GROUP THEORY FOR UNIFIED MODEL BUILDING

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Contents:

Abstract:

The results gathered here on simple Lie algebras have been selected with attention to the needs of unified model builders who study Yang–Mills theories based on simple, local-symmetry groups that contain as a subgroup the $SU_2^w \times U_1^w \times SU_3^c$ symmetry of the standard theory of electromagnetic, weak, and strong interactions. The major topics include, after a brief review of the standard model and its unification into a simple group, the use of Dynkin diagrams to analyze the structure of the group generators and to keep track of the weights (quantum numbers) of the representation vectors; an analysis of the subgroup structure of simple groups, including explicit coordinatizations of the projections in weight space; lists of representations, tensor products and branching rules for a number of simple groups; and other details about groups and their representations that are often helpful for surveying unified models, including vector-coupling coefficient calculations. Tabulations of representations, tensor products, and branching rules for E_6, SO_{10}, SU_6, F_4, SO_9, SU_5, SO_8, SO_7, SU_4, E_7, E_8, SU_8, SO_{14}, SO_{18}, SO_{22}, and for completeness, SU_3 are included. (These tables may have other applications.) Group-theoretical techniques for analyzing symmetry breaking are described in detail and many examples are reviewed, including explicit parameterizations of mass matrices.

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1. Introduction

The purpose of this review is threefold: to present a pedagogical introduction to the use of Dynkin diagrams, and especially their application to unified model building; to summarize the representations, quantum number structure, and tensor products of a number of simple Lie algebras that have attracted attention; and to describe several problems in unified model building. Table 1 and the Contents provide a fairly detailed summary of the topics covered. The choice of groups was suggested primarily by research into Yang–Mills theories [1, 2] based on simple, local-symmetry groups that are supposed to unify quantum chromodynamics (QCD) with quantum flavor dynamics (QFD). QCD is the candidate theory of the strong interactions; it hypothesizes that the strong interactions are due to the interactions of eight vector gluons with the eight symmetry currents of an SU_3 locally-symmetric Yang–Mills theory [3]. This local symmetry is denoted SU_3^c, where "c" means "color" and distinguishes this use of SU_3 from others. The gluons do not carry the charges of the flavor interactions. QFD is also a Yang–Mills theory with local symmetry G^{fl} containing the $SU_2^w \times U_1^w$ of the electromagnetic and weak interactions [4]; flavor bosons do not carry color charge. In addition to those, unified models hypothesize the existence of additional interactions. For example, the model based on a local SU_5 symmetry, which was the first example based on a simple group, has additional bosons that can mediate proton decay [5].

Section 2 contains a brief review of the "standard model" of electromagnetic, weak, and strong interactions, based on the group $SU_2^w \times U_1^w \times SU_3^c$. It is presented as background material for readers from outside particle physics. Some kinematical features of the fermion mass matrix are reviewed, and the main result, summarized in table 2, is the necessary part of the particle spectrum to be incorporated in unified models.

The proposal of unification is examined somewhat critically in section 3. It is assumed that the standard model can be embedded in a simple group G, as reviewed in ref. [6]. A qualitative review of simple Lie algebras (to be elaborated in later sections) is given. Section 3 also has an introductory description of specific models based on SU_5 [5], SO_{10} [7], and E_6 [8], and some notes on group theoretical problems to be solved when analyzing such models. Sections 2 and 3 contain "elementary" material.

Much of the analysis of the SU_5 model is easily carried out using traditional tensor techniques, but for a group as complicated as E_6, those techniques often become quite cumbersome. Thus, there is a motivation for wanting a simpler and more transparent notation. It is the contention of this review that the use of Dynkin diagrams [9] is just such a simplification; although it is hardly needed for the SU_5 work, it does make detailed discussions of larger groups like E_6 or SO_{22} quite easy. The Dynkin labels of the representation vectors, used in conjunction with the Dynkin diagram, take the whole group structure into account. In contrast, sets of tensor labels are easily made symmetric, antisymmetric, or traceless, but further algebraic structure is oftentimes expressed rather awkwardly. For example, the component-by-component analysis of the quantum number content of a representation is trivial using Dynkin's techniques; there even exist simple computer programs that do the whole job [10] for any representation of any simple group, although the results presented here were derived mostly "by hand". Consequently the use of Dynkin labels for the states simplifies the details of many group-theoretical calculations. Finally, the group theoretical structure of the symmetry breaking takes on a transparent geometrical character, especially in those cases where the concept of the "breaking direction" in weight space is applicable.

Sections 4, 5 and 6 are devoted to summarizing the group-theoretical results that are needed for understanding Dynkin's approach to representation theory. Readers familiar with unified models may wish to skip directly to section 4. The root system of a simple Lie algebra describes the effect of the raising and lowering operators of the group's algebra on the eigenvalues (or quantum numbers) of the diagonal generators, and provides a geometrical interpretation of the commutation relations. The Dynkin diagram is a convenient mnemonic for a special set of roots, called simple roots, that carry all this information. It is then described how to compute the eigenvalues of the diagonalizable generators, which form the Cartan subalgebra.

Our informal treatment of the underlying group theory is directed toward applications [11]. Thus, the usefulness of many theorems is emphasized, but no proofs are reviewed. Hopefully, this approach will supplement the many rigorous and more detailed treatments of simple Lie algebras already available in textbook form for many years [12]. The reader who desires proof in addition to an intuitive picture should look there for derivations.

Section 5 describes how Dynkin diagrams can be applied to the analysis of the finite-dimensional, unitary, irreducible representations (**irreps**) of simple Lie algebras. Each representation vector in the Hilbert space is labeled by the eigenvalues of the diagonal generators. It is demonstrated how to calculate easily and quickly the eigenvalues in any irrep of any simple algebra. As examples that are nontrivial in other notations, a complete analysis of the **27** and **78** of E_6 is given, including a calculation of the electric charge, weak charge, and color charge of each component. (This calculation is begun in section 5 and completed in section 6.)

To complete the analysis of standard-model physics, as embedded in a unified model, we study further the subgroup structure of the unifying group. Dynkin's analysis of subgroups is outlined in section 6; although for many purposes it is quite adequate to have a list of maximal subgroups, a few of the general results give important insights into model building. Explicit matrices that project the root system of a group onto the roots and weights of its subgroups are derived. The basis independent results used to establish the uniqueness (up to an equivalence transformation) of these projections are reviewed here and in ref. [6]; many examples are provided. We also discuss reflections of the generators that can be used for charge conjugation C and for CP [13]; these are associated with symmetric subgroups.

Sections 7 and 8 have discussions of the tables of irreps, tensor products, and branching rules. Section 7 is a detailed account of E_6 and its subgroups. The purpose of the text is to offer comments on the content, conventions, and applications of the tables. Then a consistent set of projection matrices from E_6 through all the physical subgroup chains to $U_1^{em} \times SU_3^c$ is derived. These matrices are helpful for calculations where explicitly labeled field operators are used. Finally a method for calculating vector-coupling coefficients is outlined and some examples are worked out.

There is some interest in using larger groups; a sketch of the groups E_7, E_8, SU_8, and the use of the complex spinor representations of the SO_{4n+6} groups [14] is given in section 8.

Section 9 covers several topics in the application of group-theoretical techniques to symmetry breaking. For the case where the breaking is done by a single irreducible representation, Michel has conjectured a classification of solutions to all possible (realistic) breaking mechanisms [15]. His solutions rely on the notion of a "maximal stability group" or "maximal little group". The breaking in unified models is done by a reducible representation; a proposal for solving this more complicated symmetry-breaking problem (without explicitly minimizing complicated Higgs potentials) is described [16]. Each candidate little group may be a minimum for a range of parameters of the Higgs potential, including radiative corrections; the minimization problem is reduced to a one-dimensional problem where a finite number of candidate answers are substituted into the (effective) potential and compared. We then look at some specific problems in Yang–Mills theories with the field operators labeled according to a convenient basis; in this context the calculation of the vector-boson and fermion mass matrices are described in a number of examples [17]. There are some interesting examples where the concept of symmetry-breaking direction in "weight space" greatly simplifies mass-matrix calculations. In conclusion there are some comments on the possible role of the charge conjugation reflection of the unifying group in symmetry breaking.

There are a number of important topics in unified model building that are not covered in this review: one that has received much attention is the use of renormalization group techniques to calculate the proton lifetime (and the mass of the bosons that mediate the decay) in terms of the experimentally measured strong and electromagnetic couplings [18–20]. There have been many other papers on the phenomenology of small effects ("rare" or "forbidden" decays, neutrino masses, and other effects not expected from the standard theory of electromagnetic, weak, and strong interactions) predicted at some level in many unified models; these too are not discussed. We have also ignored the developments in computing the spin $\frac{1}{2}$ fermion mass ratios at low Q^2 in terms of the symmetry ratios.

Many of the results contained here were derived in collaborations with M. Gell-Mann, J. Patera, P. Ramond and G. Shaw, in our investigations of unified models and Lie algebras. It is a pleasure to acknowledge their contributions and helpful conversations with J. Ginocchio. The drafts of this review were cheerfully and excellently typed by Marian Martinez. R. Roskies and H. Ruegg provided many helpful comments on the manuscript.

2. The standard model

The purpose of this section is to review briefly the standard model [3, 4] of electromagnetic, weak, and strong interactions based on the local symmetry $SU_2^w \times U_1^w \times SU_3^c$, with focus on some elementary features also basic to unified models [21]. This section is intended to provide for those outside particle physics some explanation of the language used in later sections: the relationships of the vector bosons, the adjoint representation, and interactions [2]; the structure of the representation of all left-handed, spin $\frac{1}{2}$ fermions, including particles and antiparticles and the construction of the kinetic energy and mass terms in the Lagrangian; and the symmetry breaking of $SU_2^w \times U_1^w$ down to the U_1^{em} of quantum electrodynamics (QED). The group-theoretical language relied on so heavily in unified model building is not meant to hide the physics, as it may appear at first glance, but is intended to communicate very efficiently much of the physical content of these theories; our object here is to set up the physical language so that the translation to group-theoretical language is explicit. The particle spectrum to appear in the Lagrangian of the theory is listed in table 2; the way that spectrum appears in the Lagrangian can be restated in terms of representation theory.

A Yang–Mills theory based on a local symmetry G is a field theory with the symmetry currents coupled minimally to vector-boson fields in a form analogous to QED, where the coupling of the photon field $A_\mu(x)$ to the electromagnetic current $j_\mu^{em}(x)$ has the form, $e\,j_\mu^{em}(x)\,A^\mu(x)$; e is the electric charge of a particle contributing to the current. The space integrals of the time components of the currents define formally the charges or generators of the Lie group, which, in the case of QED, is a U_1 or phase symmetry. These generators are the elements of the Lie algebra of G. Thus each generator of G is associated with a vector boson that is coupled directly to the symmetry current; it is in this fashion that Yang–Mills theories account for the interactions of Nature.

For the studies of Yang–Mills theories described here, the properties of the Lie algebra of the Lie group G are all that are needed; the global or topological properties of G are not used at this level of model building, as they are in "instanton" physics. Thus we may follow the traditional but incorrect usage, where the term "group" is often used when discussing infinitesimal transformations, which are completely described by the Lie algebra of G, and where the same symbol G is often used to denote the group and its Lie algebra. Perhaps a short, technical description of the problem will help the more demanding reader; others should skip to the next paragraph. The identification of the group with the algebra is ambiguous because usually there are several extensions of a given Lie algebra to the finite transformations of the Lie group; the choice of extension depends on the choice of discrete group elements factored out of the center of the covering group. For example, because electric charge is quantized in $\frac{1}{3}$ units, Q^{em} generates a U_1 and not its covering group, obtained if the spectrum of charges were to cover all real numbers. Similarly, the Lie algebras of $\mathrm{U}_1^{\mathrm{em}} \times \mathrm{SU}_3^{\mathrm{c}}$ and U_3 are the same, but strictly speaking, those symbols refer to different extensions of the Lie algebra to finite transformations [22]. In order to refer to the factors separately, we call the unbroken part of the theory $\mathrm{U}_1^{\mathrm{em}} \times \mathrm{SU}_3^{\mathrm{c}}$, when, in fact, the extension from the algebra to finite transformations should be called U_3, because of the connection between triality of color and electric charge. In this review we need the properties of the Lie algebra for most discussions, so all names really apply to the algebra; similarly here the term "group theoretical" almost always means "Lie-algebra theoretical". Viewed in this way, our notation is not as sloppy as it first appears.

A boson field $B_\mu^a(x)$ has the transformation properties of a gauge field and is coupled directly to the ath symmetry current $J_\mu^a(x)$; $J_\mu^a(x)$ depends on $B_\mu^a(x)$, so the theory is nonlinear. The generators of the group, and consequently the currents from which the generators are constructed, transform as the adjoint representation. In order for the coupling of the currents to the vector bosons to be invariant under G, the bosons must also transform as the adjoint irrep, since group singlets (or invariants) occur only in the products of an irrep with its complex conjugate, and adjoint irreps are always self conjugate. The vector-boson fields transforming as the adjoint irrep are a necessary part of a Yang–Mills theory.

A Yang–Mills theory may also have other particles in the Lagrangian. For example, the leptons and quarks, which are spin $\frac{1}{2}$ fermions, are usually assumed to be fundamental fields in the Lagrangian. (They may also be tightly-bound composites that behave like fundamental fields in an effective Lagrangian.) Each particle field must be assigned to an irrep of G, so that when the field is put in the Lagrangian, the invariance under G is kept manifest. Thus an important step in understanding the structure of these theories is to know the irreps and the action of the generators of G on them. This is the same as knowing the contributions of those particles to the currents and how those currents interact with the vector boson fields, which explains why the group-theoretical language is so powerful.

QCD is an unbroken SU_3-symmetric, Yang–Mills theory. (The meaning of "broken" is discussed later.) The vector bosons (called gluons) mediating the strong interactions are gauge particles coupled to the eight symmetry currents of $\mathrm{SU}_3^{\mathrm{c}}$, so the gluons transform under $\mathrm{SU}_3^{\mathrm{c}}$ transformations as the adjoint irrep or octet $\mathbf{8}^{\mathrm{c}}$ of $\mathrm{SU}_3^{\mathrm{c}}$. (In this review irreps are designated by their dimensions, with conjugate irreps marked by an over bar, e.g., $\bar{\mathbf{3}}^{\mathrm{c}}$, and other inequivalent irreps of the same dimension with primes or similar markings. Most practitioners find these conventions convenient, even though there are some justifiable objections to them. Other labelings are studied here too.) The gluons carry no flavor charges, which means that they are singlets under any transformation in the flavor group; stated more formally, the gluons transform as $(\mathbf{1}, \mathbf{8}^{\mathrm{c}})$ under $\mathrm{G}^{\mathrm{fl}} \times \mathrm{SU}_3^{\mathrm{c}}$.

Several features of QCD due to its quantum mechanical structure should be noted; they are not important for an elementary understanding of the role of QCD in model building, but they will help in forming a physical picture of it. Isolated hadronic systems are composites of quanta carrying color charges and are assumed to be color singlets. An individual color charge cannot be isolated in space and time from other color charges; that is, color is believed to be confined inside hadrons. (In spite of much effort on the problem, the confinement conjecture has been difficult to prove in QCD; it is not crucial to unified model building as described here.) As the spatial resolution of the probe of hadronic structure is shortened, the effective coupling of the gluons to the color currents decreases, so that the constituents of hadrons appear to interact more weakly. This behavior is called "asymptotic freedom", and is justified by perturbation theory calculations [23]. As a result, it appears possible to view hadrons as composites of strongly interacting elementary quanta with fractional electric charges [24] that gain an elementary identity, not by isolating them, but by probing hadrons at short distances, as is done in deep-inelastic lepton scattering [25] and high-energy electron-positron annihilation [26]. Thus quarks are elementary, not nucleons, pions, kaons, and other observed hadrons. The quarks are coupled to the gluons through their contributions to the color currents, so they must transform as a nontrivial irrep of SU_3^c: the quarks are assigned to the $\mathbf{3}^c$, the antiquarks to $\bar{\mathbf{3}}^c$. The representation theory of SU_3 is used to illustrate more general results several times later on in this paper.

In Yang–Mills models a central problem, both physically and mathematically, is relating particle states to representation vectors; each particle degree of freedom is in one-to-one correspondence with one vector of a representation. Thus one tricolored, four-component Dirac quark of a given momentum is described by $3 \times 4 = 12$ Hilbert space vectors: red, green, or blue (or whatever your favorite names for the three colors) times the label, left-handed quark, right-handed quark, left-handed antiquark, or right-handed antiquark, so the vectors have the labels, $|$color, handedness, particle or antiparticle$\rangle$. (In the limit of zero mass, left-handed means the spin projection is antiparallel to the momentum vector, and right-handed means the spin projection is parallel to the momentum; for a massive particle at rest a chiral eigenstate is a 50–50 mixture of spin projections.) The fermion field operator ψ that annihilates (i.e., removes) a particle from the state ($\psi|\text{particle}\rangle = |\text{vacuum}\rangle$) carries the same set of labels as the state, but with signs of appropriate quantum numbers reversed. The reason for using chirality (handedness) rather than some spin component of the fermion is that the chirality projections $\frac{1}{2}(1 \pm \gamma_5)$ commute with the gauge and proper Lorentz transformations; the left-handed fermions transforming as $\mathbf{f}_L$ do not mix with the right-handed fermions in $\mathbf{f}_R$ under a gauge transformation, so the set of all fermion states cannot belong to an irrep. However, $\mathbf{f}_L$ (and consequently $\mathbf{f}_R$) by itself can be an irrep.

Let us examine in general the construction of the fermion kinetic energy and the fermion mass in any Yang–Mills Lagrangian [2]. Our object is to show that the kinetic energy couples $\mathbf{f}_L$ and $\mathbf{f}_R$, that $\mathbf{f}_R$ transforms as $\bar{\mathbf{f}}_L$ (the conjugate of $\mathbf{f}_L$), and to discuss some group-theoretical aspects of the construction; we then show that the mass operator couples $\mathbf{f}_L$ to $\mathbf{f}_L$ (and $\mathbf{f}_R$ to $\mathbf{f}_R$), and it transforms as representations in the symmetric part of $\mathbf{f}_L \times \mathbf{f}_L$. The kinetic energy must be gauge invariant, but it is not necessary for $(\mathbf{f}_L \times \mathbf{f}_L + \mathbf{f}_R \times \mathbf{f}_R)_s$ to contain a singlet, since all fermion mass can arise from symmetry breaking.

The usual covariant kinetic energy has the form

$$\bar{\psi}\, \mathrm{i}\gamma^\mu (\overleftrightarrow{\partial}_\mu - \mathrm{i}gB_\mu^a T_a)\psi\,, \tag{2.1}$$

where ψ is a column vector of real (Majorana) anticommuting fields, $\bar{\psi} = \psi^\dagger \gamma_0$, T_a is an antisymmetric matrix representation of the group, so that the current contribution $\bar{\psi}\gamma^\mu T_a \psi$ is nonzero and transforms as the adjoint irrep, and $B_\mu^a(x)$ are the vector boson fields, also transforming as the adjoint irrep. (The advantages of beginning with Majorana spinors and constructing Dirac spinors later will become apparent.) Since the chirality projections $\frac{1}{2}(1 \pm \gamma_5)$ commute with $\gamma^0\gamma^\mu$, the kinetic energy can also be written $\bar{\psi}_L \mathrm{i}\gamma \cdot D\psi_L + \bar{\psi}_R \mathrm{i}\gamma \cdot D\psi_R$, where $D_\mu = \overleftrightarrow{\partial}_\mu - \mathrm{i}gB_\mu^a T_a$ and each term by itself is gauge invariant.

The field operator ψ (and any other operator in the theory) carries a definite change of quantum numbers; when ψ acts on a state, it changes the quantum numbers of that state by amount λ (called the "weight" of ψ) and a change in angular momentum for a field with spin, whether it creates or annihilates particles. The Hermitian conjugate $\psi^\dagger$ must change the quantum numbers of a state by $-\lambda$, and also make the opposite change in spin for ψ. Looking at the $\psi_L^\dagger(\gamma^0\gamma\cdot\overleftrightarrow{\partial})\psi_L$ piece of the kinetic energy, we note that the c-number pieces in parentheses cannot change quantum numbers, and the combination $\psi_L^\dagger\cdots\psi_L$ cannot either without destroying the global gauge invariance of the kinetic energy. Thus, if $\psi_L\sim\mathbf{f}_L$ ("$\sim$" means "transforms as"), then $\psi_L^\dagger\sim\bar{\mathbf{f}}_L$, since a Hermitian conjugated field operator $\psi^\dagger$ acts on the labels of the dual vector $\langle\cdots|$ in the same way that ψ acts on the labels of the ket $|\cdots\rangle$. Moreover, when $\psi_L^\dagger$ acts on the ket $|\cdots\rangle$, it changes the eigenvalues of *all* the diagonalizable generators by an amount that is the negative of the change due to ψ_L, including chirality, so $\psi_L^\dagger\sim\mathbf{f}_R$; these important results are summarized by

$$\mathbf{f}_R\sim\bar{\mathbf{f}}_L. \tag{2.2}$$

The kinetic energy can be written group theoretically as $\mathbf{f}_R(\mathrm{Op})\mathbf{f}_L+\mathbf{f}_L(\mathrm{Op})\mathbf{f}_R$.

The reflection that takes $\mathbf{f}_L$ to $\mathbf{f}_R$ includes changes of the signs of the internal quantum numbers (often done by a charge conjugation C) and the handedness (often done by parity P). Individually C and P do not have to exist in a theory, but CP must, since it is necessary to have a reflection that exchanges $\mathbf{f}_L$ and $\mathbf{f}_R$ in such a way that $\mathbf{f}_L\times\mathbf{f}_R$ contains a group invariant and an adjoint, as needed for the gauge-invariant kinetic energy (2.1). As discussed in section 6, it happens in many unified models where C exists that C reverses the signs of only some of the quantum numbers, with P reversing the remaining ones. This is because C must reflect $\mathbf{f}_L$ onto itself; if $\mathbf{f}_L$ is not self conjugate, then C cannot reflect the signs of all the quantum numbers [13].

We can now relate $\mathbf{f}_L$ to the column of N four-component spinors ψ in the case where $\mathbf{f}_L$ is irreducible; note that if the dimension of $\mathbf{f}_L$ is N, then ψ has $2N$ independent components, so there must be $2N$ constraints. The simplest example ($N=1$) is a 4 component Majorana spinor, where $\mathbf{f}_L$ is a nontrivial one-dimensional irrep of a U_1, and $\mathbf{f}_R$ has the opposite charge. A Majorana spinor has two independent components, so there are two constraints relating ψ and $\psi^\dagger$: it is $\psi=C\gamma_0\psi^{\dagger T}$, which is called the Majorana condition. The 4-by-4 matrix C is defined so C, $C\gamma_5$ and $C\gamma_5\gamma_\mu$ are antisymmetric, and $C\gamma_\mu$ and $C\sigma_{\mu\nu}$ are symmetric. Thus, $\frac{1}{2}\psi^T C\gamma\cdot D\psi=\frac{1}{2}\bar{\psi}\gamma\cdot D\psi$ as in (2.1) is Lorentz and U_1 invariant, and is a suitable kinetic energy for a Majorana spinor. It is nonzero because $C\gamma\cdot D$ is antisymmetric and fermion fields anticommute.

The only case where group-theoretical complications might occur is for the irrep $\mathbf{f}_L$ to be self conjugate, because the matrix part of the relation between ψ and $\psi^\dagger$ must be able to reverse the signs of all weights within the irrep $\mathbf{f}_L$; thus, it is the matrix part of the unitary operator, $\mathscr{C}\mathscr{P}$, $(\mathscr{C}\mathscr{P})\psi(\mathscr{C}\mathscr{P})^\dagger=C(CP)\psi$, where the unitary matrix C acts on the spin degrees of freedom and the unitary matrix (CP) acts on the weights in $\mathbf{f}_L$. (Obviously, C and (CP) commute.) The simplest generalization of the Majorana condition is $\psi^\dagger=\psi^T(CP)C\gamma_0$, and the kinetic energy (which is Hermitian) is $\frac{1}{2}\mathrm{i}\,\psi^T(CP)C\gamma\cdot D\psi\equiv\frac{1}{2}\mathrm{i}\bar{\psi}\gamma\cdot D\psi$, where $\bar{\psi}=\psi^T C(CP)=\psi^\dagger\gamma_0$; the last equality follows from the Majorana condition. Clearly, (CP) is symmetric.

This construction is completely adequate for **real** irreps: if $\mathbf{f}_L$ is real, then $(\mathbf{f}_L\times\mathbf{f}_L)_s$ contains the singlet and $(\mathbf{f}_L\times\mathbf{f}_L)_a$ contains the adjoint, so (CP), which is the matrix coupling $\mathbf{f}_L$ to $\mathbf{f}_L$ to form the singlet, must be symmetric. Moreover, if $\mathbf{f}_L$ is not self conjugate, then (CP) is outside the group and can be defined to be symmetric. However, there is also another kind of self conjugate irrep in some simple groups: if $\mathbf{f}_L$ is **pseudoreal**, then $(\mathbf{f}_L\times\mathbf{f}_L)_a$ contains the singlet and $(\mathbf{f}_L\times\mathbf{f}_L)_s$ contains the adjoint, so the

(CP) matrix is antisymmetric. In this case $\psi^{\mathrm{T}}(CP)\gamma\cdot D\psi$ vanishes identically, so the construction above must be revised. In the pseudoreal case, the kinetic energy must be written $\frac{1}{2}\mathrm{i}\psi^{\mathrm{T}}(CP)C\gamma_5\gamma\cdot D\psi\equiv\frac{1}{2}\mathrm{i}\bar{\psi}\gamma\cdot D\psi$, which is nonvanishing for the antisymmetric part of (CP). (This modification is not available for spinless particles.) The Majorana condition should now be written with a γ_5, $\psi^{\dagger}=\psi^{\mathrm{T}}(CP)C\gamma_5\gamma_0$, so that $\bar{\psi}=\psi^{\mathrm{T}}(CP)C\gamma_5=\psi^{\dagger}\gamma_0$ is again the usual Dirac conjugate. It should be emphasized that within a unified theory of this type, QCD and QED can still be transformed into their usual form [27]. However, it is not even possible to write down a kinetic energy term for spinless bosons if they transform as a single, pseudoreal irrep, at least not without making some drastic revisions in the theory [28].

The fermion mass term has the general form

$$\bar{\psi}(S+\mathrm{i}P\gamma_5)\psi\,, \tag{2.3}$$

where S and P are Hermitian matrices in the group space and need not have gauge-group singlets in a spontaneously broken theory. Since this term can be rewritten as $\bar{\psi}_{\mathrm{R}}(S-\mathrm{i}P)\psi_{\mathrm{L}}+\bar{\psi}_{\mathrm{L}}(S+\mathrm{i}P)\psi_{\mathrm{R}}$, it follows from (2.2) that the first term transforms as $\mathbf{f}_{\mathrm{L}}\times\mathbf{f}_{\mathrm{L}}$ (since $\psi_{\mathrm{R}}^{\dagger}\sim\mathbf{f}_{\mathrm{L}}$) and the second as its Hermitian conjugate, $\mathbf{f}_{\mathrm{R}}\times\mathbf{f}_{\mathrm{R}}$. Moreover, since γ_0 and $\gamma_0\gamma_5$ behave antisymmetrically between Majorana fields and fermion fields anticommute, the mass operator is in the **symmetric part**, $(\mathbf{f}_{\mathrm{L}}\times\mathbf{f}_{\mathrm{L}}+\mathbf{f}_{\mathrm{R}}\times\mathbf{f}_{\mathrm{R}})_{\mathrm{s}}$.

The mass operator of a single quark is a bilinear form on the twelve quark states. One may conclude that the mass matrix of a single quark is 12-by-12, but of course it is never written in that form since symmetry requires most of the components to be zero and the remainder to have equal magnitude; the 144 parameters are reduced to one by rotational invariance, color conservation, and the phase freedom in defining the field operators. The analysis of a single quark mass matrix is an important prototype for illustrating some of the physics of the choice of the spin $\frac{1}{2}$ fermion representation in unified models. First of all, since the local symmetry transformations commute with $1\pm\gamma_5$, the mass operator $\bar{\psi}\psi=\bar{\psi}_{\mathrm{L}}\psi_{\mathrm{R}}+\bar{\psi}_{\mathrm{R}}\psi_{\mathrm{L}}$ breaks up into two six-by-six pieces. Group theoretically, for a single quark,

$$\mathbf{f}_{\mathrm{L}}=\mathbf{3}^{\mathrm{c}}+\bar{\mathbf{3}}^{\mathrm{c}}; \tag{2.4}$$

the most general quark mass matrix has the form,

$$\begin{array}{c c} & \begin{array}{cccc} \bar{\mathrm{q}}_{\mathrm{R}} & \mathrm{q}_{\mathrm{R}} & \mathrm{q}_{\mathrm{L}} & \bar{\mathrm{q}}_{\mathrm{L}} \end{array} \\ \begin{array}{c} \mathrm{q}_{\mathrm{L}} \\ \bar{\mathrm{q}}_{\mathrm{L}} \\ \bar{\mathrm{q}}_{\mathrm{R}} \\ \mathrm{q}_{\mathrm{R}} \end{array} & \begin{pmatrix} 0 & 0 & M_{11} & M_{12} \\ 0 & 0 & M_{21} & M_{22} \\ M_{11}^{+} & M_{21}^{+} & 0 & 0 \\ M_{12}^{+} & M_{22}^{+} & 0 & 0 \end{pmatrix}, \end{array} \tag{2.5}$$

where the rows and columns of (2.5) are labeled by states in such a way that the matrix is manifestly Hermitian and the M_{ij} are elements of a **symmetric** 6-by-6 matrix M; the M_{ij} are 3-by-3 matrices in color space. (The formulation here, which may appear clumsy for a single quark, is set up to be trivially generalized to an arbitrary spin $\frac{1}{2}$-fermion mass matrix; some comments about the general case will be made. The labeling of the rows with a vertical column vector on the left and the columns by a row on top is somewhat unorthodox, but makes it possible to fit some of the examples below on a page.) The upper right-hand corner of (2.5) is the mass matrix associated with $\bar{\psi}_{\mathrm{R}}\psi_{\mathrm{L}}$; the lower left-hand matrix is the Hermitian conjugate $M^{\dagger}$, since $(\bar{\psi}_{\mathrm{R}}\psi_{\mathrm{L}})^{\dagger}=\bar{\psi}_{\mathrm{L}}\psi_{\mathrm{R}}$. The notation in (2.5) is therefore redundant; we need only to consider the upper right-hand corner. This is especially so, since the 6-by-6 symmetric matrix M with elements M_{ij} can be diagonalized by a **unitary** transformation of the form $U^{\mathrm{T}}MU$ (transpose, not adjoint). In the case that CP is conserved M is real symmetric and U is orthogonal. The generalization to the CP violating case, where M is complex symmetric even after the phase freedom of the quark states is used, is based on a theorem of Schür [29–31]: there exists a unitary matrix U such that $U^{\mathrm{T}}MU$ is real diagonal with diagonal elements being the positive square roots of the eigenvalues of the Hermitian matrix, MM^* (* is complex conjugate), **not** $\det(M-\lambda)$. (MM^* is Hermitian and positive definite.) Thus it is not necessary to diagonalize the whole matrix in (2.5), but only a "quarter of it", even in the CP violating case, as pointed out in ref. [30].

Next, we impose the requirement that the quark mass be invariant under SU_3^c transformations. Let us look at M_{11}, which must transform as the irreps in $(\mathbf{3}^c\times\mathbf{3}^c)_s=\mathbf{6}^c$. Thus M_{11} (and M_{22} for the same reason) must be zero, because no component of a $\mathbf{6}^c$ (or any other nontrivial SU_3^c irrep) is invariant under all SU_3^c transformations. The nonzero components must be in the singlet part of $(\mathrm{q_L}\times\bar{\mathrm{q}}_\mathrm{L})=\mathbf{3}^c\times\bar{\mathbf{3}}^c=\mathbf{1}^c+\mathbf{8}^c$, which reduces (2.5) to one parameter. Stated another way, ψ_L must annihilate the same color state as $\bar{\psi}_\mathrm{R}$ creates (and vice versa), so only one color state need be analyzed because the masses of each color must be equal (i.e., the mass must be a color singlet). Thus, we can write the mass matrix as (2.5) with $M_{12}=M_{21}$ nonzero numbers (times 3-by-3 unit matrices). If M_{12} is written with a phase $m\,\mathrm{e}^{\mathrm{i}\theta}$ then the phase can be eliminated by the transformation, $\psi\to\exp(\mathrm{i}\theta\gamma_5/2)\psi$, which leaves (2.1) invariant. It should now be obvious how to combine the 6 Majorana fields transforming as (2.4) into three Dirac fields of definite mass and color.

The strong interactions are believed to conserve parity; this assumption is incorporated in QCD, which is one reason for introducing the quark as a four-component Dirac spinor. Electric charge conservation also prohibits connecting $\mathrm{q_L}$ with $\mathrm{q_L}$ with a "Majorana" mass; however, neutral leptons may have Majorana masses, which violate lepton-number conservation. All quarks appear to have nonzero masses, although the "up" quark ultraviolet mass (the mass appearing in the QCD Lagrangian) may be small, on the order of an MeV or so. Moreover, parity conservation also means that the color currents must be vector currents. These requirements imply that $M_{12}=M_{21}=m$ (times a 3-by-3 unit matrix) can be defined to be real; the eigenvalues of (2.5) are $+m$, $+m$, $-m$, $-m$, just as occurs in the Dirac equation (for the same reasons), where the mass matrix is βm, that is, $m\bar{\psi}\psi=\psi^\dagger(\beta m)\psi$. These various requirements are not so redundant in the general case.

We should call attention to one more feature of this trivial example. It is helpful to be able to relate the left-handed antiquark to the left-handed quark by a charge conjugation operator C that complex conjugates the $\mathbf{3}^c$ and matches up the $\mathrm{q_L}$ with the $\bar{\mathrm{q}}_\mathrm{L}$. If we add C to SU_3^c to form the group SU_3^{c*}, then $\mathbf{f}_\mathrm{L}=\mathbf{3}^c+\bar{\mathbf{3}}^c$ is an irrep of SU_3^{c*}. In many unified models there exists an element of the gauge group G itself that acts as C, and only when it exists, is it possible to assign $\mathbf{f}_\mathrm{L}$ to an irrep of G.

The quark masses are generally believed to be due to the flavor interactions, and thus are merely parameters in the study of QCD. However, QFD is also included in unified models, so one of the main topics in the study of unified models is the attempt to understand the origin and magnitudes of the fermion masses in the theory. At present, this problem is far from solved.

At the time of this writing, five tricolored quarks have been identified by their experimental manifestations. The u (up) with electric charge $Q^{\mathrm{em}}=\frac{2}{3}$ and the d (down) with $Q^{\mathrm{em}}=-\frac{1}{3}$ are the fundamental constituents of neutrons, protons, pions, and so on. The strange quark s was hypothesized by Gell-Mann [24] to be the carrier of the strangeness quantum number. Charm (c) was originally introduced to explain the suppression of neutral strangeness changing weak currents [32], and has been substantiated as a fundamental constituent of the ψ and charmed particles. Finally a third $Q^{\mathrm{em}}=-\frac{1}{3}$

quark called b (for "bottom" in some parts of the world) is a constituent of a new set of particles, including the upsilon Υ [33].

One noteworthy feature is the large number of quarks; this proliferation is of fundamental concern, and it is one motivation for this review. There are at least 30 $\mathbf{3}^c$ and $\bar{\mathbf{3}}^c$ components of $\mathbf{f}_L$ required by phenomenology. For example, if all 30 of these are elementary and belong to one irrep of a unifying group, then that irrep must be quite large. This, among other reasons, helps motivate the development of powerful group-theoretical techniques.

The leptons are similar to quarks in that they are believed to be fundamental spin $\frac{1}{2}$ fermions. They do not interact strongly, which completes the definition of a lepton; they are assigned to singlets $\mathbf{1}^c$ of SU_3^c. Leptons and quarks both carry the charges of the weak interaction, which brings us to an introductory discussion of QFD.

One of the most significant developments of the 1970's was the confirmation that a certain $SU_2 \times U_1$ Yang–Mills field theory [4] provides an excellent description of the combined electromagnetic and weak interactions. The vector bosons mediating the charged- and neutral-current weak interactions have not yet been observed, as their masses are expected to be of order 80 to 90 GeV, beyond the range of the machines of the 1970's. Since $SU_2^w \times U_1^w$ has four generators, there are four vector bosons that are coupled to four currents. The vacuum is invariant under a subgroup of the $SU_2^w \times U_1^w$, namely the U_1 generated by the electric charge Q^{em}. The electric charge is a linear combination of the two diagonal generators of $SU_2^w \times U_1^w$

$$Q^{em} = I_3^w + Y^w/2 , \tag{2.6}$$

where I_3^w is the neutral component of the SU_2^w (where w stands for weak, and distinguishes this application of SU_2 in physics from others) and Y^w is the generator of the U_1^w. The orthogonal combination of I_3^w and Y^w is a weak interaction charge, and the current associated with it is coupled to the Z^0 boson that mediates the neutral current weak interactions. That vector boson is expected to have a mass of 90 GeV. (Although the Lagrangian of the Yang–Mills theory possesses that U_1 symmetry, the vacuum does not, so the Z^0 boson has a mass.) The charge raising and charge lowering currents in the SU_2^w are coupled to the W^- and W^+ vector bosons, which mediate the charge-current weak interactions. They are expected to have masses of about 80 GeV, which indicates that the vacuum carries all the weak charges.

The phenomenon of vector bosons becoming massive in Yang–Mills theories is called spontaneous breaking of local symmetry [34]. When we talk about broken local symmetries, we do not mean that a symmetry breaking term is added to the Lagrangian. A vector boson mass term by itself is not invariant under the local symmetry transformations. However, a vector boson mass term plus some additional interactions can be, and that is what happens in a broken symmetry. (In this paper, "broken" means spontaneously broken.) The Lagrangian is still invariant under the symmetry transformations, and the current are conserved, or in the language of quantum field theory, the Ward identities (except for anomalies) hold, which is crucial for the perturbative renormalizability of the theory. However, the charge of a broken symmetry does not annihilate the vacuum, which corresponds to the charge being spread through the vacuum. The mathematical description of electromagnetic radiation in a plasma is completely analogous to that of a vector boson in a broken vacuum [34]. The main point of this discussion is to emphasize that even for very badly broken symmetries, the Lagrangian is still invariant, and so it is still important to study the invariants that can be formed from the fields in the theory. However, certain "numbers" in the theory, like fermion mass matrix elements, will often have nontrivial transformation properties. A complete study of the symmetry properties is still needed.

Although the weak vector bosons of the standard $SU_2^w \times U_1^w$ theory have not been observed, the form of the currents of the low-mass elementary particles has been well established experimentally [35]. Aside from the photon and the gluons, the only low-mass particles that have been accessible in the laboratory are the quarks and leptons. As we discussed earlier, the left-handed quarks, antiquarks, leptons and antileptons must be assigned to representations of the gauge group and the right-handed fermions to the conjugate representations. SU_2 has singlets, doublets, triplets, and so on, and the assignment that works is to place the left-handed fermions in SU_2^w doublets and the left-handed antifermions in SU_2^w singlets. This unsymmetrical left–right assignment is rather startling, but is confirmed by a huge quantity of experimental data; the maximal parity violation in the charged current weak interactions already provided a strong hint over 20 years ago [4] that this is a sensible assignment phenomenologically. The choice of groups discussed in this paper is strongly influenced by the desire to maintain these unsymmetrical assignments in a natural fashion. Specifically, only when $\mathbf{f}_L$ is not self conjugate is it not required for every left-handed doublet to be matched with a right-handed doublet. Theories where $\mathbf{f}_L$ is self conjugate are called **vectorlike** [36]; otherwise, the theory is called **flavor chiral** [6].

We now discuss the specific assignment of the quarks to representations of $SU_2^w \times U_1^w$. The left-handed u and d′ quarks form a doublet, where the Y^w values are determined by (2.6) and the assignment that u has charge $\frac{2}{3}$ and d has charge $-\frac{1}{3}$. (The prime indicates states coupled to the currents; they are linear combinations of the mass eigenstates.) Thus, $Y_u^w = Y_d^w = \frac{1}{3}$. (All members of an SU_2^w multiplet must have the same value of Y^w, or else the Y^w generator will not commute with the SU_2^w generators, and the group will not factor.) The left-handed ū and d̄ are SU_2^w singlets, and have different values of Y^w in accordance with (2.6). The *CPT* transformation relating the left-handed fermions to the right-handed fermions just inverts this arrangement: u and d are right-handed singlets and ū and d̄ form a right-handed doublet. Since this is a basic feature of all fermion representations in local field theories, we only need to discuss the left-handed fermion representation $\mathbf{f}_L$.

This pattern of quark assignments repeats itself at least once with the c and s quarks, and is often presumed to repeat itself again with the charge $-\frac{1}{3}$ b quark and a conjectured charge $\frac{2}{3}$ t (top) quark. The b quark has charged-current weak decays [37], which is consistent with a nonzero weak isospin assignment; however, a zero weak isospin assignment is not yet ruled out.

Next we briefly indicate the relationships among these assignments to a representation of the gauge group, the currents, and the masses. For simplicity we consider first the charge $-\frac{1}{3}$ quark mass, including just the d and s quarks, and ignoring any mixing with the b quark. The mass matrix (2.5) connects (the left-handed) d̄ to the d and s, and this combination has definite, nontrivial $SU_2^w \times U_1^w$ transformation properties, with $|\Delta I^w| = \frac{1}{2}$ and $|\Delta Y^w| = 1$, so that $\Delta Q^{em} = 0$. Thus, the existence of the quark mass indicates already that $SU_2^w \times U_1^w$ is broken down to U_1^{em}. This also means that there can be mixing among all the fermions with the same color and electric charge. Thus it is important to distinguish between the states connected by the currents and those connected by the mass matrix. If we call the mass eigenstates d and s, then the charge $-\frac{1}{3}$ quarks coupled to the u and c by the charged currents of the weak interactions are called

$$d' = \cos\theta_c\, d + \sin\theta_c\, s, \qquad s' = -\sin\theta_c\, d + \cos\theta_c\, s \tag{2.7}$$

where the canonical transformation that connects the mass eigenstates to the states coupled by the currents is parameterized by θ_c. The transformation with three quarks (e.g., d′, s′ and b′) depends on four parameters, including a *CP*-violating phase [30]. [Recall the discussion below (2.5).] We summarize this discussion in table 2, again emphasizing that those designations summarize an incredible amount of phenomenology in a powerful way.

Next, we review the lepton sector. The left-handed charged leptons e^- ($m_e = 0.51100 \times 10^{-3}$ GeV), the μ^- ($m_\mu = 0.10566$ GeV) and most likely the τ^- ($m_\tau = 1.81$ GeV [38]) are all $I_3^w = -\frac{1}{2}$ members of weak isodoublets. To a high degree of accuracy, the mass eigenstates are the states in the currents, with the charge raising current transforming e_L^- to ν_{eL}, μ_L^- to $\nu_{\mu L}$ and (most likely) τ_L^- to $\nu_{\tau L}$. If the neutrinos are massless, but are only coupled to the vector bosons as in (2.1), then their current eigenstates can be defined by convention, and the charged leptons cannot mix through the neutrinos. If the neutrinos have small masses, then (2.3) need not be diagonal in the same basis as (2.1), and there can be mixing; there has been considerable effort to search for muon-number violating processes [39] and neutrino oscillations. The left-handed positron, μ^+ and τ^+ are each assigned to be weak singlets. The analogy with the quarks would be complete if there existed left-handed antineutrinos. There is no argument against them since they would be neutral, $I^w = 0$, and, consequently, $Y^w = 0$ particles, and would have no electromagnetic, weak, or strong interactions. Thus they would be hard to "see" in present day experiments, although they might have left their mark on the features of the universe, as described by the big-bang cosmology. A conservative attitude is that there are no left-handed antineutrinos, since they have not been observed. For most unified models, this attitude must be relaxed.

The model of the symmetry breaking of the standard $SU_2^w \times U_1^w$ theory down to the U_1^{em} symmetry of QED is done by giving certain scalar fields carrying the charges of the weak interactions, but no electric or color charge, a nonzero vacuum expectation value [40]. This problem can be studied in the Higgs model [41] of symmetry breaking. The boson and fermion masses are then proportional to the vacuum expected values of the scalar fields, as are the interactions induced by the symmetry breaking, at least in the classical limit. There is controversy whether the scalar fields doing the breaking of $SU_2^w \times U_1^w$ are fundamental fields in the Lagrangian of Nature, or are "effective scalars" formed, for example, as fermion–fermion bound states. In the considerations in this paper, we often treat the symmetry breaking in the Higgs model, but withhold judgement on the physical reality or origin of the scalar fields. A study of the group theoretical structure of symmetry breaking is useful, no matter what the breaking mechanism.

In the standard model, $SU_2^w \times U_1^w$ is broken by the neutral member of an $I^w = \frac{1}{2}$, $Y^w = 1$ scalar doublet $\phi(x)$. *CPT* invariance requires a conjugate doublet $\phi^\dagger(x)$ with $Y^w = -1$, so there are four spin 0 degrees of freedom. Thus, up to three vector bosons can acquire masses, using the corresponding scalar fields as the longitudinal degrees of freedom. The fourth scalar is physical and defines the symmetry breaking direction. The covariant kinetic energy of the scalars has the form $(D^\mu\phi)^\dagger D_\mu\phi$, with $D_\mu\phi = \partial_\mu\phi - \mathrm{i}g(\tau_a/2)\, B_\mu^a(x)\,\phi(x) - \mathrm{i}(g'/2)\, B_\mu(x)\,\phi(x)$, where $g'/2$ is the coupling of U_1^w and $B_\mu(x)$ is the vector boson coupled to the U_1^w current, and $B_\mu^a(x)$ ($a = 1, 2, 3$) are the bosons coupled to the SU_2^w currents with gauge coupling g; the weak isospinor representation matrices are $\tau^a/2$, where τ^a are the Pauli matrices. In the unitary gauge the constant part of ϕ, of which only the neutral component is nonzero, provides (classically) masses to the three weak bosons, while leaving the photon massless. At this point, we leave the calculation, since the details are now considered straightforward [21]. However, the discussions of symmetry breaking in section 9 do rely on a thorough knowledge of this well-known calculation.

In summary we can say that the standard model has provided an attractive framework for organizing huge quantities of experimental data. Nevertheless, from a theoretical viewpoint it is somewhat awkward, as discussed at the beginning of the next section; it does not appear to be the ultimate theory of the interactions in Nature.

3. Unification

The purpose of this section is to give a brief, but critical introduction to the unification of the standard model into a Yang–Mills theory based on a simple group. The SU_5 example, which is a prototype, is then described. Finally, the method of embedding flavor and color into the simple group, as carried out in ref. [6], is described and at the same time a brief introduction to general features of the simple algebras is given. (In this review, unified models based on simple groups are the only kind analyzed; at the present stage of development, this restriction is physically quite arbitrary, but many of the results discussed here are applicable to other kinds of gauge models.)

The "standard" $SU_2^w \times U_1^w \times SU_3^c$ Yang–Mills theory of the electromagnetic, weak, and strong interactions has provided a detailed phenomenological framework in which to analyze and correlate many experimental data. Although the constraints of this model appear to be satisfied experimentally, the choice of symmetry group, the assignment of scalars and fermions to representations, and the values of many masses and coupling constants must be deduced from experimental data. Aside from being derived from local symmetries, the three interactions are not related to each other in any specific way, and each gauge coupling for the factors SU_2^w, U_1^w and SU_3^c is a free parameter. Moreover, the standard model ignores gravity and it gives no relationships among particles of different spins. Thus, in spite of its enormous success, the standard model appears to be part of a more complete theory; it leaves too much unsaid. It is an obvious question to ask whether there are more complete theories that include the results of the standard model and also interrelate the interactions and correlate the many assignments and parameters that are put into the standard model by hand. None of the unified models discussed here is likely to be the ultimate theory either because they only partially solve some of the problems above, but one can hope that studying them will lead to a step in the right direction. These models are sometimes called "Grand Unified TheorieS", but not here.

Early attempts to find such theories were made by Pati and Salam [42], who argued that a theory with integer charged quarks [43] (which is not QCD) can be embedded into a larger theory, including new interactions that violate baryon number. Shortly after, Georgi and Glashow [5] pointed out that the standard model (including QCD) can be embedded into the simple Lie group SU_5. This means that the electromagnetic, weak and strong interactions are all contained in a larger set of interrelated interactions. Such a theory must include the color and flavor interactions plus new interactions that mix color and flavor quantum numbers. The Pati–Salam and Georgi–Glashow models have different mechanisms (in detail) for proton decay, but in these models and others like them, proton decay does result from additional interactions not in the standard model. Such theories have helped to motivate more sensitive experiments on the proton lifetime [44] and neutrino masses [45].

If there were no spontaneous symmetry breaking, all the vector bosons would be massless and all the vector boson-current coupling constants would be equal or related by group-theoretically determined constants. The symmetry breaking then distinguishes between the different interactions: the leptoquark bosons, which couple quarks to leptons and can mediate proton decay, acquire very large masses; the weak interaction bosons acquire much smaller masses; and the photon and gluons remain massless. The separation of the underlying interaction into electromagnetic, weak, strong, and other components is due to the specific pattern of spontaneous symmetry breaking. It is important to realize that this hypothesis of unification is very speculative, at least until some experimental evidence, such as proton decay, is found to support it.

Unification by a simple group means there is only one gauge coupling constant, but it also implies that the ratio of the strong and electromagnetic coupling constants is $\frac{8}{3}$ in the limit that spontaneous symmetry breaking can be ignored [18] (in the SU_5 model). Experimentally, however, the ratio of the strong coupling α_s and fine structure constant α_e is very different. At $Q^2 = 10\ \mathrm{GeV}^2$, the ratio of α_s to α_e is about 50 (with large theoretical and experimental uncertainties). As long as α_e and α_s are small and the number of fermions and spinless bosons having color is not too large, this ratio decreases as a logarithm of Q^2, and approaches $\frac{8}{3}$ around Q^2 of order $(10^{14}\ \mathrm{GeV})^2$ in the SU_5 model [19, 20]. The new interactions implied by unification are dramatic, but are so weak that very sensitive experiments are needed to detect them. Unification does have a chance of being viable.

Before looking at the general structure of unified models, it may be helpful to describe the SU_5 model proposed by Georgi and Glashow [5]. The simple group SU_5 has 24 generators, and therefore the Yang–Mills theory has 24 vector bosons coupled to 24 different currents. SU_5 contains $SU_2 \times U_1 \times SU_3$ as a subgroup, and those 12 currents are identified with those of the standard model of the electromagnetic, weak and strong interactions. The other 12 vector bosons mediate new interactions, but they are very weak because the bosons are expected to be very massive: they include a $\mathbf{3}^c$ with electric charge $-\frac{1}{3}$ in a weak isodoublet with a charge $-\frac{4}{3}$ boson, together with their $\bar{\mathbf{3}}^c$ antiparticles.

The assignment of the fermions in the SU_5 model is fairly complicated. The u, d′, electron, and its neutrino are assigned to one family of particles; the c, s′, muon, and its neutrino are assigned to a second family; and the b′, conjectured top, tau, and its neutrino are usually assigned to a third family. Each family itself is assigned to a reducible representation.

We next go through the steps of searching for an SU_5 representation to which a family can be assigned. The unifying group SU_5 contains several subgroups, but among the SU_5 generators there must be an $SU_2^w \times U_1^w \times SU_3^c$ to be identified with the standard model. In fact, as we shall discuss in great detail in section 6, $SU_2 \times U_1 \times SU_3$ is a maximal subgroup of SU_5. We may carry out the embedding by breaking up the fundamental five-dimensional, unitary, irreducible representation (irrep) of SU_5 into fundamental representations of $SU_2 \times U_1 \times SU_3$ as follows,

$$\mathbf{5} = (\mathbf{2}, \mathbf{1})(1) + (\mathbf{1}, \mathbf{3})(-2/3)\,; \tag{3.1}$$

in the entry $(\mathbf{x}, \mathbf{y})$, $\mathbf{x}$ is an irrep of SU_2 and $\mathbf{y}$ is an irrep of SU_3, and the irreps are denoted by their dimensions. The second parenthesis contains the value of the U_1 generator when acting on the states in the $(\mathbf{x}, \mathbf{y})$. It is normalized so that it can be identified with Y^w, if the SU_2 is identified with SU_2^w and the SU_3 is identified with SU_3^c. The relative value of the Y^w eigenvalues is determined from the requirement that all generators of SU_5 must be traceless. The contents of an irrep of a group in terms of the irreps of a subgroup, like (3.1), is called a **branching rule**. Knowledge of branching rules is crucial for studying the content of models, so this review contains many examples. (The SU_5 branching rules are contained in table 30.) There is no other way to fit nontrivial $SU_2 \times U_1 \times SU_3$ irreps into a five dimensional irrep aside from conjugating the $\mathbf{3}$, so the choice (3.1) is unique once the embedding is established. (Since the $\mathbf{5}$, $\mathbf{2}$ and $\mathbf{3}$ are **faithful** irreps of SU_5, SU_2 and SU_3, respectively, (3.1) already goes a long way toward establishing the embedding. What must be done is to show that the generators of SU_5 contain the generators of SU_2, U_1 and SU_3, which can be proven by looking at the branching rule for the adjoint. See below.)

The next question is whether any of the known particles can be assigned to the $\mathbf{5}$, given the above embedding of $SU_2^w \times U_1^w \times SU_3^c$ in SU_5. From (3.1) it is seen that the $\mathbf{5}$ contains a lepton weak doublet with electric charges $+1$ and 0, and a charge $-\frac{1}{3}$ quark weak singlet. Such a multiplet may be appropriate for the **right**-handed $(e^+, \bar{\nu}_e)$ doublet and the right-handed down quark, or equivalently, $\mathbf{f}_L \sim \bar{\mathbf{f}}_R$ may contain a $\bar{\mathbf{5}}$ with the $(\nu_e, e^-)_L$ doublet and the $\bar{d}_L'$ singlet. If this assignment is made, then we must find another representation of SU_5 for the left-handed e^+, ū, u and d′. Consequently, there is use for a list of irreps of SU_5 (or any other group that is a candidate for unification), such as the one in table 28, and the branching rules to see if there exists a suitable candidate. It is clear that the **10** is such a candidate. This possibility can be derived from (3.1) and the tensor product,

$$(\mathbf{5}\times\mathbf{5})_a=\mathbf{10}\,, \tag{3.2}$$

which can be found in table 29, where sub a means that the irrep is found in the antisymmetric part of the product. From (3.1) and (3.2) it follows that

$$\begin{aligned}\mathbf{10}=(\mathbf{5}\times\mathbf{5})_a&=[(\mathbf{2},\mathbf{1}^c)(1)+(\mathbf{1},\mathbf{3}^c)(-2/3)]_a^2\\&=[(\mathbf{2},\mathbf{1}^c)\times(\mathbf{2},\mathbf{1}^c)]_a(2)+[(\mathbf{2},\mathbf{1}^c)\times(\mathbf{1},\mathbf{3}^c)](1/3)+[(\mathbf{1},\mathbf{3}^c)\times(\mathbf{1},\mathbf{3}^c)]_a(-4/3)\\&=(\mathbf{1},\mathbf{1}^c)(2)+(\mathbf{2},\mathbf{3}^c)(1/3)+(\mathbf{1},\bar{\mathbf{3}}^c)(-4/3)\,,\end{aligned} \tag{3.3}$$

where the Y^w values **add**, since the Y^w is a U_1 generator. Thus it is immediately found from (3.3) and (2.6) that the $(\mathbf{1},\mathbf{1}^c)$ is a charge 1, singlet lepton, to which the positron may be assigned; the $(\mathbf{2},\mathbf{3}^c)$ is a quark doublet suitable for u and d′; and $(\mathbf{1},\bar{\mathbf{3}}^c)$ has charge $-\frac{2}{3}$ and is suitable for the ū singlet.

In summary, the electron family with $e^\pm$, ν_e, u, ū, d′ and d̄′ may be assigned to a $\bar{\mathbf{5}}+\mathbf{10}$ of SU_5. The first step of model building consists of embedding QED, SU_2^w and QCD in the unifying group, and then searching for a representation $\mathbf{f}_L$ that reproduces the standard model. With three families, $\mathbf{f}_L=\bar{\mathbf{5}}+\mathbf{10}+\bar{\mathbf{5}}+\mathbf{10}+\bar{\mathbf{5}}+\mathbf{10}$, which is not self conjugate, the theory is flavor chiral. We are certain to recover the results reviewed in section 2, as long as the additional interactions implied by unification do not modify the $SU_2^w\times U_1^w\times SU_3^c$ interactions too much.

Let us make a few comments concerning the "progress" made by going from the standard model to the SU_5 model. (1) The three independent gauge couplings of the standard model are reduced to a single coupling, and given the experimental value of the QCD and QED coupling, SU_5 gives a reasonable account of the relative amount of the vector and axial-vector weak neutral currents, which depends on the free parameter $\sin^2\theta_w$ relating the two independent couplings in the standard model [18–20]. (2) It describes the charged current weak interactions correctly. (3) If it is assumed that the symmetry breaking is lacking certain terms, the neutrino is exactly massless due to a conservation law in the theory. In more complicated models the neutrino often acquires a small but finite mass; depending on future experimental results, the prediction of massless neutrinos may (not) continue to be an advantage [45]. (4) It qualitatively predicts the mass of the b quark.

This impressive list of successes should be compared with the ambitions of unification: (1) The number of fermion families is not predicted by the theory nor are most of the masses. (2) The SU_5 model and others like it require ratios of boson masses to be of order 10^{12}. This requirement is difficult to satisfy with all symmetry breaking done with scalars appearing in the Lagrangian because quantum corrections tend to obliterate large mass ratios unless very special values of the couplings are chosen [46]. This may be a devastating criticism of the usual form of the SU_5 model. (3) Gravity has not yet been unified with the other interactions, although masses of order 10^{-4} of the Planck mass are involved. Of course, there still remain the questions, why SU_5? why families? and why the reducible choice $\bar{\mathbf{5}}+\mathbf{10}$ for each family? (Of course, not everyone considers the above to be requirements of unification, in which case, they are not shortcomings.)

Let us dwell for a moment on another issue that SU_5 and related models do not face: they do not relate particles of different spin. It would seem that a truly unified model ought to relate the gauge bosons to the other fundamental fields in the theory. Such relations do exist in theories in which the symmetry structure has been extended beyond that of Lie algebras to include fermionic generators satisfying anticommutation relations with one another and commutation relations with the members of the Lie algebra. Such algebraic systems are called "graded Lie algebras" or "supersymmetries", and model Lagrangians with global and local supersymmetries have been proposed [47].

Supergravity is perhaps the most interesting example proposed so far, especially when it is based on extended supersymmetry, where the supersymmetry charges carry an extra index that ranges from 1 to N. The $N = 8$ version Lagrangian contains explicitly the following particle spectrum: a single spin 2 graviton is accompanied by an SO_8 octet of spin $\frac{3}{2}$ particles, a set of spin 1 particles belonging to the 28-dimensional adjoint representation of SO_8, a set of Majorana spin $\frac{1}{2}$ particles belonging to the **56**, and scalar and pseudoscalar multiplets in two 35-dimensional representations of SO_8. The SO_8 representation theory is summarized in tables 36 and 37. If this particle spectrum is identified with what we call elementary particles, then the shortcomings of supergravity are severe: SO_8 is too small to include color times a sufficient flavor group and the **56** of spin $\frac{1}{2}$ fermions cannot include all three charged leptons. Moreover, there is a serious problem if the SO_8 is gauged: measured values of the gauge coupling imply a cosmological constant that is too large by a factor 10^{60}. So it is possible that this interpretation of the $N = 8$ supergravity Lagrangian is misleading, and all the particles normally considered "elementary", such as the e, μ, γ, etc., are composites on distance scales of order of the Planck mass [48], and the SO_8 is not gauged. The elementary SO_8 fields in the Lagrangian may be tied up in bound states (except the graviton), even at mass scales of order 10^{15} GeV, where the effective Lagrangian could be locally SU_5 symmetric. More will be said about an effective theory "derived" from $N = 8$ supergravity in section 8; its fate is not yet settled.

One may wish to speculate about a future unified theory of all interactions and all elementary particles that would resemble SO_8 supergravity but involve sacrificing some principle now held sacred, so that the notion of extended supergravity could be generalized. In such a hypothetical theory, an internal symmetry group G larger than SO_8 would be gauged by spin 1 bosons, and both the spin $\frac{3}{2}$ and spin $\frac{1}{2}$ fermions would be assigned to representations of G. It is then very natural to suppose that the spin $\frac{3}{2}$ fermions would belong to some basic representation of G and would include only color singlets, triplets and antitriplets. The spin $\frac{1}{2}$ particles would then presumably be assigned to a more complicated representation. These speculations are a major motivation for this review, as they were for ref. [6].

Let us discuss briefly two other rather simple models. The simple group SO_{10} contains $SU_5 \times U_1^r$ as a maximal subgroup. (The label r on the U_1 merely distinguishes it from other U_1's.) It is possible to put the $\bar{\mathbf{5}}$ and **10** of an SU_5 family together into an irrep of SO_{10}. The 16-dimensional spinor irrep of SO_{10} has a branching rule into $SU_5 \times U_1^r$ irreps,

$$\mathbf{16} = \mathbf{1}(-5) + \bar{\mathbf{5}}(3) + \mathbf{10}(-1)\,, \tag{3.4}$$

as can be found in table 43. There is a (possibly) important feature of (3.4). If a family of left-handed fermions is assigned to the **16**, then both the antiparticles and particles of a family are assigned to the same irrep in f_L. In fact, in SO_{10} there exists a group operation that exchanges the left-handed particles and antiparticles: the quarks are reflected with their antiquarks, the electron with positron, and the left-handed neutrino is exchanged with the SU_5 singlet in (3.4) [13]. This charge conjugation operation C may have great significance in (for example) coming to a systematic understanding of the symmetry breaking (and fermion masses). For example, in the SO_{10} theory, the mass matrix element connecting the SU_5 singlet $\bar{\nu}_L$ to the neutral lepton ν_L in the $\bar{\mathbf{5}}$ is even under C, but the C operation transforms the diagonal SU_5 singlet mass into the diagonal $|\Delta I^w| = 1$ neutrino mass. The ν_L mass will be small if the $I^w = 0$ mass **violates** C **maximally** (i.e., it is odd under C), so that the $\nu_L - \nu_L$ element is zero and the $\bar{\nu}_L - \bar{\nu}_L$ weak singlet term is huge. This is called a Majorana mass, since a $\bar{\nu}_L - \bar{\nu}_L$ mass term violates lepton number. The C operation in unified models is discussed in sections 6 and 9.

The third model, which we only mention for now, is based on E_6. The fermions are assigned again in families to the fundamental 27-dimensional irrep. In fact, it is one goal of this review to be able to treat the E_6 representation theory smoothly and without undo complication, and so we defer until later discussion of this example.

In each of these models, we need also to discuss the symmetry breaking and other details, which will be examimed later. Without further example or motivation, we now begin our plunge into the group-theoretical details that should ease those discussions.

There are, essentially, two methods for studying the embedding of color and flavor in a simple group G. The most direct method is to pick out the generators (or linear combinations of generators) of G that may be identified as the color and flavor generators. This method can be complicated in practice (until the short cuts are learned), but it provides so much knowledge about the structure of G that it is often obvious how to proceed to a study of symmetry breaking, masses, and so on. It is this method that is developed in some detail in sections 4–7, inclusive. The other method, advocated in ref. [6], requires listing the possible flavor and color structure of the fundamental irrep of G and then checking that the conjectured embedding gives the correct behavior of the group generators. Embedding through an irrep is completely general and is easily implemented. Both methods provide insights into unified models. The remainder of this section contains an informal restatement of the arguments and results of embedding through the fundamental irrep, and at the same time, provides an elementary introduction to the simple Lie algebras. A more formal and complete discussion of this approach can be found in ref. [6], in section II and the Appendix there.

According to the Cartan classification, which has been proven complete in numerous ways, there are four infinite series of simple Lie algebras: the algebra A_n generates the group SU_{n+1}, which is the group of transformations that leaves invariant the scalar products of vectors in an $(n+1)$-dimensional complex vector space; B_n generates SO_{2n+1}, the group of transformations leaving invariant scalar products of vectors in a real $(2n+1)$-dimensional vector space; C_n generates Sp_{2n}, the group of transformations that leaves invariant a skew-symmetric quadratic form in a real $2n$-dimensional vector space; and D_n generates SO_{2n}, which is analogous to B_n, but has a different spinor and root structure. (As warned earlier we often use the group name even when the algebraic properties are all that is needed.) In addition there are five exceptional algebras, denoted by the symbols, G_2, F_4, E_6, E_7 and E_8, where the subscripts denote the rank. The exceptional groups leave invariant certain forms with octonions. The Jacobi identity of Lie algebras requires that the commutation relations $[A, B] = C$ can be realized by associative matrices, which means that $(AB)C = A(BC)$. Since matrices of octonions can be associative only in special cases, the exceptional algebras are restricted in number. In building Yang–Mills theories, the representation theory plays the central role; although the geometrical properties of octonions are not required for analyzing theories based on exceptional groups, they can be helpful for some calculations and may also suggest a way to transcend the confines of Yang–Mills theories [49].

The embedding through the fundamental representation is done as follows. If there is an embedding of the form,

$$G \supset G^{fl} \times SU_3^c\,, \tag{3.5}$$

then the fundamental representation of G has a branching rule,

$$\mathbf{f} = \sum_i (\mathbf{x}_i, \mathbf{y}_i^c), \tag{3.6}$$

where $\mathbf{x}_i$ is an irrep of G^{fl} and $\mathbf{y}_i^c$ is an irrep of SU_3^c. Moreover, the adjoint irrep, which is the representation of dimension equal to the order of the group with matrix elements made from the structure constants, is always in the tensor product $\mathbf{f} \times \bar{\mathbf{f}}$. (This statement is true of any nontrivial irrep.)

This embedding procedure was worked out in ref. [6] only for the cases where the irreps of SU_3^c in (3.6) for some irrep are restricted to the set $\mathbf{1}^c$, $\mathbf{3}^c$ and $\bar{\mathbf{3}}^c$. If, for example, the spin $\frac{1}{2}$ fermions were restricted to leptons, quarks and antiquarks, then there would be at least one such irrep. The discussion can be made complete, due to the following theorem: If any irrep of G has a branching rule (3.6) with $\mathbf{y}_i^c$ restricted to the set $\mathbf{1}^c$, $\mathbf{3}^c$ and $\bar{\mathbf{3}}^c$, then the fundamental irrep must also satisfy the same restriction. (See the Appendix of ref. [6] for a formal proof.) Once the color content of the fundamental irrep is constructed, the color content of other irreps is found from their branching rules, which are easily derived from tensor products. Although there is no direct evidence that spin $\frac{1}{2}$ fermions other than quarks and leptons exist, it may be somewhat artificial to require no other color states from, say, 10 GeV to the unification mass. In fact, if the known quarks and leptons are fit into one irrep, then it is quite usual to expect other color states. (For exceptions, see the SO_{4n+6} models in section 8.) Nevertheless, the procedure of embedding this way is quite adequate, because embeddings where the fundamental irrep has higher color states seem quite awkward, and little appears to be gained by the new embeddings. A possible exception may be found in models based on E_8 [50], where the fundamental irrep is the adjoint, which must have an $\mathbf{8}^c$.

We now examine the simple groups, discussing their fundamental irreps, the embedding of color and flavor, the calculation of the generators, and a number of special features of the groups and their field theories. The features mentioned here can all be derived with the techniques discussed in section 4 to the end.

The unitary groups, SU_n. The fundamental irrep of SU_n (or A_{n-1}) is the $\mathbf{n}$, which is n dimensional. The most general branching rule (3.6) with the $\mathbf{1}^c$, $\mathbf{3}^c$, $\bar{\mathbf{3}}^c$ color restriction is, obviously, of the form,

$$\mathbf{n} = (\mathbf{n}_1, \mathbf{1}^c) + (\mathbf{n}_3, \mathbf{3}^c) + (\mathbf{n}_{\bar{3}}, \bar{\mathbf{3}}^c), \tag{3.7}$$

where the $\mathbf{n}_i$ are irreps of the flavor group, and $n = n_1 + 3n_3 + 3n_{\bar{3}}$. The identification of the flavor group requires a study of the group generators. The flavor group of an SU_n theory is nonsemisimple because of the U_1 factor(s). The adjoint of SU_n is computed from the $\mathbf{n}$ from the tensor product,

$$\mathbf{n} \times \bar{\mathbf{n}} = \mathbf{Adj} + \mathbf{1}, \tag{3.8}$$

where the adjoint $\mathbf{Adj}$ has dimension $n^2 - 1$. It is quite trivial mathematically to work out (3.8) from the general form of (3.7), but the most interesting cases physically appear to be those with $n_{\bar{3}} = 0$ and $n_3 = 1$, so that $n_1 = n - 3$. Then the adjoint has the branching rule of the form,

$$\begin{aligned}\mathbf{Adj} &= [(\mathbf{n-3}, \mathbf{1}^c) + (\mathbf{1}, \mathbf{3}^c)] \times [(\overline{\mathbf{n-3}}, \mathbf{1}^c) + (\mathbf{1}, \bar{\mathbf{3}}^c)] - (\mathbf{1}, \mathbf{1}^c) \\ &= (\mathbf{n-3} \times \overline{\mathbf{n-3}}, \mathbf{1}^c) + (\mathbf{n-3}, \bar{\mathbf{3}}^c) + (\overline{\mathbf{n-3}}, \mathbf{3}^c) + (\mathbf{1}, \mathbf{8}^c).\end{aligned} \tag{3.9}$$

The product $\mathbf{n-3}\times\overline{\mathbf{n-3}}$ should contain flavor generators only, just as the $(\mathbf{1},\mathbf{8}^c)$ generators of QCD are flavor singlets. Thus $\mathbf{n-3}$ must be the fundamental irrep of SU_{n-3}, or else that product would have components that do not transform as flavor-group generators and G^{fl} would not be as large as possible. The extra singlet in $\mathbf{n-3}\times\overline{\mathbf{n-3}}$ implies a U_1 factor, so (leaving U_1 values implicit)

$$\begin{aligned} &SU_n \supset SU_{n-3}\times U_1\times SU_3^c, \qquad \mathbf{n}=(\mathbf{n-3},\mathbf{1}^c)+(\mathbf{1},\mathbf{3}^c) \\ &\mathbf{Adj}(SU_n)=(\mathbf{Adj}(SU_{n-3}),\mathbf{1}^c)+(\mathbf{1},\mathbf{1}^c)+(\mathbf{1},\mathbf{8}^c)+(\mathbf{n-3},\bar{\mathbf{3}}^c)+(\overline{\mathbf{n-3}},\mathbf{3}^c)\,. \end{aligned} \tag{3.10}$$

The color content of the other irreps of SU_n can be derived from tensor products of the $\mathbf{n}$ with itself; this procedure generates all irreps of SU_n. The "basic" irreps ("basic" has a special significance when the Dynkin labeling of irreps is discussed) are obtained from $(\mathbf{n}\times\mathbf{n}\times\cdots\times\mathbf{n})_a$. In particular $(\mathbf{n}^k)_a$ is an irrep of dimension $\binom{n}{k}$ (a binomial coefficient). It is not difficult to multiply out $(\mathbf{n}^k)_a$ with $\mathbf{n}$ given in (3.10) to find that the color states in each of these irreps are restricted to the set $\mathbf{1}^c$, $\mathbf{3}^c$ and $\bar{\mathbf{3}}^c$. Any other irrep must have pieces in $(\mathbf{n}^j)_s$, and must have higher color terms. For example, $(\mathbf{n}\times\mathbf{n})_s$ is an irrep of dimension $n(n+1)/2$, containing a piece $[(\mathbf{1},\mathbf{3}^c)\times(\mathbf{1},\mathbf{3}^c)]_s=(\mathbf{1},\mathbf{6}^c)$.

The irreps of SU_3 fall into three categories, distinguished by their *triality*. The quantum numbers in one triality class never coincide with those of another triality class. In the Eightfold Way SU_3, the electric charge is a representation vector label. In the triality zero irreps, such as the **8**, **10**, **27**, etc., the electric charge eigenvalue is an integer; triality one irreps, such as the $\mathbf{3}$, $\bar{\mathbf{6}}$, **15**, etc., have charge eigenvalues $-\frac{1}{3}$ plus an integer; and triality two irreps ($\bar{\mathbf{3}}$, **6**, $\overline{\mathbf{15}}$, etc.) have charge eigenvalues $+\frac{1}{3}$ plus integer. The concept of triality for SU_3 generalizes to that of n-ality for SU_n, where it again means that the quantum numbers of the representation vectors in one class never coincide with those in another class. As a generalization (but trivial for SU_n), it is possible to distinguish irreps of **any group** in an analogous way; it is convenient then to speak of **congruency classes** [9, 51]. The number of congruency classes is mentioned in this section, and it is more fully exploited in section 5. Irreps of SU_n fall into n congruency classes, distinguished by n-ality.

The Georgi–Glashow SU_5 model is an example of (3.10), with $n=5$, $\bar{\mathbf{5}}=(\mathbf{5}^4)_a$, and $\mathbf{10}=(\mathbf{5}\times\mathbf{5})_a$. The group theory is often implemented by writing the **10** as an antisymmetric tensor A_{ij} with $i,j=1,\ldots,5$. This procedure is perfectly adequate for SU_5, but it should be noted that the **10** is a basic irrep, which means that the antisymmetric pair, ij, can in a sense be replaced by a single label. This kind of simplification of the group theory will be emphasized in the next few sections. The distinction among "basic", "simple", "fundamental", and "composite" irreps is made in section 5.

Other examples of (3.7), one with $n_{\bar{3}}=0$ and $n_3>1$, and finally, $n_{\bar{3}}>0$ and $n_3>1$ are worked out in ref. [6], and all three of these embeddings of QCD in SU_n are listed in table 3, cases 1, 2 and 3.

The tensor product of the SU_n adjoint irrep with itself contains an adjoint symmetrically; the existence of a completely symmetric d_{ijk} symbol that couples three identical adjoints (that is, three vector bosons) into an invariant occurs only for the SU_n ($n>2$) groups. (The coupling through the structure constants is completely antisymmetric. Also note that $SO_6\sim SU_4$ has a d symbol.) The existence of the d_{ijk} symbol can be a problem, because fermions loops can contribute to the triangle anomaly and destroy the theory's renormalizability. If $\mathbf{f}_L$ is self conjugate, the uncontrollable pieces cancel out automatically; in that case the theory is called vectorlike [36]. If $\mathbf{f}_L$ is complex, then it must be reducible and of a special form for the cancelation to take place. The SU_5 model with $\mathbf{f}_L=\bar{\mathbf{5}}+\mathbf{10}$ is an example where the anomaly is cancelled. However, it is noteworthy that in SU_5, there is no complex **irrep** to which $\mathbf{f}_L$ may be assigned without parity violation of the strong or electromagnetic interactions or the existence of massless charged particles. More generally, in unified models it is not difficult to avoid the triangle anomaly problem. In theories where the C operator exists as a group operation, there is never any difficulty. In any event the SO_{4n+6} and E_6 groups, although they have complex irreps, do not have a d_{ijk} symbol, so there is no anomaly problem; the adjoint occurs only in the antisymmetric part of adjoint times adjoint, as is required by the commutation relations. We place most emphasis on "flavor chiral" models, which are defined by the requirement that $\mathbf{f}_L$ is complex and the triangle anomaly problem is absent [36].

The orthogonal groups. Cartan's classification indicates two kinds of orthogonal algebras $B_n(SO_{2n+1})$ and $D_n(SO_{2n})$. In both cases (SO_m, m even or odd) the defining or vector irrep $\mathbf{m}$ is m dimensional and the adjoint irrep is constructed irreducibly from $(\mathbf{m}\times\mathbf{m})_a$. However, it is noteworthy that not all irreps of SO_m can be obtained from products of $\mathbf{m}$ with itself, but there are spinor irreps, just as fundamental as the $\mathbf{m}$, from which all irreps can be formed. The reason that $\mathbf{m}^k$ cannot contain a spinor is essentially the same reason that a $\mathbf{3}$ cannot be constructed from any number of $\mathbf{8}$'s in SU_3; the congruence of irreps of SO_m prohibit it [9, 51]. The irreps of B_n fall into two congruence classes; the irreps of D_n fall into four classes. For SO_{2n+1}, spinor times spinor has ordinary irreps only, but spinor times ordinary has spinor-like irreps only. For D_n, two classes are spinor like. This classification is helpful when deriving tensor products and is described later in detail.

The most notable difference in the spinor irreps of B_n and D_n is that B_n has only one simple spinor, of dimension 2^n and always self conjugate, but D_n has two inequivalent simple spinors, each of dimension 2^{n-1}. When n is odd, the spinors are complex and conjugate to each other, and when n is even, they are self conjugate and inequivalent. Thus SO_{10}, SO_{14}, $SO_{18}, \ldots$ all have complex spinor irreps that can be used for $\mathbf{f}_L$ to make a flavor chiral theory.

In carrying out the embedding of SU_3^c in SO_m, the $\mathbf{m}$ plays the fundamental role. If the spinors satisfy the color restriction, then so does $\mathbf{m}$, but the converse is not necessarily true. Thus, our procedure is to embed through $\mathbf{m}$ and then construct the branching rule for the spinors.

The $\mathbf{m}$ is a self conjugate irrep, which implies that the branching rule must also be self conjugate with just as many $\mathbf{3}^c$ as $\bar{\mathbf{3}}^c$, so it has the form,

$$\mathbf{m} = (\mathbf{n}_1, \mathbf{1}^c) + (\mathbf{n}_3, \mathbf{3}^c) + (\bar{\mathbf{n}}_3, \bar{\mathbf{3}}^c), \qquad m = n_1 + 6n_3\,, \tag{3.11}$$

where $\mathbf{n}_1$ in (3.11) must be self conjugate. Calculating the generators from $(\mathbf{m}\times\mathbf{m})_a$, we find easily that the flavor group must be $SO_{n_1}\times SU_{n_3}\times U_1$. If $n_3>1$, then the simple spinor has higher color states. The case that has attracted the most interest is $n_3 = 1$, in which case,

$$\begin{aligned}
&SO_m \supset SO_{m-6}\times U_1\times SU_3^c\\
&\mathbf{m} = (\mathbf{m-6}, \mathbf{1}^c) + (\mathbf{1}, \mathbf{3}^c) + (\mathbf{1}, \bar{\mathbf{3}}^c)\\
&\mathrm{Adj}(SO_m) = (\mathrm{Adj}(SO_{m-6}), \mathbf{1}^c) + (\mathbf{1}, \mathbf{1}^c) + (\mathbf{1}, \mathbf{8}^c) + (\mathbf{m-6}, \mathbf{3}^c) + (\mathbf{m-6}, \bar{\mathbf{3}}^c)\,.
\end{aligned} \tag{3.12}$$

Actually (3.12) is not a maximal subgroup; it is a subgroup of $SO_{m-6}\times SO_6$ with $\mathbf{m} = (\mathbf{m-6}, \mathbf{1}) + (\mathbf{1}, \mathbf{6})$.

There are general formulas for various tensor products and branching rules for some SO_m irreps, but the discussion becomes simpler with the Dynkin notation to be developed. Consequently, we defer that discussion to section 8.

The symplectic groups. Symplectic groups have had few applications in particle physics, with the exceptions of Sp_2, which is isomorphic to SU_2, and Sp_4, which is isomorphic to SO_5. There does appear to be a deep connection between SO_{2n+1} and Sp_{2n} [52], in addition to the fact that they have the same number of generators. The symplectic groups have played almost no role in model building, but we mention a few properties. The fundamental irrep is the **2n**, and all other irreps are found in $(\mathbf{2n})^k$, $k = 2, 3, \ldots$. The product $(\mathbf{2n}\times\mathbf{2n})_s$ gives irreducibly the adjoint, and the singlet is found in $(\mathbf{2n}\times\mathbf{2n})_a$. The embedding with the color restriction is found in table 3, case 6. There are two congruence classes, which coincide with "real" and "pseudoreal".

The exceptional groups. The exceptional groups, especially the E series, have received considerable attention from model builders. Perhaps one of the main motivations is that if one of the exceptional groups were part of a complete theory, then there might be a chance of going beyond the Yang–Mills construction and "explaining" why it is the correct choice of gauge group. At present, such hope is speculation, although the phenomenology of the E_6 models being studied appears quite adequate.

G_2 is not large enough to contain QCD and any piece of the flavor group, since it is only rank 2. Moreover, it is not likely to show up as a relevant subgroup because the fundamental irrep **7** has the branching rule into SU_3 irreps, $\mathbf{1}+\mathbf{3}+\bar{\mathbf{3}}$; the adjoint **14** branches to $\mathbf{8}+\mathbf{3}+\bar{\mathbf{3}}$. Thus if $G_2 \supset SU_3^c$, sets of $\mathbf{1}^c$, $\mathbf{3}^c$ and $\bar{\mathbf{3}}^c$ are likely to appear with equal flavor quantum numbers. Since G_2 has self conjugate irreps only, it is not likely to make a good flavor group either. Nevertheless, it is oftentimes helpful to refer to the properties of G_2 when studying general properties of other algebras and their representations. There is only one congruence class for G_2 irreps.

F_4 has rank 4 and 52 generators; its irreps are all self conjugate, so all F_4 theories are vectorlike. The embedding of color can be done through the maximal subgroup $SU_3\times SU_3^c$, with the fundamental irrep **26** branching to $(\mathbf{8},\mathbf{1}^c)+(\mathbf{3},\mathbf{3}^c)+(\bar{\mathbf{3}},\bar{\mathbf{3}}^c)$; then no other irreps of F_4 satisfy the color restriction. For other (inequivalent) embeddings of SU_3^c in F_4, there are no irreps of F_4 with color states restricted to $\mathbf{1}^c$, $\mathbf{3}^c$ and $\bar{\mathbf{3}}^c$. F_4 is investigated in some detail later on because it is a subgroup of E_6. There is only one congruence class for F_4 irreps.

E_6 has rank 6 and 78 generators, and holds a prominent position in this review. It is the only exceptional group with nonself conjugate irreps, so it is the only exceptional group for which a flavor-chiral theory is possible. Moreover, it is a generalization of SO_{10} (SO_{10} may be classified as "E_5"), which is itself a generalization of SU_5 (SU_5 may be classified as "E_4"), so the chain of subgroups $E_6 \supset SO_{10}\times U_1^t \supset SU_5\times U_1^r\times U_1^t$ contains many of the interesting flavor-chiral theories. E_6 irreps have triality.

The only maximal subgroup decomposition of E_6 containing QCD as an explicit factor is $E_6 \supset SU_3\times SU_3\times SU_3^c$, and the fundamental 27-dimensional irrep has the branching rule, $\mathbf{27} = (\bar{\mathbf{3}},\mathbf{3},\mathbf{1}^c)+(\mathbf{3},\mathbf{1},\mathbf{3}^c)+(\mathbf{1},\bar{\mathbf{3}},\bar{\mathbf{3}}^c)$. Of course, depending on the symmetry breaking hierarchy from E_6 to $U_1^{em}\times SU_3^c$, it may be that other maximal subgroups, such as F_4, $SO_{10}\times U_1$, or $SU_2\times SU_6$, could play a more significant role. Nevertheless, in each case the same generators of E_6 can be chosen to generate SU_3^c, and the same diagonal generator can be identified with Q^{em}; it is in this sense that these other subgroups do not give a new embedding of color. In each case, the **27** has three $\mathbf{3}^c$, three $\bar{\mathbf{3}}^c$, and nine $\mathbf{1}^c$, with the same distribution of electric charges. We have ignored the embeddings where the **27** has one color singlet and a color octet, since obtaining a sensible looking lepton sector is awkward. The analysis of E_6 and its subgroups is carried out in great detail in sections 6–9.

E_7 has rank 7 and 133 generators. All of its irreps are self conjugate and either real or pseudoreal. The **56** is the only irrep that can have color restricted to $\mathbf{1}^c$, $\mathbf{3}^c$ and $\bar{\mathbf{3}}^c$, and the embedding $E_7\supset SU_6\times SU_3^c$ exhibits the embedding through the branching rule, $\mathbf{56} = (\mathbf{20},\mathbf{1}^c)+(\mathbf{6},\mathbf{3}^c)+(\bar{\mathbf{6}},\bar{\mathbf{3}}^c)$. Although models based on E_7 have been suggested, they are not currently popular because those theories are vectorlike, so an explanation of the weak neutral current is tangled up with a detailed understanding of the symmetry breaking.

The fundamental irrep of E_8, which has rank 8, is its adjoint, so for any embedding of SU_3^c there must be color states beyond $\mathbf{1}^c$, $\mathbf{3}^c$ and $\bar{\mathbf{3}}^c$. It is the only group for which the adjoint cannot be constructed from some simpler irrep. The $SU_3 \times E_6$ decomposition of E_8 of the 248-dimensional adjoint is $(\mathbf{8}, \mathbf{1}) + (\mathbf{1}, \mathbf{78}) + (\mathbf{3}, \mathbf{27}) + (\bar{\mathbf{3}}, \overline{\mathbf{27}})$, so the 248 can be arranged to have one $\mathbf{8}^c$, 78 $\mathbf{1}^c$, 27 $\mathbf{3}^c$ and 27 $\bar{\mathbf{3}}^c$, which is "almost" without higher color states, and it has three families of **27**. Again, all E_8 theories are vectorlike, and the irreps of E_8 are all in a single congruence class. E_7 and E_8 will both receive some attention in section 8.

4. Dynkin diagrams

There exists much literature describing the theory and application of group theory, and especially Lie group theory, to problems in physics. Often fairly small Lie groups such as SU_2, SU_3, SO_4 or SO_5 are of interest; for these, one may get the impression that deriving the commutation relations, representations and their content, the subgroup structure, tensor products, vector coupling coefficients, recoupling coefficients, and so on is straightforward, but tedious. However, unified models are based on much larger groups (rank four or more), so there appears to be cause for anxiety over the algebraic complexity that must be faced in deriving those results. Our purpose in the next three sections is to set up the analysis of simple groups in such a way that some of the chores encountered in unified model building are not as tedious as might be expected.

Simple groups, their representations, and subgroup structure have been studied by many mathematicians and physicists, but perhaps the most convenient approach for dissecting Yang–Mills theories based on large simple groups is the one introduced by Dynkin in the early fifties [9]. Of course, it is widely understood that field theory is an especially convenient formalism for describing symmetries, and putting internal quantum number labels on field operators in a Yang–Mills theory is conceptually simple. The reason why Dynkin's labeling is so useful is that the action of a generator, or a tensor operator, on a state is designated a little more conveniently for big groups than, say, by tensor labels; it is easier to do the bookkeeping. For example, an important step in exploring a theory is identifying the color and flavor quantum numbers of a field that transforms as a component of some representation of the Yang–Mills group G, and finding out how it is transformed through its interactions with the vector bosons in the theory. This problem is reduced to **integer** arithmetic, and constant reference to the commutation relations is not needed except through the Dynkin diagram. Our account is brief, informal, and descriptive, with emphasis on the results needed to derive the many tables. Rather than proving theorems, we use examples for guidance. The mathematics can be found elsewhere [9, 11, 12].

The maximum number of simultaneously diagonalizable generators of a simple Lie algebra G is called its rank l; the total number of linearly independent generators is called its dimension. A **simple** group has no invariant subgroups, except the whole group and the identity; analogously, a simple Lie algebra contains no proper ideals. A **semi-simple** algebra can be written as a direct sum of simple algebras. Excepting the study of subalgebras, we discuss simple algebras only; U_1 is not simple.

In the standard Cartan–Weyl analysis, the generators are written in a basis where they can be divided into two sets. The Cartan subalgebra, which is the maximal Abelian subalgebra of G, contains the l diagonalizable generators H_i,

$$[H_i, H_j] = 0, \qquad i, j = 1, \ldots, l\,; \tag{4.1}$$

and the remaining generators are written so they satisfy eigenvalue equations of the form

$$[H_i, E_\alpha] = \alpha_i E_\alpha, \qquad i = 1, \ldots, l. \tag{4.2}$$

The numbers α_i in (4.2) are structure constants of the algebra in the Cartan–Weyl basis. For each operator E_α, there are l numbers α_i that can be used to designate a point in an l-dimensional Euclidean space called root space. The term "root" refers to the fact that the root vector $(\alpha_1, \ldots, \alpha_l)$ is the solution to the eigenvalue equation (4.2). A fundamental problem of Lie algebra theory is to classify **all possible root systems** for algebras of each rank, consistent with the Jacobi identity, simplicity requirement, and the antisymmetry of the commutation relations. The most elegant statement of the solution of that problem is given in terms of Dynkin diagrams, which is the topic of this section.

It is well known from the description of symmetries in quantum mechanics (or from Lie algebra theory) that the generators H_i and E_α of the symmetry group G are characterized by their actions on Hilbert space vectors, which describe the states of a physical system. A complete set of states that are necessarily interconnected by the E_α forms the basis of an irreducible representation (irrep) of G. Again, the solution to the problem of finding **all possible irreps** of any simple group G is stated most elegantly in terms of Dynkin diagrams. Irreps and Hilbert spaces are referred to rather often in this section, since most physicists think in terms of representations rather than abstract operators. However, the complete solution to the problem of enumerating irreps and their structures will not be discussed until section 5.

The physical significance of the diagonalizability of the H_i is that the Hilbert space vectors $|\lambda\rangle$ in an irrep can be labeled by the l eigenvalues (quantum numbers) of H_i: $H_i|\lambda\rangle = \lambda_i|\lambda\rangle$. Note that λ is not a complete set of labels, since we need to know which irrep of G the set λ belongs to, and in addition, often there are several Hilbert space vectors in an irrep labeled by the same set λ, so further labels of the Hilbert space vectors are usually needed. The set λ is called the **weight** of the representation vector. The solution to the problem of finding the complete list of weights of an irrep will be given in the next section; for problems in unified model building, the labeling problem can be solved by hand, so a general solution is not needed very badly . . . fortunately, since there are unsolved cases and cases where known solutions are not easily used. The labeling problem is raised again in section 7.

The only rank 1 simple algebra is SU_2; it is conventional to select J_3 to be diagonal: $J_3|m\rangle = m|m\rangle$. The commutation relations, $[J_3, J_\pm] = \pm J_\pm$, have the same structure as (4.2), so the root vectors are the one-component vectors $+1$ and -1. Thus, from the example of angular momentum theory, it should be suggestive that the E_α are ladder operators: if $|\lambda\rangle$ is an eigenfunction of the H_i with eigenvalues λ_i ($i = 1, \ldots, l$), then according to (4.2), $E_\alpha|\lambda\rangle$ is proportional to the state $|\lambda + \alpha\rangle$, which has eigenvalues $\lambda_i + \alpha_i$, assuming the proportionality constant is nonzero. In detail from (4.2),

$$H_i(E_\alpha|\lambda\rangle) = E_\alpha H_i|\lambda\rangle + \alpha_i E_\alpha|\lambda\rangle = (\lambda_i + \alpha_i)(E_\alpha|\lambda\rangle). \tag{4.2a}$$

Just as in angular momentum theory, the precise form of the linear relation between $E_\alpha|\lambda\rangle$ and $|\lambda + \alpha\rangle$ depends on phase and normalization conventions, and calculation of the proportionality is done in the same way: if $\langle\lambda|\lambda'\rangle = \delta_{\lambda\lambda'}$, then the normalization of $E_\alpha|\lambda\rangle$ can be computed in a stepwise fashion from $\langle\lambda|E_{-\alpha}E_\alpha|\lambda\rangle = \langle\lambda|E_\alpha E_{-\alpha}|\lambda\rangle - \langle\lambda|[E_\alpha, E_{-\alpha}]|\lambda\rangle$, ignoring further labels on the states, where $[E_\alpha, E_{-\alpha}]$ are further commutation relations that need to be specified, and $E_\alpha^\dagger = E_{-\alpha}$. The idea is to start the calculation with a state such that $E_\alpha|\lambda\rangle = 0$ for appropriate α. (The equation for SU_2 is $J_+|j, j\rangle = 0$.) The solution to the general problem of finding the state analogous to $|j, j\rangle$, which is called the state with **highest weight**, for any irrep of any simple group is, again, given in terms of Dynkin diagrams!

Let us finish writing down the commutation relations. If α is a root, then so is $-\alpha$. The commutator of E_α and $E_{-\alpha}$ is in the Cartan subalgebra

$$[E_\alpha, E_{-\alpha}] = \alpha^i H_i \,, \tag{4.3}$$

where the components α^i are related to the α_i by a metric tensor, to be discussed. (In SU_2, $[J_+, J_-] = 2J_3$, so the metric tensor is $\frac{1}{2}$.) The remaining commutation relations have the form,

$$[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta} \qquad (\alpha + \beta \neq 0)\,, \tag{4.4}$$

if $\alpha + \beta$ is a root, and $N_{\alpha,\beta} = 0$ if $\alpha + \beta$ is not a root. For example, 2α is never a root. (Although $[E_\alpha, E_\alpha] = 0$, it takes some effort to prove that 2α is not a root.)

If we knew all possible ladder operators E_α (or more simply, all possible root systems) for each l, then we would know all possible simple Lie algebras, since the root vectors determine the structure of the Lie algebra. In the Cartan–Weyl basis, the nonzero roots of a simple algebra are nondegenerate; there is only one E_α for each α. The Cartan subalgebra may be viewed as being associated with an l-fold degenerate zero root, by comparing (4.2) and (4.1). The derivation of the root systems uses the commutation relations (in abstract form), the Jacobi identity, and clever manipulation to derive the crucial constraints on the roots, some of which were mentioned above [53]. The possible root systems are summarized by the Dynkin diagrams. In order to motivate the results we review SU_3, since SU_2 is a little too trivial.

SU_3 has rank two, so the root vectors are defined in a two dimensional Euclidean space. The eight generators of SU_3 satisfy commutation relations [54],

$$[F_i, F_j] = \mathrm{i} f_{ijk} F_k \,, \tag{4.5a}$$

where the structure constants f_{ijk} are antisymmetric in the indices; in the Gell-Mann basis, the nonzero f_{ijk} are

$$\begin{aligned} &f_{123} = 1, \qquad f_{147} = f_{516} = f_{246} = f_{257} = f_{345} = f_{637} = \tfrac{1}{2}\,, \\ &f_{458} = f_{678} = \tfrac{1}{2}\sqrt{3}\,. \end{aligned} \tag{4.5b}$$

The structure constants are normalized so that $\Sigma_{kl} f_{ikl} f_{jkl} = 3\delta_{ij}$.

Equation (4.5) does not give the generators of SU_3 in a Cartan–Weyl basis; a Cartan–Weyl basis can be chosen as follows: select $H_1 = F_3$ and $H_2 = F_8$ to be the members of the Cartan subalgebra, and $\sqrt{2} I_\pm = F_1 \pm \mathrm{i}F_2$, $\sqrt{2} U_\pm = F_4 \pm \mathrm{i}F_5$, and $\sqrt{2} V_\pm = F_6 \pm \mathrm{i}F_7$ to be the E_α. The two component root vectors are derived from (4.5):

$$\begin{aligned} &[H_i, I_\pm] = \alpha^{\pm}_{\mathrm{I}i} I_\pm, && \alpha^{\pm}_{\mathrm{I}} = \pm(1, 0); \\ &[H_i, U_\pm] = \alpha^{\pm}_{\mathrm{U}i} U_\pm, && \alpha^{\pm}_{\mathrm{U}} = \pm(-\tfrac{1}{2}, \tfrac{1}{2}\sqrt{3})\,; \\ &[H_i, V_\pm] = \alpha^{\pm}_{\mathrm{V}i} V_\pm, && \alpha^{\pm}_{\mathrm{V}} = \pm(\tfrac{1}{2}, \tfrac{1}{2}\sqrt{3})\,. \end{aligned} \tag{4.6}$$

The well-known root diagram of SU_3 showing the vectors $\alpha^{\pm}_{\mathrm{I}}$, $\alpha^{\pm}_{\mathrm{V}}$ and $\alpha^{\pm}_{\mathrm{U}}$ is reviewed in table 4, because we want to refer to a number of features of it that generalize to any simple algebra. The axes of the root diagram are labeled by $I_3 = H_1$ and $Y = (2/\sqrt{3})H_2$, but we have fixed the scales of H_1 and H_2 to be equal in accordance with the normalization condition on the structure constants. (The metric, or Killing form, is a unit matrix.) The ladder operators of SU_3 raise and lower the values of H_1 and H_2 by the root vectors, and the action of H_i on a Hilbert space vector is to measure the value of H_i for that state, but not change the weight. Currents and field operators in a field theory have analogous effects.

Let us emphasize several features of table 4: all the nonzero roots have equal length (in a simple algebra there are nonzero roots of at most two different lengths); if α is a root so is $-\alpha$, which is completely general; the angle between any pair of roots is an integer multiple of 60°. (The angle between any pair of roots in any simple algebra is greatly restricted.)

It is useful to have a basis for the Euclidean root space, of course, but this basis should be chosen with foresight and cleverness or else the general discussion of the generators and representation theory will be a mess. Specifically for SU_3, the choice of Y and I_3 is a simple choice only because there is no difficulty in visualizing the changes in those quantum numbers due to the ladder operators. Thus, the raising operator associated with α_V^+ raises I_3 by $\frac{1}{2}$ and Y by 1. The difficulty with this basis is that for larger groups, the action of a generator usually corresponds to a complicated change of coordinates; of course this is only a difficulty in practice, but it can be quite nontrivial to deal with it. Even the orthonormality of the I_3–Y coordinate system is no real advantage for algebras of rank four or more, because visualizations must be replaced by analytical methods. A better choice of basis is rank(G) specially chosen **roots** that span the space, because then the coordinate changes due to ladder operators should have a simpler description. It is clear from table 4 that it is not possible to have an orthonormal basis for the root space of SU_3 if the basis vectors are chosen from the roots, but this is not a practical problem. A specially chosen set of roots, called **simple roots**, contain in a simple way all the information about the other roots and even about the quantum number labels of the representation vectors, which can be derived from the Dynkin diagram; a detailed picture of the l-dimensional root space is not needed.

The simple roots are identified as follows. Write the roots in any Cartesian basis; half of the nonzero roots are positive, which is defined by the requirement that the first nonzero component of a positive root in that basis is positive. Then find the positive roots that cannot be written as a linear combination with positive coefficients of the other positive roots. There are only rank(G) such roots and they are linearly independent; that defines a set of simple roots. Of course, different selections of coordinate systems will lead to a different set of simple roots; however, the members of any set are equivalent to those of any other in the sense that there is a Weyl reflection of the root diagram that relates the two sets. A Weyl reflection does not change the relative lengths or angles among the roots.

The positive roots of SU_3 from (4.6), if the coordinates are written as (Y, I_3), for example, are α_I^+, α_V^+ and α_U^+. Since $\alpha_V^+ = \alpha_I^+ + \alpha_U^+$, the simple roots in this coordinate system are α_I^+ and α_U^+. The lengths of α_I^+ and α_U^+ are the same and the angle between them is 120°. Note that for any other Cartesian coordinate system placed on table 4, the simple roots will still have the same lengths and an angle of 120° between them. That is a defining feature of the SU_3 root system; *the length and angle relations among the simple roots completely characterize any simple Lie algebra.* It will be shown soon how to compute all the roots from the simple roots; they are obviously linear combinations of the simple roots. The difference of two simple roots is not a root.

Dynkin has pointed out how the simple roots of any simple Lie algebra can be represented graphically by a two-dimensional diagram (called a Dynkin diagram). Such a diagram must indicate the relative lengths of the simple roots and the angle between each pair of simple roots. Each simple root is denoted by a dot on a diagram. In many algebras, all nonzero simple roots have the same length, so each root is designated by an open dot "○". No simple algebra has nonzero simple roots with three or more different lengths. In those cases where simple roots come with two different lengths, the longer roots are denoted by open dots ○, and the shorter roots by filled-in dots, ●.

The angle between a pair of simple roots is denoted by lines connecting the corresponding dots: no line means the angle is 90°; one line means 120°; two lines means 135°; and three lines means 150°. These are all the possibilities. Moreover, the detailed analysis shows that the ratio of lengths of two roots connected by three lines must be $\sqrt{3}$, the ratio of lengths of two roots connected by two lines is $\sqrt{2}$, and the ratio of lengths of roots connected by one line is unity. There is no constraint if two roots are not connected by lines. Returning to the SU_3 example, we see from the above conventions that its Dynkin diagram is ○—○.

The roots systems of all simple Lie algebras and the numbering conventions for the simple roots are shown in table 5. For much of the following analysis, the Dynkin diagram contains all the information we need, so we do not have to refer back to the commutation relations. Any other diagrams than those in table 5 do not give a root system of a simple Lie algebra; for example, the Jacobi identity may fail for some set of generators. An algebraic proof that table 5 is an exhaustive list of all simple root systems of simple Lie algebras can be found in ref. [12].

The simple roots do not form an orthonormal basis for root space; if a Dynkin diagram has several pieces that are completely unconnected, then the algebra is semisimple, and each connected piece is simple. The matrix that keeps track of the nonorthogonality is called the Cartan matrix. It has elements

$$A_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)\,, \tag{4.7}$$

where the vector α_i is the ith simple root (not component as in earlier equations), numbered as indicated in table 5. The matrix A can be read off of the Dynkin diagram; the Cartan matrices of simple algebras are listed in table 6. Each element of the Cartan matrix is an integer. The importance of these matrices [or the Dynkin diagram that serves as a mnemonic for (4.7)] will become obvious. The Cartan matrix is both useful for working out the entire root system and is indispensible for representation theory.

According to (4.2) the eigenvalue of H_i acting on a nonzero $E_\alpha|\lambda\rangle$ is $\lambda_i + \alpha_i$. Since α_i is a vector in root space, it is convenient to supplement the root space with points corresponding to the possible weights of the representation vectors. For example, a field operator $\phi^\dagger(\lambda)$ transforms the vacuum (with zero quantum numbers) to a state with quantum numbers λ_i: $\phi^\dagger(\lambda)|0\rangle \propto |\lambda\rangle$. Thus the weight λ can be represented as a vector in the same Euclidean space where the root vectors live.

The simple roots form a basis of root space, so each root or weight vector Λ in root space can be written as a linear combination of the simple roots α_i,

$$\Lambda = \sum_i \bar{\lambda}_i \frac{2}{(\alpha_i, \alpha_i)} \alpha_i\,, \tag{4.8}$$

where (α_i, α_i) is the length-squared of the ith simple root. The longer simple roots are normalized conventionally to a length-squared of 2, so for groups with all simple roots of the same length (SU_n, SO_{2n}, E_n), the $2/(\alpha_i, \alpha_i)$ factor is unity. Otherwise it simplifies some crucial formulas later on. The coordinates $[\bar{\lambda}, \ldots, \bar{\lambda}_l]$ give the vector Λ in the **dual** basis. The **Dynkin** components a_i of Λ (to which the $\bar{\lambda}_i$ are dual) are defined by

$$a_i = \frac{2(\Lambda, \alpha_i)}{(\alpha_i, \alpha_i)} = \sum_j \bar{\lambda}_j \frac{2}{(\alpha_j, \alpha_j)} A_{ji}\,, \tag{4.9}$$

where the last equality follows from (4.7) and (4.8). Equation (4.9) is one of the most important formulas in this review.

This selection of basis is made worthwhile by the following crucial theorem: *for any weight or root, the Dynkin labels a_i in (4.9) are integers.* (These integer coordinates were known to Weyl, but it was Dynkin who first exploited them for all their worth.) For SU_2 $A_{11} = 2$, so $a = 2m$ is always an integer; $\Lambda = m\alpha$ is the dual vector, where m is the magnetic quantum number.

Weight space is an l-dimensional Euclidean space with a scalar product that has already been used in constructing the Cartan matrix. The scalar product of any two weights can be written in the dual basis (4.8), in a mixed basis (which is extremely convenient), or in the Dynkin basis (4.9); writing out the scalar product in several forms,

$$\begin{aligned}(\Lambda, \Lambda') = (\Lambda', \Lambda) &= \sum_{ij} \bar{\lambda}'_i \frac{2}{(\alpha_i, \alpha_i)} (\alpha_i, \alpha_j) \frac{2}{(\alpha_j, \alpha_j)} \bar{\lambda}_j = \sum_{ij} \bar{\lambda}'_i \frac{2}{(\alpha_i, \alpha_i)} A_{ij} \bar{\lambda}_j \\ &= \sum_i \bar{\lambda}_i a'_i = \sum_i \bar{\lambda}'_i a_i = \sum_{ij} a'_i G_{ij} a_j\,,\end{aligned} \tag{4.10}$$

where G_{ij} is a symmetric metric tensor with elements,

$$G_{ij} = (A^{-1})_{ij} \frac{(\alpha_j, \alpha_j)}{2}\,, \tag{4.11}$$

which, comparing with (4.10) or (4.9), shows that $\bar{\lambda}_i = G_{ij} a_j$. The matrices G for each simple group can be computed from the Cartan matrices in table 6, and are listed in table 7.

The members of the Cartan subalgebra have zero roots; any linear combination Q of the H_i is also a member of the Cartan subalgebra. The diagonal generator Q is characterized by an *axis in root space.* (For SU_3, the I_3 and Y axes are marked on table 4.) A state $|\Lambda\rangle$ with weight Λ is an eigenstate of Q, $Q|\Lambda\rangle = Q(\Lambda)|\Lambda\rangle$, with eigenvalue $Q(\Lambda)$,

$$Q(\Lambda) = (Q, \Lambda)\,. \tag{4.12}$$

[Compare with (4.2).] The Q axis can be normalized to suit some set of conventions; for example, for Q^{em}, the charge of the positron is $+1$. The most convenient form for computing (4.12) is to put the Q axis in the dual basis with components $[\bar{q}_1, \ldots, \bar{q}_l]$, and the weight Λ in the Dynkin basis so that

$$Q(\Lambda) = \sum_i \bar{q}_i a_i\,. \tag{4.13}$$

The general tactic for setting up the $\bar{q}_i$ for each charge Q in unified models is to establish first that there is a Q in the Yang–Mills theory satisfying relevant physical properties (that is physics), complete the explicit embedding of Q by working backward for one irrep to get the $\bar{q}_i$ from (4.13) (choice of coordinates), and then compute $Q(\Lambda)$ for the other weights from (4.12).

The electric charge, I_3^w, and the two diagonal generators of SU_3^c are in the Cartan subalgebra of a unifying group, so finding these axes is an important step in extracting the physical content of a theory. Although the embedding of $SU_2^w \times U_1^w \times SU_3^c$ in G usually must be found before this problem can be solved (see section 6), it is easy to understand the form of its solution from an elementary discussion of the Eightfold Way, where the diagonal generators of SU_3 correspond to the third component of strong isospin and strong hypercharge. We may select the root $(2\ -1)$ (in the Dynkin basis) to correspond to the isospin raising operator, as in table 4. Recalling the elementary results of SU_3, we know that the I_3 value of $(2\,-1)$ is $+1$, of $(1\ 1)$ is $+\frac{1}{2}$ and of $(1\ -2)$ is $+\frac{1}{2}$. Thus any root or weight Λ with labels $(a_1 a_2)$ has I_3 value,

$$I_3(a_1a_2) = \bar{I}_3 \cdot \Lambda = a_1/2, \qquad \bar{I}_3 = \tfrac{1}{2}[1\ \ 0]\,. \tag{4.14}$$

As emphasized above, it is convenient to give the axes in the dual basis because of the ease of computing scalar products. In a similar fashion, the hypercharge axis, normalized in the usual way is $\bar{Y} = \frac{1}{3}[1\ \ 2]$, with

$$Y(a_1a_2) = \bar{Y} \cdot \Lambda\,. \tag{4.15}$$

Finally, the electric charge $Q^{em} = I_3 + Y/2$ can be computed by adding axes, so $\bar{Q}^{em} = \frac{1}{2}[1\ \ 0] + \frac{1}{6}[1\ \ 2] = \frac{1}{3}[2\ \ 1]$, with $Q^{em}(a_1a_2) = \bar{Q}^{em} \cdot \Lambda$. This trivial example is a prototype for more complicated ones in section 6.

The simple root α_i has Dynkin components A_{ij}, which is just the ith row of the Cartan matrix, but it is perhaps easier to remember the Dynkin diagram. For example, the two simple roots of SU_3 in the Dynkin basis are $\alpha_1 = (2\ -1)$ and $\alpha_2 = (-1\ \ 2)$, as labeled on table 4. The third positive root is $\alpha_1 + \alpha_2 = (1\ \ 1)$.

Deriving the entire root system from the simple roots requires knowing which linear combinations of simple roots are roots. The following theorems make this exercise simple. (1) In a Cartan–Weyl basis the zero root has a degeneracy equal to the rank of the algebra, and the remaining nonzero roots are **nondegenerate**. (There is just one generator per nonzero root.) (2) If α is a root, then so is $-\alpha$. The roots that can be formed by linear superpositions of simple roots with nonnegative coefficients are positive roots in some basis, so we need list only that half of the nonzero roots, which number $\frac{1}{2}[\dim(G) - \mathrm{rank}(G)]$. (3) There is a "highest root" from which the remaining roots may be computed by subtracting simple roots. The highest root for each algebra is listed in table 8, along with the number of simple roots that must be subtracted before reaching the simple roots. This procedure is slightly easier than building up the positive roots from the simple roots. The root diagrams for the rank 2 and rank 3 algebras are given in table 9, where the positive roots of SU_5 and SO_{10} are also listed. We note that the algebras B_2 and C_2 are isomorphic, as can be seen by comparing their Dynkin diagrams; D_2 is isomorphic to the semisimple algebra $SU_2 \times SU_2$ (it is not simple); and D_3 and A_3 have the same Dynkin diagrams, so they too are isomorphic.

The root system is the list of eigenvalues of the Cartan subalgebra when acting on the adjoint irrep, and the rules for working out the root diagram are a special example of the rules needed for obtaining the Dynkin labels weight-by-weight for any irrep. The reader can figure out the rules for the adjoint from table 9; a more formal statement with many examples is found in the next section.

5. Representations

The representation theory of simple algebras is summarized elegantly in terms of Dynkin diagrams. The problems (with solutions) to be discussed in this section are: (1) enumeration of all the irreps of each algebra; (2) the weight system of each irrep; (3) dimensionality and index of each irrep; and (4) the computation of tensor products. We begin with a few preliminary comments and a recapitulation of some results of section 4.

If there is an n-dimensional irrep $\mathbf{n}$ of G, then there exists an n-dimensional Hilbert space on which the generators of a simple G act faithfully. The n-by-n matrices representing the generators acting on this Hilbert space satisfy the commutation relations of the algebra. Each vector in the Hilbert space may be (partially) labeled by the eigenvalues of the diagonalized generators in the Cartan subalgebra. Each possible set of eigenvalues corresponds to a point in root space, and is called the weight vector of the Hilbert space vector of the irrep. (For example, the vector $|j, m\rangle$ of the j representation of SU_2 has weight m ($\lambda = m\alpha$) in the dual basis, or $a = 2(\lambda, \alpha)/(\alpha, \alpha) = 2m$ in the Dynkin basis.) Thus, a weight is a vector in an l-dimensional Euclidean space [l = rank (G)], and as mentioned in section 4, the points in root space can be supplemented with the set of points that can correspond to weights of the irreps. We refer to this lattice of points as weight space. Perhaps some confusion will be avoided by stating that it should be clear from context whether "vector" refers to a vector in Hilbert space or to the vector in the weight space being used to label a Hilbert space vector.

The structure constants f_{ijk} can be made into matrices with elements $(f_i)_{jk}$. These matrices satisfy the commutation relations of the algebra, which is proven from the Jacobi identity, and form an irrep called the adjoint (or regular) representation with dimension equal to the dimension of the algebra. Each ladder operator is specified by a unique root α, and is represented by a matrix that replaces the vector $|\mathbf{r}, \lambda\rangle$ by a vector $|\mathbf{r}, \lambda + \alpha\rangle$, up to normalization and phase. At the end of section 4, it was shown how to represent the roots in terms of the Dynkin basis, which is a set of **integer coordinates** in root space, and the Cartan subalgebra by **axes** in root space. There is a **highest root** from which the remaining roots may be derived by subtracting simple roots. That prescription is a special case of the general prescription for obtaining the weight system of any irrep.

The problem is to designate each irrep of any simple Lie group and its weight system in the Dynkin basis. The incredible convenience of the Dynkin basis is, as stated before, that each component of any weight λ of any irrep of any simple algebra is an integer; that is, $a_i = 2(\lambda, \alpha_i)/(\alpha_i, \alpha_i)$ is always an integer if λ is a weight. This is the generalization of the theorem in SU_2 theory that $2m$ is always an integer.

In a given irrep, some weights may be degenerate, that is, several vectors in Hilbert space may have the same weight, so that distinguishing them calls for additional labels. However, there are always some weights in an irrep that are not degenerate and one of those weights uniquely defines (up to an equivalence transformation) the irrep. That weight is called the **highest weight** Λ, and it is defined in a way similar to the way we defined the highest root in table 8, which is the highest weight of the adjoint representation. The highest weight is, as any weight, a set of integers when written in the Dynkin basis. Dynkin has shown the following crucial theorem [9 and references there]: *The highest weight of an irrep can be selected so that the Dynkin labels are non-negative integers.* Moreover, *each and every irrep is uniquely identified by a set of integers* $(a_1, \ldots, a_l)$ $(a_i \geq 0)$, *and each such set is a highest weight of one and only one irrep.*

The complete set of weights for each irrep can be derived from the highest weight and the Dynkin diagram, just as the root system was derived from the highest root in section 4, except now we shall be much more explicit. Once the weight system is in hand, the Dynkin labels can then be converted into the eigenvalues of a convenient set of diagonal generators from (4.13), including, for example, I_3^w, Y^w, Q^{em}, and the color charges. Needless to say, having a simple method for extracting the quantum number content of an irrep is an important technical tool for studying unified models. (Subgroup designations are another solution to this problem, and will be discussed later.) The irreps listed in the tables are designated by their highest weight in the Dynkin basis. As a matter of notation, a comma is not inserted between a_i and a_{i+1} when all $a_i \leq 9$.

The highest weight Λ has the same meaning as it does in SU_2, where the Hilbert space vector with highest weight in the j irrep is the one that is annihilated by J_+: $J_+|j, j\rangle = 0$. In the general case, any ladder operator belonging to a positive root acting on the Hilbert space vector $|\mathbf{r}, \Lambda\rangle$, with highest weight Λ in irrep $\mathbf{r}$, annihilates it. The vectors with lower weights are obtained by acting on the vector with highest weight with the lowering operators; in SU_2, $J_-|j, j\rangle$ is proportional to $|j, j-1\rangle$, until $J_-|j, -j\rangle = 0$. The significance of the simple roots is that the entire weight system of an irrep can be obtained in an orderly fashion. There even exists a general formula in terms of the highest weight for the maximum number of simple roots that can be subtracted from the highest weight; in SU_2, it is well known that the positive root, which is 2 in the Dynkin basis, can be subtracted $2j$ times, giving an irrep of dimension $2j+1$. By now the reader should not find it very suprising that so many results of SU_2 theory are generalized to **any** simple algebra so trivially.

As another important example showing how the well-known SU_2 results generalize, consider the following calculation. Suppose that $|\mathbf{r}, \Lambda\rangle$ has the highest weight of the irrep $\mathbf{r}$; then $|\mathbf{r}, \Lambda - \alpha\rangle$ is proportional to $E_{-\alpha}|\mathbf{r}, \Lambda\rangle$, where from (4.3) $E_{-\alpha}|\mathbf{r}, \Lambda\rangle$ has the normalization

$$|E_{-\alpha}|\mathbf{r}\Lambda\rangle|^2 = \langle \mathbf{r}, \Lambda|[E_\alpha, E_{-\alpha}]|\mathbf{r}, \Lambda\rangle = (\Lambda, \alpha)\langle \mathbf{r}, \Lambda|\mathbf{r}, \Lambda\rangle = (\Lambda, \alpha)\,. \tag{5.1}$$

This first step in a stepwise calculation of the matrix elements (up to a phase) of the ladder operators has many applications. (The first step has no labeling problems, but subsequent ones may.) One application, discussed in section 9, is to the analysis of mass matrices.

We need to know if a weight Λ' belongs to a vector (or vectors) in an irrep with highest weight Λ. The relevant theorems, stated rather informally here, simplify the procedure, and with a little practice the weight systems of quite large irreps can be produced rapidly by hand. The examples should be viewed as problems for practice.

Suppose that the highest weight has components $\Lambda = (a_1 \ldots a_l)$ in the Dynkin basis. Then the ith simple root can be subtracted from Λ a_i times. This means subtracting the ith row of the Cartan matrix from Λ a_i times, which corresponds to the fact that $E_{-\alpha_i}$, $(E_{-\alpha_i})^2, \ldots, (E_{-\alpha_i})^{a_i}$ (no sum on i) do not annihilate the state of highest weight, but $(E_{-\alpha_i})^{a_i+1}$ does. Applying this rule to the (a_1) irrep of SU_2, where the simple root in the Dynkin basis is the vector 2, we find the weight system $(a_1), (a_1 - 2), \ldots, (-a_1)$, so we identify $a_1 = 2j$ and prove that the dimensionality is $a_1 + 1 = 2j + 1$, if we ignore the possibility that some of the weights might be degenerate. (Of course the SU_2 weights are not degenerate.) Not many other rules are needed to deal with the general case.

The **level** of a weight of an irrep is the number of simple roots that must be subtracted from the highest weight to obtain it; it is a unique number, even if a weight can be obtained by several different orderings of the lowering operators. The general formula for the highest level of an irrep of highest weight $\Lambda = (a_1 \ldots a_l)$, which is called the height of the irrep, is

$$T(\Lambda) = \sum_i \bar{R}_i a_i\,, \tag{5.2}$$

where $\bar{R}$ is the level vector, written in the dual basis so that taking the scalar product in (5.2) is a single sum. The level vectors for all the simple algebras are listed in table 10. The components $\bar{R}_i$ are given by $\bar{R}_i = 2\sum_j (A^{-1})_{ij}$ [55]. Note that the level of a weight with components a'_i is $\frac{1}{2}\sum \bar{R}_i(a_i - a'_i)$, where a_i is a component of the highest weight of the irrep. The rest of the weight diagram is constructed by repeating the procedure of the previous paragraph at each level.

A useful theorem for constructing the weight systems by hand is Dynkin's theorem that the weight diagram must be "spindle shaped". This means two things: the number of weights, counting degenerate weights separately, at the kth level must be equal to the number of weights at the $(T(\Lambda)-k)$th level. (For example, the single weight at the $T(\Lambda)$th level must be nondegenerate, since the highest weight at level zero is.) Moreover, the number of weights at the $(k+1)$th level is greater than or equal to the number at the kth level, for $k < T(\Lambda)/2$.

For the low-dimensional irreps, the degeneracy problem is easily resolved by imposing the spindle shape on the weight diagram; in the examples discussed below this is so. For very large irreps a computer is helpful in finding the weight system [10], and in that instance there exists an iterative formula (the Freudenthal recursion formula) for computing the degeneracy $n_{\Lambda'}$ of a weight Λ' in terms of the degeneracies at the lower levels and the highest weight Λ:

$$[(\Lambda+\delta, \Lambda+\delta) - (\Lambda'+\delta, \Lambda'+\delta)]n_{\Lambda'} = 2 \sum_{\substack{\text{positive roots}\\ \text{positive } k}} n_{\Lambda'+k\alpha}(\Lambda'+k\alpha, \alpha)\,, \tag{5.3}$$

where $\delta = (1, 1, \ldots, 1)$ in the Dynkin basis is half the sum of all positive roots and $n_{\Lambda'+k\alpha}$ is a degeneracy already computed. For example, the degeneracy of the zero weight of the **8** of SU_3 is $n_0[((22), (22)) - ((11), (11))] = 2\sum(\alpha, \alpha) = 12$, where the sum is over positive roots, the length-squared of (22) is 8, and the length squared of (11) and the other positive roots is 2. [Recall (4.10).] Since there are 3 positive roots, $n_0 = 2$.

As familiar examples, we construct the weight diagrams for a few SU_3 irreps. From (5.2) and table 10, we find that the (1 0) has height 2. The first root can be subtracted from (1 0) to give (1 0) − (2 −1) = (−1 1); then the second root (−1 2) can be subtracted from (−1 1) to give (0 −1). Since there are no more positive Dynkin labels and no higher level was reached by subtracting a simple root several times, we are done; (1 0) is the highest weight for the **3**. Following the above rules, it is easy to find the following weight systems:

$$\begin{array}{ccc}
(0\ \ 1) & (2\ \ 0) & (1\ \ 1) \\
(1\ -1) & (0\ \ 1) & (-1\ \ 2)\quad(2\ -1) \\
(-1\ \ 0) & (-2\ \ 2)\quad(1\ -1) & (0\ \ 0)\quad(0\ \ 0) \\
 & (-1\ \ 0) & (1\ -2)\quad(-2\ \ 1) \\
 & (0\ -2) & (-1\ -1),
\end{array} \tag{5.4}$$

which the reader will recognize as $\bar{\mathbf{3}}$, **6** and **8**, respectively. The weight system of the **8** is, of course, just the root system for SU_3 already worked out in table 9. Note that (0 0) is obtained two different ways, which only places an upper limit on the degeneracy, but it must have degeneracy two, as required by the spindle-shape theorem, or the fact that the **8** is the adjoint of a rank 2 group, or by the Freundenthal recursion relation.

To show a less trivial example, the weights of the (0 0 0 0 1) of SO_{10}, which is the 16 dimensional spinor, and of the (1 0 0 0 0 0) of E_6, which is 27 dimensional and has height 16, are given in table 11, where the simple root subtracted from each weight is referred to explicitly; it is educational to check table 11. Note that the **16** and **27** have no degenerate weights, and if Λ is a weight, then $-\Lambda$ is not, which characterizes nonself-conjugate ("complex") irreps. The calculation of the electric charge, and so on, is carried out in section 7, although we have enough machinery to do it now.

The weight system of an irrep falls into one of three categories, called complex, real, or pseudoreal. These categories are characterized as follows [56]: A nonself-conjugate (which we call, somewhat loosely, complex) irrep has the feature that the weights at level k are not the negatives of those at level $T(\Lambda)-k$. For example, the 0th and second levels of the **3** of SU_3 are $(1\ 0)$ and $(0\ -1)$, respectively, so the **3** is complex. If the Dynkin diagram has a nontrivial symmetry axis, as do A_n, D_n and E_6, then there may be complex irreps. A necessary condition for $(a_1 \ldots a_l)$ to be complex is that the highest weight not be invariant under the reordering induced by the reflection. In the case of SU_{l+1}, any irrep satisfying $(a_1 \ldots a_l) \neq (a_l \ldots a_1)$ is complex, and any E_6 irrep satisfying $(a_1a_2a_3a_4a_5a_6) \neq (a_5a_4a_3a_2a_1a_6)$ is also complex. In the case of D_n, the condition $(a_1 \ldots a_{n-2}a_{n-1}a_n) \neq (a_1 \ldots a_{n-2}a_na_{n-1})$ is a sufficient condition only for n odd. All irreps of SO_{4n} are self conjugate.

Self-conjugate irreps have weight systems satisfying the requirement that the weights at level k are the negatives of those at level $T(\Lambda)-k$. If $T(\Lambda)$ is even then the representation matrices can always be brought to real form by a group transformation and the irrep is called real. However, if $T(\Lambda)$ is odd, but the irrep is self conjugate, then the irrep cannot be brought to real form and the irrep is called pseudoreal (but not complex, even though it is, in fact, complex). The $\frac{1}{2}$ integer spin irreps of SU_2 are pseudoreal. These results are tabulated in table 12; they can be confirmed by comparing the results with table 10 for the level vectors, and will be further characterized when we discuss tensor products.

As mentioned in section 3, the terms "basic", "simple", "fundamental", and "composite" are often used to describe various irreps of simple groups. (Needless to say, the choice of the term that fits a given definition tends to vary with author.) A basic irrep is one with highest weight satisfying $\Sigma\, a_i = 1$; it has only one nonzero Dynkin label. A simple irrep is a basic irrep where the nonzero a_i corresponds to an endpoint of the Dynkin diagram. All other irreps can be constructed from tensor products of the simple ones; there is one simple irrep from which all other irreps can be constructed. For example, in the orthogonal groups the vector and spinors are both simple. All irreps can be constructed from tensor products of one of the spinors, but the spinors cannot be constructed from products of the vector. The fundamental irrep is the simple irrep used to do the embedding in section 3. The composite irreps are those with highest weight satisfying $\Sigma\, a_i \geq 2$. The simple irreps from which all other irreps can be constructed by tensor products are listed in table 13, along with their dimensionalities and their Dynkin designations. Each weight of all of those irreps is nondegenerate, with the exceptions of the twofold zero weight of the **26** of F_4 and the eightfold zero weight of the **248** of E_8. Note that E_8 is the only simple algebra for which the adjoint is the fundamental irrep.

The next problem we discuss is finding the dimensionality of an irrep from the famous Weyl formula [12]:

$$N(\Lambda) = \prod_{\substack{\text{positive}\\ \text{roots}}} \frac{(\Lambda+\delta, \alpha)}{(\delta, \alpha)}\,, \tag{5.5}$$

where $\delta = (1, 1, \ldots, 1, 1)$ in the Dynkin basis, Λ is the highest weight of the irrep, and α is a positive root. This formula has a very simple structure when the positive roots are written as sums of simple roots. If $\alpha = \Sigma\, \bar{\lambda}_i\alpha_i\, 2/(\alpha_i, \alpha_i)$, where α_i is a simple root, then from (4.8) and (4.10), $(\delta, \alpha) = \Sigma\, \bar{\lambda}_i$ and $(\Lambda + \delta, \alpha) = \Sigma\, \bar{\lambda}_i(a_i + 1)$. It is easy to look at the root diagram (see table 9) and immediately write down

$N(\Lambda)$, although for a group with many positive roots, the actual computation by hand can be cumbersome. (Thus we have listed the dimensions in the tables; in many cases the dimensions and indices were copied from ref. [57].) For example, for SU_3 the positive roots are α_1, α_2 and $\alpha_1+\alpha_2$, so

$$N(a_1a_2) = (1+a_1)(1+a_2)\left(\frac{2+a_1+a_2}{2}\right). \tag{5.6}$$

It is quite clear from (5.6) that the Dynkin labels provide a compact summary of the usual tensor notation: if symmetric traceless tensors are used for the representation space, then a_1 is the number of upper indices and a_2 is the number of lower indices.

The positive roots in table 9 for G_2 are, reading from right to left, and bottom to top, α_2, α_1, $\alpha_1+\alpha_2$, $\alpha_1+2\alpha_2$, $\alpha_1+3\alpha_2$, $2\alpha_1+3\alpha_2$, which have dual coordinates $\frac{1}{3}[0\ 1]$, $[1\ 0]$, $\frac{1}{3}[3\ 1]$, $\frac{1}{3}[3\ 2]$, $[1\ 1]$ and $[2\ 1]$, respectively, so we can immediately write down the dimension formula for irreps of G_2 as

$$N(a_1a_2) = (1+a_2)(1+a_1)\left(1+\frac{3a_1+a_2}{4}\right)\left(1+\frac{3a_1+2a_2}{5}\right)\left(1+\frac{a_1+a_2}{2}\right)\left(1+\frac{2a_1+a_2}{3}\right), \tag{5.7}$$

where the factors are in the same order as the roots above.

The SU_5 formula has 10 factors and can be written down by keeping in mind the structure of the root diagram:

$$\begin{aligned} N(a_1a_2a_3a_4) = {} & (1+a_1)(1+a_2)(1+a_3)(1+a_4)\left(1+\frac{a_1+a_2}{2}\right)\left(1+\frac{a_2+a_3}{2}\right)\left(1+\frac{a_3+a_4}{2}\right) \\ & \times\left(1+\frac{a_1+a_2+a_3}{3}\right)\left(1+\frac{a_2+a_3+a_4}{3}\right)\left(1+\frac{a_1+a_2+a_3+a_4}{4}\right). \end{aligned} \tag{5.8}$$

Thus, $N(0\ 1\ 0\ 1)$ is 45 and $N(0\ 0\ 1\ 1)$ is 40.

The second-order Casimir invariant is [58]:

$$C = f^i_{jk} f^j_{il} X^k X^l = H_i G_{ij} H_j + \sum_{\substack{\text{all}\\ \text{roots}}} E_\alpha E_{-\alpha}, \tag{5.9}$$

where f^i_{jk} are structure constants, X^k are generators, and C commutes with any generator X_i. The Racah formula for the eigenvalue of C for an irrep is easily derived by letting C act on the state with highest weight Λ and manipulating the sum on roots with (4.3):

$$C(\Lambda) = (\Lambda, \Lambda + 2\delta). \tag{5.10}$$

Oftentimes $C(\Lambda)$ is not an integer; moreover, in some applications such as in the renormalization group equations, another quantity called the index is more useful:

$$l(\Lambda) = \frac{N(\Lambda)}{N(\text{adj})} C(\Lambda), \tag{5.11}$$

where $N(\text{adj})$ is the dimension of the adjoint irrep (or the order of the group) and $l(\Lambda)$ is always an integer. (Do not confuse the index $l(\Lambda)$ of an irrep with $l = \mathrm{rank(G)}$.) The index is closely related to the lengths of the weights in the irrep Λ; (5.11) can be written [59]:

$$l(\Lambda)\cdot \mathrm{rank(G)} = \sum (\lambda, \lambda) \tag{5.11a}$$

where (λ, λ) is the length-squared of a weight in irrep Λ, following the conventions below (4.8), and the sum is over **all** weights in Λ.

The index gives the one loop contribution to the μ dependence of the couplings, $\mu\ \mathrm{d}g/\mathrm{d}\mu = \beta(g)$ [23]:

$$\beta(g) = \frac{-1}{16\pi^2}\left[\frac{11}{3}c(\text{vector}) - \frac{4}{3}c(\text{Dirac fermion}) - \frac{1}{6}c(\text{spinless})\right]g^3 + \cdots \tag{5.12}$$

where, as derived below, $c(\ldots)$ is the index of the representation to which the $(\ldots)$ particles are assigned. The factor $\frac{4}{3}$ should be replaced by $\frac{2}{3}$ if (two-component) Majorana spinors are used. In the usual notation, $c(\ldots)$ is defined by [23]

$$c(\ldots)\,\delta_{ab} = \mathrm{Tr}(T_a T_b)\,, \tag{5.13}$$

where T_a $[a = 1, \ldots, N(\mathrm{adj})]$ is the ath generator in the $(\ldots)$ representation. In this same notation, the Casimir invariant (5.9) is

$$C\delta_{ij} = \sum_a (T^a T^a)_{ij}\,, \tag{5.14}$$

where $i, j = 1, \ldots, N(\Lambda)$; (5.13) and (5.14) provide the identification $c(\Lambda) = l(\Lambda)$.

The index for the SU_3 irrep $(a_1 a_2)$ is

$$l(a_1 a_2) = \tfrac{1}{12}N(a_1 a_2)\,(a_1^2 + 3a_1 + a_1 a_2 + 3a_2 + a_2^2)\,, \tag{5.15}$$

so that $l(1\ \ 1) = 6$ for the **8** and $l(1\ \ 0) = 1$ for the **3**. These results are substituted into (5.12) to obtain the famous QCD formula for the β function in the one loop approximation: $\beta(g) = -g^3(33 - 2n_f)/(24\pi^2) + \mathrm{O}(g^5)$, where n_f is the number of quark flavors. The indices of the irreps are listed in the tables.

Tensor products of irreps of a simple Lie algebra are reducible into a direct sum of irreps. Thus, the product $\mathrm{R}_1 \times \mathrm{R}_2$ of the irreps R_1 and R_2 can be written as

$$\mathrm{R}_1 \times \mathrm{R}_2 = \sum_i \mathrm{R}_i, \tag{5.16}$$

where a given irrep may occur several times in the sum.

The tensor product $\mathrm{R}_1 \times \mathrm{R}_2$ can (in principle) be computed as follows: find each weight vector a_α of R_1 and b_β of R_2, where $\alpha = 1, \ldots, N(\mathrm{R}_1)$ and $\beta = 1, \ldots, N(\mathrm{R}_2)$ (N is the dimension); form the $N(\mathrm{R}_1)\,N(\mathrm{R}_2)$ weights $a_\alpha + b_\beta$; find the highest weight, which is the weight $a_\alpha + b_\beta$ with the largest value $\bar{R}\cdot(a_\alpha + b_\beta)$ ($\bar{R}$ in table 10); calculate the weight diagram of the irrep with that highest weight and remove those weights from the set $\{a_\alpha + b_\beta\}$; find the highest weight in the remaining set; subtract the weight system of that irrep; ... and so forth. This method is cumbersome, even on a computer, but it does illustrate the important point that the reduction of a tensor product can be done in weight space.

The methods used for obtaining the products in the tables are reasonably systematic and simple, but not very illuminating. Nevertheless, it is almost inevitable that the tables will fall just short of your needs, so a description of the methods is useful:

(1) Highest weights: The highest weight in $(a_1 \dots a_l) \times (b_1 \dots b_l)$ is $(a_1 + b_1, \dots, a_l + b_l)$. There is also a rule for computing the irrep of second highest weight: subtract from the highest weight in the product the minimal sum of simple roots that connect together a nonzero component a_i in the highest weight of R_1 with a nonzero component b_j in the highest weight of R_2. As examples: if a_1 and b_1 are nonzero, subtract the first simple root from the highest weight; in E_6 $(1\ 0\ 0\ 0\ 0\ 0) \times (0\ 0\ 0\ 0\ 0\ 1)$ has highest weight $(1\ 0\ 0\ 0\ 0\ 1)$ and the second highest weight is obtained by subtracting $\alpha_6 + \alpha_3 + \alpha_2 + \alpha_1$ from $(1\ 0\ 0\ 0\ 0\ 1)$, which gives $(0\ 0\ 0\ 1\ 0\ 0)$. There are many cases where this rule gives several irreps in a product; try, for example, $\mathbf{8} \times \mathbf{8}$ in SU_3.

(2) The dimension and index sum rules:

$$N(R_1 \times R_2) = N(R_1) \cdot N(R_2) = \sum_i N(R_i)\,, \tag{5.17}$$

where $N(R_i)$ is the dimensionality of irrep R_i;

$$l(R_1 \times R_2) = l(R_1)\, N(R_2) + N(R_1)\, l(R_2) = \sum_i l(R_i)\,, \tag{5.18}$$

where $l(R_i)$ is the index (5.11) of R_i. The solution to these simultaneous Diophantine equations is very restrictive. There also exists a fourth order index sum rule [59].

(3) Crossing: If $R_1 \times R_2$ contains R_i, then $R_1 \times \bar{R}_i$ contains $\bar{R}_2$, etc., where $\bar{R}$ is the conjugate to R, which is inequivalent to R only if R is not self conjugate.

(4) $R \times \bar{R}$ always contains the singlet and adjoint representations. If R is self conjugate, then there are two cases, which depend on whether $(R \times R)$ contains the singlet in the symmetric or antisymmetric part of the product [56]. If $(R \times R)_s$ contains $\mathbf{1}$, then the adjoint representation is contained in the antisymmetric part, and it is possible to find a basis where the representation matrices are real. The defining representation of the orthogonal groups are real, as are the adjoint representations of all simple groups. If $(R \times R)_a$ contains $\mathbf{1}$, then the adjoint is contained in the symmetric part, and it is never possible to find a basis where the matrices are real, so the irrep is pseudoreal. The 2-dimensional irrep of SU_2 is the most famous example of a pseudoreal irrep. The defining representations of the symplectic groups are all pseudoreal, as is the $\mathbf{56}$ of E_7. Again, see table 12.

(5) Congruency constraints: If R_1 and R_2 are both real and not singlets, the irreps in $R_1 \times R_2$ are real or occur in $R + \bar{R}$ conjugate pairs. If R_1 and R_2 are pseudoreal, then the irreps in $R_1 \times R_2$ are real or occur in $R + \bar{R}$ conjugate pairs. If nontrivial R_1 is real and R_2 is pseudoreal, then the irreps in $R_1 \times R_2$ are pseudoreal or occur in conjugate pairs. If nontrivial R_1 is self conjugate and R_2 is complex, then all the irreps in $R_1 \times R_2$ are complex. These rules can be generalized to triality for SU_3, quadrality for SU_4, etc., and to triality for E_6. All irreps in $R_1 \times R_2$ have the same triality (or quadrality, etc.), which is equal to the sum of that for R_1 and R_2. With the concept of congruency class introduced in section 3, these constraints can be made tighter in some cases. If R_1 is in congruency class c_1 and R_2 is in c_2, then all irreps in $R_1 \times R_2$ are in congruency class $c_1 + c_2$. There is only one congruency class for G_2, F_4 and E_8, so

congruency gives no constraint. Congruency coincides with n-ality for SU_n and is defined by $c(R) = \sum ka_k \pmod n$, and triality for E_6 with $c = a_1 - a_2 + a_4 - a_5 \pmod 3$. It corresponds to "spinor" or "vector" in SO_{2n+1} so $c = a_n \pmod 2$; and to real or pseudoreal in Sp_{2n} [$c = a_1 + a_3 + a_5 + \cdots \pmod 2$] and in E_7 [$c = a_4 + a_6 + a_7 \pmod 2$]. The only case not discussed before is for SO_{2n}, where the congruency class is labeled by a two component vector, ($a_{n-1} + a_n \pmod 2$, $2a_1 + 2a_3 + \cdots + (n-2)a_{n-1} + na_n \pmod 4$). This always reduces to four classes for SO_{2n} [51].

Very rarely are these rules insufficient for giving a unique solution to the products (5.16) needed for unified model building. When they are inadequate, the confusion is often resolved by looking at simple pieces of the product, which is a form of intelligent guessing. Young tableaux can be used to solve (5.16) completely for some groups, but this method is not described here.

Let us illustrate these procedures with an almost nontrivial SU_5 example. Suppose the tensor product $\mathbf{40} \times \mathbf{10}$ is needed, where $\mathbf{40} = (0\ 0\ 1\ 1)$ and $\mathbf{10} = (0\ 1\ 0\ 0)$ from table 28. The quintality of $\mathbf{40} \times \mathbf{10}$ is -1, and the highest weight is $(0\ 1\ 1\ 1)$. The second highest weight is the highest weight minus α_2 minus α_3, or $(0\ 1\ 1\ 1) - (-1\ 2\ -1\ 0) - (0\ -1\ 2\ -1) = (1\ 0\ 0\ 2)$. Moreover, $\mathbf{10} \times \overline{\mathbf{5}}$ [$(0\ 1\ 0\ 0) \times (0\ 0\ 0\ 1)$] has a $\mathbf{5} = (1\ 0\ 0\ 0)$, so $(1\ 0\ 1\ 0)$ is likely to be in the product, and $\mathbf{10} \times \overline{\mathbf{10}}$ has a $(1\ 0\ 0\ 1)$ and a $(0\ 0\ 0\ 0)$, so the irreps $(1\ 0\ 0\ 2)$, which is the second highest weight, and $(0\ 0\ 0\ 1)$ are expected to be there. These subproducts can be found in table 29. The dimensionality of $(0\ 1\ 1\ 1) + (1\ 0\ 1\ 0) + (1\ 0\ 0\ 2) + (0\ 0\ 0\ 1)$ is $280 + 45 + 70 + 5 = 400$, which is necessary for the correct answer. The result can then be double checked with the index sum rule: $\mathbf{40} \times \mathbf{10}$ has index $40 \times 3 + 10 \times 22 = 340 = 266 + 24 + 49 + 1$, where the index for each irrep is given in table 28. Thus, we conclude,

$$\mathbf{40} \times \mathbf{10} = \overline{\mathbf{5}} + \overline{\mathbf{45}} + \overline{\mathbf{70}} + \overline{\mathbf{280}},$$

where we have defined all irreps with quintality 1 or 2 without bars and their conjugates with bars. A further confirmation comes from the crossing relation: since $\mathbf{40} \times \mathbf{10}$ contains a $\overline{\mathbf{5}}$, then $\mathbf{5} \times \mathbf{10}$ must contain a $\overline{\mathbf{40}} = (1\ 1\ 0\ 0)$, which it does. This line of reasoning is easily implemented for groups used for unified models; just look at all those tables.

6. Subgroups

A Yang–Mills theory with local symmetry G has a vector boson coupled to each current implied by G. Since G has $N(\mathrm{adj})$ generators, there are $N(\mathrm{adj})$ vector bosons in the theory. A fundamental problem of classifying and describing unified models is finding all the ways that the known interactions can be embedded in G. As reviewed in section 3, this requires finding all possible subgroups of G of the form,

$$G \supset G^{fl} \times SU_3^c \tag{6.1}$$

with G^{fl} generated by the color singlet generators of G, including $SU_2^w \times U_1^w$. The solution to the embedding problem is the spectrum of quantum numbers of the vector bosons and other particles of a Yang–Mills theory. This problem is solved explicitly in ref. [6] (in section II and the Appendix) with the proviso that the color content of at least one irrep, the fundamental one, of G be no greater than $\mathbf{1}^c$, $\mathbf{3}^c$ and $\overline{\mathbf{3}}^c$. (See table 3 and the discussion following eq. (3.5) for a summary.)

In this section the problem is studied in a slightly different fashion; of course, the answer is unchanged. Dynkin has given a method (or a set of methods) for classifying the maximal subalgebras of simple algebras. One reason for taking this approach is that it provides a detailed picture of the geometrical significance of the embedding in weight space.

Before starting the rather technical discussion, it may be helpful to look at the answer to the problem: the maximal subalgebras of the classical simple algebras of rank 8 or less are listed in table 14, and the maximal subalgebras of the exceptional algebras are listed in table 15.

A proper subalgebra G′ of G is denoted $G \supset G'$; G′ is a maximal subalgebra of G if there is no algebra G^* such that $G \supset G^* \supset G'$. The problem considered here is that of finding all maximal subalgebras of G, since the embedding of any other algebra in G can then be found in a stepwise fashion, $G \supset G' \supset G'' \ldots$. Tables 14 and 15 can be used to identify all the subgroup chains of a group of rank 8 or less that can possibly reduce to $U_1^{em} \times SU_3^c$. The nontrivial example explored in section 7 is a study of all subgroup chains of E_6 that reduce to $U_1^{em} \times SU_3^c$ with Q^{em} having $-\frac{1}{3}+$ integer eigenvalues on $\mathbf{3}^c$ states and integer eigenvalues for lepton states.

Maximal subalgebras of a simple algebra fall into two categories called (of course) regular (R) and special (S). Regular subalgebras can be obtained by looking directly at the root diagram (there exists a very simple algorithm for finding them); special subalgebras are not so obvious, and are discovered by comparing irreps of G with those of the candidate subgroups. They can be derived using methods identical to those in ref. [6].

Before getting involved in the intricacies of the general case, a review of the subgroups of SU_3 may serve as a useful orientation. By explicit examination of the SU_3 commutation relations (4.6), we can immediately find two different subalgebras. ("Different" means not equivalent; "not equivalent" means that there is no automorphism of G that relates the two embeddings.) The $SU_2 \times U_1$ subalgebra with generators F_1, F_2, F_3 and F_8, which is used in the Eightfold Way classification of hadrons, has the feature that its Cartan subalgebra, consisting of F_3 and F_8, can be identified with the Cartan subalgebra of SU_3; similarly the roots of $SU_2 \times U_1$ are a subset of the SU_3 roots. These features of the embedding of the root system of $SU_2 \times U_1$ in SU_3 are characteristic of a regular subalgebra. Note that here the subalgebra $SU_2 \times U_1 \subset SU_3$ is nonsemisimple: the U_1 factor is generated by an invariant Abelian ideal of the algebra of $SU_2 \times U_1$. U_1 factors are important in gauge theories, so nonsemisimple subalgebras of simple algebras must not be ignored.

The other maximal subalgebra, which is used in nuclear physics applications of SU_3, is of the special type: it is an SU_2 that is generated by $2F_2$, $2F_5$ and $2F_7$ [or equivalently $2F_3$, $2F_4$, $2F_5$, etc.; see the structure constants (4.5b)]. The **3** of SU_3 is projected onto the **3** of SU_2. (SU_2 irreps are labeled here by their dimension instead of their j value. Since the branching rules are to single-valued irreps only, the subgroup is actually SO_3, and not the covering group SU_2; recall section 2, where our use of group names for algebras was discussed.)

It is often useful to restate the embedding in terms of a projection matrix that takes the roots and weights of G onto the roots and weights of G′. This coordinatization of the weight space of G requires some conventions, since Weyl-reflected root diagrams should give an equivalent embedding, but with a projection matrix with different entries. Let us study $SU_3 \supset SU_2 \times U_1$. Since both algebras have rank two, we can project any root or weight onto a weight of $SU_2 \times U_1$ by a square matrix acting as

$$P(SU_3 \supset SU_2 \times U_1)\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} (b) \\ (u) \end{pmatrix}, \tag{6.3}$$

where $(a_1 a_2)$ is an SU_3 weight, (b) is $2I_3$, the weight of SU_2, and (u) is the eigenvalue of the U_1 generator. In the Eightfold Way P is given by

$$P(\mathrm{SU}_3 \supset \mathrm{SU}_2 \times \mathrm{U}_1) = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \tag{6.4}$$

with the U_1 generator normalized to three times the hypercharge; (6.4) can be easily verified from table 4. It is a general feature that P is always an integer matrix. This feature is automatic if G′ is semisimple; if there is a U_1 factor, its generator can be normalized to maintain this feature. If $P(\mathrm{SU}_3 \supset \mathrm{SU}_2 \times \mathrm{U}_1)$ is applied to the weight system of the **3**, [(1 0), (−1 1), (0 −1)], the resulting weights are (1)(1), (−1)(1) and (0)(−2), which can be identified as an SU_2 doublet with $Y = \frac{1}{3}$ and a singlet with $Y = -\frac{2}{3}$; the result is summarized as a branching rule,

$$\mathbf{3} = \mathbf{2}(1) + \mathbf{1}(-2). \tag{6.5}$$

This exercise should be carried out for other irreps of SU_3.

In the other embedding, $\mathrm{SU}_3 \supset \mathrm{SO}_3$, the subgroup has rank one, so the projection matrix is a one-by-two matrix:

$$P(\mathrm{SU}_3 \supset \mathrm{SO}_3) = (2\ 2). \tag{6.6}$$

The **8** of SU_3 is projected to the weights (4), (2), (0), (−2), (−4), (2) (0), (−2), which correspond to a **3** + **5** of SU_2; the branching rule for the **3** of SU_3 is **3** = **3**.

There is at least one projection that demonstrates explicitly an embedding; the existence of the projection matrix is trivial mathematically, since some of the roots of G, or linear combinations of them, must be identified with the roots of G′. However, for physical applications the projection matrices can be quite useful, for example, for finding the flavor and color quantum numbers of a field operator bearing a weight λ of the unifying group. The result of applying the projection P from the unifying group to the flavor (or color) subgroup on λ is its weight under flavor (or color); the quantum numbers of the operator can then be identified as described below (4.12). In other words, the projection matrices are needed (at least implicitly) whenever an explicitly labeled basis for the field operators is helpful. It is worthwhile to take a break from the formal development, to assert that $\mathrm{SU}_2 \times \mathrm{U}_1 \times \mathrm{SU}_3$ is indeed a subgroup of SU_5, and, as an example, to apply these techniques to finding the quantum numbers in the $\bar{\mathbf{5}} - \mathbf{10}$.

One convention for deriving the projection matrix is to require that the highest weight of an irrep of SU_5 is projected onto the highest weight of the $\mathrm{SU}_2 \times \mathrm{SU}_3$ representation. For example, (1 0 0 0) is the highest weight of the **5**, which branches to (**2**, **1**) + (**1**, **3**) of $\mathrm{SU}_2 \times \mathrm{SU}_3$ [recall (3.1)], and the color **3** state of highest weight can be defined to be the highest weight of this representation, so $P(1\ 0\ 0\ 0) = (0)(1\ 0)$. Similarly the highest weight of the $\bar{\mathbf{5}}$, (0 0 0 1), projects to (0)(0 1). The **10** with highest weight (0 1 0 0) branches to (**1**, **1**) + (**2**, **3**) + (**1**, $\bar{\mathbf{3}}$), so the highest weight of the subgroup is (1)(1 0), and the highest weight of the $\overline{\mathbf{10}}$, (0 0 1 0), must branch to (1)(0 1). With these conventions the projection matrix is

$$P(\mathrm{SU}_5 \supset \mathrm{SU}_2 \times \mathrm{SU}_3) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \tag{6.7}$$

where the U_1 factor will be analyzed separately, so (6.7) is a 3-by-4 matrix.

A family of left-handed fermions is assigned to a $\bar{\mathbf{5}}+\mathbf{10}$ of SU_5; (6.7) can be used to compute the $SU_2\times SU_3$ weights of those 15 states. In the $\bar{\mathbf{5}}$, we find

$$\begin{aligned}
&P(0\;0\;0\;1)=(0)(0\;1) \text{ is a charge } \tfrac{1}{3} \text{ antiquark singlet.}\\
&P(0\;0\;1\;-1)=(1)(0\;0) \text{ is a charge 0 member of a lepton doublet.}\\
&P(0\;1\;-1\;0)=(0)(1\;-1) \text{ is a charge } \tfrac{1}{3} \text{ antiquark singlet.}\\
&P(1\;-1\;0\;0)=(-1)(0\;0) \text{ is a charge } -1 \text{ member of a lepton doublet.}\\
&P(-1\;0\;0\;0)=(0)(-1\;0) \text{ is a charge } \tfrac{1}{3} \text{ antiquark singlet.}
\end{aligned}\tag{6.8}$$

These 15 equations also determine (6.7).

The weight diagram of the $\bar{\mathbf{5}}$ is quickly derived following the rules of section 5, and keeping the Dynkin diagram in mind. The diagonal quantum numbers I_3^w, Y^w and Q^{em} are described by axes, with their values for any weight given by scalar products; Y^w generates the U_1 factor in the subgroup. We can conclude immediately from the above assignments that

$$\begin{aligned}
&\bar{I}_3^w=\tfrac{1}{2}[0\;1\;1\;0] && \text{or } I_3=\tfrac{1}{2}(-1\;1\;1\;-1)\\
&\bar{Y}^w=\tfrac{1}{3}[-2\;1\;-1\;2] && \text{or } Y^w=\tfrac{5}{3}(-1\;1\;-1\;1)\\
&\bar{Q}^{em}=\bar{I}_3^w+\bar{Y}^w/2=\tfrac{1}{3}[-1\;2\;1\;1] && \text{or } Q^{em}=\tfrac{1}{3}(-4\;4\;-1\;1),
\end{aligned}\tag{6.9}$$

where the dual basis (useful for taking scalar products) and the Dynkin basis coordinatizations are given. The simple color roots are $(1\;1\;-1\;0)$ and $(0\;-1\;1\;1)$, as is found easily by applying the projection (6.7) to the SU_5 roots in table 9.

We may now calculate the weights and quantum numbers of the $\mathbf{10}$. Of course, the content of any other irrep of SU_5 can be analyzed in a similar fashion.

$$\begin{aligned}
&P(0\;1\;0\;0)=(1)(1\;0) \text{ is the u quark with } Q^{em}=\tfrac{2}{3} \text{ and } I_3^w=\tfrac{1}{2}\\
&P(1\;-1\;1\;0)=(0)(0\;1) \text{ is } \bar{u} \text{ with } Q=-\tfrac{2}{3} \text{ and } I_3^w=0\\
&P(-1\;0\;1\;0)=(1)(-1\;1) \text{ is u with } Q=\tfrac{2}{3} \text{ and } I_3^w=\tfrac{1}{2}\\
&P(1\;0\;-1\;1)=(-1)(1\;0) \text{ is d with } Q=-\tfrac{1}{3} \text{ and } I_3^w=-\tfrac{1}{2}\\
&P(-1\;1\;-1\;1)=(0)(0\;0) \text{ is } e^+ \text{ with } Q=1 \text{ and } I_3^w=0\\
&P(1\;0\;0\;-1)=(0)(1\;-1) \text{ is } \bar{u} \text{ with } Q=-\tfrac{2}{3} \text{ and } I_3^w=0\\
&P(-1\;1\;0\;-1)=(1)(0\;-1) \text{ is u with } Q=\tfrac{2}{3} \text{ and } I_3^w=\tfrac{1}{2}\\
&P(0\;-1\;0\;1)=(-1)(-1\;1) \text{ is d with } Q=-\tfrac{1}{3} \text{ and } I_3^w=-\tfrac{1}{2}\\
&P(0\;-1\;1\;-1)=(0)(-1\;0) \text{ is } \bar{u} \text{ with } Q=-\tfrac{2}{3} \text{ and } I_3^w=0\\
&P(0\;0\;-1\;0)=(-1)(0\;-1) \text{ is d with } Q=-\tfrac{1}{3} \text{ and } I_3^w=-\tfrac{1}{2}.
\end{aligned}\tag{6.10}$$

Note the ease with which the quantum number structure is analyzed from scratch in this notation. Of course, it is not very difficult to use the tensor notation to work out these results either, and it is worthwhile comparing methods. For a group as complicated as E_6, however, the Dynkin analysis does not increase in difficulty. The projections and axes for E_6 and its subgroups are worked out in section 7.

The solution to the problem of classifying all maximal subalgebras of a simple algebra is somewhat messy, although the regular subalgebras are easily found. We only summarize the solution; details can be found in Dynkin [9].

We first find all the **maximal regular subalgebras**, which are now defined: let G′ be a subalgebra of G and write G′ in any Cartan–Weyl basis with its Cartan subalgebra being the set $\{H_i'\}$ and its ladder operators being the set $\{E_\alpha'\}$. If there exists a basis of G such that $\{H_i\} \supset \{H_i'\}$ and $\{E_\alpha\} \supset \{E_\alpha'\}$, then G′ is a regular subalgebra of G.

Dynkin has derived a clever way to find all maximal regular subalgebras. The regular subalgebras fall into two categories: nonsemisimple and semisimple. Each maximal nonsemisimple subgroup is a U_1 factor times a semisimple factor obtained by removing one dot from the Dynkin diagram (table 5) for G. Clearly, there are at most $l = \mathrm{rank(G)}$ possibilities; usually there are fewer independent cases.

The maximal regular semisimple subalgebras are constructed in a similar fashion, by removing a dot from the *extended* Dynkin diagram. The extended diagram is constructed by making a simple root system that satisfies all the requirements of the simple root systems of the Dynkin diagram, except for linear independence. It is possible to add only one root to the set of simple roots that satisfies the requirement that the difference of two roots in the extended set is not a root: it is the negative of the root of highest weight [55]. The extended diagrams are listed in table 16, with the new root marked by an "x". The extended diagram with a dot removed is guaranteed to be the diagram of a semisimple Lie algebra.

There are a few cases where the method is trivial or breaks down; for example, removing a dot from the SU_n extended diagram gives back SU_n, so SU_n has no regular maximal semisimple subalgebras. (Note that the rank of G is the same as the rank of its regular maximal subalgebras.) There are five cases for the exceptional algebras where the subalgebra derived by removing a dot from the extended diagram is not maximal [60].

F_4: 3 removed is contained in the subgroup with 4 removed ($SO_9 \supset SU_2 \times SU_4$)
E_7: 3 removed is a subgroup of the subgroup with 1 or 5 removed ($SU_2 \times SO_{12} \supset SU_2 \times SU_4 \times SU_4$)
E_8: 2 removed is a subgroup of that with 7 removed; 3 removed is a subgroup of that with 6 removed; and 5 removed is a subgroup of that with 1 removed. ($SU_2 \times E_7 \supset SU_2 \times SU_8$; $SU_3 \times E_6 \supset SU_3 \times SU_2 \times SU_6$; and $SO_{16} \supset SU_4 \times SO_{10}$, respectively.)

The extended Dynkin diagrams have other interesting applications, so these exceptions do not detract too much from the beauty of this procedure.

Table 14 is a list of the maximal subalgebras of the classical simple algebras of rank 8 or less; the sets marked (R) are regular maximal subalgebras, derived from the analysis of the root diagram just discussed. We should make a number of comments on table 14.

For small rank, some Dynkin diagrams from different series become equivalent. For example,

$$A_1 \sim B_1 \sim C_1 (SU_2 \sim SO_3 \sim Sp_2)$$

all have the same root diagram, a single dot;

$$C_2 \sim B_2 (Sp_4 \sim SO_5)$$

(Sp_4 rather than SO_5 is used in discussing subgroups); and

$$A_3 \sim D_3(SU_4 \sim SO_6)$$

(we use SU_4 instead of SO_6 in labeling subgroups). Moreover $D_2 \sim A_1 \times A_1$ is semisimple. The symbol "$\sim$" means "isomorphic to".

Often a maximal nonsemisimple subgroup is strictly a subgroup of a semisimple one; nonmaximal subgroups are not listed. It also sometimes happens that a maximal subgroup contains as a subgroup just the semisimple part of a nonsemisimple subgroup, but not the U_1 part. It is best to go against the custom of dropping these nonsemisimple subgroups from the list of maximal subgroups. They are maximal, and in model building it is possible to blunder rather seriously if the list of maximal subgroups is left incomplete in this way. SU_4 has a subgroup of this type; it contains $SU_2 \times SU_2 \times U_1$ and Sp_4 as maximal subgroups, where Sp_4 itself contains an $SU_2 \times SU_2$, which is embedded in SU_4 in the same way that the $SU_2 \times SU_2$ portion of the nonsemisimple subgroup is. Table 14 includes $SU_2 \times SU_2 \times U_1$ as a regular maximal subgroup of SU_4. Perhaps a more subtle example is the list of regular subgroups of Sp_4. The branching rule of the **4** of Sp_4 to irreps of the regular maximal subgroup $SU_2 \times SU_2$ is $\mathbf{4} = (\mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{2})$. This subgroup contains a subgroup $SU_2 \times U_1$ with $\mathbf{4} = \mathbf{2}(0) + \mathbf{1}(1) + \mathbf{1}(-1)$, which is a maximal nonsemisimple subgroup obtained by removing one of the dots from the Sp_4 Dynkin diagram. However, there is another $SU_2 \times U_1$ obtained by removing the other dot with $\mathbf{4} = \mathbf{2}(1) + \mathbf{2}(-1)$. The semisimple part SU_2 is embedded in Sp_4 as the sum of the SU_2 generators of the $SU_2 \times SU_2$ subgroup. The additional U_1 is not contained in the $SU_2 \times SU_2$ subgroup, so this $SU_2 \times U_1$ is maximal, while the other one is not.

We now turn to the problem of enumerating the special subalgebras. Since the Cartan subalgebra of G' is constructed from the ladder operators of $G \supset G'$, it is clear that an analysis of the root diagram is not likely to be very convenient. The situation can be greatly simplified by looking at the representations and analyzing the possible branching rules,

$$R = \sum_i R'_i, \tag{6.11}$$

where R is an irrep of G and R'_i are irreps of $G' \subset G$. This is just the procedure discussed in section 3 and in ref. [6].

The analysis of maximal special subalgebras of a classical algebra has the significant feature that makes the discussion simple, that the branching rule for at least one of the simple representations of G into irreps of $G' \subset G$ has only one term. (Do not forget that the vector and spinors of SO_m are both simple.) In other words, there is a branching rule of the form $\mathbf{r} = \mathbf{r}'$. (This is not always so for the exceptional groups.) Thus we look for a representation of G' with these dimensions, and then check that we recover the generators of G. For example, consider $G = SO_7$. The groups with less than 21 generators, with rank three or less, *and* with a 7-dimensional irrep are SU_2 and G_2. The generators of SO_7 are given irreducibly as $(\mathbf{7} \times \mathbf{7})_a$. In SU_2, $(\mathbf{7} \times \mathbf{7})_a = \mathbf{3} + \mathbf{7} + \mathbf{11}$, which contains the SU_2 adjoint, so SU_2 is a subgroup. However it is not maximal: in G_2, $(\mathbf{7} \times \mathbf{7})_a = \mathbf{7} + \mathbf{14}$, so G_2 contains SU_2 as a maximal subgroup with $\mathbf{7} = \mathbf{7}$ and $\mathbf{14} = \mathbf{3} + \mathbf{11}$. Thus G_2 is the only maximal special subgroup of SO_7.

This approach can be used to recover the results listed in table 14. The diligent reader who derives table 14 will note several patterns emerge; there are some corresponding theorems that simplify the procedure when going beyond rank 8 (see ref. [9]), but there are a few exceptions and caveats. The discussion given above is complete enough for unified model building as now practiced, but is not exhaustive.

Finally, the special subgroups for the exceptional groups are listed in table 15. In this case, the branching rules for the simple irreps are more complicated than in the classical case. The derivation of those results is a major topic of ref. [9]; no further discussion is given here, since a fairly potent way to proceed is to guess systematically.

In applications to unified model building, the list of subgroups is needed, but it is most conveniently expressed in terms of branching rules. To derive the branching rules it remains simply to project the weights of $\mathbf{r}$ of G onto the weight space of G′ by $P(\mathrm{G} \supset \mathrm{G}')$, and then pick out (by highest weight method) the irreps of G′. Powerful computer programs have been written that implement this procedure [10, 57]. Another approach, which is very convenient for hand calculation, relies on a knowledge of tensor products of $\mathbf{R}_i$ of G and $\mathbf{r}_j$ of G′. If $\mathbf{R}_1 = \sum \mathbf{r}_{1i}$, $\mathbf{R}_2 = \sum \mathbf{r}_{2i}$ and $\mathbf{R}_1 \times \mathbf{R}_2 = \sum_j \mathbf{R}_j$, then $\sum_j \mathbf{R}_j = \sum_{ij} \mathbf{r}_{1i} \times \mathbf{r}_{2j}$. A constructive approach starting with simple irreps for R_i is usually quite fast; many of the branching rules in the tables were derived this way. A useful check is the index sum rule: for our normalization of the index (5.11), the index of R in (6.11) is the sum of the indices of the R'_i. Also see [57]. Except for working out many examples (see sections 7 and 8) this completes the discussion of the embedding of color and flavor in simple groups.

Let us analyze the possibility of defining a charge-conjugation operator [13, 61] that carries each particle into its antiparticle within $\mathbf{f}_\mathrm{L}$ and representations of other kinds of particles. The charge conjugation operator C must anticommute with electric charge and with the first, third, fourth, sixth and eighth generators of SU_3^c, while commuting with the second, fifth and seventh generators of SU_3^c in the Gell-Mann basis [54], which generate an SO_3 subgroup of SU_3^c.

For each simple group G of interest, a candidate for C may be either an element of the group (inner automorphism) or not (outer automorphism). It anticommutes with a set A of Hermitian generators of G $[C(\mathrm{A}) = -\mathrm{A}]$ and commutes with the remaining set S $[C(\mathrm{S}) = \mathrm{S}]$, which generate a symmetric subgroup $\mathrm{G}_\mathrm{S} \subset \mathrm{G}$. The commutation rules evidently must exhibit the behavior that defines a symmetric subgroup:

$$[\mathrm{S}, \mathrm{S}] \subseteq \mathrm{S}, \qquad [\mathrm{S}, \mathrm{A}] \subseteq \mathrm{A} \qquad \text{and} \quad [\mathrm{A}, \mathrm{A}] \subseteq \mathrm{S}, \tag{6.12}$$

so that C leaves the commutation relations unchanged.

When the candidate for C is inner, it carries every irrep of the group into itself. Conversely, if C carries every irrep into itself, then the phase changes under charge conjugation can be arranged so that C is inner. When C is outer, there are two possibilities:

(1) For $\mathrm{G} = \mathrm{SU}_n$, SO_{4n+2} and E_6, each complex irrep is carried into its complex conjugate.

(2) For $\mathrm{G} = \mathrm{SO}_{4n}$, which has only real and pseudoreal representations, there are candidate C's that carry representations into reflected ones that are in many cases inequivalent; for example, SO_{12} has two inequivalent 32-dimensional spinor irreps, **32** and **32′**, which could be carried into each other, while the 12-dimensional vector irrep could be taken into itself.

The mathematics of the various candidates for C is well-known [12] for all simple groups and we list the possibilities in table 17, giving the simple group G, the symmetric subgroup G_S left invariant by the candidate C, the number "rank(A)" of the generators of the Cartan subalgebra of G that anticommute with C, and the result of the action of C on an irrep R of G, whether it reflects R into itself, into $\bar{\mathrm{R}}$, or into R′. We have included, for completeness, cases in which G is too small to contain color and flavor and cases in which rank(A) is less than three.

We have assumed that there is a C that carries $\mathbf{f}_L$ into itself. Let us break up $\mathbf{f}_L$ into irreps of the group G^* formed by G and C. When C is inner, $G^* = G$ and each irrep of G is an irrep of G^*. If C is outer, but carries R into $\bar{R}$, then any self conjugate irrep of G is an irrep of G^* and the other irreps of G^* are of the form $R + \bar{R}$. If C is outer, but carries R into R′, then any self-reflected irrep of G is an irrep of G^* and the other irreps of G^* are of the form $R + R'$.

In those cases where C carries R into R and R is complex, then the irrep of G^*, also an irrep of G, is complex. In all other cases, the irrep of G^* is self conjugate. If we assign the left-handed fermions $\mathbf{f}_L$ to a self conjugate irrep of G^* or a direct sum of such irreps, then the theory is "vectorlike"; otherwise, it is "flavor chiral". The simplest flavor-chiral case, is, of course, the assignment of $\mathbf{f}_L$ to a single complex irrep of G, with C carrying every irrep into itself.

It is useful to turn at this point to another automorphism of G, namely the unitary operator CP that exchanges $\mathbf{f}_L$ and $\mathbf{f}_R$. In a Yang–Mills theory based on G, there is no freedom in the choice of CP. The kinetic energy operator has a part that creates $\mathbf{f}_L$ and $\mathbf{f}_R$ together and another part that destroys them together; since the kinetic energy is a singlet under G, $\mathbf{f}_L$ and $\mathbf{f}_R$ must have opposite values of all the operators in the Cartan subalgebra of G. Thus CP, for each G, is the unique operator in table 17 that changes the sign of the whole Cartan subalgebra, with rank(A) = rank(G). For each case, that operator carries R into $\bar{R}$ when there are complex representations, and R into R when there are not. Recall the discussion below (2.2).

The components of $\mathbf{f}_L$ may be listed in terms of Majorana spinors, or else pairs of Majorana spinors may be classed as Dirac spinors. A Dirac spinor can be made out of a pair of Majorana spinors with opposite behavior under C. In many cases, however, it turns out that there are some unpaired Majorana fermions left over, a number $|n_+ - n_-|$, where n_+ and n_- are the number of + and − (or $+i$ and $-i$) eigenvalues of the group theoretical part of C applied to a given representation of G. As a familiar example of the matrix part of C, recall the Dirac theory of the electron, where the matrix part is proportional to σ_y when acting on ψ_L or ψ_R. Leftover Majorana spinors that cannot be paired to make Dirac spinors can occur only in the case of electrical neutrality, of course. Below, when specific examples are considered, $|n_+ - n_-|$ is computed for a number of representations of various simple groups G and choices of C.

Now, just as the fermion kinetic energy operator involves $\mathbf{f}_L$ (operator) $\mathbf{f}_R$, so the fermion effective ultraviolet mass operator involves the symmetrized parts of $(\mathbf{f}_L)^2$ and $(\mathbf{f}_R)^2$. If the effective mass operator has the most general possible behavior under C and G, then studying the transformation of $\mathbf{f}_L$ under C is not very rewarding. For example, the electrically neutral color singlets can just be treated as a set of Majorana fermions, with an arbitrary mass matrix. However, it may be that the C-conserving and C-violating parts of the mass operator have special group theoretical properties. In that case, there may be important restrictions on fermion masses, particularly for neutral leptons, but also for other fermions.

Let us now proceed to study some examples. We look at the assignment of $\mathbf{f}_L$ to an irreducible complex representation of G. If G is small, we may be picking out only one "family" of fermions here; if G is sufficiently large, we can accommodate all the known and suspected families, along with other fermions. The simplest interesting example of a flavor-chiral assignment consists of putting a single family of left-handed fermions into the **16** of SO_{10}, while $\mathbf{f}_R$ goes, of course, into the $\overline{\mathbf{16}}$. Since we are assuming the existence of a C operator (which takes $\mathbf{f}_L$ into itself), we can see from table 17 that there is only one candidate that works. The one that leaves invariant the symmetric subgroup $SO_4 \times SO_6$ of SO_{10}; we note that $SO_4 \times SO_6$ has the same algebra as $SU_2 \times SU_2 \times SU_4$. The other candidates either take **16** into $\overline{\mathbf{16}}$ or else flip the signs of too few generators of the Cartan subalgebra. Thus CP, which does take **16** into $\overline{\mathbf{16}}$, leaves invariant $SO_5 \times SO_5$; there are candidates that leave invariant $SO_3 \times SO_7$ and SO_9 respectively, but they also take **16** into $\overline{\mathbf{16}}$; there are two more candidates that take **16** into **16**, the one that leaves $SU_5 \times U_1$ invariant and the one that leaves $SO_2 \times SO_8$ invariant, but they both flip the signs of only two commuting generators of G, not enough to cover the charge conjugation of color and electromagnetism.

It is important to note that the C-invariant symmetric subgroup $SO_4 \times SO_6$ is embedded in SO_{10} differently from the $SO_4 \times SO_6$ subgroup that classifies flavor and color. Using the $SU_2 \times SU_2 \times SU_4$ notation, we would normally put $\mathbf{16} = (\mathbf{2}, \mathbf{1}, \mathbf{4}) + (\mathbf{1}, \mathbf{2}, \overline{\mathbf{4}})$, where $\mathbf{4} = \mathbf{3}^c + \mathbf{1}^c$ and $\overline{\mathbf{4}} = \overline{\mathbf{3}}^c + \mathbf{1}^c$; the first SU_2 is chosen to be SU_2^w. We thus have in $(\mathbf{2}, \mathbf{1}, \mathbf{4})$ a left-handed electron and neutrino and u and d quarks, while in $(\mathbf{1}, \mathbf{2}, \overline{\mathbf{4}})$ we have the corresponding left-handed antiparticles, including $(\bar{\nu}_e)_L$. The symmetric subgroup $SO_4 \times SO_6$ (also treated as $SU_2 \times SU_2 \times SU_4$) also breaks **16** into $(\mathbf{2}, \mathbf{1}, \mathbf{4}) + (\mathbf{1}, \mathbf{2}, \overline{\mathbf{4}})$, but with a totally different basis. The common generators of the two different $SO_4 \times SO_6$ subgroups form the group $SU_2 \times SU_2$ and they consist of $2F_2^{\text{color}}$, $2F_5^{\text{color}}$, $2F_7^{\text{color}}$, $I_2^L + I_2^R$, $I_1^L - I_1^R$ and $I_3^L - I_3^R$, where $\mathbf{I}^L$ generates SU_2^w and $\mathbf{I}^R$ generates the other SU_2. We see that, in the physical $SU_2 \times SU_2 \times SU_4$ notation, C interchanges the first two SU_2's, as well as conjugating them in the usual way, and also complex conjugates the representations of SU_4. The C-invariant $SU_2 \times SU_2 \times SU_4$ is, by definition, left unaltered by C.

We now show how to calculate $|n_+ - n_-|$. Each irrep of the C-invariant subgroup $SU_2 \times SU_2 \times SU_4$ carries a + or a −. For the **16** we have $(\mathbf{2}, \mathbf{1}, \mathbf{4}) + (\mathbf{1}, \mathbf{2}, \overline{\mathbf{4}})$, giving eight +'s and eight −'s, so that $|n_+ - n_-| = 0$. The neutral fermions in the **16** are ν_L and $(\bar{\nu})_L$ only, and the condition $|n_+ - n_-| = 0$ tells us only that these can be paired to form a Dirac spinor; note that $|n_+ - n_-| = 2$ for the **10**, so the result for the **16** is not completely trivial. The mass matrix in Nature, however, if the **16** representation of SO_{10} is to be relevant, must supply a huge Majorana mass to $(\bar{\nu})_L$, so that the effective mass of ν_L comes out to be $m_D^2/m_{\text{Majorana}}$ $(\bar{\nu})_L$, where m_D is the Dirac mass connecting ν_L and $(\bar{\nu})_L$. This is discussed further in section 9.

In terms of SO_{10}, and remembering that we are treating just one family, the mass matrix must fit into the symmetric part of $\mathbf{16} \times \mathbf{16}$, which gives $\mathbf{10} + \overline{\mathbf{126}}$, where under $SU_2 \times SU_2 \times SU_4$ we have $\mathbf{10} = (\mathbf{2}, \mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{6})$ and $\mathbf{126} = (\mathbf{3}, \mathbf{1}, \mathbf{10}) + (\mathbf{1}, \mathbf{3}, \overline{\mathbf{10}}) + (\mathbf{2}, \mathbf{2}, \mathbf{15}) + (\mathbf{1}, \mathbf{1}, \mathbf{6})$. We note that the **1**, **10**, $\overline{\mathbf{10}}$ and **15** of SU_4 each contain one color singlet component, and **6** contains none, so that in terms of $SU_2 \times SU_2$ the mass matrix belonging to **10** acts like $(\mathbf{2}, \mathbf{2})$ while that belonging to **126** acts like $(\mathbf{3}, \mathbf{1})$, $(\mathbf{1}, \mathbf{3})$ and $(\mathbf{2}, \mathbf{2})$. We do not want a large term that violates $\mathbf{I}^w$ by $|\Delta \mathbf{I}^w| = 1$, so we are left with possible fermion mass terms with $(\mathbf{2}, \mathbf{2})$ from **10** and/or **126** and $(\mathbf{1}, \mathbf{3})$ from **126**. The electrically neutral component of the $(\mathbf{1}, \mathbf{3})$ from **126** is just the term that gives a Majorana mass to $(\bar{\nu})_L$; if the $(\mathbf{3}, \mathbf{1})$ term is set to zero, the $|\Delta \mathbf{I}^w| = 0$ mass violates C maximally. The electrically neutral components of the $(\mathbf{2}, \mathbf{2})$ from **10** and **126** are all C-conserving and merely give Dirac masses to e, ν, u and d. Thus an attempt to fit a single family of fermions to the **16** of SO_{10} leads to the result that **all** the C-violation in the mass matrix must lie in the $(\mathbf{1}, \mathbf{3}, \mathbf{10})$ part of the **126** representation of SO_{10} and that this piece of the mass matrix just gives a huge Majorana mass to $(\bar{\nu})_L$. This term is also the one that breaks SO_{10} down to SU_5, giving $\mathbf{16} = \mathbf{10} + \overline{\mathbf{5}} + \mathbf{1}$, where $(\bar{\nu})_L$ is precisely the singlet under SU_5.

For a less trivial example we study briefly the assignment of $\mathbf{f}_L$ to the complex **27** of E_6 [8], still keeping just one family. In the usual color-flavor decomposition to $SU_3 \times SU_3 \times SU_3^c$, where SU_2^w is in the first SU_3, the **27** contains $(\overline{\mathbf{3}}, \mathbf{3}, \mathbf{1}^c) + (\mathbf{3}, \mathbf{1}, \mathbf{3}^c) + (\mathbf{1}, \overline{\mathbf{3}}, \overline{\mathbf{3}}^c)$. We see from table 17 that the only candidate for C leaves the symmetric subgroup $SU_2 \times SU_6$ invariant, and that $SU_2 \times SU_6$ contains two generators from the Cartan subalgebra of E_6. The **27** has $(\mathbf{2}, \overline{\mathbf{6}}) + (\mathbf{1}, \mathbf{15})$ of $SU_2 \times SU_6$, so the neutral leptons in the **27** consist of a pair that may be joined together to form a Dirac spinor, and three unmatched Majorana spinors, which are eigenstates of C. Under SO_{10}, the **27** contains $\mathbf{16} + \mathbf{10} + \mathbf{1}$; as is obvious from the previous example, the three neutral leptons in **10** + **1** are the unmatched Majorana spinors, and the two neutral leptons in the **16** are connected by C.

The mass matrix, contained in the symmetric part of **27** × **27**, has parts transforming as **27** and **351'**; both E_6 irreps have both C-violating and C-conserving terms. The charge $\frac{2}{3}$ quark has only a C-conserving mass. The remaining mass matrices (charge $-\frac{1}{3}$ quarks, charged and neutral leptons) have both C-violating and C-conserving $|\Delta \mathbf{I}^{w}| = 0$ contributions in the E_6 model. Of course, physically realistic conjectures about the C behavior of the mass matrix depend on the choice of G and $\mathbf{f}_L$. Nevertheless, it may be useful to study this behavior in models; the **27** of E_6 is investigated further in section 9, where it is shown that if the $I^{w} = 0$ mass violates C maximally (that is, the $I^{w} = 0$ mass is C odd), then it leaves a $\bar{\mathbf{5}} + \mathbf{10}$ family of SU_5 out of the **27** of E_6 massless. If the $I^{w} = 0$ mass is C even, a $\mathbf{1} + \bar{\mathbf{5}} + \mathbf{10}$ is left massless.

7. E_6 and subgroups

The first topic of this section is a description of the tables of irreps, tensor products, and branching rules of E_6 and its subgroups, SU_3, SU_4, SU_5, SU_6, SO_7, SO_8, SO_9, SO_{10} and F_4 (tables 18–49). After the choice of subgroups is explained and the tables are described, a consistent set of projection matrices [recall (6.4) and (6.7)] through the different subgroup chains is derived; the matrices follow the convention that the Q^{em} axis and the QCD roots in E_6 are fixed and do not depend on the subgroup chain. This is possible because there is only one embedding of color and electric charge in E_6, as discussed near the end of section 3. The final topic is an outline of a method for calculating vector-coupling coefficients. The method is particularly effective for analyzing products of basic irreps, which are typically all that are required in unified model building today. Its simplicity is due to the use of simple roots to relate states at different levels. The labeling problem must be more directly addressed for products of composites irreps [62].

The choice of E_6 as a starting point is based on a conjecture that an E_6 (or a subgroup of E_6) Yang–Mills theory might someday be part of a realistic theory. There is little reason at present (aside from the more-or-less successful phenomenology) to feel such a commitment very strongly, and other groups (SO_{4n+6}, $n = 2, 3, 4$, SU_8, E_7 and E_8) are described briefly in section 8. At best, this kind of survey will aid in the search for more satisfactory theories. [See section 3.]

With the exception of SU_3 itself, only the simple subgroups of E_6 large enough to contain $U_1^{em} \times SU_3^c$, at least in one subgroup chain, are analyzed. In table 18, the maximal subgroups of E_6 are divided into "satisfactory" and "unsatisfactory". (Compare table 18 with tables 14 and 15.) Each satisfactory subgroup is then listed in the left-hand column, and its maximal subgroups are classified in a similar fashion; all the maximal subgroup chains that are physically acceptable for the symmetry breaking of E_6 (or one of its subgroups) can then be listed.

The subgroups that do not contain $U_1^{em} \times SU_3^c$ are called unsatisfactory; there are three ways for a subgroup to be unsatisfactory:

(1) If the maximal subgroup is rank two or less, it cannot contain $U_1^{em} \times SU_3^c$. For example, E_6 contains G_2 and SU_3 as maximal subgroups. If these subgroups contain SU_3^c, then there can be no U_1 left over for electromagnetism.

(2) It may not be possible to guarantee the usual charge assignments with fractionally charged quarks and integrally charged leptons. Consider, for example, the maximal subgroup $SU_3 \times G_2$ of E_6. The **27** of E_6 branches to $(\mathbf{3}, \mathbf{7}) + (\bar{\mathbf{6}}, \mathbf{1})$ of $SU_3 \times G_2$. If SU_3 were color with U_1^{em} in G_2, then the $(\bar{\mathbf{6}}^c, \mathbf{1})$ states would be neutral. Since $\bar{\mathbf{6}}^c$ has the same triality as $\mathbf{3}^c$, it would be impossible to maintain the usual quark

charges. If SU_3^c were in G_2, there would again be quarks and leptons with the same electric charge, since the **7** of G_2 branches into $\mathbf{1}^c + \mathbf{3}^c + \bar{\mathbf{3}}^c$ and a U_1^{em} from the SU_3 factor cannot distinguish these. Thus, $SU_3 \times G_2$ is an unsatisfactory maximal subgroup for E_6 to break down to.

(3) Sp_8 is another unsatisfactory subgroup of E_6, even though it is possible to arrange for the correct spectrum of electric charge eigenvalues. The problem is the difficulty of obtaining a satisfactory phenomenology: even if $\mathbf{f}_L$ were a **351** plus any number of **27**'s, there would be room for only two charged leptons. This is seen as follows: The **27** of E_6 branches to the **27** of Sp_8. The only maximal subgroups of Sp_8 that could contain $U_1^{em} \times SU_3^c$ are $SU_4 \times U_1$ and $SU_2 \times Sp_6$. In both cases the color content of the **27** is $\mathbf{1}^c + 3 \cdot \mathbf{3}^c + 3 \cdot \bar{\mathbf{3}}^c + \mathbf{8}^c$, where the $\mathbf{1}^c$ is neutral. The **351** then has 9 color singlet states; 2 have charge +1 and two have charge −1. In a flavor chiral assignment, this could correspond to the e and μ, but there would be no room for the τ. Thus, we list the Sp_8 subgroup as unsatisfactory, although in fact it is merely awkward, and indicate this difficulty in table 18 by putting it in square brackets. (The color content of the **27** of E_6 for the satisfactory subgroups is $9 \cdot \mathbf{1}^c + 3 \cdot \mathbf{3}^c + 3 \cdot \bar{\mathbf{3}}^c$.) With these comments in mind, the rest of table 18 is easily derived.

The explicit coordinatization of the E_6 weight space is summarized in tables 19, 20 and 21, which correspond to (6.8)–(6.10) for SU_5. Table 19 shows a choice of the color roots and the flavor axes, both in the Dynkin and dual bases. It is this choice that sets most of the conventions for the projection matrices for the subgroups listed in table 18. Table 20 lists the nonzero roots of E_6 and their flavor and color content. Table 21 provides similar information for the **27** of E_6; both tables will be discussed further.

We now describe the contents and conventions of the tables; table 22 shows the labeling of the Dynkin diagrams used in the next 31 tables in the order the algebras are analyzed. There are rank of the group independent Casimir invariants; (5.9) is the second order invariant. The orders of a complete set with the lowest possible orders, as derived by Racah [58], are also listed in table 22.

Table 23. SU_3 Irreps of Dimension Less Than 65. The convention that triality one irreps [$c = a_1 + 2a_2$ (mod 3), "c" for congruence class] are unbarred and the conjugate with Dynkin designation $(a_2\, a_1)$ with triality two is barred is followed with one exception: the symmetric tensor in $\mathbf{3} \times \mathbf{3}$, $[(1\ 0) \times (1\ 0)]_s =$ (2 0) with triality two, is called **6**, not $\bar{\mathbf{6}}$; that convention has been followed for too many years to switch now. In other tables we advocate conventions based on congruence classes, so a few traditional, but not so universal conventions in the larger groups have been switched. The index is computed from (5.15). The last two columns list the number of SU_2 (or $SU_2 \times U_1$) singlets in embedding (6.4), and SO_3 singlets in embedding (6.6).

Table 24. SU_3 Tensor Products. A table of SU_3 tensor products hardly needs any explanation, and is included here for convenience. The irreps on the right-hand side of the products are either all of triality zero or all of triality one. An irrep in the symmetric part of the product $R \times R$ carries a subscript s, and an irrep in the antisymmetric part of $R \times R$ has a sub-a.

Table 25. SU_4 Irreps of Dimension less than 180. SU_4 irreps $(a_1\, a_2\, a_3)$ have quadrality, defined as $c = a_1 + 2a_2 + 3a_3$ (mod 4). Each quadrality three irrep $(a_1\ a_2\ a_3)$ is conjugate to $(a_3\ a_2\ a_1)$ with quadrality one. The conjugate of a quadrality zero (or two) irrep also has quadrality zero (or two). The conventions are obvious from the table. The existence of four congruence classes is in accord with the number for $SO_6 \sim SU_4$. The number of SU_3 singlets in the irrep is listed in the last column; an asterisk (1*) indicates that the SU_3 singlet is also neutral (has no phase change) under the U_1 in the decomposition $SU_4 \supset SU_3 \times U_1$. (Also see table 27.)

Table 26. SU_4 Tensor Products. Quadrality conventions analogous to the triality conventions of table 24 are followed.

Table 27. SU_4 Branching Rules for $SU_4 \supset SU_3 \times U_1$. The eigenvalue of the U_1 generator in $SU_4 \supset$

$SU_3 \times U_1$ is given in parentheses and is normalized to correspond with the usual electric-charge eigenvalues. Again, note that the **3**, $\bar{\mathbf{6}}$ and **15** of SU_3 are defined to have triality one, so they have electric charges in the sequence $-\frac{1}{3}$ plus integer.

Table 28. SU_5 Irreps of Dimension less than 800. The quintality of a SU_5 irrep $(a_1\ a_2\ a_3\ a_4)$ is defined as $a_1+2a_2+3a_3+4a_4$ (mod 5). An irrep with quintality 4 (or 3) is conjugate of an irrep with quintality 1 (or 2). For dimension greater than 800, we list only those irreps with sum of the Dynkin labels less than 5. In the "SU_4 singlet" column an asterisk on 1* means the singlet is an $SU_4 \times U_1$ singlet; similarly in the last column, 1* means an $SU_2 \times SU_3 \times U_1$ singlet.

Table 29. SU_5 Tensor Products. All products between irreps with the sum of Dynkin labels less than or equal to two are included.

Table 30. Branching Rules for SU_5. The values of the U_1 generator are included in the parentheses. The normalization convention for the U_1 generator $Z = 3Y^w$ in $SU_5 \supset SU_2 \times SU_3 \times U_1$ is such that $Q^{em} = I_3 + Z/6$ in the SU_5 model.

Table 31. SU_6 Irreps of Dimension less than 1000. Sextality is defined as $a_1+2a_2+3a_3+4a_4+5a_5$ (mod 6), and self-conjugate irreps can have sextality zero or three. Sextality four and five irreps are not listed, as they are conjugate to sextality two and one irreps, respectively, that are listed. All irreps of dimension less than 1000 are given, as are the irreps with sum of Dynkin labels less than 5. The three maximal subgroups that are analyzed are all nonsemisimple; if the singlet in the compact part carries no charge of the U_1 factor, it is marked by an asterisk.

Table 32. SU_6 Tensor Products.

Table 33. SU_6 Branching Rules. The eigenvalues of the U_1 generator are normalized according to the branching rule for the **6**.

Table 34. SO_7 Irreps of Dimension less than 650 and Branching Rules for $SO_7 \supset SU_4$. The SO_7 irreps are all real, and self-conjugate. The weights in irreps with a_3 odd (spinor irreps) are in a different congruence class than the irreps with a_3 even.

Table 35. SO_7 Tensor Products.

Table 36. SO_8 Irreps of Dimension less than 1300 and the Branching Rules into SO_7 Irreps. SO_8 (and any SO_{2n}, $n > 1$) irreps fall into one of four congruence classes. For SO_8, the four classes are exemplified by the three eights and the adjoint, and can be defined by $[a_3 + a_4 \text{ (mod 2)},\ a_1 + a_3 \text{ (mod 2)}]$ [51]. (The congruency class of any SO_{4n} irrep is a similar two-component vector.) We have followed the convention of marking irreps in class (0, 1) by sub-v for "vector", in (1, 0) with sub-s for "spinor", in (1, 1) with sub-c for "conjugate", and in (0, 0) with no subscript. There is one exception: there are some (0, 0) irreps that also come in sets of threes, such as the **35**'s, **294**'s, **567**'s, etc. The convention for the v, c, s label is obvious. One of the **8**'s branches to **1** + **7** of SO_7, and the other two branch to the **8** of SO_7; this arbitrariness is due to the high symmetry of the SO_8 Dynkin diagram. If we select the convention that the **16** of SO_9 branches to the $\mathbf{8}_s + \mathbf{8}_c$, and the **16** of SO_{10} contains the same colors and electric charges as the **16** of SO_9, then it is a physical choice to select the $\mathbf{8}_s$ to branch to the **1** + **7** of SO_7. We follow those conventions in deriving the explicit projection matrices below. The $SU_4 \times U_1$ branching rules are not listed because the semisimple part is trivially derived from the embeddings $SO_8 \supset SO_7 \supset SU_4$. The branching rules for the simple SO_8 irrep to $SU_4 \times U_1$ irreps, following the conventions of the v, c and s labels just mentioned, are

$$
\begin{aligned}
\mathbf{8}_v &= \mathbf{4}(1) + \bar{\mathbf{4}}(-1)\\
\mathbf{8}_c &= \bar{\mathbf{4}}(1) + \mathbf{4}(-1)\\
\mathbf{8}_s &= \mathbf{1}(2) + \mathbf{1}(-2) + \mathbf{6}(0)\\
\mathbf{28} &= \mathbf{1}(0) + \mathbf{6}(2) + \mathbf{6}(-2) + \mathbf{15}(0),
\end{aligned}
$$

where the generator of the U_1 is in parentheses.

Table 37. SO_8 Tensor Products. The subscripts i, j, k, take on values, v, s and c, as defined in table 36.

Table 38. SO_9 Irreps of Dimension less than 5100.

Table 39. SO_9 Tensor Products.

Table 40. SO_9 Branching Rules. Note conventions discussed under table 36. To examine the $SO_9 \supset SO_7 \times U_1$ embedding, we need to look forward to the $SO_{10} \supset SO_9$ branching rule, $\mathbf{16} = \mathbf{16}$. The branching rules for $SO_9 \supset SO_7 \times U_1$ are

$$\begin{aligned}
&\mathbf{9} = \mathbf{1}(2) + \mathbf{1}(-2) + \mathbf{7}(0)\\
&\mathbf{16} = \mathbf{8}(1) + \mathbf{8}(-1)\\
&\mathbf{36} = \mathbf{1}(0) + \mathbf{7}(2) + \mathbf{7}(-2) + \mathbf{21}(0).
\end{aligned}$$

If we require that the $\mathbf{16}$ of SO_9 contain the electric charge and color spectrum of a family, then the $SO_9 \supset SO_7 \times U_1$ embedding is irrelevant.

Table 41. SO_{10} Irreps of Dimension less than 12000. The irreps of SO_{10} fall into four congruency classes, defined by $2a_1 + 2a_3 - a_4 + a_5 \pmod 4$). (The congruency class of an SO_{4n+2} irrep is a single number.) The adjoint is in the 0 class, the spinor in 1, the conjugate spinors in -1, and the vector and bispinors ($\mathbf{126}$ and $\overline{\mathbf{126}}$) are both in the 2 class. This explains why the $\mathbf{10}$ and bispinors have no zero weights. Thus, we call $(1\ 0\ 0\ 1\ 0)$ the $\mathbf{144}$, not $\overline{\mathbf{144}}$, as would be natural from a tensor product construction from $\mathbf{10} \times \overline{\mathbf{16}}$. The SU_5 singlets marked with an asterisk are $SU_5 \times U_1$ singlets.

Table 42. SO_{10} Tensor Products [63]. Note that the congruency conventions are followed.

Table 43. Branching Rules for SO_{10}. The U_1 generator, normalized by the branching rule for the $\mathbf{10}$ for $SO_{10} \supset SU_5 \times U_1^*$, is given in parenthesis; this convention is used in table 19. The $SO_{10} \supset SO_8 \times U_1$ embedding is defined by the branching rules

$$\begin{aligned}
&\mathbf{10} = \mathbf{1}(2) + \mathbf{1}(-2) + \mathbf{8}_i(0)\\
&\mathbf{16} = \mathbf{8}_j(1) + \mathbf{8}_k(-1)\\
&\overline{\mathbf{16}} = \mathbf{8}_j(-1) + \mathbf{8}_k(1)\\
&\mathbf{45} = \mathbf{1}(0) + \mathbf{8}_i(2) + \mathbf{8}_i(-2) + \mathbf{28}(0)\\
&(i \neq j \neq k \neq i,\ \ i = \mathrm{c, v\ or\ s}).
\end{aligned}$$

Note the discussions for tables 36 and 40; specifically, in $SO_{10} \supset SU_2 \times SO_7 \supset SO_7$, the $\mathbf{16} = \mathbf{8} + \mathbf{8}$, not $\mathbf{1} + \mathbf{7} + \mathbf{8}$.

Table 44. F_4 Irreps of Dimension less than 100000. All irreps of F_4 are real and fall into one congruency class.

Table 45. F_4 Tensor Products.

Table 46. F_4 Branching Rules. There are 27 $SU_3 \times SU_3$ irreps in the $\mathbf{1053}$, 19 in the $\mathbf{1053}'$, and 29 in the $\mathbf{1274}$; the lengthy lists are quickly derived if needed [57].

Table 47. E_6 Irreps of Dimension less than 100000. Congruency reduces to triality, defined as $a_1 - a_2 + a_4 - a_5 \pmod 3$. Following the kind of conventions as before, the triality two irreps are marked with a bar, so $\overline{\mathbf{27}} = (0\ 0\ 0\ 0\ 1\ 0)$ and $\overline{\mathbf{351}} = (0\ 1\ 0\ 0\ 0\ 0)$, since both have triality two.

Table 48. E_6 Tensor Products.

Table 49. E_6 Branching Rules.

It is not necessary to have a basis for weight space in order to derive the results in the tables above; for many purposes the coordinate-independent statement of the embedding is sufficient. However, for practical calculations of symmetry breaking, mass matrices, and other details of Yang–Mills theories, it is often convenient to have a labeled basis. Some examples are considered in section 9. It has already been argued that there is only one embedding of $U_1^{em} \times SU_3^c$ (up to Weyl reflections) in E_6 that is likely to be relevant for model building; it is the one where the **27** has nine $\mathbf{1}^c$, three $\mathbf{3}^c$ and three $\bar{\mathbf{3}}^c$, with the electric charge spectrum of the three $\mathbf{3}^c$ being $\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}$. The projection matrices that project a weight of E_6 down through the weights of any subgroup chain to a weight of $U_1^{em} \times SU_3^c$ can then be chosen so that the E_6 roots and axes that coincide with the SU_3^c roots and Q^{em} axis are independent of the subgroup chain. It is convenient in applications to unified models to follow this convention, rather than the standard convention, that of projecting highest weights of an irrep onto the highest weights of the subgroup representations [10].

We shall follow the convention of projecting highest weights onto highest weights for the subgroup chain,

$$E_6 \supset SO_{10} \times U_1^t \supset SU_5 \times U_1^r \times U_1^t \supset SU_2^w \times SU_3^c \times U_1^w \times U_1^r \times U_1^t, \tag{7.1}$$

which is a convention for picking out the E_6 roots to be identified with I_+^w and the color changing operators. Since it is related to any other choice by a Weyl reflection, the results of calculations based on this basis are completely general. The projection matrices for the subgroup chain in (7.1) are [10]:

$$P_1(E_6 \supset SO_{10}) = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}; \tag{7.2}$$

$$P_2(SO_{10} \supset SU_5) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}; \tag{7.3}$$

$$P_3(SU_5 \supset SU_2^w \times SU_3) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \tag{7.4}$$

[also (6.7)]

The elements are always nonnegative integers when following the highest weight to highest weight convention. The U_1 factors are in the Cartan subalgebra of E_6, and therefore correspond to axes in the root space. Those axes are identified in table 19, along with the color roots and the weak isospin root. Tables 20 and 21 contain the color and flavor content of the **78** adjoint irrep and the **27**. The calculation is the same as we did for SU_5 and SU_3 in section 6, except in addition, more attention is paid to the subgroup structure. As an example consider the root $(1\ {-1}\ 1\ {-1}\ 1\ 0)$, which is projected onto $(-1\ 0\ 1\ 0\ 0)$ by (7.2), which is a root in the **45** of SO_{10}, and can then be projected by (7.3) to the SU_5 weight $(-1\ 1\ 0\ 1)$, which is a root in the adjoint **24**. Finally (7.4) projects $(-1\ 1\ 0\ 1)$ to $(1)(0\ 1)$ of $SU_2 \times SU_3$, which identifies the $(1\ {-1}\ 1\ {-1}\ 1\ 0)$ root of E_6 as a color anti-triplet with $I_3^w = \frac{1}{2}$; it is the charge $\frac{4}{3}$, SU_5 antilepto-diquark that mediates proton decay. It is a simple computation to construct the rest of table 20 for the **78**, and also to work out the **27** of E_6, as done in table 21.

With the identification of the color and weak interactions of (7.1) and the conventions of (7.2), (7.3) and (7.4), it is possible to identify the color and weak interaction quantum numbers in E_6 directly by computing the matrix product, $P(E_6 \supset SU_2^w \times SU_3^c) = P(SU_5 \supset SU_2^w \times SU_3^c) \times P(SO_{10} \supset SU_5) \times P(E_6 \supset SO_{10})$. Thus, the Dynkin labels (a_1^c, a_2^c) of SU_3^c and a^w of SU_2^w for an E_6 weight $(a_1\ a_2\ a_3\ a_4\ a_5\ a_6)$ are, respectively,

$$\begin{aligned} a_1^c &= a_1 + 2a_2 + 2a_3 + a_4 + a_6 \\ a_2^c &= a_3 + a_4 + a_5 + a_6 \\ a^w &= a_1 + a_2 + a_3 + a_4 + a_5. \end{aligned} \tag{7.5}$$

The electric charge axis (recorded in table 19) is $Q^{em} = \frac{1}{3}(3\ -2\ 3\ -3\ 2\ -2)$ for E_6, is $Q^{em} = \frac{1}{3}(-2\ -2\ 3\ -1\ 1)$ for SO_{10}, and is $Q^{em} = \frac{1}{3}(-4\ 4\ -1\ 1)$ for SU_5. The corresponding dual axes are, respectively, $\bar{Q}^{em} = \frac{1}{3}[2\ 1\ 2\ 0\ 1\ 0]$ for E_6, $\bar{Q}^{em} = \frac{1}{3}[-1\ 0\ 3\ 1\ 2]$ for SO_{10}, and $\bar{Q}^{em} = \frac{1}{3}[-1\ 2\ 1\ 1]$ for SU_5.

There are numerous other subgroup chains from E_6 to $U_1^{em} \times SU_3^c$, as can be seen from table 18. We now list the corresponding projection matrices and make a few comments on the derivations.

The maximal subgroup $SU_3^w \times SU_3 \times SU_3^c$ is ordered so that the first SU_3^w contains SU_2^w; electric charge has contributions from the Cartan subalgebras of both of the SU_3^w and SU_3,

$$P_4(E_6 \supset SU_3^w \times SU_3 \times SU_3^c) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}. \tag{7.6}$$

As an example of a constraint on (7.6), recall from the example above (7.5) that the projection matrix must carry the $(1\ -1\ 1\ -1\ 1\ 0)$ root to $(1\ 0)(1\ 0)(0\ 1)$ of $SU_3 \times SU_3 \times SU_3^c$, since this root has $I_3^w = \frac{1}{2}$, $Q^{em} = \frac{4}{3}$, and is an antilepto-diquark. Since the **78** branches to $(\mathbf{8}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{8}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{8}) + (\mathbf{3}, \mathbf{3}, \bar{\mathbf{3}}) + (\bar{\mathbf{3}}, \bar{\mathbf{3}}, \mathbf{3})$, it is easy to continue this procedure and derive (7.6) uniquely.

The other subgroup projections must be consistent with the two above; however, there is still some freedom. A guide to the numbering of the projection matrices is given in table 50, where all the subgroup chains are shown. Let us build up to the larger subgroups of E_6, starting from $SU_4 \supset SU_3^c \times U_1$. Depending on the subgroup chain in table 50, the generator of the U_1 factor is sometimes Q^{em}, sometimes Y^w, and sometimes just a part of Y^w. A convenient choice of projection matrix is

$$P_5(SU_4 \supset SU_3^c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{7.7}$$

so the QCD Dynkin labels are a_1 and a_2, when SU_3^c is contained in an SU_4. Thus, the weights $(1\ 0\ 0)$, $(-1\ 1\ 0)$ and $(0\ -1\ 1)$ give a $\mathbf{3}^c$ in the **4**, where the bottom weight $(0\ 0\ -1)$ is the $\mathbf{1}^c$. In the case that Q^{em} generates the U_1 factor, the electric charge axis in SU_4 is

$$Q^{em} = -\tfrac{1}{3}(0\ 0\ 4), \qquad \text{or} \quad \bar{Q}^{em} = -\tfrac{1}{3}[1\ 2\ 3],$$

so the **4** has a quark of charge $-\frac{1}{3}$ and a charge one lepton, and the **6** has a charge $\frac{2}{3}$ quark and a charge $-\frac{2}{3}$ antiquark. The chain $SO_9 \supset SO_8 \supset SO_7 \supset SU_4$ is now studied; here, Q^{em} is required to be a generator in each subgroup.

The branching rule for the SO_7 **7** into a **1** + **6** of SU_4 must give a neutral lepton for the singlet; the **8** branches to $\mathbf{4}+\bar{\mathbf{4}}$, and the charges of the 15 states in **7** + **8** must coincide with those of a family. Subgroup chains with SO_7 may be relevant for analyzing symmetry breaking. A convenient projection is

$$P_6(SO_7 \supset SU_4) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \tag{7.8}$$

and the electric charge is

$$Q^{em} = -\tfrac{1}{3}(0\ \ 0\ \ 4) \qquad \text{or} \qquad \bar{Q}^{em} = -\tfrac{1}{3}[2\ \ 4\ \ 3].$$

The deviation of $P(SO_8 \supset SO_7)$ requires the same foresight mentioned for table 36; if the **16** of SO_9 branches to the $\mathbf{8}_s + \mathbf{8}_c$ of SO_8, then the $\mathbf{8}_s$ (or $\mathbf{8}_c$) must branch to **1** + **7** of SO_7, so the **16** of SO_9 branches to **1** + **7** + **8**. The choice of projection is

$$P_7(SO_8 \supset SO_7) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \tag{7.9}$$

with

$$Q^{em} = -\tfrac{1}{3}(2\ \ 0\ \ 2\ \ 0), \qquad \text{or} \qquad \bar{Q}^{em} = -\tfrac{1}{3}[3\ \ 4\ \ 3\ \ 2].$$

The projection for $SO_9 \supset SO_8$ can then be chosen as

$$P_8(SO_9 \supset SO_8) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{7.10}$$

with

$$Q^{em} = -\tfrac{1}{3}(2\ \ 0\ \ 0\ \ 2) \qquad \text{or} \qquad \bar{Q}^{em} = -\tfrac{1}{3}[3\ \ 4\ \ 5\ \ 3].$$

The color weights of SU_3^c, given in terms of the SO_9 weights $(a_1\ \ a_2\ \ a_3\ \ a_4)$, are derived from the product of projections $P_5(SU_4 \supset SU_3^c) \times P_6(SO_7 \supset SU_4) \times P_7(SO_8 \supset SO_7) \times P_8(SO_9 \supset SO_8)$ to be, simply, $a_1^c = a_2$ and $a_2^c = a_3$.

We may now derive the projections for $E_6 \supset F_4 \supset SO_9$ and $F_4 \supset SU_3 \times SU_3^c$, which must be consistent with the projection for $SO_{10} \supset SO_9$. The projection matrix $P(E_6 \supset F_4)$ (with the branching rule **27** = **1** + **26**) annihilates three weights of the **27** in giving the singlet and the two zero weights of the **26**. Those weights must be among the color singlet, electrically neutral weights as identified in table 21. Since the **16** of SO_{10} branches to the **16** of SO_9, which has no zero weights, $P(E_6 \supset F_4)$ must annihilate $(1\ {-1}\ 0\ 1\ {-1}\ 0)$, $(-1\ 0\ 1\ {-1}\ 0\ 0)$ and $(0\ 1\ {-1}\ 0\ 1\ 0)$.

There are additional constraints in this derivation: the consistency relations $P(E_6 \supset SU_3 \times SU_3^c) = P(F_4 \supset SU_3 \times SU_3^c) \times P(E_6 \supset F_4)$ and $P(E_6 \supset SO_9) = P(F_4 \supset SO_9) \times P(E_6 \supset F_4) = P(SO_{10} \supset SO_9) \times$

$P(E_6 \supset SO_{10})$, and the convention that the generators of the first SU_3 in the chain $E_6 \supset F_4 \supset SU_3 \times SU_3^c$ are the sums of the corresponding $SU_3^w \times SU_3$ generators in P_4. We may then orient the color in F_4 according to [10]:

$$P_9(F_4 \supset SU_3 \times SU_3^c) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \tag{7.11}$$

$$P_{10}(E_6 \supset F_4) = \begin{pmatrix} 0 & 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 2 & 1 & 0 & 0 \end{pmatrix}. \tag{7.12}$$

Thus, the color and flavor content of the F_4 weights can be identified and $P(F_4 \supset SO_9)$ can be obtained by matching up the corresponding weights:

$$P_{11}(F_4 \supset SO_9) = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ -2 & -4 & -3 & -1 \end{pmatrix}. \tag{7.13}$$

From P_1, P_4, and the quantum numbers of the SO_9 weights, it follows that

$$P_{12}(SO_{10} \supset SO_9) = \begin{pmatrix} -1 & -1 & -2 & -1 & -1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & -2 & -2 & -1 & -1 \end{pmatrix}. \tag{7.14}$$

The Dynkin labels for $SU_3^c \supset F_4$ are

$$\begin{aligned} a_1^c &= a_1 + 2a_2 + a_3 + a_4 \\ a_2^c &= a_1 + a_2 + a_3, \end{aligned} \tag{7.15}$$

and the electric charge axis in F_4 is

$$Q^{em} = \tfrac{1}{3}(-2 \ \ -2 \ \ 4 \ \ 2) \qquad \text{or} \qquad \bar{Q}^{em} = \tfrac{1}{3}[0 \ \ 2 \ \ 3 \ \ 2].$$

The projection from SO_{10} to $SU_2^w \times SU_2 \times SU_4$ can now be identified from the quantum numbers from the other SO_{10} subgroup chain and P_5:

$$P_{13}(SO_{10} \supset SU_2^w \times SU_2 \times SU_4) = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 0 \end{pmatrix}. \tag{7.16}$$

If the generators of the SU_2 in $SO_9 \supset SU_2 \times SU_4$ are identified as the sum of the SU_2 generators in P_{13}, we obtain

$$P_{14}(SO_9 \supset SU_2 \times SU_4) = \begin{pmatrix} -2 & -2 & -2 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -2 & -1 \end{pmatrix}. \tag{7.17}$$

The last of the SO_{10} subgroups is $SU_2 \times SO_7$, where we assume that the SU_2 is generated by the sum of the $SU_2^w \times SU_2$ generators in P_{13}. From P_5 and P_6 we then obtain

$$P_{15}(SO_{10} \supset SU_2 \times SO_7) = \begin{pmatrix} 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ -2 & -2 & -2 & -1 & -1 \end{pmatrix}, \tag{7.18}$$

where the SO_7 contains a piece of Y^w. Note that since the **10** of SO_{10} branches to $(\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{7})$ of $SU_2 \times SO_7$, the **7** has a charge $-\frac{1}{3}$ quark, and consequently Q^{em} cannot be completely contained in SO_7; the SU_2 is generated by the sum of the corresponding SU_2 generators in $SO_{10} \supset SU_2^w \times SU_2 \times SU_4$.

Finally, we examine the subgroup chains beginning with $E_6 \supset SU_2 \times SU_6$. The SU_2^w may be the explicit SU_2, or it may be buried in the SU_6. In the former case the SU_6 contains SU_3^c and the isoscalar part of Q^{em}; then the SU_3^c may be embedded in SU_6 so that the color Dynkin labels are $a_1^c = a_1$ and $a_2^c = a_2$, with $\mathbf{6} = \mathbf{1}^c + \mathbf{1}^c + \mathbf{1}^c + \mathbf{3}^c$ of SU_3^c. Then $\mathbf{27} = (\mathbf{2}, \mathbf{6}) + (\mathbf{1}, \overline{\mathbf{15}})$ is the appropriate branching rule since it contains $(\mathbf{2}, \mathbf{3}^c)$ for $(u, d)_L$, and not $(\mathbf{2}, \bar{\mathbf{3}}^c)$:

$$P'_{16}(E_6 \supset SU_2^w \times SU_6) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}. \tag{7.19}$$

The weak hypercharge generates a U_1 in SU_6 with axis $\bar{Y}^w = \frac{1}{3}[1\ 2\ 3\ 0\ 3]$. It is also possible that SU_6 contains SU_5 (with $\mathbf{6} = \mathbf{1} + \mathbf{5}$), where SU_5 contains $SU_2^w \times U_1^w \times SU_3^c$ as embedded before. Then the branching rule should have the form $\mathbf{27} = (\mathbf{2}, \bar{\mathbf{6}}) + (\mathbf{1}, \mathbf{15})$ so it contains $\bar{\mathbf{5}} + \mathbf{10}$; the projection matrix can be chosen to be

$$P_{16}(E_6 \supset SU_2 \times SU_6) = \begin{pmatrix} 0 & 0 & -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & -1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \tag{7.20}$$

where the projection of SU_6 to the standard SU_5 basis is

$$P_{17}(\mathrm{SU}_6 \supset \mathrm{SU}_5) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{7.21}$$

The $E_6 \supset SU_2 \times SU_6$ branching rule in table 49 is based on the convention, $\mathbf{27} = (\mathbf{2}, \bar{\mathbf{6}}) + (\mathbf{1}, \mathbf{15})$.

In the embedding $E_6 \supset SU_2 \times SU_6$ with the $SU_2^w \times U_1^w \times SU_3^c$ contained entirely in the SU_6, SU_6 can break down to color by two different subgroup chains besides (7.21): $SU_6 \supset SU_2^w \times U_1 \times SU_4$, where SU_4 contains color; and $SU_6 \supset SU_3^w \times U_1^w \times SU_3^c$, where SU_3^w is the same SU_3^w identified in $P_4(E_6 \supset SU_3^w \times SU_3 \times SU_3^c)$. The projection matrices are

$$P_{18}(\mathrm{SU}_6 \supset \mathrm{SU}_2^w \times \mathrm{SU}_4'') = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 \end{pmatrix}, \tag{7.22}$$

$$P_{19}(\mathrm{SU}_6 \supset \mathrm{SU}_3^w \times \mathrm{SU}_3^c) = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}. \tag{7.23}$$

Finally, SU_5 contains SU_4, and maintaining consistency with $P_5(SU_4 \supset SU_3^c)$, we identify

$$P_{20}(\mathrm{SU}_5 \supset \mathrm{SU}_4') = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{pmatrix}, \tag{7.24}$$

where SU_4' contains U_1^{em}.

This completes all the connections of the subgroup diagram shown in table 50. Some applications of these projections are considered in section 9.

It is instructive to check explicitly the effect of these projections on the weights of an irrep of E_6. Although there is only one embedding of $U_1^{em} \times SU_3^c$ in E_6, there can be several inequivalent embeddings of some subgroups between them. For example, the $\mathbf{10}$ of SO_{10} breaks up into $\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{7}$ of SO_7 in $SO_{10} \supset SU_2 \times SO_7$, but is broken up into $\mathbf{1}+\mathbf{1}+\mathbf{8}$ in the chain $SO_{10} \supset SO_9 \supset SO_8 \supset SO_7$, and described by the product $P_7P_8P_{12}$. The difference in the branching rules indicates two inequivalent embeddings. As another example SU_4 can be embedded in SO_9 with $9 = 1 + 4 + \bar{4}$ ($SO_9 \supset SO_8 \supset SO_7 \supset SU_4'$) or as $9 = 1 + 1 + 1 + 6$ ($SO_9 \supset SU_2 \times SU_4 \supset SU_4$). The latter is a regular subgroup, but the former is special. The maximal group-subgroup pairs inside E_6 are marked explicitly in table 50.

Let us beat this problem to death with an example (insisted on by somebody). Consider the $(1\ 0\ 0\ 0\ 0\ 0)$ weight of E_6, which projects to $(0\ 0\ 0\ 0\ 1)$ of the $\mathbf{16}$ of SO_{10} by P_1, to $(0\ 1\ 0\ 0)$ of the $\mathbf{10}$ of SU_5 by P_2, and to $(1)(1\ 0)$ of $SU_2^w \times SU_3^c$ by P_3. Thus, the $(1\ 0\ 0\ 0\ 0\ 0)$ is uniquely identified as the E_6 weight of a charge $\frac{2}{3}$ quark with $I_3^w = \frac{1}{2}$ and color $(1\ 0)$. This is consistent with P_4 (by construction), which gives $(1\ 0)(0\ 0)(1\ 0)$ of $SU_3^w \times SU_3 \times SU_3^c$, where $P(SU_3^w \supset SU_2^w) = (1\ 0)$. The F_4 weight from P_{10} is $(0\ 0\ 0\ 1)$ in the $\mathbf{26}$, which gives $(1\ 0)(1\ 0)$ of $SU_3 \times SU_3^c$, and $(-1\ 1\ 0\ -1)$ of the $\mathbf{16}$ of SO_9 by either P_{11} or P_{12}. In the $SO_9 \supset SU_2 \times SU_4$ projection P_{14} acting on $(-1\ 1\ 0\ -1)$ gives $(1)(1\ 0\ 0)$, consistent with the branching rule, $\mathbf{16} = (\mathbf{2}, \mathbf{4}) + (\mathbf{2}, \bar{\mathbf{4}})$. In the other subgroup chain that goes through SO_9, P_8 acting on $(-1\ 1\ 0\ -1)$ becomes the $(-1\ 1\ -1\ 0)$ of the $\mathbf{8}_s$ of SO_8; P_7 on $(-1\ 1\ -1\ 0)$ becomes the $(0\ 1\ -2)$ of the $\mathbf{7}$ of SO_7, which by P_6 becomes the $(1\ 0\ -1)$ weight of the $\mathbf{6}$ of SU_4.

Returning to the $(0\ 0\ 0\ 0\ 1)$ SO_{10} weight, we find it projected to $(1)(0\ 1\ -1)$ of $SU_2 \times SO_7$ by P_{15}, where $(0\ 1\ -1)$ projects to $(1\ 0\ 0)$ of SU_4 by P_6, which is required by the consistency condition $P_{14}P_{12} = P_6P_{15}$. By P_{13}, $(0\ 0\ 0\ 0\ 1)$ is projected to $(1)(0)(1\ 0\ 0)$ of $SU_2^w \times SU_2 \times SU_4$, consistent with the SU_2 in $SU_2 \times SO_7$ being generated by the sum of the corresponding $SU_2^w \times SU_2$ generators in $SU_2^w \times SU_2 \times SU_4$. The projection of the $(0\ 1\ 0\ 0)$ of SU_5 onto an SU_4' weight by P_{20} gives $(1\ 0\ -1)$, which is in the $\mathbf{6}$. (The $SU_4' \subset SU_5$ coincides with the $SO_{10} \supset SO_9 \supset SO_8 \supset SO_7 \supset SU_4'$ chain since $\bar{\mathbf{5}} + \mathbf{10} = \mathbf{1} + \mathbf{6} + \mathbf{4} + \bar{\mathbf{4}}$ of SU_4; thus we have chosen the P_i so that $P_{20}P_2 = P_6P_7P_8P_{12}$. Finally P_{16}' on $(1\ 0\ 0\ 0\ 0\ 0)$ is $(1)(1\ 0\ 0\ 0\ 0)$ and P_{16} on $(1\ 0\ 0\ 0\ 0\ 0)$ is $(0)(-1\ 0\ 1\ 0\ 0)$, with $(-1\ 0\ 1\ 0\ 0)$ giving $(0\ 1\ 0\ 0)$ of SU_5, $(1)(1\ 0\ 0)$ of $SU_2^w \times SU_4$, and $(1\ 0)(1\ 0)$ of $SU_3^w \times SU_3^c$. Let's go on to another topic.

This section concludes with an outline of a method for calculating the vector-coupling (VC) coefficients for groups as large as those required for unification [62]. The VC coefficients are matrix elements of the unitary transformation between the Hilbert-space basis $|\mathbf{r}_1\lambda_1\rangle\,|\mathbf{r}_2\lambda_2\rangle$ of the direct product $\mathbf{r}_1 \times \mathbf{r}_2$ and the vectors $|\mathbf{r}_i\lambda_i\rangle$ of the irreps occurring in the reduction of the product, $\mathbf{r}_1 \times \mathbf{r}_2 = \Sigma_i\, \mathbf{r}_i$; each space is $\dim(\mathbf{r}_1) \times \dim(\mathbf{r}_2)$ dimensional, so the weight λ_i implicitly carries labels needed to distinguish different vectors with the same weight. Other labels are needed if a given irrep occurs several times in $\mathbf{r}_1 \times \mathbf{r}_2$, but those too are left implicit. The VC coefficients are defined by the transformation,

$$|\mathbf{r}_3\lambda_3\rangle = \sum_{\lambda_1\lambda_2} |\mathbf{r}_1\lambda_1\rangle|\mathbf{r}_2\lambda_2\rangle\langle\mathbf{r}_1\lambda_1;\ \mathbf{r}_2\lambda_2|\mathbf{r}_3\lambda_3\rangle, \tag{7.25}$$

where $\langle\mathbf{r}_1\lambda_1; \mathbf{r}_2\lambda_2|\mathbf{r}_3\lambda_3\rangle$ is nonzero only if $\mathbf{r}_3$ is in the tensor product $\mathbf{r}_1 \times \mathbf{r}_2$ and if $\lambda_3 = \lambda_1 + \lambda_2$. Since this is a unitary transformation, the VC coefficients satisfy the orthonormality conditions,

$$\begin{aligned}
&\sum_{\lambda_1\lambda_2} \langle\mathbf{r}\lambda|\mathbf{r}_1\lambda_1;\ \mathbf{r}_2\lambda_2\rangle\langle\mathbf{r}_1\lambda_1;\ \mathbf{r}_2\lambda_2|\mathbf{r}'\lambda'\rangle = \delta_{\mathbf{r}\mathbf{r}'}\delta_{\lambda\lambda'}\\
&\sum_{\mathbf{r}\lambda} \langle\mathbf{r}_1\lambda_1; \mathbf{r}_2\lambda_2|\mathbf{r}\lambda\rangle\langle\mathbf{r}\lambda|\mathbf{r}_1\lambda_1';\ \mathbf{r}_2\lambda_2'\rangle = \delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'},
\end{aligned} \tag{7.26}$$

where $\mathbf{r}$ and $\mathbf{r}'$ are in $\mathbf{r}_1 \times \mathbf{r}_2$ and λ is a weight in $\mathbf{r}$. The right-hand sides of (7.26) should be multiplied by functions of the additional labels, which are often chosen to be delta functions.

There are methods in the literature for computing the VC coefficients. For example, Wybourne's "building up principle" involves working back and forth between the 6-j symbols of H and the isoscalar factors of G ⊃ H [11], starting at SU_2 or SU_3 and building up to some large group of interest; building up to E_6 through a physical sequence of maximal subgroups can be tedious. Sometimes, especially if the particle states are identified directly in terms of weights of the representation of the large group (as done in section 6), it is the VC coefficient that is desired (that is, the product of the isoscalar factors) and not the individual isoscalar factors through some subgroup chain. Anyhow, the isoscalar factors for G ⊃ H can be computed by "dividing" the VC coefficients of G by those of H. That relationship is often useful, so we interrupt the main discussion for a moment to review it.

The crucial theorem relating the VC coefficients of G and H (G ⊃ H) to the isoscalar factors is the Racah factorization lemma [64], which follows from the Wigner–Eckhart theorem. Let $\mathbf{g}_i$ be an irrep of G with weights λ_i; the VC coefficient to be factored is $\langle\mathbf{g}_1\lambda_1; \mathbf{g}_2\lambda_2|\mathbf{g}_3\lambda_3\rangle$. The branching rule for $\mathbf{g}$ into irreps $\mathbf{h}_i$ of H ⊂ G is $\mathbf{g} = \Sigma_i\, \mathbf{h}_i$, so a state $|\mathbf{g}\lambda\rangle$ can be relabeled as $|\mathbf{h}_i(\mathbf{g})P\lambda\rangle$, where $P \equiv P(\mathrm{G} \supset \mathrm{H})$ is the weight-space projection matrix so $P\lambda$ is a weight in the irrep $\mathbf{h}_i$ of H, and for simplicity, the embedding is arranged so $\mathbf{h}_i(\mathbf{g})$ is just one irrep in the branching rule. In cases where a given irrep occurs several times in the branching rule, this choice of basis for degenerate projected weights is not always possible, which means that the factorization is not always complete. The Racah factorization lemma states that

$$\langle \mathbf{g}_1\lambda_1;\ \mathbf{g}_2\lambda_2|\mathbf{g}_3\lambda_3\rangle = \sum_{\text{deg}} \langle \mathbf{h}_1(\mathbf{g}_1);\ \mathbf{h}_2(\mathbf{g}_2)|\mathbf{h}_3(\mathbf{g}_3)\rangle\langle \mathbf{h}_1 P\lambda_1;\ \mathbf{h}_2 P\lambda_2|\mathbf{h}_3 P\lambda_3\rangle, \tag{7.27}$$

where the sum on "deg" is over the degeneracy of $\mathbf{h}_3$ in the branching rule $\mathbf{g}_3 = \Sigma_i\ \mathbf{h}_{3i}$, $\langle \mathbf{h}_1(\mathbf{g}_1); \mathbf{h}_2(\mathbf{g}_2)|\mathbf{h}_3(\mathbf{g}_3)\rangle$ is the isoscalar factor, and $\langle \mathbf{h}_1\ P\lambda_1; \mathbf{h}_2\ P\lambda_2|\mathbf{h}_3\ P\lambda_3\rangle$ is a VC coefficient for H. It should now be clear how the isoscalar factors are computed by "dividing" a VC coefficient for $\mathbf{g}_1 \times \mathbf{g}_2 \supset \mathbf{g}_3$ by one for $\mathbf{h}_1 \times \mathbf{h}_2 \supset \mathbf{h}_3$ for an appropriate weight.

The method for computing the VC coefficients described here is a "trivial generalization" of the standard method used for SU_2, where the lowering operator $J_- = J_{1-} + J_{2-}$ is applied again and again to the states, beginning with the state of highest weight $|j_1 + j_2, j_1 + j_2\rangle = |j_1, j_1\rangle\,|j_2, j_2\rangle$, which is unique up to a phase chosen to be +1. The matrix elements of J_-, which are needed for this procedure, are well known:

$$\langle j, m-1|J_-|j, m\rangle = +[(j+m)(j-m+1)]^{1/2};$$

the convention that **all** the matrix elements for any irrep of J_- are nonnegative should be noted. The calculation then relies on the fact that the generators of a group do not mix irreps, of course, so it immediately follows that $J_-|j_1 + j_2,\ j_1 + j_2\rangle = +(2j_1 + 2j_2)^{1/2}|j_1 + j_2, j_1 + j_2 - 1\rangle$, and the VC coefficients $\langle j_1,\ j_1 - 1;\ j_2,\ j_2|j_1 + j_2,\ j_1 + j_2 - 1\rangle = [j_1/(j_1 + j_2)]^{1/2}$ and so on, follow complete with normalization. Since there are two linearly independent vectors with weight $j_1 + j_2 - 1$, there is an orthogonal vector belonging to the highest weight of another irrep, the $j_1 + j_2 - 1$ irrep in the product $j_1 \times j_2$. Thus, another phase convention is needed; in the Condon–Shortley phase convention, the minus sign is attached to

$$\langle j_1, j_1 - 1; j_2, j_2|j_1 + j_2 - 1, j_1 + j_2 - 1\rangle = -[j_2/(j_1 + j_2)]^{1/2},$$

which then sets all the phases for the rest of the $j_1 + j_2 - 1$ states in $j_1 \times j_2$. The next level is reached by acting again with J_-; if the dimensionality of the product space of some weight increases, a new irrep enters, and the phase conventions are again needed. Wigner was able to sum the iterations of all these formulas for SU_2 and give a general (but complicated) formula for the VC coefficients.

Precisely the same procedure could be used for larger simple groups, if we only knew which lowering operators $E_{-\bar{\alpha}}$ to use, had a phase convention for $\langle \mathbf{r}, \lambda - \alpha|E_{-\bar{\alpha}}|\mathbf{r}\lambda\rangle$, and could set up the phase conventions when irreps of the tensor product of $\mathbf{r}_1 \times \mathbf{r}_2$ appear in $E_{-\bar{\alpha}}(|\mathbf{r}_1\lambda_1\rangle|\mathbf{r}_2\lambda_2\rangle)$ that are not in $|\mathbf{r}_1\lambda_1\rangle|\mathbf{r}_2\lambda_2\rangle$. In fact, some of these generalizations can be made quite simply.

The weight diagram of an irrep is systematically obtained by subtracting from the highest weight the **linearly independent simple roots** in the manner prescribed in section 6; thus the VC coefficients can be calculated using only the matrix elements of $E_i \equiv E_{-\alpha_i}$, where α_i is a **simple** root. Moreover, it is not self contradictory to define the phases of all nonzero values of $\langle \mathbf{r}, \lambda - \alpha_i|E_i|\mathbf{r}\lambda\rangle$ to be positive for all irreps of G [65]; eq. (4.3) determines the matrix elements of E_i acting on a state of weight λ in terms of the scalar product of α_i and λ, which is simply the ith Dynkin label of λ, a_i, and the matrix element of E_i at the next **lower** level:

$$\langle \mathbf{r}, \lambda - \alpha_i|E_i|\mathbf{r}\lambda\rangle = +[a_i + \langle \mathbf{r}, \lambda|E_i|\mathbf{r}, \lambda + \alpha_i\rangle^2]^{1/2}\,. \tag{7.28}$$

(This expression is not adequate when there are degenerate weights; also a_i should be replaced by $a_i(\alpha_i, \alpha_i)/2$ when there are both long and short simple roots. Equation (7.28) should be viewed as schematic [62].) As a first trivial application of (7.28), note that the matrix elements of the lowering operators denoted by their simple roots and arrows in table 11 for both the **16** of SO_{10} and the **27** of E_6 are all +1. The remaining phase conventions concern the appearance of the highest weight of a new irrep after the application of E_i at some level; it is easily put in by hand in the examples below. For an extensive calculation in a labeled basis of VC coefficients of SU_3, see [66], and for SU_4, see [67].

The first example is familiar, can be done in full detail, but is not completely trivial. It is to compute the VC coefficients for $\mathbf{6}\times\mathbf{3}=\mathbf{8}+\mathbf{10}$ of SU_3. The matrix elements of the lowering operators in the **3**, **6**, **8** and **10** irreps are shown in table 51, where the lowering operator of the first simple root $(2\ -1)$ is denoted E_1 and the lowering operator associated with the second simple root $(-1\ 2)$ is denoted by E_2 on the left-hand side of the line of the states connected by E_i, and the matrix element computed from (7.28) is placed on the right-hand side of the line. It is not always necessary to work out the weight diagrams for the irreps in the reduction of $\mathbf{r}_1\times\mathbf{r}_2$, but they are convenient for obtaining the normalization (and thereby checking the rest of the calculation), for keeping track of which E_i do not annihilate a given state, for labeling degenerate weights, and sometimes for computing specific VC coefficients without reproducing the whole table. The states with zero weight in the octet, defined as $E_1|(2\ -1)\rangle = \sqrt{2}|(0\ 0)_1\rangle$ and $E_2|(-1\ 2)\rangle = \sqrt{2}|(0\ 0)_2\rangle$ are not orthonormal (see below).

Consider the coupling of the highest weights

$$|\mathbf{10}(3\ 0)\rangle = +|\mathbf{6}(2\ 0);\ \mathbf{3}(1\ 0)\rangle. \tag{7.29}$$

The next weight in the **10** is obtained by acting on both sides of (7.29) with E_1; reading the matrix elements from table 51,

$$\sqrt{3}|\mathbf{10}(1\ 1)\rangle = \sqrt{2}|\mathbf{6}(0\ 1);\ \mathbf{3}(1\ 0)\rangle + 1|\mathbf{6}(2\ 0);\ \mathbf{3}(-1\ 1)\rangle. \tag{7.30}$$

The vector with highest weight of the **8** is orthogonal to $|\mathbf{10}(1\ 1)\rangle$, which completely determines $|\mathbf{8}(1\ 1)\rangle$ up to an overall phase:

$$\sqrt{3}|\mathbf{8}(1\ 1)\rangle = 1|\mathbf{6}(0\ 1);\ \mathbf{3}(1\ 0)\rangle - \sqrt{2}|\mathbf{6}(2\ 0);\ \mathbf{3}(-1\ 1)\rangle.$$

This is the last phase convention needed for the $\mathbf{6}\times\mathbf{3}$ calculation. Occasionally it is helpful to beautify the results into a table; for example, a traditional format is

$$\begin{array}{cc|ccc} & & (0\ 1) & (2\ 0) & \mathbf{6} \\ \mathbf{6}\times\mathbf{3} & (1\ 1) & (1\ 0) & (-1\ 1) & \mathbf{3} \\ \hline & \mathbf{10} & \sqrt{2/3} & 1/\sqrt{3} & \\ & \mathbf{8} & 1/\sqrt{3} & -\sqrt{2/3} & \end{array}, \tag{7.31}$$

but we shall not bother with such niceties here.

The next level of VC coefficients are obtained by applying E_1 and E_2, respectively, first to $|\mathbf{10}(1\ 1)\rangle$ and then to $|\mathbf{8}(1\ 1)\rangle$:

$$\begin{aligned}
\sqrt{3}|\mathbf{10}(-1\ \ 2)\rangle &= |\mathbf{6}(-2\ \ 2);\ \mathbf{3}(1\ \ 0)\rangle + \sqrt{2}|\mathbf{6}(0\ \ 1);\ \mathbf{3}(-1\ \ 1)\rangle \\
\sqrt{3}|\mathbf{10}(2\ \ -1)\rangle &= \sqrt{2}|\mathbf{6}(1\ \ -1);\ \mathbf{3}(1\ \ 0)\rangle + 1|\mathbf{6}(2\ \ 0);\ \mathbf{3}(0\ \ -1)\rangle \\
\sqrt{3}|\mathbf{8}(-1\ \ 2)\rangle &= \sqrt{2}|\mathbf{6}(-2\ \ 2);\ \mathbf{3}(1\ \ 0)\rangle - 1|\mathbf{6}(0\ \ 1);\ \mathbf{3}(-1\ \ 1)\rangle \\
\sqrt{3}|\mathbf{8}(2\ \ -1)\rangle &= 1|\mathbf{6}(1\ \ -1);\ \mathbf{3}(1\ \ 0)\rangle - \sqrt{2}|\mathbf{6}(2\ \ 0);\ \mathbf{3}(0\ \ -1)\rangle\,.
\end{aligned} \tag{7.32}$$

Orthonormality provides a good check on this arithmetic. From table 51, the lowering operations to the next level are: E_1 and E_2 can act on $|\mathbf{10}(-1\ \ 2)\rangle$, and $E_2\,|\mathbf{10}(-1\ \ 2)\rangle$ must agree with $E_1|\mathbf{10}(2\ \ -1)\rangle$, since there is only one vector with $(0\ \ 0)$ weight in the **10**; according to the **8** weight diagram $E_2|\mathbf{8}(-1\ \ 2)\rangle$ and $E_1|\mathbf{8}(2\ \ -1)\rangle$ lead to zero-weight vectors that are linearly independent. These four states are, respectively,

$$\begin{aligned}
|\mathbf{10}(-3\ \ 3)\rangle &= |\mathbf{6}(-2\ \ 2);\ \mathbf{3}(-1\ \ 1)\rangle \\
\sqrt{3}|\mathbf{10}(0\ \ 0)\rangle &= |\mathbf{6}(-1\ \ 0);\ \mathbf{3}(1\ \ 0)\rangle + |\mathbf{6}(1\ \ -1);\ \mathbf{3}(-1\ \ 1)\rangle + |\mathbf{6}(0\ \ 1);\ \mathbf{3}(0\ \ -1)\rangle \\
\sqrt{6}|\mathbf{8}(0\ \ 0)_2\rangle &= 2|\mathbf{6}(-1\ \ 0);\ \mathbf{3}(1\ \ 0)\rangle - |\mathbf{6}(1\ \ -1);\ \mathbf{3}(-1\ \ 1)\rangle - |\mathbf{6}(0\ \ 1);\ \mathbf{3}(0\ \ -1)\rangle \\
\sqrt{6}|\mathbf{8}(0\ \ 0)_1\rangle &= |\mathbf{6}(-1\ \ 0);\ \mathbf{3}(1\ \ 0)\rangle + |\mathbf{6}(1\ \ -1);\ \mathbf{3}(-1\ \ 1)\rangle - 2|\mathbf{6}(0\ \ 1);\ \mathbf{3}(0\ \ -1)\rangle\,.
\end{aligned} \tag{7.33}$$

The reason for pushing the calculation this far (besides emphasizing its simplicity) is to make explicit the fact that the simple root subtraction procedure does not always leave the VC coefficients for degenerate weights orthogonal to one another, since the simple roots are often not orthogonal. In the **8**, $\langle 8(0\ \ 0)_1|8(0\ \ 0)_2\rangle = \frac{1}{2}$. However, this would seem to be a small price for the advantage that the VC coefficients can be calculated in terms of the Dynkin labeled states for any simple group, without an explicit choice of basis through some subgroup chain. The example of $\mathbf{27}\times\mathbf{27}$ in E_6 will exemplify this point.

It should be noted that, when there are many routes in the weight diagram that lead to a degenerate weight, as happens in composite irreps, the weight diagram is not trivially lifted to Hilbert space, because there is ambiguity about the best routes to label the degenerate weights. (This may be a helpful restatement of the labeling problem.) The simplest example where this problem can be found is in the calculation of the SU_3 VC coefficients for $\mathbf{6}\times\bar{\mathbf{3}} = \mathbf{15}+\mathbf{3}$ in the Dynkin basis. (Of course the calculation of these VC coefficients is trivial using the $SU_2\times U_1$ labeling, but the whole object of this method is to avoid a detailed selection of subgroup structure for the labeling.)

As a final example that also does not involve the labeling problem, consider the VC coefficients for $\overline{\mathbf{27}}\times\overline{\mathbf{27}} = \mathbf{27}_s + \mathbf{351}'_s + \mathbf{351}_a$ of E_6; this calculation was done in a single afternoon by J. Patera in our first effort to apply the above techniques. The highest weight is in the $\mathbf{351}'$, so all VC coefficients coupling $\mathbf{27}\times\mathbf{27}$ to $\mathbf{351}'$ are positive. If the $\mathbf{351}'$ weight has ±2 and 0 entries only, VC = 1; if the multiplicity of the $\mathbf{351}'$ weight is greater than 1, which means the multiplicity is 4 and the weight has the same entries as a weight in the **27**, VC $=\frac{1}{2}$; and otherwise it is $1/\sqrt{2}$, including many cases where the weight includes 0, ±2 and ±1. The VC coefficients for the **351** are $\pm\frac{1}{2}$ if the **351** weight multiplicity is greater than one (i.e., it is 5 when the weight coincides with one in the **27**), and otherwise it is $\pm1/\sqrt{2}$; the signs are such that **351** is antisymmetric in the $\overline{\mathbf{27}}\times\overline{\mathbf{27}}$ basis. The VC coefficients of the **27** are arranged symmetrically in $\overline{\mathbf{27}}\times\overline{\mathbf{27}}$, and are all of magnitude $1/\sqrt{10}$. Computation of the remaining phase is left as an exercise [62]. Displaying the answer is not as trivial as it is for small irreps of SU_2, SU_3 or SU_4.

8. Larger groups

Section 7 is a guide to E_6 and its subgroups that are large enough to contain $U_1^{em} \times SU_3^c$. A number of models based on larger groups have been suggested; this section contains a brief introduction to the irreps and tensor products of E_7, E_8, SU_8, SO_{14}, SO_{18} and SO_{22}, and some aspects of and questions about theories based on these groups.

The maximal subgroups of the exceptional groups E_7 and E_8 are listed in table 15. A short list of irreps, tensor products, and branching rules for E_7 is found in table 52 and for E_8 in table 53. The notation is the same as followed in section 7. The irreps of E_7 fall into two congruence classes, depending on whether $a_4 + a_6 + a_7$ is odd (pseudoreal) or even (real); there is only one congruence class for the irreps of E_8. (Recall table 12.)

A major objection to E_7 and E_8 is that they have self-conjugate irreps only, so it appears to take a detailed analysis of the symmetry breaking to determine whether the flavor-chiral character of the weak interactions is recovered in the low-energy limit. As an example, suppose a single family of left-handed fermions is assigned to a **56** of E_7 [**56** = (0 0 0 0 0 1 0)]. The fermion mass of a single family is symmetry breaking, since $(\mathbf{56} \times \mathbf{56})_s$ has no E_7 singlet. The $E_6 \times U_1$ content of the **56** is $\mathbf{27} + \overline{\mathbf{27}} + \mathbf{1} + \mathbf{1}$. If the low-mass fermions are in the **27**, then the corresponding states in the $\overline{\mathbf{27}}$ must acquire much larger masses, at least 17 GeV from the e^+e^- results, but not too large since those masses in $\overline{\mathbf{27}} \times \overline{\mathbf{27}}$ must have $|\Delta I^w| = \frac{1}{2}$ components, and masses above a few TeV would indicate strong contributions to the weak interactions [68]. (See ref. [69] for more on E_7 models.)

Separating the masses of such conjugate pairs is a problem in any vectorlike theory. So far, the larger mass of the "wrong-handed" doublets and singlets has been blamed on the details of the symmetry breaking, but it is not clear at this time what requirements must be satisfied for a vectorlike theory to reduce to a chiral weak-interaction theory at low energies. A nice solution to this problem would open Pandora's box, because many more groups would become likely candidates for unification. For example, a positive result would make it conceivable that the Lie algebra part of some future attractive theory is vectorlike. Since the generalization of Yang–Mills theories will probably be approached in a constructive fashion, this problem deserves more analysis.

Perhaps some insight will be gotten from a careful analysis of E_8. Not only is E_8 mathematically intriguing, but it has some possible physical interest, since its fundamental irrep **248** is also its adjoint (fermions are assigned to adjoints in some Yang–Mills theories with global supersymmetries [70]) and, moreover, there is an embedding of color and flavor where the **248** can accommodate three or four families of quarks and leptons [50]. The branching rules of the **248** to irreps of the maximal subgroups $SU_3 \times E_6$ and SO_{16} are given in tables 15 and 53. [When an irrep has not appeared in a previous table, it is listed as (Dynkin highest weight)**r**.] These subgroups contain a common rank 8 subgroup $SU_3 \times U_1 \times SO_{10}$, which is a maximal nonsemisimple subgroup of E_8 derived by removing the appropriate dot from the Dynkin diagram. Either going through $SU_3 \times E_6$ or $SU_4 \times SO_{10}$, the branching rule to $SU_3 \times SO_{10} \times U_1$ irreps is

$$\begin{aligned}\mathbf{248} = {} & (\mathbf{1}, \mathbf{16})(3) + (\mathbf{3}, \mathbf{16})(-1) + (\mathbf{1}, \overline{\mathbf{16}})(-3) + (\bar{\mathbf{3}}, \overline{\mathbf{16}})(1) + (\mathbf{3}, \mathbf{10})(2) \\ & + (\bar{\mathbf{3}}, \mathbf{10})(-2) + (\mathbf{3}, \mathbf{1})(-4) + (\bar{\mathbf{3}}, \mathbf{1})(4) + (\mathbf{8}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{45})(0) + (\mathbf{1}, \mathbf{1})(0)\,, \end{aligned} \tag{8.1}$$

where the U_1 generator is in parenthesis and color and flavor are embedded in SO_{10} as described in section 6; the explicit SU_3 is a family group in this example.

In order for the **248** to be flavor chiral at low energies, both the fermions and their antifermions (when the antiparticle exists at low energy) are assigned to the $(\mathbf{3}, \mathbf{16}) + (\mathbf{1}, \mathbf{16})$. It may be helpful to study C and CP for E_8, following the formalism of section 6, in formulating the mass problem.

The CP operation in E_8 is the reflection of the group generations that leaves an $SO_{16} \subset E_8$ invariant. As always, CP reverses the signs of all weights, and in this case, it is an inner automorphism. From table 17, SO_{16} and $SU_2 \times E_7$ are the only two symmetric subgroups of E_8, and the reflection leaving $SU_2 \times E_7$ invariant flips the signs of just four elements of the Cartan subalgebra of E_8. Thus there are just two candidate schemes for defining C and P; in terms of the $(\mathbf{3}, \mathbf{16})$ fermionic sector of (8.1), they are:

$$\begin{tikzcd} (\mathbf{3}, \mathbf{16})_L \arrow[r, "C'"] \arrow[d, "CP"'] & (\bar{\mathbf{3}}, \overline{\mathbf{16}})_L \arrow[dl, "P'"'] \\ (\bar{\mathbf{3}}, \overline{\mathbf{16}})_R & \end{tikzcd} \qquad \begin{tikzcd} (\mathbf{3}, \mathbf{16})_L \arrow[loop above, "C"] \arrow[d, shift right, "CP"'] \arrow[d, shift left, "P"] \\ (\bar{\mathbf{3}}, \overline{\mathbf{16}})_R \end{tikzcd} \tag{8.2}$$

Scheme 1 Scheme 2

where C' leaves the SO_{16} invariant and C leaves $SU_2 \times E_7$ invariant. Thus P' in scheme 1 simply exchanges L and R, but P in scheme 2 exchanges L and R **and** has a nontrivial action on the E_8 weights. (C and P each leave an $SU_2 \times E_7$ invariant, but the product CP leaves SO_{16} invariant.)

The more physical candidate for C appears to be scheme 2, where C commutes with the SU_3, U_1 and one of the SO_{10} Cartan subalgebra generators of $SU_3 \times U_1 \times SO_{10} \subset E_8$, and anticommutes with the same members of the SO_{10} Cartan subalgebra as the C for SO_{10} described in section 6. Perhaps C or C' plays an important role in classifying and analyzing the possible symmetry breaking patterns. For example, C' conserving masses match $(\mathbf{3}, \mathbf{16})$ to $(\bar{\mathbf{3}}, \overline{\mathbf{16}})$ (and so on) in a vectorlike fashion, while C conserving masses match $(\mathbf{3}, \mathbf{16})$ to itself (and so on). C conservation at some level seems to help obtain a flavor chiral breaking pattern. The analysis does not end here, since the neutrino mass problem also needs consideration; the $I^w = 0$ and $|\Delta I^w| = \frac{1}{2}$ breaking terms may also have definite C properties. The C in (8.2) is closely related to C in the E_6 flavor chiral models, which is discussed in more detail in section 9.

There is another aspect of C properties that deserves comment. Often researchers make up rules for giving fermion masses at various stages of the spontaneous breaking: the most popular is that at a given stage all singlet combinations get masses. This may be sensible in chiral theories, but it is a disaster in E_8, where already $(\mathbf{248} \times \mathbf{248})_s$ has a singlet. The rules must be modified for vectorlike theories, perhaps to state that the nonzero masses are only those singlets that conserve (or violate) the C properties in a particular way. The E_8 singlet violates C of (8.2), while conserving C'. These fragmentary comments will have to suffice here.

Some hopes and shortcomings of extended supergravity were briefly described in section 3. (Also see [47] for a review.) In the traditional interpretation, the elementary fields of $N = 8$ extended supergravity transform as representations of SO_8 and are identified with the elementary particles. The helicity 2 field transforms as an SO_8 singlet, and is identified with the graviton; helicity $\frac{3}{2}$ as $\mathbf{8}_v = (1\ 0\ 0\ 0)$, helicity 1 as $\mathbf{28} = (0\ 1\ 0\ 0)$, helicity $\frac{1}{2}$ as $\mathbf{56}_v = (0\ 0\ 1\ 1)$, helicity 0 as $\mathbf{35}_s + \mathbf{35}_c = (0\ 0\ 0\ 2) + (0\ 0\ 2\ 0)$, and the negative helicities as the appropriate conjugates. Tables 36 and 37 review the irreps and tensor products of SO_8. If the SO_8 is gauged (the vector bosons do transform correctly) and if SU_3^c is contained in this SO_8, then the largest possible flavor group is $U_1 \times U_1$, and the charged weak interactions cannot be mediated by elementary gauge particles. The elementary fermion sector also has problems: besides being vectorlike, there is room for only two charge -1 leptons; the e, μ and τ cannot all be treated as elementary. A gauge coupling of about 1/50 implies a cosmological constant so huge that the universe should be 10^{-30} cm in diameter, a prediction needing change or reinterpretation. Either $N = 8$ extended supergravity appears irrelevant as a physical theory, or this interpretation is wrong. The theory is attractive enough to attempt other interpretations [48]. Speculations about the interpretation are based on hidden symmetries of the global $N = 8$ Lagrangian; Cremmer and Julia found it to have a hidden SU_8 local symmetry [71].

The crucial conjecture is that the kinetic energy terms for the vector bosons associated with a formal, local symmetry are generated dynamically. For example, in certain two-dimensional field theories with hidden symmetries, the propagators of the currents of those symmetries, which are at least bilinear in the fundamental fields, do acquire poles in the quantum field theory [72]. Thus, the currents become dynamically generated, composite gauge particles. The generalization to the $N = 8$ supersymmetry currents, where all particles should be gauge particles, was conjectured in ref. [48]; the composite particles are identified with the supercurrents. The composite particles of each helicity fall into representations of SU_8. The helicity $-\frac{3}{2}$ states transform as a $\bar{\mathbf{8}} = (0\ 0\ 0\ 0\ 0\ 0\ 1)$, the helicity -1 as $\mathbf{63} = (1\ 0\ 0\ 0\ 0\ 0\ 1)$, helicity $-\frac{1}{2}$ as $\mathbf{216} = (0\ 1\ 0\ 0\ 0\ 0\ 1)$, helicity 0 as $\mathbf{420} = (0\ 0\ 1\ 0\ 0\ 0\ 1)$, helicity $\frac{1}{2}$ as $\mathbf{504} = (0\ 0\ 0\ 1\ 0\ 0\ 1)$, helicity 1 as $\mathbf{378} = (0\ 0\ 0\ 0\ 1\ 0\ 1)$, helicity $\frac{3}{2}$ as $\mathbf{168} = (0\ 0\ 0\ 0\ 0\ 1\ 1)$, helicity 2 as $\overline{\mathbf{36}} = (0\ 0\ 0\ 0\ 0\ 0\ 2)$, and helicity $\frac{5}{2}$ as $\bar{\mathbf{8}} = (0\ 0\ 0\ 0\ 0\ 0\ 1)$, plus *CPT* conjugates. This spectrum is overly chiral and leads to anomalies and other nonrenormalizable diseases. Thus, SU_8 must be broken at, say, the Planck mass to something smaller, for example, SU_5 or $SU_5 \times SU_3$. Some speculations on the symmetry breakdown and the efforts to make sense out of this spectrum are found in ref. [48]; the SU_8 representation theory, tensor products, and branching rules are listed in table 54.

We conclude this section with a brief introduction to yet another class of models, intended to unify the family structure with color and flavor, in which $\mathbf{f}_L$ is assigned to a simple complex spinor of SO_{4n+6} (n is number of low-mass families): SO_{14} has two families in the 64-dimensional spinor; SO_{18} has three families in the **256**, and SO_{22} has four families in the **1024**. The vector bosons in SO_{4n+6} are divided into the sets in $SO_{4n-4} \times SO_{10}$, where the SO_{10} is generated by color and flavor and SO_{4n-4} contains a gauged SU_2^f family group [14], and the bosons that mix family with color and flavor. The group theory of SO_{14} is reviewed in table 55, of SO_{18} in table 56, and of SO_{22} in table 57. There are many regularities in the tables that should not have to be mentioned explicitly.

Even if such ungainly groups appear far fetched, the systematics of the masses and mixing angles of the families could have wide generality. In order to give another example of the power of Dynkin diagrams (and raise some more questions), we show the initial steps in setting up the mass matrix for SO_{18}.

Let us break up $SO_{18} \supset SO_8 \times SO_{10}$, where SO_{10} contains the usual color and flavor $SU_2^w \times U_1^w \times SU_3^c$. If it is supposed that SO_{18} is broken to $SO_8 \times SU_2^w \times U_1^w \times SU_3^c$ at some large mass scale, and the SO_8 is broken to its maximal subgroup $SU_2^f \times Sp_4^t$ at some smaller scale, then it is conceivable that the Sp_4^t, if unbroken, may be confining and have a scale parameter much larger than the one in QCD; Sp_4^t is the extended color group (or "color prime" or "technicolor") in this theory. Then it is usually assumed that $\bar{Q}Q$ bound states, where Q is a fermion carrying the Sp_4^t charge, do the weak breaking in these kinds of theories [73]. (We should not lean too heavily on this breaking scenario, because even if it does survive certain criticisms [74], we have no simple example to support it.) We assume that the fermions with nonzero Sp_4^t charges are much heavier than the ordinary quarks and leptons. Most of the important physics issues still need to be faced, even with these assumptions, and we don't face any of them here. The discussion below is intended as examples in manipulation the Dynkin notation.

The **256** of SO_{18} has the $SO_8 \times SO_{10}$ branching rule $\mathbf{256} = (\mathbf{8}_s, \mathbf{16}) + (\mathbf{8}_c, \overline{\mathbf{16}})$; the nine-by-nine $SO_8 \times SO_{10}$ projection matrix can be selected as

$$P(\mathrm{SO}_{18}\supset \mathrm{SO}_8\times \mathrm{SO}_{10})=\begin{pmatrix}1&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0\\0&0&1&2&2&2&2&1&1\\0&0&0&0&1&0&0&0&0\\0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&1\end{pmatrix}, \tag{8.3}$$

which is derived from the branching rule, $\mathbf{18}=(\mathbf{8}_v,\mathbf{1})+(\mathbf{1},\mathbf{10})$ and the conventions of section 7. This is a regular subgroup embedding.

The embedding of $\mathrm{SU}_2^f\times \mathrm{Sp}_4^t$ in the SO_8 can be derived from the branching rules, $\mathbf{8}_v=(\mathbf{2},\mathbf{4})$, $\mathbf{8}_c=(\mathbf{2},\mathbf{4})$, or $\mathbf{8}_s=(\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{5})$. The choice that $\mathbf{8}_s$ has the "different" branching rule is analogous to similar conventions made in section 7. The projection matrix is

$$P(\mathrm{SO}_8\supset \mathrm{SU}_2\times \mathrm{Sp}_4)=\begin{pmatrix}1&2&1&0\\1&0&1&0\\0&1&0&1\end{pmatrix}. \tag{8.4}$$

The weight diagram of the $\mathbf{4}$ of Sp_4 is (1 0), (−1 1), (1 −1), (−1 0) and the $\mathbf{5}$, which is the vector of $\mathrm{SO}_5\sim \mathrm{Sp}_4$, is (0 1), (2 −1), (0 0), (−2 1), (0 −1). $\mathrm{SU}_2\times \mathrm{Sp}_4$ is a special subgroup of SO_8. There are two weights in the $\mathbf{8}_s$ that project to the same $\mathrm{SU}_2\times \mathrm{Sp}_4$ weight. Specifically, one linear combination of states with weights (−1 0 1 0) and (1 0 −1 0) of $\mathbf{8}_s$ is the zero weight in $(\mathbf{3},\mathbf{1})$ and the orthogonal combination is the zero weight in $(\mathbf{1},\mathbf{5})$. Clearly, the full analysis of the mass matrix requires computing some vector-coupling coefficients using, for example, the techniques in section 7; the $I_3^f=0$ family in the $\mathbf{256}$ is a linear combination of states with different weights. For example, the (0 0 0 0 1) SO_{10} quark is a linear combination of the states with weights (−1 0 1 −1 0 0 0 0 1) and (1 0 −1 0 0 0 0 0 1).

This situation does not occur in the SO_{14} model, but it is too small for phenomenology anyhow. In the embedding $\mathrm{SO}_{14}\supset \mathrm{SU}_2^f\times \mathrm{SU}_2^t\times \mathrm{SO}_{10}$, which is a regular subgroup, the $\mathbf{64}$ branches to $(\mathbf{2},\mathbf{1},\mathbf{16})+(\mathbf{1},\mathbf{2},\overline{\mathbf{16}})$. It does happen in SO_{22}, where the embedding of $\mathrm{SU}_2^f\times \mathrm{Sp}_6^t$ in SO_{12} is special. The spinor $\mathbf{32}'$ for SO_{12} is arranged to branch to $(\mathbf{3},\mathbf{6})+(\mathbf{1},\mathbf{14}')$ [$\mathbf{14}'=(0\ 0\ 1)$], so that the $\overline{\mathbf{16}}$'s of SO_{10}, which occur in $\mathbf{1024}$ as $(\mathbf{32},\mathbf{16})+(\mathbf{32}',\overline{\mathbf{16}})$, all carry extended color. The $\mathbf{32}$ then branches to $(\mathbf{4},\mathbf{1})+(\mathbf{2},\mathbf{14})$ [the weight system of the $\mathbf{14}$ is (0 1 0), (1 −1 1), (−1 0 1), (1 1 −1), (−2 2 −1), (2 −1 0), (0 0 0), (0 0 0), and the negatives of the 6 nonzero weights] and the (±1)(0 0 0) weights occur in both of the $\mathrm{SU}_2^f\times \mathrm{Sp}_6^t$ irreps in the branching rule of the $\mathbf{32}$.

From the projection matrices for SO_{18} and SO_{10}, it is easy to derive the dual coordinates of the roots and axes. They are

$$\begin{aligned}2\bar I_3^f&=[1\ 2\ 1\ 0\ 0\ 0\ 0\ 0\ 0] & \bar a_2^c&=[0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 0]\\ \bar a_1^t&=[1\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0] & 2\bar I_3^w&=[0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1]\\ \bar a_2^t&=[0\ 1\ 1\ 2\ 2\ 2\ 2\ 1\ 1] & 3\bar Q^{em}&=[0\ 0\ 0\ 0\ -1\ 0\ 3\ 1\ 2].\\ \bar a_1^c&=[0\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 1] & &\end{aligned} \tag{8.5}$$

Thus, the quantum numbers of an SO_{18} weight can be immediately identified.

Writing down the 256 weights of **256** is tedious, but the calculation can be focused on the 64 weights that are relevant for the low-mass fermions. The ordinary low-mass quarks and leptons are contained in the (**3**, **1**, **16**) of $SU_2^f \times Sp_4^t \times SO_{10}$ and the heavy extended colored objected are in (**1**, **5**, **16**) and (**2**, **4**, $\overline{\mathbf{16}}$). If $Sp_4^t \times SU_3^c \times U_1^{em}$ is conserved, each extended color, color, and electric charge sector is decoupled from the others in the mass matrix, and the (**3**, **1**, **16**) can be analyzed by itself. This then requires extracting the 48 states, and for that, we need to find the relevant 64 weights.

Let us find the four SO_{18} weights that contribute to the Sp_4^t singlet, charge $\frac{2}{3}$ quarks with SO_{10} weight (0 0 0 0 1). Each SO_{18} weight then has the form $(a\ b\ c\ d\ 0\ 0\ 0\ 0\ 1)$, so the $SO_8 \times SO_{10}$ weight from (8.3) is $(a, b, c, c+2d+1)(0\ 0\ 0\ 0\ 1)$, and the $SU_2^f \times Sp_4^t$ weight from (8.4) is $(a+2b+c)(a+c, b+c+2d+1)$. The Sp_4 weight must be (0 0), so $c=-a$ and $2d = a-b-1$; thus, $I_3^f = b$. The I_3^f values are +1, 0 and −1. For $b=1$, $a=0$, since an SO_{2n} spinor never has a weight with Dynkin label ±2. The weight of this state is (0 1 0 −1 0 0 0 0 1). The $I_3^f = -1$ state has weight (0 −1 0 0 0 0 0 0 1), and the weights for the $I_3^f = 0$ states are (1 0 −1 0 0 0 0 0 1) and (−1 0 1 −1 0 0 0 0 1), the former giving the (1 0 −1 0) and the latter giving the (−1 0 1 0) weight of the $\mathbf{8}_s$ of SO_8.

Let us compare this with the traditional way of computing the weight diagram. From (5.2), and with $\bar{R} = [16, 30, 42, 52, 60, 66, 70, 36, 36]$ (table 10), we find the level of (0 1 0 −1 0 0 0 0 1) to be 11 (it is, with this information, easy to see that the simple roots subtracted from the highest weight are $\alpha_9 + \alpha_8 + 2\alpha_7 + 2\alpha_6 + 2\alpha_5 + 2\alpha_4 + \alpha_3$), of (0 −1 0 0 0 0 0 0 1) is 15, of (1 0 −1 0 0 0 0 0 1) is 13, and of (−1 0 1 −1 0 0 0 0 1) is also 13; since the **256** is quite broad by level 15, it takes some writing to obtain all those weights. The rest of the calculation is simply a matter of subtracting the SO_{10} weights, with $\alpha_1 = (0\ 0\ 0\ {-1}\ 2\ {-1}\ 0\ 0\ 0)$, $\alpha_2 = (0\ 0\ 0\ 0\ {-1}\ 2\ {-1}\ 0\ 0)$, etc., where the only change from the procedure in table 11 is the first SO_{10} simple roots, which affects a_3. The projection worked out in table 21 shows that the charge $-\frac{2}{3}$ antiquarks with SU_3^c Dynkin labels (−1 0) are (0 1 0 1 0 0 −1 1 0), (0 −1 0 1 0 0 −1 1 0) and the $I_3^f = 0$ weights are (1 0 −1 1 0 0 −1 1 0) and (−1 0 1 0 0 0 −1 1 0).

This procedure can be carried out for other states of the **16**. The weights of the mass matrix elements can then be derived and their transformation properties analyzed, but such analyses require more knowledge about the symmetry breaking than we have here.

9. Symmetry breaking

Our purpose in this section is to analyze Yang–Mills theories with a representation of (effective) spinless fields $\phi(\mathbf{r})$, transforming as **r**, with components that acquire vacuum expectation values $\langle\phi(\mathbf{r})\rangle$ that break the symmetry group G to a subgroup H. We then give some simple applications to the analysis of various mass matrices. Let us begin with some general comments.

Since Yang–Mills theories are constructed on Lie groups and rely in essential ways on the structure of Lie groups [2], it might be possible to classify the candidate symmetry breaking directions in the Hilbert space for **r** without detailed study of specific symmetry breaking mechanisms. This would be fortunate, because the origin and determination of $\langle\phi(\mathbf{r})\rangle$ (given **r**) is not completely specified in the Yang–Mills formalism (except for very special choices of **r**). If, for example, the symmetry breaking is done by the Higgs mechanism, then $\langle\phi(\mathbf{r})\rangle$ depends on the arbitrary parameters in the Higgs potential, and the symmetry breaking pattern does also; in present theories they are evaluated phenomenologically. The Higgs potential is a fourth order, invariant polynomial in $\phi(\mathbf{r})$ (although radiative corrections bring in higher-order invariants) that has its absolute minimum at a non-zero value of $\langle\phi(\mathbf{r})\rangle$. The usual approach to symmetry breaking has been to minimize the Higgs potential (or an effective potential), that is, solve for $\langle\phi(\mathbf{r})\rangle$ in terms of those parameters, then find the subgroups that leave $\langle\phi(\mathbf{r})\rangle$ invariant for various values of the parameters. Finally the desired subgroup is selected, which usually leaves $\langle\phi(\mathbf{r})\rangle$ invariant for a range of parameters, and the parameters are adjusted within that range so various masses agree as well as possible with experiment [75, 76]. When ambiguities arise, they are often resolved by looking at the radiative corrections [77, 78]. If accurate relations among the parameters are needed phenomenologically, it is important to check whether radiative corrections modify them [46]. In any event, the ultimate goal of the calculation is to determine $\langle\phi(\mathbf{r})\rangle$ and the subgroup $\mathrm{H}\subset\mathrm{G}$ that leaves $\langle\phi(\mathbf{r})\rangle$ invariant, where $\langle\phi(\mathbf{r})\rangle$ minimizes the Higgs potential.

Let us dwell on the Higgs problem a moment longer. The minima for many classical potentials have been found. L.-F. Li already studied many cases with $\mathbf{r}$ irreducible in 1973 [75]. His approach was to set up a canonical form for $\phi(\mathbf{r})$ (which reduces the number of variables, but often requires foresight), substitute it into $V(\phi(\mathbf{r}))$, and then explicitly minimize it. This approach was extended by Ruegg and his collaborators to a number of cases where $\mathbf{r}$ is reducible [76]. The algebraic complexity of this program grows rapidly with the complexity of the representations, since essentially, it requires minimizing a function in a dim($\mathbf{r}$)-dimensional space, but it does have the advantage of being terribly explicit. It is also adequately powerful for some cases of physical interest. However, there are difficulties that suggest "explicit Higgsism" may not account for all the symmetry breaking. For example, consider the minimum of the Higgs potential for the $\mathbf{5}+\overline{\mathbf{5}}+\mathbf{24}$ breaking of SU_5. The parameters can be chosen so $\langle\phi(\mathbf{24})\rangle$ breaks SU_5 to $\mathrm{SU}_2^{\mathrm{w}}\times\mathrm{U}_1^{\mathrm{w}}\times\mathrm{SU}_3^{\mathrm{c}}$; in order to agree with experiment, $\langle\phi(\mathbf{24})\rangle$ must have magnitude of order 10^{15} GeV. The $\mathbf{5}+\overline{\mathbf{5}}$ breaks $\mathrm{SU}_2^{\mathrm{w}}\times\mathrm{U}_1^{\mathrm{w}}$ to $\mathrm{U}_1^{\mathrm{em}}$, where $\langle\phi(\mathbf{5})\rangle$ is of order 300 GeV. This breaking pattern holds for a range of parameters, but, unfortunately, the huge ratio $|\langle\phi(\mathbf{24})\rangle|/|\langle\phi(\mathbf{5})\rangle|$ is maintained over a tiny range of parameters only; it requires that the renormalized coefficient of the $(\phi^{\dagger}(\mathbf{5})\,\phi(\mathbf{5}))(\phi^{\dagger}(\mathbf{24})\,\phi(\mathbf{24}))$ term is tiny, even much smaller than the several loop radiative corrections. Thus, the radiative corrections to the parameters in the Higgs potential tend to obliterate huge mass ratios, unless that coefficient is carefully chosen (in an "unnatural" fashion) so its renormalized value is nearly zero. This predicament is called the "hierarchy problem" [46].

The technical problems and dilemmas of explicit Higgsism will not be discussed further here; dynamical symmetry-breaking mechanisms that can resolve some of these difficulties are not discussed either. Instead, we give an introductory group theoretical discussion of the possible breaking directions that a general class of breaking mechanisms can give. Of course we are guided by the Higgs problem, where $\langle\phi(\mathbf{r})\rangle$ is determined by minimizing an invariant function of $\phi(\mathbf{r})$. Because of the parameter dependence of the minimum, the most general answer is a list of subgroups H for the $\langle\phi(\mathbf{r})\rangle$ that can minimize some class of invariant functions of $\phi(\mathbf{r})$. Specifically, we expect that any realistic breaking mechanism is determined by a symmetry breaking direction that minimizes a nontrivial invariant function of $\phi(\mathbf{r})$. Let us restate the problem more formally.

For a group G and representation $\mathbf{r}$ of G, we want to find the components of $\mathbf{r}$ that can extremize nontrivial invariant functions of $\mathbf{r}$, and then to find the subgroups of G that leave each of these components invariant. If we can do so, then finding the breaking direction is reduced to a one-dimensional problem, which can be solved by substituting each candidate answer into the specific function to minimized, and selecting the minimum.

L. Michel has conjectured a solution to this problem in the case that the function is a Higgs potential and $\mathbf{r}$ is real and irreducible, or $\mathbf{r}=\mathbf{r}'+\overline{\mathbf{r}}'$ where the irrep $\mathbf{r}'$ is complex [15]. The presentation here of his results is intuitive and not quite as general mathematically as is possible; the reader should refer to [15] and [79] for references and proofs. His conjecture does not cover the breaking of the group of a unified model, which must be done by a reducible representation. However, there is reason to speculate that Michel's conjecture can be formulated to cover all possible "realistic" breaking schemes. An algebraic (or geometrical) solution to the symmetry breaking problem would be attractive. Unfortunately, some conditions (not necessarily satisfied for an arbitrary Higgs potential) should be satisfied before such solutions can be guaranteed.

The Michel–Radicati theorems on symmetry breaking motivate the above conjectures [80]. They are restated here in a somewhat restricted form in terms of a Yang–Mills theory based on the simple group G with a representation of (effective) spinless fields $\phi(\mathbf{r})$ that breaks G to H by an effective Higgs mechanism, in the simple case that $\mathbf{r}$ is a real irrep or a self-conjugate pair of complex irreps. (Irreducibility is not an essential requirement of the theorems, but it is essential for Michel's conjecture.) The vacuum value $\langle\phi(\mathbf{r})\rangle$ is a point (or ray) in the Hilbert space of $\mathbf{r}$, which is a dim($\mathbf{r}$)-dimensional space. The Michel–Radicati theorems connect properties of the space of $\langle\phi(\mathbf{r})\rangle$ to the subgroups H leaving $\langle\phi(\mathbf{r})\rangle$ invariant, and to the existence of stationary points of invariant functions of $\phi(\mathbf{r})$, such as a Higgs potential. Specifically, their theorems relate a subset of those subgroups to the existence of stationary points of the functions. (Of course the real problem is to relate stationary points, or more specifically the extrema, of the function to the subgroups, which is the converse of the theorems.) Let us first state the theorems, complete with new jargon, and then define the new terms in detail.

(1) If $\mathrm{H}\subset\mathrm{G}$ is a **maximal little group** of $\mathbf{r}$, then the $\langle\phi(\mathbf{r})\rangle$ that are invariant under the transformations in H are in a **closed stratum** of the dim($\mathbf{r}$) space.

(2) If $\langle\phi(\mathbf{r})\rangle$ is in a closed stratum of the dim($\mathbf{r}$) space, then $\langle\phi(\mathbf{r})\rangle$ is a stationary point of some (or in some cases, all) smooth real invariant functions of $\phi(\mathbf{r})$. (This theorem will be restated below more completely.)

The largest subgroup of G that leaves a nonzero $\langle\phi(\mathbf{r})\rangle$ invariant is called the little group (or stability group) of $\langle\phi(\mathbf{r})\rangle$, H; the generators in G/H do not annihilate $\langle\phi(\mathbf{r})\rangle$. As $\langle\phi(\mathbf{r})\rangle$ goes through all possible directions in the dim($\mathbf{r}$) space, the corresponding little groups go over a subset of the subgroups of G. For each little group, the branching rule of $\mathbf{r}$ into irreps of H must have at least one singlet: $\mathbf{r}=\mathbf{1}+\cdots$. H is a maximal little group if there is no $\langle\phi(\mathbf{r})\rangle$ in the dim($\mathbf{r}$) space with little group H* satisfying $\mathrm{G}\supset\mathrm{H}^*\supset\mathrm{H}$, where, of course, H and H* both have branching rules of the form $\mathbf{r}=\mathbf{1}+\cdots$. In this problem we hold the length $|\langle\phi(\mathbf{r})\rangle|$ fixed, and vary its direction only.

It is worthwhile repeating an elementary but important point: the embedding of the little groups of $\mathbf{r}$ in G can be specified by the branching rule for $\mathrm{H}\subset\mathrm{G}$ as $\mathbf{r}=\mathbf{1}+\cdots$, or the branching rule for some other faithful irrep of G. Just as for the maximal subgroups, a little group of a given name in some instances can have several inequivalent embeddings in G. For example, E_8 has three inequivalent SU_2 maximal subgroups; there are less academic examples in symmetry breaking problems.

The total list of little groups often includes many of the subgroups of G, ranging from maximal or nearly maximal subgroups down to nothing. Of course some of these breaking patterns may not be possible for a given irrep in a Yang–Mills theory: if $\mathbf{r}$ does not have enough degrees of freedom to give masses to all the bosons in G/H, then that breaking is excluded; and more trivially, if $\mathbf{r}$ has no singlets in some (nearly) maximal subgroup, that subgroup is never the unbroken Yang–Mills theory after the symmetry breaking. The group theoretical solution to the symmetry breaking problem keeps track of both of these features of a Yang–Mills field theory.

Next, we examine the relation of the little groups to the regions in the dim($\mathbf{r}$) space of the $\langle\phi(\mathbf{r})\rangle$. Suppose H is the little group of $\langle\phi(\mathbf{r})\rangle$. The subgroup constructed from $g\mathrm{H}g^{-1}$, where g is a transformation in G, is isomorphic to H; since the gauge freedom of G allows $\langle\phi(\mathbf{r})\rangle$ to be transformed to $\Delta(g)\langle\phi(\mathbf{r})\rangle$ ($\Delta(g)$ is the matrix representation of g on $\mathbf{r}$), this transformation makes no change in the physical content of the theory. The "line" traced out in the dim($\mathbf{r}$) space by $\Delta(g)\langle\phi(\mathbf{r})\rangle$ for all g is called the **orbit** of $\langle\phi(\mathbf{r})\rangle$. Except in a few special cases, the orbit covers only part of the dim($\mathbf{r}$) space. (Recall $\langle\phi(\mathbf{r})\rangle$ is of fixed length.)

Next we gather together sets of orbits into **strata**. Let us examine the little group of a point in the dim($\mathbf{r}$) space that is an infinitesimal distance away from the orbit of $\langle\phi(\mathbf{r})\rangle$, asking whether the little group has changed; if it has not changed up to a conjugation of the little group, that orbit belongs to the same stratum as $\langle\phi(\mathbf{r})\rangle$. Thus, the dim($\mathbf{r}$) space can be divided into strata, which are distinguished by their little groups.

One of the most interesting characteristics of a stratum is the nature of its "edge", whether it is open or closed in the topological sense. An orbit is in a **generic** stratum if there is an orbit within every infinitesimal distance of each orbit (that is, the stratum is open), so the boundary is itself in a different stratum, which is closed. Actually, the boundary orbit of a generic stratum is in a stratum with a larger little group. This statement can be tightened up considerably: if the little group of $\langle\phi(\mathbf{r})\rangle$ is a maximal little group, then $\langle\phi(\mathbf{r})\rangle$ is in a closed stratum, which is the first theorem above.

It may be helpful to use this language on a trivial situation. Let the little group be the set of reflections $x+a\leftrightarrow -x+a$ that leaves $V(x)=x^2$ invariant. For $a\neq 0$ the little group is "nothing"; the intervals (strata) $a\neq 0$ are open. At the boundary point $a=0$, which is a "closed stratum", there is a twofold reflection symmetry $x\leftrightarrow -x$. This illustrates the first theorem. The feature that $V(x)$ is also minimum at $a=x=0$ corresponds to the second theorem. The feature that there are no other minima corresponds to the Michel conjecture.

The second theorem relates the topological characteristics of the edge of the strata to the existence of stationary points of invariant functions of $\phi(\mathbf{r})$. Theorem 2 above can be broken down into several cases [80]. Let the magnitude of $\langle\phi(\mathbf{r})\rangle$, which is the second-order invariant of $\mathbf{r}$, be fixed. Michel and Radicatti then prove that:

(1) No higher-order invariant of $\mathbf{r}$ has an extremum on an orbit in a generic stratum. (However, **functions** of invariants may have extrema in generic strata without the individual invariants being extreme.)

(2) If there is only one orbit in a closed stratum (called a critical orbit), then all real smooth functions of the invariants of $\mathbf{r}$ are stationary (i.e., $\mathrm{d}V(\phi(\mathbf{r}))=0$ for any $V(\phi)$ at $\phi(\mathbf{r})=\langle\phi(\mathbf{r})\rangle$). Orbits small distances away from a critical orbit have smaller little groups.

(3) If there is more than one orbit in a closed stratum, any function of the invariants is stationary on at least two orbits (a relative minimum and maximum), and for any given orbit there is some function of the invariants that is stationary on it.

Let us summarize the line of argument just followed: H is a maximal little group implies $\langle\phi(\mathbf{r})\rangle$ is in a closed stratum, which in turn, implies that $V(\phi(\mathbf{r}))$ is stationary for $\langle\phi(\mathbf{r})\rangle$ on some orbit in that stratum. This line of argument is converse to the one needed for minimizing the Higgs potential, where the little groups should be determined by minimizing $V(\phi(\mathbf{r}))$. Michel shows [15] for a special case that the converse is, indeed, true: If $V(\phi(\mathbf{r}))$ is a 4th order Higgs potential and $\mathbf{r}$ is an irrep or a conjugate pair of complex irreps, then the little group of any $\langle\phi(\mathbf{r})\rangle\neq 0$ that minimizes $V(\phi)$ is maximal. When extended to include radiative corrections, we call this the Michel conjecture.

This conjecture is probably not obvious to many model builders, because there are some popular, but trivial counter examples. If the Higgs potential depends only on the second order invariant $\phi^\dagger\phi$, then $\langle\phi(\mathbf{r})\rangle$ of fixed magnitude may have nonmaximal little groups. For example, octet breaking of SU_3 with a reflection symmetry $\mathbf{8} \leftrightarrow -\mathbf{8}$ classically breaks SU_3 to $SU_2 \times U_1$ or $U_1 \times U_1$. Even without the reflection symmetry the adjoint **78** can break E_6 in the classical Higgs problem down to $(U_1)^6$. However, the value of the minimum of $V(\langle \phi(\mathbf{r})\rangle)$ is the same for all subgroups (afterall, $V(\phi(\mathbf{r}))$ depends only on $|\langle\phi(\mathbf{r})\rangle|$, so it is constant), and it is necessary to examine the radiative corrections to make a choice. After the quantum corrections are completed, the only remaining stationary points do have maximal little groups. (The concern that $V(\phi(\mathbf{r}))$ may have additional minima can be seen from studying an SU_3 invariant sixth order function of the **8** of the form $(\mathbf{8}^2 - a)^2 + (\mathbf{8}^3 - b)^2$ the ratio of c_3/c_2 at the minimum depends on the coefficients, and so it is arbitrary. (c_i is the ith order invariant.) Only for a special value of the ratio is the little group $SU_2 \times U_1$; otherwise it is the nonmaximal little group $U_1 \times U_1$ [81].) Of course, renormalizability restricts the potential to 4th order, so this example is academic. However, it makes it clear that the Michel conjecture could be wrong: extreme values of $V(\phi)$ can be due to extremizing the function **or** to extremizing the invariants. The evidence is that, for a Higgs potential plus radiative corrections, the extreme values of $V(\phi)$ results from extremizing the invariants (not the function), so Michel's conjecture holds for cases of physical interest.

The symmetry breaking patterns derived from Michel's conjecture coincide with the patterns derived by explicitly minimizing Higgs potentials with radiative corrections. Thus, the list of maximal little groups gives a complete list of possible minima of the Higgs potential. This list gives as complete an answer as possible, at least until there is a theory for the arbitrary parameters in the (effective) potential. When there is, the minimum can be found by substitution. It must be noted, however, that there may be some maximal little groups of $\langle\phi\rangle$ that cannot minimize a Higgs potential. For example, adjoint breaking of SO_{2n+1} yields $SO_{2n-1} \times U_1$ or $SU_n \times U_1$ only, depending on parameters [75], where the other maximal little groups are merely stationary points.

A few examples of the above procedure will establish its simplicity in applications; the answers to well-known problems and many new ones can be obtained with essentially no work. One of the prettiest examples is single adjoint breaking; it is easy to verify the following rule [82]: each maximal little group of the adjoint irrep of G is found by removing one dot from the Dynkin diagram; the group of the resulting Dynkin diagram times a U_1 factor is a maximal little group. [Recall the discussion below (6.10).] (To get a complete list of maximal stability groups this procedure should be repeated for each dot in the Dynkin diagram of G.) Thus the **78** of E_6 can break it to $SO_{10} \times U_1$, $SU_2 \times SU_5 \times U_1$, $SU_2 \times SU_3 \times SU_3 \times U_1$, or $SU_6 \times U_1$. As mentioned before, in the Higgs problem without radiative corrections, the potential depends on the invariant $\phi^\dagger(\mathbf{78})\ \phi(\mathbf{78})$ only, so the minimum does not depend on direction; the symmetry group of $V(\phi)$ is $SO_{78} \supset E_6$ until quantum corrections are included. However, the one-loop corrections bring in higher-order invariants; for a model with adjoints of scalars and vectors, $SO_{10} \times U_1$ is the stability group of the absolute minimum [78].

As another example, SU_n can be broken by $\mathbf{n} + \bar{\mathbf{n}}$ to SU_{n-1} only, since $\mathbf{n}$ has no singlets in other subgroups of SU_n that are not also subgroups of SU_{n-1}. Similarly, $\mathbf{n}$ breaks SO_n to SO_{n-1}. The breaking of SO_{10} by $\mathbf{16} + \overline{\mathbf{16}}$ is a quite nontrivial Higgs problem. The singlet of the **16** in SU_5 is already familiar, where $\mathbf{16} = \mathbf{1} + \bar{\mathbf{5}} + \mathbf{10}$ and $\mathbf{10} = \mathbf{5} + \bar{\mathbf{5}}$. There is another maximal little group, which can be found by the chain $SO_{10} \supset SO_9 \supset SO_8 \supset SO_7$ with $\mathbf{16} = \mathbf{1} + \mathbf{7} + \mathbf{8}$ of SO_7 and $\mathbf{10} = \mathbf{1} + \mathbf{1} + \mathbf{8}$. Clearly SO_7 with rank 3 is not a subgroup of SU_5: either look at table 14 or note that there are no candidate branching rules of the **5** of SU_5 to irreps of SO_7. The numerics of other conceivable branching rules of the form, $\mathbf{16} = \mathbf{1} + \cdots$, of other maximal little groups are clearly impossible, as can be seen from table 14 and a rudimentary knowledge of the irreps of those subgroups of SO_{10} or their subgroups. So the complete list of maximal little groups of the $\mathbf{16} + \overline{\mathbf{16}}$ of SO_{10} are SU_5 and SO_7. Michel's conjecture yields the same list of breaking patterns as explicit Higgsism [76].

Finally, consider some E_6 examples. The maximal stability groups of $\mathbf{27} + \overline{\mathbf{27}}$ are SO_{10} and F_4; all other singlets of **27** are in subgroups of SO_{10} and F_4. The $\mathbf{351'}+\overline{\mathbf{351'}}$ [$\mathbf{351'}=(0\ 0\ 0\ 0\ 2\ 0)$] has more maximal stability groups: SO_{10}, F_4, Sp_8, SU_3, $SU_2\times SU_4$, and G_2. This example makes it clear that **all** the maximal subgroups are of interest for a complete analysis of symmetry breaking, so the more lengthy tables of ref. [57] are indispensable. The shortcoming of our tables, that they are restricted mainly to physical embeddings of color and flavor, is partially rectified in table 58, where the branching rules to the irreps of all the maximal subgroups of the first few irreps of each simple group up to rank 6 are listed. The reader should enjoy finding little groups of other irreps; a long list will appear in [16].

There are a number of candidate extensions of Michel's conjecture to reducible representations, but without a statement of the physical constraints on the symmetry breaking in unified models, they may appear rather ad hoc. Without some restrictions any little group appears possible, including nothing. As another example, if a strictly maximal little group is required so that the singlets of the different irreps are "lined up" as well as possible, then E_6 and SO_{10} cannot be broken to just QCD and QED. Besides there are many Higgs-model counter examples to such a stringent requirement. (See ref. [76], or consider the algebraically trivial example of $\mathbf{3}+\bar{\mathbf{3}}+\mathbf{8}$ breaking of SU_3 with $\mathbf{8}\leftrightarrow-\mathbf{8}$ symmetry, up to dimension 4 terms, and nonzero vacuum values of $\langle\bar{\mathbf{3}},\mathbf{3}\rangle$ and $\langle\mathbf{8},\mathbf{8}\rangle$. SU_3 is then broken to either SU_2 or U_1, depending on the sign of the $\bar{\mathbf{3}}(\mathbf{8}^2)_s\mathbf{3}$ invariant, where $(\mathbf{8}^2)_s$ is coupled to the $\mathbf{8}$ in $\bar{\mathbf{3}}\times\mathbf{3}$. Apparently, proper attention must be paid to the "mixed" invariants, made from different irreps in the Higgs problem, so attention to this is required in general.)

A possible method for finding the little group in the reducible case can be physically motivated as follows. Consider a two representation problem with $\mathbf{r}_1+\mathbf{r}_2$ (or fields $\phi(\mathbf{r}_1)$ and $\phi(\mathbf{r}_2)$), where $\mathbf{r}_1$ and $\mathbf{r}_2$ are each real irreps or complex-conjugate pairs of irreps of G. (Generalization to more irreps will be obvious.) Since it is likely that the breakings $\langle\phi(\mathbf{r}_1)\rangle$ and $\langle\phi(\mathbf{r}_2)\rangle$ are due to different physical mechanisms, there is no reason a priori to believe that the vacuum values have the same order of magnitude. (The mass hierachies required for unified models to be viable support this viewpoint.) Thus, the symmetry breaking should be treated sequentially, not simultaneously, with $\langle\phi(\mathbf{r}_1)\rangle>\langle\phi(\mathbf{r}_2)\rangle$.

The first breaking can be analyzed by Michel's conjecture: $\langle\phi(\mathbf{r}_1)\rangle$ breaks G to H_1, where H_1 is a maximal little group of $\mathbf{r}_1$; $\mathbf{r}_2$ is broken into a sum of irreps $\Sigma_i\,\mathbf{r}_{2i}$ of H_1, where each $\mathbf{r}_{2i}$ is a real irrep of H_1 or a complex-conjugate pair. At first sight, the sequential picture does not seem to help, since the next problem, that of breaking H_1 to H_2 by the reducible representation $\Sigma_i\,\mathbf{r}_{2i}$, looks like the original problem. However, at mass scales much below $\langle\phi(\mathbf{r}_1)\rangle$ the effective Lagrangian with symmetry H_1 and spinless bosons $\Sigma_i\,\phi(\mathbf{r}_{2i})$ in an effective Higgs potential still knows about the symmetry G. For example, the invariants of $\mathbf{r}_2$ break up into specific sums of the invariants of H_1. In many cases, *the nonzero component of* $\langle\phi(\mathbf{r}_2)\rangle$ *is in only one* $\mathbf{r}_{2i}$ *of* H_1, *so that* H_2 *is a maximal little group of* $\mathbf{r}_{2i}$. This proposal correctly reproduces the symmetry breaking patterns of the Higgs potentials that have been solved; however, $\langle\phi(\mathbf{r}_1)\rangle$ can be moved from its orbit in an H_2 invariant direction, so although the group H_2 may be correct, $\langle\phi(\mathbf{r}_1)\rangle$ is no longer invariant under H_1. Thus, the feature that the subgroups work out correctly could be a coincidence [83].

A simple example will illustrate the conjecture and its difficulties. Reconsider the $\mathbf{3}+\bar{\mathbf{3}}+\mathbf{8}$ of SU_3 problem. If the $\mathbf{3}$ breaking is first, its maximal little group is SU_2 only. Then, with $\mathbf{8}=\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{3}$ of SU_2, the final possibilities are SU_2, U_1, and nothing. In the other order, $\mathbf{8}$ can break SU_3 to $SU_2\times U_1$ with $\mathbf{3}=\mathbf{2}(-1)+\mathbf{1}(2)$. The $\mathbf{3}$ then breaks $SU_2\times U_1$ to SU_2 or U_1. Only in the case of SU_2 do both the $\bar{\mathbf{3}}\,\mathbf{8}^2\,\mathbf{3}$ and $\bar{\mathbf{3}}\,\mathbf{8}\,\mathbf{3}$ invariants have stationary points (the stationary point of the $\bar{\mathbf{3}}\,\mathbf{8}\,\mathbf{3}$ invariant is a saddlepoint); in the other two cases only the $\bar{\mathbf{3}}\ \mathbf{8}^2\,\mathbf{3}$ is stationary. If there is the notion of a critical orbit in the reducible representation breaking case, it seems to require the breaking directions of the various representations to line up as well as possible. However, at least in this example, the Higgs mechanism can select other cases, so the notion of a critical orbit is not as attractive as for the irreducible case. Even with these concerns in mind, it is interesting to apply the rule to cases that have been solved explicitly.

The above rules can be used to rederive rather quickly the breaking patterns of the $\mathbf{n}+\bar{\mathbf{n}}+\mathbf{n}+\bar{\mathbf{n}}$ of SU_n, which breaks it to SU_{n-2}; the $\mathbf{n}+\mathbf{n}$ of SO_n, which breaks it to SO_{n-2}; the $\mathbf{n}+\bar{\mathbf{n}}+$ adjoint of SU_n, which gives the long list in ref. [76]. The reader might enjoy checking their answer for $\mathbf{16}+\overline{\mathbf{16}}+\mathbf{45}$ breaking of SO_{10}.

As a final simple example, consider how far the difermions in a one-family E_6 model with $\mathbf{f}_L=\mathbf{27}$ can break E_6 along the chain, $SO_{10}\supset SU_5\supset SU_2\times U_1\times SU_3^c\supset U_1^{em}\times SU_3^c$. The difermions are spinless bosons in $(\overline{\mathbf{27}}\times\overline{\mathbf{27}})_s=\mathbf{27}+\mathbf{351}'$. There exist vacuum values in the $\mathbf{351}'$ that can break E_6 to $SU_2\times U_1\times SU_3^c$, with a vacuum value in the $\mathbf{27}$ completing the physical breaking chain, up to additional U_1 factors. However, our question is, is there a Higgs potential that can do this? In fact, this complicated Higgs problem has been solved by a computer program written by Stech and collaborators [8], and their preliminary results are that E_6 can at best be broken to $SU_2\times U_1\times SU_3^c$.

As stated before, the maximal little groups of the $\mathbf{27}$ are SO_{10} and F_4, and the maximal little groups of the $\mathbf{351}'$ are SO_{10}, F_4, Sp_8, G_2, SU_3, and $SU_2\times SU_4$. (In the last case, the branching rule of the $\mathbf{27}$ is $(\mathbf{2},\mathbf{6})+(\mathbf{1},\mathbf{15})$.) Thus, it is clear that the greatest flexibility is gotten if the $\mathbf{27}$ does the first breaking. Then the $\mathbf{351}'$ breaks up into $\mathbf{1}+\mathbf{10}+\overline{\mathbf{16}}+\mathbf{54}+\overline{\mathbf{126}}+\mathbf{144}$ of SO_{10}. The $\mathbf{126}$ can break SO_{10} to SU_5, and the $\mathbf{144}$ can break it to $SU_2\times U_1\times SU_3^c$. Thus, $U_1^{em}\times SU_3^c$ is not a solution of $\mathbf{27}+\mathbf{351}'$ breaking. Further exploration and application of these techniques will not be described here; they are planned to be the topics of [16]. Also, family physics and other efforts to be more realistic are not discussed here.

Exploitation of the correspondence between weight space and Hilbert space of an irrep greatly simplifies representation theory, and if one is willing to select a basis for the embedding of flavor and color in G, this correspondence can often be used to simplify greatly explicit calculations in Yang–Mills theories. For example, there are several interesting cases where the concept of symmetry breaking direction in weight space (in addition to the Hilbert space direction) is useful, and can be used to analyze the breaking by directly computing the vector-boson mass matrix. The generality of this procedure is vindicated by the above analysis; we discuss this in more detail now.

The vector-boson mass matrix of a Yang–Mills gauge theory with local symmetry G can be obtained in the lowest-order approximation to the Higgs symmetry-breaking mechanism often in a very simple way once the vacuum expectation values of the spin zero fields are known. Its group theoretical structure is abstracted from the term in the Lagrangian, $(D_\mu\phi)^\dagger D^\mu\phi$, where $D_\mu=\partial_\mu-igA_\mu^aT_a$, T_a is a representation matrix of the scalar fields, and $\langle\phi(\mathbf{r})\rangle=\phi_v$ is a set of constant fields in the unitary gauge corresponding to the spontaneously broken vacuum state of the quantum field theory. The mass matrix, obtained from $(D_\mu\phi_v)^\dagger D^\mu\phi_v=M^{ab}A_{\mu a}A_b^\mu$ is often written in the notation,

$$(M^2)_{ab}=g^2(\phi_v^*)_i\;(T_{-a})_{ij}\;(T_b)_{jk}\;(\phi_v)_k, \tag{9.1}$$

where $(T_a)_{ij}$ is the ij matrix element of the ath generator in the representation $\mathbf{r}$ to which the spinless fields are assigned, so $a=1,\ldots,N(\mathrm{adj})$ and $i=1,\ldots,N(\mathbf{r})$. T_a is a ladder matrix with root a, so $(T_a)^\dagger=T_{-a}$. The notation in (9.1) is rather schematic, since, i, j, k are weights plus whatever other labels are needed for the representation vectors. In a more explicit notation $(T_a)_{ij}=\langle\mathbf{r},\lambda|X_a|\mathbf{r},\lambda'\rangle$, where X_a is the generator with root a, and the matrix element of X_a is nonzero when $\lambda=a+\lambda'$. Since Hermitian conjugation (or CP conjugation) reverses the signs of the weight of a field and takes $\mathbf{r}$ to $\bar{\mathbf{r}}$, (9.1) may be written,

$$(M^2)_{ab}=g^2\sum_{\substack{\lambda\lambda'\lambda''\\ \mathbf{r}}}\phi_v^\dagger(\bar{\mathbf{r}},-\lambda'')\;\langle\bar{\mathbf{r}},-\lambda''|X_{-a}|\bar{\mathbf{r}},-\lambda\rangle\;\langle\mathbf{r},\lambda|X_b|\mathbf{r},\lambda'\rangle\;\phi_v(\mathbf{r},\lambda')\,. \tag{9.2}$$

This notation has the advantage of making it explicit that $\phi_v(\mathbf{r},\lambda)$ is a tensor operator and $\phi_v^\dagger(\bar{\mathbf{r}},-\lambda)$ is an adjoint tensor operator, with $\phi_v^\dagger(\bar{\mathbf{r}},-\lambda)=$ phase $\times\,(\phi_v(\mathbf{r},\lambda))^\dagger$.

In the (classical) Higgs model, (9.2) is evaluated by computing the vacuum values $\phi_v(\mathbf{r},\lambda)$ by minimizing the Higgs potential for some choice of the parameters, and then diagonalizing (9.2) to find the vector boson masses. The little group is determined by the eigenstates of (9.2) with zero eigenvalues, or, equivalently, by the set of transformations leaving $\phi_v(\mathbf{r},\lambda)$ invariant. We now look at some special cases where a choice of basis makes the analysis of (9.2) easy.

There are cases where the mass matrix is greatly simplified if the definition of a tensor operator is substituted into (9.2). A tensor operator $T(\mathbf{r},\lambda)$ is defined by the commutation relations,

$$[X_a, T(\mathbf{r},\lambda)]=\sum_{\lambda'}\langle\mathbf{r},\lambda'|X_a|\mathbf{r},\lambda\rangle\; T(\mathbf{r},\lambda'), \tag{9.3}$$

and the adjoint of the operator changes weights when acting on kets by $-\lambda$. Thus (9.2) may be written without loss of generality as [84]

$$(M^2)_{ab}=-g^2\sum_{\lambda}\mathrm{Tr}\{[X_{-a},\phi_v^\dagger(\bar{\mathbf{r}},-\lambda)]\;[X_b,\phi_v(\mathbf{r},\lambda)]\}, \tag{9.4}$$

which is a convenient form in cases where $\phi_v(\mathbf{r},\lambda)$ is easily expanded as a polynomial of the generators. Of course, the simplest case is adjoint breaking, where ϕ_v is just a generator.

Suppose that $\mathbf{r}$ is the adjoint of G and $\phi_v(\mathbf{r},\lambda)$ has zero weight, so that ϕ_v can be expanded in terms of the generators in the Cartan subalgebra as

$$\phi_v(\mathbf{r},\lambda=0)=\sum_{i=1}^{l}\bar{c}_iH_i; \tag{9.5}$$

the axis defined by $[\bar{c}_1,\ldots,\bar{c}_l]$ is the symmetry breaking direction in *weight space*, as is now shown. If (9.5) is substituted into (9.4), along with the commutation relation (4.2) ($[H_i,E_a]=a_iE_a$, where a is a root), then (9.4) becomes

$$(M^2)_{ab}=g^2\,\mathrm{Tr}\Big\{\Big(\sum_i a_i\bar{c}_i\Big)\;X_{-a}\;\Big(\sum_j b_j\bar{c}_j\Big)\;X_b\Big\}=g^2\,\delta_{ab}\;\Big(\sum_i\bar{c}_ia_i\Big)^2, \tag{9.6}$$

where an obvious normalization convention on the group generators is used in the last step. Note that $\Sigma\, a_i\bar{c}_i=(c,a)$ is a scalar product in weight space, so the mass is proportional to the projection of the weight of the vector boson onto the symmetry breaking axis c; no boson in the Cartan subalgebra gets a mass by this breaking.

We use this result to answer explicitly and quickly, can **24** adjoint breaking take SU_5 to $SU_2\times U_1\times SU_3$. The breaking must be in the Cartan subalgebra since the rank of the two groups is the same. An affirmative answer is required by the "dot removing procedure", so it must be possible to find a vector c that does so. Any adjoint breaking must leave the gluons massless. Thus c must be perpendicular to the color roots, identified from the action of (7.4) on the roots in table 9 to be (1 0 0 1), (1 1 −1 0) and (0 −1 1 1). Thus, $\bar{c}$ must have the form $[a,b,a+b,-a]$. For the big $I^w=0$ breaking, c must also be perpendicular to the I_+^w root (−1 1 1 −1), so $a=-2b$. Thus $\bar{c}$ is proportional to [−2 1 −1 2] and c is proportional to $(1\ -1\ 1\ -1)$. The magnitudes of $(c,\ a)$ for the lepto-diquark roots are equal and nonzero; this gives an explicit representation of the breaking of SU_5 to $SU_2 \times U_1 \times SU_3$ by the **24**.

It is easy to carry out the analogous analysis for adjoint breaking of E_6. The breaking direction must be perpendicular to the simple color roots, $(0\ 1\ 0\ 0\ -1\ 1)$ and $(0\ -1\ 0\ 0\ 1\ 1)$ (see table 20), so

$$\bar{c} = [-c\ \ d\ \ a\ \ b\ \ d\ \ 0], \tag{9.7}$$

where the embedding of color and flavor follows the conventions of section 7. If, in addition, c is perpendicular to the I^{w}_{+} root, as is necessary for the very large components of the $I^{\mathrm{w}} = 0$ breaking, the big breaking B has the form

$$\bar{B} = [-c\ \ c\ \ a\ \ b\ \ c\ \ 0]. \tag{9.8}$$

The vector-boson mass eigenvalues are then parameterized in terms of a, b and c; the scalar products of B with the roots are listed in table 20.

Three of the six vector bosons associated with the Cartan subalgebra of E_6 must acquire a mass by other means, since the adjoint breaking alone leaves all six massless. The weak breaking has $|\Delta \mathbf{I}^{\mathrm{w}}| = \frac{1}{2}$, so its E_6 weight is nonzero. Nonzero components $\phi_{\mathrm{v}}(\mathbf{r}, \lambda)$ with λ nonzero also contribute to the mass of bosons in the Cartan subalgebra.

Let us suppose for simplicity that there is a $\phi_{\mathrm{v}}(\mathbf{r})$ on an orbit with a ϕ_{v} that is nonzero for just one, nondegenerate value of λ. The portion of $(M^2)_{ab}$ referring to the Cartan subalgebra is usually not diagonal, but the part for the bosons corresponding to the nonzero roots is diagonal, since the nonzero roots are not degenerate. In this simple case the sums in (9.2) reduce to

$$(M^2)_{ab} = g^2 \delta_{ab} [|\langle \mathbf{r}, \lambda + a | X_a | \mathbf{r}, \lambda \rangle\ \ \phi_{\mathrm{v}}(\mathbf{r}, \lambda)|^2 + |\langle \bar{\mathbf{r}}, -\lambda + a | X_a | \bar{\mathbf{r}}, -\lambda \rangle\ \ \phi^{\dagger}_{\mathrm{v}}(\bar{\mathbf{r}}, -\lambda)|^2]. \tag{9.9}$$

The matrix elements of the generators may be computed in a stepwise fashion from (4.2a), as discussed at the end of section 7. Consider the simple yet useful case where the λ for which ϕ_{v} is nonzero is an "extreme" weight. An extreme weight is one where for any root a, if $\lambda + a$ is a weight, $\lambda - a$ is not. From (9.9) it is clear that if neither $\lambda + a$ nor $\lambda - a$ is a weight, then $M_{aa} = 0$; it is always true that $(\lambda, a) = 0$ in such a case. Next, let us select the sign of root a so that $\lambda - a$ is a weight of $\mathbf{r}$; then

$$(M^2)_{aa} = g^2 |\langle \mathbf{r}, \lambda - a | X_{-a} | \mathbf{r}, \lambda \rangle|^2\ \ |\phi_{\mathrm{v}}(\mathbf{r}, \lambda)|^2 = (M^2)_{-a-a}\,. \tag{9.10}$$

The value of $|\langle \mathbf{r}, \lambda - a | X_{-a} | \mathbf{r}, \lambda \rangle|^2$ is (λ, a), which follows immediately from $\langle \mathbf{r}, \lambda | \mathbf{r}, \mu \rangle = \delta_{\lambda\mu}$, $\langle \mathbf{r}, \lambda | X_a = 0$, and $[X_a, X_{-a}] = a_i H_i$; cf. (5.1).

Suppose the $|\Delta \mathbf{I}^{\mathrm{w}}| = \frac{1}{2}$ breaking of E_6 is due to a single, nonzero, extreme weight, L. Then $(Q^{\mathrm{em}}, L) = 0$, $(L, I^{\mathrm{w}}_3) = \pm\frac{1}{2}$, and L is perpendicular to the color roots, which requires $\bar{L}$ to be of the form,

$$\bar{L} = [-d, d \pm 1, d + e, e, d \pm 1, 0], \tag{9.11a}$$

or from (4.9),

$$L = (-3d \mp 1, 2d - e \pm 2,\ d + e \mp 1,\ e - 2d \mp 1,\ 2d - e \pm 2,\ -d - e). \tag{9.11b}$$

All the $Q^{\mathrm{em}} = 0$, $|\Delta I^{\mathrm{w}}_3| = \frac{1}{2}$ weights of the **27** (table 21) are of this type; the three candidates are: $(0\ 0\ 0\ 1\ 0\ -1)$, $(-1\ 0\ 1\ -1\ 0\ 0)$ and $(0\ 1\ -1\ 0\ 1\ 0)$. A linear combination of the bosons coupled to

Q^{t}, Q^{r} and Y^{w} (see table 19) gets a mass from any one of these breakings, so $SU_2^w \times U_1^w \times U_1^r \times U_1^t$ is broken to $U_1^{em} \times U_1 \times U_1$. A boson with nonzero weight, a, gets a mass-squared contribution proportional (L, a).

In the above E_6 examples, ϕ_v is in a stratum where it can be brought to a form $\phi_v(\mathbf{r}, \lambda)$, which is nonzero for just one weight: in the first case $\lambda = 0$ and in the second $\lambda \neq 0$. Then there is a geometrical interpretation of the boson masses as scalar products in weight space, and it is easy to identify the maximal little group solutions in terms of this parameterization. However, there are closed strata where the ϕ_v are not equivalent to a single-weight $\phi_v(\mathbf{r}, \lambda)$. (The existence of such strata corresponds to, in the classical Higgs problem, the problem of finding a canonical form for the vacuum value [75, 76].) Often (perhaps always) ϕ_v in any closed stratum can be brought to a two weight form $\phi_v(\mathbf{r}, \lambda_1) + \phi_v(\mathbf{r}, \lambda_2)$, if it cannot be brought to a one weight form. As an example of a stratum where the canonical form must have two weights, recall the breaking of SO_{10} to SO_7 by $\mathbf{16} + \overline{\mathbf{16}}$. Here the canonical two-weight form has λ_1 related to λ_2 by the C reflection (see section 6), so $C\phi_v(\mathbf{r}, \lambda_1)\, C^{-1} = \phi_v(\mathbf{r}, \lambda_2)$. A vacuum value in the $\mathbf{27} + \overline{\mathbf{27}}$ that breaks E_6 to F_4 has the same property. These C conserving breakings transform a chiral into a vectorlike theory. Thus the two-weight form of (9.2) may be irrelevant; it is not recorded here because it is messy, but easily derived from (9.2).

We return to the E_6 model in order to make a few more comments.

Suppose the $I^w = 0$ breaking belongs to just one nonzero weight of the $\mathbf{78}$. Then B has the form (9.8), further constrained by the requirement that $(B, Q^{em}) = 0$ from QED. Using table 19, (9.8) is further simplified by the constraint $a = b + c$. Note from table 20, though, that the SU_5 leptodiquarks then remain massless. In fact, the only nonzero $I^w = 0$ weight is $(0\ -1\ 1\ 1\ -1\ -1)$, which is left invariant by an entire SU_6. (This would not necessarily be so for a nonextreme weight, since both terms in (9.9) then contribute.) This would have the unwanted implication of a large proton decay rate, so presumably the $I^w = 0$ breaking has a component with zero weight or several nonzero weights. If we require that B have one zero weight, then the big breaking must transform as a component of the $\mathbf{78}$, $\mathbf{650}$, $\mathbf{2430}$, $\mathbf{2925}$, or higher-dimensional, triality-zero representation. If the fermions are assigned to triality 1 or 2 representations, those triality zero irreps do not contribute to the fermion masses directly. Consequently, models with a big $I^w = 0$ breaking of zero weight only are not adequate, since they do not solve the neutrino mass problem, at least not in the tree approximation. Thus, we conclude that the breaking of E_6 is more complicated than these naive guesses.

Suppose that B is due to an explicit adjoint representation of Higgs scalars alone. The only independent Casimir invariants of E_6 are of order 2, 5, 6, 8, 9 and 12, so the Higgs potential can depend on the length of the $\mathbf{78}$ only, since the fourth order invariant is proportional to the square of the second order one. Thus, in the tree approximation, a, b and c in (9.8) are not restricted, which is an example where Michel's conjecture does not apply for E_6 since the symmetry is SO_{78}. The one-loop corrections to the effective potential select $a = 0$ and $b = -c$, so an entire $SO_{10} \times U_1$ is left unbroken [78]. There is, however, no reason to believe that, when the fermions and scalars (which contribute with the opposite sign of the vector masses in the one-loop approximation) are included, these radiative corrections would dominate the determination of B. It is even conceivable that B is determined by the weak breaking.

In simple Higgs models with a reducible representation of scalars, it often happens that scalars in different irreps get vacuum values and, for a range of parameters, their directions are perpendicular in weight space. (Recall the SU_3 Higgs model with scalars transforming as $\mathbf{3} + \overline{\mathbf{3}} + \mathbf{8}$.) The weak breaking in the standard SU_5 model transforms as $\mathbf{5} + \overline{\mathbf{5}}$, or the $\mathbf{10}$ of the SO_{10} model, which suggests that the weak breaking L have weights $(0\ -1\ 1\ 0\ -1\ 0)$ and $(1\ 0\ -1\ 1\ 0\ 0)$ of the $\overline{\mathbf{27}}$ E_6; see table 21. L is perpendicular to B if $a = 2c$ and $b = 3c$. If B is in this direction, then it breaks E_6 down to $SU_2^w \times SU_2 \times U_1 \times U_1^t \times SU_3^c$.

We now continue the study of the fermion mass terms begun in section 2, applying the results of the C analysis of section 6. Consider the SO_{10} model with a single family $\mathbf{f}_L = \mathbf{16}$. Following the embedding conventions of section 7, we identify the physical significance of each of the 45 SO_{10} roots. The nonzero color roots are $(0\ 1\ 0\ 0\ 0)$, $(1\ 0\ 0\ -1\ 1)$ and $(-1\ 1\ 0\ 1\ -1)$, and their negatives, and the electric charge axis, properly normalized, is $\frac{1}{3}(-2\ -2\ 3\ -1\ 1)$. The action of C on the generators is to flip the signs of these roots and the Q^{em} axis. The remaining equations can be gotten from the generators, but it is slightly simpler to study the weights in the **10**: write out the weights of the **10**, compute their flavor and color content according to (7.3) and (7.4), and then require that the action of C (an inner automorphism) on the weights do what it must to color and electric charge. It follows that the action of C on the SO_{10} weights in the Dynkin basis is

$$C(SO_{10}) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{9.12}$$

Thus C leaves invariant the axis with Dynkin labels $(0\ 0\ 1\ -1\ -1)$, which corresponds to the diagonal generator $(3Y^w + 4Q^r - 10I_3^w)/5$; C inverts the SU_3^c roots, electric charge, and $2Q^r + Y^w$, where the Y^w axis is $\frac{1}{3}(-4\ -1\ 6\ -5\ -1)$.

The action of C on the weights in the **16** is as follows: the u quark weights, $(0\ 0\ 0\ 0\ 1)$, $(-1\ 0\ 0\ 1\ 0)$ and $(0\ -1\ 0\ 0\ 1)$ are reflected to the $\bar{u}$ weights, $(0\ 0\ -1\ 1\ 0)$, $(1\ 0\ -1\ 0\ 1)$ and $(0\ 1\ -1\ 1\ 0)$, respectively; the d quark weights $(0\ 1\ 0\ -1\ 0)$, $(-1\ 1\ 0\ 0\ -1)$ and $(0\ 0\ 0\ -1\ 0)$ are reflected to the $\bar{d}$ weights, $(0\ -1\ 1\ 0\ -1)$, $(1\ -1\ 1\ -1\ 0)$ and $(0\ 0\ 1\ 0\ -1)$, respectively; and the e^- $(1\ 0\ 0\ 0\ -1)$ is reflected to the e^+ weight $(-1\ 0\ 1\ -1\ 0)$. Finally, the ν_L with weight $(1\ -1\ 0\ 1\ 0)$ is reflected to $(-1\ 1\ -1\ 0\ 1)$, which is the SU_5 singlet and is called the $(\bar{\nu})_L$.

The weights of the neutral lepton mass matrix are the sums of the weights of the corresponding states. Thus, the ν_L mass matrix element $\langle\nu_L|M|\nu_L\rangle$ has weight $(2\ -2\ 0\ 2\ 0)$ with $|\Delta\mathbf{I}^w| = 1$: It is reflected by C onto $\langle(\bar{\nu})_L|M|(\bar{\nu})_L\rangle$, which has weight $(-2\ 2\ -2\ 0\ 2)$ and is a weak isospin singlet. The off-diagonal element $\langle\nu_L|M|\bar{\nu}_L\rangle$ and its transpose have weight $(0\ 0\ -1\ 1\ 1)$, $|\Delta\mathbf{I}^w| = \frac{1}{2}$, and are invariant under C. The neutral-lepton mass matrix can be written in the form,

$$\begin{array}{rl}
 & \begin{array}{cc} \qquad\nu_L & \qquad\qquad\bar{\nu}_L \\ (1\ -1\ 0\ 1\ 0) & \qquad(-1\ 1\ -1\ 0\ 1) \end{array} \\
\begin{array}{c} \nu_L \\ (1\ -1\ 0\ 1\ 0) \\ \\ \bar{\nu}_L \\ (-1\ 1\ -1\ 0\ 1) \\ \\ \end{array} &
\left(\begin{array}{ll} (2\ -2\ 0\ 2\ 0) & [(0\ 0\ -1\ 1\ 1)] \\ |\Delta\mathbf{I}^w| = 1 & |\Delta\mathbf{I}^w| = \frac{1}{2} \\ & \\ [(0\ 0\ -1\ 1\ 1)] & (-2\ 2\ -2\ 0\ 2) \\ |\Delta\mathbf{I}^w| = \frac{1}{2} & |\Delta\mathbf{I}^w| = 0 \end{array}\right)
\end{array}, \tag{9.13}$$

where the [. . .] signify that the mass matrix element is reflected onto itself by C.

The $|\Delta\mathbf{I}^w| = \frac{1}{2}$ mass has the same weight $(0\ 0\ -1\ 1\ 1)$ as the u quark, so it is expected to have a value of a few MeV. In order for the small eigenvalue of (9.13) to be a few eV or less, the $|\Delta\mathbf{I}^w| = 0$ term must be huge, and if we ignore the $|\Delta\Gamma^{w}| = 1$ term, the mass matrix has the form [14],

$$\begin{pmatrix} 0 & m \\ m & M \end{pmatrix}, \tag{9.14}$$

which has small eigenvalue m^2/M, approximately. Note that (9.14) can be restated as: *the weak isospin conserving mass violates C maximally*, that is, it is odd under C, while the $|\Delta\Gamma^{w}| = \frac{1}{2}$ mass conserves C. Analogous observations for larger groups or larger irreps are not as trivial as they appear here.

The second example is less trivial: the unifying group is E_6 and a single family is assigned to a **27**. The **27** has two charge $-\frac{1}{3}$ quarks and their antiparticles, so there is an opportunity to study the C properties of the quark masses in this example.

The symmetric subgroups of E_6 are Sp_8, $SU_2\times SU_6$, $SO_{10}\times U_1$, and F_4. Of these, the reflection associated with Sp_8 and F_4 reflect **27** to $\overline{\mathbf{27}}$; CP is associated with Sp_8. We have already argued that C must be associated with $SU_2\times SU_6$, since it is inner and flips the signs of enough diagonal generators; C leaves invariant two of the six diagonal quantum numbers in E_6.

The embedding of color and flavor in E_6 can be described by the subgroup chain $E_6 \supset SO_{10}\times U_1^{t} \supset SU_5\times U_1^{r}\times U_1^{t} \supset SU_2^{w}\times U_1^{w}\times SU_3^{c}\times U_1^{r}\times U_1^{t}$, with the projection of the E_6 to SO_{10} weights given by (7.2) and the remaining projections are given by (7.3) and (7.4).

The C reflection is constructed in the same fashion as (9.12) for SO_{10}. It is

$$C(E_6) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{9.15}$$

It inverts color roots and reverses the signs of electric charge and $2Q^{r} + Y^{w}$, while leaving $3Y^{w} + 4Q^{r} - 10I_3^{w}$ and Q^{t} invariant.

Three of the neutral lepton weights in the **27** are eigenvectors of $C(E_6)$ with eigenvalues +1: $(-1\ 0\ 1\ -1\ 0\ 0)$, $(0\ 1\ -1\ 0\ 1\ 0)$ and $(1\ -1\ 0\ 1\ -1\ 0)$. The other two neutral weights $(0\ 0\ 0\ 1\ 0\ -1)$ and $(1\ 0\ -1\ 0\ 0\ 1)$ are transformed into one another by C. The remaining weights carry electric charge and transform under C as required: for example, the charge $\frac{2}{3}$ u quark has weights $(1\ 0\ 0\ 0\ 0\ 0)$, $(1\ -1\ 0\ 0\ 1\ 0)$ and $(1\ 0\ 0\ 0\ 0\ -1)$, which are reflected by C in (9.15) to the $\bar{u}$ weights, $(0\ 0\ -1\ 1\ 0\ 0)$, $(0\ 1\ -1\ 1\ -1\ 0)$, and $(0\ 0\ -1\ 1\ 0\ 1)$, respectively. The u quark mass carries weight $(1\ 0\ -1\ 1\ 0\ 0)$, which is a C conserving, $|\Delta\Gamma^{w}| = \frac{1}{2}$ mass.

The mass matrices of the charge $-\frac{1}{3}$ quarks and the charged leptons have precisely the same weight structure, so we consider the quarks only. The charge $-\frac{1}{3}$ quarks in the SU_5 **10** of the SO_{10} **16**, to be denoted **10(16)**, have the weights, $(0\ 0\ 0\ 0\ -1\ 1)$, $(0\ -1\ 0\ 0\ 0\ 1)$ and $(0\ 0\ 0\ 0\ -1\ 0)$; the C partners $(0\ -1\ 1\ 0\ 0\ -1)$, $(0\ 0\ 1\ 0\ -1\ -1)$ and $(0\ -1\ 1\ 0\ 0\ 0)$, respectively, are in $\bar{\mathbf{5}}$**(16)**. The other charge $-\frac{1}{3}$ quark is in **5(10)**, with weights $(-1\ 1\ 0\ 0\ 0\ 0)$, $(-1\ 0\ 0\ 0\ 1\ 0)$ and $(-1\ 1\ 0\ 0\ 0\ -1)$, and with C partners $(0\ 0\ 0\ -1\ 1\ 0)$, $(0\ 1\ 0\ -1\ 0\ 0)$ and $(0\ 0\ 0\ -1\ 1\ 1)$, respectively, in $\bar{\mathbf{5}}$**(10)**. The mass matrix M for the color state $(1\ 0)$ for quarks and $(-1\ 0)$ for antiquarks is, following the notation of (2.5),

$$
\begin{array}{c}
\\
\\
\mathrm{D}\ \mathbf{5(10)} \\ (-1\ 1\ 0\ 0\ 0\ 0) \\
\\
\mathrm{d}\ \mathbf{10(16)} \\ (0\ 0\ 0\ 0\ -1\ 1) \\
\\
\bar{\mathrm{d}}\ \bar{\mathbf{5}}\mathbf{(10)} \\ (0\ 0\ 0\ -1\ 1\ 0) \\
\bar{\mathrm{D}}\ \bar{\mathbf{5}}\mathbf{(16)} \\ (0\ -1\ 1\ 0\ 0\ -1)
\end{array}
\begin{array}{cccc}
\mathrm{D}\ \mathbf{5(10)} & \mathrm{d}\ \mathbf{10(16)} & \bar{\mathrm{d}}\ \bar{\mathbf{5}}\mathbf{(10)} & \bar{\mathrm{D}}\ \bar{\mathbf{5}}\mathbf{(16)} \\
(-1\ 1\ 0\ 0\ 0\ 0) & (0\ 0\ 0\ 0\ -1\ 1) & (0\ 0\ 0\ -1\ 1\ 0) & (0\ -1\ 1\ 0\ 0\ -1) \\
\end{array}
$$

$$
\begin{pmatrix}
0 & 0 & [(-1\ 1\ 0\ -1\ 1\ 0)] & (-1\ 0\ 1\ 0\ 0\ -1) \\
 & & |\Delta \mathbf{I}^{w}| = 0 & |\Delta \mathbf{I}^{w}| = 0 \\
0 & 0 & (0\ 0\ 0\ -1\ 0\ 1) & [(0\ -1\ 1\ 0\ -1\ 0)] \\
 & & |\Delta \mathbf{I}^{w}| = \tfrac{1}{2} & |\Delta \mathbf{I}^{w}| = \tfrac{1}{2} \\
[(-1\ 1\ 0\ -1\ 1\ 0)] & (0\ 0\ 0\ -1\ 0\ 1) & 0 & 0 \\
(-1\ 0\ 1\ 0\ 0\ -1) & [(0\ -1\ 1\ 0\ -1\ 0)] & 0 & 0
\end{pmatrix}. \tag{9.16}
$$

There are two candidate assignments with extreme C behavior for the weak isospin conserving mass: either the $(-1\ 0\ 1\ 0\ 0\ -1)$ mass is nonzero, the d state is left massless (before the weak breaking), and C is maximally violated; or the $(-1\ 1\ 0\ -1\ 1\ 0)$ mass is nonzero, the D is massless, and the mass is C conserving. For the purposes of studying the charged particle masses, these situations appear interchangeable, although the precise identification of the $\bar{\mathbf{5}}+\mathbf{10}$ left massless in the limit of no weak breaking differs in the two cases. In the first case (d massless), the $\bar{\mathbf{5}}$ belongs to the SO_{10} **10**; in the second case (D massless), the $\bar{\mathbf{5}}$ comes from the SO_{10} **16**. The same result holds for the two charged leptons in the **27**.

In order to decide which assignment is more attractive, we turn to a study of the neutral lepton mass matrix, which can be written as a matrix of weights where the labels on the rows and columns is given in (9.17a) below:

$$
\begin{pmatrix}
(0\ 0\ 0\ 2\ 0\ -2) & [(1\ 0\ -1\ 1\ 0\ 0)] & (1\ -1\ 0\ 2\ -1\ -1) & (\mathbf{-1\ 0\ 1\ 0\ 0\ -1}) & (0\ 1\ -1\ 1\ 1\ -1) \\
 & & \updownarrow & \updownarrow & \updownarrow \\
[(1\ 0\ -1\ 1\ 0\ 0)] & (\mathbf{2\ 0\ -2\ 0\ 0\ 2}) & (\mathbf{2\ -1\ -1\ 1\ -1\ 1}) & (0\ 0\ 0\ -1\ 0\ 1) & (1\ 1\ -2\ 0\ 1\ 1) \\
(1\ -1\ 0\ 2\ -1\ 1) \leftrightarrow & (\mathbf{2\ -1\ -1\ 1\ -1\ 1}) & [(\mathbf{2\ -2\ 0\ 2\ -2\ 0})] & [(0\ -1\ 1\ 0\ -1\ 0)] & [(1\ 0\ -1\ 1\ 0\ 0)] \\
(\mathbf{-1\ 0\ 1\ 0\ 0\ -1}) \leftrightarrow & (0\ 0\ 0\ -1\ 0\ 1) & [(0\ -1\ 1\ 0\ -1\ 0)] & [(-2\ 0\ 2\ -2\ 0\ 0)] & [(\mathbf{-1\ 1\ 0\ -1\ 1\ 0})] \\
(0\ 1\ -1\ 1\ 1\ -1) \leftrightarrow & (1\ 1\ -2\ 0\ 1\ 1) & [(1\ 0\ -1\ 1\ 0\ 0)] & [(\mathbf{-1\ 1\ 0\ -1\ 1\ 0})] & [(0\ 2\ -2\ 0\ 2\ 0)]
\end{pmatrix} \tag{9.17}
$$

where the I_3^{w} value of the mass matrix element is one-half the sum of the first five Dynkin labels; the weights are in $\overline{\mathbf{27}}$, $\overline{\mathbf{351}'}$, or both; the bold faced weights have $I_3^{w}=0$; and the bracketed ones are C eigenstates. It may be helpful to write (9.17) in the more transparent notation of table 21:

	$\bar{\mathbf{5}}$(16)	1(16)	1(1)	5(10)	$\bar{\mathbf{5}}$(10)
$\bar{\mathbf{5}}$(16)	$\overline{\mathbf{15}}(\overline{\mathbf{126}})$	$\bar{\mathbf{5}}$(126) +$\bar{\mathbf{5}}$(10)	$\bar{\mathbf{5}}$(16)	$\mathbf{24}(\overline{\mathbf{144}})$ $+\mathbf{1}(\overline{\mathbf{16}})$	$\overline{\mathbf{15}}(\overline{\mathbf{144}})$
1(16)	$\bar{\mathbf{5}}$(126) +5(10)	1(126)	1(16)	$\mathbf{5}(\overline{\mathbf{144}})$ $+\mathbf{5}(\overline{\mathbf{16}})$	$\bar{\mathbf{5}}(\overline{\mathbf{144}})$
1(1)	$\bar{\mathbf{5}}$(16)	1(16)	$\mathbf{1}(\mathbf{1}(\overline{\mathbf{351}'}))$	5(10)	$\bar{\mathbf{5}}$(10)
5(10)	$\mathbf{24}(\overline{\mathbf{144}})$ $+\mathbf{1}(\overline{\mathbf{16}})$	$\mathbf{5}(\overline{\mathbf{144}})$ $+\mathbf{5}(\overline{\mathbf{16}})$	5(10)	15(54)	$\mathbf{1}(\mathbf{1}(\overline{\mathbf{27}}))$ +24(54)
$\bar{\mathbf{5}}$(10)	$\overline{\mathbf{15}}(\overline{\mathbf{144}})$	$\bar{\mathbf{5}}(\overline{\mathbf{144}})$	$\bar{\mathbf{5}}$(10)	$\mathbf{1}(\mathbf{1}(\overline{\mathbf{27}}))$ +24(54)	$\overline{\mathbf{15}}$(54)

(9.17a)

where the matrix elements correspond directly with those in (9.17). The $\overline{\mathbf{144}}$, **126**, **54** and **16** of SO_{10} occur only in the $\overline{\mathbf{351}}'$ of E_6; the **5(10)** and $\bar{\mathbf{5}}$**(10)** have components in both the $\overline{\mathbf{27}}$ and $\overline{\mathbf{351}}'$; the $\overline{\mathbf{16}}$ is always in the $\overline{\mathbf{27}}$; and additional selection rules from U_1^t and U_1^r restrict the origin of **1(1)** as shown explicitly. Let us first assume that the weak isospin conserving part of (9.17) is maximally C violating, so that only the entries with weights (2 0 −2 0 0 2) $[\mathbf{1}(\mathbf{126}(\overline{\mathbf{351}}'))]$, (2 −1 −1 1 −1 1) $[\mathbf{1}(\mathbf{16}(\overline{\mathbf{351}}'))]$, and (−1 0 1 0 0 −1) $[\mathbf{24}(\overline{\mathbf{144}}(\overline{\mathbf{351}}')) + \mathbf{1}(\overline{\mathbf{16}}(\overline{\mathbf{27}}))]$ are nonzero. For a general choice of parameters, (9.17) has four nonzero eigenvalues and one zero eigenvalue; the massless fermion has weight (0 1 −1 0 1 0), which is in $\bar{\mathbf{5}}$**(10)**. Thus, with maximal C violation, the massless fermions at the weak isospin conserving level are classified by $\bar{\mathbf{5}} + \mathbf{10}$.

In the case of C conservation, the elements with weights (2 −2 0 2 −2 0) $[\mathbf{1}(\mathbf{1}(\overline{\mathbf{351}}'))]$ and (−1 1 0 −1 1 0) $[\mathbf{1}(\mathbf{1}(\overline{\mathbf{27}})) + \mathbf{24}(\mathbf{54}(\overline{\mathbf{351}}'))]$ are nonzero, and the neutrals in the SO_{10} $\mathbf{1} + \mathbf{10}$ get masses. Both neutral states in the **16** remain massless, at least until some C violation is introduced at the SO_{10} level. Thus, the C conservation hypothesis leaves a $\mathbf{1} + \bar{\mathbf{5}} + \mathbf{10}$ of SU_5 to get masses from other sources, such as the weak interactions. If the four component ν mass comes from the weak interactions, then its mass is of order the u mass, not in accord with experience.

Stated in a slightly different way, all the C conserving weak isosinglet masses leave SO_{10} invariant, so the fermions occur in SO_{10} irreps, **16**'s in this case, but the C violating masses leave just SU_5 invariant, while violating SO_{10}, and the low mass fermions in the **27** occur in a $\bar{\mathbf{5}}(\mathbf{10}) + \mathbf{10}(\mathbf{16})$ pattern.

In summary, we find that the hypothesis of maximal C violation of the weak isospin invariant masses can lead to a satisfactory fermion spectrum in several one-family flavor-chiral models. However, this formulation needs more analysis, since at least two of the $I^w = 0$, C-violating matrix elements must be nonzero, which requires both the $\overline{\mathbf{27}}$ or $\overline{\mathbf{351}}'$ irreps; the weak breaking must have another source in models of this type. Consideration of C and application of the technology described here should help in the search for a more satisfactory solution of the symmetry breaking problem; there is at present much physics to be done.

Tables

Table 1
Table of tables

Table 1 (continued)

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21	Weights and content of the **27** of E_6
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24	SU_3 tensor products
25	SU_4 irreps of dimension less than 180
26	SU_4 tensor products
27	Branching rules for $SU_4 \supset SU_3 \times U_1$
28	SU_5 irreps of dimension less than 1200
29	SU_5 tensor products
30	Branching rules for SU_5
31	SU_6 irreps of dimension less than 1000
32	SU_6 tensor products
33	SU_6 branching rules
34	SO_7 irreps of dimension less than 650 and $SO_7 \supset SU_4$ branching rules
35	SO_7 tensor products
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37	SO_8 tensor products
38	SO_9 irreps of dimension less than 5100
39	SO_9 tensor products
40	Branching rules for SO_9 irreps
41	SO_{10} irreps of dimension less than 12000
42	SO_{10} tensor products
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44	F_4 irreps of dimension less than 100000
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50	Guide to E_6 projection matrices
51	Matrix elements for SU_3 vector-coupling coefficient examples
52	Irreps, products, and branching rules for E_7
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54	Irreps, products, and branching rules for SU_8
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58	Branching rules to all maximal subgroups

Table 2
Model builders' view of the elementary particle spectrum

Spin 2: One graviton, considered in supergravity, but usually ignored in models that unify just color and flavor.

Spin 3/2: An intriguing "hole" in the spectrum; ignored in unified models, but supergravity Lagrangians suggest it should be filled.

Spin 1: Vector bosons mediating Nature's interactions, including the photon of QED, the charged and neutral weak bosons, and the eight gluons of the strong interactions. Unified models suggest additional vector bosons; for example, in some models there are bosons that mediate proton decay.

Spin 1/2: Quarks and leptons (only the left-handed states are listed)

$$\begin{pmatrix} \mathrm{u} \\ \mathrm{d}' \end{pmatrix}_{\mathrm{L}} \quad \begin{pmatrix} \mathrm{c} \\ \mathrm{s}' \end{pmatrix}_{\mathrm{L}} \quad \begin{pmatrix} \mathrm{t}(?) \\ \mathrm{b}' \end{pmatrix}_{\mathrm{L}} \qquad \text{Weak doublets}$$

$$\bar{\mathrm{u}}_{\mathrm{L}} \quad \bar{\mathrm{d}}_{\mathrm{L}} \quad \bar{\mathrm{c}}_{\mathrm{L}} \quad \bar{\mathrm{s}}_{\mathrm{L}} \quad \bar{\mathrm{b}}_{\mathrm{L}} \quad \bar{\mathrm{t}}_{\mathrm{L}}(?) \qquad \text{Weak singlets}$$

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_{\mathrm{L}} \quad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_{\mathrm{L}} \quad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_{\mathrm{L}} \qquad \text{doublets}$$

$$e^+_{\mathrm{L}} \quad \mu^+_{\mathrm{L}} \quad \tau^+_{\mathrm{L}} \qquad \text{singlets}$$

Are there additional quarks and leptons, or other fermions with higher colors?

Spin 0: None are known for certain. The weak breaking follows a $|\Delta \mathbf{I}^{\mathrm{w}}| = \frac{1}{2}$ rule and the superstrong breaking, $|\Delta \mathbf{I}^{\mathrm{w}}| = 0$, but it is not known whether either of these are associated with explicit scalar particles. One possibility is that the superstrong breaking is due to explicit scalars in the Lagrangian, but the weak breaking is due to composites. The origin of the symmetry breaking is a major puzzle in today's particle theory.

Table 3
Embeddings of SU_3^c in Simple Groups G, subject to the constraint that at least one irrep of G has no more than $\mathbf{1}^c$, $\mathbf{3}^c$ and $\bar{\mathbf{3}}^c$. G^{fl} (fl for flavor) is the largest subgroup defined by $G \supset G^{fl} \times SU_3^c$. The irreps of G satisfying the restriction to the set $\mathbf{1}^c$, $\mathbf{3}^c$, $\bar{\mathbf{3}}^c$, are listed, along with their dimensionality. See ref. [6]

Case	G	G^{fl}	f	Dimensionality
1.	SU_n	$SU_{n_1} \times SU_{n_3} \times U_1$	$\mathbf{n}$	$n = n_1 + 3n_3$
2.	SU_n	$SU_{n-3} \times U_1$	$(\mathbf{n}^k)_a$	$\binom{n}{k}$
3.	SU_n	$SU_{n_1} \times SU_{n_3} \times SU_{n_{\bar{3}}} \times U_1 \times U_1$	$\mathbf{n}$	$n = n_1 + 3n_3 + 3n_{\bar{3}}$
4.	SO_n	$SO_{n_1} \times SU_{n_3} \times U_1$	$\mathbf{n}$	$n = n_1 + 6n_3$
5.	SO_n	$SO_{n-6} \times U_1$	$\mathbf{n}$	$n = n_1 + 6$
			σ, σ' or $\bar{\sigma}$	$2^{[(n-1)/2]}$
6.	Sp_{2n}	$Sp_{2n_1} \times SU_{n_3} \times U_1$	$\mathbf{2n}$	$2n = 2n_1 + 6n_3$
7.	F_4	SU_3	$\mathbf{26}$	26
8.	E_6	$SU_3 \times SU_3$	$\mathbf{27}$	27
9.	E_7	SU_6	$\mathbf{56}$	56

Table 4

Roots of SU_3. The α_i are the conventional choice of simple roots; (ij) are the Dynkin labels of the roots; and the axes associated with the diagonal generators Y, Q and I_3 are indicated and normalized

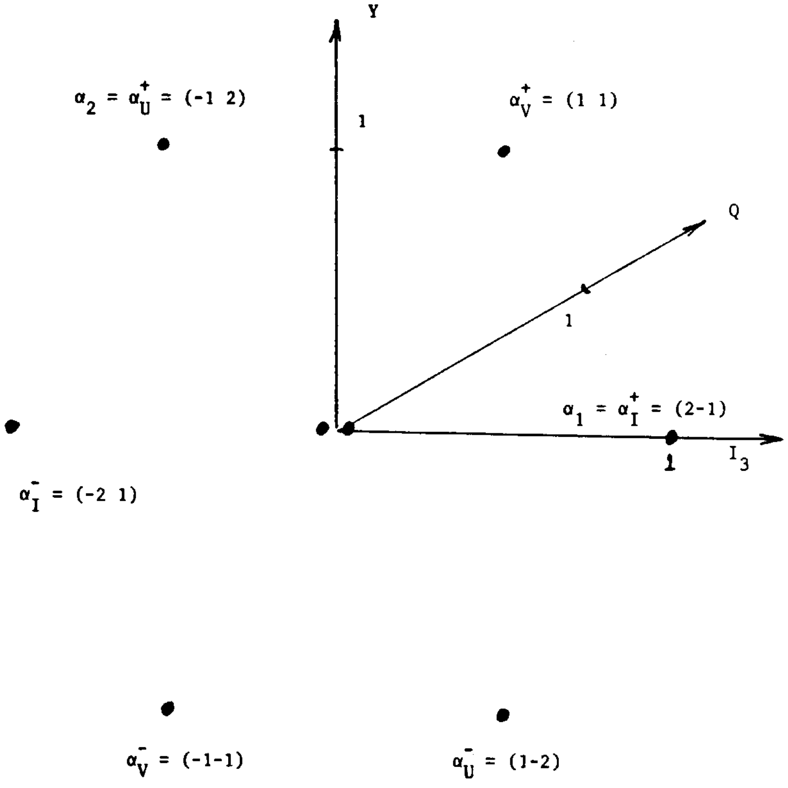


Table 5

Dynkin diagrams for simple Lie algebras. (Black dots represent shorter roots.)

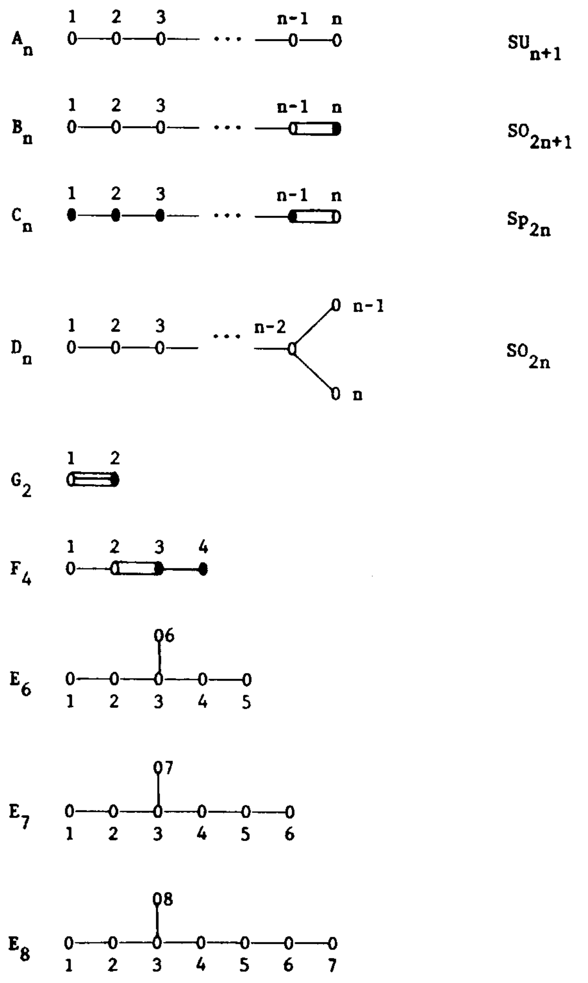


Table 6
Cartan matrices of simple Lie algebras

$$A(\mathrm{A}_n)=\begin{pmatrix} 2 & -1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ -1 & 2 & -1 & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & -1 & 2 \end{pmatrix} \qquad A(\mathrm{G}_2)=\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

$$A(\mathrm{F}_4)=\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$$A(\mathrm{B}_n)=\begin{pmatrix} 2 & -1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ -1 & 2 & -1 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & -1 & 2 & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 2 & -2 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & -1 & 2 \end{pmatrix} \qquad A(\mathrm{E}_6)=\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

$A(\mathrm{C}_n)$ is the transpose of $A(\mathrm{B}_n)$, since the short and long roots are interchanged.

$$A(\mathrm{D}_n)=\begin{pmatrix} 2 & -1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & & & 2 & -1 & -1 \\ 0 & 0 & 0 & & & & -1 & 2 & 0 \\ 0 & 0 & 0 & & & & -1 & 0 & 2 \end{pmatrix} \qquad A(\mathrm{E}_7)=\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$A(\mathrm{E}_8)=\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Table 7
Metric tensors G for weight space

$$G(\mathrm{A}_n) = \frac{1}{n+1}\begin{pmatrix} 1\cdot n & 1\cdot(n-1) & 1\cdot(n-2) & \cdots & 1\cdot 2 & 1\cdot 1 \\ 1\cdot(n-1) & 2\cdot(n-1) & 2\cdot(n-2) & \cdots & 2\cdot 2 & 2\cdot 1 \\ 1\cdot(n-2) & 2\cdot(n-2) & 3\cdot(n-2) & \cdots & 3\cdot 2 & 3\cdot 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1\cdot 2 & 2\cdot 2 & 3\cdot 2 & \cdots & (n-1)\cdot 2 & (n-1)\cdot 1 \\ 1\cdot 1 & 2\cdot 1 & 3\cdot 1 & \cdots & (n-1)\cdot 1 & n\cdot 1 \end{pmatrix}$$

$$G(\mathrm{B}_n) = \frac{1}{2}\begin{pmatrix} 2 & 2 & 2 & \cdots & 2 & 1 \\ 2 & 4 & 4 & \cdots & 4 & 2 \\ 2 & 4 & 6 & \cdots & 6 & 3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 2 & 4 & 6 & \cdots & 2(n-1) & n-1 \\ 1 & 2 & 3 & \cdots & n-1 & n/2 \end{pmatrix}$$

$$G(\mathrm{C}_n) = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 2 & \cdots & 2 & 2 \\ 1 & 2 & 3 & \cdots & 3 & 3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & 3 & \cdots & n-1 & n-1 \\ 1 & 2 & 3 & \cdots & n-1 & n \end{pmatrix}$$

$$G(\mathrm{D}_n) = \frac{1}{2}\begin{pmatrix} 2 & 2 & 2 & \cdots & 2 & 1 & 1 \\ 2 & 4 & 4 & \cdots & 4 & 2 & 2 \\ 2 & 4 & 6 & \cdots & 6 & 3 & 3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 2 & 4 & 6 & \cdots & 2(n-2) & n-2 & n-2 \\ 1 & 2 & 3 & \cdots & n-2 & n/2 & (n-2)/2 \\ 1 & 2 & 3 & \cdots & n-2 & (n-2)/2 & n/2 \end{pmatrix}$$

$$G(\mathrm{E}_6) = \frac{1}{3}\begin{pmatrix} 4 & 5 & 6 & 4 & 2 & 3 \\ 5 & 10 & 12 & 8 & 4 & 6 \\ 6 & 12 & 18 & 12 & 6 & 9 \\ 4 & 8 & 12 & 10 & 5 & 6 \\ 2 & 4 & 6 & 5 & 4 & 3 \\ 3 & 6 & 9 & 6 & 3 & 6 \end{pmatrix}$$

$$G(\mathrm{E}_7) = \frac{1}{2}\begin{pmatrix} 4 & 6 & 8 & 6 & 4 & 2 & 4 \\ 6 & 12 & 16 & 12 & 8 & 4 & 8 \\ 8 & 16 & 24 & 18 & 12 & 6 & 12 \\ 6 & 12 & 18 & 15 & 10 & 5 & 9 \\ 4 & 8 & 12 & 10 & 8 & 4 & 6 \\ 2 & 4 & 6 & 5 & 4 & 3 & 3 \\ 4 & 8 & 12 & 9 & 6 & 3 & 7 \end{pmatrix}$$

$$G(\mathrm{E}_8) = \begin{pmatrix} 4 & 7 & 10 & 8 & 6 & 4 & 2 & 5 \\ 7 & 14 & 20 & 16 & 12 & 8 & 4 & 10 \\ 10 & 20 & 30 & 24 & 18 & 12 & 6 & 15 \\ 8 & 16 & 24 & 20 & 15 & 10 & 5 & 12 \\ 6 & 12 & 18 & 15 & 12 & 8 & 4 & 9 \\ 4 & 8 & 12 & 10 & 8 & 6 & 3 & 6 \\ 2 & 4 & 6 & 5 & 4 & 3 & 2 & 3 \\ 5 & 10 & 15 & 12 & 9 & 6 & 3 & 8 \end{pmatrix}$$

$$G(\mathrm{G}_2) = \frac{1}{3}\begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}$$

$$G(\mathrm{F}_4) = \begin{pmatrix} 2 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 3 & 3/2 \\ 1 & 2 & 3/2 & 1 \end{pmatrix}$$

Table 8
Root diagrams in the Dynkin basis. "Level of simple roots" is the number of simple roots that must be subtracted from the highest root in order to obtain the simple roots; the next level has the n zero roots corresponding to the Cartan subalgebra

Algebra	Highest root	Level of simple roots	Dimension
A_n	(1 0 0 ... 0 0 1)	$n-1$	$n(n+2)$
B_n	(0 1 0 ... 0 0 0)	$2n-2$	$n(2n+1)$
C_n	(2 0 0 ... 0 0 0)	$2n-2$	$n(2n+1)$
D_n	(0 1 0 ... 0 0 0)	$2n-4$	$n(2n-1)$
G_2	(1 0)	4	14
F_4	(1 0 0 0)	10	52
E_6	(0 0 0 0 0 1)	10	78
E_7	(1 0 0 0 0 0 0)	16	133
E_8	(0 0 0 0 0 0 1 0)	28	248

Table 9
Positive roots in the Dynkin basis of rank 2 and 3 simple algebras, of SU_5 (rank 4) and of SO_{10} (rank 5)

SU_3
(1 1)
(2 −1)(−1 2)

Sp_4
(2 0)
(0 1)
(2 −1)(−2 2)

G_2
(1 0)
(−1 3)
(0 1)
(1 −1)
(2 −3)(−1 2)

SU_4
(1 0 1)
(1 1 −1)(−1 1 1)
(2 −1 0)(−1 2 −1)(0 −1 2)

SO_7
(0 1 0)
(1 −1 2)
(1 0 0)(−1 0 2)
(1 1 −2)(−1 1 0)
(2 −1 0)(−1 2 −2)(0 −1 2)

Sp_6
(2 0 0)
(0 1 0)
(1 −1 1)(−2 2 0)
(1 1 −1)(−1 0 1)
(2 −1 0)(−1 2 −1)(0 −2 2)

SU_5
(1 0 0 1)
(1 0 1 −1)(−1 1 0 1)
(1 1 −1 0)(−1 1 1 −1)(0 −1 1 1)
(2 −1 0 0)(−1 2 −1 0)(0 −1 2 −1)(0 0 −1 2)

SO_{10}
(0 1 0 0 0)
(1 −1 1 0 0)
(−1 0 1 0 0)(1 0 −1 1 1)
(−1 1 −1 1 1)(1 0 0 −1 1)(1 0 0 1 −1)
(0 −1 0 1 1)(−1 1 0 −1 1)(−1 1 0 1 −1)(1 0 1 −1 −1)
(0 −1 1 −1 1)(0 −1 1 1 −1)(−1 1 1 −1 −1)(1 1 −1 0 0)
(0 0 −1 0 2)(0 0 −1 2 0)(0 −1 2 −1 −1)(−1 2 −1 0 0)(2 −1 0 0 0)

Table 10

Level vectors of simple groups. The ordering follows the conventions of table 5

SU_{n+1}	$\bar{R} = [n, 2(n-1), 3(n-2), \ldots, (n-1)2, n]$
SU_5	$\bar{R} = [4, 6, 6, 4]$
SU_6	$\bar{R} = [5, 8, 9, 8, 5]$
SO_{2n+1}	$\bar{R} = [2n, 2(2n-1), 3(2n-2), 4(2n-3), \ldots, (n-1)(n+2), n(n+1)/2]$
SO_9	$\bar{R} = [8, 14, 18, 10]$
Sp_{2n}	$\bar{R} = [(2n-1), 2(2n-2), 3(2n-3), \ldots, (n-1)(n+1), n^2]$
SO_{2n}	$\bar{R} = [(2n-2), 2(2n-3), 3(2n-4), \ldots, (n-2)(n+1), n(n-1)/2, n(n-1)/2]$
SO_8	$\bar{R} = [6, 10, 6, 6]$
SO_{10}	$\bar{R} = [8, 14, 18, 10, 10]$
G_2	$\bar{R} = [10, 6]$
F_4	$\bar{R} = [22, 42, 30, 16]$
E_6	$\bar{R} = [16, 30, 42, 30, 16, 22]$
E_7	$\bar{R} = [34, 66, 96, 75, 52, 27, 49]$
E_8	$\bar{R} = [92, 182, 270, 220, 168, 114, 58, 136]$

Table 11a

Weight diagram for the **16** of SO_{10}

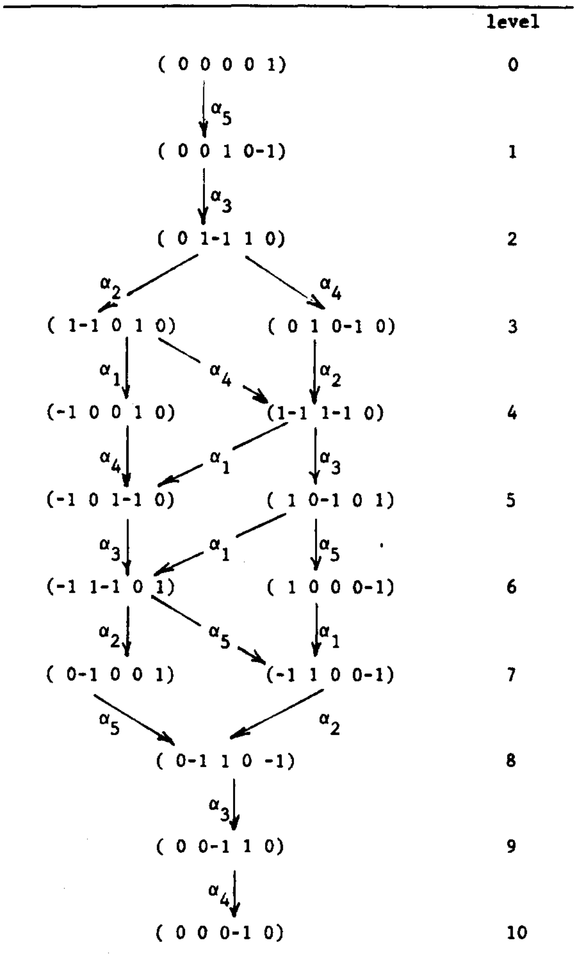


Table 11b
Weight diagram for the 27 of E_6

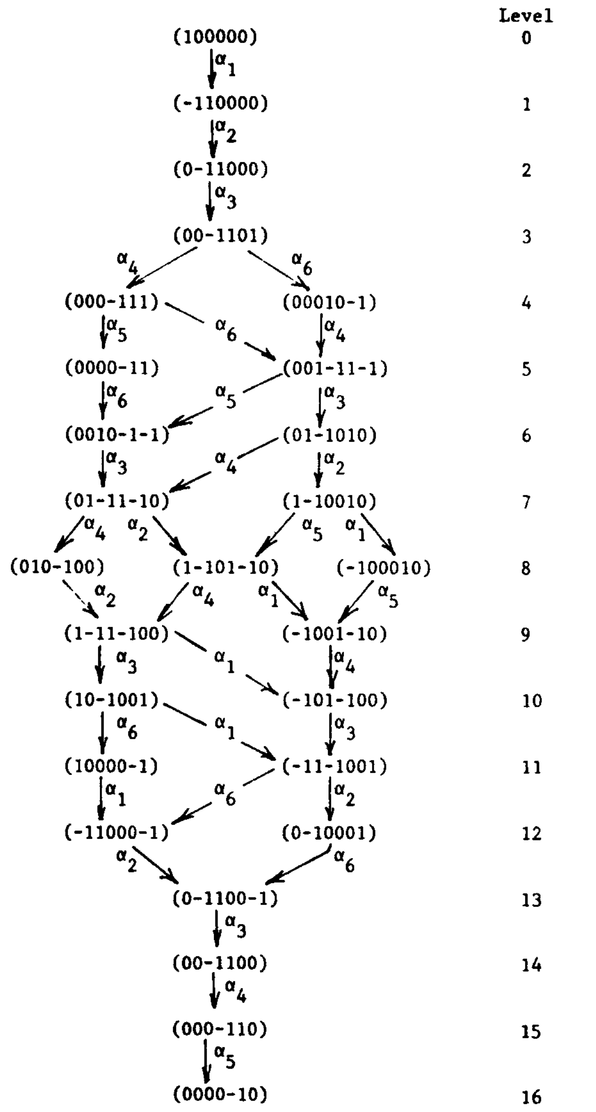


Table 12
Self-conjugate representations of simple groups

Group	Restriction	Real/Pseudoreal (PR)
SU_{l+1} $l = 2, 3, 4; 6, 7, 8; \ldots$	$(a_1, \ldots, a_l) =$ $(a_l, \ldots, a_1)$	Real only
SU_{l+1} $l = 1, 5, 9, 13, \ldots$	$(a_1, \ldots, a_l) =$ $(a_l, \ldots, a_1)$	Real if $a_{(l+1)/2}$ even PR if $a_{(l+1)/2}$ odd
SO_{2l+1} $l = 3, 4; 7, 8; 11, 12; \ldots$	none	Real only
SO_{2l+1} $l = 1, 2; 5, 6; 9, 10; \ldots$	none (α_l is the short root)	Real if a_l even PR if a_l odd
Sp_{2l}	none (α_l is the long root)	Real if $\sum_{i\,\mathrm{odd}} a_i$ even PR if $\sum_{i\,\mathrm{odd}} a_i$ odd
SO_{2l} l odd	$a_{l-1} = a_l$	Real only
SO_{2l} l even, $\frac{1}{2}l$ even	none	Real only
SO_{2l} l even, $\frac{1}{2}l$ odd	none (α_{l-1} and α_l are the spinor roots)	Real if $a_{l-1} + a_l$ even PR if $a_{l-1} + a_l$ odd
G_2	none	Real only
F_4	none	Real only
E_6	$(a_1, a_2, \ldots, a_5, a_6) =$ $(a_5, a_4, \ldots, a_1, a_6)$	Real only
E_7	none [Note, **56** = (0000010)]	Real if $a_4 + a_6 + a_7$ even PR if $a_4 + a_6 + a_7$ odd
E_8	none	Real only

Table 13
Simple irreps of simple Lie algebras

Algebra	Dynkin designation	Dimensionality
A_n	(10 ... 0)	$n+1$
	or (0 ... 01)*	$\overline{n+1}$
B_n	(10 ... 0)*	$2n+1$
	(000 ... 01)	2^n
C_n	(10 ... 0)	$2n$
D_n	(10 ... 0)*	$2n$
	(00 ... 01)	2^{n-1}
	or (00 ... 010)*	2^{n-1}
G_2	(01)	7
F_4	(0001)	26
E_6	(100000)	27
	or (000010)*	$\overline{27}$
E_7	(0000010)	56
E_8	(00000010)	248

* This irrep can be constructed from products of the unstarred irrep.

Table 14
Maximal subalgebras of classical simple Lie algebras with rank 8 or less

Rank 1		
SU_2	$\supset U_1$	(R)
(SU_2, SO_3, Sp_2, all isomorphic)		
Rank 2		
SU_3	$\supset SU_2\times U_1$	(R)
	$\supset SU_2$	(S)
Sp_4	$\supset SU_2\times SU_2$; $SU_2\times U_1$	(R)
	$\supset SU_2$	(S)
(SO_5 isomorphic to Sp_4, $SO_4\sim SU_2\times SU_2$)		
Rank 3		
SU_4	$\supset SU_3\times U_1$; $SU_2\times SU_2\times U_1$	(R)
	$\supset Sp_4$; $SU_2\times SU_2$	(S)
SO_7	$\supset SU_4$; $SU_2\times SU_2\times SU_2$; $Sp_4\times U_1$	(R)
	$\supset G_2$	(S)
Sp_6	$\supset SU_3\times U_1$; $SU_2\times Sp_4$	(R)
	$\supset SU_2$; $SU_2\times SU_2$	(S)
(SO_6 is isomorphic to SU_4)		
Rank 4		
SU_5	$\supset SU_4\times U_1$; $SU_2\times SU_3\times U_1$	(R)
	$\supset Sp_4$	(S)
SO_9	$\supset SO_8$; $SU_2\times SU_2\times Sp_4$; $SU_2\times SU_4$; $SO_7\times U_1$	(R)
	$\supset SU_2$; $SU_2\times SU_2$	(S)
Sp_8	$\supset SU_4\times U_1$; $SU_2\times Sp_6$; $Sp_4\times Sp_4$	(R)
	$\supset SU_2$; $SU_2\times SU_2\times SU_2$	(S)
SO_8	$\supset SU_2\times SU_2\times SU_2\times SU_2$; $SU_4\times U_1$	(R)
	$\supset SU_3$; SO_7; $SU_2\times Sp_4$	(S)

Table 14 (continued)

Rank 5		
SU_6	$\supset SU_5\times U_1$; $SU_2\times SU_4\times U_1$; $SU_3\times SU_3\times U_1$	(R)
	$\supset SU_3$; SU_4; Sp_6; $SU_2\times SU_3$	(S)
SO_{11}	$\supset SO_{10}$; $SU_2\times SO_8$; $Sp_4\times SU_4$; $SU_2\times SU_2\times SO_7$; $SO_9\times U_1$	(R)
	$\supset SU_2$	(S)
Sp_{10}	$\supset SU_5\times U_1$; $SU_2\times Sp_8$; $Sp_4\times Sp_6$	(R)
	$\supset SU_2$; $SU_2\times Sp_4$	(S)
SO_{10}	$\supset SU_5\times U_1$; $SU_2\times SU_2\times SU_4$; $SO_8\times U_1$	(R)
	$\supset Sp_4$; SO_9; $SU_2\times SO_7$; $Sp_4\times Sp_4$	(S)
Rank 6		
SU_7	$\supset SU_6\times U_1$; $SU_2\times SU_5\times U_1$; $SU_3\times SU_4\times U_1$	(R)
	$\supset SO_7$	(S)
SO_{13}	$\supset SO_{12}$; $SU_2\times SO_{10}$; $Sp_4\times SO_8$; $SU_4\times SO_7$; $SU_2\times SU_2\times SO_9$; $SO_{11}\times U_1$	(R)
	$\supset SU_2$	(S)
Sp_{12}	$\supset SU_6\times U_1$; $SU_2\times Sp_{10}$; $Sp_4\times Sp_8$; $Sp_6\times Sp_6$	(R)
	$\supset SU_2$; $SU_2\times SU_4$; $SU_2\times Sp_4$	(S)
SO_{12}	$\supset SU_6\times U_1$; $SU_2\times SU_2\times SO_8$; $SU_4\times SU_4$; $SO_{10}\times U_1$	(R)
	$\supset SU_2\times Sp_6$; $SU_2\times SU_2\times SU_2$; SO_{11}; $SU_2\times SO_9$; $Sp_4\times SO_7$	(S)
Rank 7		
SU_8	$\supset SU_7\times U_1$; $SU_2\times SU_6\times U_1$; $SU_3\times SU_5\times U_1$; $SU_4\times SU_4\times U_1$	(R)
	$\supset SO_8$; Sp_8; $SU_2\times SU_4$	(S)
SO_{15}	$\supset SO_{14}$; $SU_2\times SO_{12}$; $Sp_4\times SO_{10}$; $SO_7\times SO_8$; $SU_4\times SO_9$; $SU_2\times SU_2\times SO_{11}$; $SO_{13}\times U_1$	(R)
	$\supset SU_2$; SU_4; $SU_2\times Sp_4$	(S)
Sp_{14}	$\supset SU_7\times U_1$; $SU_2\times Sp_{12}$; $Sp_4\times Sp_{10}$; $Sp_6\times Sp_8$	(R)
	$\supset SU_2$; $SU_2\times SO_7$	(S)
SO_{14}	$\supset SU_7\times U_1$; $SU_2\times SU_2\times SO_{10}$; $SU_4\times SO_8$; $SO_{12}\times U_1$	(R)
	$\supset Sp_4$; Sp_6; G_2; SO_{13}; $SU_2\times SO_{11}$; $Sp_4\times SO_9$; $SO_7\times SO_7$	(S)
Rank 8		
SU_9	$\supset SU_8\times U_1$; $SU_2\times SU_7\times U_1$; $SU_3\times SU_6\times U_1$; $SU_4\times SU_5\times U_1$	(R)
	$\supset SO_9$; $SU_3\times SU_3$	(S)
SO_{17}	$\supset SO_{16}$; $SU_2\times SO_{14}$; $Sp_4\times SO_{12}$; $SO_7\times SO_{10}$; $SO_8\times SO_9$; $SU_4\times SO_{11}$; $SU_2\times SU_2\times SO_{13}$; $SO_{15}\times U_1$	(R)
	$\supset SU_2$	(S)
Sp_{16}	$\supset SU_8\times U_1$; $SU_2\times Sp_{14}$; $Sp_4\times Sp_{12}$; $Sp_6\times Sp_{10}$; $Sp_8\times Sp_8$	(R)
	$\supset SU_2$; Sp_4; $SU_2\times SO_8$	(S)
SO_{16}	$\supset SU_8\times U_1$; $SU_2\times SU_2\times SO_{12}$; $SU_4\times SO_{10}$; $SO_8\times SO_8$; $SO_{14}\times U_1$	(R)
	$\supset SO_9$; $SU_2\times Sp_8$; $Sp_4\times Sp_4$; SO_{15}; $SU_2\times SO_{13}$; $Sp_4\times SO_{11}$; $SO_7\times SO_9$	(S)

Table 15
Maximal subalgebras of exceptional algebras; branching rules for the fundamental representation

$G_2 \supset SU_3$	$7 = 1 + 3 + \overline{3}$	(R)
$\supset SU_2 \times SU_2$	$7 = (2, 2) + (1, 3)$	(R)
$\supset SU_2$	$7 = 7$	(S)
$F_4 \supset SO_9$	$26 = 1 + 9 + 16$	(R)
$\supset SU_3 \times SU_3$	$26 = (8, 1) + (3, 3) + (\overline{3}, \overline{3})$	(R)
$\supset SU_2 \times Sp_6$	$26 = (2, 6) + (1, 14)$	(R)
$\supset SU_2$	$26 = 9 + 17$	(S)
$\supset SU_2 \times G_2$	$26 = (5, 1) + (3, 7)$	(S)
$E_6 \supset SO_{10} \times U_1$	$27 = 1 + 10 + 16$	(R)
$\supset SU_2 \times SU_6$	$27 = (2, \overline{6}) + (1, 15)$	(R)
$\supset SU_3 \times SU_3 \times SU_3$	$27 = (\overline{3}, 3, 1^c) + (3, 1, 3) + (1, \overline{3}, \overline{3})$	(R)
$\supset SU_3$	$27 = 27$	(S)
$\supset G_2$	$27 = 27$	(S)
$\supset Sp_8$	$27 = 27$	(S)
$\supset F_4$	$27 = 1 + 26$	(S)
$\supset SU_3 \times G_2$	$27 = (\overline{6}, 1) + (3, 7)$	(S)
$E_7 \supset E_6 \times U_1$	$56 = 1 + 1 + 27 + \overline{27}$	(R)
$\supset SU_8$	$56 = 28 + \overline{28}$	(R)
$\supset SU_2 \times SO_{12}$	$56 = (2, 12) + (1, 32)$	(R)
$\supset SU_3 \times SU_6$	$56 = (3, 6) + (\overline{3}, \overline{6}) + (1, 20)$	(R)
$\supset SU_2$	$56 = 10 + 18 + 28$	(S)
$\supset SU_2$	$56 = 6 + 12 + 16 + 22$	(S)
$\supset SU_3$	$56 = 28 + \overline{28}$	(S)
$\supset SU_2 \times SU_2$	$56 = (5, 2) + (3, 6) + (7, 4)$	(S)
$\supset SU_2 \times G_2$	$56 = (4, 7) + (2, 14)$	(S)
$\supset SU_2 \times F_4$	$56 = (4, 1) + (2, 26)$	(S)
$\supset G_2 \times Sp_6$	$56 = (1, 14') + (7, 6)$	(S)
$E_8 \supset SO_{16}$	$248 = 120 + 128$	(R)
$\supset SU_5 \times SU_5$	$248 = (24, 1) + (1, 24) + (10, 5) + (\overline{10}, \overline{5}) + (5, \overline{10}) + (\overline{5}, 10)$	(R)
$\supset SU_3 \times E_6$	$248 = (8, 1) + (1, 78) + (3, 27) + (\overline{3}, \overline{27})$	(R)
$\supset SU_2 \times E_7$	$248 = (3, 1) + (1, 133) + (2, 56)$	(R)
$\supset SU_9$	$248 = 80 + 84 + \overline{84}$	(R)
$\supset SU_2$	$248 = 3 + 15 + 23 + 27 + 35 + 39 + 47 + 59$	(S)
$\supset SU_2$	$248 = 3 + 11 + 15 + 19 + 23 + 27 + 29 + 35 + 39 + 47$	(S)
$\supset SU_2$	$248 = 3 + 7 + 11 + 15 + 17 + 19 + 23 + 23 + 27 + 29 + 35 + 39$	(S)
$\supset G_2 \times F_4$	$248 = (14, 1) + (1, 52) + (7, 26)$	(S)
$\supset SU_2 \times SU_3$	$248 = (3, 1) + (1, 8) + (7, 8) + (5, 10) + (5, \overline{10}) + (3, 27)$	(S)
$\supset Sp_4$	$248 = 10 + 84 + 154$	(S)

Table 16
Extended Dynkin diagrams for simple Lie algebras. (The extended root is marked by x; black dots represent shorter roots.)

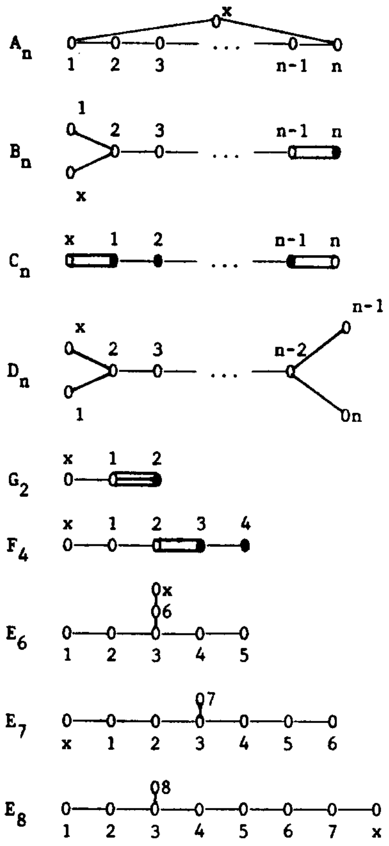


Table 17
Symmetric subgroups of simple groups

G	G_S	rank(A)	Action of C on irrep R
SU_n	SO_n	$n-1$	$\bar{R}$
SU_{p+q}	$SU_p \times SU_q \times U_1$	min(p, q)[a]	R
SU_{2n}	Sp_{2n}	$n-1$	$\bar{R}$
SO_{p+q}	$SO_p \times SO_q$	min(p, q)[a]	R (p or q even)
			$\bar{R}$ or R′ (p and q odd)[b]
SO_{2n}	$SU_n \times U_1$	[n/2]	R
Sp_{2n}	$SU_n \times U_1$	n	R
Sp_{2p+2q}	$Sp_{2p} \times Sp_{2q}$	min(p, q)[a]	R
G_2	$SU_2 \times SU_2$	2	R
F_4	$SU_2 \times Sp_6$	4	R
	SO_9	1	R
E_6	Sp_8	6	$\bar{R}$
	$SU_2 \times SU_6$	4	R
	$SO_{10} \times U_1$	2	R
	F_4	2	$\bar{R}$
E_7	SU_8	7	R
	$SU_2 \times SO_{12}$	4	R
	$E_6 \times U_1$	3	R
E_8	SO_{16}	8	R
	$SU_2 \times E_7$	4	R

[a] The case p or q equal unity defines a symmetric subgroup with SU_1 or SO_1 empty; the Lie algebra of Sp_2 is isomorphic to that of SU_2.

[b] If $p+q=4n+2$, C reflects complex spinor irreps into their conjugates; if $p+q=4n$, C reflects the real or pseudoreal spinor irreps into the nonequivalent spinor of the same dimension.

Table 18
E_6 and its subgroups with $U_1^{em} \times SU_3^c$

Group	No. max. subgroups	Satisfactory maximal subgroups	Unsatisfactory maximal subgroups
E_6	8	F_4, $SO_{10} \times U_1$, $SU_2 \times SU_6$, $SU_3 \times SU_3 \times SU_3$	G_2, SU_3, $SU_3 \times G_2$, [Sp_8]
F_4	5	SO_9, $SU_3 \times SU_3$	SU_2, $SU_2 \times G_2$, [$SU_2 \times Sp_6$]
SO_9	5	SO_8, $SU_2 \times SU_4$	SU_2, $SU_2 \times SU_2$, $SU_2 \times SU_2 \times Sp_4$, $SO_7 \times U_1$*
SO_8	4	SO_7, $SU_4 \times U_1$	SU_3, $SU_2 \times Sp_4$, $SU_2 \times SU_2 \times SU_2 \times SU_2$
SO_7	3	SU_4	G_2, $SU_2 \times SU_2 \times SU_2$, $Sp_4 \times U_1$
SU_4	3	$SU_3 \times U_1$	Sp_4, $SU_2 \times SU_2$
SO_{10}	6	$SU_5 \times U_1$, $SU_2 \times SU_2 \times SU_4$, SO_9, $SU_2 \times SO_7$, $SO_8 \times U_1$	Sp_4, $Sp_4 \times Sp_4$
SU_6	7	$SU_5 \times U_1$, $SU_2 \times U_1 \times SU_4$, $SU_3 \times SU_3 \times U_1$	SU_3, $SU_2 \times SU_3$, [SU_4], [Sp_6]
SU_5	3	$SU_4 \times U_1$, $SU_2 \times U_1 \times SU_3$	Sp_4

* See discussion for table 40.

Table 19
Physical roots and axes in E_6 weight space

	Dynkin basis	Dual basis		Dynkin basis	Dual basis
	(1 1) (0 0 0 0 0 1)	[1 2 3 2 1 2]	I_3^w axis	$\frac{1}{2}$(1 0 0 0 1 −1)	$\frac{1}{2}$[1 1 1 1 1 0]
Color roots	(2 −1) (0 1 0 0 −1 0)	[1 2 2 1 0 1]	Y^w axis	$\frac{1}{3}$(3 −4 6 −6 1 −1)	$\frac{1}{3}$[1 −1 1 −3 −1 0]
	(−1 2) (0 −1 0 0 1 1)	[0 0 1 1 1 1]	Q^t axis	3(1 −1 0 1 −1 0)	[1 −1 0 1 −1 0]
Weak isospin root	(1 0 0 0 1 −1)	[1 1 1 1 1 0]	Q^r axis	(−3 −1 4 1 −1 −4)	[−1 1 4 3 1 0]
Q^{em} axis	$\frac{1}{3}$(3 −2 3 −3 2 −2)	$\frac{1}{3}$[2 1 2 0 1 0]	$\frac{1}{5}(3Y^w + 4Q^r - 10I_3)$	(−1 −1 2 −1 −1 0)	[−1 −1 0 −1 −10]

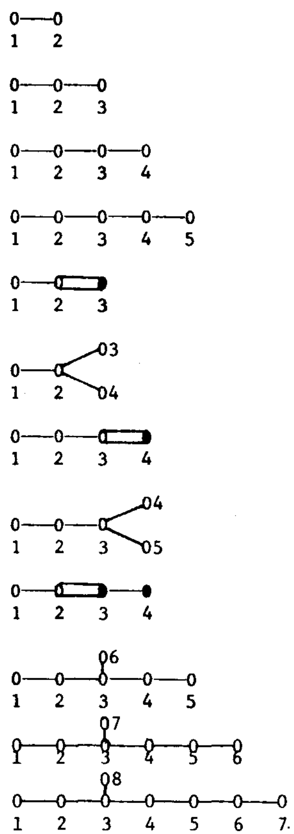
Table 20
Nonzero E_6 roots

Root	Level	Color	Q^{em}	I_3^w	Q^t	$SU_5(SO_{10})$	$\bar{B}\cdot\alpha$	$\bar{L}\cdot\alpha$
Color SU_3 roots								
(0 0 0 0 0 1)	0	(1 1)	0	0	0	24(45)	0	0
(0 1 0 0 −1 0)	4	(2 −1)	0	0	0	24(45)	0	0
(0 −1 0 0 1 1)	7	(−1 2)	0	0	0	24(45)	0	0
Left-handed SU_3 roots								
(1 0 0 0 1 −1)	6	(0 0)	1	1	0	24(45)	0	± 1
(−1 1 0 0 1 −1)	7	(0 0)	0	1/2	−3	$\bar{5}(16)$	$3c$	$3d \pm 2$
(−2 1 0 0 0 0)	12	(0 0)	−1	−1/2	−3	$\bar{5}(16)$	$3c$	$3d \pm 1$
Right-handed SU_3 roots								
(0 −1 1 1 −1 −1)	9	(0 0)	0	0	3	$1(\overline{16})$	$a+b-2c$	$-d+2e\mp 2$
(0 0 −1 2 −1 0)	10	(0 0)	−1	0	3	$\overline{10}(\overline{16})$	$-a+2b-c$	$-2d+e\mp 1$
(0 −1 2 −1 0 −1)	10	(0 0)	1	0	0	10(45)	$2a-b-c$	$d+3\mp 1$
SU_5 antilepto-diquarks								
(1 −1 1 −1 1 0)	4	(0 1)	4/3	1/2	0	24(45)	$a-b-c$	0
(1 0 1 −1 0 −1)	8	(1 −1)	4/3	1/2	0	24(45)	$a-b-c$	0
(1 −1 1 −1 1 −1)	15	(−1 0)	4/3	1/2	0	24(45)	$a-b-c$	0
(0 −1 1 −1 0 1)	9	(0 1)	1/3	−1/2	0	24(45)	$a-b-c$	∓ 1
(0 0 1 −1 −1 0)	13	(1 −1)	1/3	−1/2	0	24(45)	$a-b-c$	∓ 1
(0 −1 1 −1 0 0)	20	(−1 0)	1/3	−1/2	0	24(45)	$a-b-c$	∓ 1
SO_{10}/SU_5 leptoquarks								
(0 0 1 0 0 −1)	1	(1 0)	2/3	1/2	0	10(45)	a	$d+e$
(0 −1 1 0 1 −1)	8	(−1 1)	2/3	1/2	0	10(45)	a	$d+e$
(0 0 1 0 0 −2)	12	(0 −1)	2/3	1/2	0	10(45)	a	$d+e$
(−1 0 1 0 −1 0)	6	(1 0)	−1/3	−1/2	0	10(45)	a	$d+e\mp 1$
(−1 −1 1 0 0 0)	13	(−1 1)	−1/3	−1/2	0	10(45)	a	$d+e\mp 1$
(−1 0 1 0 −1 −1)	17	(0 −1)	−1/3	−1/2	0	10(45)	a	$d+e\mp 1$
(−1 0 0 1 0 0)	4	(0 1)	−2/3	0	0	10(45)	$b+c$	$d+e$
(−1 1 0 1 −1 −1)	8	(1 −1)	−2/3	0	0	10(45)	$b+c$	$d+e$
(−1 0 0 1 0 −1)	15	(−1 0)	−2/3	0	0	10(45)	$b+c$	$d+e$
E_6/SO_{10} leptoquarks								
(0 1 0 −1 1 0)	3	(1 0)	2/3	1/2	−3	10(16)	$-b+2c$	$2d-e\pm 2$
(0 0 0 −1 2 0)	10	(−1 1)	2/3	1/2	−3	10(16)	$-b+2c$	$2d-e\pm 2$
(0 1 0 −1 1 −1)	14	(0 −1)	2/3	1/2	−3	10(16)	$-b+2c$	$2d-e\pm 2$
(−1 1 0 −1 0 1)	8	(1 0)	−1/3	−1/2	−3	10(16)	$-b+2c$	$2d-e\pm 1$
(−1 0 0 −1 1 1)	15	(−1 1)	−1/3	−1/2	−3	10(16)	$-b+2c$	$2d-e\pm 1$
(−1 1 0 −1 0 0)	19	(0 −1)	−1/3	−1/2	−3	10(16)	$-b+2c$	$2d-e\pm 1$
(−1 0 1 −1 1 0)	5	(0 1)	1/3	0	−3	$\bar{5}(16)$	$a-b+2c$	$3d\pm 1$
(−1 1 1 −1 0 −1)	9	(1 −1)	1/3	0	−3	$\bar{5}(16)$	$a-b+2c$	$3d\pm 1$
(−1 0 1 −1 1 −1)	16	(−1 0)	1/3	0	−3	$\bar{5}(16)$	$a-b+2c$	$3d\pm 1$
(−1 1 −1 0 1 1)	6	(0 1)	−2/3	0	−3	10(16)	$-a+3c$	$2d-e\pm 2$
(−1 2 −1 0 0 0)	10	(1 −1)	−2/3	0	−3	10(16)	$-a+3c$	$2d-e\pm 2$
(−1 1 −1 0 1 0)	17	(−1 0)	−2/3	0	−3	10(16)	$-a+3c$	$2d-e\pm 2$

Table 21
Weights and content of the **27** of E_6

Weight	Level	Color	Q^{em}	I_3^W	Q'	$SU_5(SO_{10})$	SO_{10} weight
(0 0 0 1 0 −1)	4	(0 0)	0	1/2	1	$\bar{5}$(16)	(1 −1 0 1 0)
(−1 0 0 1 −1 0)	9	(0 0)	−1	−1/2	1	$\bar{5}$(16)	(1 0 0 0 −1)
(1 −1 1 −1 0 0)	9	(0 0)	1	0	1	10(16)	(−1 0 1 −1 0)
(1 0 −1 0 0 1)	10	(0 0)	0	0	1	1(16)	(−1 1 −1 0 1)
(0 0 1 −1 1 −1)	5	(0 0)	1	1/2	−2	5(10)	(0 −1 1 0 0)
(−1 0 1 −1 0 0)	10	(0 0)	0	−1/2	−2	5(10)	(0 0 1 −1 −1)
(0 1 −1 0 1 0)	6	(0 0)	0	1/2	−2	$\bar{5}$(10)	(0 0 −1 1 1)
(−1 1 −1 0 0 1)	11	(0 0)	−1	−1/2	−2	$\bar{5}$(10)	(0 1 −1 0 0)
(1 −1 0 1 −1 0)	8	(0 0)	0	0	4	1(1)	(0 0 0 0 0)
(1 0 0 0 0 0)	0	(1 0)	2/3	1/2	1	10(16)	(0 0 0 0 1)
(1 −1 0 0 1 0)	7	(−1 1)	2/3	1/2	1	10(16)	(−1 0 0 1 0)
(1 0 0 0 0 −1)	11	(0 −1)	2/3	1/2	1	10(16)	(0 −1 0 0 1)
(0 0 0 0 −1 1)	5	(1 0)	−1/3	−1/2	1	10(16)	(0 1 0 −1 0)
(0 −1 0 0 0 1)	12	(−1 1)	−1/3	−1/2	1	10(16)	(−1 1 0 0 −1)
(0 0 0 0 −1 0)	16	(0 −1)	−1/3	−1/2	1	10(16)	(0 0 0 −1 0)
(−1 1 0 0 0 0)	1	(1 0)	−1/3	0	−2	5(10)	(1 0 0 0 0)
(−1 0 0 0 1 0)	8	(−1 1)	−1/3	0	−2	5(10)	(0 0 0 1 −1)
(−1 1 0 0 0 −1)	12	(0 −1)	−1/3	0	−2	5(10)	(1 −1 0 0 0)
(0 0 0 −1 1 1)	4	(0 1)	1/3	0	−2	$\bar{5}$(10)	(−1 1 0 0 0)
(0 1 0 −1 0 0)	8	(1 −1)	1/3	0	−2	$\bar{5}$(10)	(0 0 0 −1 1)
(0 0 0 −1 1 0)	15	(−1 0)	1/3	0	−2	$\bar{5}$(10)	(−1 0 0 0 0)
(0 −1 1 0 0 0)	2	(0 1)	1/3	0	1	$\bar{5}$(16)	(0 0 1 0 −1)
(0 0 1 0 −1 −1)	6	(1 −1)	1/3	0	1	$\bar{5}$(16)	(1 −1 1 −1 0)
(0 −1 1 0 0 −1)	13	(−1 0)	1/3	0	1	$\bar{5}$(16)	(0 −1 1 0 −1)
(0 0 −1 1 0 1)	3	(0 1)	−2/3	0	1	10(16)	(0 1 −1 1 0)
(0 1 −1 1 −1 0)	7	(1 −1)	−2/3	0	1	10(16)	(1 0 −1 0 1)
(0 0 −1 1 0 0)	14	(−1 0)	−2/3	0	1	10(16)	(0 0 −1 1 0)

Table 22
Ordering of simple roots of Dynkin diagrams; orders of independent Casimir invariants for tables 23 to 53. (Shorter roots are denoted by black dots.)



	Orders of Independent Casimir Invariants
SU_3	2,3
SU_4	2,3,4
SU_5	2,3,4,5
SU_6	2,3,4,5,6
SO_7	2,4,6
SO_8	2,4,4,6
SO_9	2,4,6,8
SO_{10}	2,4,5,6,8
F_4	2,6,8,12
E_6	2,5,6,8,9,12
E_7	2,6,8,10,12,14,18
E_8	2,8,12,14,18,20,24,30

Table 23
SU_3 irreps of dimension less than 65

Dynkin label	Dimension (name)	l (index)	Triality	SU_2 singlets	SO_3 singlets
(10)	3	1	1	1	0
(20)	6	5	2*	1	1
(11)	8	6	0	1**	0
(30)	10	15	0	1	0
(21)	15	20	1	1	0
(40)	15′	35	1	1	1
(05)	21	70	1	1	0
(13)	24	50	1	1	0
(22)	27	54	0	1**	1
(60)	28	126	0	1	1
(41)	35	105	0	1	0
(70)	36	210	1	1	0
(32)	42	119	1	1	0
(08)	45	330	1	1	1
(51)	48	196	1	1	0
(90)	55	495	0	1	0
(24)	60	230	1	1	1
(16)	63	336	1	1	0
(33)	64	240	0	1**	0

*Note standard convention that $6 = (20)$.
**$SU_2 \times U_1$ singlet.

Table 24
SU_3 tensor products; triality 0 and 1 combinations shown

$\bar{3}\times\bar{3} = 3_a + \bar{6}_s$
$3\times\bar{3} = 1 + 8$
$6\times 3 = 8 + 10$
$6\times\bar{3} = 3 + 15$
$6\times 6 = \bar{6}_s + 15_a + 15'_s$
$6\times\bar{6} = 1 + 8 + 27$
$8\times 3 = 3 + \bar{6} + 15$
$8\times\bar{6} = 3 + \bar{6} + 15 + 24$
$8\times 8 = 1_s + 8_s + 8_a + 10_a + \overline{10}_a + 27_s$
$10\times 3 = 15 + 15'$
$\overline{10}\times 3 = \bar{6} + 24$
$10\times\bar{6} = 3 + 15 + 42$
$\overline{10}\times\bar{6} = 15 + 21 + 24$
$10\times 8 = 8 + 10 + 27 + 35$
$10\times 10 = 10_a + 27_s + 28_s + 35_a$
$\overline{10}\times 10 = 1 + 8 + 27 + 64$
$\overline{15}\times\bar{3} = \bar{6} + 15 + 24$
$15\times\bar{3} = 8 + 10 + 27$
$\overline{15}\times 6 = 3 + \bar{6} + 15 + 24 + 42$
$15\times 6 = 8 + 10 + \overline{10} + 27 + 35$
$15\times 8 = 3 + \bar{6} + 15_1 + 15_2 + 15' + 24 + 42$
$15\times 10 = \bar{6} + 15 + 15' + 24 + 42 + 48$
$15\times\overline{10} = 3 + \bar{6} + 15' + 24 + 42 + 60$
$\overline{15}\times\overline{15} = 3_a + \bar{6}_s + 15_s + 15_a + 15'_s + 21_a + 24_s + 24_a + 42_a + 60_s$
$15\times\overline{15} = 1 + 8_1 + 8_2 + 10 + \overline{10} + 27_1 + 27_2 + 35 + \overline{35} + 64$

Table 25
SU_4 irreps of dimension less than 180

Dynkin label	Dimension (name)	l (index)	Quadrality	SU_3 singlets
(100)	4	1	1	1
(010)	6	2	2	0
(200)	10	6	2	1
(101)	15	8	0	1*
(011)	20	13	1	0
(020)	20′	16	0	0
(003)	20″	21	1	1
(400)	35	56	0	1
(201)	36	33	1	1
(210)	45	48	0	0
(030)	50	70	2	0
(500)	56	126	1	1
(120)	60	71	1	0
(111)	64	64	2	0
(301)	70	98	2	1
(202)	84	112	0	1*
(310)	84′	133	1	0
(600)	84″	252	2	1
(040)	105	224	0	0
(104)	120	238	1	1
(007)	120′	462	1	1
(220)	126	210	2	0
(112)	140	203	1	0
(031)	140′	259	1	0
(410)	140″	308	2	0
(302)	160	296	1	1
(800)	165	792	0	1
(121)	175	280	0	0

*$SU_3 \times U_1$ singlet.

Table 26
SU_4 tensor products; quadrality 0, 1 and 2 shown

$4 \times 4 = 6_a + 10_s$
$4 \times \bar{4} = 1 + 15$
$6 \times \bar{4} = 4 + 20$
$6 \times 6 = 1_s + 15_a + 20'_s$
$\overline{10} \times \bar{4} = 20 + 20''$
$10 \times \bar{4} = 4 + 36$
$10 \times 6 = 15 + 45$
$10 \times 10 = 20'_s + 35_s + 45_a$
$10 \times \overline{10} = 1 + 15 + 84$
$15 \times 4 = 4 + 20 + 36$
$15 \times 6 = 6 + 10 + \overline{10} + 64$
$15 \times 10 = 6 + 10 + 64 + 70$
$15 \times 15 = 1_s + 15_s + 15_a + 20'_s + 45_a + \overline{45}_a + 84_s$
$\overline{20} \times 4 = 15 + 20' + 45$
$20 \times 4 = 6 + \overline{10} + 64$
$\overline{20} \times 6 = 4 + 20 + 36 + 60$
$\overline{20} \times 10 = 20 + 36 + 60 + 84'$
$\overline{20} \times \overline{10} = 4 + 20 + 36 + 140$
$20 \times 15 = 4 + 20_1 + 20_2 + 20'' + 36 + 60 + 140$
$20 \times 20 = 6_a + 10_s + \overline{10}_s + 50_a + 64_s + 64_a + 70_a + \overline{126}_s$
$20 \times \overline{20} = 1 + 15_1 + 15_2 + 20' + 45 + \overline{45} + 84 + 175$
$20' \times 4 = 20 + 60$
$20' \times 6 = 6 + 50 + 64$
$20' \times 10 = 10 + 64 + 126$
$20' \times 15 = 15 + 20' + 45 + \overline{45} + 175$
$20' \times 20 = 4 + 20 + 36 + 60 + 140 + 140'$
$20' \times 20' = 1_s + 15_a + 20'_s + 84_s + 105_s + 175_a$

Table 27
Branching rules for $SU_4 \supset SU_3 \times U_1$

(100) = $4 = 1(1) + 3(-1/3)$ (establishes normalization of U_1 generator)
(010) = $6 = 3(2/3) + \bar{3}(-2/3)$
(200) = $10 = 1(2) + 3(2/3) + 6(-2/3)$
(101) = $15 = 1(0) + 3(-4/3) + \bar{3}(4/3) + 8(0)$
(011) = $20 = 3(-1/3) + \bar{3}(-5/3) + \bar{6}(-1/3) + 8(1)$
(020) = $20' = \bar{6}(-4/3) + 6(4/3) + 8(0)$
(003) = $20'' = 1(-3) + \bar{3}(-5/3) + \bar{6}(-1/3) + \overline{10}(1)$
(400) = $35 = 1(4) + 3(8/3) + 6(4/3) + 10(0) + 15'(-4/3)$
(201) = $36 = 1(1) + 3(-1/3) + \bar{3}(7/3) + 6(-5/3) + 8(1) + 15(-1/3)$
(210) = $45 = 3(8/3) + \bar{3}(4/3) + 6(4/3) + 6(4/3) + 8(0) + 10(0) + 15(-4/3)$
(030) = $50 = 10(2) + \overline{10}(-2) + 15(2/3) + \overline{15}(-2/3)$
(500) = $56 = 1(5) + 3(11/3) + 6(7/3) + 10(1) + 15'(-1/3) + \overline{21}(-5/3)$
(120) = $60 = \bar{6}(-1/3) + 6(7/3) + 8(1) + 10(1) + 15(-1/3) + \overline{15}(-5/3)$
(111) = $64 = 3(2/3) + \bar{3}(-2/3) + \bar{6}(2/3) + 6(-2/3) + 8(2) + 8(-2) + 15(2/3) + \overline{15}(-2/3)$

Table 28
SU_5 irreps of dimension less than 800

Dynkin label	Dimension (name)	l (index)	Quintality	SU_4 singlets	$SU_2 \times SU_3$ singlets
(1000)	5	1	1	1	0
(0100)	10	3	2	0	1
(2000)	15	7	2	1	0
(1001)	24	10	0	1*	1*
(0003)	35	28	2	1	0
(0011)	40	22	2	0	0
(0101)	45	24	1	0	0
(0020)	50	35	1	0	1
(2001)	70	49	1	1	0
(0004)	70′	84	1	1	0
(0110)	75	50	0	0	1*
(0012)	105	91	1	0	0
(2010)	126	105	0	0	0
(5000)	126′	210	0	1	0
(3001)	160	168	2	1	0
(1101)	175	140	2	0	1
(1200)	175′	175	0	0	0
(0300)	175″	210	1	0	1
(2002)	200	200	0	1*	1*
(1020)	210	203	2	0	0
(6000)	210′	462	1	1	0
(3100)	224	280	0	0	0
(1110)	280	266	1	0	0
(3010)	280′	336	1	0	0
(0210)	315	357	2	0	1
(1004)	315′	462	2	1	0
(7000)	330	924	2	1	0
(2200)	420	574	1	0	0
(4100)	420′	714	1	0	0
(1012)	450	510	2	0	0
(3002)	450′	615	1	1	0
(1102)	480	536	1	0	0
(0040)	490	882	2	0	1
(0008)	495	1716	2	1	0
(4010)	540	882	2	0	0
(0202)	560	728	2	0	0
(1300)	560′	868	2	0	0
(1005)	560″	1092	1	1	0
(2110)	700	910	2	0	0
(1030)	700′	1050	0	0	0
(0009)	715	3003	1	1	0
(1021)	720	924	1	0	1
(5100)	720′	1596	2	0	0

(0130)	980	1666	1	0	1
(1111)	1024	1280	0	0	1*
(0121)	1050	1540	2	0	0
(0211)	1120	1624	1	0	0
(0220)	1176	1960	0	0	1*

Table 29

SU_5 tensor products

$5\times 5 = 10_a + 15_s$

$5\times \overline{5} = 1 + 24$

$\overline{10}\times \overline{5} = 10 + 40$

$10\times \overline{5} = 5 + 45$

$\overline{10}\times \overline{10} = 5_s + 45_a + 50_s$

$\overline{10}\times 10 = 1 + 24 + 75$

$\overline{15}\times \overline{5} = 35 + 40$

$15\times \overline{5} = 5 + 70$

$\overline{15}\times \overline{10} = 45 + 105$

$15\times \overline{10} = 24 + 126$

$\overline{15}\times \overline{15} = 50_s + 70'_s + 105_a$

$\overline{15}\times 15 = 1 + 24 + 200$

$24\times 5 = 5 + 45 + 70$

$24\times 10 = 10 + 15 + 40 + 175$

$24\times 15 = 10 + 15 + 160 + 175$

$24\times 24 = 1_s + 24_s + 24_a + 75_s + 126_a + \overline{126_a} + 200_s$

$\overline{40}\times \overline{5} = 10 + 15 + 175$

$40\times \overline{5} = 45 + 50 + 105$

$\overline{40}\times 10 = 24 + 75 + 126 + 175'$

$\overline{40}\times \overline{10} = 5 + 45 + 70 + 280$

$\overline{40}\times 15 = 75 + 126 + 175' + 224$

$\overline{40}\times \overline{15} = 5 + 45 + 70 + 480$

$40\times 24 = 10 + 35 + 40_1 + 40_2 + 175 + 210 + 450$

$\overline{40}\times \overline{40} = 45_a + 50_s + 70_s + 175''_a + 280_a + 280_s + 280'_a + 420_s$

$40\times \overline{40} = 1 + 24_1 + 24_2 + 75 + 126 + \overline{126} + 200 + 1024$

$45\times 5 = 10 + 40 + 175$

$\overline{45}\times 5 = 24 + 75 + 126$

$\overline{45}\times 10 = 5 + 45 + 50 + 70 + 280$

$\overline{45}\times \overline{10} = 10 + 15 + 40 + 175 + 210$

$\overline{45}\times 15 = 45 + 70 + 280 + 280'$

$\overline{45}\times \overline{15} = 10 + 40 + 175 + 450$

$45\times 24 = 5 + 45_1 + 45_2 + 50 + 70 + 105 + 280 + 480$

$\overline{45}\times \overline{40} = 10 + 15 + 40 + 160 + 175_1 + 175_2 + 210 + 315 + 700$

$\overline{45}\times 40 = 45 + 50_1 + 50_2 + 70 + 105 + 280 + 480 + 720$

$45\times 45 = 10_a + 15_s + 35_s + 40_s + 40_a + 175_s + 175_a + 210_s + 315_a + 450_a + 560_s$

$\overline{45}\times 45 = 1 + 24_1 + 24_2 + 75_1 + 75_2 + 126 + \overline{126} + 175' + \overline{175}' + 200 + 1024$

$50\times 5 = 40 + 210$

$\overline{50}\times 5 = 75 + 175'$

$\overline{50}\times \overline{10} = 10 + 175 + 315$

$\overline{50}\times 10 = 45 + 175'' + 280$

$\overline{50}\times \overline{15} = 15 + 175 + 560$

$\overline{50}\times 15 = 50 + 280 + 420$

$50\times 24 = 45 + 50 + 105 + 280 + 720$

$\overline{50}\times \overline{40} = 40 + 175 + 210 + 315 + 560' + 700$

$\overline{50}\times 40 = 5 + 45 + 70 + 280 + 480 + 1120$

$50\times 45 = 10 + 40 + 175 + 210 + 315 + 450 + 1050$

$50\times \overline{45} = 24 + 75 + 126 + \overline{126} + \overline{175}' + 700' + 1024$

$50\times 50 = 15_s + 175_a + 210_s + 490_s + 560_s + 1050_a$

$\overline{50}\times 50 = 1 + 24 + 75 + 200 + 1024 + 1176$

$75\times 5 = 45 + 50 + 280$

$75\times 10 = 10 + 40 + 175 + 210 + 315$

$75\times 15 = 40 + 175 + 210 + 700$

$75\times 24 = 24 + 75_1 + 75_2 + 126 + \overline{126} + 175' + \overline{175}' + 1024$

$75\times 40 = 10 + 15 + 40 + 175_1 + 175_2 + 210 + 315 + 450 + 560 + 1050$

$75\times 45 = 5 + 45_1 + 45_2 + 50 + 70 + 105 + 175'' + 280_1 + 280_2 + 480 + 720 + 1120$

$75\times 50 = 5 + 45 + 50 + 70 + 280 + 480 + 720 + 980 + 1120$

$75\times 75 = 1_s + 24_s + 24_a + 75_s + 75_a + 126_a + \overline{126_a} + 175'_s + \overline{175}'_s + 200_s + 700'_s + \overline{700}'_s + 1024_s + 1024_a + 1176_s$

Table 30
Branching rules for SU_5

$SU_5 \supset SU_4 \times U_1$
$(1000) = 5 = 1(4) + 4(-1)$
$(0100) = 10 = 4(3) + 6(-2)$
$(2000) = 15 = 1(8) + 4(3) + 10(-2)$
$(1001) = 24 = 1(0) + 4(-5) + \bar{4}(5) + 15(0)$
$(0003) = 35 = 1(-12) + \bar{4}(-7) + \overline{10}(-2) + 20''(3)$
$(0011) = 40 = \bar{4}(-7) + 6(-2) + \overline{10}(-2) + 20(3)$
$(0101) = 45 = 4(-1) + 6(-6) + 15(4) + 20(-1)$
$(0020) = 50 = \overline{10}(-6) + 20(-1) + 20'(4)$
$(2001) = 70 = 1(4) + 4(-1) + \bar{4}(9) + 10(-6) + 15(4) + 36(-1)$
$(0004) = 70' = 1(-16) + \bar{4}(-11) + \overline{10}(-6) + 20''(-1) + \overline{35}(4)$
$(0110) = 75 = 15(0) + 20(-5) + \overline{20}(5) + 20'(0)$

$SU_5 \supset SU_2 \times SU_3 \times U_1$
$5 = (2, 1)(3) + (1, 3)(-2)$
$10 = (1, 1)(6) + (1, \bar{3})(-4) + (2, 3)(1)$
$15 = (3, 1)(6) + (2, 3)(1) + (1, 6)(-4)$
$24 = (1, 1)(0) + (3, 1)(0) + (2, 3)(-5) + (2, \bar{3})(5) + (1, 8)(0)$
$35 = (4, 1)(-9) + (3, \bar{3})(-4) + (2, \bar{6})(1) + (1, \overline{10})(6)$
$40 = (2, 1)(-9) + (2, 3)(1) + (1, \bar{3})(-4) + (3, \bar{3})(-4) + (1, 8)(6) + (2, \bar{6})(1)$
$45 = (2, 1)(3) + (1, 3)(-2) + (3, 3)(-2) + (1, \bar{3})(8) + (2, \bar{3})(-7) + (1, \bar{6})(-2) + (2, 8)(3)$
$50 = (1, 1)(-12) + (1, 3)(-2) + (2, \bar{3})(-7) + (3, \bar{6})(-2) + (1, 6)(8) + (2, 8)(3)$
$70 = (2, 1)(3) + (4, 1)(3) + (1, 3)(-2) + (3, 3)(-2) + (3, \bar{3})(8) + (2, 6)(-7) + (2, 8)(3) + (1, 15)(-2)$
$70' = (5, 1)(-12) + (4, \bar{3})(-7) + (3, \bar{6})(-2) + (2, \overline{10})(3) + (1, \overline{15}')(8)$
$75 = (1, 1)(0) + (1, 3)(10) + (2, 3)(-5) + (1, \bar{3})(-10) + (2, \bar{3})(5) + (2, \bar{6})(-5) + (2, 6)(5) + (1, 8)(0) + (3, 8)(0)$

Table 31
SU_6 irreps of dimension less than 1000

Dynkin label	Dimension	l (index)	Sextality	SU_5 singlets	$SU_2 \times SU_4$ singlets	$SU_3 \times SU_3$ singlets
(10000)	6	1	1	1	0	0
(01000)	15	4	2	0	1	0
(00100)	20	6	3	0	0	2
(20000)	21	8	2	1	0	0
(10001)	35	12	0	1*	1*	1*
(30000)	56	36	3	1	0	0
(11000)	70	33	3	0	0	0
(01001)	84	38	1	0	0	0
(00101)	105	52	2	0	0	0
(00020)	105′	64	2	0	1	0
(20001)	120	68	1	1	0	0
(00004)	126	120	2	1	0	0
(00200)	175	120	0	0	0	2+1*
(01010)	189	108	0	0	1*	1*
(00110)	210	131	1	0	0	0
(00012)	210′	152	2	0	0	0
(00005)	252	330	1	1	0	0
(20010)	280	192	0	0	0	0
(30001)	315	264	2	1	0	0
(00102)	336	248	1	0	0	0
(11001)	384	256	2	0	1	0
(20002)	405	324	0	1*	1*	1*

Table 31 (continued)

Dynkin label	Dimension	l (index)	Sextality	SU_5 singlets	$SU_2 \times SU_4$ singlets	$SU_3 \times SU_3$ singlets
(00021)	420	358	1	0	0	0
(00006)	462	792	0	1	0	0
(00030)	490	504	0	0	1	0
(00013)	504	516	1	0	0	0
(10101)	540	378	3	0	0	2
(02001)	560	456	3	0	0	0
(40001)	700	810	3	1	0	0
(30010)	720	696	1	0	0	0
(70000)	792	1716	1	1	0	0
(11010)	840	668	1	0	0	0
(10200)	840′	764	1	0	0	0
(30100)	840″	864	0	0	0	0
(11100)	896	768	0	0	0	0
(00300)	980	1134	3	0	0	4

(10110)	1050	880	2	0	0	0
(21001)	1134	1053	3	0	0	0
(22000)	1134′	1296	0	0	0	0
(02010)	1176	1120	2	0	1	0
(02100)	1176′	1204	1	0	0	0
(11002)	1260	1146	1	0	0	0
(01200)	1470	1568	2	0	0	0
(10102)	1701	1620	2	0	0	0
(13000)	1764	2310	1	0	0	0
(04000)	1764′	2688	2			
(02002)	1800	1920	2	0	0	0
(01110)	1960	1932	3	0	0	2
(10021)	2205	2352	2	0	1	0
(21010)	2430	2592	2	0	0	0
(21100)	2520	2868	1	0	0	0
(20200)	2520′	2976	2	0	0	0
(10030)	2520″	3156	1	0	0	0
(01102)	3240	3564	3	0	0	0
(11011)	3675	3780	0	0	1*	1*
(10201)	3969	4536	0	0	0	2 + 1*
(00400)	4116	7056	0	0	0	4 + 1*
(10111)	4410	4767	1	0	0	0
(12100)	4410′	5712	2	0	0	0
(00301)	4410″	6216	2	0	0	0
(01021)	4536	5508	3	0	0	0
(00130)	4704	7056	3	0	0	0
(02011)	5040	6024	1			
(01030)	5040′	7104	2			
(02101)	5670	7128	0			
(00211)	5880	7812	3			
(02020)	6720	9216	0			
(01201)	6804	8910	1			
(00220)	7056	10752	2			
(00310)	7056′	11256	1			
(01111)	8064	9984	2			
(01120)	10080	14352	1			
(01210)	11340	16848	0			

Table 32
SU_6 tensor products

$6\times 6 = 15_a + 21_s$

$6\times \bar{6} = 1 + 35$

$15\times 6 = 20 + 70$

$15\times \bar{6} = 6 + 84$

$\overline{15}\times 15 = 1 + 35 + 189$

$\overline{15}\times \overline{15} = 15_s + 105_a + 105'_s$

$20\times \bar{6} = 15 + 105$

$20\times \overline{15} = 6 + 84 + 210$

$20\times 20 = 1_a + 35_s + 175_s + 189_a$

$21\times 6 = 56 + 70$

$21\times \bar{6} = 6 + 120$

$\overline{21}\times \overline{15} = 105 + 210'$

$21\times \overline{15} = 35 + 280$

$\overline{21}\times 20 = 84 + 336$

$\overline{21}\times 21 = 1 + 35 + 405$

$\overline{21}\times \overline{21} = 105'_s + 126_s + 210'_a$

$35\times 6 = 6 + 84 + 120$

$35\times 15 = 15 + 21 + 105 + 384$

$35\times 20 = 20 + 70 + \overline{70} + 540$

$35\times 21 = 15 + 21 + 315 + 384$

$35\times 35 = 1_s + 35_s + 35_a + 189_s + 280_a + \overline{280}_a + 405_s$

$70\times \bar{6} = 15 + 21 + 384$

$\overline{70}\times \bar{6} = 105 + 105' + 210'$

$70\times \overline{15} = 6 + 84 + 120 + 840$

$\overline{70}\times \overline{15} = 84 + 210 + 336 + 420$

$70\times 20 = 35 + 189 + 280 + 896$

$70\times \overline{21} = 6 + 84 + 120 + 1260$

$\overline{70}\times \overline{21} = 210 + 336 + 420 + 504$

$70\times 35 = 20 + 56 + 70_1 + 70_2 + 540 + 560 + 1134$

$70\times 70 = 175_s + 189_a + 280_s + 490_a + 840''_a + 896_s + 896_a + 1134'_s$

$\overline{70}\times 70 = 1 + 35_1 + 35_2 + 189 + 280 + \overline{280} + 405 + 3675$

$84\times 6 = 15 + 105 + 384$

$\overline{84}\times 6 = 35 + 189 + 280$

$84\times 15 = 20 + 70 + \overline{70} + 540 + 560$

$\overline{84}\times 15 = 6 + 84 + 120 + 210 + 840$

$\overline{84}\times 20 = 15 + 21 + 105 + 105' + 384 + 1050$

$84\times 21 = 20 + 70 + 540 + 1134$

$\overline{84}\times 21 = 84 + 120 + 840 + 720$

$84\times 35 = 6 + 84_1 + 84_2 + 120 + 210 + 336 + 840 + 1260$

$\overline{84}\times 70 = 15 + 21 + 105 + 315 + 384_1 + 384_2 + 1050 + 1176 + 2430$

$\overline{84}\times \overline{70} = 15\times 105_1 + 105_2 + 105' + 210' + 384 + 1050 + 1701 + 2205$

$84\times 84 = 15_a + 21_s + 105_s + 105_a + 105'_a + 210'_s + 384_s + 384_a + 1050_s + 1176_a + 1701_a + 1800_s$

$\overline{84}\times 84 = 1 + 35_1 + 35_2 + 175 + 189_1 + 189_2 + 280 + \overline{280} + 405 + 896 + \overline{896} + 3675$

Table 33
SU_6 branching rules

$SU_6 \supset SU_5 \times U_1$
$(10000) = 6 = 1(-5) + 5(1)$
$(01000) = 15 = 5(-4) + 10(2)$
$(00100) = 20 = 10(-3) + \overline{10}(3)$
$(20000) = 21 = 1(-10) + 5(-4) + 15(2)$
$(10001) = 35 = 1(0) + 5(6) + \bar{5}(-6) + 24(0)$
$(30000) = 56 = 1(-15) + 5(-9) + 15(-3) + \overline{35}(3)$
$(11000) = 70 = 5(-9) + 10(-3) + 15(-3) + \overline{40}(3)$
$(01001) = 84 = 5(1) + 10(7) + 24(-5) + 45(1)$
$(00101) = 105 = 10(2) + \overline{10}(8) + 40(2) + 45(-4)$
$(00020) = 105' = \overline{15}(8) + 40(2) + 50(-4)$

$SU_6 \supset SU_2 \times SU_4 \times U_1$
$6 = (2, 1)(2) + (1, 4)(-1)$
$15 = (1, 1)(4) + (1, 6)(-2) + (2, 4)(1)$
$20 = (1, 4)(3) + (1, \bar{4})(-3) + (2, 6)(0)$
$21 = (3, 1)(4) + (2, 4)(1) + (1, 10)(-2)$
$35 = (1, 1)(0) + (3, 1)(0) + (1, 15)(0) + (2, 4)(-3) + (2, \bar{4})(3)$
$56 = (4, 1)(6) + (3, 4)(3) + (2, 10)(0) + (1, \overline{20''})(-3)$
$70 = (2, 1)(6) + (1, 4)(3) + (3, 4)(3) + (2, 6)(0) + (2, 10)(0) + (1, \overline{20})(-3)$
$84 = (2, 1)(2) + (1, 4)(5) + (1, \bar{4})(-1) + (3, 4)(-1) + (2, 6)(-4) + (1, 20)(-1) + (2, 15)(2)$
$105 = (2, 4)(1) + (2, \bar{4})(-5) + (1, 6)(-2) + (3, 6)(-2) + (1, \overline{10})(-2) + (1, 15)(4) + (2, 20)(1)$
$105' = (1, 1)(-8) + (1, 6)(-2) + (2, \bar{4})(-5) + (3, \overline{10})(-2) + (1, 20')(4) + (2, 20)(1)$

$SU_6 \supset SU_3 \times SU_3 \times U_1$
$6 = (3, 1)(1) + (1, 3)(-1)$
$15 = (\bar{3}, 1)(2) + (1, \bar{3})(-2) + (3, 3)(0)$
$20 = (1, 1)(3) + (1, 1)(-3) + (3, \bar{3})(-1) + (\bar{3}, 3)(1)$
$21 = (6, 1)(2) + (1, 6)(-2) + (3, 3)(0)$
$35 = (1, 1)(0) + (8, 1)(0) + (1, 8)(0) + (3, \bar{3})(2) + (\bar{3}, 3)(-2)$
$56 = (10, 1)(3) + (1, 10)(-3) + (3, 6)(-1) + (6, 3)(1)$
$70 = (8, 1)(3) + (1, 8)(-3) + (3, \bar{3})(-1) + (\bar{3}, 3)(1) + (3, 6)(-1) + (6, 3)(1)$
$84 = (3, 1)(1) + (1, 3)(-1) + (\bar{6}, 1)(1) + (1, \bar{6})(-1) + (\bar{3}, \bar{3})(3) + (\bar{3}, \bar{3})(-3) + (3, 8)(1) + (8, 3)(-1)$
$105 = (\bar{3}, 1)(2) + (\bar{3}, 1)(-4) + (1, \bar{3})(-2) + (1, \bar{3})(4) + (3, 3)(0) + (\bar{6}, 3)(0) + (3, \bar{6})(0) + (8, \bar{3})(-2) + (\bar{3}, 8)(2)$
$105' = (6, 1)(-4) + (1, 6)(4) + (3, 3)(0) + (8, \bar{3})(-2) + (\bar{3}, 8)(2) + (\bar{6}, \bar{6})(0)$

Table 34
SO_7 irreps of dimension less than 650 and branching rules ($SO_7 \supset SU_4$)

Dynkin label	Dimension (name)	$l/2$ (index)	Branching into SU_4 irreps
(100)	7	1	$1 + 6$
(001)	8	1	$4 + \bar{4}$
(010)	21	5	$6 + 15$
(200)	27	9	$1 + 6 + 20'$
(002)	35	10	$10 + \overline{10} + 15$
(101)	48	14	$4 + \bar{4} + 20 + \overline{20}$
(300)	77	44	$1 + 6 + 20' + 50$
(110)	105	45	$6 + 15 + 20' + 64$
(011)	112	46	$20 + \overline{20} + 36 + \overline{36}$
(003)	112'	54	$20'' + \overline{20''} + 36 + \overline{36}$
(020)	168	96	$20' + 64 + 84$
(201)	168'	85	$4 + \bar{4} + 20 + \overline{20} + 60 + \overline{60}$
(400)	182	156	$1 + 6 + 20' + 50 + 105$
(102)	189	90	$10 + \overline{10} + 15 + 45 + \overline{45} + 64$
(004)	294	210	$35 + \overline{35} + 70 + \overline{70} + 84$
(210)	330	220	$6 + 15 + 20' + 50 + 64 + 175$
(012)	378	234	$45 + \overline{45} + 64 + 70 + \overline{70} + 84$
(500)	378'	450	$1 + 6 + 20' + 50 + 105 + 196$
(301)	448	344	$4 + \bar{4} + 20 + \overline{20} + 60 + \overline{60} + 140' + \overline{140'}$
(111)	512	320	$20 + \overline{20} + 36 + \overline{36} + 60 + \overline{60} + 140 + \overline{140}$
(103)	560	390	$20'' + \overline{20''} + 36 + \overline{36} + 84' + \overline{84'} + 140 + \overline{140}$
(202)	616	440	$10 + \overline{10} + 15 + 45 + \overline{45} + 64 + 126 + \overline{126} + 175$

Table 35
SO_7 tensor products

$7\times 7 = 1_s + 21_a + 27_s$
$8\times 7 = 8 + 48$
$8\times 8 = 1_s + 7_a + 21_a + 35_s$
$21\times 7 = 7 + 35 + 105$
$21\times 8 = 8 + 48 + 112$
$21\times 21 = 1_s + 21_a + 27_s + 35_s + 168_s + 189_a$
$27\times 7 = 7 + 77 + 105$
$27\times 8 = 48 + 168'$
$27\times 21 = 21 + 27 + 189 + 330$
$27\times 27 = 1_s + 21_a + 27_s + 168_s + 182_s + 330_a$
$35\times 7 = 21 + 35 + 189$
$35\times 8 = 8 + 48 + 112 + 112'$
$35\times 21 = 7 + 21 + 35 + 105 + 189 + 378$
$35\times 27 = 35 + 105 + 189 + 616$
$35\times 35 = 1_s + 7_a + 21_a + 27_s + 35_s + 105_s + 168_s + 189_a + 294_s + 378_a$
$48\times 7 = 8 + 48 + 112 + 168'$
$48\times 8 = 7 + 21 + 27 + 35 + 105 + 189$
$48\times 21 = 8 + 48_1 + 48_2 + 112 + 112' + 168' + 512$
$48\times 27 = 8 + 48 + 112 + 168' + 448 + 512$
$48\times 35 = 8 + 48_1 + 48_2 + 112_1 + 112_2 + 112' + 168' + 512 + 560$
$48\times 48 = 1_s + 7_a + 21_{a1} + 21_{a2} + 27_s + 35_{s1} + 35_{s2} + 77_a + 105_s + 105_a + 168_s + 189_s + 189_a + 330_a + 378_a + 616_s$

Table 36
SO_8 irreps of dimension less than 1300

Dynkin label	Dimension (name)	Congruency class	$l/8$ (index)	Branching into SO_7 irreps
(1000)	8_v	(01)	1	8
(0001)	8_s	(10)	1	$1+7$
(0010)	8_c	(11)	1	8
(0100)	28	(00)	6	$7+21$
(2000)	35_v	(00)	10	35
(0002)	35_s	(00)	10	$1+7+27$
(0020)	35_c	(00)	10	35
(0011)	56_v	(01)	15	$8+48$
(1010)	56_s	(10)	15	$21+35$
(1001)	56_c	(11)	15	$8+48$
(3000)	112_v	(01)	54	$112'$
(0003)	112_s	(10)	54	$1+7+27+77$
(0030)	112_c	(11)	54	$112'$
(1100)	160_v	(01)	60	$48+112$
(0101)	160_s	(10)	60	$7+21+27+105$
(0110)	160_c	(11)	60	$48+112$
(1002)	224_{sv}	(01)	100	$8+48+168'$
(1020)	224_{cv}	(01)	100	$112+112'$
(2001)	224_{vs}	(10)	100	$35+189$
(2010)	224_{vc}	(11)	100	$112+112'$
(0012)	224_{sc}	(11)	100	$8+48+168'$
(0021)	224_{cs}	(10)	100	$35+189$

Table 36 (continued)

Dynkin label	Dimension (name)	Congruency class	$l/8$ (index)	Branching into SO_7 irreps
(4000)	294_v	(00)	210	294
(0004)	294_s	(00)	210	$1+7+27+77+182$
(0040)	294_c	(00)	210	294
(0200)	300	(00)	150	$27+105+168$
(1011)	350	(00)	150	$21+35+105+189$
(2100)	567_v	(00)	324	$189+378$
(0102)	567_s	(00)	324	$7+21+27+77+105+330$
(0120)	567_c	(00)	324	$189+378$
(3001)	672_{vs}	(11)	444	$112'+560$
(3010)	672_{vc}	(10)	444	$294+378$
(1003)	672_{sv}	(11)	444	$8+48+168'+448$
(1030)	672_{cv}	(10)	444	$294+378$
(0013)	672_{sc}	(01)	444	$8+48+168'+448$
(0031)	672_{cs}	(01)	444	$112'+560$
(5000)	$672'_v$	(01)	660	672
(0005)	$672'_s$	(10)	660	$1+7+27+77+182+378'$
(0050)	$672'_c$	(11)	660	672
(0111)	840_v	(01)	465	$48+112+168'+512$
(1110)	840_s	(10)	465	$105+168+189+378$
(1101)	840_c	(11)	465	$48+112+168'+512$
(0022)	$840'_v$	(00)	540	$35+189+616$
(2020)	$840'_s$	(00)	540	$168+294+378$
(2002)	$840'_c$	(00)	540	$35+189+616$
(2011)	1296_v	(01)	810	$112+112'+512+560$
(1012)	1296_s	(10)	810	$21+35+105+189+330+616$
(1021)	1296_c	(11)	810	$112+112'+512+560$

Table 37
SO_8 tensor products

$8_i \times 8_i = 1_s + 28_a + (35_i)_s$ (i = v, s, or c)

$8_i \times 8_j = 8_k + 56_k$ (i, j, k cyclic)

$28 \times 8_i = 8_i + 56_i + 160_i$

$28 \times 28 = 1_s + 28_a + (35_v)_s + (35_s)_s + (35_c)_s + 300_s + 350_a$

$35_i \times 8_i = 8_i + 112_i + 160_i$

$35_i \times 8_j = 56_j + 224_{ij}$ $(i \neq j)$

$35_i \times 28 = 28 + 35_i + 350 + 567_i$

$35_i \times 35_i = 1_s + 28_a + (35_i)_s + (294_i)_s + 300_s + (567_i)_a$

$35_i \times 35_j = 35_k + 350 + 840'_k$ (i, j, k cyclic)

$56_i \times 8_i = 28 + 35_j + 35_k + 350$ $(i \neq j \neq k \neq i)$

$56_i \times 8_j = 8_k + 56_k + 160_k + 224_{jk}$

$56_i \times 28 = 8_i + 56_{i1} + 56_{i2} + 160_i + 224_{ji} + 224_{ki} + 840_i$

$56_i \times 35_i = 56_i + 160_i + 224_{ji} + 224_{ki} + 1296_i$ $(i \neq j \neq k \neq i)$

$56_i \times 35_j = 8_i + 56_i + 160_i + 224_{ji} + 672_{jk} + 840_i$ $(i \neq j \neq k \neq i)$

$56_i \times 56_i = 1_s + 28_{a1} + 28_{a2} + (35_v)_s + (35_s)_s + (35_c)_s + 300_s + 350_s + 350_a + (567_j)_a + (567_k)_a + (840'_i)_s$ $(i \neq j \neq k \neq i)$

$56_i \times 56_j = 8_k + 56_{k1} + 56_{k2} + 112_k + 160_{k1} + 160_{k2} + 224_{ik} + 224_{jk} + 840_k + 1296_k$ (i, j, k cyclic)

$(28^3)_s = 28_1 + 28_2 + 28_3 + 350 + 567_v + 567_s + 567_c + 1925$

Table 38
SO_9 irreps of dimension less than 5100

Dynkin label	Dimension (name)	$l/2$ (index)	SO_8 singlets	$SU_2 \times SU_4$ singlets
(1000)	9	1	1	0
(0001)	16	2	0	0
(0100)	36	7	0	0
(2000)	44	11	1	1
(0010)	84	21	0	1
(0002)	126	35	0	0
(1001)	128	32	0	0
(3000)	156	65	1	0
(1100)	231	77	0	0
(0101)	432	150	0	0
(4000)	450	275	1	1
(0200)	495	220	0	1
(2001)	576	232	0	0
(1010)	594	231	0	0
(0003)	672	308	0	0
(0011)	768	320	0	0
(2100)	910	455	0	0
(1002)	924	385	0	0
(5000)	1122	935	1	0
(0110)	1650	825	0	0
(3001)	1920	1120	0	0
(0020)	1980	1155	0	1
(2010)	2457	1365	0	1
(6000)	2508	2717	1	1
(1101)	2560	1280	0	0
(1200)	2574	1573	0	0
(0102)	2772	1463	0	0
(0004)	2772′	1848	0	0
(3100)	2772″	1925	0	0
(2002)	3900	2275	0	0
(0300)	4004	3003	0	0
(0012)	4158	2541	0	0
(1003)	4608	2816	0	0
(0201)	4928	3080	0	0
(1011)	5040	2870	0	0

Table 39
SO_9 tensor products

$$9\times 9 = 1_s + 36_a + 44_s$$
$$16\times 9 = 16 + 128$$
$$16\times 16 = 1_s + 9_s + 36_a + 84_a + 126_s$$
$$36\times 9 = 9 + 84 + 231$$
$$36\times 16 = 16 + 128 + 432$$
$$36\times 36 = 1_s + 36_a + 44_s + 126_s + 495_s + 594_a$$
$$44\times 9 = 9 + 156 + 231$$
$$44\times 16 = 128 + 576$$
$$44\times 36 = 36 + 44 + 594 + 910$$
$$44\times 44 = 1_s + 36_a + 44_s + 450_s + 495_s + 910_a$$
$$84\times 9 = 36 + 126 + 594$$
$$84\times 16 = 16 + 128 + 432 + 768$$
$$84\times 36 = 9 + 84 + 126 + 231 + 924 + 1650$$
$$84\times 44 = 84 + 231 + 924 + 2457$$
$$84\times 84 = 1_s + 36_a + 44_s + 84_a + 126_s + 495_s + 594_a + 924_s + 1980_s + 2772_a$$
$$126\times 9 = 84 + 126 + 924$$
$$126\times 16 = 16 + 128 + 432 + 672 + 768$$
$$126\times 36 = 36 + 84 + 126 + 594 + 924 + 2772$$
$$126\times 44 = 126 + 594 + 924 + 3900$$
$$126\times 84 = 9 + 36 + 84 + 126 + 231 + 594 + 924 + 1650 + 2772 + 4158$$
$$126\times 126 = 1_s + 9_s + 36_a + 44_s + 84_a + 126_s + 231_a + 495_s + 594_a + 924_s + 1650_s + 1980_s + 2772_a + 2772'_s + 4158_a$$
$$128\times 9 = 16 + 128 + 432 + 576$$
$$128\times 16 = 9 + 36 + 44 + 84 + 126 + 231 + 594 + 924$$
$$128\times 36 = 16 + 128_1 + 128_2 + 432 + 576 + 768 + 2560$$
$$128\times 44 = 16 + 128 + 432 + 576 + 1920 + 2560$$
$$128\times 84 = 16 + 128_1 + 128_2 + 432_1 + 432_2 + 576 + 672 + 768 + 2560 + 5040$$
$$128\times 126 = 16 + 128_1 + 128_2 + 432_1 + 432_2 + 576 + 672 + 768_1 + 768_2 + 2560 + 4608 + 5040$$
$$128\times 128 = 1_s + 9_s + 36_{a1} + 36_{a2} + 44_s + 84_{a1} + 84_{a2} + 126_{s1} + 126_{s2} + 156_s + 231_s + 231_a + 495_a + 594_s + 594_a + 910_a + 924_s + 924_a + 1650_s + 2457_a + 2772_a + 3900_s$$

Table 40
Branchings of SO_9 representations

$SO_9 \supset SO_8$

$$9 = 1 + 8_v$$
$$16 = 8_c + 8_s$$
$$36 = 8_v + 28$$
$$44 = 1 + 8_v + 35_v$$
$$84 = 28 + 56_v$$
$$126 = 35_c + 35_s + 56_v$$
$$128 = 8_c + 8_s + 56_c + 56_s$$
$$156 = 1 + 8_v + 35_v + 112_v$$
$$231 = 8_v + 28 + 35_v + 160_v$$
$$432 = 56_c + 56_s + 160_c + 160_s$$
$$450 = 1 + 8_v + 35_v + 112_v + 294_v$$
$$495 = 35_v + 160_v + 300$$
$$576 = 8_c + 8_s + 56_c + 56_s + 224_{vc} + 224_{vs}$$
$$594 = 28 + 56_v + 160_v + 350$$

$SO_9 \supset SU_2 \times SU_4$

$$9 = (3,1) + (1,6)$$
$$16 = (2,4) + (2,\bar{4})$$
$$36 = (3,1) + (1,15) + (3,6)$$
$$44 = (1,1) + (5,1) + (3,6) + (1,20')$$
$$84 = (1,1) + (1,10) + (1,\overline{10}) + (3,6) + (3,15)$$
$$126 = (1,6) + (1,15) + (3,10) + (3,\overline{10}) + (3,15)$$
$$128 = (2,4) + (2,\bar{4}) + (4,4) + (4,\bar{4}) + (2,20) + (2,\overline{20})$$
$$156 = (3,1) + (1,6) + (7,1) + (5,6) + (1,50) + (3,20')$$
$$231 = (3,1) + (5,1) + (1,6) + (3,6) + (5,6) + (3,15) + (3,20') + (1,64)$$
$$432 = (2,4) + (2,\bar{4}) + (4,4) + (4,\bar{4}) + (2,20) + (2,\overline{20}) + (4,20) + (4,\overline{20}) + (2,36) + (2,\overline{36})$$
$$450 = (1,1) + (5,1) + (9,1) + (3,6) + (7,6) + (1,20') + (5,20') + (3,50) + (1,105)$$
$$495 = (1,1) + (5,1) + (3,6) + (5,6) + (1,20') + (5,20') + (3,15) + (3,64) + (1,84)$$
$$576 = (2,4) + (2,\bar{4}) + (4,4) + (4,\bar{4}) + (6,4) + (6,\bar{4}) + (2,20) + (2,\overline{20}) + (4,20) + (4,\overline{20}) + (2,60) + (2,\overline{60})$$
$$594 = (3,1) + (1,6) + (3,6) + (5,6) + (1,15) + (3,15) + (5,15) + (3,10) + (3,\overline{10}) + (3,20') + (1,45) + (1,\overline{45}) + (3,64)$$

Table 41
SO_{10} irreps of dimension less tan 12000

Dynkin label	Dimension (name)	Congruency class	$l/2$ (index)	SU_5 singlets	$SU_2 \times SU_2 \times SU_4$ singlets	SO_9 singlets	$SU_2 \times SO_7$ singlets
(10000)	10	2	1	0	0	1	0
(00001)	16	1	2	1	0	0	0
(01000)	45	0	8	1*	0	0	0
(20000)	54	0	12	0	1	1	1
(00100)	120	2	28	0	0	0	1
(00002)	126	2	35	1	0	0	0
(10010)	144	1	34	0	0	0	0
(00011)	210	0	56	1	1	0	0
(30000)	210′	2	77	0	0	1	0
(11000)	320	2	96	0	0	0	0
(01001)	560	1	182	1	0	0	0
(40000)	660	0	352	0	1	1	1
(00030)	672	1	308	1	0	0	0
(20001)	720	1	266	0	0	0	0
(02000)	770	0	308	1*	1	0	1
(10100)	945	0	336	0	0	0	0
(10002)	1050	0	420	0	0	0	0
(00110)	1200	1	470	0	0	0	0
(21000)	1386	0	616	0	0	0	0
(00012)	1440	1	628	1	0	0	0
(10011)	1728	2	672	0	0	0	0
(50000)	1782	2	1287	0	0	1	0
(30010)	2640	1	1386	0	0	0	0
(00004)	2772	0	1848	1	0	0	0
(01100)	2970	2	1353	0	0	0	0
(01002)	3696	2	1848	1	0	0	0
(11010)	3696′	1	1694	0	0	0	0
(00200)	4125	0	2200	0	1	0	1
(60000)	4290	0	4004	0	1	1	1
(20100)	4312	2	2156	0	0	0	1
(12000)	4410	2	2401	0	0	0	0
(31000)	4608	2	2816	0	0	0	0
(20002)	4950	2	2695	0	0	0	0
(10003)	5280	1	3124				
(01011)	5940	0	2904				
(00102)	6930	0	4004				
(00013)	6930′	2	4389				
(03000)	7644	0	5096				
(40001)	7920	1	5566				
(02001)	8064	1	4592				
(20011)	8085	0	4312				
(10101)	8800	1	4620				
(00022)	8910	0	5544				
(70000)	9438	2	11011				
(00005)	9504	1	8580				
(00111)	10560	2	5984				
(10021)	11088	1	6314				

Table 42
SO_{10} tensor products

$10 \times 10 = 1_s + 45_a + 54_s$

$\overline{16} \times 10 = 16 + 144$

$16 \times 16 = 10_s + 120_a + 126_s$

$\overline{16} \times 16 = 1 + 45 + 210$

$45 \times 10 = 10 + 120 + 320$

$45 \times 16 = 16 + 144 + 560$

$45 \times 45 = 1_s + 45_a + 54_s + 210_s + 770_s + 945_a$

$54 \times 10 = 10 + 210' + 320$

$54 \times 16 = 144 + 720$

$54 \times 45 = 45 + 54 + 945 + 1386$

$54 \times 54 = 1_s + 45_a + 54_s + 660_s + 770_s + 1386_a$

$120 \times 10 = 45 + 210 + 945$

$120 \times \overline{16} = 16 + 144 + 560 + 1200$

$120 \times 45 = 10 + 120 + 126 + \overline{126} + 320 + 1728 + 2970$

$120 \times 54 = 120 + 320 + 1728 + 4312$

$120 \times 120 = 1_s + 45_a + 54_s + 210_s + 210_a + 770_s + 945_a + 1050_s + \overline{1050}_s + 4125_s + 5940_a$

$126 \times 10 = 210 + 1050$

$\overline{126} \times \overline{16} = 144 + 672 + 1200$

$126 \times \overline{16} = 16 + 560 + 1440$

$126 \times 45 = 120 + 126 + 1728 + 3696$

$126 \times 54 = \overline{126} + 1728 + 4950$

$126 \times 120 = 45 + 210 + 945 + 1050 + 5940 + 6930$

$126 \times 126 = 54_s + 945_a + 1050_s + 2772_s + 4125_s + 6930_a$

$\overline{126} \times 126 = 1 + 45 + 210 + 770 + 5940 + 8910$

$\overline{144} \times 10 = 16 + 144 + 560 + 720$

$\overline{144} \times 16 = 45 + 54 + 210 + 945 + 1050$

$\overline{144} \times \overline{16} = 10 + 120 + 126 + 320 + 1728$

$144 \times 45 = 16 + 144_1 + 144_2 + 560 + 720 + 1200 + 3696'$

$144 \times 54 = 16 + 144 + 560 + 720 + 2640 + 3696'$

$\overline{144} \times 120 = 16 + 144_1 + 144_2 + 560_1 + 560_2 + 720 + 1200 + 1440 + 3696' + 8800$

$\overline{144} \times 126 = 144 + 560 + 720 + 1200 + 1440 + 5280 + 8800$

$\overline{144} \times \overline{126} = 16 + 144 + 560 + 1200 + 1440 + 3696' + 11088$

$144 \times 144 = 10_s + 120_{a1} + 120_{a2} + 126_s + \overline{126}_s + 210'_s + 320_s + 320_a + 1728_s + 1728_a + 2970_s + 3696_a + 4312_a + 4950_s$

$\overline{144} \times 144 = 1 + 45_1 + 45_2 + 54 + 210_1 + 210_2 + 770 + 945_1 + 945_2 + 1050 + \overline{1050} + 1386 + 5940 + 8085$

$210 \times 10 = 120 + 126 + \overline{126} + 1728$

$210 \times 16 = 16 + 144 + 560 + 1200 + 1440$

$210 \times 45 = 45 + 210_1 + 210_2 + 945 + 1050 + \overline{1050} + 5940$

$210 \times 54 = 210 + 945 + \overline{1050} + 1050 + 8085$

$210 \times 120 = 10 + 120_1 + 120_2 + 126 + \overline{126} + 320 + 1728_1 + 1728_2 + 2970 + 3696 + \overline{3696} + 10560$

$210 \times 126 = 10 + 120 + 126 + 320 + 1728 + 2970 + 3696 + 6930' + 10560$

$210 \times 144 = 16 + 144_1 + 144_2 + 560_1 + 560_2 + 672 + 720 + 1200_1 + 1200_2 + 1440 + 3696' + 8800 + 11088$

$210 \times 210 = 1_s + 45_s + 45_a + 54_s + 210_s + 210_a + 770_s + 945_{a1} + 945_{a2} + \overline{1050}_s + 1050_s + 4125_s + 5940_s + 5940_a + 6930_a + \overline{6930}_a + 8910_s$

Table 43
Branching rules for SO_{10}

$SO_{10} \supset SU_5 \times U_1$
$(10000) = 10 = 5(2) + \bar{5}(-2)$
$(00001) = 16 = 1(-5) + \bar{5}(3) + 10(-1)$
$(01000) = 45 = 1(0) + 10(4) + \overline{10}(-4) + 24(0)$
$(20000) = 54 = 15(4) + \overline{15}(-4) + 24(0)$
$(00100) = 120 = 5(2) + \bar{5}(-2) + 10(-6) + \overline{10}(6) + 45(2) + \overline{45}(-2)$
$(00002) = 126 = 1(-10) + \bar{5}(-2) + 10(-6) + \overline{15}(6) + 45(2) + \overline{50}(-2)$
$(10010) = 144 = \bar{5}(3) + 5(7) + 10(-1) + 15(-1) + 24(-5) + 40(-1) + \overline{45}(3)$
$(00011) = 210 = 1(0) + 5(-8) + \bar{5}(8) + 10(4) + \overline{10}(-4) + 24(0) + 40(-4) + \overline{40}(4) + 75(0)$
$(30000) = 210' = 35(-6) + \overline{35}(6) + 70(2) + \overline{70}(-2)$
$(11000) = 320 = 5(2) + \bar{5}(-2) + 40(-6) + \overline{40}(6) + 45(2) + \overline{45}(-2) + 70(2) + \overline{70}(-2)$
$(01001) = 560 = 1(-5) + \bar{5}(3) + \overline{10}(-9) + 10(-1)_1 + 10(-1)_2 + 24(-5) + 40(-1) + 45(7) + \overline{45}(3) + \overline{50}(3) + \overline{70}(3) + 75(-5) + 175(-1)$

$SO_{10} \supset SU_2 \times SU_2 \times SU_4$
$10 = (2, 2, 1) + (1, 1, 6)$
$16 = (2, 1, 4) + (1, 2, \bar{4})$
$45 = (3, 1, 1) + (1, 3, 1) + (1, 1, 15) + (2, 2, 6)$
$54 = (1, 1, 1) + (3, 3, 1) + (1, 1, 20') + (2, 2, 6)$
$120 = (2, 2, 1) + (1, 1, 10) + (1, 1, \overline{10}) + (3, 1, 6) + (1, 3, 6) + (2, 2, 15)$
$126 = (1, 1, 6) + (3, 1, \overline{10}) + (1, 3, 10) + (2, 2, 15)$
$144 = (2, 1, 4) + (1, 2, \bar{4}) + (3, 2, \bar{4}) + (2, 3, 4) + (2, 1, 20) + (1, 2, \overline{20})$
$210 = (1, 1, 1) + (1, 1, 15) + (2, 2, 6) + (3, 1, 15) + (1, 3, 15) + (2, 2, 10) + (2, 2, \overline{10})$
$210' = (2, 2, 1) + (1, 1, 6) + (4, 4, 1) + (3, 3, 6) + (2, 2, 20') + (1, 1, 50)$
$320 = (2, 2, 1) + (1, 1, 6) + (4, 2, 1) + (2, 4, 1) + (3, 1, 6) + (1, 3, 6) + (2, 2, 15) + (3, 3, 6) + (1, 1, 64) + (2, 2, 20')$
$560 = (2, 1, 4) + (1, 2, \bar{4}) + (4, 1, 4) + (1, 4, \bar{4}) + (2, 3, 4) + (3, 2, \bar{4}) + (2, 1, 20) + (1, 2, \overline{20}) + (2, 1, 36) + (1, 2, \overline{36}) + (2, 3, 20) + (3, 2, \overline{20})$

$SO_{10} \supset SO_9$
$10 = 1 + 9$
$16 = 16$
$45 = 9 + 36$
$54 = 1 + 9 + 44$
$120 = 36 + 84$
$126 = 126$
$144 = 16 + 128$
$210 = 84 + 126$
$210' = 1 + 9 + 44 + 156$
$320 = 9 + 36 + 44 + 231$
$560 = 128 + 432$

$SO_{10} \supset SU_2 \times SO_7$
$10 = (3, 1) + (1, 7)$
$16 = (2, 8)$
$45 = (3, 1) + (1, 21) + (3, 7)$
$54 = (1, 1) + (5, 1) + (3, 7) + (1, 27)$
$120 = (1, 1) + (3, 7) + (1, 35) + (3, 21)$
$126 = (1, 21) + (3, 35)$
$144 = (2, 8) + (4, 8) + (2, 48)$
$210 = (1, 7) + (1, 35) + (3, 21) + (3, 35)$
$210' = (3, 1) + (7, 1) + (1, 7) + (5, 7) + (3, 27) + (1, 77)$
$320 = (3, 1) + (5, 1) + (1, 7) + (3, 7) + (5, 7) + (3, 21) + (3, 27) + (1, 105)$
$560 = (2, 8) + (4, 8) + (2, 48) + (4, 48) + (2, 112)$

Table 44
F_4 irreps of dimension less than 100000

Dynkin label	Dimension (name)	$l/6$ (index)	SO_9 singlets	$SU_3 \times SU_3$ singlets
(0001)	26	1	1	0
(1000)	52	3	0	0
(0010)	273	21	0	1
(0002)	324	27	1	1
(1001)	1053	108	0	0
(2000)	1053′	135	0	1
(0100)	1274	147	0	1
(0003)	2652	357	1	1
(0011)	4096	512	0	0
(1010)	8424	1242	0	1
(1002)	10829	1666	0	1
(3000)	12376	2618		
(0004)	16302	3135	1	1
(2001)	17901	3213		
(0101)	19278	3213	0	1
(0020)	19448	3366	0	2
(1100)	29172	5610		
(0012)	34749	6237	0	1
(1003)	76076	16093		
(0005)	81081	20790		

Table 45
F_4 tensor products

$26\times 26 = 1_s + 26_s + 52_a + 273_a + 324_s$
$52\times 26 = 26 + 273 + 1053$
$52\times 52 = 1_s + 52_a + 324_s + 1053'_s + 1274_a$
$273\times 26 = 26 + 52 + 273 + 324 + 1053 + 1274 + 4096$
$273\times 52 = 26 + 273 + 324 + 1053 + 4096 + 8424$
$273\times 273 = 1_s + 26_s + 52_a + 273_{a1} + 273_{a2} + 324_{s1} + 324_{s2} + 1053_s + 1053_a + 1053'_s + 1274_a + 2652_s + 4096_s + 4096_a + 8424_s + 10829_a + 19278_a + 19448_s$
$324\times 26 = 26 + 273 + 324 + 1053 + 2652 + 4096$
$324\times 52 = 52 + 273 + 324 + 1274 + 4096 + 10829$
$324\times 273 = 26 + 52 + 273_1 + 273_2 + 324 + 1053_1 + 1053_2 + 1274 + 2652 + 4096_1 + 4096_2 + 8424 + 10829 + 19278 + 34749$
$324\times 324 = 1_s + 26_s + 52_a + 273_a + 324_{s1} + 324_{s2} + 1053_a + 1053'_s + 1274_a + 2652_s + 4096_s + 4096_a + 8424_s + 10829_a + 16302_s + 19448_s + 34749_a$

Table 46
Branchings of F_4 representations

$F_4 \supset SO_9$
$(0001) = 26 = 1 + 9 + 16$
$(1000) = 52 = 16 + 36$
$(0010) = 273 = 9 + 16 + 36 + 84 + 128$
$(0002) = 324 = 1 + 9 + 16 + 44 + 126 + 128$
$(1001) = 1053 = 16 + 36 + 84 + 126 + 128 + 231 + 432$
$(2000) = 1053' = 126 + 432 + 495$
$(0100) = 1274 = 36 + 84 + 128 + 432 + 594$

$F_4 \supset SU_3 \times SU_3$
$26 = (8,1) + (3,3) + (\bar{3},\bar{3})$
$52 = (8,1) + (1,8) + (6,\bar{3}) + (\bar{6},3)$
$273 = (1,1) + (8,1) + (3,3) + (\bar{3},\bar{3}) + (10,1) + (\overline{10},1) + (6,\bar{3}) + (\bar{6},3) + (3,\bar{6}) + (\bar{3},6) + (15,3) + (\overline{15},\bar{3}) + (8,8)$
$324 = (1,1) + (8,1) + (1,8) + (3,3) + (\bar{3},\bar{3}) + (6,\bar{3}) + (\bar{6},3) + (27,1) + (6,6) + (\bar{6},\bar{6}) + (15,3) + (\overline{15},\bar{3}) + (8,8)$

Table 47
E_6 irreps of dimension less than 100000

Dynkin label	Dimension (name)	$l/6$ (index)	Triality	F_4 singlets	SO_{10} singlets	$SU_2 \times SU_6$ singlets	$SU_3 \times SU_3 \times SU_3$ singlets
(100000)	27	1	1	1	1	0	0
(000001)	78	4	0	0	1*	0	0
(000100)	351	25	1	0	0	0	0
(000020)	351′	28	1	1	1	0	0
(100010)	650	50	0	1	1*	1	2
(100001)	1728	160	1	0	1	0	0
(000002)	2430	270	0	0	1*	1	1
(001000)	2925	300	0	0	0	0	1
(300000)	3003	385	0	1	1	0	1
(000110)	5824	672	0	0	0	0	0
(010010)	7371	840	1	0	0	0	0
(200010)	7722	946	1	1	1	0	0
(000101)	17550	2300	1	0	0	0	0
(000021)	19305	2695	1	0	1	0	0
(400000)	19305′	3520	1	1	1	0	0
(020000)	34398	5390	1	0	0	0	0
(100011)	34749	4752	0	0	1*	0	
(000003)	43758	7854	0				
(100002)	46332	7260	1				
(101000)	51975	7700	1	0	0	0	0
(210000)	54054	8932	1	0	0	0	0
(100030)	61425	10675	1				
(010100)	70070	10780	0	0	0	1	
(010020)	78975	12825	0	0	0	0	0
(200020)	85293	14580	0	1	1*	1	2
(100110)	112320	18080	1				

*$SO_{10} \times U_1$ singlet.

Table 48

E_6 tensor products

$$\overline{27} \times \overline{27} = 27_s + 351_a + 351'_s$$
$$\overline{27} \times 27 = 1 + 78 + 650$$
$$78 \times 27 = 27 + 351 + 1728$$
$$78 \times 78 = 1_s + 78_a + 650_s + 2430_s + 2925_a$$
$$351 \times \overline{27} = 78 + 650 + 2925 + 5824$$
$$\overline{351} \times \overline{27} = 27 + 351 + 1728 + 7371$$
$$351 \times 78 = 27 + 351 + 351' + 1728 + 7371 + 17550$$
$$\overline{351} \times \overline{351} = 27_s + 351_a + 351'_s + 1728_s + 1728_a + 7371_a + 7722_s + 17550_s + 34398_s + 51975_a$$
$$351 \times \overline{351} = 1 + 78 + 650_1 + 650_2 + 2430 + 2925 + 5824 + \overline{5824} + 34749 + 70070$$
$$\overline{351'} + 27 = 650 + 3003 + \overline{5824}$$
$$\overline{351'} + \overline{27} = 27 + 1728 + 7722$$
$$351' \times 78 = 351 + 351' + 7371 + 19305$$
$$\overline{351'} \times \overline{351} = 351 + 1728 + 7371 + 7722 + 51975 + 54054$$
$$351' \times \overline{351} = 78 + 650 + 2925 + 5824 + 34749 + 78975$$
$$\overline{351'} \times \overline{351'} = 351'_s + 7371_a + 7722_s + 19305'_s + 34398_s + 54054_a$$
$$351' \times \overline{351'} = 1 + 78 + 650 + 2430 + 34749 + 85293$$
$$650 \times 27 = 27 + 351 + 351' + 1728 + 7371 + 7722$$
$$650 \times 78 = 78 + 650_1 + 650_2 + 2925 + 5824 + \overline{5824} + 34749$$
$$650 \times 351 = 27 + 351_1 + 351_2 + 351' + 1728_1 + 1728_2 + 7371_1 + 7371_2 + 7722 + 17550 + 19305 + 51975 + 112320$$
$$650 \times 351' = 27 + 351 + 351' + 1728 + 7371 + 7722 + 17550 + 19305 + 61425 + 112320$$
$$650 \times 650 = 1_s + 78_s + 78_a + 650_{s1} + 650_{s2} + 650_a + 2430_s + 2925_{a1} + 2925_{a2} + 3003_s + \overline{3003}_s + 5824_s + \overline{5824}_s + 5824_a + \overline{5824}_a + 34749_s + 34749_a + 70070_s + 78975_a + \overline{78975}_a + 85293_s$$
$$\overline{1728} \times \overline{27} = 351 + 351' + 1728 + 7371 + 17550 + 19305$$
$$\overline{1728} \times 27 = 78 + 650 + 2430 + 2925 + 5824 + 34749$$
$$1728 \times 78 = 27 + 351 + 1728_1 + 1728_2 + 7371 + 7722 + 17550 + 46332 + 51975$$
$$\overline{1728} \times \overline{351} = 27 + 351' + 351_1 + 351_2 + 1728_1 + 1728_2 + 7371_1 + 7371_2 + 7722 + 17550_1 + 17550_2 + 19305 + 46332 + 51975 + 112320 + 314496$$
$$\overline{1728} \times \overline{1728}_a = 351_1 + 351_2 + 1728 + 7371_1 + 7371_2 + 17550 + 19305_1 + 19305_2 + 51975_1 + 51975_2 + 112320 + 314496 + 393822 + 494208$$
$$\overline{1728} \times \overline{1728}_s = 27 + 351'_1 + 351'_2 + 1728 + 7371_1 + 7371_2 + 7722 + 17550_1 + 17550_2 + 19305 + 34398 + 46332 + 61425 + 112320 + 314496 + 386100 + 459459$$

Table 49
Branchings of E_6 representations

$E_6 \supset F_4$

$(100000) = 27 = 1 + 26$

$(000001) = 78 = 26 + 52$

$(000100) = 351 = 26 + 52 + 273$

$(000020) = 351' = 1 + 26 + 324$

$(100010) = 650' = 1 + 26_1 + 26_2 + 273 + 324$

$(100001) = 1728 = 26 + 52 + 273 + 324 + 1053$

$(000002) = 2430 = 324 + 1053 + 1053'$

$(001000) = 2925 = 52 + 273_1 + 273_2 + 1053 + 1274$

$E_6 \supset SO_{10} \times U_1$ (Value of U_1 generator in parenthesis)

$27 = 1(4) + 10(-2) + 16(1)$

$78 = 1(0) + 45(0) + 16(-3) + \overline{16}(3)$

$351 = 10(-2) + \overline{16}(-5) + 16(1) + 45(4) + 120(-2) + 144(1)$

$351' = 1(-8) + 10(-2) + \overline{16}(-5) + 54(4) + \overline{126}(-2) + 144(1)$

$650 = 1(0) + 10(6) + 10(-6) + 16(-3) + \overline{16}(3) + 45(0) + 54(0) + 144(-3) + \overline{144}(3) + 210(0)$

$1728 = 1(4) + 10(-2) + 16_1(1) + 16_2(1) + \overline{16}(7) + 45(4) + 120(-2) + 126(-2) + 144(1) + \overline{144}(-5) + 210(4) + 320(-2) + 560(1)$

$2430 = 1(0) + 16(-3) + \overline{16}(3) + 45(0) + 126(-6) + \overline{126}(6) + 210(0) + 560(-3) + \overline{560}(3) + 770(0)$

$2925 = 16(-3) + \overline{16}(3) + 45_1(0) + 45_2(0) + 120_1(6) + 120_2(-6) + 144(-3) + \overline{144}(3) + 210(0) + 560(-3) + \overline{560}(3) + 945(0)$

$E_6 \supset SU_2 \times SU_6$

$27 = (2, \bar{6}) + (1, 15)$

$78 = (3, 1) + (1, 35) + (2, 20)$

$351 = (2, \bar{6}) + (1, 21) + (3, 15) + (1, 105) + (2, \overline{84})$

$351' = (1, 15) + (3, 21) + (2, \overline{84}) + (1, 105')$

$650 = (1, 1) + (1, 35) + (2, 20) + (3, 35) + (2, 70) + (2, \overline{70}) + (1, 189)$

$1728 = (2, \bar{6}) + (1, 15) + (4, \bar{6}) + (3, 15) + (1, 105) + (2, \overline{84}) + (2, \overline{120}) + (3, 105) + (1, 384) + (2, \overline{210})$

$2430 = (1, 1) + (5, 1) + (2, 20) + (4, 20) + (3, 35) + (1, 189) + (1, 405) + (3, 175) + (2, 540)$

$2925 = (3, 1) + (1, 35) + (2, 20) + (3, 35) + (4, 20) + (2, 70) + (2, \overline{70}) + (1, 175) + (1, 280) + (1, \overline{280}) + (3, 189) + (2, 540)$

$E_6 \supset SU_3^w \times SU_3 \times SU_3^c$

$27 = (\bar{3}, 3, 1) + (3, 1, 3) + (1, \bar{3}, \bar{3})$

$78 = (8, 1, 1) + (1, 8, 1) + (1, 1, 8) + (3, 3, \bar{3}) + (\bar{3}, \bar{3}, 3)$

$351 = (\bar{3}, 3, 1) + (\bar{3}, \bar{6}, 1) + (6, 3, 1) + (3, 1, 3) + (\bar{6}, 1, 3) + (3, 8, 3) + (1, \bar{3}, \bar{3}) + (1, 6, \bar{3}) + (8, \bar{3}, \bar{3}) + (3, 1, \bar{6}) + (1, \bar{3}, 6) + (\bar{3}, 3, 8)$

$351' = (\bar{3}, 3, 1) + (6, \bar{6}, 1) + (3, 1, 3) + (3, 8, 3) + (1, \bar{3}, \bar{3}) + (8, \bar{3}, \bar{3}) + (\bar{6}, 1, \bar{6}) + (1, 6, 6) + (\bar{3}, 3, 8)$

$650 = (1, 1, 1) + (1, 1, 1) + (8, 1, 1) + (1, 8, 1) + (1, 1, 8) + (3, 3, \bar{3}) + (3, 3, \bar{3}) + (\bar{3}, \bar{3}, 3) + (\bar{3}, \bar{3}, 3) + (6, \bar{3}, 3) + (\bar{3}, 6, 3) + (\bar{6}, 3, \bar{3}) + (3, \bar{6}, \bar{3})$
$+ (\bar{3}, \bar{3}, \bar{6}) + (3, 3, 6) + (8, 8, 1) + (8, 1, 8) + (1, 8, 8)$

$1728 = (\bar{3}, 3, 1) + (\bar{3}, 3, 1) + (\bar{3}, \bar{6}, 1) + (6, 3, 1) + (\bar{3}, 15, 1) + (\overline{15}, 3, 1) + (3, 1, 3) + (3, 1, 3) + (\bar{6}, 1, 3) + (3, 8, 3) + (3, 8, 3) + (15, 1, 3) + (\bar{6}, 8, 3)$
$+ (1, \bar{3}, \bar{3}) + (1, \bar{3}, \bar{3}) + (1, 6, \bar{3}) + (8, \bar{3}, \bar{3}) + (8, \bar{3}, \bar{3}) + (1, \overline{15}, \bar{3}) + (8, 6, \bar{3}) + (3, 1, \bar{6}) + (3, 8, \bar{6}) + (1, \bar{3}, 6) + (8, \bar{3}, 6) + (\bar{3}, 3, 8) + (\bar{3}, 3, 8) + (\bar{3}, \bar{6}, 8)$
$+ (6, 3, 8) + (3, 1, 15) + (1, \bar{3}, \overline{15})$

Table 50

Guide to projection matrices for $E_6 \supset \cdots \supset U_1^{em} \times SU_3^c$. The U_1 factors may be found in table 18. The factor X in $P(X \supset Y)$ is chosen to be simple; it is underlined when ambiguity is possible

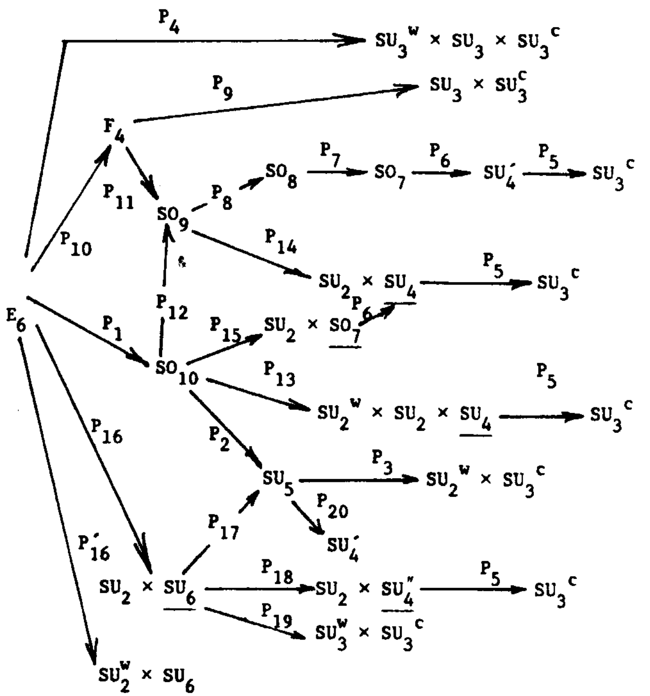


Table 51

Matrix elements of the simple root lowering operators for the **3**, **6**, **8** and **10** of SU_3. Internal lines in the octet each have value $1/\sqrt{2}$

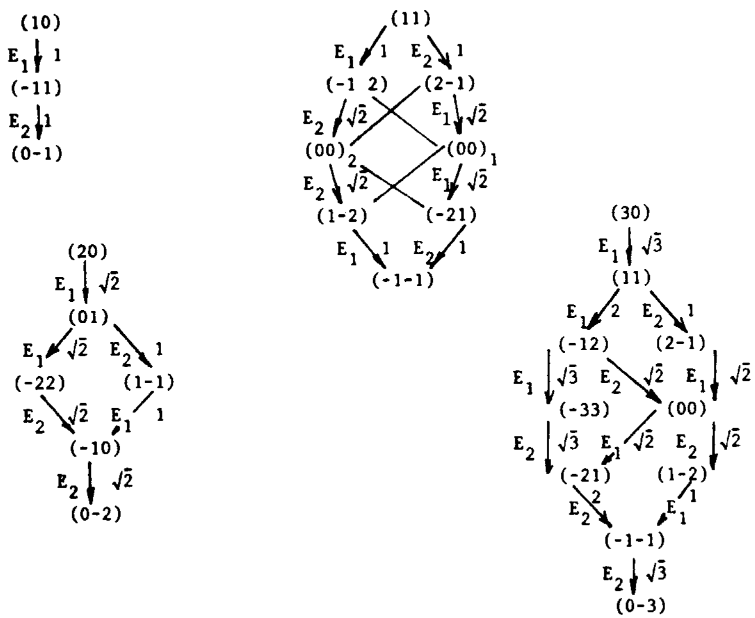


Table 52
Irreps, products and branching rules for E_7

Dynkin label	Dimension (name)	$l/12$ index	Branching into E_6 irreps U_1 factors suppressed
(0000010)	56	1	$1+1+27+\overline{27}$
(1000000)	133	3	$1+27+\overline{27}+78$
(0000001)	912	30	$27+\overline{27}+78+78+351+\overline{351}$
(0000020)	1463	55	$1+1+1+27+27+\overline{27}+\overline{27}+351'+\overline{351}'+650$
(0000100)	1539	54	$1+27+27+\overline{27}+\overline{27}+78+351+\overline{351}+650$
(1000010)	6480	270	Branching rules to other
(2000000)	7371	351	regular subgroups below.
(0100000)	8645	390	
(0000030)	24320	1440	
(0001000)	27664	1430	
(0000011)	40755	2145	
(0000110)	51072	2832	
(1000001)	86184	4995	

$56\times 56 = 1_a+133_s+1463_s+1539_a$
$133\times 56 = 56+912+6480$
$133\times 133 = 1_s+133_a+1539_s+7371_s+8645_a$
$912\times 56 = 133+1539+8645+40755$
$912\times 133 = 56+912+6480+27664+86184$
$912\times 912 = 1_a+133_s+1463_s+1539_a+7371_a+8645_s+40755_a+152152_s+253935_s+365750_a$
$1463\times 56 = 56+6480+24320+51072$
$1463\times 133 = 1463+1539+40755+150822$
$1463\times 912 = 912+6480+27664+51072+362880+885248$
$1463\times 1463 = 1_s+133_a+1463_a+1539_s+7371_s+150822_s+152152_a+293930_s+617253_s+915705_a$
$1539\times 56 = 56+912+6480+27664+51072$
$1539\times 133 = 133+1463+1539+8645+40755+152152$
$1539\times 912 = 56+912+6480_1+6480_2+27664+51072+86184+362880+861840$

$E_7 \supset SU_8$
$(0000010) = 56 = 28+\overline{28}$
$(1000000) = 133 = 63+70$
$(0000001) = 912 = 36+\overline{36}+420+\overline{420}$
$(0000020) = 1463 = 1+70+336+\overline{336}+720$
$(0000100) = 1539 = 63+378+\overline{378}+720$

$E_7 \supset SU_2\times SO_{12}$
$56 = (2,12)+(1,32)$
$133 = (3,1)+(2,32')+(1,66)$
$912 = (2,12)+(3,32)+(1,352)+(2,220)$
$1463 = (1,66)+(3,77)+(1,462)+(2,352')$
$1539 = (1,1)+(2,32')+(1,77)+(3,66)+(1,495)+(2,352')$

$E_7 \supset SU_3\times SU_6$
$56 = (3,6)+(\overline{3},\overline{6})+(1,20)$
$133 = (8,1)+(1,35)+(3,\overline{15})+(\overline{3},15)$
$912 = (3,6)+(\overline{3},\overline{6})+(6,\overline{6})+(\overline{6},6)+(1,70)+(1,\overline{70})+(8,20)+(3,84)+(\overline{3},\overline{84})$
$1463 = (1,1)+(1,35)+(3,\overline{15})+(\overline{3},15)+(6,21)+(\overline{6},\overline{21})+(1,175)+(8,35)+(3,\overline{105})+(\overline{3},105)$
$1539 = (1,1)+(8,1)+(1,35)+(3,\overline{15})+(\overline{3},15)+(3,\overline{21})+(\overline{3},21)+(6,15)+(\overline{6},\overline{15})+(1,189)+(3,\overline{105})+(\overline{3},105)+(8,35)$

Table 53
Irreps, products and branching rules for E_8

Dynkin label	Dimension (name)	$l/60$ (index)
(00000010)	248	1
(10000000)	3875	25
(00000020)	27000	225
(00000100)	30380	245
(00000001)	147250	1425
(10000010)	779247	8379
(00000030)	1763125	22750
(00100000)	2450240	29640
(00000110)	4096000	51200
(20000000)	4881384	65610
(01000000)	6696000	88200
(00000011)	26411008	372736

$248 \times 248 = 1_s + 248_a + 3875_s + 27000_s + 30380_a$
$3875 \times 248 = 248 + 3875 + 30380 + 147250 + 779247$
$3875 \times 3875 = 1_s + 248_a + 3875_s + 27000_s + 30380_a + 147250_s + 779247_a + 2450240_s + 4881384_s + 6696000_a$

Branching rules to regular maximal subgroups; SO_{16} and SU_9 Dynkin labels given.

$E_8 \supset SO_{16}$
$248 = (01000000)120 + (00000001)128$
$3875 = (20000000)135 + (00010000)1820 + (10000010)1920$

$E_8 \supset SU_9$
$248 = (10000001)80 + (00100000)84 + (00000100)\overline{84}$
$3875 = (10000001)80 + (11000000)240 + (00000011)\overline{240} + (00010001)1050 + (10001000)\overline{1050} + (01000010)1215$

$E_8 \supset SU_2 \times E_7$
$248 = (3, 1) + (1, 133) + (2, 56)$
$3875 = (1, 1) + (2, 56) + (3, 133) + (1, 1539) + (2, 912)$

$E_8 \supset SU_3 \times E_6$
$248 = (8, 1) + (1, 78) + (3, 27) + (\bar{3}, \overline{27})$
$3875 = (1, 1) + (8, 1) + (3, 27) + (\bar{3}, \overline{27}) + (\bar{6}, 27) + (6, \overline{27}) + (8, 78) + (1, 650) + (3, 351) + (\bar{3}, \overline{351})$

$E_8 \supset SU_5 \times SU_5$
$248 = (1, 24) + (24, 1) + (5, \overline{10}) + (\bar{5}, 10) + (10, 5) + (\overline{10}, \bar{5})$
$3875 = (1, 1) + (1, 24) + (24, 1) + (5, \overline{10}) + (\bar{5}, 10) + (10, 5) + (\overline{10}, \bar{5}) + (1, 75) + (75, 1) + (5, \overline{15}) + (\bar{5}, 15) + (15, 5) + (\overline{15}, \bar{5}) + (5, \overline{40}) + (\bar{5}, 40) + (40, 5) + (\overline{40}, \bar{5})$
$\quad + (10, 45) + (\overline{10}, \overline{45}) + (45, \overline{10}) + (\overline{45}, 10) + (24, 24)$

Table 54
Irreps, products and branching rules for SU_8

Dynkin (name)	Dimension (name)	Octality	l (index)	Branching into SO_8 irreps
(1000000)	8	1	1	
(0100000)	28	2	6	28
(2000000)	36	2	10	$1+35_v$
(0010000)	56	3	15	56_v
(1000001)	63	0	16	$28+35_v$
(0001000)	70	4	20	35_s+35_c
(3000000)	120	3	55	8_v+112_v
(1100000)	168	3	61	8_v+160_v
(0100001)	216	1	75	56_v+160_v
(2000001)	280	1	115	$8_v+112_v+160_v$
(4000000)	330	4	220	$1+35_v+294_v$
(0200000)	336	4	160	$1+35_v+300$
(1010000)	378	4	156	$28+350$
(0010001)	420	2	170	35_s+35_c+350
(0001001)	504	3	215	$56_v+224_{sv}+224_{cv}$
(2100000)	630	4	340	$28+35_v+567_v$
(0100010)	720	0	320	$35_s+35_c+300+350$
(0000005)	792	3	715	$8_v+112_v+672'_v$
(3000001)	924	2	550	$28+35_v+294_v+567_v$
(2000010)	945	0	480	$28+350+567_v$
(0000110)	1008	3	526	$8_v+160_v+840_v$
(0000200)	1176	2	700	$1+35_v+300+840'_v$

(Note that the projection of 8 to 8_v is a convention and may be changed to 8 to 8_s or 8 to 8_c.)

$8\times 8 = 28_a+36_s$
$\overline{8}\times 8 = 1+63$
$28\times 8 = 56+168$
$28\times \overline{8} = 8+216$
$28\times 28 = 70_s+336_s+378_a$
$\overline{28}\times 28 = 1+63+720$
$36\times 8 = 120+168$
$36\times \overline{8} = 8+280$
$36\times 28 = 378+630$
$36\times \overline{28} = 63+945$
$36\times 36 = 330_s+336_s+630_a$
$\overline{36}\times 36 = 1+63+1232$
$56\times 8 = 70+278$
$56\times \overline{8} = 28+420$
$\overline{56}\times \overline{28} = 56+504+1008$
$56\times \overline{28} = 8+216+1344$
$\overline{56}\times \overline{36} = 504+1512'$
$56\times \overline{36} = 216+1800$
$\overline{56}\times \overline{56} = 28_a+420_s+1176_s+1512_a$
$\overline{56}\times 56 = 1+63+720+2352$
$63\times 8 = 8+216+280$
$63\times 28 = 28+36+420+1280$
$63\times 36 = 28+36+924+1280$
$63\times 56 = 56+168+504+2800$
$63\times 63 = 1_s+63_s+63_a+720_s+945_a+\overline{945}_a+1232_s$

Branching rules to $SU_3\times SU_5\times U_1$ irreps; U_1 generator in parentheses:
$(1000000) = 8 = (3,1)(-5)+(1,5)(3)$
$(0100000) = 28 = (\overline{3},1)(-10)+(1,10)(6)+(3,5)(-2)$

Table 54 (continued)

$(2000000) = 36 = (6, 1)(-10) + (1, 15)(6) + (3, 5)(-2)$

$(0010000) = 56 = (1, 1)(-15) + (1, \overline{10})(9) + (\overline{3}, 5)(-7) + (3, 10)(1)$

$(1000001) = 63 = (1, 1)(0) + (8, 1)(0) + (3, \overline{5})(-8) + (\overline{3}, 5)(8) + (1, 24)(0)$

$(0001000) = 70 = (1, 5)(-12) + (1, \overline{5})(12) + (3, \overline{10})(4) + (\overline{3}, 10)(-4)$

$(3000000) = 120 = (10, 1)(-15) + (6, 5)(-7) + (3, 15)(1) + (1, \overline{35})(9)$

$(1100000) = 168 = (8, 1)(-15) + (\overline{3}, 5)(-7) + (6, 5)(-7) + (3, 10)(1) + (1, \overline{40})(9) + (3, 15)(1)$

$(0100001) = 216 = (3, 1)(-5) + (1, 5)(3) + (\overline{6}, 1)(-5) + (\overline{3}, \overline{5})(-13) + (\overline{3}, 10)(11) + (8, 5)(3) + (1, 45)(3) + (3, 24)(-5)$

$(2000001) = 280 = (3, 1)(-5) + (1, 5)(3) + (15, 1)(-5) + (6, \overline{5})(-13) + (8, 5)(3) + (3, 24)(-5) + (\overline{3}, 15)(11) + (1, 70)(3)$

$(4000000) = 330 = (15', 1)(-20) + (10, 5)(-12) + (1, \overline{70}')(12) + (6, 15)(-4) + (3, \overline{35})(4)$

$(0200000) = 336 = (\overline{6}, 1)(-20) + (\overline{3}, 10)(-4) + (8, 5)(-12) + (1, \overline{50})(12) + (3, \overline{40})(4) + (6, 15)(-4)$

$(1010000) = 378 = (3, 1)(-20) + (1, 5)(-12) + (3, \overline{10})(4) + (\overline{3}, 10)(-4) + (8, 5)(-12) + (1, \overline{45})(12) + (\overline{3}, 15)(-4) + (6, 10)(-4) + (3, \overline{40})(4)$

$(0010001) = 420 = (\overline{3}, 1)(-10) + (1, \overline{5})(-18) + (1, 10)(6) + (3, 5)(-2) + (\overline{6}, 5)(-2) + (\overline{3}, \overline{10})(14) + (1, 40)(6) + (\overline{3}, 24)(-10) + (8, 10)(6) + (3, 45)(-2)$

$(0001001) = 504 = (1, \overline{10})(9) + (\overline{3}, 5)(-7) + (\overline{3}, \overline{5})(17) + (1, \overline{15})(9) + (1, 24)(-15) + (3, 10)(1) + (\overline{6}, 10)(1) + (8, \overline{10})(9) + (3, 40)(1) + (\overline{3}, 45)(-7)$

Table 55
Irreps, products and branching rules for SO_{14}

Irreps and $SO_{14} \supset SU_2 \times SU_2 \times SO_{10}$ branching rules:

$(1000000) = 14 = (2, 2, 1) + (1, 1, 10)$

$(0100000) = 91 = (3, 1, 1) + (1, 3, 1) + (1, 1, 45) + (2, 2, 10)$

$(0010000) = 364 = (2, 2, 1) + (3, 1, 10) + (1, 3, 10) + (1, 1, 120) + (2, 2, 45)$

$(0001000) = 1001 = (1, 1, 1) + (2, 2, 10) + (3, 1, 45) + (1, 3, 45) + (1, 1, 210) + (2, 2, 120)$

$(0000100) = 2002 = (1, 1, 10) + (1, 1, 126) + (1, 1, \overline{126}) + (2, 2, 45) + (3, 1, 120) + (1, 3, 120) + (2, 2, 210)$

$(0000011) = 3003 = (1, 1, 45) + (1, 1, 120) + (2, 2, 120) + (3, 1, 210) + (1, 3, 210) + (2, 2, 126) + (2, 2, \overline{126})$

$(0000002) = 1716 = (1, 1, 120) + (3, 1, 126) + (1, 3, \overline{126}) + (2, 2, 210)$

$(0000001) = 64 = (2, 1, 16) + (1, 2, \overline{16})$

Products of spinors:

$64 \times 64 = 14_a + 364_s + 1716_s + 2002_a$

$64 \times \overline{64} = 1 + 91 + 1001 + 3003$

Table 56
Irreps, products and branching rules for SO_{18}

Irreps and $SO_{18} \supset SO_8 \times SO_{10}$ branching rules:

$(100000000) = 18 = (8_v, 1) + (1, 10)$

$(010000000) = 153 = (28, 1) + (1, 45) + (8_v, 10)$

$(010000000) = 816 = (56_v, 1) + (1, 120) + (28, 10) + (8_v, 45)$

$(000100000) = 3060 = (35_s, 1) + (35_c, 1) + (1, 210) + (56_v, 10) + (28, 45) + (8_v, 120)$

$(000010000) = 8568 = (56_v, 1) + (1, 126) + (1, \overline{126}) + (35_s, 10) + (35_c, 10) + (56_v, 45) + (28, 120) + (8_v, 210)$

$(000001000) = 18564 = (28, 1) + (1, 210) + (56_v, 10) + (8_v, 126) + (8_v, \overline{126}) + (35_s, 45) + (35_c, 45) + (28, 210) + (56_v, 120)$

$(000000100) = 31824 = (8_v, 1) + (1, 120) + (28, 10) + (8_v, 210) + (28, 126) + (28, \overline{126}) + (56_v, 45) + (56_v, 210) + (35_s, 120) + (35_c, 120)$

$(000000011) = 43758 = (1, 1) + (1, 45) + (8_v, 10) + (8_v, 120) + (28, 45) + (28, 210) + (56_v, 120) + (56_v, 126) + (56_v, \overline{126}) + (35_s, 210) + (35_c, 210)$

$(000000002) = 24310 = (1, 10) + (8_v, 45) + (28, 120) + (56_v, 210) + (35_s, 126) + (35_c, \overline{126})$

$(000000001) = 256 = (8_s, 16) + (8_c, \overline{16})$

Products of spinors:

$256 \times 256 = 18_s + 816_a + 8568_s + 31824_a + 24310_s$

$256 \times \overline{256} = 1 + 153 + 3060 + 18564 + 43758$

Table 57

Irreps, products and branching rules for SO_{22}

Irreps and $SO_{22} \supset SO_{12} \times SO_{10}$ branching rules:

(10000000000) = 22 = (1, 10) + (12, 1)

(01000000000) = 231 = (1, 45) + (66, 1) + (12, 10)

(00100000000) = 1540 = (1, 210) + (220, 1) + (66, 10) + (12, 45)

(00010000000) = 7315 = (1, 210) + (495, 1) + (12, 120) + (220, 10) + (66, 45)

(00001000000) = 26334 = (1, 126) + $(1, \overline{126})$ + (792, 1) + (12, 210) + (495, 10) + (66, 120) + (220, 45)

(00000100000) = 74613 = (1, 210) + (462, 1) + (462′, 1) + (12, 126) + $(12, \overline{126})$ + (792, 10) + (66, 210) + (495, 45) + (220, 120)

(00000010000) = 170544 = (1, 120) + (792, 1) + (12, 210) + (462, 10) + (462′, 10) + (66, 126) + $(66, \overline{126})$ + (792, 45) + (220, 210) + (495, 120)

(00000001000) = 319770 = (1, 45) + (495, 1) + (12, 120) + (792, 10) + (66, 210) + (462, 45) + (462′, 45) + (220, 126) + $(220, \overline{126})$ + (792, 120) + (495, 210)

(00000000100) = 497420 = (1, 10) + (220, 1) + (12, 45) + (495, 10) + (66, 120) + (792, 45) + (220, 210) + (462, 120) + (462′, 120) + (495, 126) + $(495, \overline{126})$
+ (792, 210)

(00000000011) = 646646 = (1, 1) + (12, 10) + (66, 1) + (66, 45) + (220, 10) + (220, 120) + (495, 45) + (495, 210) + (792, 120) + (792, 126) + $(792, \overline{126})$
+ (462, 210) + (462′, 210)

(00000000002) = 352716 = (12, 1) + (66, 10) + (220, 45) + (495, 120) + (792, 210) + (462, 126) + $(462', \overline{126})$

(00000000001) = 1024 = (32, 16) + $(32', \overline{16})$

Products of spinors:

$1024 \times 1024 = 22_a + 1540_s + 26334_a + 170544_s + 352716_s + 497420_a$

$1024 \times \overline{1024} = 1 + 231 + 7315 + 74613 + 319770 + 646646$

Table 58

Branching rules to all maximal subgroups

This table is designed to facilitate analyses such as the search for maximal little groups, and also it represents a summary of the group theory aspects of the review. The branching rules of a few low-lying irreps to the irreps of *every* maximal subgroup are listed for simple groups up to rank 6. Many results repeat those in tables 14, 15, and the branching rule tables, but here there is no restriction to subgroups that can contain flavor and color. When this table or the preceding ones are insufficient, the reader should refer to the much longer tables of ref. [57], although in many practical cases a quick calculation based on the results of this table will fill in the missing information. The format is to give both the Dynkin designation and the dimensionality (name) as (Dynkin)*r*, except when the subgroup is SU_2, $SU_2 \times SU_2$, or more products of SU_2's, in which case only the dimensionality is listed. The eigenvalues of the U_1 generator, when relevant, are given in parentheses after the irrep names, and are normalized to be integers.

<u>Rank 2:</u> $SU_3 \supset SU_2 \times U_1$ (R)

(10)3 = 1(−2) + 2(1)

(20)6 = 1(−4) + 2(−1) + 3(2)

(11)8 = 1(0) + 2(3) + 2(−3) + 3(0)

$SU_3 \supset SU_2$ (S)

(10)3 = 3

(20)6 = 1 + 5

(11)8 = 3 + 5

$Sp_4 \supset SU_2 \times SU_2$ (R)

(10)4 = (2, 1) + (1, 2)

(01)5 = (1, 1) + (2, 2)

(20)10 = (3, 1) + (1, 3) + (2, 2)

$Sp_4 \supset SU_2 \times U_1$ (R)

(10)4 = 2(1) + 2(−1)

(01)5 = 1(2) + 1(−2) + 3(0)

(20)10 = 1(0) + 3(0) + 3(2) + 3(−2)

Table 58 (continued)

$Sp_4 \supset SU_2$ (S)
$(10)4 = 4$
$(01)5 = 5$
$(20)10 = 3 + 7$

$G_2 \supset SU_3$ (R)
$(01)7 = (00)1 + (10)3 + (01)\bar{3}$
$(10)14 = (10)3 + (01)\bar{3} + (11)8$

$G_2 \supset SU_2 \times SU_2$ (R)
$(01)7 = (1, 3) + (2, 2)$
$(10)14 = (1, 3) + (3, 1) + (2, 4)$

$G_2 \supset SU_2$ (S)
$(01)7 = 7$
$(10)14 = 3 + 11$

Rank 3: $SU_4 \supset SU_3 \times U_1$ (R)
$(100)4 = (00)1(3) + (10)3(-1)$
$(010)6 = (10)3(2) + (01)\bar{3}(-2)$
$(101)15 = (00)1(0) + (10)3(-4) + (01)\bar{3}(4) + (11)8(0)$

$SU_4 \supset SU_2 \times SU_2 \times U_1$ (R)
$(100)4 = (2, 1)(1) + (1, 2)(-1)$
$(010)6 = (1, 1)(2) + (1, 1)(-2) + (2, 2)(0)$
$(101)15 = (1, 1)(0) + (3, 1)(0) + (1, 3)(0) + (2, 2)(2) + (2, 2)(-2)$

$SU_4 \supset Sp_4$ (S)
$(100)4 = (10)4$
$(010)6 = (00)1 + (01)5$
$(101)15 = (01)5 + (20)10$

$SU_4 \supset SU_2 \times SU_2$ (S)
$(100)4 = (2, 2)$
$(010)6 = (1, 3) + (3, 1)$
$(101)15 = (1, 3) + (3, 1) + (3, 3)$

$SO_7 \supset SU_4$ (R)
$(100)7 = (000)1 + (010)6$
$(001)8 = (100)4 + (001)\bar{4}$
$(010)21 = (010)6 + (101)15$

$SO_7 \supset SU_2 \times SU_2 \times SU_2$ (R)
$(100)7 = (1, 1, 3) + (2, 2, 1)$
$(001)8 = (1, 2, 2) + (2, 1, 2)$
$(010)21 = (1, 1, 3) + (1, 3, 1) + (3, 1, 1) + (2, 2, 3)$

$SO_7 \supset Sp_4 \times U_1$ (R)
$(100)7 = (00)1(2) + (00)1(-2) + (01)5(0)$
$(001)8 = (10)4(1) + (10)4(-1)$
$(010)21 = (00)1(0) + (01)5(2) + (01)5(-2) + (20)10(0)$

$SO_7 \supset G_2$ (S)
$(100)7 = (01)7$
$(001)8 = (00)1 + (01)7$
$(010)21 = (01)7 + (10)14$

Table 58 (continued)

$Sp_6 \supset SU_3 \times U_1$ (R)

$(100)6 = (10)3(1) + (01)\bar{3}(-1)$
$(010)14 = (10)3(-2) + (01)\bar{3}(2) + (11)8(0)$
$(001)14' = (00)1(3) + (00)1(-3) + (20)6(-1) + (02)\bar{6}(1)$
$(200)21 = (00)1(0) + (20)6(2) + (02)\bar{6}(-2) + (11)8(0)$

$Sp_6 \supset SU_2 \times Sp_4$(R)

$(100)6 = (1)(00)(2, 1) + (0)(10)(1, 4)$
$(010)14 = (0)(00)(1, 1) + (0)(01)(1, 5) + (1)(10)(2, 4)$
$(001)14' = (0)(10)(1, 4) + (1)(01)(2, 5)$
$(200)21 = (2)(0)(3, 1) + (0)(20)(1, 10) + (1)(10)(2, 4)$

$Sp_6 \supset SU_2$ (S)

$(100)6 = 6$
$(010)14 = 5 + 9$
$(001)14' = 4 + 10$
$(200)21 = 3 + 7 + 11$

$Sp_6 \supset SU_2 \times SU_2$ (S)

$(100)6 = (2, 3)$
$(010)14 = (1, 5) + (3, 3)$
$(001)14' = (4, 1) + (2, 5)$
$(200)21 = (1, 3) + (3, 1) + (3, 5)$

Rank 4: $SU_5 \supset SU_4 \times U_1$ (R)

$(1000)5 = (000)1(4) + (100)4(-1)$
$(0100)10 = (100)4(3) + (010)6(-2)$
$(1001)24 = (000)1(0) + (100)4(-5) + (001)\bar{4}(5) + (101)15(0)$

$SU_5 \supset SU_2 \times SU_3 \times U_1$ (R)

$(1000)5 = (1)(00)(2, 1)(3) + (0)(10)(1, 3)(-2)$
$(0100)10 = (0)(00)(1, 1)(6) + (0)(01)(1, \bar{3})(-4) + (1)(10)(2, 3)(1)$
$(1001)24 = (0)(00)(1, 1)(0) + (2)(00)(3, 1)(0) + (1)(10)(2, 3)(-5) + (1)(01)(2, \bar{3})(5) + (0)(11)(1, 8)(0)$

$SU_5 \supset Sp_4$ (S)

$(1000)5 = (01)5$
$(0100)10 = (20)10$
$(1001)24 = (20)10 + (02)14$

$SO_9 \supset SO_8$(R)

$(1000)9 = (0000)1 + (1000)8_v$
$(0001)16 = (0010)8_c + (0001)8_s$
$(0100)36 = (1000)8_v + (0100)28$

$SO_9 \supset SU_2 \times SU_2 \times Sp_4$ (R)

$(1000)9 = (1)(1)(00)(2, 2, 1) + (0)(0)(01)(1, 1, 5)$
$(0001)16 = (0)(1)(10)(1, 2, 4) + (1)(0)(10)(2, 1, 4)$
$(0100)36 = (2)(0)(00)(3, 1, 1) + (0)(2)(00)(1, 3, 1) + (0)(0)(20)(1, 1, 10) + (1)(1)(01)(2, 2, 5)$

$SO_9 \supset SU_2 \times SU_4$ (R)

$(1000)9 = (2)(000)(3, 1) + (0)(010)(1, 6)$
$(0001)16 = (1)(100)(2, 4) + (1)(001)(2, \bar{4})$
$(0100)36 = (2)(000)(3, 1) + (0)(101)(1, 15) + (2)(010)(3, 6)$

Table 58 (continued)

$SO_9 \supset SO_7 \times U_1$ (R)
$(1000)9 = (000)1(2) + (000)1(-2) + (100)7(0)$
$(0001)16 = (001)8(1) + (001)8(-1)$
$(0100)36 = (000)1(0) + (100)7(2) + (100)7(-2) + (010)21(0)$

$SO_9 \supset SU_2$ (S)
$(1000)9 = 9$
$(0001)16 = 5 + 11$
$(0100)36 = 3 + 7 + 11 + 15$

$SO_9 \supset SU_2 \times SU_2$ (S)
$(1000)9 = (3, 3)$
$(0001)16 = (2, 4) + (4, 2)$
$(0100)36 = (1, 3) + (3, 1) + (3, 5) + (5, 3)$

$Sp_8 \supset SU_4 \times U_1$ (R)
$(1000)8 = (100)4(1) + (001)\bar{4}(-1)$
$(2000)36 = (000)1(0) + (200)10(2) + (002)\overline{10}(-2) + (101)15(0)$
$(0001)42 = (000)1(4) + (000)1(-4) + (200)10(-2) + (002)\overline{10}(2) + (020)20'(0)$

$Sp_8 \supset SU_2 \times Sp_6$ (R)
$(1000)8 = (1)(000)(2, 1) + (0)(100)(1, 6)$
$(2000)36 = (2)(000)(3, 1) + (0)(200)(1, 21) + (1)(100)(2, 6)$
$(0001)42 = (0)(010)(1, 14) + (1)(001)(2, 14')$

$Sp_8 \supset Sp_4 \times Sp_4$ (R)
$(1000)8 = (00)(10)(1, 4) + (10)(00)(4, 1)$
$(2000)36 = (00)(20)(1, 10) + (20)(00)(10, 1) + (10)(10)(4, 4)$
$(0001)42 = (00)(00)(1, 1) + (10)(10)(4, 4) + (01)(01)(5, 5)$

$Sp_8 \supset SU_2$ (S)
$(1000)8 = 8$
$(2000)36 = 3 + 7 + 11 + 15$
$(0001)42 = 5 + 9 + 11 + 17$

$Sp_8 \supset SU_2 \times SU_2 \times SU_2$ (S)
$(1000)8 = (2, 2, 2)$
$(2000)36 = (1, 1, 3) + (1, 3, 1) + (3, 1, 1) + (3, 3, 3)$
$(0001)42 = (1, 1, 5) + (1, 5, 1) + (5, 1, 1) + (3, 3, 3)$

$SO_8 \supset SU_2 \times SU_2 \times SU_2 \times SU_2$ (R)
$(1000)8_v = (2, 2, 1, 1) + (1, 1, 2, 2)$
$(0001)8_s = (1, 2, 1, 2) + (2, 1, 2, 1)$
$(0010)8_c = (1, 2, 2, 1) + (2, 1, 1, 2)$
$(0100)28 = (1, 1, 1, 3) + (1, 1, 3, 1) + (1, 3, 1, 1) + (3, 1, 1, 1) + (2, 2, 2, 2)$

$SO_8 \supset SU_4 \times U_1$ (R)
$(1000)8_v = (100)4(1) + (001)\bar{4}(-1)$
$(0001)8_s = (000)1(2) + (000)1(-2) + (010)6(0)$
$(0010)8_c = (100)4(-1) + (001)\bar{4}(1)$
$(0100)28 = (000)1(0) + (010)6(2) + (010)6(-2) + (101)15(0)$

$SO_8 \supset SU_3$ (S)
$(1000)8_v = (11)8$
$(0001)8_s = (11)8$
$(0010)8_c = (11)8$
$(0100)28 = (11)8 + (30)10 + (03)\overline{10}$

Table 58 (continued)

$SO_8 \supset SO_7$ (S)

$(1000)8_v = (001)8$
$(0001)8_s = (000)1 + (100)7$
$(0010)8_c = (001)8$
$(0100)28 = (100)7 + (010)21$

$SO_8 \supset SU_2 \times Sp_4$ (S)

$(1000)8_v = (1)(10)(2, 4)$
$(0010)8_c = (1)(10)(2, 4)$
$(0001)8_s = (0)(01)(1, 5) + (2)(00)(3, 1)$
$(0100)28 = (2)(00)(3, 1) + (0)(20)(1, 10) + (2)(01)(3, 5)$

$F_4 \supset SO_9$ (R)

$(0001)26 = (0000)1 + (1000)9 + (0001)16$
$(1000)52 = (0001)16 + (0100)36$

$F_4 \supset SU_3 \times SU_3$ (R)

$(0001)26 = (11)(00)(8, 1) + (10)(10)(3, 3) + (01)(01)(\bar{3}, \bar{3})$
$(1000)52 = (11)(00)(8, 1) + (00)(11)(1, 8) + (20)(01)(6, \bar{3}) + (02)(10)(\bar{6}, 3)$

$F_4 \supset SU_2 \times Sp_6$ (R)

$(0001)26 = (1)(100)(2, 6) + (0)(010)(2, 14)$
$(1000)52 = (2)(000)(3, 1) + (0)(200)(1, 21) + (1)(001)(2, 14')$

$F_4 \supset SU_2$ (S)

$(0001)26 = 9 + 17$
$(1000)52 = 3 + 11 + 15 + 23$

$F_4 \supset SU_2 \times G_2$ (S)

$(0001)26 = (4)(00)(5, 1) + (2)(01)(3, 7)$
$(1000)52 = (2)(00)(3, 1) + (0)(10)(1, 14) + (4)(01)(5, 7)$

Rank 5: $SU_6 \supset SU_5 \times U_1$ (R)

$(10000)6 = (0000)1(-5) + (1000)5(1)$
$(00100)20 = (0100)10(-3) + (0010)\overline{10}(3)$
$(10001)35 = (0000)1(0) + (1000)5(6) + (0001)\bar{5}(-6) + (1001)24(0)$

$SU_6 \supset SU_2 \times SU_4 \times U_1$ (R)

$(10000)6 = (1)(000)(2, 1)(2) + (0)(100)(1, 4)(-1)$
$(00100)20 = (0)(100)(1, 4)(3) + (0)(001)(1, \bar{4})(-3) + (1)(010)(2, 6)(0)$
$(10001)35 = (0)(000)(1, 1)(0) + (2)(000)(3, 1)(0) + (0)(101)(1, 15)(0) + (1)(100)(2, 4)(-3) + (1)(001)(2, \bar{4})(3)$

$SU_6 \supset SU_3 \times SU_3 \times U_1$ (R)

$(10000)6 = (00)(10)(1, 3)(-1) + (10)(00)(3, 1)(1)$
$(00100)20 = (00)(00)(1, 1)(3) + (00)(00)(1, 1)(-3) + (10)(01)(3, \bar{3})(-1) + (01)(10)(\bar{3}, 3)(1)$
$(10001)35 = (00)(00)(1, 1)(0) + (00)(11)(1, 8)(0) + (11)(00)(8, 1)(0) + (10)(01)(3, \bar{3})(2) + (01)(10)(\bar{3})(-2)$

$SU_6 \supset SU_3$ (S)

$(10000)6 = (20)6$
$(00100)20 = (30)10 + (03)\overline{10}$
$(10001)35 = (11)8 + (22)27$

$SU_6 \supset SU_4$ (S)

$(10000)6 = (010)6$
$(00100)20 = (200)10 + (002)\overline{10}$
$(10001)35 = (101)15 + (020)20'$

Table 58 (continued)

$SU_6 \supset Sp_6$ (S)
(10000)6 = (100)6
(00100)20 = (100)6 + (001)14′
(10001)35 = (010)14 + (200)21

$SU_6 \supset SU_2 \times SU_3$ (S)
(10000)6 = (1)(10)(2, 3)
(00100)20 = (3)(00)(4, 1) + (1)(11)(2, 8)
(10001)35 = (2)(00)(3, 1) + (0)(11)(1, 8) + (2)(11)(3, 8)

$SO_{11} \supset SO_{10}$ (R)
(10000)11 = (00000)1 + (10000)10
(00001)32 = (00001)16 + (00010)$\overline{16}$
(01000)55 = (10000)10 + (01000)45

$SO_{11} \supset SU_2 \times SO_8$ (R)
(10000)11 = (2)(0000)(3, 1) + (0)(1000)(1, 8_v)
(00001)32 = (1)(0001)(2, 8_s) + (1)(0010)(2, 8_c)
(01000)55 = (2)(0000)(3, 1) + (0)(0100)(1, 28) + (2)(1000)(3, 8_v)

$SO_{11} \supset Sp_4 \times SU_4$ (R)
(10000)11 = (01)(000)(5, 1) + (00)(010)(1, 6)
(00001)32 = (10)(100)(4, 4) + (10)(001)(4, $\bar{4}$)
(01000)55 = (20)(000)(10, 1) + (00)(101)(1, 15) + (01)(010)(5, 6)

$SO_{11} \supset SU_2 \times SU_2 \times SO_7$ (R)
(10000)11 = (1)(1)(000)(2, 2, 1) + (0)(0)(100)(1, 1, 7)
(00001)32 = (0)(1)(001)(1, 2, 8) + (1)(0)(001)(2, 1, 8)
(01000)55 = (2)(0)(000)(3, 1, 1) + (0)(2)(000)(1, 3, 1) + (0)(0)(010)(1, 1, 21) + (1)(1)(100)(2, 2, 7)

$SO_{11} \supset SO_9 \times U_1$ (R)
(10000)11 = (0000)1(2) + (0000)1(−2) + (1000)9(0)
(00001)32 = (0001)16(1) + (0001)16(−1)
(01000)55 = (0000)1(0) + (1000)9(2) + (1000)9(−2) + (0100)36(0)

$SO_{11} \supset SU_2$ (S)
(10000)11 = 11
(00001)32 = 6 + 10 + 16
(01000)55 = 3 + 7 + 11 + 15 + 19

$Sp_{10} \supset SU_5 \times U_1$ (R)
(10000)10 = (1000)5(1) + (0001)$\bar{5}$(−1)
(20000)55 = (0000)1(0) + (2000)15(2) + (0002)$\overline{15}$(−2) + (1001)24(0)

$Sp_{10} \supset SU_2 \times Sp_8$ (R)
(10000)10 = (1)(0000)(2, 1) + (0)(1000)(1, 8)
(20000)55 = (2)(0000)(3, 1) + (0)(2000)(1, 36) + (1)(1000)(2, 8)

$Sp_{10} \supset Sp_4 \times Sp_6$ (R)
(10000)10 = (10)(000)(4, 1) + (00)(100)(1, 6)
(20000)55 = (20)(000)(10, 1) + (00)(200)(1, 21) + (10)(100)(4, 6)

$Sp_{10} \supset SU_2$ (S)
(10000)10 = 10
(20000)55 = 3 + 7 + 11 + 15 + 19

Table 58 (continued)

$Sp_{10} \supset SU_2 \times Sp_4$ (S)
(10000)10 = (1)(01)(2, 5)
(20000)55 = (2)(00)(3, 1) + (0)(20)(1, 10) + (2)(02)(3, 14)

$SO_{10} \supset SU_5 \times U_1$ (R)
(10000)10 = (1000)5(2) + (0001)$\bar{5}$(−2)
(00001)16 = (0000)1(−5) + (0001)$\bar{5}$(3) + (0100)10(−1)
(01000)45 = (0000)1(0) + (0100)10(4) + (0010)$\overline{10}$(−4) + (1001)24(0)

$SO_{10} \supset SU_2 \times SU_2 \times SU_4$ (R)
(10000)10 = (1)(1)(000)(2, 2, 1) + (0)(0)(010)(1, 1, 6)
(00001)16 = (1)(0)(100)(2, 1, 4) + (0)(1)(001)(1, 2, $\bar{4}$)
(01000)45 = (2)(0)(000)(3, 1, 1) + (0)(2)(000)(1, 3, 1) + (0)(0)(101)(1, 1, 15) + (1)(1)(010)(2, 2, 6)

$SO_{10} \supset SO_8 \times U_1$ (R)
(10000)10 = (0000)1(2) + (0000)1(−2) + (1000)8_v(0)
(00001)16 = (0010)8_c(1) + (0001)8_s(−1)
(01000)45 = (0000)1(0) + (1000)8_v(2) + (1000)8_v(−2) + (0100)28(0)

$SO_{10} \supset Sp_4$ (S)
(10000)10 = (20)10
(00001)16 = (11)16
(01000)45 = (20)10 + (21)35

$SO_{10} \supset SO_9$(S)
(10000)10 = (0000)1 + (1000)9
(00001)16 = (0001)16
(01000)45 = (1000)9 + (0100)36

$SO_{10} \supset SU_2 \times SO_7$ (S)
(10000)10 = (2)(000)(3, 1) + (0)(100)(1, 7)
(00001)16 = (1)(001)(2, 8)
(01000)45 = (2)(000)(3, 1) + (0)(010)(1, 21) + (2)(100)(3, 7)

$SO_{10} \supset Sp_4 \times Sp_4$ (S)
(10000)10 = (00)(01)(1, 5) + (01)(00)(5, 1)
(00001)16 = (10)(10)(4, 4)
(01000)45 = (00)(20)(1, 10) + (20)(00)(10, 1) + (01)(01)(5, 5)

<u>Rank 6</u>: $SU_7 \supset SU_6 \times U_1$ (R)
(100000)7 = (00000)1(6) + (10000)6(−1)
(100001)48 = (00000)1(0) + (10001)35(0) + (10000)6(−7) + (00001)$\bar{6}$(7)

$SU_7 \supset SU_2 \times SU_5 \times U_1$ (R)
(100000)7 = (1)(0000)(2, 1)(5) + (0)(1000)(1, 5)(−2)
(100001)48 = (0)(0000)1(0) + (2)(0000)(3, 1)(0) + (0)(1001)(1, 24)(0) + (1)(1000)(2, 5)(−7) + (1)(0001)(2, $\bar{5}$)(7)

$SU_7 \supset SU_3 \times SU_4 \times U_1$ (R)
(100000)7 = (10)(000)(3, 1)(4) + (00)(100)(1, 4)(−3)
(100001)48 = (00)(000)(1, 1)(0) + (11)(000)(8, 1)(0) + (00)(101)(1, 15)(0) + (10)(001)(3, $\bar{4}$)(7) + (01)(100)($\bar{3}$, 4)(−7)

$SU_7 \supset SO_7$ (S)
(100000)7 = (100)7
(100001)48 = (010)21 + (200)27

Table 58 (continued)

$SO_{13} \supset SO_{12}$ (R)
(100000)13 = (000000)1 + (100000)12
(000001)64 = (000001)32 + (000010)32′
(010000)78 = (100000)12 + (010000)66

$SO_{13} \supset SU_2 \times SO_{10}$ (R)
(100000)13 = (2)(00000)(3, 1) + (0)(10000)(1, 10)
(000001)64 = (1)(00001)(2, 16) + (1)(00010)(2, $\overline{16}$)
(010000)78 = (2)(00000)(3, 1) + (0)(01000)(1, 45) + (2)(10000)(3, 10)

$SO_{13} \supset Sp_4 \times SO_8$ (R)
(100000)13 = (01)(0000)(5, 1) + (00)(1000)(8_v, 1)
(000001)64 = (10)(0001)(4, 8_s) + (10)(0010)(4, 8_c)
(010000)78 = (20)(0000)(10, 1) + (00)(0100)(1, 28) + (01)(1000)(5, 8_v)

$SO_{13} \supset SU_4 \times SO_7$ (R)
(100000)13 = (010)(000)(6, 1) + (000)(100)(1, 7)
(000001)64 = (100)(001)(4, 8) + (001)(001)($\bar{4}$, 8)
(010000)78 = (000)(010)(1, 21) + (101)(000)(15, 1) + (010)(100)(6, 7)

$SO_{13} \supset SU_2 \times SU_2 \times SO_9$ (R)
(100000)13 = (1)(1)(0000)(2, 2, 1) + (0)(0)(1000)(1, 1, 9)
(000001)64 = (0)(1)(0001)(1, 2, 16) + (1)(0)(0001)(2, 1, 16)
(010000)78 = (2)(0)(0000)(3, 1, 1) + (0)(2)(0000)(1, 3, 1) + (0)(0)(0100)(1, 1, 36) + (1)(1)(1000)(2, 2, 9)

$SO_{13} \supset SO_{11} \times U_1$ (R)
(100000)13 = (00000)1(2) + (00000)1(−2) + (10000)11(0)
(000001)64 = (00001)32(1) + (00001)32(−1)
(010000)78 = (00000)1(0) + (10000)11(2) + (10000)11(−2) + (01000)55(0)

$SO_{13} \supset SU_2$ (S)
(100000)13 = 13
(000001)64 = 4 + 10 + 12 + 16 + 22
(010000)78 = 3 + 7 + 11 + 15 + 19 + 23

$Sp_{12} \supset SU_6 \times U_1$ (R)
(100000)12 = (10000)6(1) + (00001)$\bar{6}$(−1)
(200000)78 = (00000)1(0) + (10001)35(0) + (20000)21(2) + (00002)$\overline{21}$(−2)

$Sp_{12} \supset SU_2 \times Sp_{10}$ (R)
(100000)12 = (1)(00000)(2, 1) + (0)(10000)(1, 10)
(200000)78 = (2)(00000)(3, 1) + (0)(20000)(1, 55) + (1)(10000)(2, 10)

$Sp_{12} \supset Sp_4 \times Sp_8$ (R)
(100000)12 = (10)(0000)(4, 1) + (00)(1000)(1, 8)
(200000)78 = (20)(0000)(10, 1) + (00)(2000)(1, 36) + (10)(1000)(4, 8)

$Sp_{12} \supset Sp_6 \times Sp_6$ (R)
(100000)12 = (100)(000)(6, 1) + (000)(100)(1, 6)
(200000)78 = (200)(000)(21, 1) + (000)(200)(1, 21) + (100)(100)(6, 6)

$Sp_{12} \supset SU_2$ (S)
(100000)12 = 12
(200000)78 = 3 + 7 + 11 + 15 + 19 + 23

Table 58 (continued)

$Sp_{12} \supset SU_2 \times SU_4$ (S)
(100000)12 = (1)(010)(2, 6)
(200000)78 = (2)(000)(3, 1) + (0)(101)(1, 15) + (2)(020)(3, 20′)

$Sp_{12} \supset SU_2 \times Sp_4$ (S)
(100000)12 = (2)(10)(3, 4)
(200000)78 = (2)(00)(3, 1) + (0)(20)(1, 10) + (2)(01)(3, 5) + (4)(20)(5, 10)

$SO_{12} \supset SU_6 \times U_1$ (R)
(100000)12 = (10000)6(1) + (00001)$\overline{6}$(−1)
(000001)32 = (10000)6(−2) + (00001)$\overline{6}$(2) + (00100)20(0)
(000010)32′ = (00000)1(3) + (00000)1(−3) + (01000)15(−1) + (00010)$\overline{15}$(1)
(010000)66 = (00000)1(0) + (01000)15(2) + (00010)$\overline{15}$(−2) + (10001)35(0)

$SO_{12} \supset SU_2 \times SU_2 \times SO_8$ (R)
(100000)12 = (1)(1)(0000)(2, 2, 1) + (0)(0)(1000)(1, 1, 8_v)
(000001)32 = (0)(1)(0001)(1, 2, 8_s) + (1)(0)(0010)(2, 1, 8_c)
(000010)32′ = (0)(1)(0010)(1, 2, 8_c) + (1)(0)(0001)(2, 1, 8_s)
(010000)66 = (2)(0)(0000)(3, 1, 1) + (0)(2)(0000)(1, 3, 1) + (0)(0)(0100)(1, 1, 28) + (1)(1)(1000)(2, 2, 8_v)

$SO_{12} \supset SU_4 \times SU_4$ (R)
(100000)12 = (010)(000)(6, 1) + (000)(010)(1, 6)
(000001)32 = (100)(100)(4, 4) + (001)(001)($\overline{4}$, $\overline{4}$)
(000010)32′ = (100)(001)(4, $\overline{4}$) + (001)(100)($\overline{4}$, 4)
(010000)66 = (101)(000)(15, 1) + (000)(101)(1, 15) + (010)(010)(6, 6)

$SO_{12} \supset SO_{10} \times U_1$ (R)
(100000)12 = (00000)1(2) + (00000)1(−2) + (10000)10(0)
(000001)32 = (00001)16(1) + (00010)$\overline{16}$(−1)
(000010)32′ = (00001)16(−1) + (00010)$\overline{16}$(1)
(010000)66 = (00000)1(0) + (10000)10(2) + (10000)10(−2) + (01000)45(0)

$SO_{12} \supset SU_2 \times Sp_6$ (S)
(100000)12 = (1)(100)(2, 6)
(000001)32 = (3)(000)(4, 1) + (1)(010)(2, 14)
(000010)32′ = (2)(100)(3, 6) + (0)(001)(1, 14′)
(010000)66 = (2)(000)(3, 1) + (0)(200)(1, 21) + (2)(010)(3, 14)

$SO_{12} \supset SU_2 \times SU_2 \times SU_2$ (S)
(100000)12 = (3, 2, 2)
(000001)32 = (1, 4, 1) + (3, 2, 3) + (5, 2, 1)
(000010)32′ = (1, 1, 4) + (3, 3, 2) + (5, 1, 2)
(010000)66 = (3, 1, 1) + (1, 3, 1) + (1, 1, 3) + (3, 3, 3) + (5, 3, 1) + (5, 1, 3)

$SO_{12} \supset SO_{11}$ (S)
(100000)12 = (00000)1 + (10000)11
(000001)32 = (00001)32
(000010)32′ = (00001)32
(010000)66 = (100000)11 + (01000)55

$SO_{12} \supset SU_2 \times SO_9$ (S)
(100000)12 = (2)(0000)(3, 1) + (0)(1000)(1, 9)
(000001)32 = (1)(0001)(2, 16)
(000010)32′ = (1)(0001)(2, 16)
(010000)66 = (2)(0000)(3, 1) + (0)(0100)(1, 36) + (2)(1000)(3, 9)

Table 58 (continued)

$SO_{12} \supset Sp_4 \times SO_7$ (S)
(100000)12 = (01)(000)(5, 1) + (00)(100)(1, 7)
(000001)32 = (10)(001)(4, 8)
(000010)32′ = (10)(001)(4, 8)
(010000)66 = (20)(000)(10, 1) + (00)(010)(1, 21) + (01)(100)(5, 7)

$E_6 \supset SO_{10} \times U_1$ (R)
(100000)27 = (00000)1(4) + (10000)10(−2) + (00001)16(1)
(000001)78 = (00000)1(0) + (00001)16(−3) + (00010)$\overline{16}$(3) + (01000)45(0)

$E_6 \supset SU_2 \times SU_6$ (R)
(100000)27 = (1)(00001)(2, $\bar{6}$) + (0)(01000)(1, 15)
(000001)78 = (2)(00000)(3, 1) + (0)(10001)(1, 35) + (1)(00100)(2, 20)

$E_6 \supset SU_3 \times SU_3 \times SU_3$ (R)
(100000)27 = (01)(10)(00)($\bar{3}$, 3, 1) + (10)(00)(10)(3, 1, 3) + (00)(01)(01)(1, $\bar{3}$, $\bar{3}$)
(000001)78 = (11)(00)(00)(8, 1, 1) + (00)(11)(00)(1, 8, 1) + (00)(00)(11)(1, 1, 8) + (10)(10)(01)(3, 3, $\bar{3}$) + (01)(01)(10)($\bar{3}$, $\bar{3}$, 3)

$E_6 \supset SU_3$ (S)
(100000)27 = (22)27
(000001)78 = (11)8 + (41)35 + (14)$\overline{35}$

$E_6 \supset G_2$ (S)
(100000)27 = (02)27
(000001)78 = (10)14 + (11)64

$E_6 \supset Sp_8$(S)
(100000)27 = (0100)27
(000001)78 = (2000)36 + (0001)42

$E_6 \supset F_4$ (S)
(100000)27 = (0000)1 + (0001)26
(000001)78 = (0001)26 + (1000)52

$E_6 \supset SU_3 \times G_2$ (S)
(100000)27 = (02)(00)($\bar{6}$, 1) + (10)(01)(3, 7)
(000001)78 = (11)(00)(8, 1) + (00)(10)(1, 14) + (11)(01)(8, 7)

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