means the pendulum is not moving, which means that there's no period at all. In this way, Will's common sense reasoning in evaluating this solution is coherent with his epistemological stance that "logical reasoning" is appropriate, even when you lack prior knowledge.

In contrast to his work on the Pendulum problem, his epistemological stance towards problems in history, where facts can't be reasoned out and must be known, aligns with a *recalling formal knowledge* approach, further supporting the idea that the absence of *recalling formal knowledge* on the Pendulum problem is, in part, epistemological. In the next section, I elaborate on Will's epistemological stance towards the Arctan problem and its alignment with a *recalling formal knowledge* approach.

Mathematical Reasoning Is Not Natural

While trying to recall ideas from his calculus class, Will reflects on what is difficult about the Arctan problem, revealing his epistemological stance that the reasoning required for this problem is "not normal reasoning."

> [01:03:14]
> W: Um, but yeah. It's 'cause I, I just don't remember those, uh, that information. I don't think that, and I couldn't, this is, I hate, it's, that's why I hated these problems so much, that I couldn't reason through them. I couldn't think, “oh well, infinity, you know, this happens.” I just can't, it doesn't make any sense.
> E: It's not like the previous problem [i.e. the Pendulum problem], is what you're saying?
> W: No, not at all. That's like concrete. That's like, “ok, pendulum moving. I can see that.” That's why I hated Taylor series so much, is 'cause you can't see it. It's, it's just, it's like pure mathematical reasoning that's, like, not normal reasoning. It's, you think about it a different way. You can't just think as a person like, “oh yeah it's, pendulum swings to a certain point, this happens.” You have to think about it in terms of, like, infinity and what happens when you go to infinity. That's like, I don't think that's like in, humans don't think like that naturally, so you have to learn it.
> E: Ok.
> W: So I wouldn't be able to really, I, like, I couldn't get partial credit on this problem. I'd be like, uh, uh, ok.

Will articulates a difference between “pure mathematical reasoning” and “normal reasoning.” The pure mathematical reasoning, such as reasoning about Taylor series or infinity, is more abstract and cannot be envisioned in the way that a pendulum’s motion can. Pure mathematical reasoning does not make sense and isn’t how people naturally think. Therefore, the ways of reasoning required for these problems *must* be learned in and recalled from class. In this way, intuitive ways of reasoning or logical assumptions one might make, such as drawing conclusions from intuitions about a pendulum, are not appropriate on these types of problems. This aligns with his previous description of the difference between problems like the Pendulum problem, where logical reasoning is appropriate, and history problems that require knowledge of certain facts that cannot simply be derived or intuited. Here, because this type of pure mathematical reasoning cannot be reasoned out, Will would not succeed in getting partial credit in this problem, because he lacks knowledge of some requisite facts required for solving this problem.

This epistemological stance that the pure mathematical reasoning required here doesn’t make sense coheres with his approach to the Arctan problem. M*aking sense and exploring,* finding ways to make sense of arctangent and the series equation and exploring and evaluating ways to determine the bounds of a “good approximation,” does not make sense as an approach if one views the mathematical content on the Arctan problem as not able to be reasoned through. Instead, one must rely on *recalling formal knowledge* from calculus class.

In fact, this epistemological stance towards the Arctan problem explains why Will would persist in *recalling formal knowledge,* even after he fails to recall the ideas he seeks. If Will’s epistemological stance towards this problem is that the application of specific learned methods are required, then “logical reasoning” will not be productive – as Will says, he wouldn’t get any partial credit.

At the end of the interview, Will reflects again on the difference between the Pendulum and Arctan problems, describing an epistemological difference – that these problems require different kinds of reasoning and approaches.

> [01:15:59]
> W: But [the Pendulum problem] is just more fundamentally easy to grasp, because this, [the Arctan problem], you're dealing with polynomials. You know, approximating them as it goes to infinity and breaking into a million pieces. You know, just, it's not logical. It, it requires a different method of reasoning than the physical world, because it's not physically real, you know?
> I: Ok.
> W: So this one [points to the Arctan problem], I would be completely lost if I had never been in calc 2. I mean, I couldn't even look at it.

Importantly, these epistemological stances towards these two problems do not necessarily represent context-independent epistemological beliefs about learning and reasoning in math and physics. The fact that Pendulum is a relatively simple physical system with relatively simple mathematics likely contributes to Will's epistemological stance that logical reasoning is appropriate. One could imagine that Will may take a stance that more abstract physics problems are unintuitive and require formal knowledge of the relevant physical concepts and equations. Conversely, Will, in *making sense and exploring* on the Pendulum problem, does mention a connection between the series expression to ideas from power series in calculus, showing that calculus content does not always represent "unnatural reasoning" for Will.

The point of this analysis is not to make broad claims about what epistemologically stances Will takes towards problems in calculus and physics but to point out the particular in-the-moment stances that Will takes towards these two problems. These stances likely draw on some aspects of Will's experiences with physics and calculus, but that does not necessarily imply that these particular epistemological stances are the only ones that Will can take when dealing with physics and calculus.

HOW WILL'S SENSES OF ACCOUNTABILITY TOWARDS THE CONTENT OF THESE PROBLEMS SUPPORT DIFFERENT APPROACHES

In addition to Will's epistemological stances toward the problems, Will's sense that he has seen problems like the Arctan problem before in his calculus class drives the search for those previously seen ideas. I propose that Will's *sense of accountability* towards ideas relevant to the Arctan problem supports his persistence in trying to recall those ideas, whereas a lack of accountability towards canonical physics ideas relevant to the Pendulum problem supports an approach of *making sense and exploring*.

Will Does Not Feel Accountable For Knowing About the Pendulum Problem

One could assume that Will's lack of exposure to the relevant content and to the type of problem presented in the Pendulum problem would mean that he could not feel accountable to those ideas. However, one possible interpretation of this interview situation is that it is testing how quickly a correct approach can be applied to this problem. I argue that it is the initial interview interaction between Will and I that helps signal that Will is not being held accountable for the content of the problem.

After being presented with the problem, Will reads the problem silently for about 20 seconds, after which he checks in to see if he needs to have seen this type of problem before:

> [00:02:54]
> W: And it's ok if you haven't seen this type of problem before, right?
> E: Uh, yeah. That's fine.
> W: Ok.

After this explicit confirmation that Will need not have seen this problem before, he continues to read silently for about one minute, after which he starts to confirm his understanding of the problem with me.

> [00:03:59]
> W: So this is the, this is the equation for the small angle oscillation, right?
> E: Mhm.
> W: So, but there's no angle in it.
> E: In this?
> W: In this one. Right.
> E: Uh.
> W: But they're asking how big, how big can the angle of the pendulum be before that equation is no longer accurate?
> E: Yes.
> W: Ok. [pause] So you don't want me to calc...you just want me to answer the question, right? I don't have to do anything?
> E: Uh, yeah. Yeah, I mean, does it make sense, what they're asking?
> W: Yeah, it makes sense. I mean, I can understand what they're trying to say.
> E: Ok.
> W: I haven't done oscillations, but I can understand the idea that they're trying to get through.

Here, Will poses questions to the me, both confirming his understanding of the problem and asking what his role in the interview should be ("So you don't want me to calc-, you just want me to answer the question, right? I don't have to do anything?"). I twice confirm Will's questions. These confirmations may support the interview interaction as me engaging to help Will make sense of this unfamiliar problem rather than testing what he already knows, a view that could have been supported had I withheld information. Furthermore, I provided time for Will to silently read the problem, uninterrupted. After this exchange, Will reads silently for about another minute. This could also contribute to a sense that Will is being given time to make sense of an unfamiliar problem, rather than having to immediately recall an answer. Will's final statement in this section indicates that he hasn't seen this topic in his courses before, in part positioning himself as unfamiliar with this content.

After the extended silence at the end of this section, Will starts to check in again, now checking more specifically about features of the problem. I reiterate that I want to hear Will's understanding of the problem, whether correct or incorrect:

> [00:05:47]
> W: Well, I would guess that these T's should be equal, right? If they're going to be, they're both going to be accurate approximations.
> E: Ok.
> W: So at a certain point, when they're no longer roughly approximate is when your angle is getting too big that this one [the small angle approximation] breaks down, and this is the one [series equation] you have to use. Is that correct?
> E: Um, well I mean, I mean, I just sort of want to figure out how you would think about it.
> W: Ok.
> E: So whether or not it's correct.
> W: Ok, so you want me to explain how I would think about it.
> E: Yeah.

In response to this, Will's talk switches from asking questions to confirm his understanding of the problem to stating his approach to making sense of the problem, and his understanding of the two expressions for the period. This shift in Will's talk suggests that Will is taking up an interpretation of the interview situation as one where his ideas, rather than the canonical physics ideas, take primacy.

This interpretation of the interview situation plausibly supports a *making sense and exploring* approach to the Pendulum problem. If the purpose of the interview is to hear how Will would think about the problem, he is not accountable to particular canonical ways of understanding and approaching this problem. This likely supports the development of his own ideas on the problem and exploration and evaluation of possible algebraic solutions to the problem.

Will Feels Accountable for Recalling Taylor Series Knowledge

Given that the interview has thus far supported Will's *sense making and exploration* and that I have explicitly and implicitly communicated that the goal of the interview is to find out how Will thinks about problems rather than testing whether he knows the canonically correct methods, it would be reasonable to expect that this lack of accountability would carry through to the Arctan problem. However, this is not the case.

At the very start of the problem, Will signals that he is familiar with Taylor series and this type of problem:

> [00:46:15]
> W: Try to simplify it before you even start. Um, I, don't you need to know how to formulate the Taylor series for, I don't know, you don't, I guess. I don't even know how to formulate the Taylor series, so it's a good thing they kind of gave it to me. Um, ok, I absolutely hate this stuff, but I have done this one before. I have taken, like, calc 2, so I have done this before. [reads the problem to himself again].

In this initial reaction to the problem, Will signals not only familiarity with this type of problem from calculus, but also that he dislikes this kind of problem – possibly, because, as discussed earlier, Will dislikes that you can't reason logically through the pure mathematical reasoning of Taylor series.

Throughout the problem, Will signals that he feels accountable for this Taylor series knowledge by indicating his frustration at not being able to remember the relevant facts from math class:

> [00:58:27]
> W: And it's annoying, this one's really annoying, because I definitely have done this or something like it, so I should know how to do this one. It's been in my mind before, um, but I did get a 40 on this test, so, didn't know it that well. Uh, [laughs] let me think.

Here, Will reiterates that he has seen the problem before so he *should* know the canonical method for solving it. Throughout his work on the Arctan problem, Will holds himself accountable to this knowledge, and this accountability drives *frustration* at not being able to remember those ideas from calculus class.

Later on, Will notes that it's "upsetting" that he can't remember the ideas from class. I ask Will what information he would look up or ask someone, given the chance. My attention on what canonical pieces of knowledge Will does not know likely supports feelings of accountability towards those pieces of knowledge. Similarly, receiving that knowledge later on from me may tacitly signal that recalling this information was crucial.

At the end of the Arctan problem, he reiterates that not knowing how to do this problem is more upsetting than not knowing the correct approach to the Pendulum problem, because he is familiar with this type of problem:

> [01:11:52]
> W: Um, but yeah. This one is more upsetting to me than the other one, because I did actually do these kinds of problems before. And like, I don't really have, or see how to do these right, but that's how I would do it at this point, not remembering much.

This accountability and frustration are compounded by the fact that he feels that this problem would be very simple to do if he could just remember the relevant method:

> W: Um, so maybe if I brushed up a bit, I'd go, "oh, it's easy to do." It's very possible that it only takes, like, two seconds. This is a frust-, it's frustrating because I know that this is not difficult or doesn't seem that difficult, looks like a fairly, uh, commonplace, uh, Taylor series problem. But since I don't remember how to do, do Taylor series at all, it's really hard for me to do.

This sense of accountability and frustration towards not being able to remember how to solve these types of problems is distinct from, although related to, his epistemological stance towards the type of knowledge required for this problem: pure mathematical reasoning has to be learned in class and can’t be reasoned. Identifying the type of knowledge required for a problem is separate from feels about whether or not you should be responsible for and accountable to that knowledge. For Will, this sense of accountability to the Taylor series knowledge required for this problem is connected to his history and familiarity with these ideas and drives his affective feelings of frustration in trying to recall the relevant Taylor series ideas.

This sense of accountability towards Taylor series knowledge and, specifically, knowledge of the relevant method for solving this problem, adds to an understanding of why Will persists throughout the interview to try to remember facts from calculus class, rather than recognizing that he has forgotten the relevant knowledge, leading to a change in his approach. One explanation is that his in-the-moment epistemological stance towards this problem supports the view that this reasoning cannot be intuitively understood. Rather, it can only be learned in calculus class, so recalling those ideas is crucial to solving the problem correctly. Will’s sense of accountability towards this knowledge and his associated frustration at not being able to recall that knowledge may also support the activity of *recalling formal knowledge*.

At the end of the interview, I directly ask Will how the general formula for Taylor series, the equation he sought throughout the interview and that was eventually supplied to him, was helpful. Will connects this search for the general formula as (at least partially) fueled by his frustration at not being able to recall it:

> [01:13:01]
> W: Uh, it didn't, it ended up not really helping me. Uh, I guess I just wanted to remember the basic, it was so frustrating to me that I couldn't remember the basic formula for Taylor series. And when they ask you a question on, you know, Taylor series, uh, or anything, you just want to remember it. And I know I'd done it before. It was just frustrating me that I didn't remember the basic idea of it.

Will’s sense of accountability is connected to the notion of authorship in expansive framing. Taking an expansive framing lens towards understanding transfer, student authorship of knowledge supports adaptation of that knowledge to new situations, supporting transfer. Here, I posit that a *lack of accountability* towards classroom knowledge provides Will with space to make sense and explore ways to adapt his prior knowledge of algebraic equations and physical motion of pendula in order to make sense of the Pendulum problem through mechanisms similar to the ones described for authorship. The reason I don't use the language of authorship and expansive framing to explain Will's behavior is that, on the Arctan problem, Will attempts to transfer-in his prior knowledge of Taylor series, knowledge over which he feels a *lack of authorship*. Rather than making sense of the ideas in the problem, Will feels accountable to recalling those ideas and methods "from that chapter of the book." This is a different type of transfer than that sought by Engle and colleagues – the goal is not adaptation of knowledge, but accurate recall. In this case, Will's attempt to bring in his prior knowledge of Taylor series is supported by his accountability towards that knowledge, and the associated frustration at not being able to recall that knowledge, not a sense of authorship.

A Role for Affect

Coupled with this sense of accountability towards these two problems are also Will's affective responses to the problems. For example, on the Pendulum problem, one role I could have played in the opening moments of Will's work was to put Will at ease and to set up a comfortable situation in which he felt free to explore different approaches. On the Arctan problem, Will's affect relates to both his history with the topic of Taylor series (that he hated those kinds of problems because he couldn't make sense of them, and that his low test performance in his calculus class could support anxiety and a lack of self-efficacy with the topic) as well as his frustration in the interview towards not being able to recall the Taylor series facts he seeks to remember. Just as this work seeks to emphasize the advantages of incorporating individuals' epistemological stances and senses of accountability towards the relevant knowledge to understand successes and failures in transfer, future work may incorporate other factors, such as students' affective responses, to add onto our understanding of the factors that affect students reasoning in different contexts.

DISCUSSION

Factors that Influence the Transfer of Knowledge Across the Disciplines

Attempts to foster transfer across disciplines typically focus on particular content that is common to those two disciplines. The method of supporting this transfer is to align the content of different courses such that students are exposed to the same content in different disciplinary contexts. This study contributes by showing that student reasoning on similar problems across two disciplines depends not only the similarities in the structure and content of the two problems, but also on factors beyond these considerations – in this case, students' epistemological stances towards the problems and their senses of accountability towards the particular content of the problems. That is, students may see and experience different kinds of differences between the problems where experts see only deep similarity in content and structure.

How Can Transfer Across Disciplines Be Fostered?

The classical transfer perspective typically aims to foster transfer by providing sufficient examples to help the learners identify the common core structure in different problem contexts. In interdisciplinary efforts to foster such transfer, this typically involves learning the same content knowledge in different disciplinary contexts at the same time (Al-Holou et al., 1999; Dunn & Barbanel, 2000; Plomer et al., 2010), mirroring transfer experiments that provide many analogous problems to help learners form a generalized schema for recognizing a common deep structure.

Just as these interdisciplinary course reform efforts attend to content alignment, they could similarly attend to students' epistemological stances towards what kinds of knowledge or approaches are needed to answer questions across these disciplines. This could help students view problems such as the Pendulum and Arctan problems as similar. Beyond the structural similarities in asking students to make judgments of when a good approximation becomes bad, instruction could also attend to what kinds of knowledge students see different disciplinary problems as requiring.

These interdisciplinary reforms could also attend to differential levels of accountability towards the relevant content knowledge. Supporting student authorship of ideas by asking students to explain *their own ideas* using an authoritative text as a resource and revoicing students' ideas to credit the student with authorship has been shown to support the transformation of prior knowledge for application in new situations within the same discipline (Engle et al., 2012). Similarly supporting authorship of ideas across the disciplines, rather than accountability to being able to recall and apply particular disciplinary content, could foster a willingness to transform those ideas learned in particular disciplinary contexts for use in another.

Future work may expand the study of transfer by performing well-defined transfer experiments to explore the effect of these different factors on transfer across the disciplines. Furthermore, studies of students' epistemological stances and senses of authorship in interdisciplinary learning situations may aid in understanding the successes and failures of those situations in fostering transfer across disciplines.

Chapter 6: Summary and Future Directions

SUMMARY

In chapter 1, I describe how, although it is very focused on qualitative conceptual understanding, the PER community has not attended nearly as much to how physics equations can be conceptually understood. This dissertation consists of three separate studies, relating to how physics students make intuitive or conceptual sense with symbolic equations, as well as studying factors that support and suppress this type of reasoning.

Chapter 3 began by studying the nature of how undergraduate physics students conceptually understand physics equations. I show that *symbolic forms use* provides a good description of one way this conceptual understanding of physics equations is instantiated. Furthermore, a symbolic forms-based understanding of the equation can support heuristic problem-solving shortcuts that leverage the equation in ways that avoid explicit algorithmic computation. Symbolic forms use in reasoning with equations is one component of problem-solving expertise in physics.

Chapter 4 investigates connections between how students reason conceptually with equations and their in-the-moment epistemological stances towards what kinds of knowledge and reasoning are appropriate. A novel coding scheme shows the plausibility of a connection between symbolic forms use and epistemological stances that value *coherence,* either between (1) formal physics reasoning and everyday reasoning or (2) physics equations and physics concepts. Beyond this, I show that individuals are not static in how they reason about equations or in their epistemological stances towards learning and understanding equations. Even students who don't appear to know how to use symbolic forms in interpreting physics equations may be able to do so in other contexts. Studying the dynamics in how students reason with physics equations bolsters both theoretical understanding in conceptual/epistemological dynamics and instructional intuitions about how to develop this problem-solving expertise by starting from what students already know.

Chapter 5 uses different disciplinary problem contexts to investigate what factors affect how students reason with mathematics. Although students' reasoning likely depends on the particular content and structure of the problems, students perceive additional differences between the problems that are not obvious. Students' different epistemological stances towards those problems, as well as different degrees of accountability towards the particular content knowledge in the problems can support the development of different reasoning trajectories while working on different disciplinary problems.

FUTURE DIRECTIONS

This dissertation starts work in how symbolic forms can be leveraged in problem solving for the *blended processing* of equations. Although this dissertation thoroughly deals with *blended processing* with the *Base + Change* symbolic form on a small set of similarly structured problems, more examples of *blended processing* need to be documented to better understand how symbolic forms can be leveraged for heuristic *blended processing* solutions.

At the same time, this study of symbolic forms is limited to algebraic equations, with which undergraduate students have a lot of experience (and plausibly also a lot of intuitions about). As shown with Devon, helping students use symbolic forms in introductory physics can involve showing students that their intuitive understandings of other algebraic equations are useful. But physics goes beyond algebra and includes difficult advanced mathematics: calculus, vector calculus, differential equations, and linear algebra, for example. Work needs to be done to understand what it looks like to intuitively understand this advanced mathematics in physics, and how to support the development of these forms of reasoning in students when the level of math becomes more complex. Certainly more students will see the intuitive ideas in the money equation than then intuitive ideas in second order differential equations.

In a different direction, future work could extend the claims of consequentiality for how a student's reasoning with equations develops as they work on a problem. Although I identified two factors that play roles in the development of a student's in-the-moment reasoning, these are not the only relevant factors. The interaction with the interviewer in these situations also plays an important role. I explicitly touch on this issue in chapter 5, where I argue that the development of accountability in the interview is partially supported through interaction between interviewer and interviewee. More work needs to be done to continue to understand the role the interviewer plays in these case studies of Devon and Will, and how the interviewer's choices contribute to expressed reasoning in that situation.

Symbolic Forms Are Valuable Tools, But Only in the Service of Sensemaking

Another area for future study is how symbolic forms can be incorporated into the physics curriculum. Although this dissertation suggests symbolic forms as an instructional target, I will suggest that the way that they are taught is also important. Consider the explicit problem-solving procedures that direct students to start problem solving by drawing a diagram. This move makes sense as the problem-solving research showed that experts start by drawing physics diagrams of the situation. From this, it seems that teaching students to begin problem-solving episodes by drawing a diagram would plausibly support problem-solving expertise.

One example of prompting students to draw free-body diagrams shows that this is not always the case. Heckler (2010) showed that students who were explicitly directed to draw a free body diagram less frequently used solutions that were aligned with those diagrams than students for whom no diagram was prompted. Case study interviews showed that students prompted to draw diagrams may view the production of the diagram as a required step divorced from seeking a solution to the problem. In contrast, students who were not directed to draw diagrams often still would, but as a step that contributes to making sense of and solving the problem. Although most students started the problem-solving episode by drawing a diagram, not all did so as a part of making sense of and finding a solution to the problem. In fact, explicitly prompting students to draw a diagram led to it being a disconnected step from solving the problem.

Similarly, explicit teaching of symbolic forms as a rote and required problem-solving tool may not be desirable. Symbolic forms use evidences problem-solving expertise only if used for authentic sense-making practices. Said another way, symbolic forms are valuable as useful tools in the epistemological pursuit of *making sense* of physical phenomena and formal physics content. Care is needed to prevent symbolic forms from becoming another rote requirement.

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