SEMI-CLASSICAL LIMITS OF
SIMPLICIAL QUANTUM GRAVITY

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Abstract.

We consider the simplicial state-sum model of Ponzano and Regge as a path integral for quantum gravity in three dimensions.

We examine the Lorentzian geometry of a single 3-simplex and of a simplicial manifold, and interpret an asymptotic formula for $6j$-symbols in terms of this geometry. This extends Ponzano and Regge's similar interpretation for Euclidean geometry.

We give a geometric interpretation of the stationary points of this state-sum, by showing that, at these points, the simplicial manifold may be mapped locally into flat Lorentzian or Euclidean space. This lends weight to the interpretation of the state-sum as a path integral, which has solutions corresponding to both Lorentzian and Euclidean gravity in three dimensions.

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1. Introduction.

The simplicial approach to gravity dates back to Regge [2], who was the first to formulate a discrete form of general relativity on a n-dimensional simplicial manifold. He showed that n-dimensional curved space-time could be built up from joining together flat n-dimensional simplicial blocks, where the curvature is measured by the deficit angle ε_l around an $(n-2)$-dimensional bone (simplex) of area A_l. The Regge action,

$$S_R = \frac{1}{8\pi} \sum_{\text{bones},l} A_l \, \varepsilon_l, \qquad (1.1)$$

is the discrete action which approximates the Einstein-Hilbert action of general relativity.

Ponzano and Regge [1] studied a model in which the simplicial blocks are 3-dimensional tetrahedra. Each edge of a tetrahedron is labelled by a half-integer j, corresponding to the $(2j+1)$-dimensional fundamental representation of the group $SU(2)$, and the length of that edge is assigned to be $(j + \frac{1}{2}) \, \hbar$. A tetrahedron is then a geometric representation (Figure 2) of the Racah co-efficient, or $6j$-symbol, which is the recoupling co-efficient between four angular momenta (see Figure 1). Regge and Ponzano proposed expressions for different semi-classical limits of a $6j$-symbol. These limits correspond to taking $\hbar \to 0$, while $j \to \infty$, such that each edge length $(j + \frac{1}{2}) \, \hbar$ remains constant.

Regge and Ponzano then preceded to build up a general 3-dimensional simplicial manifold out of tetrahedra. Their crucial insight was to consider the state sum over labellings j_l of the internal edges of the product of $6j$ symbols, one for each tetrahedron t, weighted by a specified function $f(j_l)$. This defines the partition function

$$Z_{RP} = \sum_{\text{labellings},j_l} \left(f(j_l) \prod_{\text{tetra},t} \left\{ \begin{matrix} j_1^{(t)} & j_2^{(t)} & j_3^{(t)} \\ j_4^{(t)} & j_5^{(t)} & j_6^{(t)} \end{matrix} \right\} \right). \qquad (1.2)$$

By taking a certain semi-classical limit of this partition function, they argued that it has many of the features of a path integral for 3-dimensional gravity, i.e.,

$$\int d\mu \exp\left(iS_R\right), \qquad (1.3)$$

where S_R is the Regge action (1.1) for discrete Euclidean gravity in $(3+0)$ dimensions.

Stationary phase solutions of the semi-classical limit of the partition function should then correspond to classical geometries. Regge and Ponzano showed that some cases of stationary solutions correspond to flat Euclidean 3-space. Further details of this model and its relation to Penrose's spin networks [3,4] were elucidated by Hasslacher and Perry [5].

These ideas have been recently rediscovered in the context of topological field theories. In particular, the theory of Turaev and Viro [6] is the generalization [7,8] of the Regge-Ponzano model, in which spins are replaced by representations of the quantum group $U_q(sl(2))$. The semi-classical limit of the sum-over-representations in the Turaev-Viro model has been shown [9] to be related to a path-integral for 3-d Euclidean gravity with a cosmological constant, which is related to the level of the Turaev-Viro theory.

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In this paper we complete Regge and Ponzano's analysis of the stationary points of the actions which arise in the semi-classical limit, giving each of them a geometrical interpretation. Two types of stationary points arise, having Euclidean geometry or Lorentzian geometry. We analyse the case of Lorentzian geometry in detail, as the basic facts are not well-known. The case of Euclidean geometry is similar.

In sections 2 and 3 we analyse the Lorentzian geometry of a simplex, and show that Regge and Ponzano's exponential asymptotic formula has an interpretation in terms of the Lorentzian geometry of a single simplex. This result is entirely analogous to Regge and Ponzano's geometric interpretation of their oscillatory asymptotic formula in terms of Euclidean geometry.

In sections 4 and 5 we consider the relation of some of the stationary points of the state sum model for a simplicial manifold to Lorentzian geometry. Our main result is in section 5. This is that the stationary points of the action, in the semiclassical limit, correspond to a generalised flat Lorentzian space. In this generalisation, there is locally a mapping of a neighbourhood of the simplicial manifold to flat Minkowski space. This is a generalisation of the usual notion of a flat Lorentzian metric on a manifold which determines a local 1-1 mapping of the simplicial manifold to a region of Minkowski space. The condition that it is locally 1-1 is dropped for our generalised flat solutions. In the semiclassical limits which correspond to Euclidean space, there is an entirely analogous analysis.

The physical interpretation of these results is discussed briefly in section 6.

2. Lorentzian geometry of a simplex.

In this section, we consider the Lorentzian geometry of a tetrahedron. We define the analogues of angles between faces of a tetrahedron in Lorentzian space, for the particular cases which arise in the Regge-Ponzano model. As described in the next section, a condition imposed in this model is that the values on the edges around a face satisfy the triangle inequalities. This implies that the tetrahedra which arise are those in which all faces, and hence all edges, are spacelike.

First consider the usual definition of angles in Euclidean space. Consider a vertex of a triangle. The interior angle is the angle between the two sides, and the exterior angle is the angle between the two normals. These are both numbers between 0 and π and it follows from the definitions that

$$\text{interior angle } = \pi - \text{exterior angle .} \tag{2.1}$$

Now consider a vertex of a triangle in 2-dimensional Minkowski space which has all its edges spacelike. Suppose the vertex is at the origin, for convenience. There are two cases to consider

(1) Thin wedge. The triangle does not contain any of the light cone of the vertex (See Figure 3, which depicts the triangle, light cone and the normals to two of the edges). There is an element of $SO(1,1)^+$ which takes one edge into another. The Lorentzian angle at the vertex is defined to be the boost parameter of this element, which is a positive real number. This is an interior angle; an exterior angle is not defined.

(2) Thick wedge. The triangle contains all of the future, or all of the past portion of the light cone (Figure 4). In this case there is an element of $SO(1,1)^+$ which takes one

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outward normal into the other outward normal. The Lorentzian angle at the vertex is defined to be minus the boost parameter of this element, and so is a negative real number. This is an exterior angle; an interior angle is not defined.

Thick and thin wedge angles are related by the following construction. Starting with a thick wedge (say), produce one edge a distance past the vertex. The line segment so added forms two sides of a thin wedge triangle with the other edge of the original triangle (Figure 4). It is straightforward to see that the thick and the thin wedges have opposite angles, i.e., one is minus the other.

Now consider a tetrahedron in 3-dimensional Minkowski space, which has all edges and all faces spacelike. Taking an orthogonal cross-section through an edge yields one of the wedges discussed above. The Lorentzian angle between the two faces which meet at the edge is defined to be the angle of this wedge. It follows that these angles are also one of two types, interior or exterior.

The Lorentzian angle between two faces of a tetrahedron meeting at an edge (hk) can be expressed in terms of the normals n_h and n_k to the faces (using the convention for the signature of the metric that n is timelike with norm $n^2 = -1$). For an interior angle (Figure 3), it is given by

$$\Theta_{hk} = \cosh^{-1}(n_h.n_k), \qquad (2.2)$$

and for an exterior angle, (Figure 4), by

$$\Theta_{hk} = -\cosh^{-1}(-n_h.n_k). \qquad (2.3)$$

Only certain patterns of interior and extrerior angles can occur over a given simplex. Since the faces are spacelike, their normals are timelike and hence can be classified as future-pointing or past-pointing. Two different types of Lorentzian simplex can now be distinguished. One type has two future-pointing normals to faces, and two past-pointing. The other type has three future-pointing and one past-pointing, or vice versa. The angle between two faces with normals both pointing in the same sense is exterior, whereas if one normal is future-pointing and one past pointing, the angle is interior. Thus, the first type of Lorentzian simplex has three exterior and three interior angles, and the second type has two exterior and four interior angles.

3. Asymptotic formulae for a 3-simplex.

Regge and Ponzano [1] proposed two asymptotic formulae for $6j$-symbols. They interpreted one of these in terms of the geometry of a simplex in Euclidean space. In this section, we show how the other case may be interpreted in terms of the Lorentzian geometry discussed in the previous section.

Regge and Ponzano associated a tetrahedron (3-simplex) to a $6j$-symbol, by defining the edge-lengths to be $j_{12} = (a + \frac{1}{2}) \hbar$, $j_{13} = (b + \frac{1}{2}) \hbar$, etc. (Figure 2), corresponding to the $6j$-symbol

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}.$$

The $6j$-symbol is only defined when the values of the angular momenta which correspond to the edges around a face satisfy the triangle inequalities, $|b - c| \le a \le b + c$. This implies that the edge-lengths around a face satisfy the triangle inequalities, up to additional terms of $\pm \frac{1}{2}$. Also, a, b, c are required to satisfy $a + b + c =$ integer, for each face. If these constraints are not satisfied, the value of the $6j$-symbol is defined to be zero.

The vertices and edges of a tetrahedron define a graph. The six edge lengths j_{hk}, for $h < k = 1, \ldots 4$, determine distances between the vertices of the graph. The geometric problem is to find an embedding of this graph into flat Euclidean or Minkowski space.

The volume V of the tetrahedron is given by the Cayley determinant

$$V^2 = \frac{1}{2^3 \, (3!)^2} \begin{vmatrix} 0 & j_{34}^2 & j_{24}^2 & j_{23}^2 & 1 \\ j_{34}^2 & 0 & j_{14}^2 & j_{13}^2 & 1 \\ j_{24}^2 & j_{14}^2 & 0 & j_{12}^2 & 1 \\ j_{23}^2 & j_{13}^2 & j_{12}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}. \tag{3.1}$$

The tetrahedon may be embedded in a 3-dimensional Euclidean space if and only if $V^2 > 0$. However, the triangle inequalities around each face are not sufficient to imply this. In fact, it is also possible to have $V^2 < 0$, in which case the tetrahedron may be embedded in 3-dimensional Minkowski space.

Note that the volume V is related to the length j_{hk}, the normals n_h and n_k and the areas A_h and A_k of the faces meeting at this edge, by the formula

$$V^2 = \frac{4 A_h^2 A_k^2}{9 j_{hk}^2} (1 - (n_h . n_k)^2), \tag{3.2}$$

implying that $V^2 \le 0$, if any of, and hence all of, the pairs of normals satisfy $|n_h . n_k| > 1$.

Regge and Ponzano proposed expressions for the asymptotic limit of a $6j$ symbol in which all the angular momenta become large, for each of these two cases. If the angular momenta are scaled by a factor λ, this limit corresponds to taking $\lambda \to \infty$, while $\hbar \to 0$, such that the edge lengths $(\lambda a + \frac{1}{2}) \hbar$ remain constant. They argued that the asymptotic limit for $V^2 > 0$ is

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} \simeq \frac{1}{\sqrt{12\pi V}} \cos\left(\sum_{h<k} j_{hk} \, \theta_{hk} + \frac{\pi}{4} \right), \tag{3.3}$$

where θ_{hk} is the exterior angle between the outward normals of the faces meeting at edge (hk), j_{hk} is the length of this edge, and the sum is over the six edges of the tetrahedron, labelled by (hk), with $h < k$.

By setting up a formal analogy with the WKB method, Regge and Ponzano were able to continue their asymptotic formula to the region $V^2 < 0$, to obtain

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} \simeq \frac{1}{2\sqrt{12\pi \, |V|}} \; \cos\phi \; \exp\left(-\left|\sum_{h<k} j_{hk} \, \mathrm{Im}\, \theta_{hk}\right|\right), \tag{3.4}$$

where θ_{hk} is still defined by the Euclidean formula

$$\cos\theta_{hk} = \frac{n_h.n_k}{n_h^2} = -n_h.n_k. \tag{3.5}$$

This implies that θ_{hk} has the form

$$\theta_{hk} = m_{hk}\pi + i \, \mathrm{Im}\, \theta_{hk}, \tag{3.6}$$

for an integer m_{hk}, which is fixed by putting

$$m_{hk} = \begin{cases} 0, & \text{exterior angle about } (hk), \\ 1, & \text{interior angle about } (hk). \end{cases} \tag{3.7}$$

The number ϕ is defined by

$$\phi = \sum_{h<k} (j_{hk} - \frac{1}{2}) \, m_{hk}\pi. \tag{3.8}$$

These definitions imply that for allowed values of j_{hk}, i.e. where the $6j$-symbols are non-zero,

$$\cos\phi = (-1)^{\left(\sum (j_{hk}-\frac{1}{2}) \, m_{hk}\right)}. \tag{3.9}$$

We shall now show that the formula (3.4) for the asymptotic limit has a natural interpretation in terms of the embedding of a tetrahedron with $V^2 < 0$ in Lorentzian space, in which the imaginary parts of angles are interpreted in terms of the Lorentzian angles, defined in section 2.

From the definitions of Lorentzian angles Θ_{hk} in equations (2.2) and (2.3),

$$n_h.n_k = -(-1)^{m_{hk}} \, \cosh\Theta_{hk} \, . \tag{3.10}$$

However, we want to identify the Lorentzian angle with the imaginary part of the Euclidean angle (3.6).

From equation (3.6),

$$\cos\theta_{hk} = \cos\left(m_{hk}\pi + i \, \mathrm{Im}\, \theta_{hk}\right)$$
$$= (-1)^{m_{hk}} \, \cosh\left(\mathrm{Im}\, \theta_{hk}\right). \tag{3.11}$$

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Thus, we may put
$$\Theta_{hk} = \operatorname{Im} \theta_{hk} . \tag{3.12}$$

The asymptotic limit for the $6j$-symbol, in which all the angular momentum become large, for the case $V^2 < 0$ is therefore equal to

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} \simeq \frac{1}{2\sqrt{12\pi \, |V|}} \, \cos \phi \, \exp\left(-\left|\sum_{h<k} j_{hk} \, \Theta_{hk}\right|\right). \tag{3.13}$$

This formula for the asymptotic limit in the Lorentzian case is now in the same form as that for the limit in the Euclidean case (3.3). In the Euclidean case, Regge and Ponzano interpreted $\sum j\,\theta$ as the Regge action for the simplex. It seems reasonable for us to interpret $\sum j\,\Theta$ as the Regge action for a Lorentzian simplex, as we shall show in the next section.

4. Lorentzian geometry of a simplicial manifold.

We now want to consider the Lorentzian geometry of a 3-dimensional simplicial manifold, consisting of 3-simplices of the type which may be embedded in Lorentzian space. The curvature of the simplicial manifold is measured by the deficit angles around 1-dimensional edges in the complex, in terms of which the Regge action is defined.

Firstly, recall that for a n-dimensional Euclidean simplicial manifold, Regge defined the deficit angle ε_l, around an $(n-2)$-dimensional 'bone' of area A_l, (see Figure 5), by

$$\varepsilon_l = 2\pi - \sum_{\text{tetra},t} \alpha_l^t, \tag{4.1}$$

where α_l^t are the interior angles in the simplices around the bone. For flat space, the sum of the interior angles around the bone will be 2π, implying that the deficit angle is zero.

For the case of a Lorentzian simplicial manifold, we define the Lorentzian deficit angle ε_l to be (see Figure 6)

$$\varepsilon_l = \sum_{\text{tetra},t} \Theta_l^t, \tag{4.2}$$

where Θ_l^t are the Lorentzian angles in the simplices around the bone, which we defined in section 1. With this definition, the deficit angle for flat Lorentzian space is also zero.

The Regge action is then defined for either case by

$$S_R = \kappa \sum_{\text{edges},l} A_l \, \varepsilon_l. \tag{4.3}$$

This is the discrete action given by evaluating the Einstein-Hilbert action on the simplicial manifold. The constant κ is $\frac{1}{8\pi}$ for general relativity. We shall take $\kappa = 1$ for convenience.

For the 3-dimensional case, the bones are the 1-dimensional edges of length j_l, so that the 3-dimensional Regge action for a Lorentzian simplicial manifold is

$$S_R = \sum_{\text{edges},l} j_l \, \varepsilon_l,$$

$$= \sum_{\text{tetra},t} \left(\sum_{\text{edges},l} j_l \Theta_l^t\right). \tag{4.4}$$

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This means that $\sum j\Theta$ may be interpreted as the Regge action for a single tetrahedron.

5. 3-dimensional simplicial gravity.

As explained in the introduction, our motivation for considering Lorentzian simplicial geometry is to extend the work of Regge and Ponzano [1] to the case of 3-dimensional Lorentzian simplicial gravity. Regge and Ponzano showed that the semi-classical limit of the state-sum over representations of products of $6j$-symbols, corresponding to tetrahedra for which $V^2 > 0$, may be interpreted as a path-integral for 3-dimensional Euclidean gravity. This state-sum is thought of as a discrete version of the sum over metrics in the standard approach to path-integral quantum gravity [15].

We have shown that the semi-classical limit of a single $6j$-symbol, corresponding to a tetrahedron for which $V^2 < 0$, has a natural interpretation in terms of embedding that 3-simplex in a 3-dimensional Lorentzian space. A complex of such 3-simplexes forms a simplicial manifold, on which the Regge action for gravity is defined. We now want to ask if the semi-classical limit of the state-sum of products of the corresponding $6j$-symbols may be interpreted as a path-integral for 3-dimensional Lorentzian gravity with this action.

The Regge-Ponzano model consists of a simplicial manifold formed by a complex of 3-simplexes. A labelling of this simplicial manifold is defined by assigning a representation a of $SU(2)$ to each internal and external edge of the complex, such that the $6j$-symbol associated to each 3-simplex is well-defined, by virtue of satisfying the necessary conditions described in section 2. For a given labelling, the length of each edge is taken to be $j = (a + \frac{1}{2})\hbar$. The initial data for the model consists of a particular labelling of all edges in the boundary. To relate this to gravity, recall [12] that specifying all external edge-lengths is equivalent to specifying a 2-metric on the boundary.

For a given initial labelling of the boundary, there are an (infinite) number of possible ways of consistently labelling the internal edges of the complex. A state of the model is one such labelling of the n internal edges, denoted by $\{x_l\}$, $l = 1, \ldots, n$. The Regge-Ponzano partition function is defined as the sum, over all possible states, of products of $6j$-symbols $T_t = \left\{ \begin{smallmatrix} \cdots \\ \cdots \end{smallmatrix} \right\}$, which correspond to each tetrahedron t. It is given by

$$Z_{RP} = \sum_{\text{state}, \{x_l\}} A(x_1, \ldots, x_n), \tag{5.1}$$

where

$$A(x_1, \ldots, x_n) = \prod_{\text{int.edges}, l} (2x_l + 1)\,(-1)^{\chi} \prod_{\text{tetra}, t} T_t, \tag{5.2}$$

$$\chi = \sum_{\text{edges}, l} (n_l - 2)\,x_l, \tag{5.3}$$

and n_l is the number of simplexes meeting at edge l.

The key point is that this partition function is independent of the way that the interior of the manifold is triangulated, and depends only on the initial data on the boundary and the topology of the interior. Hence, it formally defines a topological field theory.

However, if there are any internal edges, then the partition function is infinite and must be regularised to give finite answers. A simple regularisation procedure was demonstrated by Regge and Ponzano [1]. The partition function is finite for cases where the manifold has a boundary, the vertices are all in the boundary and any loop in the manifold can be deformed to a loop on the boundary. This is because the triangle inequalities constrain the labellings of the interior edges to a finite set of values. Examples of such manifolds are the handlebodies, which are the solid sphere, the solid torus, and solid surfaces of higher genus. It is remarkable that Witten's discussion [13] of quantised gravity, from a different viewpoint, also arrived at the conclusion that the partition function is well-defined for these manifolds but requires regularisation to make sense in the more general cases.

We shall not consider the regularisation in what follows. A more rigorous approach would be to consider the Turaev-Viro model, in which the state sum is always finite, as mentioned in the introduction.

Our basic assumption is that the partition function (5.1) is dominated by stationary phase points or saddle points, for which a WKB-type approximation is valid. Regge and Ponzano's asymptotic formulae for $6j$-symbols, (3.3), (3.13), give two distinct regions in which we can reasonably look for such stationary points. The first such region is where all j's are large and every simplex is Euclidean, as considered by Regge and Ponzano. The second region is where all j's are large and every simplex is Lorentzian. We consider the Lorentzian region first.

Taking the Lorentzian semi-classical limit (3.13) of each $6j$-symbol, the partition function becomes a sum over terms of the form

$$A(x_1, \ldots, x_n) = \prod_{\text{int.edges},l} (2x_l + 1)\, (-1)^\chi \prod_{\text{tetra},t} \frac{\cos \phi_t}{2\sqrt{12\pi\, |V_t|}}\, \exp\left(-\left|\sum_{\text{edges},k} j_k\, \Theta_k^t\right|\right). \quad (5.4)$$

The important feature is the behaviour of $A(x_1, \ldots, x_n)$ as each x_l varies. Because the sum of the angular momenta around a face is an integer, the x_l may only vary by integers. Under $x_l \mapsto x_l + 1$, the term

$$(-1)^\chi \prod_t \cos \phi_t, \quad (5.5)$$

varies by multiplication by $(-1)^{N_k}$, where N_k is the number of thick wedges around the k-th edge. Stationary points are therefore only possible when N_k is even. The remaining terms are all positive, so the sum will be dominated by points where the exponent in

$$\exp\left(-\sum_{\text{tetra},t} \left|\sum_{\text{edges},k} j_k\, \Theta_k^t\right|\right) \quad (5.6)$$

is greatest and is slowly varying. In such a situation a number of neighbouring terms are all large. This is the discrete version of the steepest descent approximation to a contour integral. The exponent in (5.6) defines an action S_Q, which has the same form as the

Regge action (4.4), except that, here, the modulus of the action for each tetrahedron is taken.

The first question is whether this modulus sign is really necessary. We have checked that it is. Numerical experiments show that action $\sum j\Theta$ is negative for the tetrahedra with two future pointing faces, and positive for tetrahedra with one or three future pointing faces. However, it is not known whether this is true in general. Also, we checked that the Regge-Ponzano formula is a good approximation in both cases; surprisingly, all the numerical tables in their paper have only tetrahedra of one type. This is because it is quite hard to find small half-integers for which the tetrahedron has three or one future pointing normals.

The next question is to examine the stationary points of the action S_Q and discuss their geometric significance. For each tetrahedron t, define

$$\sigma_t = \text{sign}\Big(\sum_{\text{edges},k} j_k\, \Theta_k^t \Big). \tag{5.7}$$

According to our numerical experiments, these signs do not change as the j's are varied within one type of Lorentzian simplex. The variation of the action

$$S_Q = \sum_{\text{tetra},t} \sigma_t \Big(\sum_{\text{edges},k} j_k\, \Theta_k^t \Big) \tag{5.8}$$

with respect to the length of edge l is therefore

$$\frac{\delta S_Q}{\delta j_l} = \sum_{\text{tetra},t} \sigma_t\, \Theta_l^t + \sum_{\text{tetra},t} \sigma_t \Big(\sum_{\text{edges},k} j_k\, \frac{\partial \Theta_k^t}{\partial j_l} \Big), \tag{5.9}$$

The second term is zero by the Lorentzian version of Regge's calculation [2]. Hence, the stationary points of the action are given by one equation for each edge l

$$\sum_{\text{tetra},t} \sigma_t\, \Theta_l^t = 0. \tag{5.10}$$

We claim that this is the required condition for the simplicial manifold to be mapped locally into a flat Minkowski space. This shows that the stationary configurations are locally flat in a generalised sense. The normal definition of a flat manifold is that a neighbourhood of each point is mapped by a 1-1 mapping to a region of flat space. Here, the generalisation is that the mapping is not required to be 1-1. The singular points of this mapping, i.e., the folds, lie on the 2-simplexes of the simplicial manifold. Each individual simplex is mapped isometrically, but the singular nature of the map is manifested in that several points, each lying in different simplexes, may be mapped to the same point of Minkowski space.

The argument that this condition is true is as follows. Consider the neighbourhood of an edge in the simplicial manifold. This consists of a number of wedges from the tetrahedra that meet at that edge, arranged in a circle about the edge. From (5.7), there is a sign

$\sigma_t = \pm 1$ associated to each tetrahedron, and a Lorentzian angle Θ_l^t in each wedge at the edge. The neighbourhood is mapped to flat $(2+1)$-dimensional Lorentzian space by the following rule (see Figure 7). Pick one wedge and map it isometrically into flat space. The wedges are now treated in the order in which they circulate the edge. The second wedge to be mapped into flat space is adjacent to the first wedge in the manifold, and it shares a common face. There are two possible ways of mapping this into the flat space so that the face is still common. In one of these ways, the wedges meet only on the common face. This is chosen if the signs σ_t are equal for the two tetrahedra. In the other way of mapping the second wedge to flat space the two wedges overlap, and this is chosen if the signs differ. In a sense, the second wedge backtracks the first. This process is then repeated for the third and succesive wedges in a similar way.

Consider the two-dimensional normal cross-section to the edge, as in Figure 7. The Lorentz transformation which takes the outer edge of the first wedge to the outer edge of the last wedge is

$$(-1)^{N_l} \begin{pmatrix} \cosh \varepsilon_l' & \sinh \varepsilon_l' \\ \sinh \varepsilon_l' & \cosh \varepsilon_l' \end{pmatrix} \tag{5.11}$$

(or its inverse), where

$$\varepsilon_l' = \sum_{\text{tetra},t} \sigma_t \, \Theta_l^t. \tag{5.12}$$

Therefore, when all of the wedges have been mapped into flat space the first and the last wedges meet on their common face to form a continuous whole if and only if this matrix is the identity matrix. This condition is equivalent to the vanishing of ε_l' and the requirement that N_l be even. These are the two conditions for stationary points in the state sum model.

We may regard ε_l' as a generalised deficit angle and (5.10) as the condition for this generalised deficit parameter to be zero. Our conclusion is that the stationary points of the partition function (5.1), in the region of large j on Lorentzian tetrahedra, correspond to a Lorentzian space which is locally flat in the generalised sense. This means that there is a map from the manifold to flat Minkowski space.

In the Euclidean region for stationary points, the approximation is obtained by putting formula (3.3) in the state sum (5.1). This is discussed by Moussouris [10] and Ooguri [11]. Each cosine in (3.3) may be written as a sum

$$\cos \alpha = e^{i\alpha} + e^{(-i\alpha)}. \tag{5.13}$$

This leads to stationary phase points, as opposed to maxima which were encountered for the Lorentzian case. However, the same interpretation may be given. Each simplex may be assigned a sign, $\sigma_t = \pm 1$, and the geometry of a stationary phase point is a generalised flat Euclidean space. There are terms for each possible assignment of signs to each tetrahedron, and each of these may give rise to a stationary point. The stationary points do not just correspond to taking all of the terms with the same sign, $e^{i\alpha}$, for example.

6. Discussion.

Our interpretation of the stationary points for the Regge-Ponzano partition function is that they correspond to generalised flat, three-dimensional Lorenzian or Euclidean spaces. This lends weight to the idea that this partition function may be considered as a discrete version of the path integral for quantum gravity in three dimensions. Obviously, this idea is by no means complete. In particular, the state sum is infinite for the Regge-Ponzano model. It would be interesting to see if our interpretation goes through to the Turaev-Viro model [6], in which the partition function is given by a state sum over representations of a quantum group at a root of unity, and so is naturally finite. For that model, we would expect the stationary points to correspond to generalised Lorentzian or Euclidean spaces of constant curvature in three dimensions. However, it is encouraging that a discrete model can be found with a partition function which has stationary solutions corresponding to flat space, as we would expect a path integral for 3-dimensional quantum gravity to be concentrated on these solutions. Penrose has argued forcefully [3,4] that the best approach to quantum gravity is to look for a discrete model from which space-time emerges in some limit.

The generalised flat solutions for the Einstein equations were discussed by Horowitz [14]. They arise naturally as the solutions to the Einstein equations when the Einstein action is written in terms of the frame field and connection form. In three dimensions, such a stationary point is precisely a flat $ISO(2,1)$ connection and a section of the associated bundle with fibre affine Minkowski space. This data is equivalent to a map from the space-time manifold to a flat Lorentzian space. As in this paper, the mapping need not be 1-1. The target manifold need not have the same topology, and of course the mapping will induce a degenerate metric on the original space-time. Ooguri [11] suggested that the Regge-Ponzano theory should be interpreted directly in terms of the path integral for the action in this form, i.e., with frame fields and connection forms. However, our interpretation and Horowitz' results only imply such an interpretation for the stationary points.

It is interesting to consider whether the stationary points considered in this paper have an interpretation as a map to another manifold which has a flat Lorentzian metric, in other words whether our local characterisation extends to this global characterisation.

In the standard approach to quantum gravity using path integrals [15], the path integral is the sum over metrics on the interior of a manifold, consistent with the metric on the boundary. However, in this case, the path integral is not well-defined since the Einstein action is unbounded below for conformal rescalings. This problem may be solved by "the rotation of the contour for the conformal factor" [16]. This is the formal (i.e. informal) process of redefining the path integral for the conformal fluctuations separately, which has the effect of changing the sign of the fluctuation of the action for these modes. This process renders the action bounded below. However, this is precisely what happens at our Lorentzian semiclassical approximation: the action is strictly positive, because of the modulus in equation (5.6). The price of maintaining a geometric description of the action is that the classical solutions have the feature that there are regions in which "time goes backwards". Indeed, since each face of each tetrahedron is spacelike, if two neighbouring tetrahedra have opposite signs σ_t, then they have opposite induced time orientations from

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the Minkowski space.

The state sum model can be regarded as a definition of a path integral for quantum gravity. The terms in the state sum (5.1) which are not stationary do not correspond to classical geometries. This is a typical feature of path integrals. However, path integrals have approximations around stationary points which have a geometric interpretation. The fluctuations in the state-sum model around a Lorentzian stationary point may be written, according to the asymptotic approximation, as

$$Z_{RP} = \int \prod_l \mathrm{d}\mu(j_l) \, \exp(-S_Q), \tag{6.1}$$

where the generalised Regge action is

$$S_Q = \sum_{\text{edges},k} j_k \, \varepsilon'_k \, . \tag{6.2}$$

This looks like a path integral in the sense of integration of a space of metrics with an action. Note, however, that contrary to what we would normally expect for a Lorentzian path-integral, the integral is over terms of the form e^{-S} rather than e^{iS}. Also, in the Euclidean case the corresponding result is an integral over e^{iS} terms.

As a final thought, this makes the state-sum model rather similar to quantum cosmology models [17], in that the one path integral contains both Lorentzian and Euclidean regimes. However, here, the Euclidean regime is oscillatory and the Lorentzian regime exponential.

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