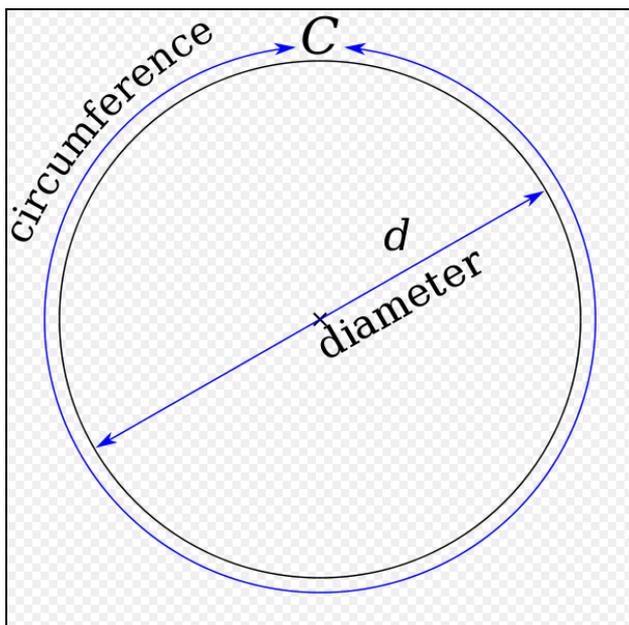
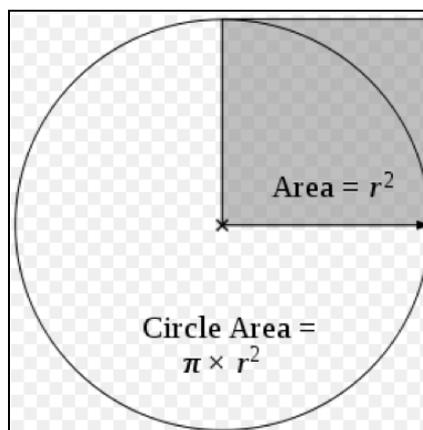


# Pi



Circumference of circle equals  $\pi$  times diameter.



Area of circle equals  $\pi$  times shaded area.

$\pi$  (sometimes written **pi**) is a [mathematical constant](#) whose value is the [ratio](#) of any [circle's](#) circumference to its diameter; this is the same value as the ratio of a circle's area to the square of its radius.  $\pi$  is approximately equal to 3.14159 in the usual [decimal positional notation](#). Many formulae from mathematics, [science](#), and [engineering](#) involve  $\pi$ , which makes it one of the most important mathematical constants.[1]

$\pi$  is an [irrational number](#), which means that its value cannot be expressed exactly as a [fraction](#), the numerator and denominator of which are [integers](#). Consequently, its [decimal representation](#) never ends or repeats.  $\pi$  is also a [transcendental number](#), which implies, among other things, that no finite sequence of algebraic operations on integers (powers, roots, sums, etc.) can be equal to its value; proving this was a late achievement in mathematical history and a significant result of 19th century German mathematics. Throughout the history of mathematics, there has been much effort to determine  $\pi$  more accurately and to understand its nature; fascination with the number has even carried over into non-mathematical culture.

Probably because of the simplicity of its definition, the concept of  $\pi$  has become entrenched in popular culture to a degree far greater than almost any other mathematical construct.[2] It is, perhaps, the most common ground between mathematicians and non-mathematicians.[3] Reports on the latest, most-precise calculation of  $\pi$  are common news items.[4][5][6] The current record for the decimal expansion of  $\pi$ , if verified, stands at 5 trillion digits.[7]

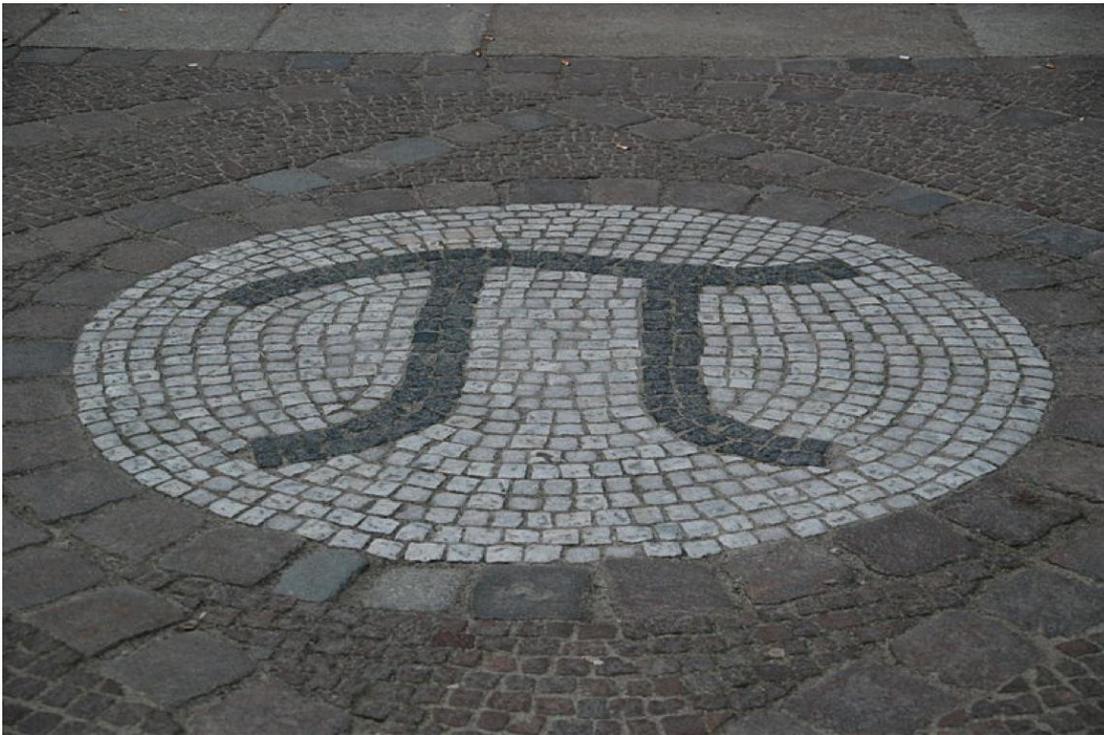
The Greek letter  $\pi$  was first adopted for the number as an abbreviation of the Greek word for [perimeter](#) ( $\pi\epsilon\rho\acute{\iota}\mu\epsilon\tau\rho\varsigma$ ), or as an abbreviation for "periphery/diameter", by [William Jones](#) in 1706. The constant is also known as **Archimedes' Constant**, after [Archimedes of Syracuse](#) who provided an approximation of the number, although this name for the constant is uncommon in modern English-speaking contexts.



"perimeter", referring to the ratio of the perimeter to the diameter, which is constant for all circles[10]. [William Jones](#) was the first to use the Greek letter in this way, in 1706,[11] and it was later popularized by [Leonhard Euler](#) in 1737.[12][13] [William Jones](#) wrote:

There are various other ways of finding the Lengths or Areas of particular Curve Lines, or Planes, which may very much facilitate the Practice; as for instance, in the Circle, the Diameter is to the Circumference as 1 to ... 3.14159, etc. =  $\pi$  ...[14]

When used as a symbol for the [mathematical constant](#), the Greek letter ( $\pi$ ) is not capitalized at the beginning of a sentence. The capital letter  $\Pi$  (Pi) has a completely different mathematical meaning; it is used for expressing [the product of a sequence](#).



Lower-case  $\pi$  denotes the constant. This [mosaic](#) is outside the mathematics building at the [Technische Universität Berlin](#).

### Geometric definition

In [Euclidean plane geometry](#),  $\pi$  is defined as the [ratio](#) of a [circle's circumference](#)  $C$  to its [diameter](#)  $d$ : [10]

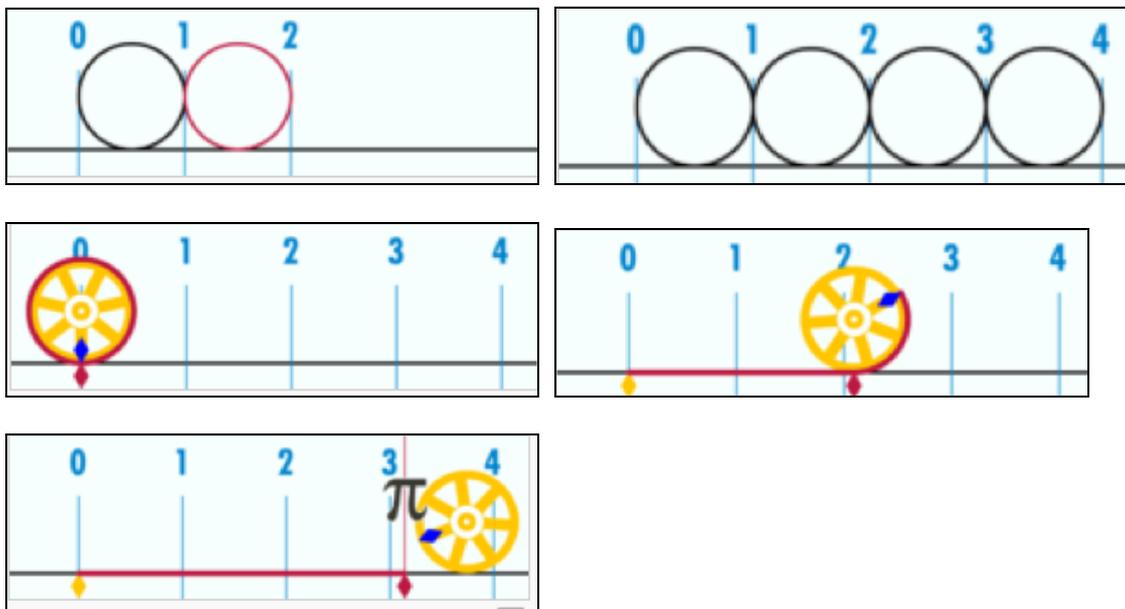
$$\pi = \frac{C}{d}.$$

The ratio  $C/d$  is constant, regardless of a circle's size. For example, if a circle has twice the diameter  $d$  of another circle it will also have twice the circumference  $C$ , preserving the ratio  $C/d$ .

Alternatively  $\pi$  can be defined as the ratio of a circle's [area](#)  $A$  to the area of a square whose side is equal to the [radius](#)  $r$  of the circle: [10][15]

$$\pi = \frac{A}{r^2}.$$

These definitions depend on results of Euclidean geometry, such as the fact that all circles are **similar**, and the fact that the right-hand-sides of these two equations are equal to each other (i.e. the **area of a disk** is  $Cr/2$ ). These two geometric definitions can be considered a problem when  $\pi$  occurs in areas of mathematics that otherwise do not involve geometry. For this reason, mathematicians often prefer to define  $\pi$  without reference to geometry, instead selecting one of its **analytic** properties as a definition. A common choice is to define  $\pi$  as twice the smallest positive  $x$  for which the **trigonometric function**  $\cos(x)$  equals zero.[16]



When a circle's **diameter** is 1 **unit**, its **circumference** is  $\pi$  units.

## Irrationality and transcendence

*Main article: [Proof that  \$\pi\$  is irrational](#)*

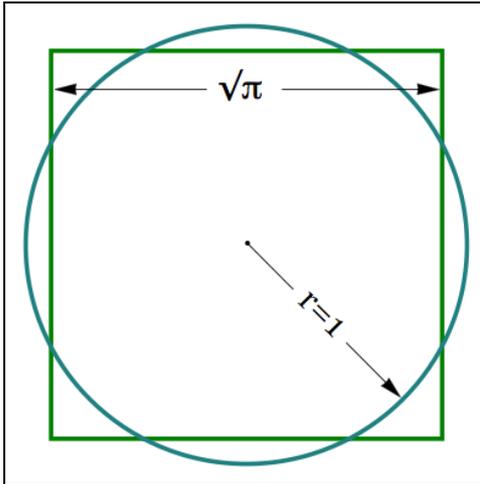
$\pi$  is an **irrational number**, meaning that it cannot be written as the **ratio of two integers**.  $\pi$  is also a **transcendental number**, meaning that there is no **polynomial** with **rational** coefficients for which  $\pi$  is a **root**.[17] An important consequence of the transcendence of  $\pi$  is the fact that it is not **constructible**. Because the coordinates of all points that can be constructed with **compass and straightedge** are constructible numbers, it is impossible to **square the circle**: that is, it is impossible to construct, using compass and straightedge alone, a square whose area is equal to the area of a given circle.[18] This is historically significant, for squaring a circle is one of the easily understood elementary geometry problems left to us from antiquity. Many amateurs in modern times have attempted to solve each of these problems, and their efforts are sometimes ingenious, but in this case, doomed to failure: a fact not always understood by the amateur involved.[19]

## Decimal representation

*See also: [Approximations of  \$\pi\$](#)*

The decimal representation of  $\pi$  **truncated** to 50 **decimal places** is:[20]

$\pi = 3.14159265358979323846264338327950288419716939937510\dots$



Because  $\pi$  is a [transcendental number](#), [squaring the circle](#) is not possible in a finite number of steps using the classical tools of [compass and straightedge](#)

Various online web sites provide  $\pi$  to many more digits.[21] While the decimal representation of  $\pi$  has been computed to more than a [trillion](#) ( $10^{12}$ ) digits,[22] elementary [applications](#), such as estimating the circumference of a circle, will rarely require more than a dozen decimal places. For example, the decimal representation of  $\pi$  truncated to 11 decimal places is good enough to estimate the circumference of any circle that fits inside the Earth with an [error](#) of less than one millimetre, and the decimal representation of  $\pi$  truncated to 39 decimal places is sufficient to estimate the circumference of any circle that fits in the [observable universe](#) with precision comparable to the radius of a [hydrogen atom](#).[23][24]

Because  $\pi$  is an [irrational number](#), its decimal representation does not [repeat](#), and therefore does not terminate. This sequence of non-repeating digits has fascinated mathematicians and laymen alike, and much effort over the last few centuries has been put into computing ever more of these digits and investigating  $\pi$ 's properties.[25] Despite much analytical work, and [supercomputer](#) calculations that have determined over 1 [trillion](#) digits of the decimal representation of  $\pi$ , no simple [base-10](#) pattern in the digits has ever been found.[26] Digits of the decimal representation of  $\pi$  are available on many web pages, and there is [software for calculating the decimal representation of  \$\pi\$](#)  to billions of digits on any [personal computer](#).

Number system	Approximation of $\pi$
<a href="#">Binary</a>	11.00100100001111110110...[27]
<a href="#">Octal</a>	3.11037552421026430215...
<a href="#">Decimal</a>	3.14159265358979323846264338327950288...
<a href="#">Hexadecimal</a>	3.243F6A8885A308D31319...[28]
<a href="#">Rational approximations</a>	$3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{52163}{16604}, \frac{103993}{33102}, \dots$ [29] (listed in order of increasing accuracy)
<a href="#">Continued fraction</a>	[3;7,15,1,292,1,1,1,2,1,3,1,14,2,1,1...][30] (This fraction is not <a href="#">periodic</a> . Shown in <a href="#">linear notation</a> )
<a href="#">Generalized C.F.</a>	$\frac{4}{1} + \frac{1}{\frac{1}{3} + \frac{4}{5} + \frac{9}{7} + \frac{16}{9} + \frac{25}{11} + \dots}$

### Estimating the value

*Main article: [Approximations of  \$\pi\$](#)*

The earliest numerical approximation of  $\pi$  is almost certainly the value 3. In the Bible

verse 1 Kings 7:23 the value is stated as 3 - "He made the Sea of cast metal, circular in shape, measuring ten cubits from rim to rim and five cubits high. It took a line of thirty cubits[a] to measure around it." Which gives an accurate enough value of pi for the purpose it was trying to serve. Footnotes: [32] In cases where little precision is required, it may be an acceptable substitute. That 3 is an underestimate follows from the fact that it is the ratio of the [perimeter](#) of an [inscribed regular hexagon](#) to the [diameter](#) of the [circle](#).

$\pi$  can be empirically estimated by drawing a large circle, then measuring its diameter and circumference and dividing the circumference by the diameter. Another geometry-based approach, attributed to [Archimedes](#), [33] is to calculate the [perimeter](#),  $P_n$ , of a [regular polygon](#) with  $n$  sides [circumscribed](#) around a circle with diameter  $d$ . Then compute the [limit of a sequence](#) as  $n$  increases to [infinity](#):

$$\pi = \lim_{n \rightarrow \infty} \frac{P_n}{d}.$$

This sequence converges because the more sides the polygon has, the smaller its maximum distance from the circle. Archimedes determined the accuracy of this approach by comparing the perimeter of the circumscribed polygon with the perimeter of a regular polygon with the same number of sides [inscribed](#) inside the circle. Using a polygon with 96 sides, he computed the fractional range: [34]

$$3\frac{10}{71} < \pi < 3\frac{10}{70}.$$

$\pi$  can also be calculated using purely mathematical methods. Due to the transcendental nature of  $\pi$ , there are no closed form expressions for the number in terms of algebraic numbers and functions. [17] Formulas for calculating  $\pi$  using elementary arithmetic typically include [series](#) or [summation notation](#) (such as "..."), which indicates that the formula is really a formula for an infinite sequence of approximations to  $\pi$ . [35] The more terms included in a calculation, the closer to  $\pi$  the result will get.

Most formulae used for calculating the value of  $\pi$  have desirable mathematical properties, but are difficult to understand without a background in [trigonometry](#) and [calculus](#). However, some are quite simple, such as this form of the [Gregory–Leibniz series](#): [36]

$$\pi = 4 \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{4}{1} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \dots$$

While that series is easy to write and calculate, it is not immediately obvious why it yields  $\pi$ . In addition, this series converges so slowly that nearly 300 terms are needed to calculate  $\pi$  correctly to two decimal places. [37] However, by computing this series in a somewhat more clever way by taking the midpoints of partial sums, it can be made to converge much faster. Let the [sequence](#)

$$\pi_{0,1} = \frac{4}{1}, \pi_{0,2} = \frac{4}{1} - \frac{4}{3}, \pi_{0,3} = \frac{4}{1} - \frac{4}{3} + \frac{4}{5}, \pi_{0,4} = \frac{4}{1} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} \dots$$

and then define

$$\pi_{i,j} = \frac{\pi_{i-1,j} + \pi_{i-1,j+1}}{2} \text{ for all } i, j \geq 1$$

then computing  $\pi_{10,10}$  will take similar computation time to computing 150 terms of the original series in a brute-force manner, and  $\pi_{10,10} = 3.141592653589793$ , correct to 9 decimal places. This computation is an example of the [van Wijngaarden](#)

transformation.[38]

```
3.1415926535 8979323846 2643383279 5028841971 6939937510
5820974944 5923078164 0628620899 8628034825 3421170679
8214808651 3282306647 0938446095 5058223172 5359408128
4811174502 8410270193 8521105559 6446229489 5493038196
4428810975 6659334461 2847564823 3786783165 2712019091
4564856692 3460348610 4543266482 1339360726 0249141273
7245870066 0631558817 4881520920 9628292540 9171536436
7892590360 0113305305 4882046652 1384146951 9415116094
3305727036 5759591953 0921861173 8193261179 3105118548
0744623799 6274956735 1885752724 8912279381 8301194912
9833673362 4406566430 8602139494 6395224737 1907021798
6094370277 0539217176 2931767523 8467481846 7669405132
0005681271 4526356082 7785771342 7577896091 7363717872
1468440901 2249534301 4654958537 1050792279 6892589235
4201995611 2129021960 8640344181 5981362977 4771309960
5187072113 4999999837 2978049951 0597317328 1609631859
5024459455 3469083026 4252230825 3344685035 2619311881
7101000313 7838752886 5875332083 8142061717 7669147303
5982534904 2875546873 1159562863 8823537875 9375195778
1857780532 1712268066 1300192787 6611195909 2164201989
3809525720 1065485863 2788659361 5338182796 8230301952
0353018529 6899577362 2599413891 2497217752 8347913151
5574857242 4541506959
```

An estimate of  $\pi$  accurate to 1120 decimal digits was obtained using a [gear-driven calculator](#) in 1948, by [John Wrench](#) and Levi Smith. This was the most accurate estimate of  $\pi$  before electronic computers came into use.[31]

For many purposes, 3.14 or  $\frac{22}{7}$  is close enough, although engineers often use 3.1416 (5 [significant figures](#)) or 3.14159 (6 significant figures) for more precision.<sup>[[citation needed](#)]</sup> The approximations  $\frac{22}{7}$  and  $\frac{355}{113}$ , with 3 and 7 significant figures respectively, are obtained from the simple [continued fraction](#) expansion of  $\pi$ . The approximation  $\frac{355}{113}$  (3.1415929...) is the best one that may be expressed with a three-digit or four-digit [numerator and denominator](#); the next good approximation  $\frac{52163}{16604}$  (3.141592387...), which is also accurate to 7 significant figures, requires much bigger numbers, due to the large number 292 in the continued fraction expansion of  $\pi$ .<sup>[29]</sup> For extremely accurate approximations, either Ramanujan's approximation of  $\frac{25118864}{8549136000} \sqrt{10}$  <sup>[39]</sup> (3.14159265258...) or  $\frac{103993}{33102}$  <sup>[29]</sup> (3.14159265301...) are used, which are both accurate to 10 significant figures.

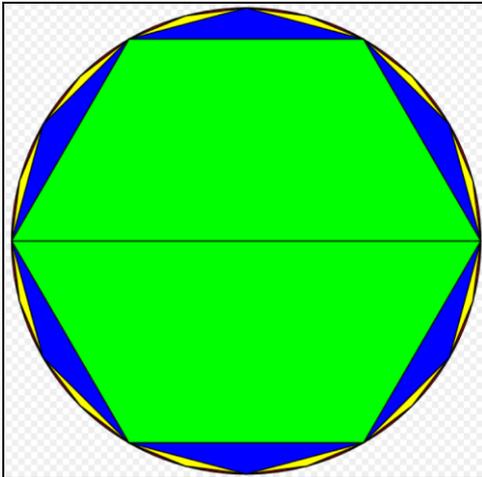
## History

See also: *Chronology of computation of  $\pi$  and Approximations of  $\pi$*

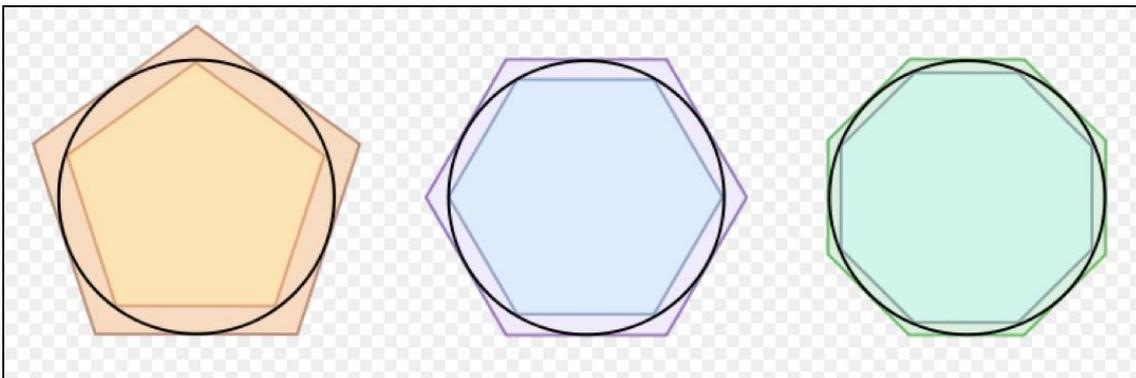
The [Great Pyramid](#) at Giza, constructed c.2589–2566 BC, was built with a perimeter of 1760 cubits and a height of 280 cubits giving the ratio  $1760/280 \approx 2\pi$ . The same [apotropaic](#) proportions were used earlier at the Pyramid of [Meidum](#) c.2613-2589 BC and later in the pyramids of Abusir c.2453-2422. Some Egyptologists consider this to have been the result of deliberate design proportion. Verner wrote, "We can conclude that although the ancient Egyptians could not precisely define the value of  $\pi$ , in practice they used it".<sup>[40]</sup> Petrie, author of *Pyramids and Temples of Gizeh* concluded: "but these relations of areas and of circular ratio are so systematic that we should grant that

they were in the builders design".[41] Others have argued that the Ancient Egyptians had no concept of  $\pi$  and would not have thought to encode it in their monuments. They argued that creation of the pyramid may instead be based on simple ratios of the sides of right-angled triangles (the seked).[42]

The early history of  $\pi$  from textual sources roughly parallels the development of mathematics as a whole.[43]



Estimating  $\pi$  with inscribed polygons



Estimating  $\pi$  with circumscribed and inscribed polygons

## Antiquity

The earliest known textually evidenced approximations of pi date from around 1900 BC. They are found in the Egyptian [Rhind Papyrus](#)  $256/81 \approx 3.160$  and on Babylonian tablets  $25/8 = 3.125$ , both within 1 percent of the true value.[10]

The Indian text *Shatapatha Brahmana* (composed between the 8th to 6th centuries BCE, [Iron Age India](#))[44] gives  $\pi$  as  $339/108 \approx 3.139$ . It has been suggested that passages in the [1 Kings 7:23](#) and [2 Chronicles 4:2](#) discussing a ceremonial pool in the temple of [King Solomon](#) with a diameter of ten cubits and a circumference of thirty cubits show that the writers considered  $\pi$  to have had an approximate value of three, which various authors have tried to explain away through various suggestions such as a hexagonal pool or an outward curving rim.[45]

[Archimedes](#) (287–212 BC) was the first to estimate  $\pi$  rigorously. He realized that its

magnitude can be bounded from below and above by inscribing circles in [regular polygons](#) and calculating the outer and inner polygons' respective perimeters:[32] By using the equivalent of 96-sided polygons, he proved that [32] The average of these values is about 3.14185.

[Ptolemy](#), in his [Almagest](#), gives a value of 3.1416, which he may have obtained from [Apollonius of Perga](#).[46]

Around AD 265, the [Wei Kingdom](#) mathematician [Liu Hui](#) provided a simple and rigorous [iterative algorithm](#) to calculate  $\pi$  to any degree of accuracy. He himself carried through the calculation to a 3072-gon (i.e. a 3072-sided polygon) and obtained an approximate value for  $\pi$  of 3.1416.[47] Later, Liu Hui invented a quick method of calculating  $\pi$  and obtained an approximate value of 3.14 with only a 96-gon,[47] by taking advantage of the fact that the difference in area of successive polygons forms a geometric series with a factor of 4.

Around 480, the Chinese mathematician [Zu Chongzhi](#) demonstrated that  $\pi \approx 355/113$  ( $\approx 3.1415929$ ), and showed that  $3.1415926 < \pi < 3.1415927$ [47] using Liu Hui's algorithm applied to a 12288-gon. This value would remain the most accurate approximation of  $\pi$  available for the next 900 years.

[Maimonides](#) mentions with certainty the irrationality of  $\pi$  in the 12th century.[48] This was proved in 1768 by [Johann Heinrich Lambert](#).[49] In the 20th century, proofs were found that require no prerequisite knowledge beyond [integral calculus](#). One of those, due to [Ivan Niven](#), is widely known.[50][51] A somewhat earlier similar proof is by [Mary Cartwright](#).[52]

## Second millennium AD

Until the [second millennium AD](#), estimations of  $\pi$  were accurate to fewer than 10 decimal digits. The next major advances in the study of  $\pi$  came with the development of [infinite series](#) and subsequently with the discovery of [calculus](#), which permit the estimation of  $\pi$  to any desired accuracy by considering sufficiently many terms of a relevant series. Around 1400, [Madhava of Sangamagrama](#) found the first known such series:

$$\pi = \sqrt[4]{\sum_{k=0}^{\infty} \frac{3^k + 1}{(-1)^k}} = \frac{1}{\sqrt[4]{1}} - \frac{3}{\sqrt[4]{1}} + \frac{2}{\sqrt[4]{1}} - \frac{1}{\sqrt[4]{1}} + \frac{0}{\sqrt[4]{1}} - \frac{11}{\sqrt[4]{1}} + \dots$$

This is now known as the [Madhava–Leibniz series](#)[53][54] or Gregory–Leibniz series since it was rediscovered by [James Gregory](#) and [Gottfried Leibniz](#) in the 17th century. Unfortunately, the rate of convergence is too slow to calculate many digits in practice; about 4,000 terms must be summed to improve upon Archimedes' estimate. However, by transforming the series into

$$= \sqrt[4]{15} \left( 1 - \frac{3 \cdot 3}{1} + \frac{2 \cdot 3^3}{1} - \frac{1 \cdot 3^5}{1} + \dots \right),$$

$$\pi = \sqrt[4]{15} \sum_{k=0}^{\infty} \frac{3^k + 1}{(-3)^{-k}} = \sqrt[4]{15} \sum_{k=0}^{\infty} \frac{3^k + 1}{(-\frac{3}{1})^k}$$

[Madhava](#) was able to estimate  $\pi$  as 3.14159265359, which is correct to 11 decimal places. The record was beaten in 1424 by the [Persian mathematician](#), [Jamshīd al-Kāshī](#), who gave an estimate  $\pi$  that is correct to 16 decimal digits.[55]

The first major European contribution since Archimedes was made by the German mathematician [Ludolph van Ceulen](#) (1540–1610), who used a geometric method to give an estimate of  $\pi$  that is correct to 35 decimal digits. He was so proud of the calculation, which required the greater part of his life, that he had the digits engraved into his tombstone.<sup>[56]</sup>  $\pi$  is sometimes called "Ludolph's Constant", though not as often as it is called "Archimedes' Constant."<sup>[57]</sup>

Around the same time, the methods of calculus and determination of infinite series and products for geometrical quantities began to emerge in Europe. The first such representation was the [Viète's formula](#),

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \dots$$

found by [François Viète](#) in 1593. Another famous result is [Wallis' product](#),

$$\frac{\pi}{2} = \prod_{k=1}^{\infty} \frac{(2k)^2}{(2k)^2 - 1} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \dots = \frac{4}{3} \cdot \frac{16}{15} \cdot \frac{36}{35} \cdot \frac{64}{63} \dots$$

by [John Wallis](#) in 1655. [Isaac Newton](#) derived the [arcsin](#) series for  $\pi$  in 1665–66 and calculated 15 digits:

$$\begin{aligned} \pi &= 6 \arcsin \frac{1}{2} \\ &= 6 \left( \frac{1}{2^1 \cdot 1} + \left(\frac{1}{2}\right) \frac{1}{2^3 \cdot 3} + \left(\frac{1 \cdot 3}{2 \cdot 4}\right) \frac{1}{2^5 \cdot 5} + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right) \frac{1}{2^7 \cdot 7} + \dots \right) \\ &= 3 \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{16^n (2n+1)} \\ &= 3 + \frac{1}{8} + \frac{9}{640} + \frac{15}{7168} + \frac{35}{98304} + \frac{189}{2883584} + \frac{693}{54525952} + \frac{429}{167772160} + \dots \end{aligned}$$

although he later confessed: "I am ashamed to tell you to how many figures I carried these computations, having no other business at the time."<sup>[58]</sup> It converges linearly to  $\pi$ , with a [rate of convergence](#)  $\mu$  that adds at least three decimal places for every five terms. As  $n$  approaches [infinity](#),  $\mu$  approaches  $1/4$  and  $1/\mu$  approaches  $4$ :

$$\mu = \frac{(2n-1)^2}{8n(2n+1)}; \frac{1}{\mu} = \frac{8n(2n+1)}{(2n-1)^2}$$

In 1706 [John Machin](#) was the first to compute 100 decimals of  $\pi$ , using the [arctan](#) series in the formula

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}$$

with

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

Formulas of this type, now known as [Machin-like formulas](#), were used to set several successive records and remained the best known method for calculating  $\pi$  well into the age of computers. A remarkable record was set by the calculating prodigy [Zacharias Dase](#), who in 1844 employed a Machin-like formula to calculate 200 decimals of  $\pi$  in his head at the behest of [Gauss](#). The best value at the end of the 19th century was due to [William Shanks](#), who took 15 years to calculate  $\pi$  with 707 digits, although due to a mistake only the first 527 were correct. (To avoid such errors, modern record calculations of any kind are often performed twice, with two different formulas. If the results are the same, they are likely to be correct.)

Theoretical advances in the 18th century led to insights about  $\pi$ 's nature that could not

be achieved through numerical calculation alone. [Johann Heinrich Lambert](#) proved the irrationality of  $\pi$  in 1761, and [Adrien-Marie Legendre](#) also proved in 1794  $\pi^2$  to be irrational. When [Leonhard Euler](#) in 1735 solved the famous [Basel problem](#), finding the exact value of the [Riemann zeta function](#) of 2,

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

which is  $\pi^2/6$ , he established a deep connection between  $\pi$  and the [prime numbers](#). Both Legendre and Euler speculated that  $\pi$  might be [transcendental](#), which was finally proved in 1882 by [Ferdinand von Lindemann](#).

## Computation in the computer age

Practically, one needs only 39 digits of  $\pi$  to make a circle the size of the observable universe accurate to the size of a hydrogen atom.[\[23\]](#)

The advent of digital computers in the 20th century led to an increased rate of new  $\pi$  calculation records. [John von Neumann](#) et al. used [ENIAC](#) to compute 2037 digits of  $\pi$  in 1949, a calculation that took 70 hours.[\[59\]](#) Additional thousands of decimal places were obtained in the following decades, with the million-digit milestone passed in 1973. Progress was not only due to faster hardware, but also new algorithms. One of the most significant developments was the discovery of the [fast Fourier transform](#) (FFT) in the 1960s, which allows computers to perform arithmetic on extremely large numbers quickly.

In the beginning of the 20th century, the Indian mathematician [Srinivasa Ramanujan](#) found many new formulas for  $\pi$ , some remarkable for their elegance, mathematical depth and rapid convergence.[\[60\]](#) One of his formulas is the series,

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}}$$

where  $k!$  is the [factorial](#) of  $k$ .

A collection of some others are in the table below:[\[61\]](#)

$\pi = \frac{1}{Z}$	$Z = \sum_{n=0}^{\infty} \frac{((2n)!)^3(42n+5)}{(n!)^6 16^{3n+1}}$
$\pi = \frac{4}{Z}$	$Z = \sum_{n=0}^{\infty} \frac{(-1)^n(4n)!(21460n+1123)}{(n!)^4 441^{2n+1} 2^{10n+1}}$
$\pi = \frac{4}{Z}$	$Z = \sum_{n=0}^{\infty} \frac{(6n+1) \left(\frac{1}{2}\right)_n^3}{4^n (n!)^3}$
$\pi = \frac{32}{Z}$	$Z = \sum_{n=0}^{\infty} \frac{\left(\frac{\sqrt{5}-1}{2}\right)^{8n} (42n\sqrt{5}+30n+5\sqrt{5}-1) \left(\frac{1}{2}\right)_n^3}{64^n (n!)^3}$
$\pi = \frac{27}{4Z}$	$Z = \sum_{n=0}^{\infty} \frac{\left(\frac{2}{27}\right)^n (15n+2) \left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(n!)^3}$
$\pi = \frac{15\sqrt{3}}{2Z}$	$Z = \sum_{n=0}^{\infty} \frac{\left(\frac{4}{125}\right)^n (33n+4) \left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(n!)^3}$
$\pi = \frac{85\sqrt{85}}{18\sqrt{3}Z}$	$Z = \sum_{n=0}^{\infty} \frac{\left(\frac{4}{85}\right)^n (133n+8) \left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(n!)^3}$
$\pi = \frac{5\sqrt{5}}{2\sqrt{3}Z}$	$Z = \sum_{n=0}^{\infty} \frac{\left(\frac{4}{125}\right)^n (11n+1) \left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(n!)^3}$

$\pi = \frac{2\sqrt{3}}{Z}$	$Z = \sum_{n=0}^{\infty} \frac{(8n+1) \left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3 9^n}$
$\pi = \frac{\sqrt{3}}{9Z}$	$Z = \sum_{n=0}^{\infty} \frac{(40n+3) \left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3 49^{2n+1}}$
$\pi = \frac{2\sqrt{11}}{11Z}$	$Z = \sum_{n=0}^{\infty} \frac{(280n+19) \left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3 99^{2n+1}}$
$\pi = \frac{\sqrt{2}}{4Z}$	$Z = \sum_{n=0}^{\infty} \frac{(10n+1) \left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3 9^{2n+1}}$
$\pi = \frac{4\sqrt{5}}{5Z}$	$Z = \sum_{n=0}^{\infty} \frac{(644n+41) \left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3 5^n 7^{2n+1}}$
$\pi = \frac{4\sqrt{3}}{3Z}$	$Z = \sum_{n=0}^{\infty} \frac{(-1)^n (28n+3) \left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3 3^n 4^{n+1}}$
$\pi = \frac{4}{Z}$	$Z = \sum_{n=0}^{\infty} \frac{(-1)^n (20n+3) \left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3 2^{2n+1}}$
$\pi = \frac{72}{Z}$	$Z = \sum_{n=0}^{\infty} \frac{(-1)^n (4n)! (260n+23)}{(n!)^4 4^n 18^{2n}}$
$\pi = \frac{3528}{Z}$	$Z = \sum_{n=0}^{\infty} \frac{(-1)^n (4n)! (21460n+1123)}{(n!)^4 4^n 882^{2n}}$

where

$$(x)_n$$

is the [Pochhammer symbol](#) for the falling factorial.

The related one found by the [Chudnovsky brothers](#) in 1987 is

$$\frac{1}{\sqrt{10002}} = \sum_{n=0}^{\infty} \frac{(3n)! (n!)^3 (-0.70350)_{3n}}{(n!)^4 4^n 18^{2n}}$$

which delivers 14 digits per term.[60] The Chudnovskys used this formula to set several  $\pi$  computing records in the end of the 1980s, including the first calculation of over one billion (1,011,196,691) decimals in 1989. It remains the formula of choice for  $\pi$  calculating software that runs on personal computers, as opposed to the [supercomputers](#) used to set modern records. On August 6, 2010, PhysOrg.com reported that Japanese and American computer experts Shigeru Kondo and Alexander Yee said they've calculated the value of  $\pi$  to 5 trillion decimal places on a personal computer, double the previous record.[62]

Whereas series typically increase the accuracy with a fixed amount for each added term, there exist iterative algorithms that *multiply* the number of correct digits at each step, with the downside that each step generally requires an expensive calculation. A breakthrough was made in 1975, when [Richard Brent](#) and [Eugene Salamin](#) independently discovered the [Brent–Salamin algorithm](#), which uses only arithmetic to double the number of correct digits at each step.[63] The algorithm consists of setting

$$a_0 = 1 \quad b_0 = \frac{1}{\sqrt{2}} \quad t_0 = \frac{1}{4} \quad p_0 = 1$$

and iterating

$$\begin{aligned} a_{n+1} &= \frac{a_n + b_n}{2} & b_{n+1} &= \sqrt{a_n b_n} \\ t_{n+1} &= t_n - p_n (a_n - a_{n+1})^2 & p_{n+1} &= 2p_n \end{aligned}$$

until  $a_n$  and  $b_n$  are close enough. Then the estimate for  $\pi$  is given by

$$\pi \approx \frac{(a_n + b_n)^2}{4t_n}.$$

Using this scheme, 25 iterations suffice to reach 45 million correct decimals. A similar algorithm that quadruples the accuracy in each step has been found by [Jonathan and Peter Borwein](#).[\[64\]](#) The methods have been used by [Yasumasa Kanada](#) and team to set most of the  $\pi$  calculation records since 1980, up to a calculation of 206,158,430,000 decimals of  $\pi$  in 1999. As of January 2010, the record was almost 2.7 trillion digits.[\[65\]](#) This beats the previous record of 2,576,980,370,000 decimals, set by Daisuke Takahashi on the T2K-Tsukuba System, a supercomputer at the University of Tsukuba northeast of Tokyo.[\[66\]](#)

Another method for fast calculation of the constant  $\pi$  is the method for fast summing series of special form [FEE](#). To calculate the  $\pi$  it's possible to use the Euler formula

$$\frac{\pi}{4} = \arctan \frac{1}{2} + \arctan \frac{1}{3},$$

and apply [the FEE](#) to sum the Taylor series for

$$\arctan \frac{1}{2} + \arctan \frac{1}{3}.$$

One can apply the same procedure also to the other special series approximating the constant  $\pi$ . Besides the formulas representing the  $\pi$  via arctangents, the new formulas for  $\pi$  derived in the 1990s by [S. Plouffe](#), [F. Bellard](#) and some other computer scientists, are good for fast summing via the [FEE](#) and fast computation of the constant  $\pi$ .

An important recent development was the [Bailey–Borwein–Plouffe formula](#) (BBP formula), discovered by [Simon Plouffe](#) and named after the authors of the paper in which the formula was first published, [David H. Bailey](#), [Peter Borwein](#), and [Simon Plouffe](#).[\[67\]](#) The formula,

$$\pi = \sum_{k=0}^{\infty} \frac{10^k}{1} \left( \frac{8^k + 1}{4} - \frac{8^k + 4}{5} - \frac{8^k + 2}{1} - \frac{8^k + 0}{1} \right),$$

is remarkable because it allows extracting any individual [hexadecimal](#) or [binary](#) digit of  $\pi$  without calculating all the preceding ones.[\[67\]](#) Between 1998 and 2000, the [distributed computing](#) project [PiHex](#) used a modification of the BBP formula due to [Fabrice Bellard](#) to compute the [quadrillionth](#) (1,000,000,000,000,000:th) bit of  $\pi$ , which turned out to be 0.[\[68\]](#)

If a formula of the form

$$\pi = \sum_{k=0}^{\infty} \frac{1}{b^{ck}} \frac{p(k)}{q(k)},$$

were found where  $b$  and  $c$  are positive integers and  $p$  and  $q$  are polynomials with fixed degree and integer coefficients (as in the BPP formula above), this would be one the most efficient ways of computing any digit of  $\pi$  at any position in base  $bc$  without computing all the preceding digits in that base, in a time just depending on the size of the integer  $k$  and on the fixed degree of the polynomials. Plouffe also describes such formulas as the interesting ones for computing numbers of class [SC\\*](#), in a logarithmically polynomial space and almost linear time, depending only on the size (order of magnitude) of the integer  $k$ , and requiring modest computing resources. The previous formula (found by Plouffe for  $\pi$  with  $b = 2$  and  $c = 4$ , but also found for  $\log(9/10)$  and for a few other irrational constants), implies that  $\pi$  is a [SC\\*](#) number.

In September 2010, [Yahoo!](#) employee Nicholas Sze used the company's [Hadoop](#)

production application to compute 256 bits of  $\pi$  starting at a position a little before the two-quadrillion<sup>th</sup> (2,000,000,000,000,000<sup>th</sup>) bit, doubling the previous record by PiHex. The record was broken on 1,000 of Yahoo!'s computers over a 23-day period. The formula is used to compute a single bit of  $\pi$  in a small set of mathematical steps.[69][70]

In 2006, [Simon Plouffe](#), using the [integer relation algorithm](#) PSLQ, found a series of formulas.[71] Let  $q = e\pi$  (Gelfond's constant), then

$$\frac{\pi}{24} = \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{3}{q^n - 1} - \frac{4}{q^{2n} - 1} + \frac{1}{q^{4n} - 1} \right)$$

$$\frac{\pi^3}{180} = \sum_{n=1}^{\infty} \frac{1}{n^3} \left( \frac{4}{q^n - 1} - \frac{5}{q^{2n} - 1} + \frac{1}{q^{4n} - 1} \right)$$

and others of form,

$$\pi^k = \sum_{n=1}^{\infty} \frac{1}{n^k} \left( \frac{a}{q^n - 1} + \frac{b}{q^{2n} - 1} + \frac{c}{q^{4n} - 1} \right)$$

where  $k$  is an [odd number](#), and  $a, b, c$  are [rational numbers](#).

In the previous formula, if  $k$  is of the form  $4m + 3$ , then the formula has the particularly simple form,

$$\pi^k = \sum_{n=1}^{\infty} \frac{1}{n^k} \left( \frac{a_n - 1}{\delta_{q^n - 1}} - \frac{a_{3n} - 1}{\delta_{q^{3n} - 1}} + \frac{a_{4n} - 1}{\delta_{q^{4n} - 1}} \right)$$

for some rational number  $p$  where the [denominator](#) is a highly factorable number. General expressions for these kinds of sums are known.[72]

## Representation as a continued fraction

The sequence of partial denominators of the simple [continued fraction](#) of  $\pi$  does not show any obvious pattern:[30]

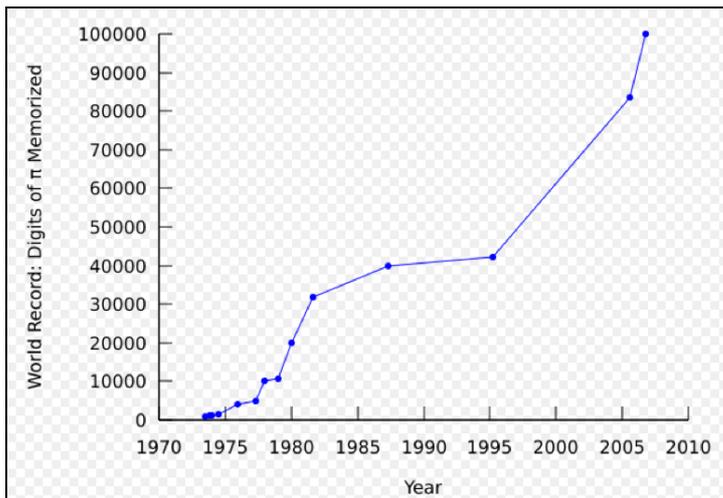
$$\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, \dots]$$

or

$$\pi = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}}}}}}$$

However, there are [generalized continued fractions](#) for  $\pi$  with a perfectly regular structure, such as:[73]

$$\pi = \frac{4}{1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \frac{7^2}{2 + \frac{9^2}{2 + \dots}}}}}} = 3 + \frac{1^2}{6 + \frac{3^2}{6 + \frac{5^2}{6 + \frac{7^2}{6 + \frac{9^2}{6 + \dots}}}} = \frac{4}{1 + \frac{1^2}{3 + \frac{2^2}{5 + \frac{3^2}{7 + \frac{4^2}{9 + \dots}}}}}$$



Recent decades have seen a surge in the record for number of digits memorized.

## Memorizing digits

*Main article:* [Piphilology](#)

Well before computers were used in calculating  $\pi$ , memorizing a *record* number of digits had become an obsession for some people. In 2006, [Akira Haraguchi](#), a retired Japanese engineer, claimed to have recited 100,000 decimal places.<sup>[74]</sup> This, however, has yet to be verified by [Guinness World Records](#). The Guinness-recognized record for remembered digits of  $\pi$  is 67,890 digits, held by [Lu Chao](#), a 24-year-old graduate student from [China](#).<sup>[75]</sup> It took him 24 hours and 4 minutes to recite to the 67,890th decimal place of  $\pi$  without an error.<sup>[76]</sup>

There are many ways to memorize  $\pi$ , including the use of "piems", which are poems that represent  $\pi$  in a way such that the length of each word (in letters) represents a digit. Here is an example of a piem, originally devised by [Sir James Jeans](#): *How I want a drink, alcoholic of course, after the heavy lectures involving quantum mechanics*.<sup>[77]</sup><sup>[78]</sup> The first word has three letters, the second word has one, the third has four, the fourth has one, the fifth has five, and so on. The *Cadaeic Cadenza* contains the first 3835 digits of  $\pi$  in this manner.<sup>[79]</sup> Piems are related to the entire field of humorous yet serious study that involves the use of [mnemonic techniques](#) to remember the digits of  $\pi$ , known as [piphilology](#). In other languages there are similar methods of memorization. However, this method proves inefficient for large memorizations of  $\pi$ . Other methods include remembering patterns in the numbers and the [method of loci](#).<sup>[80]</sup><sup>[81]</sup>

## Open questions

One open question about  $\pi$  is whether it is a [normal number](#)—whether any digit block occurs in the expansion of  $\pi$  just as often as one would statistically expect if the digits had been produced completely "randomly", and that this is true in *every* integer base, not just base 10.<sup>[82]</sup> Current knowledge on this point is very weak; e.g., it is not even known which of the digits 0,...,9 occur infinitely often in the decimal expansion of  $\pi$ ,<sup>[83]</sup> although it is clear that at least two such digits must occur infinitely often, since

otherwise  $\pi$  would be rational, which it is not.

Bailey and Crandall showed in 2000 that the existence of the above mentioned [Bailey–Borwein–Plouffe formula](#) and similar formulas imply that the normality in base 2 of  $\pi$  and various other constants can be reduced to a plausible [conjecture of chaos theory](#).<sup>[84]</sup>

It is also unknown whether  $\pi$  and  $e$  are [algebraically independent](#), although Yuri Nesterenko proved the algebraic independence of  $\{\pi, e\pi, \Gamma(1/4)\}$  in 1996.<sup>[85]</sup>

## Use in mathematics and science

*Main article: [List of formulae involving  \$\pi\$](#)*

$\pi$  is ubiquitous in mathematics, science, and engineering.<sup>[86]</sup>

### Geometry and trigonometry

*See also: [Area of a disk](#)*

For any circle with radius  $r$  and diameter  $d = 2r$ , the circumference is  $\pi d$  and the area is  $\pi r^2$ . Further,  $\pi$  appears in formulas for areas and volumes of many other geometrical shapes based on circles, such as [ellipses](#), [spheres](#), [cones](#), and [tori](#).<sup>[87]</sup> Accordingly,  $\pi$  appears in [definite integrals](#) that describe circumference, area or volume of shapes generated by circles. In the basic case, half the area of the [unit disk](#) is given by the [integral](#):<sup>[88]</sup>

$$\int_{-1}^1 \sqrt{1-x^2} dx = \frac{\pi}{2}$$

and

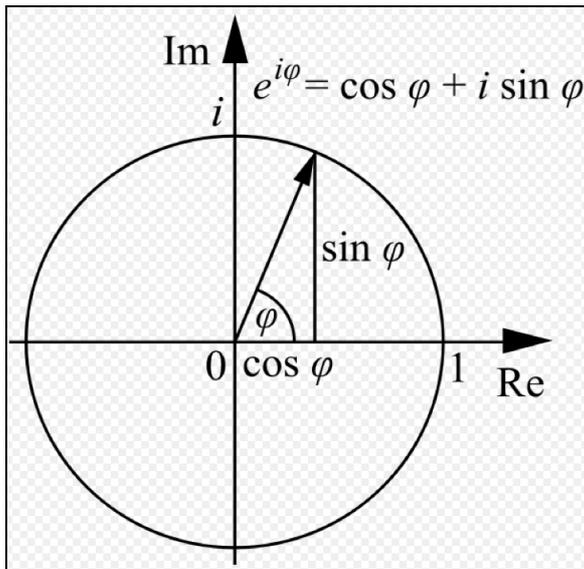
$$\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} dx = \pi$$

gives half the circumference of the [unit circle](#).<sup>[87]</sup> More complicated shapes can be integrated as [solids of revolution](#).<sup>[89]</sup>

From the unit-circle definition of the [trigonometric functions](#) also follows that the sine and cosine have period  $2\pi$ . That is, for all  $x$  and integers  $n$ ,  $\sin(x) = \sin(x + 2\pi n)$  and  $\cos(x) = \cos(x + 2\pi n)$ . Because  $\sin(0) = 0$ ,  $\sin(2\pi n) = 0$  for all integers  $n$ . Also, the angle measure of  $180^\circ$  is equal to  $\pi$  radians. In other words,  $1^\circ = (\pi/180)$  radians.

In modern mathematics,  $\pi$  is often *defined* using trigonometric functions, for example as the smallest positive  $x$  for which  $\sin x = 0$ , to avoid unnecessary dependence on the subtleties of Euclidean geometry and integration. Equivalently,  $\pi$  can be defined using the [inverse trigonometric functions](#), for example as  $\pi = 2 \arccos(0)$  or  $\pi = 4 \arctan(1)$ . Expanding inverse trigonometric functions as [power series](#) is the easiest way to derive infinite series for  $\pi$ .

## Complex numbers and calculus



Euler's formula depicted on the complex plane. Increasing the angle  $\varphi$  to  $\pi$  radians ( $180^\circ$ ) yields Euler's identity.

A complex number  $z$  can be expressed in polar coordinates as follows:

$$z = r \cdot (\cos \theta + i \sin \theta)$$

The frequent appearance of  $\pi$  in complex analysis can be related to the behavior of the exponential function of a complex variable, described by Euler's formula

$$e^{i\varphi} = \cos \varphi + i \sin \varphi$$

where  $i$  is the imaginary unit satisfying  $i^2 = -1$  and  $e \approx 2.71828$  is Euler's number. This formula implies that imaginary powers of  $e$  describe rotations on the unit circle in the complex plane; these rotations have a period of  $360^\circ = 2\pi$ . In particular, the  $180^\circ$  rotation  $\varphi = \pi$  results in the remarkable Euler's identity

$$e^{i\pi} = -1.$$

There are  $n$  different  $n$ -th roots of unity

$$\epsilon_{\mathbb{Z}_n} = \{1, \zeta, \zeta^2, \dots, \zeta^{n-1}\} \quad (\zeta = 0, 1, \zeta^2, \dots, \zeta^{n-1})$$

The Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

A consequence is that the gamma function of a half-integer is a rational multiple of  $\sqrt{\pi}$ .

## Physics

Although not a physical constant,  $\pi$  appears routinely in equations describing fundamental principles of the Universe, due in no small part to its relationship to the nature of the circle and, correspondingly, spherical coordinate systems. Using units such

as **Planck units** can sometimes eliminate  $\pi$  from formulae.

**Heisenberg's uncertainty principle**, which shows that the uncertainty in the measurement of a particle's position ( $\Delta x$ ) and **momentum** ( $\Delta p$ ) can not both be arbitrarily small at the same time:[90]

$$\Delta x \Delta p \geq \frac{\hbar}{4\pi} = \frac{\hbar}{2}$$

**Einstein's field equation of general relativity**:[91]

$$R_{ik} - \frac{g_{ik}R}{2} + \Lambda g_{ik} = \frac{8\pi G}{c^4} T_{ik}$$

The **cosmological constant**  $\Lambda$  from Einstein's field equation is related to the intrinsic **energy density** of the **vacuum**  $\rho_{\text{vac}}$  via the **gravitational constant**  $G$  as follows:[92]

$$\Lambda = 8\pi G \rho_{\text{vac}}$$

**Coulomb's law** for the **electric force**, describing the force between two **electric charges** ( $q_1$  and  $q_2$ ) separated by distance  $r$  (with  $\epsilon_0$  representing the **vacuum permittivity** of free space):[93]

$$F = \frac{|q_1 q_2|}{4\pi \epsilon_0 r^2}$$

**Magnetic permeability of free space** relates the production of a magnetic field in a vacuum by an electric current in units of **Newtons** (N) and **Amperes** (A):[94]

$$\mu_0 = \frac{4\pi \times 10^{-7} \text{ N/A}^2}{\text{V}^2}$$

**Kepler's third law constant**, relating the **orbital period** ( $P$ ) and the **semi-major axis** ( $a$ ) to the **masses** ( $M$  and  $m$ ) of two co-orbiting bodies:

$$\left(\frac{2\pi}{P}\right)^2 a^3 = \omega^2 a^3 = G(M + m)$$

## Probability and statistics

In **probability** and **statistics**, there are many **distributions** whose formulas contain  $\pi$ , including:

the **probability density function** for the **normal distribution** with mean  $\mu$  and **standard deviation**  $\sigma$ , due to the **Gaussian integral**:[95]

$$f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

the probability density function for the (standard) **Cauchy distribution**:[96]

$$f(x) = \frac{1}{\pi} \frac{1}{1+x^2}$$

Note that since  $\int_{-\infty}^{\infty} f(x) dx = 1$  for any probability density function  $f(x)$ , the above formulas can be used to produce other integral formulas for  $\pi$ . [97]

**Buffon's needle** problem is sometimes quoted as an empirical approximation of  $\pi$  in

"popular mathematics" works. Consider dropping a needle of length  $L$  repeatedly on a surface containing parallel lines drawn  $S$  units apart (with  $S > L$ ). If the needle is dropped  $n$  times and  $x$  of those times it comes to rest crossing a line ( $x > 0$ ), then one may approximate  $\pi$  using the [Monte Carlo method](#):[\[98\]](#)[\[99\]](#)[\[100\]](#)[\[101\]](#)

$$\pi \approx \frac{2nL}{xS}.$$

Though this result is mathematically impeccable, it cannot be used to determine more than very few digits of  $\pi$  *by experiment*. Reliably getting just three digits (including the initial "3") right requires millions of throws,[\[98\]](#) and the number of throws grows [exponentially](#) with the number of digits desired. Furthermore, any error in the measurement of the lengths  $L$  and  $S$  will transfer directly to an error in the approximated  $\pi$ . For example, a difference of a single [atom](#) in the length of a 10-centimeter needle would show up around the 9th digit of the result. In practice, uncertainties in determining whether the needle actually crosses a line when it appears to exactly touch it will limit the attainable accuracy to much less than 9 digits.

## Geomorphology and chaos theory

Under ideal conditions (uniform gentle slope on an homogeneously erodible substrate), the ratio between the actual length of a river and its straight-line from source to mouth length tends to approach  $\pi$ .[\[102\]](#) [Albert Einstein](#) was the first to suggest that rivers have a tendency towards an ever more loopy path because the slightest curve will lead to faster currents on the outer side, which in turn will result in more erosion and a sharper bend. The sharper the bend, the faster the currents on the outer edge, the more the erosion, the more the river will twist and so on. However, increasing loopiness will result in rivers doubling back on themselves and effectively short-circuiting, creating an [ox-bow lake](#). The balance between these two opposing factors leads to an average ratio of  $\pi$  between the actual length and the direct distance between source and mouth.[\[103\]](#)

## Criticism

*Main article: [Tau \(2π\)](#)*

Over the years, various mathematicians and scientists have been critical of the use of  $\pi$ . In particular a constant representing  $2\pi$  has been proposed because a circle is  $2\pi$  radians around, and  $2\pi$  is found in many equations such as the [Gaussian distribution](#), the [Fourier transform](#), and the [reduced Planck constant](#). Where  $\pi$  is most often seen alone, computing the area of a circle with  $A = \pi r^2$ , is a [quadratic form](#); while  $\pi r^2$  is simpler than , quadratic forms generally do have a term, as in [energy of a spring](#) or [kinetic energy](#) being and respectively (the term resulting from integration). [Hermann Laurent](#) in *Traité D'Algebra* wrote equations using  $2\pi$  as a single symbol.[\[104\]](#) Robert Palais proposed to use a "pi with three legs" ( $\pi$ ) to denote 1 turn,[\[105\]](#) while physicist [Michael Hartl](#) proposed to use the Greek letter  $\tau$  ([tau](#)) to refer to the constant  $2\pi$ .[\[106\]](#)[\[107\]](#)

QuickTime™ and a  
decompressor  
are needed to see this picture.

A "Pi pie" to celebrate [Pi Day](#)

## In popular culture

Probably because of the simplicity of its definition, the concept of  $\pi$  has become entrenched in popular culture to a degree far greater than almost any other mathematical construct.[2] It is, perhaps, the most common ground between mathematicians and non-mathematicians.[3] Reports on the latest, most-precise calculation of  $\pi$  are common news items.[4][5][108]

Nobel prize winning poet [Wisława Szymborska](#) wrote a poem about  $\pi$ , and here is an excerpt:[109]

*The caravan of digits that is pi does not stop at the edge of the page, but runs off the table and into the air, over the wall, a leaf, a bird's nest, the clouds, straight into the sky, through all the bloatedness and bottomlessness. Oh how short, all but mouse-like is the comet's tail!*

Many schools around the world observe [Pi Day](#) (March 14, from 3.14).[110] Several college [cheers](#) at the [Georgia Institute of Technology](#)[111] and the [Massachusetts Institute of Technology](#)[112] include "3.14159!"

On November 7, 2005, [alternative](#) musician [Kate Bush](#) released the album *Aerial*. The album contains the song "Pi" whose lyrics consist principally of Bush singing the digits of  $\pi$  to music, beginning with "3.14".[113]

In [Carl Sagan's](#) novel *Contact*,  $\pi$  played a key role in the story. The novel suggested that there was a message buried deep within the digits of  $\pi$  placed there by the creator of the universe.[114] This part of the story was omitted from the [film](#) adaptation of the novel.

In the *Star Trek: The Original Series* episode "Wolf in the Fold", after a murderous alien entity (which had once been [Jack the Ripper](#)) takes over the *Enterprise's* main

computer with the intention of using it to slowly kill the crew, [Kirk](#) and [Spock](#) draw the entity out of the computer by forcing it to compute pi to the nonexistent last digit, causing the creature to abandon the computer, allowing it to be beamed into space.

In the *Stargate SG-1* season 2 episode "[Thor's Chariot](#)", [Daniel Jackson](#) and [Samantha Carter](#) and Cimmeria local Gairwyn are transported to the Hall of Thor's Might, in which one of the walls has four runes, while another has four simple geometric figures. After Daniel Jackson mentions the fact that the runes on the wall also represented the numbers 3, 14, 15 and 9, Samantha Carter realizes that this sequence of numbers corresponds to  $\pi$ . The team then correctly solves this puzzle by marking the radius on the circle on the second wall.

In *The Simpsons* season 12 episode "[Bye Bye Nerdie](#)", [Professor Frink](#) exclaims "pi is exactly three!" to get the attention of the attendees to the "12th Annual Big Science Thing" contest.

[Darren Aronofsky](#)'s film *Pi* deals with a [number theorist](#).

In the fictional movie, *Night at the Museum: Battle of the Smithsonian*,  $\pi$  is the answer to the combination that will allow the Tablet of Akh-man-Ra to open the gates to the underworld.

A style of writing called [Pilish](#) has been developed, in which the lengths of consecutive words match the digits of the number  $\pi$ .

List of numbers – Irrational and suspected irrational numbers

$\gamma - \zeta(3) - \sqrt{2} - \sqrt{3} - \sqrt{5} - \varphi - \rho - \delta S - \alpha - e - \pi - \delta$

## See also

- The [Feynman point](#), a sequence of six 9s that appears at the 762nd through 767th decimal places of  $\pi$
- [Indiana Pi Bill](#)
- [List of topics related to pi](#)
- [Proof that 22/7 exceeds  \$\pi\$](#)

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- [a b](#) See [Alfred S. Posamentier](#) and [Ingmar Lehmann](#), *Pi: A Biography of the World's Most Mysterious Number*. Prometheus Books, 2004. ISBN 978-1-59102-200-8.

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## External links

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- Formulas for  $\pi$  at MathWorld
- 1 Trillion Digits of Pi at DigitsofPi.org
- Representations of Pi at Wolfram Alpha
- Pi at PlanetMath
- Determination of  $\pi$  at Cut-the-knot
- Pi on In Our Time at the BBC.
- Statistical Distribution Information on Pi based on 1.2 trillion digits of  $\pi$
- $\pi$  Search Engine (2 billion digits, including  $e$  and  $\sqrt{2}$ )
- Pi is Wrong!, Robert Palais abstracts the history of alternatives to  $\pi$ .
- The Tau Manifesto, physicist Michael Hartl outlines a proposal to replace  $\pi$  with  $\tau = 2\pi$  (tau).
- A000796 *Decimal expansions of Pi* and related links at the On-Line Encyclopedia of Integer Sequences
- <http://numbers.computation.free.fr/Constants/Pi/pi.html>

For 1 million digits see <http://newton.ex.ac.uk/research/qsystems/collabs/pi/pi6.txt>

