

Exchange Rate Anomalies Under Model Misspecification: A Mixed Optimal/Robust
Approach

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Abstract

Many asset pricing anomalies imply the existence of time-varying predictable excess returns, as well as short-run underreaction and long-run overreaction to news. The presence of short-sales constraints or of missperception has been often invoked to explain such patterns. In this paper I present a ‘mixed optimal/robust’ model with rational agents that hold no missperception and face no borrowing constraints. In equilibrium, agents do not borrow as much as they could, and filter news in ways that are consistent with the anomalies mentioned above. We use the model to rationalize the forward premium puzzle.

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1. Introduction

Many asset pricing anomalies imply the existence of time-varying predictable excess returns, as well as short-run underreaction and long-run overreaction to news. The presence of short-sales constraints or of missperception has been often invoked to explain such patterns. In this paper I present a ‘mixed optimal/robust’ model with rational agents that hold no missperception and face no borrowing constraints. In equilibrium, agents do not borrow as much as they could, and filter news in ways that are consistent with the anomalies mentioned above. We use the model to rationalize the forward premium puzzle.

The idea behind this approach is that agents maximize expected utility as in standard models. However, they fear misspecification. Thus, they subject their optimization problem to a robustness constraint (RC). This constraint imposes an upper bound on the rate at which returns can deteriorate as the amount of misspecification increases. The results are that even agents who maximize expected wealth and who face no short-sales constraints will: (a) choose their portfolio within a closed set even in the presence of predictable excess returns (i.e., ‘limits to arbitrage’ arise in equilibrium); and (b) will use robust forecasting formulas that are consistent with the misperception found in the data.

We use this setup to explain two anomalies in international financial markets: the forward premium puzzle (FPP) and delayed overshooting (DO). According to the FPP countries with higher than usual interest rates, relative to the US, tend to experience a subsequent appreciation of their currency relative to the US dollar. The DO anomaly is less prevalent than the FPP. It implies that the exchange rate often follows a hump-shape pattern in response to a monetary shock.

The empirical literature has shown that these anomalies cannot be explained only in terms of time-varying risk premia. Empirical evidence indicates that systematic forecast errors on the part of market participants play an important role. Although uncovering this link is an important step, it generates intriguing questions: why do agents systematically hold misperception? Can these misperception be arbitrated away by savvier investors? These questions can be addressed by the mixed optimal/robust setup.

In formulating the RC and solving the agent’s problem I use ‘robust control’ methods which have been developed during the last decade. The starting point is the recognition that the models we use are simply approximations to the ‘true model,’ and there is no reason to maintain that the resulting misspecification patterns can be parametrized by a well specified probability distribution, as is

done in the stochastic approach. A remarkable result is that for a very large class of *true models*, one can represent misspecifications via *unknown-but-bounded* disturbance sequences. An attractive property of robust control is that one can obtain closed form solutions, which I use to compute the feasible portfolio set and the robust forecasts.

In a mixed optimal/robust (\mathcal{O}/\mathcal{R}) setup investors have a well specified model of the economy, but they take seriously into account that there might be model misspecification. As in rational expectations models, investors try to take advantage of all profit opportunities under a *well specified* probabilistic model of the economy. However, as in robust models, they choose their portfolios subject to a robustness constraint that requires that small, *but unknown*, misspecifications don't lead to large losses; and also requires that the performance of the portfolio does not deteriorate too fast as the norm of uncertainty increases.¹

In this setup, investors who face no short-sales constraints, who use the correct model to form expectations, and whose objective is to maximize expected profits may choose not to take infinite positions. We will show that under some parameter restrictions investors will choose their portfolio within a closed and bounded set (the Q_t -set). That is, 'limits to arbitrage' will arise endogenously from a desire for robustness. Thus, when faced with a seemingly profitable opportunity, an investor will behave conservatively and not choose an overly large position.

The interesting aspect of the model is that, if it exists, the boundary of the Q_t -set is state-dependent: it is a function of the exchange rate realization, the history of interest rate differentials, and the degree of required robustness. As a result, the exchange rate has two functions: it determines the sign of expected excess returns, as well as the boundary of the Q_t -set.

Clearly, in situations in which the representative investor selects his portfolio at the boundary of the Q_t -set, expected excess returns can be positive. Furthermore, if the degree of required robustness declines over time, there can be a negative covariance between exchange rate changes and the interest rate differential, and an unconditional delayed response of the exchange rate to interest rate shocks. This rationalizes the foreign exchange anomalies alluded to above.

We formalize the preceding ideas by considering a 'mixed \mathcal{O}/\mathcal{R} ' economy in which agents select their portfolios by wearing two hats. They wear a robust hat to

¹As we shall see, the problem solved by agents does not require that profits be greater than a certain level under all circumstances, so it is not a constraint that only considers the worst case scenario.

construct the Q_t -set of admissible portfolios under an ‘uncertain model’ that allows for misspecification.² They then wear an optimizing hat to select a portfolio within that set in order to maximize expected profits under a well specified ‘probabilistic model.’ In other words, agents select portfolios in a standard optimizing fashion within a certain ‘ Q_t -set’. However, they do not consider portfolios outside this Q_t -set.³ This setup is similar to the ‘mixed $\mathcal{H}_2/\mathcal{H}_\infty$ ’ approach developed in the control literature.

Since we are considering a multiperiod asset pricing problem, the probabilistic model and the uncertain model used by an investor must contain conjectures of how future exchange rates will be determined, as well as descriptions of the interest rate differential process. This raises the issue of what is a reasonable equilibrium concept in a mixed \mathcal{O}/\mathcal{R} economy? We propose one which is practically identical to the standard competitive equilibrium of rational expectations. The only difference is that we impose *two*, instead of just one, consistency requirement on the agent’s conjectures.

The existing economics literature has dealt with uncertainty in two contrasting ways. In the rational expectations literature all uncertainty is represented probabilistically. In contrast, in the robust literature, uncertainty is represented by sequences of totally unknown disturbances that satisfy certain norm bounds.⁴ Our methodological contribution is to pose and solve a multiperiod asset pricing problem in which both optimizing and robustness considerations are present. This approach is a compromise between standard rational expectations models, which assume away any potential misspecification, and robust models, which force agents to be overly conservative.

The structure of the paper is as follows. In Section 2 we present a brief outline of the argument. In Section 3 we present a simple rational expectations model that will serve as a benchmark. In Section 4 we consider the mixed \mathcal{O}/\mathcal{R} economy and solve for the equilibrium. In Section 5 we show how the equilibrium exchange rate process can rationalize the foreign exchange market anomalies. In Section 6 we present a review of the literature. Lastly, we present the proofs in the Appendix.

²The main difference between robust models and rational expectations models is that in the former consider misspecifications that need not be parametrized in a probabilistic way.

³As we shall see, the desire for robustness does not imply that agents are inactive in the market and simply stay in bed. The problem that agents solve in order to compute the R_t -set does not require that excess returns be greater than a certain level under all circumstances, so it is not a constraint that considers a simple minded worst case scenario.

⁴See the next subsection for a review of the literature.

2. Outline

We start in Section 3 with a simple benchmark model in which the exchange rate is determined by the uncovered interest parity condition. Agents can borrow and lend freely a domestic bond as well as a foreign bond, and their objective is to maximize expected profits. In this setup the expected depreciation of domestic currency must equal the interest rate differential

$$E_t(f_{t+1}) - f_t = r_t - r_t^f \quad (2.1)$$

where f_t is the log exchange rate. Like in rational expectations models, in order to allow the agent to compute $E_t(f_{t+1})$, it is assumed that the interest rate differential $r_t - r_t^f := y_t$ follows a well specified stochastic process. Thus, agents can use Bayes law in combination with knowledge of the model to compute $E_t(f_{t+1})$. The equilibrium f_t is then obtained by solving (2.1) recursively.

It is well known that foreign exchange market anomalies cannot be explained by standard rational expectation models like the one we have just described. If one introduces risk aversion, the same holds true unless risk aversion is unreasonable high. In this paper we take a different route to generate time varying predictable excess returns.

The mixed \mathcal{O}/\mathcal{R} economy of Section 4 is meant to capture the fact that real world investors frequently refrain from taking overly large positions even if expected excess returns are high. To make the analysis simple and sharp, we consider practically the same setup as in Section 3: an agent that faces no short-selling constraints maximizes expected profits subject to a robustness constraint (RC), that we will describe below.

The basic methodological departure from the existing literature is that we endow the agent with two models of the economy: a ‘probabilistic model’ under which he estimates the mathematical expectation of excess returns and chooses the optimal portfolio, and an ‘uncertain model’ under which he computes what we call the Q_t -set of admissible portfolios that satisfies the RC .

The uncertain model resembles the benchmark model of Section 3, except that it includes dynamic misspecification patterns that are taken to be unknown. These misspecification sequences are only required to be square summable.

In contrast, the probabilistic model parametrize uncertainty. The interest rate differential as well as the exchange rate follow well specified stochastic processes. That is, under the probabilistic model the agent knows how Q_{t+i} -sets will be computed in the future and knows that the $t+i$ ’s exchange rates may be determined

at the boundary or in the interior of the Q_{t+i} -set.

The *RC* is defined by Problem \mathcal{R} . Loosely speaking, the *RC* requires that small misspecification do not lead to large losses, but permits large losses in the presence of huge misspecification. In other words, the *RC* requires that as the norm of uncertainty grows, excess returns deteriorate at a rate no greater than a number $\gamma > 0$.

During each period the agent observes the exchange rate f_t and past interest rate differentials y^t and solves the so called Problem \mathcal{O}/\mathcal{R} : maximize expected profits under the probabilistic model, subject to the constraint that the portfolio satisfies the *RC* under the uncertain model.

Since we are departing from the traditional rational expectations framework, we must address the issue of an appropriate equilibrium concept in a mixed \mathcal{O}/\mathcal{R} economy. The equilibrium concept we will propose is practically identical to the standard competitive equilibrium of rational expectations except for one difference. Typically in rational expectations models there is a consistency requirement that the conjecture of future prices must be confirmed by the future equilibrium price function. In a mixed \mathcal{O}/\mathcal{R} economy there are *two* consistency requirements, stemming from the fact that the agent can choose a portfolio in the interior or on the boundary of the Q_t -set, which in turn is derived under the uncertain model.

We ask three questions. First, when will an agent in a mixed \mathcal{O}/\mathcal{R} economy choose not to take infinite positions? Second, under which circumstances can there be predictable excess returns in equilibrium? Finally, can we generate time-varying patterns of predictable excess returns that rationalize the foreign exchange market anomalies we listed in the Introduction?

To address these issues we construct Markov equilibria in two steps. First, we characterize the Q_t -set and determine the conditions under which it is closed and bounded. Then, we construct an equilibrium exchange rate path that ensures market clearing at all times.

In order for ‘limits to arbitrage’ to arise from a desire for robustness, and the intuitive story we described above to go through, the Q_t -set must be non-empty, closed and bounded. Proposition 4.1 provides necessary and sufficient conditions on primitive parameters for this to be the case. Furthermore, it characterizes the boundary of the Q_t -set in terms of a quadratic equation. This will turn out to be key in finding a closed-form solution for the exchange rate.

Whether or not the agent will choose his portfolio in the interior or at the boundary of the Q_t -set depends on the sign of the expected excess returns. The expectation is computed using the equilibrium exchange rate process, which is

characterized by Proposition 4.3.

In order to rationalize the foreign exchange market anomalies it is necessary to have time-varying predictable excess returns along the equilibrium path. This is possible because the boundary of the Q_t -set is state-dependent. It is determined by the current exchange rate, the interest rate differential's history, the robust forecasts made by agents, and by the required degree of robustness.

Proposition 5.1 identifies sufficient conditions for expected excess returns to be time varying and strictly positive over a certain time interval. That is, the upper boundary of the Q_t -set equals the supply of the domestic bond (S) and expected excess returns are positive. One of these restrictions is that the degree of required robustness declines over time.

In Section 5 we present simulations that show that if agents behave in the way we have just described, and f_t is as characterized in Proposition 5.1, then it is possible for the exchange rate path that clears the market to be consistent with the foreign exchange market anomalies we alluded to above.

An appealing property of the model is that as the degree of robustness is reduced, the robust forecasts converge to the rational expectations forecasts and the boundaries of the Q_t -set vanish. As a result, in the limit, the equilibrium exchange rate function converges to the rational expectations one.

3. The Benchmark Economy

In this section we present a simple rational expectations model in which the exchange rate is determined by the celebrated uncovered interest parity condition. Our objective is to have a familiar benchmark. In latter sections we will use this benchmark to evaluate the effects of introducing model uncertainty and robustness considerations, as well as to illustrate the issues involved in defining an equilibrium in a mixed \mathcal{O}/\mathcal{R} setup. Some readers may wish to skip this section as the material is standard.

Consider a representative agent who can invest in either a domestic bond or a foreign bond. A domestic bond purchased at time t pays $\exp(r_t)$ units of domestic currency at $t + 1$, while a foreign bond pays $\exp(r_t^*)$ units of foreign currency. There is a fixed supply of the domestic bond equal to $S \geq 0$, while the foreign bond has a perfectly elastic supply.

Uncertainty originates from the fact that the interest rate differential ($y_t := r_t - r_t^*$) is random. In this section we will assume that this uncertainty can be parametrized in terms of the following stochastic process.

$$\begin{aligned} y_j &= x_j + \sigma_v \tilde{v}_j, & j &= \{1, \dots, T\} \\ x_{j+1} &= ax_j + \sigma_w \tilde{w}_j, & x_0 &= 0, \quad |a| < 1 \end{aligned} \quad (3.1)$$

The disturbances are independent and identically distributed. For each j , the disturbance \tilde{v}_j is a realization of a random variable that has a standard normal distribution. Furthermore, $\tilde{v}_j \in \mathbb{R}$. The same holds true for \tilde{w}_j .

According to (3.1) the interest rate differential is hit by transitory as well as persistent disturbances. However, the agent cannot distinguish one from the other. At time t he observes only the history $\{y_j\}_{j=1}^t := y^t$, and additionally knows that the effect of a persistent disturbance on y_t decays at rate a .⁵

We will denote the log exchange rate by f_t . As usual, an increase in f_t corresponds to a depreciation of the domestic currency. We close the model by postulating that there is a final time $T + 1$, at which $y_{T+1} = 0$ and f_{T+1} is exogenously given by $\beta_{T+1} \in \mathbb{R}$. We assume that β_{T+1} is a realization of a random variable with a finite first moment.

Like in standard rational expectations models, to ensure that the representative agent can compute the mathematical expectation of excess returns, in addition to knowledge about (3.1), we endow him with a model of how future exchange rates will be determined. We assume that the conjecture has a Markovian form:

$$\tilde{f}_{t+i}^{\mathcal{B}}(b) = b_{1,t+i}y_{t+i} + b_{2,t+i}E_{t+i}(x_{t+i+1}) + b_{3,t+i}, \quad t + i \leq T \quad (3.2)$$

In equilibrium the parameter vector b^* must be such that the conjecture is consistent with the equilibrium exchange rate function.

For further reference we state the problem solved by the representative agent.

Problem \mathcal{O} . *Given the current exchange rate f_t and history $\{y_j\}_{j=1}^t$, choose a portfolio, $q_t \in \mathbb{R}$, in order to maximize the expected value of next period's wealth, under the interest rate differential process (3.1), and the conjectured next period's exchange rate function.*

In the benchmark economy an equilibrium consists of a conjecture $\tilde{f}_{t+1}^{\mathcal{B}}(b_{t+1})$, an exchange rate function $f_t^{\mathcal{B}}(y^t; \tilde{f}_{t+1}^{\mathcal{B}}(b_{t+1}))$, and a portfolio strategy $q_t^*(f_t, y^t)$, such that during every period, taking the exchange rate as given $q_t^*(f_t, y^t)$ solves Problem \mathcal{O} ; the domestic bond market clears; and the conjecture is consistent:

⁵This state-space representation of the interest rate differential can be interpreted in terms of the Dornbusch (1976) model.

$f_t^{\mathcal{B}}(y^t; \tilde{f}_{t+1}^{\mathcal{B}}(b_{t+1}^*)) = \tilde{f}_t^{\mathcal{B}}(b_t^*)$ for all $t \leq T$. That is, there is a fixed point of the mapping from the agent's conjecture to the model that generates exchange rates. This is simply the definition of equilibrium used in standard rational expectations models.

The solution to Problem \mathcal{O} entails taking infinitely large short or long positions unless there are no expected excess returns:

$$E_t(\rho_{t+1}, f_t) := f_t - E_t(f_{t+1}) + y_t = 0 \quad (3.3)$$

That is, in a risk-neutral setup expected devaluation equals the forward premium: $E_t(f_{t+1}^{\mathcal{B}}) - f_t = y_t$. This is the uncovered interest parity condition.⁶

In order to compute $E_t(f_{t+1})$ the agent uses the fact that y^t is generated by stochastic process (3.1), and that next period's exchange rate will be given by $\tilde{f}_{t+1}^{\mathcal{B}}(b_{t+1})$. Bayes law then implies that $E(y_{t+i}|y_1, \dots, y_t) = a^{i-1}\hat{x}_{t+1}$ where \hat{x}_{t+1} is given by the Kalman filter recursion⁷

$$\hat{x}_{j+1} = a\hat{x}_j + ak_j[y_j - \hat{x}_j], \quad \hat{x}_1 = 0. \quad (3.4)$$

The gain and the variance of the estimator are given, respectively, by:

$$k_j = \frac{Z_j}{Z_j + \sigma_v^2}, \quad Z_{j+1} = \frac{a^2}{Z_j^{-1} + \sigma_v^{-2}} + \sigma_w^2, \quad Z_1 = \sigma_w^2$$

We show in the Appendix that the consistency requirement $f_t^{\mathcal{B}}(y^t; \tilde{f}_{t+1}^{\mathcal{B}}) = \tilde{f}_t^{\mathcal{B}}(y^t)$ is satisfied if and only if $\beta_t^* = E_t(\beta_{t+1}^*)$, $b_{1,t}^* = -1$, and $b_{2,t}^* = -\frac{1-a^{T-t}}{1-a}$. Thus, the equilibrium exchange rate function is

$$f_t^{\mathcal{B}}(y^t) = -y_t - \phi_t \hat{x}_{t+1} + \beta_t \quad (3.5)$$

⁶ A foreign bond purchased at time t pays $\left(1 + \frac{r^*}{n}\right)^n$ units of the foreign currency at $t+1$, while a domestic bond pays $\left(1 + \frac{r_t}{n}\right)^n$ units of domestic currency. Under continuous compounding the uncovered interest parity condition is: $\exp(r_t) = \frac{F_{t+1}}{F_t} \exp(r^*)$, where $\lim_{n \rightarrow \infty} \left(1 + \frac{r_t}{n}\right)^n = \exp(r_t^i)$, and F_t is the exchange rate. Equation (3.3) follows by taking logs and setting $y_t := r_t - r_t^*$.

⁷ To see this, note that each disturbance \tilde{w}_j and \tilde{v}_j belongs to the set of all possible realizations of a random variable $z \sim N(0, 1)$, except $\{-\infty, \infty\}$. Moreover, $\Pr(z = -\infty) = \Pr(z = +\infty) = 0$.

where,

$$\beta_t = E_t(\beta_{t+1}) \quad \text{and} \quad \phi_t = \frac{1 - a^{T-t}}{1 - a}$$

This function says that the exchange rate appreciates if there is an increase either in the current interest rate differential or in the forecast of future differentials (i.e., $\sum_{i=1}^{T-t} E_t(y_{t+i}) = \sum_{i=1}^{T-t} a^{i-1} \hat{x}_{t+1} = \phi_t \hat{x}_{t+1}$). Since $|a| < 1$, (3.5) converges to the familiar formula $f_t^B = \beta_t - y_t - \frac{1}{1-a} \hat{x}_{t+1}$ for large T .

As is well known, this exchange rate determination model cannot explain the most salient foreign exchange anomalies.

4. The ‘Mixed \mathcal{O}/\mathcal{R} ’ Economy

In the rational expectations model of Section 3 agents take an ‘on the average’ approach in selecting portfolios. Given the means and variances of disturbances, agents make forecasts of future interest rates by setting future shocks equal to their expected value (zero in our benchmark case). Thus, forecasts are $E_t(y_{t+j}) = a^j \hat{x}_t$. The underlying assumption is that although future shocks might be very large in absolute value, they will wash out and are uncorrelated.

The point of departure of the robust approach is that agents recognize the possibility that there might be model misspecification. This might arise from modeling errors, parameter variation, etc. In a robust setup agents make forecasts using a ‘guaranteed relative performance’ approach instead of an on the average approach. An important method to tackle robustness issues that was developed during the 1980s is the so called \mathcal{H}_∞ -control.

In this paper we consider a hybrid optimal/robust setup in which agents, who can borrow and lend freely *optimally* select their portfolio from a set that satisfies a certain *robustness constraint*. The idea is that agents want to ensure that small but unknown misspecifications don’t lead to large losses. Thus, when faced with a seemingly profitable opportunity, they will behave conservatively and will not choose an overly large position. As we shall see, the desire for robustness does not imply that agents are inactive in the market and simply stay in bed.⁸

We implement this idea by posing a mixed \mathcal{O}/\mathcal{R} problem in which an agent has two models of the economy: an ‘uncertain model’ under which he computes what

⁸We would like to emphasize at the outset that the robustness constraint will not require that realized profits be greater than a certain level under all circumstances, so it is not a constraint that only considers the worst case scenario.

we will call the Q_t -set of admissible portfolios; and a ‘probabilistic model’ under which he estimates the mathematical expectation of excess returns and chooses the optimal portfolio. The uncertain model allows for the existence of unknown misspecifications. In contrast, the probabilistic model parametrizes uncertainty.

4.1. The Uncertain Model

Agents construct the robust portfolio set by using an uncertain model of the economy that allows for a quite general model uncertainty. In contrast to the Bayesian approach this model uncertainty is not confined to cases that can be parametrized in a probabilistic way.

Since we are considering a multiperiod asset pricing problem, the agent’s uncertain model must include representations of the interest rate differential process, as well as of future exchange rates under the existence of unknown time-varying misspecifications.

Consider first the representation of the interest rate differential. The agent takes the view that (3.1) is simply a linear approximation to the true process, which might be a high order non-linear dynamic process. Since the true model is unknown, there is no reason to believe that misspecification patterns can be represented by sequences of i.i.d. disturbances $\{\tilde{v}_j, \tilde{w}_j\}$ as in (3.1). Instead, the agent represents the interest rate differential process by adding to (3.1) *totally unknown sequences* $\{\Delta_j^a, \Delta_j^w, \Delta_j^v\}_{j=0}^{T+1}$, that allow for a very wide range of time-varying misspecifications in the disturbances, as well as in the trend component. That is, instead of (3.1) we now have $y_j = x_j + \Delta_j^v + \sigma_v \tilde{v}_j$ and $x_{j+1} = a[1 + \Delta_j^a]x_j + \Delta_j^w + \sigma_w \tilde{w}_j$. Thus, under the uncertain model⁹

$$\begin{aligned} y_j &= x_j + \sigma_v v_j, & v_j &:= \tilde{v}_j + \Delta_j^v \sigma_v^{-1} \\ x_{j+1} &= ax_j + \sigma_w w_j, & w_j &:= \tilde{w}_j + a\Delta_j^a x_j + \Delta_j^w \sigma_w^{-1} \\ & & x_0 &= 0 \end{aligned} \tag{4.1}$$

Since there is a myriad of potential well-behaved true models, the agent allows the misspecification sequences to be unknown. The only requirement imposed on

⁹Throughout this paper we will consider unstructured uncertainty. We assume that the parameters (a, σ_w, σ_v) are known and that the initial value of the unobservable state x_1 is unknown. We represent this uncertainty by setting $x_0 = 0$, so that x_1 is given by the first element of the disturbance sequence: $x_1 = \sigma_w w_0$.

$\{\Delta_j^a, \Delta_j^w, \Delta_j^v\}_{j=0}^{T+1}$ is that they be square summable:

$$\|\Delta^s\|_{2,[0,T]}^2 = \sum_{j=0}^{T+1} (\Delta_j^s)^2 < \infty, \quad \text{for } s = \{a, v, w\} \quad (4.2)$$

This is actually a weak requirement and allows for quite a large set of misspecifications, including those with complicated dynamic patterns. Note that if we set $\Delta_j^a = \Delta_j^w = \Delta_j^v = 0$ for all j , then (4.1) reduces to the benchmark model (3.1).¹⁰

Let us consider now the representation of future exchange rates. In order to ensure that it is not ‘anything goes’ under the uncertain model, the conjecture is restricted to have the same form as the equilibrium exchange rate function of the benchmark economy (3.5). That is, for $t+i \leq T$

$$\tilde{f}_{t+i}^u(y^{t+i}) = -y_{t+i} - \phi_{t+i} \mathcal{F}_{t+i}(x_{t+i+1}) + u_{t+i}, \quad \|u\|_{2,[1,T]} < \infty \quad (4.3)$$

The first term is equal to the one in (3.5), and the second term simply replaces the conditional expectation $E_{t+i}(x_{t+i+1})$ in (3.5) by the robust state forecast $\mathcal{F}_{t+i}(x_{t+i+1})$.¹¹ The agent recognizes that the first two terms in (4.3) are simply an approximation, or that several events that he does not anticipate might take place (institutional changes, supply shocks, etc.). Since he is using a robust method, he does not represent this uncertainty in a probabilistic way. Instead, he represents this unmodelled uncertainty with an *unknown disturbance sequence* u^T , which is only required to be square-summable. Lastly, in order to forecast next period’s exchange rate, the agent represents next period’s state forecast as follows

$$\mathcal{F}_t(\mathcal{F}_{t+1}(x_{t+2})) = a\mathcal{F}_t(x_{t+1}) + \sigma_w w_{t+1} \quad (4.4)$$

where w_{t+1} is an unknown disturbance with finite energy.

Armed with the uncertain model (4.1)-(4.4), the agent determines the robust portfolio set (\mathbf{Q}_t -set) by solving the following feasibility problem.

¹⁰To illustrate what we mean by unknown time-varying misspecifications suppose that the true interest rate differential process has two state variables: $\bar{y}_t = \bar{x}_{1,t} + \bar{x}_{2,t}$, where $\bar{x}_{1,t+1} = a_{11}\bar{x}_{1,t} + a_{12}\bar{x}_{2,t} + \tilde{w}_t$ and $\bar{x}_{2,t+1} = a_{21}\bar{x}_{1,t} + a_{22}\bar{x}_{2,t} + \tilde{w}_t$. Although this system is linear, the misspecification sequences $\{\Delta_j^a, \Delta_j^w, \Delta_j^v\}$ associated with the univariate representation in (4.1) are quite complicated and highly correlated with the state.

¹¹We could have also considered a conjecture with undetermined coefficients: $\tilde{f}_{t+i}^u(y^{t+i}) = \kappa_{1,t+i}y_{t+i} + \kappa_{2,t+i}\mathcal{F}_{t+i}(x_{t+i+1}) + u_{t+i}$, and then determine the vectors κ that belong to an equilibrium. As will become clear latter, doing that does not add any insights and would complicate notation.

Problem \mathcal{R} . Find the set of portfolios that satisfy the following ‘robustness constraint’ (RC):

$$\frac{m - q_t[f_t - f_{t+1}(\omega) + y_t]}{\|\omega\|_{2,[0,t+1]}^2} \leq \gamma_t^2 \quad (4.5)$$

for all non-zero disturbance sequences $\{\omega_{j+1}\}_{j=0}^t \in l_{2,[0,t+1]}$ that are consistent with observations $\{y^t, f_t\}$ under the uncertain model (4.1)-(4.4).

The norm of the unknown disturbances is given by¹² $\|\omega\|_{2,[0,t+1]}^2 := \sum_{j=0}^{t+1} [w_j^2 + v_j^2] + u_{t+1}^2$.

We would like to emphasize that by imposing the RC we are not assuming the existence of self-imposed short-selling constraints (i.e., limits to arbitrage). In fact, as we shall see, the \mathbf{Q}_t -set might be unbounded. Proposition 4.1 characterizes this set and specifies when is that limits to arbitrage will indeed arise.

It is not transparent what the \mathbf{Q}_t -set is by simple inspection because (4.5) depends on disturbance sequences $\{\omega_j\}_{j=0}^{t+1}$ that are not known. Unlike the benchmark model, there is no probabilistic characterization of these disturbances. Thus, we cannot use standard Bayesian filtering. Instead, we will characterize the \mathbf{Q}_t -set using \mathcal{H}_∞ -control techniques.

Before solving this problem we will provide the economic intuition. Roughly speaking, the RC requires that small misspecifications don’t lead to large profit losses, but permits bad portfolio performance in the presence of large misspecifications. In other words, the RC simply requires that the performance of the portfolio does not deteriorate too fast as the amount of uncertainty increases. This is much milder than requiring that the portfolio has a guaranteed performance under all circumstances.

The term $q_t[f_t - f_{t+1}(\omega) + y_t] \equiv q_t \rho_{t+1}(\omega)$ can be interpreted as an index of *realized excess returns*. For instance, in the event of an exchange rate depreciation in excess of the interest rate differential ($f_{t+1} - f_t > y_t$), realized excess returns will be positive if the portfolio is short in the domestic bond and long in the foreign bond (i.e., $q_t < 0$ and $q_t^f > 0$). The parameter m can be interpreted as an index of *desired excess returns*. It can either be negative, positive or zero. Lastly, the denominator in (4.5) is a measure of the *amount of uncertainty*. This uncertainty

¹²For any finite T , the unknown disturbance sequence $\{v_j, w_j\}_{j=0}^T$ defined by (4.1) belongs to the $l_{2,[0,T]}$ space. This follows from the fact that $l_{2,[0,T]}$ is a linear space, $(\tilde{v}_j, \tilde{w}_j) \in l_{2,[0,T]}$ and $\Delta_j^s \in l_{2,[0,T]}$.

can reflect misspecification in either the interest rate differential process or in the exchange rate formation mechanism. Unlike the benchmark model (3.1)-(3.2), there might exist unknown misspecifications that cannot be parametrized (i.e., (4.1)-(4.4)).

It is important to note that the RC does *not* imply that under all circumstances the realized excess returns index must exceed the desired level m ! Obviously, the RC holds if $q_t \rho_{t+1}(\omega) \geq m$. However, the RC requirement is much milder: when there is very little misspecification (i.e., $\|\omega\|_{2,[0,t+1]}^2 \rightarrow 0$) the excess returns index should not be too far below the desired level m . In contrast, when there is a lot of uncertainty, the excess returns index can indeed be much lower than m . In this latter case the RC simply requires that the index of realized excess returns should not deteriorate at a rate greater than γ_t^2 as the disturbances' norm increases. This is why γ_t^{-1} is an index of *required robustness*. As we shall see, this index will play a key role. In the limit when $\gamma_t \rightarrow \infty$, the equilibrium exchange rate function will converge to the benchmark function (3.5).

For illustrative purposes let us interpret $\|\omega\|_{2,[0,t+1]}^2$ as an index of market turmoil or uncertainty faced by all agents in a given market. Under this view there is a sense in which the RC is consistent with the existence of certain payment schemes for money managers based on relative performance. When the market is in turmoil or in a trading range, many money managers will perform badly. In these circumstances, an individual manager will not be penalized by his principal if his portfolio does not perform well.

Second, under this view we can see that the RC *does not* imply that the investor will 'stay everyday in bed' and take no positions whatsoever, fearing that no matter what he does, he will always lose. In fact, the opposite is true. On the one hand, if q_t were set to zero and $\|\omega\|_{2,[0,t+1]}^2 \rightarrow 0$, then (4.5) would be violated: a money manager that unilaterally kept out of a 'clearly good market' might have difficulties keeping his job. On the other hand, RC permits realized excess returns to be quite low, and even negative if there is a lot of turmoil in the market.

Third, the RC is consistent with the notion that investors are unwilling to take very large positions that might bankrupt them in normal times. As we shall see, under some restrictions on parameters, the RC will be satisfied exactly when q_t belongs to a certain closed and bounded set $Q_t^\gamma(f_t, y^t)$.

4.1.1. Differences Between Problem \mathcal{R} and Standard \mathcal{H}_∞ –Control

The name \mathcal{H}_∞ -control derives from the fact that in the robust control literature the objective often considered is an \mathcal{H}_∞ -norm.¹³ Using the notation of our model, the \mathcal{H}_∞ -norm typically considered in that literature would be: $H_t^{rc}(\gamma) := \sup_{\omega \in \Omega_t} \frac{\sum_{j=1}^{t+1} [h_j(x_j, y_j, q_j)]^2}{\|\omega\|_{2,[0,t+1]}^2} \leq \gamma^2$, where $h(\cdot)$ is a C^1 function. In order to establish a link between H_t^{rc} and Problem \mathcal{R} , note that if $\{y_j\}_{j=1}^t \neq 0$, the *robustness constraint* in Problem \mathcal{R} is equivalent to $H_t^e := \sup_{\omega \in \Omega_t} \frac{m - q_t \rho_{t+1}(\omega_{t+1})}{\|\omega\|_{2,[0,t+1]}^2} \leq \gamma_{t+1}^2$. These two objects, although similar, differ in several respects.

First, in standard \mathcal{H}_∞ –control the cost index $h_j(\cdot)$ does not contain forward looking variables. In contrast, in a portfolio selection problem it is essential to include future prices. Since prices next period depend on prices during the period following the next and so on, it is necessary to specify how it is that agents represent forward looking variables. This is why we had to specify the so called uncertain model (4.1)-(4.4).

Second, the summation in the numerator H_t^{rc} starts at $j = 1$, while the numerator in H_t^e only contains current and future variables. This is because standard \mathcal{H}_∞ –control considers the cumulative cost. In contrast, in macroeconomics and finance typically the past does not enter into the objective function of agents. Thus, it does not make sense to start the summation in H_t^e 's numerator at $j = 1$ if the problem is being solved at time $j = t$.

Third, the numerator in H_t^e is not squared like in H_t^{rc} . We could square it, but that would not make a lot sense in a portfolio selection context. The idea behind the RC is to ensure that realized excess returns are not too low in normal times. Squaring the numerator would imply that the investor would not like profits to be too high in normal times! In engineering and in some economic problems the objective is typically to minimize the distance to a certain target, or the effort used to control a system. Thus it makes sense to square the numerator.

¹³In particular, let \mathcal{G}_χ be a linear operator that maps an input sequence $\{\omega_j\}_{j=0}^{t+1}$ to a certain objective under control policy χ . The \mathcal{H}_∞ induced norm of the operator \mathcal{G}_χ is then defined as

$$\|\mathcal{G}_\chi \omega\|_\infty = \sup_{\omega \in l_{2,[0,t+1]}, \omega \neq 0} \frac{\|\mathcal{G}_\chi \omega\|_{2,[0,t+1]}}{\|\omega\|_{2,[0,t+1]}}$$

where $\|\omega\|_{2,[0,t+1]}$ is the l_2 -norm of the sequence ω . The ‘optimal \mathcal{H}_∞ Problem’ is to determine $\underline{\gamma} = \inf_{\chi \in \mathcal{Q}} \|\mathcal{G}_\chi \omega\|_\infty$, and to find the control policy χ^* that attains $\underline{\gamma}$.

Last, we have posed the problem as a *feasibility problem*. This is different from the so called ‘optimal \mathcal{H}_∞ Problem’ which consists of finding the lowest possible γ (i.e., $\underline{\gamma} = \inf H_t^{rc}$), and the control policy that attains $\underline{\gamma}$.

4.2. The Probabilistic Model

As we have seen, the uncertain model allows for misspecification in the representation of the interest rate differential as well as in the representation of next period’s exchange rate. In contrast, the probabilistic model we are now going to describe resembles the rational expectations model of Section 3. It parametrizes all uncertainty in terms of *known probability distributions*, and it endows the agent with knowledge of the process that generates futures exchange rates.¹⁴

Consider an agent who assumes that the interest rate differential is generated by the benchmark process (i.e., (3.1)), and knows that during each period the demand for the domestic bond may be set either at the boundary or in the interior of the Q_t -set. Furthermore, he knows the likelihood of each event. That is, he knows that with some probability α_t the exchange rate at time t will be such that the boundary of the Q_t -set will equal the supply of the domestic bond. Denoting this exchange rate by $f_t^{\mathcal{H}_\infty}(y^t)$, we have that

$$\partial Q_t^\gamma(f_t^{\mathcal{H}_\infty}, y^t) = S \quad (4.6)$$

Meanwhile, with probability $1 - \alpha_t$ the exchange rate at time t , $f_t^\mathcal{E}(y^t)$, will be such that expected excess returns are zero:

$$E_t(\rho_{t+1}, f_t^\mathcal{E}) = 0 \quad (4.7)$$

where $E_t(\rho_{t+1}, f_t^\mathcal{E}) := f_t^\mathcal{E} - E_t(f_{t+1}) + y_t$. Like the rational expectations model of Section 3, in order for the agent to be able to compute this expectation we need to endow him with a conjecture of how future exchange rates will be determined. We consider a conjecture that has the same Markovian form as the one in the benchmark model.

$$\tilde{f}_{t+i}^{pr}(\varphi, \psi) = \begin{cases} \tilde{f}_{t+i}^{\mathcal{H}_\infty}(\varphi) = \varphi_{1,t+i}y_{t+i} + \varphi_{2,t+i}\hat{x}_{t+i+1} + \varphi_{3,t+i} & \text{pr. } \alpha_{t+i} \\ \tilde{f}_{t+i}^\mathcal{E}(\psi) = \psi_{1,t+i}y_{t+i} + \psi_{2,t+i}\hat{x}_{t+i+1} + \psi_{3,t+i} & \text{pr. } 1 - \alpha_{t+i} \end{cases} \quad (4.8)$$

¹⁴Recall that the agent estimates the expectation of excess returns and selects his portfolio from the robust portfolio set under the probabilistic model.

In equilibrium, the vectors φ and ψ must be such that the consistency requirement listed below, in the definition of equilibrium, is satisfied.¹⁵

Summing up, under the probabilistic model the interest rate differential process is given by (3.1) and the representation of future exchange rates is (4.8). Thus, under the probabilistic model the expectation of $t + 1$'s exchange rate using information available at t is

$$E_t[f_{t+1}(\varphi, \psi)] = \alpha_{t+1}E_t[\tilde{f}_{t+1}^{\mathcal{H}_\infty}(\varphi; y^{t+1})] + [1 - \alpha_{t+1}]E_t[\tilde{f}_{t+1}^{\mathcal{E}}(\psi; y^{t+1})].$$

4.3. The Agent's Problem

The problem solved by the representative investor is a combination of the two auxiliary problems we have defined.

Problem \mathcal{O}/\mathcal{R} . *Given the exchange rate f_t and the history of interest rate differentials $\{y_j\}_{j=1}^t$, choose a portfolio that solves Problem \mathcal{O} under the probabilistic model, subject to the constraint that the portfolio solves Problem \mathcal{R} under the uncertain model (i.e., it belongs to the \mathcal{Q}_t -set).*

Here \mathcal{H}_∞ -control is used to characterize the set of feasible portfolios, while classical optimal control is used to select the 'optimal portfolio.' We consider that this problem captures the way in which real world money managers make decisions. They exploit profit opportunities as long as the portfolio is contained within reasonable bounds. However, they will not contemplate taking overly large positions, even though there are positive expected excess returns under some baseline model. This is specially true in new markets where expected returns can be very high, but they are unknown territory. Problem \mathcal{O}/\mathcal{R} allows the investor to choose his portfolio in order to maximize his expected utility as long as the portfolio he chooses is contained within the \mathcal{Q}_t -set. However, any portfolio outside this set is not admissible to the investor even though it might have a higher expected return under the probabilistic model. Note that Problem \mathcal{O}/\mathcal{R} implicitly defines a class of lexicographic preferences for the representative investor.

4.4. Equilibrium Concept

We will consider an equilibrium concept that is practically identical to the standard competitive equilibrium of rational expectations models, like the one in Sec-

¹⁵As we shall see, in general it is not possible for the equilibrium exchange rate function to equal $\tilde{f}_t^{\mathcal{H}_\infty}(y^t)$ for all t , or equal $\tilde{f}_t^{\mathcal{E}}(y^t)$ for all t .

tion 3, except for an additional consistency requirement.

Definition *An equilibrium of the \mathcal{O}/\mathcal{R} economy is an exchange rate function $f_t^*(y^t)$, two conjectures: $\tilde{f}_{t+i}^{\mathcal{H}\infty}(y^{t+i}, \psi_{t+i}^*)$ and $\tilde{f}_{t+i}^{\mathcal{E}}(y^{t+i}, \phi_{t+i}^*)$, a sequence α^T with $\alpha_j \in [0, 1]$, and a portfolio strategy $q_t^*(f_t, y^t)$, such that for all $t \in [1, T]$:*

1. *Taking the exchange rate as given, $q_t^*(f_t, y^t)$ solves Problem \mathcal{O}/\mathcal{R} .*
2. *There is market clearing: $q_t^*(f_t^*(y^t), y^t) = S$.*
3. *The exchange rate function $f_t^*(y^t)$ satisfies*

$$f_t^*(y^t) = \begin{cases} f_t^{\mathcal{H}\infty}(y^t) & \text{pr. } \alpha_t \\ f_t^{\mathcal{E}}(y^t) & \text{pr. } 1 - \alpha_t \end{cases} \quad (4.9)$$

where $\alpha_t \in [0, 1]$, $f_t^{\mathcal{H}\infty}(y^t)$ is defined by (4.6) and $f_t^{\mathcal{E}}(y^t)$ by (4.7).

4. *The following consistency requirements are satisfied.¹⁶*
 $\tilde{f}_t^{\mathcal{H}\infty}(\varphi^*; y^t) = f_t^{\mathcal{H}\infty}(y^t; \tilde{f}_{t+1}^u)$ and $\tilde{f}_t^{\mathcal{E}}(\psi^*; y^t) = f_t^{\mathcal{E}}(y^t; \tilde{f}_{t+1}^{\mathcal{H}\infty}, \tilde{f}_{t+1}^{\mathcal{E}})$.

The new element in this definition is point (4). It says that exchange rate realizations must confirm the conjectures under which $E_t(\rho_{t+1}, f_t)$ is computed. Since the exchange rate function will equal either $f_t^{\mathcal{H}\infty}$ or $f_t^{\mathcal{E}}$, two consistency requirements are needed.

For illustrative purposes note that if, for all t , the robustness constraint were totally relaxed ($\gamma_t = \infty$), then $\alpha_t = 0$ for all t . Consequently, the consistency requirement would simply be $f_t^*(y^t, \tilde{f}_{t+1}^{\mathcal{E}}) = \tilde{f}_t^{\mathcal{E}}(y^t, \psi_t^*)$. Clearly, in this case $f_t^{\mathcal{E}}(y^t)$ would equal the benchmark $f_t^{\mathcal{B}}(y^t)$ in (3.5).

4.5. The Robust Portfolio Set

In this subsection we solve Problem \mathcal{R} . This entails estimating the set that contains the unknown disturbance sequence $\omega^t := (w^t, v^t)$. We do this by making use of

¹⁶In order to make explicit the fact that the agent constructs the R_t -set under the uncertain model, and computes $E_t(\rho_{t+1}, f_t)$ under the probabilistic model, we will write $f_t^{\mathcal{H}\infty}(y^{t+1})$ as $f_t^{\mathcal{H}\infty}(y^t; \tilde{f}_{t+1}^u)$ and $f_t^{\mathcal{E}}(y^{t+1})$ as $f_t^{\mathcal{E}}(y^t; \tilde{f}_{t+1}^{\mathcal{H}\infty}, \tilde{f}_{t+1}^{\mathcal{E}})$.

the fact that if $\{y_j\}_{j=1}^t \neq 0$, the RC is satisfied if and only if

$$J(q_t, y^t) : = \sup_{\substack{\{\omega_j\}_{j=0}^{t+1} \in l_{2,[0,t+1]} \\ \{\omega_j\}_{j=0}^{t+1} \neq \mathbf{0}}} \{m - q_t \rho_{t+1}(\omega_{t+1}) - \gamma_t^2 \|\omega\|_{2,[0,t+1]}^2\} \leq 0$$

(4.10)

subject to : $x_j + \sigma_v v_j = y_j, \quad j = 1, \dots, t$

Note that if the supremum in (4.10) is bounded and the maximizing sequence $\{\omega_j\}_{j=0}^{t+1}$ is unique, then we can generate an \mathcal{H}_∞ estimate of the unobservable state x_{t+1} . This will allow us characterize the Q_t -set in terms of only the current exchange rate and past y_j 's. As it stands, Problem (4.10) seems quite complicated because disturbance sequences are not restricted to follow any specific process, and can be highly correlated. We solve this problem in the Appendix by breaking it into three simple sub-problems as in Tornell (2000), and Basar and Bernhard (1991).

The basic idea behind the solution method is to assume temporarily that the value of the state at $t+1$ (x_{t+1}) equals a certain value x . The first sub-problem is to find the maximally malevolent sequence of past disturbances (from the perspective of objective (4.10)) that are consistent with history $\{y_j\}_{j=1}^t$ and that bring the unobservable state from $x_0 = 0$ to the certain value x at time $t+1$. The second sub-problem determines the disturbance ω_{t+1} under the assumption that the unobservable state x_{t+1} takes the value x . Lastly, the third sub-problem generates the estimate of next period's state $\mathcal{F}_t(x_{t+1})$ and exchange rate $\mathcal{F}_t(f_{t+1})$. This decomposition can be carried out because the dynamic system we are considering is Markovian.

The solution to problem (4.10) is given by the following two propositions. The first characterizes the Q_t -set for a given forecast of the state. Proposition 4.2 provides the robust forecast of the state.

Proposition 4.1 (R_t -set). *Given the exchange rate realization f_t , and the interest rate differential observations y^t , a portfolio satisfies the robustness constraint under the uncertain model, if and only if q_t belongs to the Q_t -set, defined by¹⁷*

$$Q_t^\gamma(f_t, y^t) = \left\{ q_t \in \mathbb{R} \mid J(q_t, f_t) \equiv \frac{1}{4} \Gamma_{t+1} q_t^2 - \Lambda_t q_t + M_{t+1} \leq 0 \right\},$$

¹⁷The function $l_{t+1}(y^t)$ is given by equation (7.9). It is part of the forward dynamic programming value function that determines the \mathcal{H}_∞ estimates of $\{\omega_{j+1}\}_{j=0}^{t-1} : W_{t+1}(x) = -k_t[x - \hat{x}_t]^2 - l_{t+1}$.

$$\begin{aligned}
M_{t+1} &\equiv m - l_{t+1}(y^t) \\
\Gamma_{t+1} &\equiv \gamma_{t+1}^{-2} [1 + \sigma_v^2 + \phi_t^2 \sigma_w^2 - \phi_t^2 Z_{t+1}] \\
\Lambda_t &\equiv f_t + y_t + \phi_t \mathcal{F}_t(x_{t+1}; q_t), \quad \phi_t \equiv \frac{1 - a^{T-t}}{1 - a}
\end{aligned} \tag{4.11}$$

where Z_{t+1} is given by recursion (3.4).

- The Q_t -set is non-empty if and only if $\Lambda_t^2 \geq \Gamma_{t+1} M_{t+1}$.
- The Q_t -set is closed and bounded if and only if $\Gamma_{t+1} \geq 0$.

This Proposition will prove to be quite useful because it has converted the RC in Problem \mathcal{R} into a simple condition that q_t must satisfy. Namely, the Q_t -set consists of all q_t 's such that the parabola $J(q_t, f_t)$ is non-positive, as illustrated in Figure A.

Two points are worth highlighting. First, Proposition 4.1 makes clear that imposing the RC is not the same as assuming that there are limits to arbitrage. In fact, when parameters are such that $\Gamma_{t+1} < 0$, the agent can take infinite positions and still satisfy the RC (panel (b) of Figure A). Second, staying in bed (i.e., having a zero position) is, in general, not compatible with the RC. Clearly, whenever $M_{t+1} > 0$, the Q_t -set does not contain $q_t = 0$ (panel (a) of Figure A).

The quadratic equation $J(q_t, f_t)$ has very attractive properties. First, whether it is convex in q_t only depends on parameters, through Γ_{t+1} . Second, the current exchange rate only enters through Λ_t . This implies that, if it exists, the boundary of the Q_t -set is a monotonic function of f_t . This property will allow us to pin down uniquely the equilibrium exchange rate.

Note that Λ_t also contains the robust forecast of the unobservable state $\mathcal{F}_t(x_{t+1}; q_t)$. This is because in order to solve Problem (4.10), the agent must have a robust estimate of next period's exchange rate $\mathcal{F}_t(f_{t+1})$. Recall that under the uncertain model, $\mathcal{F}_t(f_{t+1}) = -y_{t+1} - a\phi_{t+1}\mathcal{F}_t(x_{t+1}) + u_{t+1}$, where u_{t+1} is an unknown disturbance. Thus, to fully characterize the Q_t -set the agent needs to have a robust estimate of next period's state. The next proposition provides such an estimate.

Proposition 4.2. *The agent's robust forecast of the unobservable component of the interest rate differential is*

$$\mathcal{F}_t(x_{t+1}; q_t) = \hat{x}_{t+1} - \frac{1}{2} \gamma_{t+1}^{-2} \phi_{t+1} Z_{t+1} q_t \tag{4.12}$$

where \hat{x}_{t+1} and Z_{t+1} are given by recursion (3.4).

Interestingly, $\mathcal{F}_t(x_{t+1}; q_t)$ contains \hat{x}_{t+1} and Z_{t+1} which also appear in the updating formulas of the benchmark economy. Recall that under the benchmark model, \hat{x}_{t+1} is the conditional expectation of the interest rate differential $E_t(y_{t+1})$, and Z_{t+1} is the variance of this estimate. The formal similarity of the two forecasting formulas is noteworthy, given the diametrically different specifications of uncertainty.

In contrast to standard rational expectations estimates of latent variables, robust estimates are functions of the portfolio chosen by the investor. Equation (4.12) says that if the agent is long in the domestic bond ($q_t > 0$), his \mathcal{H}_∞ forecast about the forward premium is more pessimistic than the rational expectations forecast. In contrast, his \mathcal{H}_∞ forecast would be more bullish if he were short ($q_t < 0$).¹⁸

An attractive property of the model is that if the agent does not care about robustness, the \mathcal{H}_∞ and rational expectations forecasts are equal: if we let $\gamma_{t+1} \rightarrow \infty$, (4.12) equals the Kalman filter (3.4).

4.6. Solution to Problem \mathcal{O}/\mathcal{R}

Proposition 4.1 has defined the Q_t -set in terms of a quadratic equation that depends only on observable variables. The solution to Problem \mathcal{O}/\mathcal{R} is now straightforward. Namely, if expected excess returns are positive, the agent buys as much domestic bonds as allowed by the Q_t -set. If they are negative, he goes as short as possible. That is,

$$q_t^*(f_t, y^t) = \begin{cases} \bar{q}_t^\gamma(f_t, y^t) & \text{if } E_t(\rho_{t+1}; f_t) > 0 \\ \underline{q}_t^\gamma(f_t, y^t) & \text{if } E_t(\rho_{t+1}; f_t) < 0 \\ q \in Q_t^\gamma(f_t, y^t) & \text{if } E_t(\rho_{t+1}; f_t) = 0 \end{cases} \quad (4.13)$$

where $\bar{q}_t^\gamma(f_t, y^t)$ and $\underline{q}_t^\gamma(f_t, y^t)$ are the upper and lower boundaries of the Q_t -set $Q_t^\gamma(f_t, y^t)$.

Expectations are formed under the probabilistic model. That is, agents compute $E_t(\rho_{t+1}, f_t)$ under the view that the interest rate differential is generated by

¹⁸Note that in any equilibrium $q_t = S$. Thus, along any equilibrium path the robust forecast is $\mathcal{F}_t(x_{t+1}; S) := \mathcal{F}_t(x_{t+1})$.

the benchmark process (3.1) with known probability distribution. Moreover, they know the model that generates future exchange rates (4.8).

Note that if the RC were totally relaxed for all periods, the Q_t -set would always be unbounded and the expectations operator would be the same as in the benchmark economy. Thus, the solution of Problem \mathcal{O}/\mathcal{R} would coincide with that of problem \mathcal{O} in the benchmark economy.

4.7. Markov Equilibria

In a ‘Markov equilibrium’ the exchange rate depends only on the current value of y_t and the current estimate of the state ($E_t(x_{t+1})$ or $\mathcal{F}_t(x_{t+1})$). The next Proposition identifies conditions on parameters under which a Markov equilibrium exists and it exhibits the exchange rate function in closed form. The proof is in the Appendix.

Proposition 4.3 (Markov Equilibria). *Consider a mixed \mathcal{O}/\mathcal{R} economy.*

- *There exist ME along which expected excess returns can be different from zero if the Q_t -set is bounded ($\Gamma_{t+1} > 0$) and the domestic bond’s supply satisfies $|S| > |\underline{S}_t(y^t)|$ for some t . The exchange rate function is $f_T^*(y^T) = f_T^\varepsilon(y^T)$ and for $t < T$*

$$f_t^*(y^t) = \begin{cases} f_t^{\mathcal{H}_\infty}(y^t) & \text{if } \begin{cases} E_t(\rho_{t+1}, f_t^{\mathcal{H}_\infty}) \geq 0, \Gamma_{t+1} > 0 \text{ and } \bar{q}_t(f_t^{\mathcal{H}_\infty}) = S \\ E_t(\rho_{t+1}, f_t^{\mathcal{H}_\infty}) \leq 0, \Gamma_{t+1} > 0 \text{ and } \underline{q}_t(f_t^{\mathcal{H}_\infty}) = S \end{cases} \\ f_t^\varepsilon(y^t) & \text{otherwise} \end{cases} \quad (4.14)$$

where $f_t^{\mathcal{H}_\infty}(y^t)$, $f_t^\varepsilon(y^t)$ and the threshold \underline{S}_t are given by (4.15), (4.16) and (7.13a), respectively. The conjectures that support this equilibrium are (4.3) and (4.8) with $\psi_{1,t}^* = \varphi_{1,t}^* = -1$, $\psi_{2,t}^* = \varphi_{2,t}^* = -\frac{1-a^{T-t}}{1-a}$, and $\varphi_{3,t}^* = \frac{M_{t+1}}{S} + \zeta_t \gamma_{t+1}^{-2} S$.

- *If $\Gamma_{t+1} \geq 0$ and $|S| \leq |\underline{S}_t(y^t)|$ for all t , a Markov equilibrium need not exist.*
- *If $\Gamma_{t+1} < 0$ for all t , there are zero excess returns in all ME and the exchange rate is given by (3.5).*

This Proposition demonstrates the existence of equilibria along which there can be some periods during which there are non-zero *predictable excess returns*. In the next section we will impose further parameter restrictions that will ensure the existence of either negative or *positive predictable excess returns* on a given

time interval. This will allow us to rationalize the forward premium puzzle and delayed overshooting.

In the remainder of this subsection we provide an heuristic derivation of Proposition 4.3. In equilibrium, taking the exchange rate f_t as given, agents choose $q_t^*(f_t, y^t)$ according to (4.13), and the market for domestic bonds clears. The key point is that the exchange rate has two functions: it determines the boundary of the Q_t -set as well as the sign of expected excess returns. In order to ensure market clearing at time t , f_t must adjust so that one of three events occurs: (i) $\bar{q}_t^\gamma(f_t^*, y^t) = S$ and $E_t(\rho_{t+1}, f_t^*) \geq 0$; (ii) $\underline{q}_t^\gamma(f_t^*, y^t) = S$ and $E_t(\rho_{t+1}, f_t^*) \leq 0$; or $E_t(\rho_{t+1}, f_t^*) = 0$ and $S \in Q_t^\gamma(f_t^*, y^t)$.

It follows that a necessary condition for expected excess returns to be different from zero is that the Q_t -set is non-empty and either its lower or upper boundary is finite. Since the Q_t -set is defined by a quadratic equation (i.e., (4.11)), this condition holds only if $\Gamma_{t+1} \geq 0$. Otherwise, $\bar{q}_t(f_t, y^t) = \infty$ and $\underline{q}_t(f_t, y^t) = -\infty$ for all f_t .

Figure A, drawn for the case $\Gamma_{t+1} > 0$ and $S > 0$, helps understand the intuition. The Q_t -set is composed of all q_t 's such that the parabola $J(q_t, f_t) = 0$ is non-positive. When $\Gamma_{t+1} > 0$, this parabola is strictly convex. Thus, the Q_t -set is connected and its two boundaries are finite. They are equal to the real roots of $J(q_t, f_t) = 0$.¹⁹ Note that by varying f_t we generate a family of parabolas that intersect only once: at the vertical axis. Since an increase in f_t makes the parabola rotate clockwise, there is a unique exchange rate (call it $f_t^{\mathcal{H}\infty}(y^t)$) for which a boundary of the Q_t -set equals S .

The fact that there is a function $f_t^{\mathcal{H}\infty}(y^t)$ that equalizes a boundary of the Q_t -set to S does not mean that $f_t^{\mathcal{H}\infty}(y^t)$ belongs to an equilibrium. In addition we need to ensure that when the upper(lower) boundary is equal to S , the agent is willing to demand as much(little) as possible of the domestic bond (i.e., expected excess returns must be non-negative(non-positive) if $f_t = f_t^{\mathcal{H}\infty}$). Can we determine which boundary of the Q_t -set will equal S ? The answer is yes. It is evident from Figure A that at each point in time only one of the boundaries of the Q_t -set can be equal to S . In fact, there is a threshold $\underline{S}_t(y^t)$, such that if $S \geq \underline{S}_t(y^t)$, then $\bar{q}_t^\gamma(f_t^{\mathcal{H}\infty}, y^t) = S$. In contrast, if $S \leq \underline{S}_t(y^t)$, then $\underline{q}_t^\gamma(f_t^{\mathcal{H}\infty}, y^t) = S$.

It follows that when $\Gamma_{t+1} > 0$ the exchange rate function $f_t^{\mathcal{H}\infty}(y^t)$ belongs to an equilibrium if either (i) $S \geq \underline{S}_t(y^t)$ and $E_t(\rho_{t+1}, f_t^{\mathcal{H}\infty}) \geq 0$; or (ii) $S < \underline{S}_t(y^t)$

¹⁹This equation has real roots if and only if $\Lambda_t(f_t)^2 \geq M_{t+1}\Gamma_{t+1}$. If this condition is violated, the R_t -set is empty.

and $E_t(\rho_{t+1}, f_t^{\mathcal{H}\infty}) \leq 0$. Unfortunately, there is no guarantee that either of these conditions will be satisfied. In principle, there may be some histories y^t for which the sign of expected excess returns might be the opposite of what is required in (i) or (ii). To ensure market clearing at all times, it is necessary (although not sufficient) to introduce a second auxiliary exchange rate function, $f_t^{\mathcal{E}}(y^t)$, with the property that expected excess returns under the probabilistic model are zero (i.e., $E_t(\rho_{t+1}, f_t^{\mathcal{E}}) = 0$). This is why Proposition 4.3 allows the equilibrium exchange rate be equal to either $f_t^{\mathcal{H}\infty}(y^t)$ or $f_t^{\mathcal{E}}(y^t)$.

Proposition 4.3 requires that $|S| > |\underline{S}_t(y^t)|$ when $\Gamma_{t+1} \geq 0$ to ensure that the supply of the domestic bond S belongs to the Q_t -set. This, in turn, ensures that exchange rate function (4.14) induces market clearing at all times. To see why this is so consider the case $S > \underline{S}_t(y^t) > 0$ illustrated in panel (a) of Figure B, and denote by f_t^a the exchange rate associated with the parabola that crosses the horizontal axis at S . If $E_t(\rho_{t+1}, f_t^a) \geq 0$, f_t^a is part of an equilibrium because the agent sets his demand for the domestic bond equal to the upper boundary of the Q_t -set, which is equal to the supply: $q_t^* = \bar{q}_t^{\gamma}(f_t^a, y^t) = S$. If instead $E_t(\rho_{t+1}, f_t^a) < 0$, let the equilibrium exchange rate be f_t^b , defined by $E_t(\rho_{t+1}, f_t^b) = 0$. Since $f_t^b > f_t^a$, the f_t^b -parabola is obtained by rotating the f_t^a -parabola clockwise, and S is in the interior of the set $Q_t^{\gamma}(y^t, f_t^b)$. Combining this with the fact that the agent is indifferent about the value of q_t^* if expected excess returns are zero, we conclude that f_t^b belongs to an equilibrium. That is, any $S > \underline{S}_t(y^t) > 0$ belongs to the Q_t -set whenever $E_t(\rho_{t+1}, f_t^{\mathcal{H}\infty}) < 0$ and $f_t^*(y^t) = f_t^{\mathcal{E}}(y^t)$. The reader can verify that the same argument can be made for the case $S < \underline{S}_t(y^t) \leq 0$.

To see why a Markov equilibrium may not exist if $|S| \leq |\underline{S}_t(y^t)|$ for some t , consider panel (b) of Figure B. Now f_t^a is the exchange rate that makes the lower bound of the Q_t -set equal to S (i.e., $\underline{q}_t^{\gamma}(f_t^a, y^t) = S$). If $E_t(\rho_{t+1}, f_t^a) > 0$, we cannot set the exchange rate at a more appreciated level (i.e., $f_t^b < f_t^a$) to ensure zero expected excess returns and still clear the market. This is because the relevant portion of the f_t^b -parabola would lay above the f_t^a -parabola. Thus, S would not belong to the set $Q_t^{\gamma}(y^t, f_t^b)$.

It is straightforward to derive the functions $f_t^{\mathcal{H}\infty}(y^t)$ and $f_t^{\mathcal{E}}(y^t)$. Since the conjecture of future exchange rates that the representative agent uses to generate the Q_t -set is given by (4.3)-(4.4), it can be verified that for $t < T$, $f_t^{\mathcal{H}\infty}(y^t)$ is given

by

$$\begin{aligned} f_t^{\mathcal{H}_\infty}(y^t) &= -y_t - \phi_t \mathcal{F}_t(x_{t+1}) + \frac{1}{4} \Gamma_{t+1} S + \frac{M_{t+1}}{S} \\ &= -y_t - \phi_t \hat{x}_{t+1} + \zeta_t \gamma_{t+1}^{-2} S + \frac{M_{t+1}}{S} \end{aligned} \quad (4.15)$$

where $\phi_t := \frac{1-a^{T-t}}{1-a}$ and

$$\zeta_t := \frac{1}{4} [1 + \sigma_v^2 + \phi_{t+1}^2 \sigma_w^2 + \phi_{t+1}^2 Z_{t+1}] > 0$$

The second line in (4.15) follows by replacing $\mathcal{F}_t(x_{t+1})$ by (4.12). It is important that we can represent $f_t^{\mathcal{H}_\infty}(y^t)$ in two equivalent ways in order to show that the two consistency requirements are satisfied. To determine $f_t^\mathcal{E}(y^t)$ recall that since expectations are computed under the probabilistic model, (4.9) implies that $f_t^\mathcal{E}(y^t) = \alpha_{t+1} E_t(\tilde{f}_{t+1}^{\mathcal{H}_\infty}) + [1 - \alpha_{t+1}] E_t(\tilde{f}_{t+1}^\mathcal{E}) - y_t$. Using conjecture (4.8) and imposing the consistency requirement we have that in any Markov equilibrium

$$f_t^\mathcal{E}(y^t; \tilde{f}_{t+1}^{\mathcal{H}_\infty}, \tilde{f}_{t+1}^\mathcal{E}) = -y_t - \phi_t \hat{x}_{t+1} + \psi_{3,t}^* \quad (4.16)$$

$$\psi_{3,t}^* = \begin{cases} E_t \left(\alpha_{t+1} \left[S \gamma_{t+1}^{-2} \zeta_t + \frac{M_{t+1}}{S} \right] + [1 - \alpha_{t+1}] \psi_{3,t+1}^* \right) & t < T \\ \beta_T & t = T \end{cases}$$

As a closing remark, note that when constructing the \mathbb{Q}_t -set the representative agent knows that $f_{t+1}^*(y^{t+1})$ will be given by either (4.15) or (4.16). However, he is not sure which function will realize at $t+1$ and about the process that generates y_{t+1} . Since he is using a robust method, he does not represent this uncertainty in a probabilistic way (i.e., using the sequence α^t and (3.1)). Instead, he considers process (4.1) for y_{t+1} and his forecast about next period's forecast of the unobservable state is given by (4.4). He then represents $t+1$'s exchange rate with $\tilde{f}_{t+1}^u(y^{t+1})$ defined in (4.3).

Figure 1 illustrates the performance of a portfolio under the benchmark rational expectations economy of Section 3 and the mixed economy of this section. The graphs in Figure 1 summarize the results of the simulations we describe in the appendix. For each period, panel (a) depicts the worst performance, across all simulations, of $H_{t+1}^e = \frac{m-S[f_t - f_{t+1} + y_t]}{\sum_{h=0}^{t+1} [w_h^2 + v_h^2]}$. As we can see, according to this criterion, at all times the normalized excess returns are greater under the mixed

economy than in the rational expectations economy. That is, for all t the index H_{t+1}^e corresponding to the mixed economy lays below the corresponding index for the rational expectations economy. The same pattern is observed in panel (b), which depicts the worst performance, across simulations, of the difference between the indexes of desired returns and realized returns $m - S[f_t - f_{t+1} + y_t]$.

5. Rationalizing Foreign Exchange Market Anomalies

In this section we will show that if there is a time interval in which the exchange rate is determined by the upper boundary of the Q_t -set (i.e., $f_t^* = f_t^{\mathcal{H}_\infty}$), then it is possible to rationalize the anomalies mentioned in the Introduction. When this occurs *positive expected excess returns* exist in equilibrium, and it is possible to generate a negative Fama regression coefficient, as well as unconditional delayed overshooting.

Although Proposition 4.3 characterizes an equilibrium along which expected excess returns can be non-zero, it does not specify the conditions under which there is a given time interval during which expected excess returns will actually be either positive or negative. In this section we provide sufficient conditions for expected excess returns to be strictly positive or negative on a given time interval. The sufficient condition is that S be large and that required robustness (γ_t^{-1}) decreases over time:

$$\gamma_{t+1} = g\gamma_t, \quad g > 1 \quad (5.1)$$

We can think of a market where agents are initially very uncertain about what is the ‘true model,’ and they try to be very robust against ‘unknown misspecifications’. Over time, required robustness is reduced and eventually converges to zero. As a result, *ceteris paribus*, the size of the Q_t -set increases allowing agents to better exploit the profit opportunities that their benchmark model indicates exist. This view is consistent with the notion that investors tend to be very cautious when investing in new markets. Over time the degree of conservatism declines.

There is a deeper issue: what determines the level of γ_t ? This we leave for future research.

The following Proposition exhibits an equilibrium along which there is a time interval in which the RC binds (so $f_t^* = f_t^{\mathcal{H}_\infty}$) and expected excess returns are guaranteed to be strictly positive.

Proposition 5.1 (Switching Equilibrium). *If the degree of robustness (γ_t^{-1}) decreases fast enough and the bond's supply $S > 0$ is large enough, there exists a switching time $\tau > 0$ such that, for all $t < \tau < \bar{\tau}$ the equilibrium exchange rate is determined by the Q_t -set's upper boundary and expected excess returns are positive*

$$f_t^*(y^t) = -y_t - \phi_t \hat{x}_{t+1} + \frac{M_{t+1}}{S} + \gamma_{t+1}^{-2} \zeta_t S \quad (5.2)$$

Meanwhile, for $t \geq \tau$, the exchange rate is given by (4.14). The upper bound on the switching time is

$$\bar{\tau} = \inf \left\{ t < T - 1 \mid S < \underline{S}_t, \gamma_{t+1}^{-2} \zeta_t S + \frac{M_{t+1}}{S} < \psi_{3t} \text{ or } E_t(\rho_{t+1}, f_t^{\mathcal{H}_\infty} | f_{t+1}^* = f_{t+1}^{\mathcal{H}_\infty}) < 0 \right\}$$

This Proposition is a special case of Proposition 4.3. It says that as long as none of the events listed in $\bar{\tau}$'s definition has occurred, $f_t^*(y^t)$ is determined by the Q_t -set's upper boundary. However, once one of these events occurs $f_t^*(y^t)$ reverts to the function defined in Proposition 4.3, where expected excess returns can have any sign.

The intuition for why expected excess returns will be positive during a long time interval is as follows. An increasing path for γ_t means that required robustness falls over time. This implies that, ceteris paribus, the upper boundary of the Q_t -set will increase. As a result, the $t + 1$ demand for domestic bonds will be greater than the demand at time t . Since the supply of the domestic bond is fixed, the $t + 1$ exchange rate will have to appreciate (i.e., $f_{t+1} < f_t$). If this expected appreciation is big enough, expected excess returns at time t will be positive. Agents at time t know that at $t + 1$ the demand for the domestic bond will be set at the upper boundary of the Q_{t+1} -set because they know that $\gamma_{t+2} > \gamma_{t+1}$. Equation (5.3) below makes precise the conditions on g and S under which this argument holds.

We would like to note that since γ_t grows over time, if terminal time T is large enough, the equilibrium exchange rate function will converge to the standard rational expectations formula (3.5). This is an attractive property of the model.

Proposition 5.1 follows directly from Proposition 4.3. Suppose that at any $t \leq \tau$ the exchange rate is $f_t^{\mathcal{H}_\infty}(y^t)$ as given by (4.15). Since $S \geq \underline{S}_t$, the representative agents sets his demand for domestic bonds equal to the upper boundary of the Q_t -set if expected excess returns are non-negative (i.e., $E_t(\rho_{t+1}, f_t^{\mathcal{H}_\infty}) \geq 0$). It is straightforward to compute this expectation because along the equilibrium characterized by Proposition 5.1, $\alpha_{t+1} = 1$. That is, at $t + 1$ the exchange rate

will also be determined by the upper boundary of the Q_{t+1} -set. Consequently, $E_t(\rho_{t+1}, f_t^{\mathcal{H}_\infty}) = f_t^{\mathcal{H}_\infty} - E_t(\tilde{f}_{t+1}^{\mathcal{H}_\infty}(\varphi_{t+1}^*) + y_t)$. Using (4.8) and (5.2) we have that

$$E_t(\rho_{t+1}, f_t^{\mathcal{H}_\infty} \mid f_{t+1}^* = \tilde{f}_{t+1}^{\mathcal{H}_\infty}(\varphi_{t+1}^*)) = \gamma_t^{-2}[\zeta_t - g^{-2}\zeta_{t+1}]S + \frac{E_t(l_{t+2}) - l_{t+1}}{S} \quad (5.3)$$

Note that (5.3) has been derived under the probabilistic model. That is, agents set $\alpha_{t+1} = 1$, and compute the mathematical expectation $E_t(\tilde{f}_{t+1}^{\mathcal{H}_\infty}(\varphi_{t+1}^*))$ under the benchmark model for the interest rate differential (3.1). Then, using Bayes' updating equation (3.4), agents set $E_t(y_{t+1}) = \hat{x}_{t+1}$ and $E_t(\hat{x}_{t+2}) = a\hat{x}_{t+1}$.

In order for agents' conjecture to be confirmed in equilibrium (i.e., $f_{t+1}^* = \tilde{f}_{t+1}^{\mathcal{H}_\infty}$), it is necessary that expected excess returns at $t+1$ be non-negative. Here is where the decreasing robustness requirement kicks in: if g is high enough, the term in brackets is positive for all t . This term can be negative only if $Z_{t+1} < g^{-2}Z_{t+2}$. Recall that Z_j is given by the Kalman filter and it converges at a decreasing rate to some positive number $Z = \frac{a^2}{Z^{-1} + \sigma_v^{-2}} + \sigma_w^2$. Thus, it is sufficient to set $g^2 > \frac{Z_1}{Z_0}$ to ensure that $[\zeta_t - g^{-2}\zeta_{t+1}]$ is positive for all $t > 0$. Furthermore, since the first term in (5.3) is increasing in S , while the second is decreasing in S , there is a positive switching time τ such that (5.3) is positive if $t < \tau - 1$ and S is large enough.

Consider now period $t = \tau - 1$. Since the exchange rate will be given by (4.14) for all $t \geq \tau$, expected excess returns at time $\tau - 1$ are: $E_{\tau-1}(\rho_\tau, f_{\tau-1}^{\mathcal{H}_\infty}) = \frac{M_\tau}{S} + \gamma_{\tau-1}^{-2}\zeta_{\tau-1}S - \psi_{3\tau}$. From the definition of τ , it follows that this expression is non-negative.

5.1. The Forward Premium Puzzle

The typical 'Fama regression' regresses the exchange rate depreciation on the forward premium

$$f_{t+1} - f_t = \beta_0 + \beta_{fama}y_t + \varepsilon_t \quad (5.4)$$

Since the uncovered interest parity condition (3.3) holds in standard rational expectations models, like that of Section 3, the estimate of β_{fama} should not be statistically different from one ($\hat{\beta}_{fama} = 1$).²⁰

The forward premium puzzle is that in almost all data sets the estimates of β_{fama} are less than one, and in many cases they are negative. A negative $\hat{\beta}_{fama}$ implies that there is a negative covariance between exchange rate changes and the

²⁰The estimate of β_0 might be different from zero if agents are risk averse.

forward premium (i.e., $cov(f_{t+1} - f_t, y_t) < 0$). This is puzzling because it means, for instance, that when the U.S. interest rate is above the German one, the Dollar tends to *appreciate* relative to the Mark.

In this subsection we investigate whether the exchange rate process (5.2) can generate negative coefficients in the Fama regression.

In the simulations shown in Figures 1 and 2 artificial forward premium sequences (y_t) are generated using the benchmark process (3.1). For each of the hundred forward premium sequences we compute two exchange rate sequences (f_t): one corresponding to the rational expectations model (3.5) and the other corresponding to the mixed economy (3.5). In both cases the agent uses the parameters of the data generating process (a, σ_v, σ_w) to make forecasts. That is, there is *no missperception* in either the benchmark model or the probabilistic model.

Figure 1 graphs $f_{t+1} - f_t$ against y_t . Panel (a) corresponds to the benchmark model in which uncovered interest parity holds and the exchange rate is given by (3.5). As expected, one can see a positive correlation between the two variables. Panel (b) corresponds to the exchange rate characterized in Proposition 5.1, in which required robustness falls over time. Here we can see a negative correlation between $f_{t+1} - f_t$ and y_t .

Figure 2 plots the estimates of β_{fama} that correspond to each of the 100 simulations. In Panel (a), which corresponds to the rational expectations model, a majority of $\hat{\beta}_{fama}^{re}$'s are in a neighborhood of one (the average estimate is +0.843). In contrast, a majority of $\hat{\beta}_{fama}^{mixed}$'s corresponding to our mixed model are negative (the average estimate is -0.301!).

The exchange rate process (3.5) in Proposition 5.1 can generate positive expected excess returns and a negative β_{fama} because agents set their demand for the domestic bond at the upper boundary of the Q_t -set at all times, and this boundary is increasing (on average) over time. This is because the degree of required robustness is declining (i.e., γ_t is increasing).

We would like to note that an increasing γ_t is not necessary for positive and time-varying expected excess returns. Recall that Proposition 4.3 tells us that even when γ_t is constant, there are equilibria in which expected excess returns are positive if S is sufficiently large. The additional restrictions on S and the growth of γ_t imposed in Proposition 5.1 make the simulations simpler because they ensure that expected excess returns will be strictly positive on a certain time interval.

5.2. Delayed Overshooting

Another anomaly is the so called delayed overshooting puzzle. Under rational expectations, an increase in the U.S. interest rate induces an immediate appreciation of the Dollar. If the interest rate is expected to mean-revert, the exchange rate must overshoot at impact in order to generate expected depreciation along the transition path and ensure that the uncovered interest parity condition is satisfied. Eichenbaum and Evans (1985) have found that the typical impulse response of exchange rates to a monetary shock does not follow this path. Instead, after the initial appreciation the exchange rate continues to appreciate for several months in response to a contractionary monetary shock that increases the interest rate differential.

Figure 3 depicts three impulse response functions (IRF) to an interest rate shock, associated with exchange rate function (3.5). As we can see, the unconditional IRF has the hump shape found by Eichenbaum and Evans (1985) and others.

The IRF to a persistent shock of size δ (f_t^{pers}) is generated by feeding into exchange rate function (5.2) the interest rate differential sequence generated by a persistent shock of size $w_0 = \frac{\delta}{\sigma_w}$. Similarly, the IRF to a transitory shock (f_t^{tr}) is generated by setting the transitory shock $v_1 = \frac{\delta}{\sigma_v}$. Lastly, the IRF to an unconditional shock to y_1 of size δ is given by

$$f_t^{unc} = q[f_t^{pers}] + [1 - q][f_t^{tr}], \quad q = \frac{\sigma_w^2}{\sigma_w^2 + \sigma_v^2}$$

Figure 3 graphs the average across 100 simulations of these functions.

6. Review of the Literature

Robust Control has been a very active area of research since the 1980s. The point of departure of the robust approach is the recognition that even in physics there is no such thing as the correct model. Thus, one has to recognize that inevitably any model has some misspecification, and representing it in a probabilistic way does not guarantee robustness.

Robust Control was developed in order to tackle control problems in which attaining some sort of ‘guaranteed performance’ is important. This stands in contrast to stochastic optimal control that takes an ‘on the average’ approach. There are several approaches to robust control, like for instance \mathcal{H}_∞ -control, risk-

sensitive control, and minimum entropy.²¹ While the \mathcal{H}_∞ framework is appropriate to ensure robust stability, it might entail some sacrifice in performance. Therefore, a continuing research effort has sought to bring an optimality criterion back to the picture and combine it with robust considerations. Such a framework is the so called ‘mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control.’ The basic problem considered is that of choosing a control policy in order to minimize an upper bound of the expected loss under the assumption that the disturbances are Gaussian (the \mathcal{H}_2 -norm), subject to the constraint that the controlled system satisfies a robustness constraint under the assumption that the disturbances are square summable (the \mathcal{H}_∞ -norm). See Bernstein and Haddad (1989), Khargonekar and Rotea (1991), and Zhou, Glover, Bodenheimer, and Doyle (1994).

The approach of this paper is similar to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ approach. We use \mathcal{H}_∞ -control techniques to characterize the admissible portfolio set, not to determine the optimal portfolio policy. We then allow agents to choose a portfolio to maximize their expected utility under a probabilistic model, subject to the portfolio belonging to the admissible set. The advantage of this mixed approach is that it does not force agents to be overly conservative.

In economics, the notion that not all uncertainty can be parametrized in a probabilistic way goes back to Knight (1921), who distinguished between quantifiable ‘risks’ and unknown ‘uncertainties’. Gilboa and Schmeidler (1989) present an axiomatic decision making framework where this distinction is made. Epstein and Schneider (2001) extend this framework to a dynamic setup. Dow and Ribeiro (1992), Epstein and Wang (1994), and Epstein and Zin (1989) have used this framework to analyze investment decisions and asset pricing. Recently, Lars Hansen and Tom Sargent have considered the robust control approach in economics. See for instance Hansen, Sargent and Tallarini (1999) and Hansen, Sargent and Wang (2000). The latter paper, as well as Tornell (2000) share with the present paper the fact that there is a latent variable that determines agents’ payoffs, and agents must estimate it using available information.

The forward premium puzzle has been documented for many data sets over different countries and time periods. Surveys are provided by Lewis (1995) and Engel (199X).

The delayed overshooting puzzle was documented by Eichenbaum and Evans (1985). Gourinchas and Tornell (2000) have shown that this puzzle can be rationalized by invoking another puzzle present in the data: there is a systematic

²¹Basar and Bernhard (1995) and Zhou, Doyle and Glover (1996) are excellent references.

missperception in the forecasts of interest rate differentials. The forecasts implicitly assume that shocks are more transitory than what they actually are. The results in this paper suggest that there is a sense in which this *missperception* can be rationalized in a setup where agents try to be robust against *misspecification*.

7. Appendix

Derivation of (3.5).

For $t = T$, (3.2) and (3.3) imply that $f_t^B(y^T) = \beta_T - y_T$. Thus, the consistency requirement is satisfied if and only if $b_{3,T}^* = \beta_T$, $b_{1,T} = -1$, and $b_{2,T}^* = 0$. For $t = T - 1$, we have that $f_{T-1}^B(y^{T-1}) = E_{T-1}(\tilde{f}_T^B(b_T^*)) - y_{T-1}$. Thus, $b_{3,T-1}^* = E_{T-1}(\beta_T)$, $b_{1,T-1}^* = -1$, and $b_{2,T-1}^* = -1$. Since $E_t(y_{t+1}) = \hat{x}_{t+1}$ and $E_{t+1}(\hat{x}_{t+2}) = a\hat{x}_{t+1}$, (3.3) implies that for any $t < T - 1$, $f_t^B(y^t; \tilde{f}_{t+1}^B(b_{t+1}^*)) = E_t(b_{3,t+1}^*) + [b_{1,t+1}^* + ab_{2,t+1}^*]\hat{x}_{t+1} - y_t$. It follows that the consistency requirement $f_t^B(y^t; \tilde{f}_{t+1}^B(b_{t+1}^*)) = \tilde{f}_t^B(b_t^*)$ holds for all $t \leq T$ if and only if $b_{3,t}^* = E_t(b_{3,t+1}^*)$, $b_{1,t}^* = -1$, $b_{2,t}^* = ab_{2,t+1}^* - 1$ and $b_{2,T}^* = 0$. This implies $b_{2,t}^* = \frac{1-a^{T-t}}{1-a}$. ■

Proof of Proposition 4.1.

Throughout this proof we will denote the state's and interest rate differential's trajectories generated by the disturbance subsequence $v^s := \{w_j, v_j\}_{j=0}^s$ as follows:

$$x_s = X_s(v^{s-1}), \quad y_s = Y_s(v^s), \quad s \geq 1 \quad (7.1)$$

We can then define the following subsets of disturbance sequences

$$\Omega_s := \left\{ \{v_i\}_{i=0}^{t+1} \left| \sum_{j=0}^{t+1} [w_j^2 + v_j^2] < \infty \text{ and } Y_j(v^j) = y_j, \quad \forall j \leq s \right. \right\} \quad (7.2)$$

That is, Ω_s is the set of square summable disturbance sequences $\{v_j\}_{j=0}^{t+1}$ that are compatible with history $\{y_j\}_{j=1}^s$.

For any sequence $\{y_j\}_{j=1}^s \neq 0$ the robustness constraint is satisfied if and only if $J(q_t, f_t) \leq 0$, defined by problem (4.10). We will derive $J(q_t, f_t)$ following the same procedure as Tornell (2000). First, we will assume temporarily that the unobservable state x_{t+1} takes a specific value x , and break problem (4.10) in two parts: the 'cost-to-come function' that considers the terms in (4.10) indexed $j = 0, \dots, t$; and the 'cost-to-go function' that includes the terms with $j = t + 1$.

Then, we determine the \mathcal{H}_∞ estimate of the state $\mathcal{F}_t(x_{t+1})$.

Cost-to-come value function

Using (7.2) and (4.10) we can express the first sub-problem as

$$\begin{aligned} W_{t+1}(x) = & \sup_{\{v_j\}_{j=0}^t \in \Omega_t} -\gamma_{t+1}^2 \sum_{j=0}^t [w_j^2 + v_j^2] \\ & \text{subject to} \quad x_{t+1} = x \end{aligned} \quad (7.3)$$

The only information an agent has about the disturbances $\{v_j\}_{j=0}^t$ is that they are square summable sequences, and that they have been generated by a dividend history $\{y_j\}_{j=1}^t$ according to the dynamic system (4.1). The solution to (7.3) is an intermediate step that allows the agent to estimate x_{t+1} . It characterizes the disturbance sequences that make (4.10) less likely to hold, given that they bring the state from $x_0 = 0$ to $x_{t+1} = x$, and are consistent with history. We will derive $W_{t+1}(x)$ by representing (7.3) as a recursive problem. In order to do this let $\Omega_s(x|y^s)$ be the set of admissible disturbance sequences that bring the state to level x at time $s + 1$, and that are consistent with history $\{y_j\}_{j=1}^s$

$$\Omega_s(x|y^s) := \{v \in \Omega_s \mid x = X_s(v^{s-1})\}, \quad s \in \{1, \dots, t\} \quad (7.4)$$

Analogously to (7.3) we can define the cost-to-come value function, conditional on information up to time s as

$$W_{s+1}(x) = \sup_{v^s \in \Omega_s(x|y^s)} -\gamma_{t+1}^2 \sum_{j=0}^s [w_j^2 + v_j^2], \quad s \in \{1, \dots, t\} \quad (7.5)$$

If (7.5) has a finite solution, it satisfies the following *forward dynamic programming* equation

$$\begin{aligned} W_{s+1}(x) = & \begin{cases} \max_{(\xi, w, v)} \{W_s(\xi) - \gamma_{t+1}^2 [w^2 + v^2]\} \\ \text{subject to} \quad x = a_s \xi + \sigma_w w \\ \quad \quad \quad y_s = \xi + \sigma_v v \end{cases} \quad s \in \{1, \dots, t\} \quad (7.6) \\ W_1(x) = & -x^2 \gamma_{t+1}^2 \sigma_w^{-2} \end{aligned}$$

Note that (ξ, w, v) corresponds to (x_s, w_s, v_s) and that x corresponds to x_{s+1} . *Forward dynamic programming* problems are solved in a similar way to standard *back-*

ward DP problems. The difference is that they are solved starting at the initial time, not the terminal time. To solve problem (7.6) we need to find a closed form solution for the value function $W_{s+1}(x)$. Since $W_{s+1}(x)$ is the supremum of a quadratic function subject to an affine constraint, it is quadratic in x . The next Lemma, which is proved in Tornell (2000), provides the solution.

Lemma 7.1 (Cost-to-Come Value Function). *For any $\gamma_{t+1} \in (0, \infty)$ the solution to (7.6) is*

$$W_{s+1}(x) = -K_{s+1}[x - \tilde{x}_{s+1}]^2 - l_{s+1}, \quad s \in \{1, \dots, t\} \quad (7.7)$$

where and $(\tilde{x}_{s+1}, K_{s+1}, l_{s+1})$ satisfy the recursion: $\tilde{x}_1 = 0$, $K_1 = \gamma_{t+1}^2 \sigma_w^{-2}$, $l_1 = 0$; and for $s \in \{1, \dots, t\}$

$$\begin{aligned} \tilde{x}_{s+1} &= a_s \tilde{x}_s + \frac{a_s}{1 + \sigma_v^2 \gamma_{t+1}^{-2} K_s} [y_s - \tilde{x}_s] \\ K_{s+1} &= \left[\frac{a_s^2}{P_s} + \gamma_{t+1}^{-2} \sigma_w^2 \right]^{-1}, \quad \text{with } P_s = \gamma_{t+1}^2 \sigma_v^{-2} + K_s \end{aligned} \quad (7.8)$$

$$\begin{aligned} l_{s+1} &= l_s + \lambda_s(\gamma_s, \hat{x}_{s+1}, \hat{x}_s, y_s, Z_s), \quad \text{where } k_i = \gamma_t^2 Z_i^{-1}, \\ \lambda_s &= \frac{-\gamma_s^2}{\frac{a^2}{\sigma_w^2} + \frac{1}{\sigma_v^2} + Z_s^{-1}} \left[\frac{a}{\sigma_w^2} \left(\frac{y_s^2}{\sigma_v^2} + Z_s^{-1} \hat{x}_s \right) \hat{x}_{s+1} + 2 \frac{y_s Z_s^{-1} \hat{x}_s}{\sigma_v^2} - \frac{y_s^2}{\sigma_v^2} \left(\frac{a^2}{\sigma_w^2} + Z_s^{-1} \right) - \hat{x}_s^2 \left[Z_s^{-1} \left(\frac{1}{\sigma_v^2} + \frac{a^2}{\sigma_w^2} \right) \right] \right] \end{aligned} \quad (7.9)$$

The cost-to-go value function

Here we maintain the assumption that $x_{t+1} = x$ and determine the \mathcal{H}_∞ forecast of $\omega_{t+1} := (v_{t+1}, w_{t+1}, u_{t+1})$. In an \mathcal{H}_∞ setup even if an agent knew the value of the unobservable state x_{t+1} , he would not forecast that either y_{t+1} will be x_{t+1} , or that the time $t+1$ estimate of x_{t+2} will be $a\hat{x}_{t+1}$. This is because in the presence of misspecification there is no reason to believe that the disturbance ω_{t+1} will be identically zero. In fact the \mathcal{H}_∞ forecast of ω_{t+1} is dependent on the portfolio chosen. This is formalized by the following problem in which the unobservable state x_{t+1} is again assumed to take the value x .

$$\begin{aligned} V_{t+1}(x, q_t, f_t) &= \sup_{\omega \in \mathcal{V}_x} \{ m - \rho_{t+1} q_t - \gamma_{t+1}^2 [w^2 + v^2 + u^2] \} \\ &\text{subject to} \quad x_{t+1} = x \\ &\quad y_{t+1} = x_{t+1} + \sigma_v v \end{aligned} \quad (7.10)$$

The set of admissible strategies for the disturbance (i.e., \mathcal{V}_x) consists of the Markov strategies $\omega_{t+1} = \omega(x, q_t)$ with $\omega(\cdot) : \mathcal{R}^2 \mapsto \mathcal{R}^3$.²² Three observations are in order. First, a major simplification relative to the original problem (4.10) is that strategies are functions of (x_{t+1}, q_t) , not of the entire history of observations $(\{y_j, q_j, f_j\}_{j=1}^t)$. In this sub-problem agents make forecasts, acting as if the state x_{t+1} takes the specific value x . Second, the disturbance ω_{t+1} has access to the realization of q_t in order to ensure robustness. Third, no hard bound has been imposed on the disturbance ω_{t+1} . As we shall see, if problem (7.10) has a solution, the disturbance will be bounded in equilibrium.

In order to solve (7.10) we use (4.3) to make the substitution $\rho_{t+1} = f_t - [\phi_0 + \phi_1(x + \sigma_v v) + \phi_{2t}(ax + \sigma_w w) + u] + y_t$, where

$$\phi_0 = 0, \quad \phi_1 = -1, \quad \phi_{2t} = \frac{1 - a^{T-t}}{1 - a}$$

Since the first order conditions are sufficient for a maximum, replacing the optimized value of ω in (7.10), it follows that the ‘cost-to-go value function’ is given by

$$V_{t+1}(x, q_t, f_t) = m - \bar{\Lambda}_t(x)q_t + \frac{1}{4}\gamma_{t+1}^{-2}[1 + \sigma_v^2\phi_1^2 + \sigma_w^2\phi_{2t}^2]q_t^2 \quad (7.11)$$

where $\bar{\Lambda}_t(x) \equiv f_t + y_t - [\phi_1 + a\phi_2]x - \phi_0$.

Determination of $\mathcal{F}_t(x_{t+1})$.

In order to determine the \mathcal{H}_∞ forecast of x_{t+1} and of the interest rate differential $\mathcal{F}_t(y_{t+1})$ we use the following Theorem (see Tornell (2000) for a proof).²³

Theorem 7.2. *There exists a solution to Problem \mathcal{R} if and only if there are bounded functions $W_{t+1}(x)$ and $V_{t+1}(x)$, defined by (7.5) and (7.10), which satisfy*

$$\sup_x \Upsilon(x) := \sup_x \{V_{t+1}(x) + W_{t+1}(x)\} < \infty \quad (7.12)$$

²²Markov strategies are also known as feedback strategies. These strategies are closed-loop strategies in which history matters only through its effect on the current state. See Basar and Olsder (1995).

²³This Theorem implies that one can determine whether an equilibrium exists by considering only Markov equilibria (i.e., where strategies and exchange rates only depend on the estimate of the state x_{t+1} , y_t and f_t). If a Markov equilibrium does not exist, then there exists no other equilibrium in which portfolio and disturbance strategies are more complicated functions of history $\{y_j\}_{j=1}^t$. Note, however that this Theorem does not say that there is a unique equilibrium. If a Markov equilibrium exists, there might exist other equilibria in which strategies have the same open-loop representation as the Markov strategies.

The feedback estimate of x_{t+1} is given by $\mathcal{F}_t(x_{t+1}) = \arg \max \Upsilon(x)$. Furthermore, if (7.12) holds and $\Upsilon(x)$ is strictly concave, $\mathcal{F}_t(x_{t+1})$ is unique.

The expression for $\mathcal{F}_t(x_{t+1})$ in (4.12) is obtained by replacing (7.7) and (7.11) in (7.12), by making the change of variable $Z_t := \gamma_{t+1}^{-2} K_t$, and by noting that $\tilde{x}_t = \hat{x}_t$, where \hat{x}_t is given by (3.4).²⁴ ■

Proof of Proposition 4.3.

In equilibrium, taking the exchange rate f_t as given, agents choose $q_t^*(f_t, y^t)$ according to (4.13), and the market for domestic bonds clears: $q_t^*(f_t^*, y^t) = S$. We will construct an equilibrium exchange rate function $f_t^*(y^t)$ by combining $f_t^{\mathcal{H}\infty}(y^t)$ and $f_t^{\mathcal{E}}(y^t)$ defined in (4.15) and (4.16), respectively.

Since $f_t^{\mathcal{H}\infty}$ is an exchange rate that equalizes a boundary of the Q_t -set to the domestic bond's supply (i.e., $\partial Q_t^\gamma(f_t^{\mathcal{H}\infty}, y^t) = S$), equation (4.11) implies that $f_t^{\mathcal{H}\infty}$ must satisfy $0 = J(S, f_t^{\mathcal{H}\infty}) = \frac{1}{4}\Gamma_{t+1}S_t^2 - \Lambda_t(f_t)S_t + M_{t+1}$. Since only Λ_t depends on f_t , it follows that $f_t^{\mathcal{H}\infty}(y^t)$ is given by (4.15).

In order to determine when is it that the function $f_t^{\mathcal{H}\infty}(y^t)$ can be part of an equilibrium we need to answer the following questions: (i) when will the Q_t -set be non-empty, and its largest (\bar{q}_t) and/or smallest (\underline{q}_t) boundaries be finite? (ii) Is there a unique $f_t^{\mathcal{H}\infty}$ such that $\partial Q_t^\gamma(f_t^{\mathcal{H}\infty}, y^t) = S$? (iii) For a given $S \in \mathbb{R}$, can we determine whether the highest or the lowest boundary of the Q_t -set equals S : $\bar{q}_t(f_t^{\mathcal{H}\infty}, y^t) = S$ or $\underline{q}_t(f_t^{\mathcal{H}\infty}, y^t) = S$? The answer to the last question is important because $f_t^{\mathcal{H}\infty}(y^t)$ can be part of an equilibrium only if either $\bar{q}_t(f_t^{\mathcal{H}\infty}; y^t) = S$ and $E_t(\rho_{t+1}, f_t^{\mathcal{H}\infty}) \geq 0$, or $\underline{q}_t(f_t^{\mathcal{H}\infty}; y^t) = S$ and $E_t(\rho_{t+1}, f_t^{\mathcal{H}\infty}) \leq 0$.

To answer these questions it is useful to refer to Figure A. For a given f_t , (a) the Q_t -set is non-empty when the roots of $J(q_t; f_t) = 0$ are real, and (b) at least one of the boundaries of the Q_t -set is finite when $J(q_t; f_t)$ is a convex function of q_t . Since the roots of $J(q_t, f_t) = 0$ are $\left[\Lambda_t(f_t) \pm \sqrt{\Lambda_t(f_t)^2 - \Gamma_{t+1}M_{t+1}} \right] 2\Gamma_{t+1}^{-1}$, (a) holds if and only if $\Lambda_t(f_t)^2 \geq \Gamma_{t+1}M_{t+1}$. Condition (b) holds if and only if $\Gamma_{t+1} \geq 0$, in which case the Q_t -set is connected and its boundaries are the roots of $J(q_t, f_t) = 0$:

$$\bar{q}_t(f_t; y^t) = \frac{\Lambda_t(f_t) + \sqrt{\Lambda_t(f_t)^2 - \Gamma_{t+1}M_{t+1}}}{\frac{1}{2}\Gamma_{t+1}}, \quad \underline{q}_t(f_t; y^t) = \frac{\Lambda_t(f_t) - \sqrt{\Lambda_t(f_t)^2 - \Gamma_{t+1}M_{t+1}}}{\frac{1}{2}\Gamma_{t+1}}$$

²⁴In general the kernel in (7.10) is an indefinite quadratic form in ω_{t+1} . Depending on the value of γ_{t+1} , it might or might not be concave in ω_{t+1} . This is not the case here because the numerator in the robustness constraint is linear.

Note that if $\Gamma_{t+1} < 0$, then $\bar{q}_t(f_t, y^t) = \infty$ and $\underline{q}_t(f_t, y^t) = -\infty$ for all f_t and y^t . As we shall show, the answer to (ii) is in the affirmative if $S \neq 0$, and the answer to (iii) depends on whether S is greater or smaller than a certain threshold \underline{S}_t . That is, $\bar{q}_t(f_t^{\mathcal{H}_\infty}, y^t) = S$ if $S \geq \underline{S}_t$, while $\underline{q}_t(f_t^{\mathcal{H}_\infty}, y^t) = S$ if $S \leq \underline{S}_t$. In order to determine \underline{S}_t , we consider four cases.

Case i. ($\Gamma_{t+1} > 0$ and $M_{t+1} > 0$). Since $\Lambda_t(f_t^{\mathcal{H}_\infty}) = \frac{1}{4}\Gamma_{t+1}S + \frac{M_{t+1}}{S}$, the Q_t -set is non-empty if $f_t = f_t^{\mathcal{H}_\infty}(y^t)$. This is because $\min_S[(\Lambda_t(f_t^{\mathcal{H}_\infty}))^2] = \Gamma_{t+1}M_{t+1}$, and so condition $\Lambda_t(f_t^{\mathcal{H}_\infty})^2 \geq \Gamma_{t+1}M_{t+1}$ is satisfied for all S . When $S > 0$, we have that $\Lambda_t(f_t^{\mathcal{H}_\infty}) > 0$ and both $\bar{q}_t(f_t^{\mathcal{H}_\infty}; y^t)$ and $\underline{q}_t(f_t^{\mathcal{H}_\infty}; y^t)$ are positive. Since Λ_t is increasing in f_t , it follows that \bar{q}_t is increasing in f_t , while \underline{q}_t is decreasing in f_t . Furthermore, $\bar{q}_t \geq 2\Lambda_t\Gamma_{t+1}^{-1}$ and $\underline{q}_t \leq 2\Lambda_t\Gamma_{t+1}^{-1}$. Thus, there is a unique $f_t^{\mathcal{H}_\infty}$ such that $\bar{q}_t(f_t^{\mathcal{H}_\infty}, y^t) = S > 0$ if and only if $S \geq \underline{S}_t = 2\Gamma_{t+1}^{-1}\Lambda_t(f_t^{\mathcal{H}_\infty})$. Similarly, there is a unique $f_t^{\mathcal{H}_\infty}$ such that $\underline{q}_t(f_t^{\mathcal{H}_\infty}, y^t) = S > 0$ if and only if $0 < S \leq \underline{S}_t = 2\Gamma_{t+1}^{-1}\Lambda_t(f_t^{\mathcal{H}_\infty})$. When $S < 0$ we have that $\Lambda_t(f_t^{\mathcal{H}_\infty}) < 0$ and both $\bar{q}_t(f_t^{\mathcal{H}_\infty}; y^t)$ and $\underline{q}_t(f_t^{\mathcal{H}_\infty}; y^t)$ are negative. Since now \bar{q}_t is decreasing in f_t , while \underline{q}_t is increasing in f_t , the same argument establishes that there is a unique $f_t^{\mathcal{H}_\infty}$ such that $\bar{q}_t(f_t^{\mathcal{H}_\infty}, y^t) = S < 0$ if and only if $0 > S \geq \underline{S}_t = 2\Gamma_{t+1}^{-1}\Lambda_t(f_t^{\mathcal{H}_\infty})$. Similarly, there is a unique $f_t^{\mathcal{H}_\infty}$ such that $\underline{q}_t(f_t^{\mathcal{H}_\infty}, y^t) = S < 0$ if and only if $S \leq \underline{S}_t = 2\Gamma_{t+1}^{-1}\Lambda_t(f_t^{\mathcal{H}_\infty})$. Lastly, if $S = 0$, there does not exist an $f_t^{\mathcal{H}_\infty}$.

Case ii. ($\Gamma_{t+1} > 0$ and $M_{t+1} < 0$). For all f_t the roots of $J(q_t; f_t) = 0$ are real and of opposite sign. Since $J(0, f_t) < 0$ and the largest root is increasing in f_t , there is a unique $f_t^{\mathcal{H}_\infty}$ such that $\bar{q}_t(f_t^{\mathcal{H}_\infty}, y^t) = S$ for any $S > \underline{S}_t = 0$. Similarly, there is a unique $f_t^{\mathcal{H}_\infty}$ such that $\underline{q}_t(f_t^{\mathcal{H}_\infty}, y^t) = S$ for any $S < \underline{S}_t = 0$. Lastly, if $S = 0$, there does not exist an $f_t^{\mathcal{H}_\infty}$.

Case iii. ($\Gamma_{t+1} > 0$ and $M_{t+1} = 0$). For all f_t the roots of $J(q_t; f_t) = 0$ are real; one root is zero, and the other $4\Lambda_t(f_t)\Gamma_{t+1}^{-1}$. Therefore, if $S > 0$ and $f_t = f_t^{\mathcal{H}_\infty}$, the boundaries of the Q_t -set are $\bar{q}_t(f_t^{\mathcal{H}_\infty}; y^t) = 4\Lambda_t(f_t^{\mathcal{H}_\infty})\Gamma_{t+1}^{-1}$ and $\underline{q}_t(f_t^{\mathcal{H}_\infty}; y^t) = 0$. Since $\Lambda_t(f_t^{\mathcal{H}_\infty}) = \frac{1}{4}\Gamma_{t+1}S + \frac{M_{t+1}}{S}$, we have that $\Lambda_t(f_t^{\mathcal{H}_\infty})\Gamma_{t+1}^{-1} = \frac{1}{4}S$. Thus, the market clears: $\bar{q}_t(f_t^{\mathcal{H}_\infty}; y^t) = S > 0$. Similarly, if $S < 0$, $\bar{q}_t(f_t^{\mathcal{H}_\infty}; y^t) = 0$ and $\underline{q}_t(f_t^{\mathcal{H}_\infty}; y^t) = 4\Lambda_t(f_t^{\mathcal{H}_\infty})\Gamma_{t+1}^{-1} = S$. Lastly, if $S = 0$, any exchange rate clears the market. In this case the threshold \underline{S}_t equals zero.

Case iv. ($\Gamma_{t+1} = 0$). In this case $J(q_t, f_t)$ is linear in q_t . Thus, at most one of the boundaries of the Q_t -set is finite. Since $\Lambda_t(f_t^{\mathcal{H}_\infty}) = M_{t+1}/S$, we have that $J(q_t, f_t^{\mathcal{H}_\infty}) = -\frac{M_{t+1}}{S}q_t + M_{t+1}$. When $S > 0$, $\bar{q}_t(f_t^{\mathcal{H}_\infty}) = S$ if $M_{t+1} < 0$, while $\underline{q}_t(f_t^{\mathcal{H}_\infty}) = S$ if $M_{t+1} > 0$. When $S < 0$, $\bar{q}_t(f_t^{\mathcal{H}_\infty}) = S$ if $M_{t+1} > 0$, while $\underline{q}_t(f_t^{\mathcal{H}_\infty}) = S$ if $M_{t+1} < 0$. An $f_t^{\mathcal{H}_\infty}$ does not exist if either $M_{t+1} = 0$ or $S = 0$.

Collecting the observations we have just made, it follows that if $\Gamma_{t+1} > 0$, the threshold \underline{S}_t is:

$$|\underline{S}_t| = 2\sqrt{\max\{0, M_{t+1}\Gamma_{t+1}^{-1}\}} \quad (7.13a)$$

For case i we substitute $\Lambda_t(f_t^{\mathcal{H}_\infty}) = \frac{1}{4}\Gamma_{t+1}S + \frac{M_{t+1}}{S}$ in $\underline{S}_t = 2\Gamma_{t+1}^{-1}\Lambda_t(f_t^{\mathcal{H}_\infty})$ and obtain $\underline{S}_t^2 = 4M_{t+1}\Gamma_{t+1}^{-1}$. In cases ii and iii it is clear that $\underline{S}_t = 0$.

Now, we derive $f_t^\varepsilon(y^t)$, which is defined by $E_t(\rho_{t+1}, f_t^\varepsilon) = 0$. Expectations are computed under the probabilistic model: (3.1) and (4.8). In any equilibrium conjecture (4.8) must satisfy the consistency requirements: $\tilde{f}_t^{\mathcal{H}_\infty}(y^t) = f_t^{\mathcal{H}_\infty}(y^t; \tilde{f}_{t+1}^u)$ and $\tilde{f}_t^\varepsilon(y^t) = f_t^\varepsilon(y^t; \tilde{f}_{t+1}^{\mathcal{H}_\infty}, \tilde{f}_{t+1}^\varepsilon)$. Since $f_t^{\mathcal{H}_\infty}(y^t)$ can be represented in two equivalent ways (see (4.15)), the first requirement is satisfied if $\varphi_{1,t} = -1$ and $\varphi_{2,t} = -\phi(t) := -\frac{1-a^{T-t}}{1-a}$. For the second requirement note that Proposition 4.3 exhibits an equilibrium where $\alpha_T = 0$. This implies that for $t \geq T-1$ the functions f_t^ε and \tilde{f}_t^ε are equal to those in the benchmark model: (3.5) and (3.2). That is, $\psi_{1,T}^* = -1, \psi_{2,T}^* = 0, \psi_{3,T}^* = \beta_T, \psi_{1,T-1}^* = -1, \psi_{2,T-1}^* = -1$, and $\psi_{3,T-1}^* = E_{T-1}(\psi_{3,T}^*)$. For $t < T-1$, the uncovered interest parity condition (3.3) implies $f_t^\varepsilon = \alpha_{t+1}E_t(\tilde{f}_{t+1}^{\mathcal{H}_\infty}) + [1 - \alpha_{t+1}]E_t(\tilde{f}_{t+1}^\varepsilon) - y_t$. Using (4.8) and letting $\xi_t := \frac{1}{4}\Gamma_{t+1}S + \frac{M_{t+1}}{S}$, we have that

$$\begin{aligned} f_t^\varepsilon &= \alpha_{t+1}E_t[-y_{t+1} + \phi_{t+1}\mathcal{F}_{t+1}(x_{t+2}) + \xi_{t+1}] + \\ &\quad [1 - \alpha_{t+1}]E_t[\psi_{1t+1}y_{t+1} + \psi_{2t+1}E_{t+1}(x_{t+2}) + \psi_{3t+1}] - y_t \end{aligned}$$

Since (3.4) and (4.12) imply that $\mathcal{F}_t(x_{t+1}) = \hat{x}_{t+1} - \frac{1}{2}\phi_t\gamma_{t+1}^{-2}Z_{t+1}S$, $E_t(y_{t+1}) = E_t(x_{t+1}) = \hat{x}_{t+1}$, and $E_t(E_{t+1}(\hat{x}_{t+2})) = a\hat{x}_{t+1}$, we have that

$$\begin{aligned} f_t^\varepsilon &= \alpha_{t+1}\left[-\hat{x}_{t+1} + \phi_{t+1}\left(a\hat{x}_{t+1} - \frac{\phi_{t+1}\gamma_{t+1}^{-2}Z_{t+1}S}{2}\right) + \xi_{t+1}\right] + \\ &\quad [1 - \alpha_{t+1}][\psi_{1t+1}\hat{x}_{t+1} + \psi_{2t+1}a\hat{x}_{t+1} + \psi_{3t+1}] - y_t \\ &= -y_t + \hat{x}_{t+1}[\alpha_{t+1}[-1 + a\phi_{t+1}] + [1 - \alpha_{t+1}][\psi_1 + a\psi_{2t+1}]] + \\ &\quad E_t\left(\alpha_{t+1}\left[S\gamma_{t+1}^{-2}\zeta_t + \frac{M_{t+1}}{S}\right] + [1 - \alpha_{t+1}]\psi_{3,t+1}^*\right) \end{aligned}$$

It follows from (4.8) that $\tilde{f}_t^\varepsilon(y^t) = f_t^\varepsilon(y^t; \tilde{f}_{t+1}^\varepsilon, \tilde{f}_{t+1}^{\mathcal{H}_\infty})$ for all $t \leq T$ if $\psi_{1,t}^* = -1$, $\psi_{2,t}^* = \varphi_{2,t} = -\frac{1-a^{T-t}}{1-a}$ and $\psi_{3,t}^*$ is given by (4.16). We obtain $f_t^\varepsilon(y^t)$ by replacing the vector ψ^* in (4.8).

Now, we construct the equilibrium exchange rate function $f_t^*(y^t)$. It follows

from the agent's portfolio strategy (4.13) that we can set $f_t^*(y^t)$ equal to $f_t^{\mathcal{H}_\infty}(y^t)$ if either $\bar{q}_t(f_t^{\mathcal{H}_\infty}; y^t) = S$ and $E_t(\rho_{t+1}, f_t^{\mathcal{H}_\infty}) \geq 0$, or $\underline{q}_t(f_t^{\mathcal{H}_\infty}; y^t) = S$ and $E_t(\rho_{t+1}, f_t^{\mathcal{H}_\infty}) \leq 0$. If neither of these conditions is satisfied, we can set $f_t^*(y^t)$ equal to $f_t^\varepsilon(y^t)$, provided that $S \in Q_t^\gamma$ (so that there is market clearing). As we discuss in the text, $S \in Q_t^\gamma(f_t^\varepsilon, y^t)$ if $\bar{q}_t(f_t^{\mathcal{H}_\infty}; y^t) = S$ and $E_t(\rho_{t+1}, f_t^{\mathcal{H}_\infty}) < 0$. However, $S \notin Q_t^\gamma(f_t^\varepsilon, y^t)$ if $\underline{q}_t(f_t^{\mathcal{H}_\infty}; y^t) = S$ and $E_t(\rho_{t+1}, f_t^{\mathcal{H}_\infty}) > 0$. This is because $f_t^\varepsilon < f_t^{\mathcal{H}_\infty}$ and the J-parabola in Figure B would shift upwards.

Lastly, we verify that under the uncertain model the disturbance sequences belong to $l_{2,[0,T]}$. The \mathcal{H}_∞ estimates of the unknown disturbances in (4.1) and (4.3) are $w_t^{\mathcal{H}_\infty}(q_t) = -\frac{1}{2} \frac{1-a^{T-t}}{1-a} \gamma_t^{-2} \sigma_w q_t$, $v_t^{\mathcal{H}_\infty}(q_t) = -\frac{1}{2} \gamma_t^{-2} \sigma_v q_t$, and $u_t^{\mathcal{H}_\infty}(q_t) = -\frac{1}{2} \gamma_t^{-2} q_t$. Clearly, for any finite T these sequences are square summable. ■

Simulations

The simulations corresponding to Figures 1-4 are done using the following procedure. First, we draw I sequences of disturbances $\{w_j, v_j\}_{j=0}^{\bar{T}}$ from a standard normal distribution, where $I = \bar{T} = 100$. Then, for each disturbance sequence, we generate an artificial interest rate differential sequence $\{y_j\}_{j=1}^{\bar{T}}$ using the following process: $y_j = x_j + \sigma_v v_j$, $x_{j+1} = (a + \Delta_j^a) x_j + \sigma_w w_j$, where $x_0 = 0$, $a = 0.9345$, $\sigma_v^2 = 0.05882$ and $\sigma_w^2 = 0.50756$. For Figure 1 the misspecification Δ_j^a is drawn from a normal distribution: $N(0.035, 0.012^2)$. For Figures 2-4 we set $\Delta_j^a = 0$.

We generate the interest rate differential forecasts for the benchmark and mixed models ($E_t(y_{t+1})$ and $\mathcal{F}_t(y_{t+1})$, respectively), as well as the exchange rate functions (f_t^B and $f_t^{\mathcal{H}_\infty}$) using the formulas in the text by setting $S = 10$, $m = 9$, $\gamma_0 = 1.14$, $g = 1.?$, $x_0 = 0$, $a = 0.9345$, $\sigma_v^2 = 0.05882$ and $\sigma_w^2 = 0.50756$. Since converge takes place quite fast, we set for all j , $Z_j = Z$ (the converged value). Similarly, we take terminal time $T(> \bar{T})$ to be large so that $a^{T-t} \rightarrow 0$. Thus, we set $\phi_j = \frac{1}{1-a}$.

In panel (a) of Figure 1 we plot $\max_i \left\{ \frac{m - S[f_j^{i,RE} - f_{j+1}^{i,RE} + y_j^i]}{\sum_{h=0}^{j+1} [w_h^{i2} + v_h^{i2}]} \right\}$ and $\max_i \left\{ \frac{m - S[f_j^{i,H^\infty} - f_{j+1}^{i,H^\infty} + y_j^i]}{\sum_{h=0}^{j+1} [w_h^{i2} + v_h^{i2}]} \right\}$.

In panel (b) we plot $\max_i \{m - S[f_j^{i,RE} - f_{j+1}^{i,RE} + y_j^i]\}$ and $\max_i \{m - S[f_j^{i,H^\infty} - f_{j+1}^{i,H^\infty} + y_j^i]\}$. In Figures 2-4 we compute $(f_{j+1}^{RE} - f_j^{RE})$ and $f_j^{H^\infty}$ using (3.5) and (5.2), respectively. In Figure 3 we plot the averages across the 100 simulations of $f_{j+1}^{RE} - f_j^{RE}$ and $f_{j+1}^{H^\infty} - f_j^{H^\infty}$.

References

Basar, T. and P. Bernhard, 1995, *\mathcal{H}_∞ - Optimal Control and Related Minimax Design Problems*, Birkhauser.

- Bernstein, D., and W. Haddad, 1989, "LQG Control with an \mathcal{H}_∞ Performance Bound: A Riccati Equation Approach," *IEEE Transactions on Automatic Control*, 34:293-305.
- Daw, J. and S. Ribeiro, 1992, "Uncertainty Aversion, Risk Aversion and the Optimal Choice of Portfolio," *Econometrica* 60(1), 197-204.
- Dornbusch, R., 1976, "Expectations and Exchange Rate Dynamics," *Journal of Political Economy*, 84, 1161-1176.
- Epstein, L.G., and T. Wang, 1994, "Intertemporal Asset Pricing Under Knightian Uncertainty," *Econometrica* 62(3), 283-322.
- Epstein, L. and M. Schneider, 2001, "Recursive Multiple Priors," mimeo.
- Epstein, L. and S. Zin, 1989, "Intertemporal Substitution, Risk Aversion, and the Temporal Behavior of Asset Returns: A Theoretical Framework," *Econometrica*, 57, 937-969.
- Eichenbaum, M. and C. Evans, 1985, "Some Empirical Evidence on the Effects of Monetary Policy Shocks on Exchange Rates," *Quarterly Journal of Economics*, 100, 975-1009.
- Gilboa, I., and D. Schmeidler, 1989, "Maxmin expected utility with non-unique prior," *Journal of Mathematical Economics* 18, 141-143.
- Gourinchas, P., and A. Tornell, 2000, "Exchange Rate Dynamics and Missperception," mimeo UCLA.
- Hansen, L., T. Sargent and T. Tallarini, 1999, "Robust Permanent Income and Pricing," *Review of Economic Studies*, 66(4), pp.873-907.
- Hansen, L., T. Sargent and N.E. Wang, 2000, "Robust Permanent Income and Pricing with Filtering," mimeo.
- Khargonekar, P. and M. Rotea, 1991, "Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control: A Convex Optimization Approach," *IEEE Transactions on Automatic Control*, 36:824-837.
- Knight, F., 1921, *Risk, Uncertainty and Profit*.
- Lewis, K., 1998, "Puzzles in International Financial Markets," in Grossman G., and K. Rogoff eds. *Handbook of International Economics*, North-Holland.
- Tornell, A., 2000, "Asset Pricing Anomalies in a \mathcal{H}_∞ -robust Economy", NBER working paper.
- Zhou, K., J. Doyle and K. Glover, 1996, *Robust and Optimal Control*, Prentice Hall.
- Zhou, K., K. Glover, B. Bodenheimer and J. Doyle, 1994, "Mixed \mathcal{H}_2 and \mathcal{H}_∞ Performance Objectives I: Robust performance analysis," *IEEE Transactions on Automatic Control*, 39:1564-1574.

Figure B: Market Clearing

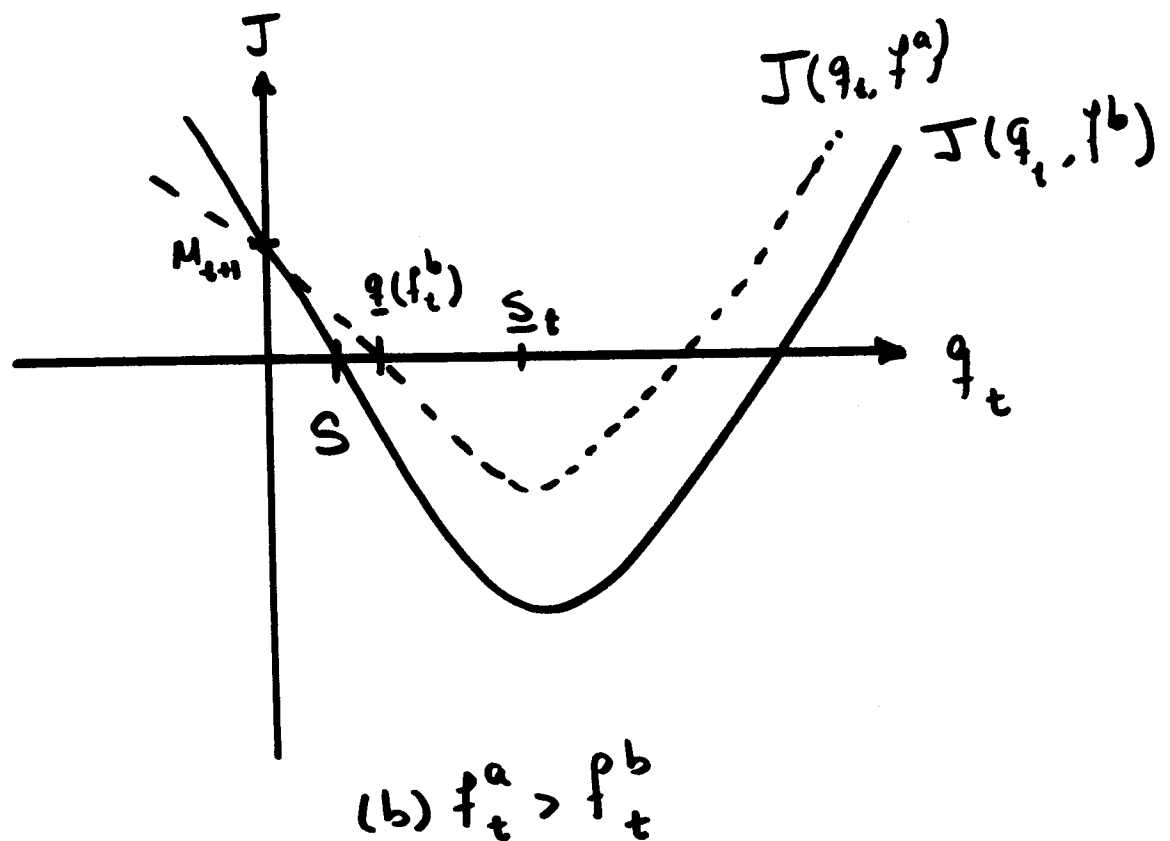
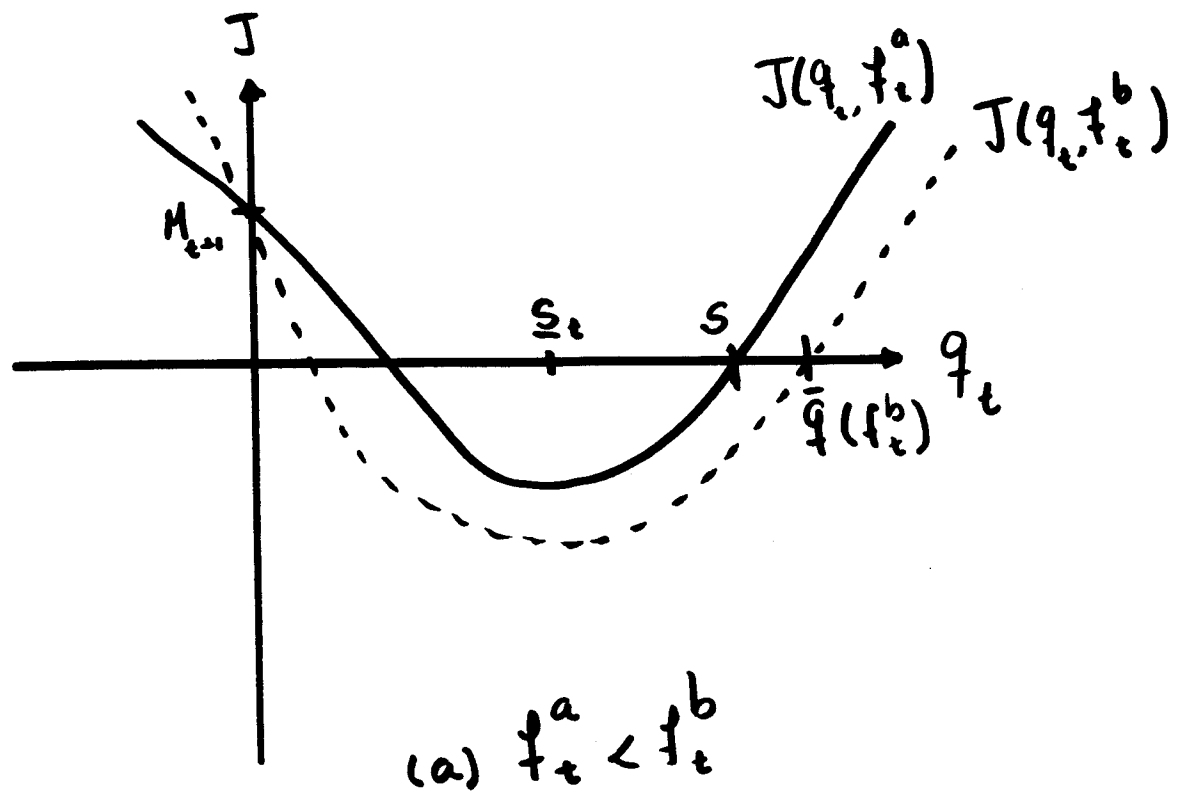
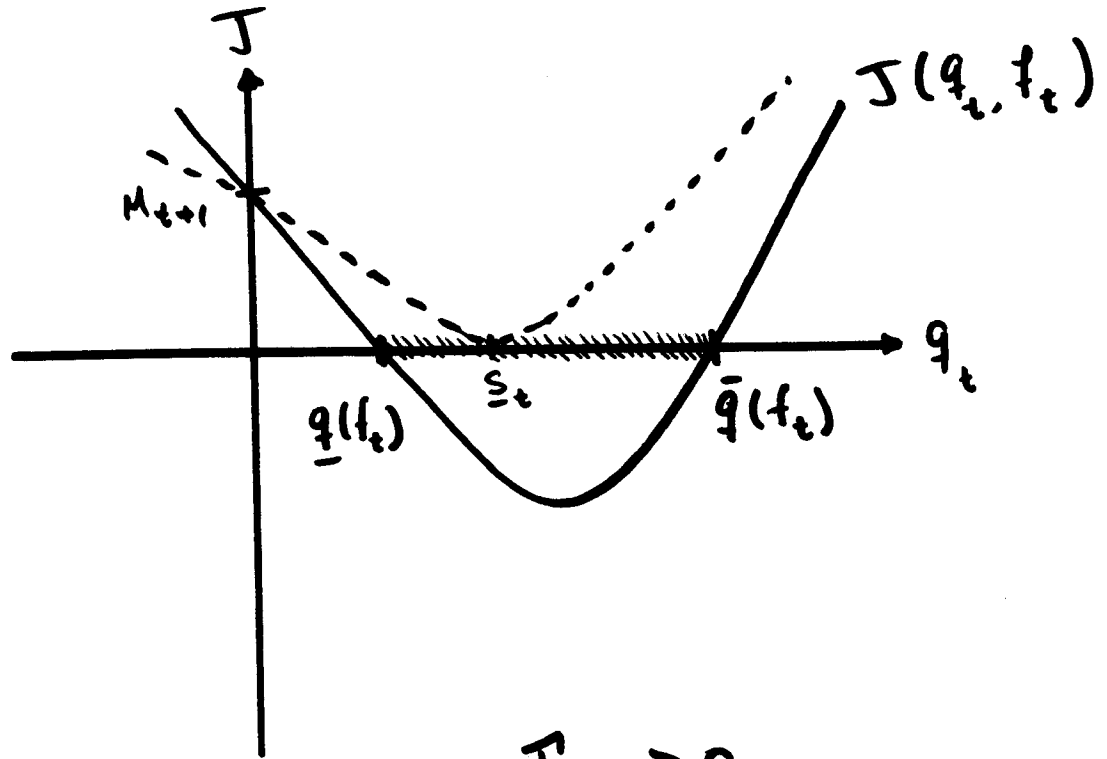
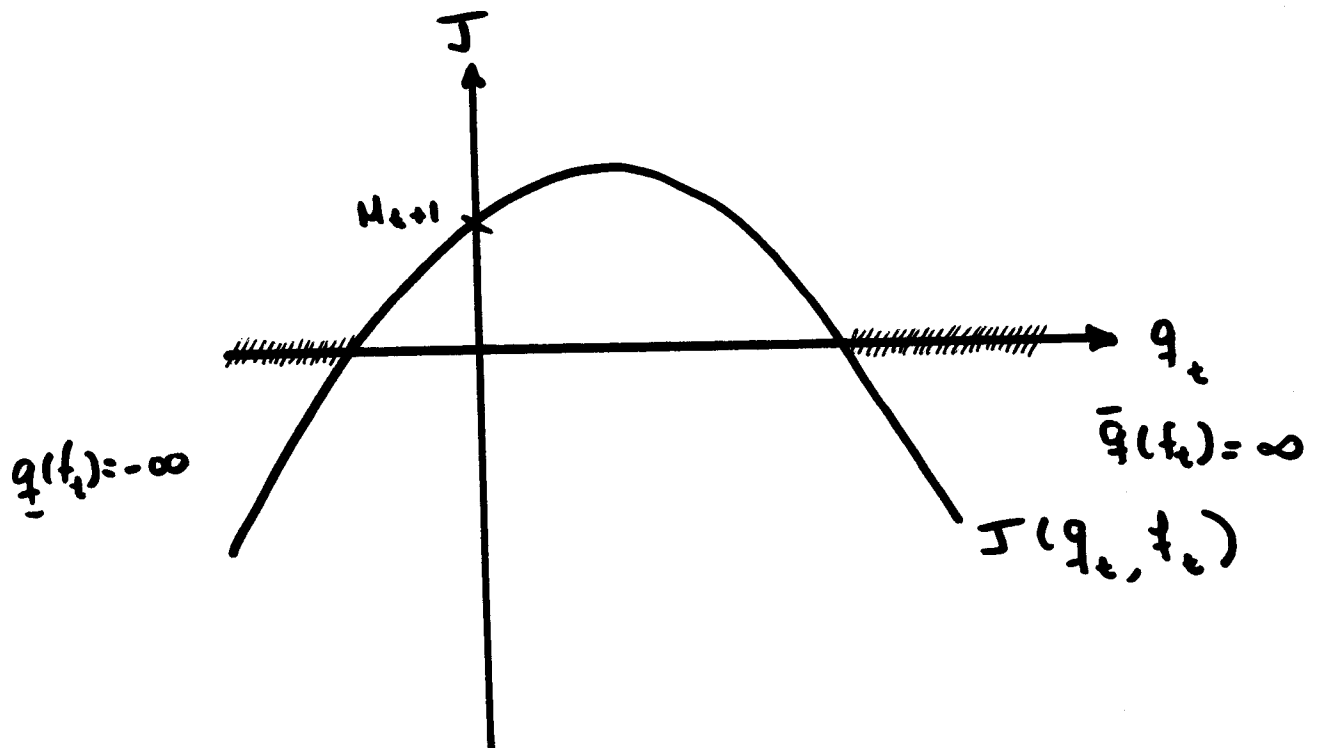


Figure A: The R_t -set



(a) $T_{t+1} > 0$

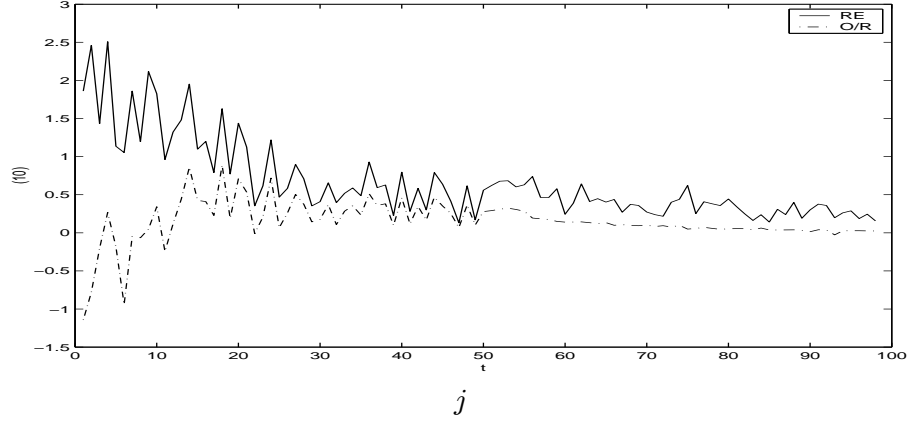


(b) $T_{t+1} < 0$

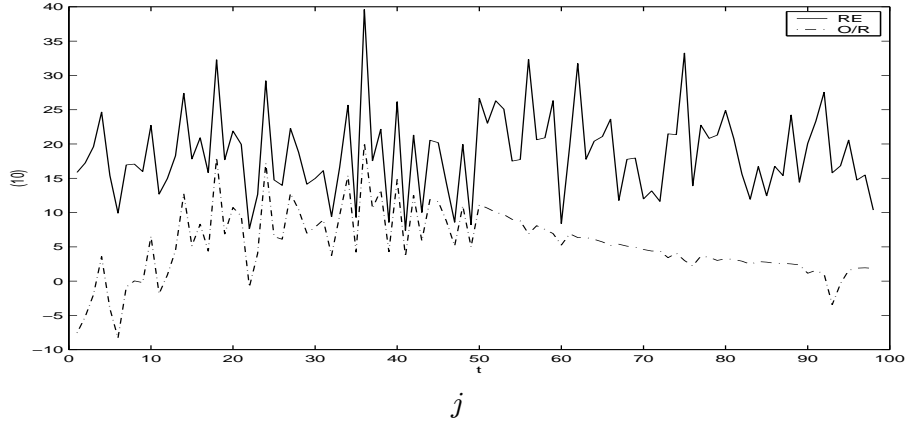
Figure 1: Realized Excess Returns Under Misspecification

$$(x_{j+1} = (a + \Delta_j^a)x_j + \sigma_w w_j)$$

(a) $\max_i \left[\frac{m - S[f_j^i - f_{j+1}^i + y_j^i]}{\sum_{h=0}^{j+1} [w_h^{i2} + v_h^{i2}]} \right]$



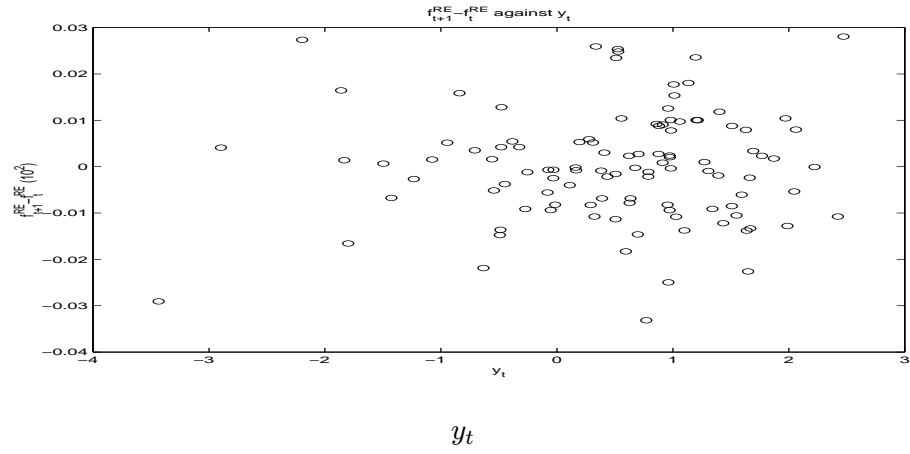
(b) $\max_i [m - S[f_j^i - f_{j+1}^i + y_j^i]]$



Note: The solid line corresponds to Benchmark Rational Expectations Economy. The dotted line corresponds to the Mixed O/R Economy.

Figure 2: Exchange Rate Depreciation and the Interest Rate Differential

(a) Benchmark Rational Expectations Economy.



(b) Mixed O/R Economy.

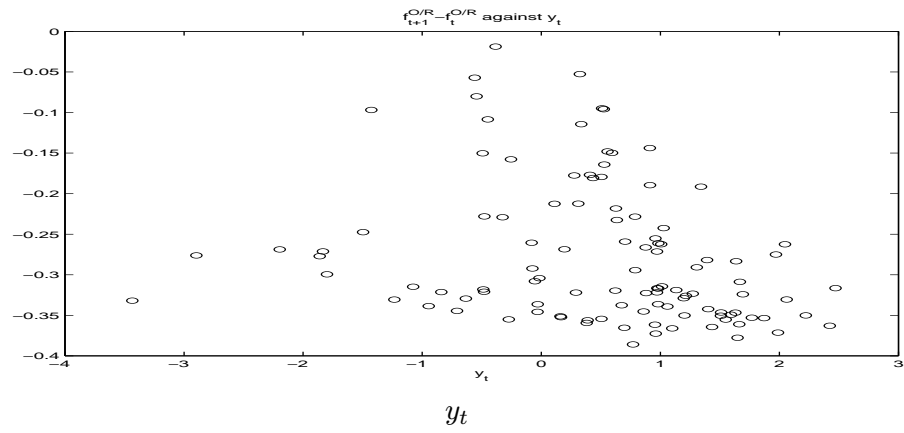
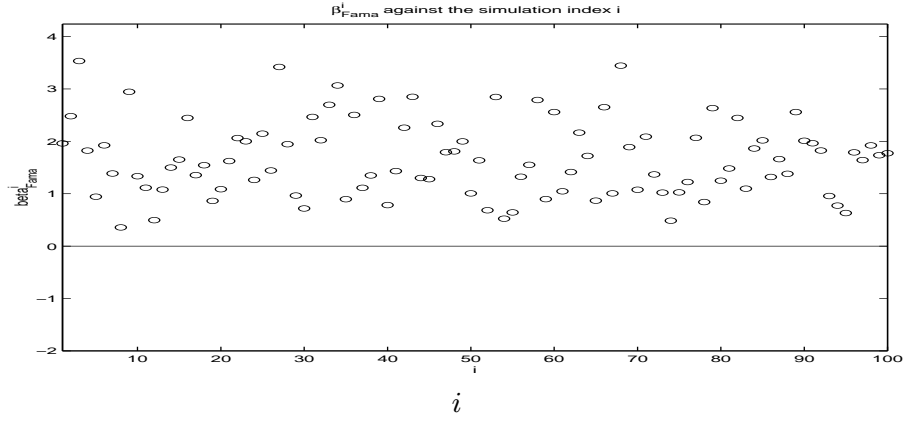


Figure 3: The Forward Discount Puzzle

For each simulated sequence of interest rate differentials y_j^i , we report the point estimate of β^i in the regression $f_{j+1}^i - f_j^i = \alpha^i + \beta^i y_j^i + \varepsilon_t^i$

(a) Benchmark Rational Expectations Economy.



(b) Mixed O/R Economy.

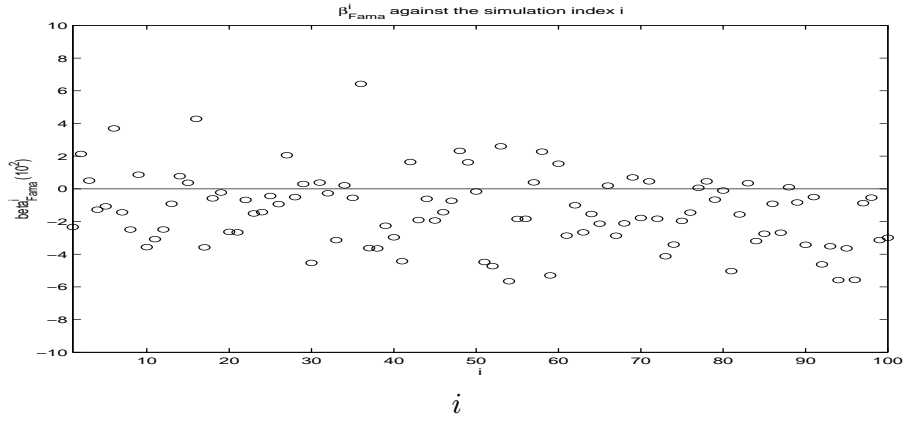
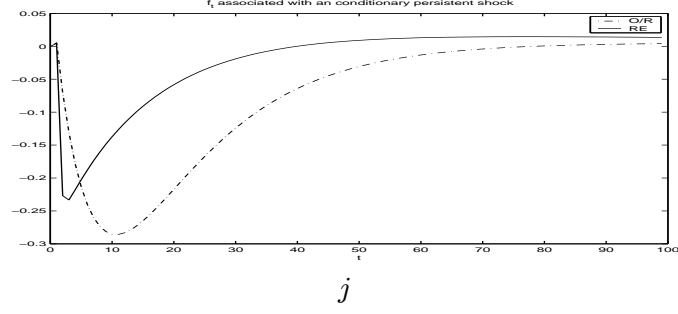
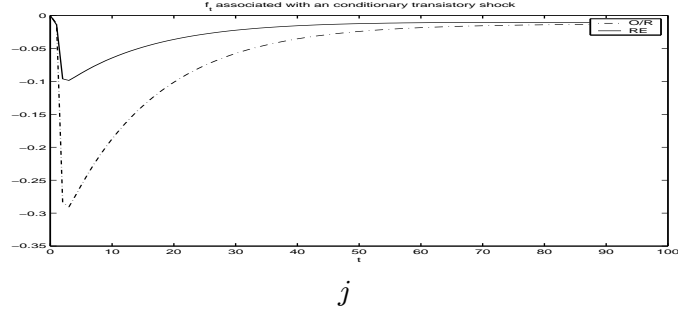


Figure 4. Impulse Responses

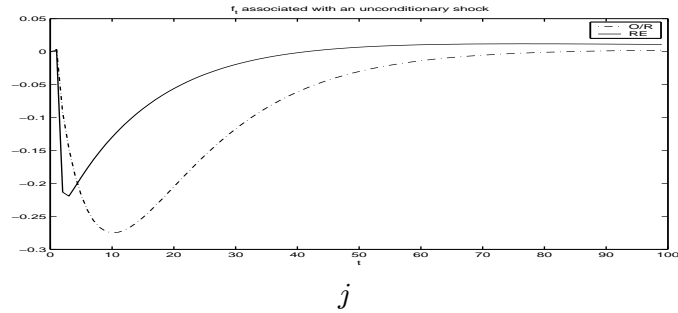
f_j associated with a persistent shock of size $\frac{\delta}{\sigma_w}$ at time 0



f_j associated with a transitory shock of size $\frac{\delta}{\sigma_v}$ at time 1



f_j associated with an unconditional shock



Note: The solid line corresponds to Benchmark Rational Expectations Economy. The dotted line corresponds to the Mixed O/R Economy.