Yvette Kosmann-Schwarzbach
Translated by Stephanie Frank Singer

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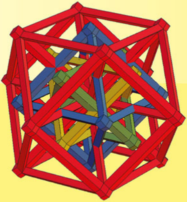
Groups and Symmetries

From Finite Groups to Lie Groups



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From Finite Groups to Lie Groups

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Springer

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Cover Illustration: The figure on the front cover represents weights for the fundamental representation of sl(4). Courtesy of Adrian Ocneanu.

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Sophus Lie (1842–1899), around 1865, at the end of his studies at the University of Christiana (Oslo), approximately seven years before his first work on continuous groups, later known as "Lie groups."

(*Photo Frederik Klem/Joronn Vogt, with the kind permission of Joronn Vogt and Arild Stubhaug*)

Contents

Introduction

Johnny walked to a corner of the room,
faced the wall, and started mumbling to himself.
After a while he turned around and said:
"You need the theory of group characters."
Whereupon von Neumann went to Issai Schur,
obtained reprints of two of his papers
and gave them to Wigner.[1]

Symmetries of geometric figures, crystals, and other macroscopic physical items have been observed and analyzed for centuries. In modern terms, the symmetries of an object form a group. The idea of a group emerged slowly in the middle of the nineteenth century. Afterward, though, the study of groups accelerated markedly. Due to the work of Sophus Lie (1842–1899), Georg Frobenius (1849–1917), Wilhelm Killing (1847–1923), Élie Cartan (1869–1951), Issai Schur (1875–1941), Hermann Weyl (1885–1955), and many, many others, group theory expanded enormously. Applications to quantum mechanics and elementary particle theory were developed throughout the twentieth century. If this history interests you, read the introduction to *Group Theory and Physics* by Shlomo Sternberg (1994), and consult the books by Charles Curtis (1999), Thomas Hawkins (2000), and Armand Borel (2001), or the essays and transcriptions of round tables between physicists and mathematicians in the volume *Symmetries in Physics.*[2]

Sophus Lie wrote in an 1877 letter to the mathematician Adolph Mayer[3] that he had "created group theory" in January 1873. He was certainly referring to the groups that he called "continuous" and that have since become known

[1] A. Pais, *The Genius of Science*, Oxford University Press, Oxford, 2000, p. 335. This scene took place in 1926 in Berlin. "Johnny" was the mathematician and physicist John von Neumann (1903–1957), speaking to Eugene Wigner (1902–1995). The papers by I. Schur were probably "Neue Begründung der Theorie der Gruppencharaktere," *Sitzungsberichte Preussischen Akademie der Wissenschaften zu Berlin* (1905), 406–432, and F. G. Frobenius and I. Schur, "Über die reellen Darstellungen der endlichen Gruppen," *ibid.* (1906), 186–208.

[2] M. G. Doncel, A. Hermann, L. Michel, and A. Pais, eds., *Symmetries in Physics (1600–1980)*, Seminari d'Història de les Ciencies, Universitat Autònoma de Barcelona, Bellaterra, 1987.

[3] Mayer (1839–1908) was a professor at the University of Heidelberg whose work on the calculus of variations and partial differetial equations was close to that of Lie.

as “Lie groups” and in French as “groupes de Lie.”[4] Lie was trying to extend the application of groups from the study of algebraic equations, where Évariste Galois had introduced them, to the study of differential equations. Inspired by the work of Camille Jordan and by a collaboration with Felix Klein, Lie published an article titled *Über Gruppen von Transformationen* in 1874. As early as 1871, the idea of an infinitesimal generator of a one-parameter group of transformations had already appeared in his work;[5] the set of infinitesimal generators of one-parameter subgroups of a continuous group forms what today is called a “Lie algebra.”[6] Between 1888 and 1890, Killing established a classification of semisimple Lie algebras over the field of complex numbers, which was corrected and completed by Cartan in his 1894 thesis. Cartan would succeed in classifying Lie algebras over the field of real numbers, a more difficult problem, in 1914.

Frobenius, in response to a question that had been posed by Dirichlet,[7] invented the theory of characters of finite groups in 1896, and Schur, beginning in 1901, developed the representation theory of finite and infinite groups. Their joint work, *Über die reellen Darstellungen der endlichen Gruppen*, was published in the proceedings of the Berlin Academy in 1906. The theory of characters was used by William Burnside in the second edition of his treatise *Theory of Groups of Finite Order*, which appeared in 1911.

It was Eugene Wigner and Weyl who showed the pertinence of group theory, particularly group representations, for the quantum mechanics being developed in the 1920s by Werner Heisenberg and Paul Dirac. Wigner was the first to introduce group theory into physics in two articles in the *Zeitschrift für Physik* in 1927. His book *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren*, which appeared in 1931, was followed in 1939 by a fundamental article in the *Annals of Mathematics* in which he determined the representations of the Poincaré group. At the end of 1927, Weyl published an article in the *Zeitschrift für Physik* in which he introduced what is now called “Weyl quantization,” and the first edition of his *Gruppentheorie und Quantenmechanik* came out in 1928. In 1932 Bartel van der Waerden’s *Die gruppentheoretische Methode in der Quantenmechanik* appeared. These three books marked the beginning of the use of group theory in theoretical physics, which continues to this day.

After Heisenberg had solved a two-electron problem related to the spectrum of the helium atom, Wigner become interested in the more-than-two-identical-particles problem. He first solved the case of three elements, then went

4 The term “groupes de Lie” first appeared in French in 1893 in the thesis of Lie’s student Arthur Tresse. See p. 46.

5 This point of view was essential to Emmy Noether’s 1918 article in the *Göttinger Nachrichten* establishing the relationship between symmetries of a variational problem and conservation laws. About ten years later, Noether published a very important article situating the representation theory of finite groups and of algebras in the context of noncommutative rings.

6 This name was suggested much later by Weyl in his 1933–1934 lectures at the Institute for Advanced Study in Princeton.

7 Peter Gustav Lejeune Dirichlet (1805–1859) taught in Paris, Breslau (now Wrocław), Berlin, and finally Göttingen. His works on number theory and on analysis are equally fundamental.

to consult von Neumann on how to extend his already very complicated computations to the case of an arbitrary number of electrons. In 1983, he recalled,[8]

> As to representation theory: I realized that there must be such a theory but I had no knowledge of it. Dr. von Neumann to whom I presented the problem (and I presented to him the representations of the permutation groups of three and four elements because those I could establish by long hand calculations) gave me a reprint of the article of Frobenius and Schur [1906]. And that was wonderful!

One may consider group representation theory as a vast generalization of Fourier analysis. It has been developed continuously and has, since the middle of the twentieth century, seen innumerable applications to differential geometry, ergodic theory, probability theory, number theory, the theory of automorphic forms, and dynamical systems, as well as to physics, chemistry, molecular biology, and signal processing. Currently it is the basis of several branches of mathematics and physics.

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The aim of this course is to introduce and illustrate the basic notions of finite group theory and, more generally, compact topological groups, to introduce the notions of Lie algebras and Lie groups, at least in the case of linear Lie groups (closed subgroups of the linear groups), to study the Lie groups SO(3) and SU(2) in detail, and then the Lie group SU(3) and its application to the theory of quarks. The idea of a group representation, an action on a vector space by linear transformations, plays the most fundamental role in this study.

In Chapter 1, we recall some general facts about groups and group actions, and we give examples of finite and infinite groups.

To study the representations of finite groups, we exploit the properties of the characters of these representations, that is, the traces of the linear transformations that define the representation in question. This is done in Chapter 2, where we define irreducible representations and intertwining operators. We state Schur's lemma which, though simple, will be essential to the proofs of important consequences such as the orthogonality of the characters of irreducible representations. We study the regular representation. A brief paragraph will introduce induced representations.

Chapter 3 extends certain results proved for finite groups to compact groups, using the existence of an invariant measure on each compact group, the Haar measure. In this chapter, some topological and analytic results are stated without proof and references to the literature are supplied. A table summarizes the properties of representations of compact groups.

The beginning of Chapter 4 consists of an introduction to the general notion of a Lie algebra and a review of exponentiation of matrices. Next we study the Lie algebra of a linear Lie group, that is, the set of infinitesimal generators of one-parameter subgroups, with the operation of matrix commutation, and we show the relationship between the representations of a Lie group and the representations of its Lie algebra.

[8] Round Table on the Evolution of Symmetries, chaired by Louis Michel, in Doncel et al., p. 633.

In Chapter 5 we study the Lie groups SO(3) and SU(2), proving the fundamental property that SU(2) is the universal cover of the rotation group.

In Chapter 6 we determine all the irreducible representations, first of $\mathfrak{sl}(2, \mathbb{C})$, and then of SU(2) and SO(3). These results are as important for the general theory of group representations as they are for quantum mechanics.

Chapter 7 consists of a study of spherical harmonics, which appear in the representation theory of the group of rotations of Euclidean space of three dimensions. This is an introduction to the theory of special functions.

In Chapter 8 we begin the study of irreducible representations of SU(3) by considering examples, and we show that quark theory appears as a consequence of the mathematical properties of this group. In this chapter we introduce the roots and weights that are, more generally, the essential tools the representation theory of a large class of Lie algebras, the semisimple Lie algebras.

***

We have introduced much notation and many concepts used in physics in the hope that this book will be useful to students in the physical sciences as well as in mathematics. Except for Chapters 3 and 8, complete proofs of all results are given. The symbol □ denotes the end of a proof. At the end of each chapter we give bibliographic references to the principal sources of the material treated in the chapter and references for further study. For each chapter, there are exercises of various degrees of difficulty, many of which introduce additional definitions and results. The eight chapters are followed by a series of nine problems with detailed solutions, which constitute both an application of results proved in this book and in introduction to several more advanced topics. The book ends with a bibliography and an index.

***

This textbook arose from a course taught for several years at the École Polytechnique to third-year students majoring in mathematics or physics. This course, roughly corresponding to an advanced undergraduate or beginning graduate course, was taught in nine weeks as ninety-minute lectures followed by two-hour exercise sessions. The problems that complement the text are drawn from the examinations that were administered at the end of the term.

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This book is designed to serve as an introduction to more advanced texts and to encourage the reader to develop and use the ideas sketched here. The bibliography at the end of this work contains some very classical texts, several books on the history of the subject, recent textbooks on diverse aspects of group and representation theory, books that will be useful as reference for results not proved here, and books that further develop harmonic analysis or representation theory in various ways, chosen from an extremely vast literature. Among these works, we have starred those that are most directly related to the matter treated in this text. Others, perhaps in spite of their titles, go "beyond an introduction." We hope that our readers will find instruction and pleasure in this book.

Paris, July 2009

Acknowledgments

This English edition is a revised and adapted version of *Groupes et Symétries: Groupes finis, groupes et algèbres de Lie, représentations*, Éditions de l'École Polytechnique, 2006, itself a revised edition of the first edition (2004). I am very grateful to the translator, Dr. Stephanie Singer, for her excellent translation, to Ann Kostant and the editors of the series *Universitext* for accepting this short textbook for publication, and to the staff of Springer, New York, for their help in its preparation. I also thank the institutions and colleagues who kindly allowed me to reproduce the portraits of mathematicians and physicists that illustrate this book.

I wish to reiterate my thanks to all those who helped me in the preparation of the French editions: Alain Guichardet, for his numerous comments on the notes that preceded this book, André Rougé for his clarification of the history of the eightfold way, my colleagues Nicole Berline and Pascale Harinck, as well as Jean-Paul Blaizot and Jean-Claude Tolédano, for their encouragement and useful feedback, and the students of the École Polytechnique from whose remarks I have learned a great deal. I am grateful to Shlomo Sternberg, whose book on group theory and physics inspired me greatly, and to Adrian Ocneanu, of Pennsylvania State University, who sent me diagrams illustrating the theory of roots of Lie algebras, one of which appears on the cover of *Groupes et Symétries.*

Wilhelm Killing (1847–1923) around 1890, shortly after the publication of his innovative work on the properties of infinitesimal isometries of "space forms," generalizations of Euclidean space. This study brought him to the idea that much later was called "Lie algebra." Eventually he classified semisimple Lie algebras.

(*Portrait collection, Universitäts- und Landesbibliothek Münster*)

Chapter 1

General Facts About Groups

1 Review of Definitions

A *group* is a set equipped with an associative composition law, containing an identity element and such that each element has an inverse. The identity element, also called the unit element, may be denoted by e, 1, or, in the case of a matrix group, I.

A group is called *commutative* or *abelian* if its composition law is commutative. In this case the composition is generally denoted by $+$ and the identity element is generally denoted by 0.

We denote by $|X|$ the cardinality of a finite set X. The *order of a finite group* G is the number $|G|$ of elements of the group.

An element $g \in G$ is said to be of order n ($n \geq 1$) if n is the smallest integer such that

$$g^n = e.$$

Example. The rotation through an angle $2\pi/n$ is an element of order n in the group of rotations of the plane.

A *subgroup* H of G is a subset satisfying the conditions $e \in H$, $g \in H$ implies $g^{-1} \in H$, if $g \in H$ and $g' \in H$ then $gg' \in H$.

The *subgroup generated* by a subset of a group G is the smallest subgroup of G containing the subset.

A group is called *cyclic* if it is generated by a single element. Such a group is abelian.

Let H be a subgroup of G. The *left H-cosets* are the sets gH, $g \in G$. The set of left H-cosets is the quotient set G/H. The *right H-cosets* are the sets Hg, $g \in G$. The set of right H-cosets is the quotient set $H\backslash G$. The set G is the union of left (respectively, right) H-cosets, each of which has $|H|$ elements. We deduce that the order of H divides the order of G, and that the number of left H-cosets equals the number of right H-cosets. We can now state the following result.

Theorem 1.1 (Lagrange's Theorem). *Let G be a finite group and let H be a subgroup of G. Then $|H|$ divides $|G|$ and*

$$|G|/|H| = |H\backslash G| = |G/H| \ .$$

The integer $|G|/|H|$ is called the *index* of H in G.

P.Y. Kosmann-Schwarzbach, *Groups and Symmetries*, Universitext,
DOI 10.1007/978-0-387-78866-1_1,

A *morphism of groups* (or *group morphism*) $\varphi : G_1 \to G_2$ is a mapping of a group G_1 into a group G_2 such that for every g, $g' \in G_1$, $\varphi(gg') = \varphi(g)\varphi(g')$, which implies that $\varphi(e_1) = e_2$, where e_i is the identity element of G_i $(i = 1, 2)$, and $\varphi(g^{-1}) = (\varphi(g))^{-1}$. Note that most authors use the term *homomorphism*.

An *isomorphism of groups* is a bijection that is also a morphism. Its inverse is then also a morphism.

An *automorphism of a group* is an isomorphism of a group onto itself. In particular, for each $g \in G$, the isomorphism $\mathcal{C}_g$ of G defined by

$$\mathcal{C}_g : h \mapsto ghg^{-1}$$

is called an *inner automorphism* or a *conjugation*.

A subgroup of G is called *normal* or *invariant* if it is stable under conjugation by every element of G.

If a subgroup H of G is normal, then for each $g \in G$, $gH = Hg$, and the set of what we may now call the set of H-cosets is then a group, called the *quotient group* of G by H, and denoted by G/H. The projection $p : G \to G/H$, defined by $g \mapsto gH$, is a morphism of groups.

The kernel of a morphism of groups $\varphi : G_1 \to G_2$ is a normal subgroup of G_1. Conversely, if H is a normal subgroup of a group G, then it is the kernel of a group morphism, the projection $p : G \to G/H$.

The *center* of G is by definition the set $\{h \in G \mid \forall g \in G, hg = gh\}$. The center of G is a normal subgroup of G.

If G_1 and G_2 are groups, their *direct product* is the product $G_1 \times G_2$ with group law $(g_1, g_2).(g'_1, g'_2) = (g_1 g'_1, g_2 g'_2)$.

The notion of a *semidirect product* of groups is introduced in Exercise 1.4.

2 Examples of Finite Groups

2.1 Cyclic Group of Order n

For n a positive integer, the set of integer multiples of n is denoted by $n\mathbb{Z}$. It is a subgroup of the abelian group $\mathbb{Z}$; in fact, it is a normal subgroup, since all subgroups of an abelian group are normal.

The following groups are isomorphic and are called *cyclic groups* of order n:

- $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, in particular $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, denoted additively by $\{0, 1\}$ or multiplicatively by $\{1, -1\}$.
- The group of rotations of the plane around the origin through angles $2k\pi/n, 0 \le k \le n-1$, under composition.
- The group of complex numbers $\{e^{2ik\pi/n} \mid 0 \le k \le n-1\}$ under multiplication.
- The subgroup $\{1, g, g^2, \ldots, g^{n-1}\}$ if g is an element of order n in a group G.

A cyclic group is generated by a single element.

2.2 Symmetric Group $\mathfrak{S}_n$

The permutation group of a set of cardinality n is denoted by $\mathfrak{S}_n$ and called the *symmetric group on n elements*. The order of $\mathfrak{S}_n$ is $n!$.

Every element of $\mathfrak{S}_n$ can be written as a product of transpositions. To every element $\sigma \in \mathfrak{S}_n$, we associate the number 1 or -1 according to the parity of the number of transpositions, which parity is independent of the decomposition. This number is denoted by $(-1)^\sigma$ and called the *signature* of σ. The map $\sigma \in \mathfrak{S}_n \mapsto (-1)^\sigma \in \mathbb{Z}_2$ is a group morphism, called the *signature morphism*.

The *alternate group* $\mathfrak{A}_n$ is the kernel of the signature morphism. If $n \geq 2$, it is a normal subgroup of index 2 of $\mathfrak{S}_n$.

2.3 Dihedral Group

The *dihedral group* $D_{(n)}$ is the group of rotations and symmetries of the plane preserving a regular n-sided polygon ($n \geq 3$). It is a subgroup of $\mathfrak{S}_n$ of order $2n$. Some authors use the notation D_{2n} for this group. (See Exercise 1.6.)

2.4 Other Examples

We denote by $\mathrm{O}(3)$ the group of linear isometries of $\mathbb{R}^3$, and by $\mathrm{SO}(3)$ the group of rotations of $\mathbb{R}^3$, which is the kernel of the determinant map on $\mathrm{O}(3)$, and thus a normal subgroup of $\mathrm{O}(3)$. For each regular polyhedron—tetrahedron, cube, octahedron, icosahedron (fullerene), dodecahedron—we define the corresponding groups of symmetries, that is, the subgroup of $\mathrm{SO}(3)$ and the subgroup of $\mathrm{O}(3)$ leaving the solid globally invariant. These are finite groups, and they are known as *crystallographic groups*. The first is of index 2 in the second:

	SO(3)	O(3)
tetrahedron	$\mathfrak{A}_4$ order 12	$\mathfrak{S}_4$ order 24
cube	$\mathfrak{S}_4$ order 24	$\mathfrak{S}_4 \times \mathbb{Z}_2$ order 48
octahedron	$\mathfrak{S}_4$ order 24	$\mathfrak{S}_4 \times \mathbb{Z}_2$ order 48
icosahedron	$\mathfrak{A}_5$ order 60	$\mathfrak{A}_5 \times \mathbb{Z}_2$ order 120
dodecahedron	$\mathfrak{A}_5$ order 60	$\mathfrak{A}_5 \times \mathbb{Z}_2$ order 120

The classification of all finite subgroups of $\mathrm{SO}(3)$ and of $\mathrm{O}(3)$, which is well known, is of great importance in physics, in particular in crystallography.

3 Examples of Infinite Groups

Among the groups with an infinite number of elements, there are *discrete groups* such as, for example, the abelian group $\mathbb{Z}$. But we are mainly interested in the nondiscrete groups called "continuous groups," of which we now give examples.

Let $\mathbb{K}$ denote the field $\mathbb{R}$ or $\mathbb{C}$. We denote by $\mathrm{GL}(n, \mathbb{K})$ the group of linear isomorphisms of $\mathbb{K}^n$, called the *linear group* of dimension n. This is the group of invertible $n \times n$ matrices with coefficients in $\mathbb{K}$, of which we shall consider several subgroups. We denote the transpose of a matrix A by ${}^tA$. The complex conjugate of a complex number or matrix is denoted by a bar over that number or matrix.

- Special linear group over $\mathbb{K}$:

$$\mathrm{SL}(n,\mathbb{K}) = \{A \in \mathrm{GL}(n,\mathbb{K}) \mid \det A = 1\}.$$

- Orthogonal group over $\mathbb{K}$:

$$\mathrm{O}(n,\mathbb{K}) = \{A \in \mathrm{GL}(n,\mathbb{K}) \mid A\ {}^t A = I\}.$$

In particular, we call the real orthogonal group the *orthogonal group* and we denote it simply by $\mathrm{O}(n)$.

- Special orthogonal group over $\mathbb{K}$:

$$\mathrm{SO}(n,\mathbb{K}) = \{A \in \mathrm{O}(n,\mathbb{K}) \mid \det A = 1\}.$$

In particular, we call the real special orthogonal group the *special orthogonal group*, and we denote it simply by $\mathrm{SO}(n)$.

More generally, if $p + q = n$, we denote by J_{pq} the diagonal matrix with p diagonal elements 1 followed by q diagonal elements -1, and we set

$$\mathrm{O}(p,q) = \{A \in \mathrm{GL}(n,\mathbb{R}) \mid A\ J_{pq}\ {}^t A = J_{pq}\}$$

and

$$\mathrm{SO}(p,q) = \{A \in \mathrm{O}(p,q) \mid \det A = 1\}.$$

In particular, $\mathrm{O}(3,1)$, the group of isometries of Minkowski space, is called the *Lorentz group*.

- Unitary group:

$$\mathrm{U}(n) = \{A \in \mathrm{GL}(n,\mathbb{C}) \mid A\ {}^t\overline{A} = I\}.$$

The matrix ${}^t\overline{A}$ is the adjoint of the matrix A, also denoted by A^*.

- Special unitary group:

$$\mathrm{SU}(n) = \{A \in \mathrm{U}(n) \mid \det A = 1\}.$$

Definition 3.1. *A* topological group *is a group G that is a Hausdorff topological space such that multiplication $(g, g') \mapsto gg'$ is a continuous map of $G \times G$ into G and inversion $g \mapsto g^{-1}$ is a continuous map of G into G.*

The linear group $\mathrm{GL}(n,\mathbb{K})$ with its usual topology (as an open subset of $\mathbb{K}^{n^2}$) is a locally compact topological group, and each of the groups listed above is a closed subgroup of a linear group. The groups $\mathrm{O}(n)$ and $\mathrm{U}(n)$, as well as $\mathrm{SO}(n)$ and $\mathrm{SU}(n)$, are compact.

Below we give the definition of real and complex Lie groups, which brings in the notion of *manifolds*, an abstraction of the notion of a surface in Cartesian space.

A (*real*) *Lie group* of (real) dimension N is a group that is a smooth (real) manifold of dimension N such that multiplication and inversion are smooth differentiable maps.

A *complex Lie group* of complex dimension N is a group that is a complex-analytic manifold of complex dimension N such that multiplication and inversion are analytic maps.

All closed subgroups of $\mathrm{GL}(n, \mathbb{K})$ are real Lie groups, called *linear Lie groups.* This is a theorem that we state without proof. The linear Lie groups are the only ones we consider, and we shall call them simply *Lie groups.* Thus all of the examples above are examples of Lie groups. There are other examples of Lie groups: the symplectic groups $\mathrm{Sp}(n)$ and $\mathrm{Sp}(n, \mathbb{C})$, the spinor groups, etc. In this text we study the Lie groups $\mathrm{SO}(3)$ and $\mathrm{SU}(2)$ in detail.

Remark. Every complex Lie group of complex dimension N is a real Lie group of real dimension $2N$. The group $\mathrm{GL}(n, \mathbb{C})$ and some of its subgroups are also complex Lie groups. This is the case for $\mathrm{SL}(n, \mathbb{C})$, $\mathrm{O}(n, \mathbb{C})$, and $\mathrm{SO}(n, \mathbb{C})$ but, not for $\mathrm{U}(n)$, nor for $\mathrm{SU}(n)$, which are real Lie groups only.

4 Group Actions and Conjugacy Classes

Definition 4.1. *Let G be be a group and let M be a set. A* group action (*or simply an* action) *of G on M is a mapping $\alpha : G \times M \to M$, denoted by $(g, m) \mapsto g \cdot m$, such that for every $m \in M$, $e \cdot m = m$, and for every g and $g' \in G$, $g \cdot (g' \cdot m) = (gg') \cdot m$. In this case we say that G acts on M.*

In other words, α defines a group morphism of G into the group of bijections of M onto itself.

If G is a topological group and if M is a topological space, we shall assume that the action α of G on M is continuous. If G is a Lie group and if M is a smooth manifold, we shall similarly assume that α is smooth.

The *orbit* of $m \in M$ under the action of G is the set $\{g \cdot m \mid g \in G\}$. The orbits define a partition of M.

Examples.

- The trivial action of G on a set M consists in sending each element of the group to the identity mapping of M onto itself. In this case, the orbits are the points of M.
- Under the action of $G = \mathrm{O}(2)$ on the unit sphere $M = S^2 \subset \mathbb{R}^3$ by rotations around the axis Oz, the orbit of $m \in S^2$ is a point if m is the north pole or the south pole, and is otherwise a circle around the axis Oz (a one-dimensional submanifold).
- The action of G on G by left (respectively, right) translation is the action

$$(g, m) \in G \times G \mapsto gm \in G$$

(respectively, $(g, m) \in G \times G \mapsto mg^{-1} \in G$). In other words, we associate to $g \in G$ the left (respectively, right) translation l_g (respectively, $r_{g^{-1}}$) in G, thus defining a morphism of G into the bijections of G onto G (but not into the automorphisms of G).

We remark that the left coset gH of $g \in G$ relative to a subgroup H of G is the orbit of g under the action of $H \subset G$ acting by right translation.

- The action of G on G by conjugation, denoted by $(g,h) \mapsto \mathcal{C}_g(h)$ and defined by
$$\mathcal{C}_g(h) = ghg^{-1},$$

is an action of G on itself by automorphisms. The orbit of $h \in G$,
$$C_h = \{ghg^{-1} \mid g \in G\},$$
is called the *conjugacy class* of h.

Examples.

- The conjugacy class of e is $\{e\}$.
- If G is abelian, the conjugacy class of $g \in G$ is $\{g\}$. This is the case, for example, for $\mathrm{SO}(2)$.
- In $\mathfrak{S}_n$, the number of conjugacy classes is the number of *partitions* of n. (See Exercise 1.3.)
- In $\mathrm{GL}(n,\mathbb{K})$, the conjugacy class of a matrix A is the set of all matrices *similar* to A, $\{PAP^{-1} \mid P \in \mathrm{GL}(n,\mathbb{K})\}$.
- In $\mathrm{SO}(3)$, two rotations are conjugate by a rotation if and only if their angles are equal or opposite. The conjugacy classes are in bijective correspondence with the interval $[0,\pi]$.

In fact, by a direct orthonormal change of basis, any rotation can be written as $g(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$, with $\theta \in [0,2\pi]$. From the equality $g(-\theta) = g_0 g(\theta) g_0^{-1}$, where $g_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, it is easy to see that $g(\theta')$ is conjugate to $g(\theta)$ if and only if $\theta' = \theta$ or $\theta' = -\theta$.

References

For the classification of the finite subgroups of $\mathrm{SO}(3)$, see Sternberg (1994), Artin (1991), or Ledermann–Weir (1996). Sternberg also analyzes the subgroups of $\mathrm{O}(3)$ and identifies, for each subgroup, a crystal possessing that symmetry.

Basic notions in topology such as discrete spaces, Hausdorff spaces, compact and locally compact spaces, as well as connected spaces, are recalled in many books on representation theory. See, e.g., Stillwell (2008) or Naimark–Štern (1982).

One can find an introduction to the theory of differentiable manifolds and Lie groups in Sagle–Walde (1973) and Rossmann (2002), and, with applications to mechanics, in Marsden–Ratiu (1999). For the study of the differential geometry of Lie groups considered as manifolds see, for example, Warner (1983).

Exercises

Exercise 1.1 *Application of Lagrange's Theorem.*

Show that in a finite group G of order n, for each element $a \in G$, $a^n = e$. Conclude that every morphism of G into the group $\mathrm{GL}(1,\mathbb{C}) = \mathbb{C}^*$ takes values in the group of nth roots of unity.

Exercise 1.2 *Center of $\mathfrak{S}_n$.*

Show that for $n \geq 3$, the only element in the center of the symmetric group $\mathfrak{S}_n$ is the identity.

Exercise 1.3 *Conjugacy classes of $\mathfrak{S}_n$.*

An element in $\mathfrak{S}_n$ is called a *cycle* of length k ($1 \leq k \leq n$) if it is a circular permutation of k elements that leaves the other $n-k$ elements invariant.

(a) Show that each permutation is the composition of disjoint cycles. Is this decomposition unique?
(b) Show that in $\mathfrak{S}_n$ two elements are conjugate if and only if they have the same decomposition into cycles. Deduce that the number of conjugacy classes of $\mathfrak{S}_n$ is the number of partitions of the integer n, that is, of finite sequences of integers $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$, and $\sum_{i=1}^{\ell} \lambda_i = n$.

Exercise 1.4 *Semidirect products of groups.*

If a group H acts on a group N by group automorphisms, we define a multiplication on $G = N \times H$ by

$$(n,h)(n',h') = (n(h \cdot n'), hh').$$

(a) Show that, equipped with this multiplication, G is a group. This group, denoted by $N \rtimes H$, is called the *semidirect product* of N and H. What is the inverse of (n,h) in G? Show that N is a normal subgroup of G.
(b) Show that the group $\mathfrak{S}_3$ is the semidirect product of the alternate group $\mathfrak{A}_3$ and $\mathbb{Z}_2$. Is the group $\mathfrak{S}_n$ a semidirect product of the alternate group $\mathfrak{A}_n$ and $\mathbb{Z}_2$, for $n \geq 3$?
(c) Find the standard definitions of the Galilean group and the Poincaré group in any textbook. Show that each of these groups is a semidirect product.

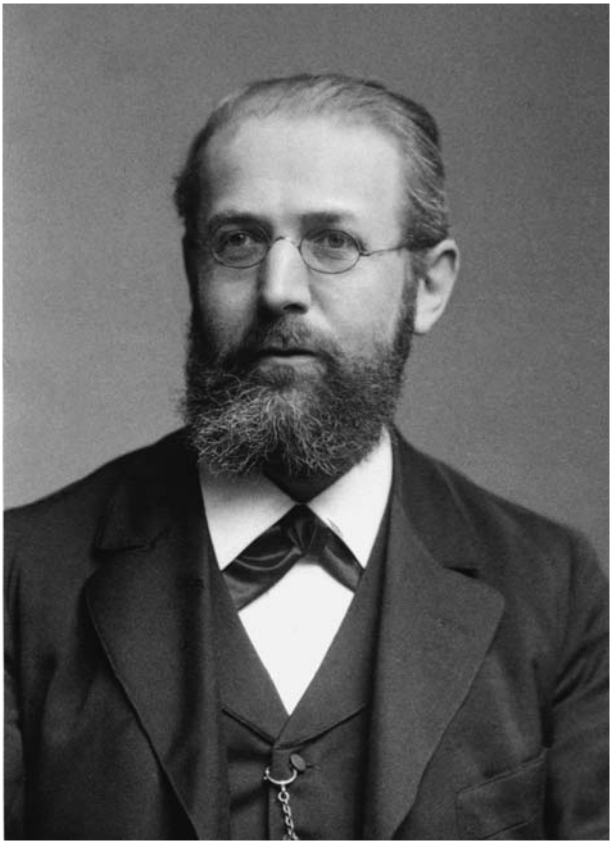
Exercise 1.5 *Group of isometries of the plane.*

(a) Show that the group $\mathbb{Z}_2 = \{1, -1\}$ acts by group automorphisms on $\mathrm{SO}(2)$ via $\varepsilon \cdot g = g^\varepsilon$, where $\varepsilon = \pm 1$. Show that $\mathrm{O}(2)$ is the semidirect product $\mathrm{SO}(2) \rtimes \mathbb{Z}_2$.
(b) Determine the conjugacy classes of $\mathrm{SO}(2)$ and of $\mathrm{O}(2)$.

Exercise 1.6 *The dihedral group $D_{(n)}$.*

Let $D_{(n)}$ be the dihedral group of order $2n$, i.e., the group of rotations and symmetries of the plane that leave invariant a regular n-sided polygon ($n \geq 3$).

(a) Show that $D_{(n)} = \Gamma_n \rtimes \mathbb{Z}_2$, where Γ_n is the cyclic group of order n and the group $\mathbb{Z}_2 = \{1, -1\}$ acts on Γ_n by $\varepsilon \cdot g = g^\varepsilon$, where $\varepsilon = \pm 1$.
(b) Show that $D_{(n)}$ is isomorphic to $\mathfrak{S}_n$ if and only if $n = 3$.
(c) The set $\{\pm 1, \pm i, \pm j, \pm k\}$, with multiplication law $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$, is a group, called the *quaternion group*. Show that $D_{(4)}$ is not isomorphic to the quaternion group.



Ferdinand Georg Frobenius (1849–1917), who created the representation theory of finite groups around 1896, was a professor in Berlin and Zurich.

(*Collection of the Mittag-Leffler Institute, Swedish Royal Academy of Science*)

Chapter 2

Representations of Finite Groups

In mathematics and physics, the notion of a group representation is fundamental. The idea is to study the different ways that groups can act on vector spaces by *linear transformations.*

In this chapter, unless otherwise indicated, we shall consider only representations of finite *groups in* complex, finite-dimensional *vector spaces.*

1 Representations

1.1 General Facts

Let G be a finite group. If E is a vector space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we denote by $\mathrm{GL}(E)$ the group of $\mathbb{K}$-linear isomorphisms of E. (The group $\mathrm{GL}(E)$ is not finite unless $E = \{0\}$.)

Definition 1.1. *A* representation *of a group G is a finite-dimensional complex vector space E along with a group morphism of groups $\rho : G \to \mathrm{GL}(E)$.*

Thus, for every $g, g' \in G$,

$$\rho(gg') = \rho(g)\rho(g'), \quad \rho(g^{-1}) = (\rho(g))^{-1}, \quad \rho(e) = \mathrm{Id}_E.$$

The vector space E is called the *support* of the representation, and the dimension of E is called the *dimension of the representation.* We denote such a representation by (E, ρ) or simply ρ.

If in particular $E = \mathbb{C}^n$, we say that the representation is a *matrix representation* of dimension n.

The *fundamental representation* of a subgroup G of $\mathrm{GL}(E)$ is the representation of G on E defined by the canonical injection of G into $\mathrm{GL}(E)$.

Any representation such that $\rho(g) = \mathrm{Id}_E$ for each $g \in G$ is called a *trivial representation.*

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Example 1.2. Here is a first example of a representation of a nonabelian group. Let $t \in \mathfrak{S}_3$ be the transposition $123 \mapsto 132$ and c the cyclic permutation $123 \mapsto 231$ that generate $\mathfrak{S}_3$. We set $j = e^{2i\pi/3}$, so that $j^2 + j + 1 = 0$. We can represent $\mathfrak{S}_3$ on $\mathbb{C}^2$ by defining

$$\rho(e) = I, \quad \rho(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho(c) = \begin{pmatrix} j & 0 \\ 0 & j^2 \end{pmatrix}.$$

Definition 1.3. *Let* $(\ |\ )$ *be a scalar product on* E*. We say that the representation* ρ *is* unitary *if* $\rho(g)$ *is unitary for every* g*, that is, if*

$$\forall g \in G,\ \forall x, y \in E, \quad (\rho(g)x \mid \rho(g)y) = (x \mid y).$$

A representation (E, ρ) is called *unitarizable* if there is a scalar product on E such that ρ is unitary.

In order to prove the following theorem, as well as many other propositions, we shall use a fundamental property:

Lemma 1.4. *Let* G *be a finite group. For every function* φ *on* G *taking values in a vector space,*

$$\forall g \in G, \quad \sum_{h \in G} \varphi(gh) = \sum_{h \in G} \varphi(hg) = \sum_{k \in G} \varphi(k). \tag{1.1}$$

Proof. In fact, once g is chosen, every element of G can be written uniquely in the form gh (or hg), where $h \in G$. □

Theorem 1.5. *Every representation of a finite group is unitarizable.*

Proof. Let (E, ρ) be a representation of a finite group G, and let $(\ \mid\ )$ be a scalar product on E. We consider

$$(x \mid y)' = \frac{1}{|G|} \sum_{g \in G} (\rho(g)x \mid \rho(g)y),$$

which is a scalar product on E. In fact, suppose that $(x \mid x)' = 0$, that is, $\sum_{g \in G} (\rho(g)x \mid \rho(g)x) = 0$. Then for each $g \in G$, $(\rho(g)x \mid \rho(g)x) = 0$, and in particular, $(x \mid x) = 0$, whence $x = 0$.

This scalar product on E is invariant under ρ. In fact,

$$\begin{aligned}(\rho(g)x \mid \rho(g)y)' &= \frac{1}{|G|} \sum_{h \in G} (\rho(h)\rho(g)x \mid \rho(h)\rho(g)y) \\ &= \frac{1}{|G|} \sum_{h \in G} (\rho(hg)x \mid \rho(hg)y) = (x \mid y)',\end{aligned}$$

where we have used the fundamental equation (1.1), which holds for any function φ on G. Thus ρ is a unitary representation of G on $(E, (\ \mid\ )')$. □

1.2 Irreducible Representations

Let (E, ρ) be a representation of G. A vector subspace $F \subset E$ is called *invariant* (or *stable*) under ρ (or under G, if the name of the representation is understood) if for every $g \in G$, $\rho(g)F \subset F$. (since F is finite-dimensional, the condition $\rho(g)F \subset F$ implies $\rho(g)F = F$.) We can then speak of the representation ρ restricted to F, which is a representation of G on F. We denote it by $\rho|_F$. Such a representation restricted to an invariant subspace is also called a *subrepresentation.*

Definition 1.6. *A representation (E, ρ) of G is called* irreducible *if $E \neq \{0\}$ and if the only vector subspaces of E invariant under ρ are $\{0\}$ and E itself.*

Example. The representation of dimension 2 of $\mathfrak{S}_3$ defined in Example 1.2 is irreducible, since the eigenspaces of $\rho(t)$ and $\rho(c)$ have trivial intersection.

Proposition 1.7. *Every irreducible representation of a finite group is finite-dimensional.*

Proof. Let (E, ρ) be an irreducible representation of a finite group G and let $x \in E$. Because the subset $\{\rho(g)x \mid g \in G\}$ is finite, it generates a finite-dimensional vector subspace of E. If $x \neq 0$, this vector subspace of E is not equal to $\{0\}$. Because this subspace is invariant under ρ, it coincides with E, which is thus finite-dimensional. □

1.3 Direct Sum of Representations

Definition 1.8. *Let (E_1, ρ_1) and (E_2, ρ_2) be representations of G. Then*

$$(E_1 \oplus E_2, \rho_1 \oplus \rho_2),$$

where $(\rho_1 \oplus \rho_2)(g)(x_1, x_2) = (\rho_1(g)(x_1), \rho_2(g)(x_2))$, for $g \in G, x_1 \in E_1, x_2 \in E_2$, is a representation of G called the direct sum *of the representations (E_1, ρ_1) and (E_2, ρ_2).*

Clearly a direct sum of representations of strictly positive dimensions cannot be irreducible, even if the summands are irreducible. For matrix representations ρ_1 and ρ_2, the matrices of the direct sum representation of ρ_1 and ρ_2 are block-diagonal matrices

$$\begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix}.$$

More generally, if m is a strictly positive integer, we can use recursion to define the direct sum of m representations $\rho_1 \oplus \cdots \oplus \rho_m$. If (E, ρ) is a representation of G we denote by $m\rho$ the representation $\rho \oplus \cdots \oplus \rho$ (direct sum of m terms) on the vector space $E \oplus \cdots \oplus E$ (m terms).

A representation is called *completely reducible* if it is a direct sum of irreducible representations.

Lemma 1.9. *Let ρ be a unitary representation of G on $(E, (\ \mid\ ))$. If $F \subset E$ is invariant under ρ, then $F^\perp = \{y \in E \mid \forall x \in F, (x \mid y) = 0\}$ is also invariant under ρ.*

Proof. Let $y \in F^\perp$. Then, because F is invariant under ρ, for every $g \in G$ and $x \in F, (x \mid \rho(g)y) = (\rho(g^{-1})x \mid y) = 0$. Thus $\rho(g)y \in F^\perp$. □

Theorem 1.10 (Maschke's Theorem). *Every finite-dimensional representation of a finite group is completely reducible.*

Proof. Let (E, ρ) be be a representation of G. By Theorem 1.5, one may suppose this representation to be unitary. If ρ is not irreducible, let F be a vector subspace of E invariant under ρ such that $F \neq \{0\}$ and $F \neq E$. Then $E = F \oplus F^\perp$, where F (by hypothesis) and $F^\perp$ (by Lemma 1.9) are invariant under ρ, and $\dim F < \dim E$, $\dim F^\perp < \dim E$. By induction on the dimension of E, we obtain the desired result. □

In fact, this theorem is true under more general conditions. (See the study of compact groups in Chapter 3.)

1.4 Intertwining Operators and Schur's Lemma

Definition 1.11. *Let (E_1, ρ_1) and (E_2, ρ_2) be representations of G. We say that a linear map $T : E_1 \to E_2$* intertwines *ρ_1 and ρ_2 if*

$$\forall g \in G,\ \ \rho_2(g) \circ T = T \circ \rho_1(g),$$

in which case T is called an intertwining operator *for ρ_1 and ρ_2.*

The definition can be expressed in the commutativity of the following diagram for each $g \in G$:

$$\begin{array}{ccc} E_1 & \xrightarrow{T} & E_2 \\ {\scriptstyle \rho_1(g)}\downarrow & & \downarrow{\scriptstyle \rho_2(g)} \\ E_1 & \xrightarrow{T} & E_2 \end{array}$$

The following expressions are often used to express the same property:

- T is equivariant under ρ_1 and ρ_2,
- T is a morphism of G-vector spaces,
- T is a G-morphism,
- $T \in \mathrm{Hom}_G(E_1, E_2)$.

If $E_1 = E_2 = E$ and if $\rho_1 = \rho_2 = \rho$, an intertwining operator for ρ_1 and ρ_2 is just an operator that *commutes* with ρ.

Definition 1.12. *The representations ρ_1 and ρ_2 are called* equivalent *if there is a* bijective *intertwining operator for ρ_1 and ρ_2.*

If T is such a bijective intertwining operator, then

$$\forall g \in G,\ \rho_2(g) = T \circ \rho_1(g) \circ T^{-1}.$$

The existence of an intertwining operator is an equivalence relation on representations, which leads to the notion of an *equivalence class of representations.* We let $\sim$ denote this equivalence relation.

Two representations (E_1, ρ_1) and (E_2, ρ_2) are equivalent if and only if there is a basis B_1 of E_1 and a basis B_2 of E_2 such that for every $g \in G$, the matrix of $\rho_1(g)$ in the basis B_1 is equal to the matrix of $\rho_2(g)$ in the basis B_2. In particular, if the representations (E_1, ρ_1) and (E_2, ρ_2) are equivalent, then E_1 is isomorphic to E_2.

For equivalent matrix representations, we thus obtain *similar matrices*: if $E_1 = E_2 = \mathbb{C}^n$, and if ρ_1 and ρ_2 are equivalent, then the matrices $\rho_1(g)$ and $\rho_2(g)$ are similar via the same similarity matrix for every g.

If ρ_0 is an n-dimensional representation of G on E, the choice of a basis (e_i) of E determines a matrix representation $(\mathbb{C}^n, \rho)$; by changing to the basis (e'_i) via a matrix T, one obtains the equivalent representation $(\mathbb{C}^n, \rho')$,

$$\rho'(g) = T \circ \rho(g) \circ T^{-1}.$$

Lemma 1.13. *If T intertwines ρ_1 and ρ_2, then the kernel of T,* Ker T*, is invariant under ρ_1, and the image of T,* ImT*, is invariant under ρ_2.*

Proof. If $x \in E_1$ and $Tx = 0$, then $T(\rho_1(g)x) = \rho_2(g)(Tx) = 0$. Thus Ker T is a subspace of E_1 invariant under ρ_1.

Let $y \in$ Im T. Then, there exists $x \in E_1$ such that $y = Tx$. Therefore $\rho_2(g)y = \rho_2(g)(Tx) = T(\rho_1(g)x)$, and hence Im T is a subspace of E_2 invariant under ρ_2. □

Lemma 1.14. *If T commutes with ρ, each eigenspace of T is invariant under ρ.*

Proof. In fact, if $Tx = \lambda x, \lambda \in \mathbb{C}$, then $T(\rho(g)x) = \lambda\rho(g)x$. Thus the eigenspace of T corresponding to the eigenvalue λ is invariant under ρ. □

Theorem 1.15 (Schur's Lemma). *Let T be an operator intertwining irreducible representations (E_1, ρ_1) and (E_2, ρ_2) of G.*

- *If ρ_1 and ρ_2 are not equivalent, then $T = 0$.*
- *If $E_1 = E_2 = E$ and $\rho_1 = \rho_2 = \rho$, then T is a scalar multiple of the identity of E.*

Proof. If ρ_1 and ρ_2 are not equivalent, T is not bijective. Hence either Ker$T \neq \{0\}$, or Im $T \neq E_2$. By Lemma 1.13, Ker T is invariant under ρ_1. Because ρ_1 is irreducible, if Ker $T \neq \{0\}$, then Ker $T = E_1$; hence $T = 0$. By Lemma 1.13, Im T is invariant under ρ_2. Because ρ_2 is irreducible, if Im $T \neq E_2$, then Im $T = \{0\}$, and hence $T = 0$.

If $E_1 = E_2 = E$ and $\rho_1 = \rho_2 = \rho$, then for every $g \in G, \rho(g) \circ T = T \circ \rho(g)$, and T commutes with the representation ρ. Let λ be an eigenvalue of T, which must exist because T is an endomorphism of E, a vector space over $\mathbb{C}$, and let

E_λ be the eigenspace associated to λ. By Lemma 1.14, E_λ is invariant under ρ. By hypothesis $E_\lambda \neq \{0\}$, therefore, since ρ is irreducible, $E_\lambda = E$, which means that $T = \lambda \operatorname{Id}_E$. We remark that the proof of the second part of the theorem uses the hypothesis that the vector space of the representation is a complex vector space. □

Conversely, if each operator commuting with the representation ρ is a scalar multiple of the identity, then ρ is irreducible. In fact, if ρ were not irreducible, the projection onto a nontrivial invariant subspace would be a nonscalar operator commuting with ρ.

Remark. Lemma 1.14 has very important consequences in quantum mechanics. The symmetry operators of a system represented by a Hamiltonian $\widehat{H}$ (an operator acting on a Hilbert space) are precisely the operators that commute with $\widehat{H}$. For each energy level, that is, for each eigenvalue of the Hamiltonian, there is a corresponding eigenspace. By this lemma, each eigenspace is the support of a representation of the group of symmetries of the system. *Wigner's principle* then states that for each energy level, the corresponding representation is an irreducible representation of the full symmetry group of the system. The dimension of the representation corresponding to the given energy level is called the degree of *degeneracy* of the energy level.

2 Characters and Orthogonality Relations

2.1 Functions on a Group, Matrix Coefficients

We shall denote by $\mathcal{F}(G)$, or sometimes by $\mathbb{C}[G]$, the vector space of functions on G taking values in $\mathbb{C}$. When this vector space is equipped with the scalar product defined below, we call the resulting Hilbert space $L^2(G)$. (This definition will be extended to compact groups.)

We adopt the convention that a *scalar product* is antilinear in the first argument and linear in the second.

Definition 2.1. *On $L^2(G)$, the scalar product is defined by*

$$(f_1 \mid f_2) = \frac{1}{|G|} \sum_{g \in G} \overline{f_1(g)} f_2(g).$$

We shall be interested in the matrix coefficients of representations.

Definition 2.2. *If ρ is a representation of G on $\mathbb{C}^n$, then for every ordered pair (i,j), $1 \leq i \leq n$, $1 \leq j \leq n$, the function $\rho_{ij} \in L^2(G)$ defined for each $g \in G$ to be the coefficient of the matrix $\rho(g)$ in the ith row and the jth column, $(\rho(g))_{ij} \in \mathbb{C}$, is called a* matrix coefficient *of ρ.*

For a representation ρ on a vector space E, we define the matrix coefficients ρ_{ij} relative to a basis (e_i) satisfying

$$\rho(g) e_j = \sum_i \rho_{ij}(g) e_i,$$

where i is the row index and j is the column index. If ρ is a unitary representation on a finite-dimensional Hilbert space, then

$$\rho(g^{-1}) = (\rho(g))^{-1} = {}^t\overline{(\rho(g))}.$$

Hence, in an orthonormal basis,

$$\rho_{ij}(g^{-1}) = \overline{\rho_{ji}(g)},$$

and in particular, the diagonal coefficients of $\rho(g)$ and $\rho(g^{-1})$ are complex conjugates.

2.2 Characters of Representations and Orthogonality Relations

We denote by Tr the trace of an endomorphism.

Definition 2.3. *Let (E, ρ) be a representation of G. The character of ρ is the function χ_ρ on G taking complex values defined by*

$$\forall g \in G,\ \chi_\rho(g) = \mathrm{Tr}\,(\rho(g)).$$

Equivalent representations have the same character.
For a matrix representation of dimension n,

$$\chi_\rho(g) = \sum_{i=1}^{n} (\rho(g))_{ii}. \tag{2.1}$$

On each conjugacy class of G, the function χ_ρ is constant.

Definition 2.4. *A* class function *on G is a function constant on each conjugacy class.*

Thus characters of representations are class functions on the group.

Proposition 2.5. *The following are elementary properties of characters:*

- $\chi_\rho(e) = \dim \rho$.
- $\forall g \in G,\ \chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$.
- *The character of a direct sum of representations is the sum of the characters, $\chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2}$.*

Proof. The first property is a consequence of formula (2.1). To prove the second formula, we may assume that ρ is unitary in a certain scalar product and choose an orthonormal basis. The direct sum property is obvious. □

If (E_1, ρ_1) and (E_2, ρ_2) are representations of the same group G, we define their *tensor product* to be $(E_1 \otimes E_2, \rho_1 \otimes \rho_2)$, where

$$(\rho_1 \otimes \rho_2)(g) = \rho_1(g) \otimes \rho_2(g),$$

for each $g \in G$. (See Exercise 2.5 for a review of the relevant definitions.) The following is an important property of characters.

Proposition 2.6. *The character of a tensor product of representations is the product of the characters,*

$$\chi_{\rho_1 \otimes \rho_2} = \chi_{\rho_1}\ \chi_{\rho_2}. \tag{2.2}$$

Proof. The equality follows from the fact that the trace of a tensor product of matrices is the product of the traces. □

By Proposition 2.5, for representations ρ_1 and ρ_2 of G,

$$(\chi_{\rho_1} \mid \chi_{\rho_2}) = \frac{1}{|G|} \sum_{g \in G} \chi_{\rho_1}(g^{-1}) \chi_{\rho_2}(g). \tag{2.3}$$

We shall show that the characters of inequivalent irreducible representations are orthogonal and that the character of an irreducible representation is of norm 1.

Proposition 2.7. *Let (E_1, ρ_1) and (E_2, ρ_2) be representations of G and let $u : E_1 \to E_2$ be a linear map. Then the linear map $T_u : E_1 \to E_2$ defined by*

$$T_u = \frac{1}{|G|} \sum_{g \in G} \rho_2(g)\ u\ \rho_1(g)^{-1} \tag{2.4}$$

intertwines ρ_1 and ρ_2.

Proof. We calculate

$$\begin{aligned} \rho_2(g)\ T_u &= \frac{1}{|G|} \sum_{h \in G} \rho_2(gh)\ u\ \rho_1(h^{-1}) \\ &= \frac{1}{|G|} \sum_{k \in G} \rho_2(k)\ u\ \rho_1(k^{-1}g), \end{aligned}$$

by the fundamental equation (1.1). Hence,

$$\rho_2(g)\ T_u = T_u\ \rho_1(g).$$

The operator T_u is thus an intertwining operator for ρ_1 and ρ_2. □

Proposition 2.8. *Let (E_1, ρ_1) and (E_2, ρ_2) be irreducible representations of G, let $u : E_1 \to E_2$ be a linear map, and define T_u by equation* (2.4).

(*i*) *If ρ_1 and ρ_2 are inequivalent, then $T_u = 0$.*
(*ii*) *If $E_1 = E_2 = E$ and $\rho_1 = \rho_2 = \rho$, then*

$$T_u = \frac{\operatorname{Tr} u}{\dim E}\ \mathrm{Id}_E.$$

Proof. The first assertion is clear by Schur's lemma (Theorem 1.15). For the second, we need only calculate λ given that $T_u = \lambda \, \mathrm{Id}_E$. So we obtain $\operatorname{Tr} T_u = \frac{1}{|G|} \sum_{g \in G} \operatorname{Tr} u = \operatorname{Tr} u$, and thus $\lambda = \frac{\operatorname{Tr} u}{\dim E}$. □

Proposition 2.9. *Let (E_1, ρ_1) and (E_2, ρ_2) be irreducible representations of G. We choose bases in E_1 and E_2.*

(*i*) *If ρ_1 and ρ_2 are inequivalent, then*

$$\forall i, j, k, \ell, \sum_{g \in G} (\rho_2(g))_{k\ell} (\rho_1(g^{-1}))_{ji} = 0.$$

(*ii*) *If $E_1 = E_2 = E$ and $\rho_1 = \rho_2 = \rho$, then*

$$\frac{1}{|G|} \sum_{g \in G} (\rho(g))_{k\ell} (\rho(g^{-1}))_{ji} = \frac{1}{\dim E} \delta_{ki} \delta_{\ell j}.$$

Proof. We use a basis (e_j) of E_1, $1 \leq j \leq \dim E_1$, and a basis (f_ℓ) of E_2, $1 \leq \ell \leq \dim E_2$. For $u : E_1 \to E_2$, T_u is defined by (2.4). We have, for $1 \leq i \leq \dim E_1, 1 \leq k \leq \dim E_2$,

$$(T_u)_{ki} = \frac{1}{|G|} \sum_{g \in G} \sum_{m=1}^{\dim E_1} \sum_{p=1}^{\dim E_2} (\rho_2(g))_{kp} \; u_{pm} \; (\rho_1(g^{-1}))_{mi}.$$

Let us choose our linear map u to be the map $u_{(\ell j)} : E_1 \to E_2$ defined by $u_{(\ell j)}(e_k) = \delta_{jk} f_\ell$. Then

$$(u_{(\ell j)})_{pm} = \delta_{\ell p} \delta_{jm},$$

and consequently,

$$(T_{u_{(\ell j)}})_{ki} = \frac{1}{|G|} \sum_{g \in G} (\rho_2(g))_{k\ell} (\rho_1(g^{-1}))_{ji}.$$

Next we apply Proposition 2.8. If ρ_1 and ρ_2 are inequivalent, then $T_{u_{(\ell j)}}$ is always zero, whence (i). If $E_1 = E_2 = E$ and $\rho_1 = \rho_2 = \rho$, then

$$\frac{1}{|G|} \sum_{g \in G} (\rho(g))_{k\ell} (\rho(g^{-1}))_{ji} = (T_{u_{(\ell j)}})_{ki} = \frac{\operatorname{Tr} u_{(\ell j)}}{\dim E} \delta_{ki} = \frac{\delta_{ki} \delta_{\ell j}}{\dim E},$$

which proves (ii). □

Corollary 2.10. *Let (E_1, ρ_1) and (E_2, ρ_2) be unitary irreducible representations of G. We choose orthonormal bases in E_1 and E_2.*

(*i*) *If ρ_1 and ρ_2 are inequivalent, then for every i, j, k, l,*

$$((\rho_1)_{ij} \mid (\rho_2)_{k\ell}) = 0.$$

(*ii*) *If $E_1 = E_2 = E$ and $\rho_1 = \rho_2 = \rho$, then for every i, j, k, l,*

$$(\rho_{ij} \mid \rho_{k\ell}) = \frac{1}{\dim E} \delta_{ik} \delta_{j\ell}.$$

Proof. In fact, if ρ_1 is unitary for a scalar product on E_1 and if the chosen basis in E_1 is orthonormal, then

$$\frac{1}{|G|}\sum_{g\in G}(\rho_2(g))_{k\ell}(\rho_1(g^{-1}))_{ji} = \frac{1}{|G|}\sum_{g\in G}(\rho_2(g))_{k\ell}\overline{(\rho_1(g))_{ij}} = ((\rho_1)_{ij} \mid (\rho_2)_{k\ell}).$$

Proposition 2.9 thus implies (i) and (ii). □

Theorem 2.11 (Orthogonality Relations). *Let G be a finite group.*

(*i*) *If ρ_1 and ρ_2 are inequivalent irreducible representations of G, then*

$$(\chi_{\rho_1} \mid \chi_{\rho_2}) = 0.$$

(*ii*) *If ρ is an irreducible representation of G, then*

$$(\chi_\rho \mid \chi_\rho) = 1.$$

Proof. By the equality (2.3) and the preceding proposition, if ρ_1 and ρ_2 are inequivalent irreducible representations, then $(\chi_{\rho_1} \mid \chi_{\rho_2}) = 0$. If $\rho_1 = \rho_2 = \rho$, then $\frac{1}{|G|}\sum_{g\in G}\rho(g)_{ii}\rho(g^{-1})_{jj} = \frac{\delta_{ij}}{\dim E}$, whence $(\chi_\rho \mid \chi_\rho) = 1$. □

We define the *irreducible characters* of G to be the set of characters of inequivalent irreducible representations of G. We write χ_{ρ_i} or even χ_i to denote the character of an irreducible representation ρ_i. The preceding results can be formulated as follows.

Theorem 2.12. *The irreducible characters of G form an orthonormal set in $L^2(G)$.*

Corollary 2.13. *The inequivalent irreducible representations of a finite group G are finite in number.*

We shall denote by $\widehat{G}$ the set of equivalence classes of irreducible representations of G.

2.3 Character Table

"Character table" is the name given to the table whose columns correspond to conjugacy classes of a group and whose rows correspond to inequivalent irreducible representations of the group. At the intersection of the row and the column one writes the value of the character of the representation, evaluated on an element (any element) of the conjugacy class. Let N be the number of conjugacy classes of the group G. (In other words, N is the number of columns; we shall show that it is also the number of rows.) Let g_i be an element of G in the conjugacy class C_{g_i}, $1 \le i \le N$, which consists of $|C_{g_i}|$ elements. Let ρ_k and ρ_ℓ be irreducible representations of G. Then

$$(\chi_{\rho_k} \mid \chi_{\rho_\ell}) = \frac{1}{|G|}\sum_{i=1}^{N}|C_{g_i}|\ \overline{\chi_{\rho_k}(g_i)}\ \chi_{\rho_\ell}(g_i) = \delta_{k\ell}.$$

This formula can be restated as the following result.

Proposition 2.14. *If the ith column is given weight $|C_{g_i}|$, the rows of the character table are orthogonal and of norm $\sqrt{|G|}$.*

We write character tables in the following form:

	$\lvert C_{g_1}\rvert$ g_1	$\ldots\ldots$ $\ldots\ldots$	$\lvert C_{g_N}\rvert$ g_N
$\ldots$	$\ldots$	$\ldots\ldots$	$\ldots$
χ_{ρ_k}	$\chi_{\rho_k}(g_1)$	$\ldots\ldots$	$\chi_{\rho_k}(g_N)$
$\ldots$	$\ldots$	$\ldots\ldots$	$\ldots$
χ_{ρ_ℓ}	$\chi_{\rho_\ell}(g_1)$	$\ldots\ldots$	$\chi_{\rho_\ell}(g_N)$
$\ldots$	$\ldots$	$\ldots\ldots$	$\ldots$

2.4 Application to the Decomposition of Representations

We denote by $\rho_1, \ldots, \rho_N$ the inequivalent irreducible representations of G. (We shall see in Corollary 3.7 that this number N equals the number of conjugacy classes of G.) More precisely, we choose from each equivalence class of representations of G a representative that we denote by ρ_i.

In the equalities below, the equal sign denotes membership in the same equivalence class.

Theorem 2.15. *Let ρ be any representation of G and let χ_ρ be its character. Then*

$$\rho = \bigoplus_{i=1}^{N} m_i \rho_i,$$

where

$$m_i = (\chi_{\rho_i} \mid \chi_\rho).$$

Proof. We know by Theorem 1.10 that ρ is direct sum of irreducible representations. We can group the terms corresponding to the same equivalence class of irreducible representations ρ_i, and we obtain $\rho = \oplus_{i=1}^N m_i \rho_i$, for some nonnegative integers m_i. One sees then that $\chi_\rho = \sum_{i=1}^N m_i \, \chi_{\rho_i}$, and hence by orthogonality $(\chi_{\rho_i} \mid \chi_\rho) = m_i \, (\chi_{\rho_i} \mid \chi_{\rho_i}) = m_i$. □

Definition 2.16. *If ρ admits the decomposition*

$$\rho = m_1 \rho_1 \oplus m_2 \rho_2 \oplus \cdots \oplus m_N \rho_N,$$

then the nonnegative integer m_i is the multiplicity *of ρ_i in ρ, and $m_i \rho_i$ is the* isotypic component *of type ρ_i of ρ.*

Corollary 2.17. *The decomposition into isotypic components is unique up to order.*

Corollary 2.18. *Two representations with the same character are equivalent.*

By the previous theorem,

$$(\chi_\rho \mid \chi_\rho) = \sum_{i=1}^{N} m_i^2.$$

Hence we have the following result.

Theorem 2.19 (Irreducibility Criterion). *A representation ρ is irreducible if and only if $(\chi_\rho \mid \chi_\rho) = 1$.*

3 The Regular Representation

3.1 Definition

In general, if a group G acts on a set M, then G acts linearly on the space $\mathcal{F}(M)$ of functions on M taking values in $\mathbb{C}$ by $(g, f) \in G \times \mathcal{F}(M) \mapsto g \cdot f \in \mathcal{F}(M)$, where

$$\forall x \in M,\ (g \cdot f)(x) = f(g^{-1}x).$$

We can see immediately that this gives us a representation of G on $\mathcal{F}(M)$.

Take $M = G$, the group acting on itself by left multiplication. One obtains a representation R of G on $\mathcal{F}(G)$ called the *left regular representation* (or simply *regular representation*) of G. Thus, by definition,

$$\forall g, h \in G,\ (R(g)f)(h) = f(g^{-1}h).$$

In the same way one can define the *right regular representation* R', associated to the right action of G on itself, by $(R'(g)f)(h) = f(hg)$. The right and left regular representations are equivalent. For a finite group G the vector space $\mathcal{F}(G)$ of maps of G into $\mathbb{C}$ is finite-dimensional, of dimension $|G|$. The regular representation is thus of dimension $|G|$.

We use the basis $(\epsilon_g)_{g \in G}$ of $\mathcal{F}(G)$ defined by

$$\epsilon_g : G \to \mathbb{C} \begin{cases} \epsilon_g(g) = 1, \\ \epsilon_g(h) = 0, \ \text{if } h \neq g. \end{cases}$$

The regular representation of G satisfies

$$\forall g,\ h \in G,\ R(g)(\epsilon_h) = \epsilon_{gh}.$$

In fact, for every $k \in G$, $(R(g)\epsilon_h)(k) = \epsilon_h(g^{-1}k)$, and $\epsilon_h(g^{-1}k) = 1$ if $k = gh$, while $\epsilon_h(g^{-1}k) = 0$ otherwise. (In the right regular representation, $\epsilon_h \mapsto \epsilon_{hg^{-1}}$.)

Proposition 3.1. *On $L^2(G) = \mathcal{F}(G)$ with scalar product $(\ \mid\ )$, the regular representation is unitary.*

Proof. For f_1 and $f_2 \in L^2(G)$ we have, for every $g \in G$,

$$\begin{aligned}(R(g)f_1 \mid R(g)f_2) &= \frac{1}{|G|}\sum_{h\in G}\overline{(R(g)f_1)(h)}(R(g)f_2)(h)\\ &= \frac{1}{|G|}\sum_{h\in G}\overline{f_1(g^{-1}h)}f_2(g^{-1}h)\\ &= \frac{1}{|G|}\sum_{k\in G}\overline{f_1(k)}f_2(k) = (f_1 \mid f_2).\end{aligned}$$

The operator $R(g)$ is thus unitary for every $g \in G$. □

3.2 Character of the Regular Representation

On the one hand,

$$\chi_R(e) = \mathrm{Tr}(R(e)) = \dim \mathcal{F}(G) = |G|.$$

On the other hand, if $g \neq e$, then

$$\chi_R(g) = \mathrm{Tr}(R(g)) = 0$$

because in this case, for every $h \in G$, $R(g)\epsilon_h \neq \epsilon_h$.

The regular representation R is reducible because $\sum_{g\in G}\epsilon_g$ generates a vector subspace W of $\mathcal{F}(G)$ of dimension 1 that is invariant under R. In fact, for every $g \in G$, $R(g)(\sum_{h\in G}\epsilon_h) = \sum_{h\in G}\epsilon_{gh} = \sum_{k\in G}\epsilon_k$. Furthermore, $R\mid_W$ is equivalent to the trivial representation, since for every $x \in W$, $R(g)(x) = x$. We shall show that, in fact, the regular representation contains each irreducible representation of G with multiplicity equal to its dimension.

Example 3.2. The regular representation of $\mathfrak{S}_3$ on $\mathbb{C}[\mathfrak{S}_3]$ is of dimension 6. It decomposes into the direct sum of the one-dimensional trivial representation, the one-dimensional sign representation, and two copies of the two-dimensional irreducible representation studied in Example 1.2.

3.3 Isotypic Decomposition

We now use the notation introduced in Section 2.4.

Proposition 3.3. *The decomposition of the regular representation of G into isotypic components is $R = \oplus_{i=1}^N n_i\rho_i$, where $\rho_i, i = 1,\dots,N$, are the irreducible representations of G, and $n_i = \dim\rho_i$.*

Proof. We know that

$$\chi_R(g) = \begin{cases}|G| & \text{if } g = e,\\ 0 & \text{if } g \neq e,\end{cases}$$

and hence $(\chi_{\rho_i} \mid \chi_R) = \chi_{\rho_i}(e) = \dim\rho_i$. □

Theorem 3.4. *We have*

$$\sum_{i=1}^{N}(n_i)^2 = |G|,$$

where $n_i = \dim \rho_i$.

Proof. We have $|G| = \chi_R(e) = \sum_{i=1}^N n_i \chi_{\rho_i}(e) = \sum_{i=1}^N (n_i)^2$. □

The equality $\sum_{i=1}^N (n_i)^2 = |G|$ is often used, for example, in order to determine the dimension of a "missing" irreducible representation when one already knows $N-1$ representations.

3.4 Basis of the Vector Space of Class Functions

The vector space of class functions on G taking values in $\mathbb{C}$ has for dimension the number of conjugacy classes of G. We shall show that this is also the number of equivalence classes of irreducible representations.

Let (E,ρ) be be a representation of G, and let f be a function on G. We consider the endomorphism ρ_f of E defined by

$$\rho_f = \sum_{g\in G} f(g)\rho(g). \tag{3.1}$$

Thus, by definition, for every $x \in E$, $\rho_f(x) = \sum_{g\in G} f(g)\rho(g)(x)$.

Lemma 3.5. *The endomorphism* ρ_f *has the following properties:*

(*i*) *If* f *is a class function,* ρ_f *commutes with* ρ.
(*ii*) *If* f *is a class function and if* ρ *is irreducible, then*

$$\rho_f = \frac{|G|(\overline{f} \mid \chi_\rho)}{\dim \rho} Id_E.$$

Proof. For every function f, we have

$$\begin{aligned}\rho_f \circ \rho(g) &= \sum_{h\in G} f(h)\rho(h)\rho(g) = \sum_{h\in G} f(h)\rho(hg) \\ &= \sum_{k\in G} f(kg^{-1})\rho(k) = \sum_{h\in G} f(ghg^{-1})\rho(gh).\end{aligned}$$

If f is assumed to be a class function, we obtain

$$\rho_f \circ \rho(g) = \rho(g)\sum_{h\in G} f(h)\rho(h) = \rho(g)\circ\rho_f.$$

Let us prove (ii). By (i) and Schur's lemma (Theorem 1.15), there is a $\lambda \in \mathbb{C}$ such that $\rho_f = \lambda Id_E$. On the other hand, $\mathrm{Tr}\rho_f = \sum_{g\in G} f(g)\mathrm{Tr}\rho(g) = \sum_{g\in G} f(g)\chi_\rho(g) = |G|(\overline{f} \mid \chi_\rho)$, from which the result follows. □

Theorem 3.6. *The irreducible characters form an orthonormal basis of the vector space of class functions.*

Proof. We know that the characters $\rho_1, \ldots, \rho_N$ of inequivalent irreducible representations of G form an orthonormal set in $L^2(G)$ (Theorem 2.12). Let us show that this set spans the vector subspace of class functions. Let f be a class function such that for $1 \leq i \leq N$, $(f|\chi_{\rho_i}) = 0$. We consider $(\rho_i)_{\overline{f}} = \sum_{g\in G} \overline{f}(g)\rho_i(g)$. By the previous lemma, $(\rho_i)_{\overline{f}} = 0$, and we deduce, by decomposition, that for any representation ρ we have $\rho_{\overline{f}} = 0$. In particular, $R_{\overline{f}} = 0$, where R is the regular representation. Thus,

$$0 = R_{\overline{f}}(\epsilon_g) = \sum_{h\in G} \overline{f}(h)R(h)(\epsilon_g) = \sum_{h\in G} \overline{f}(h)\epsilon_{hg},$$

for $g \in G$, and, in particular,

$$0 = R_{\overline{f}}(\epsilon_e) = \sum_{h\in G} \overline{f}(h)\epsilon_h = \overline{f},$$

so $f = 0$. □

Corollary 3.7. *The number of equivalence classes of irreducible representations of a finite group is equal to the number of conjugacy classes of that group.*

In other words, *the character table is square.*

Proposition 3.8. *The columns of the character table of a finite group G are orthogonal and of norm $\sqrt{|G|/|C_g|}$, where $|C_g|$ denotes the number of elements of the conjugacy class of g. Explicitly,*

$$\sum_{i=1}^{N} \overline{\chi_{\rho_i}(g)}\chi_{\rho_i}(g') = 0, \ \textit{if } g \textit{ and } g' \textit{ are not conjugate,}$$

$$\frac{1}{|G|}\sum_{i=1}^{N} \overline{\chi_{\rho_i}(g)}\chi_{\rho_i}(g) = \frac{1}{|C_g|}.$$

In particular, when $g = e$, we recover the equation $\sum_{i=1}^{N}(\dim \rho_i)^2 = |G|$.

Proof. By Theorem 3.6, if f is a class function, then

$$f = \sum_{i=1}^{N} (\chi_{\rho_i} \mid f)\chi_{\rho_i}.$$

For $g \in G$, consider the class function f_g that takes the value 1 on g and the value 0 on every other conjugacy class of G. We have

$$\begin{aligned}(\chi_{\rho_i} \mid f_g) &= \frac{1}{|G|}\sum_{h\in G} \overline{\chi_{\rho_i}(h)} f_g(h) \\ &= \frac{|C_g|}{|G|}\overline{\chi_{\rho_i}(g)}\ ,\end{aligned}$$

and thus $f_g = \frac{|C_g|}{|G|} \sum_{i=1}^N \overline{\chi_{\rho_i}(g)} \chi_{\rho_i}$. In particular, if $g' \notin C_g$, then

$$0 = f_g(g') = \frac{|C_g|}{|G|} \sum_{i=1}^N \overline{\chi_{\rho_i}(g)} \chi_{\rho_i}(g'),$$

which proves the first formula and hence the orthognality of the columns of the character table. On the other hand, $1 = f_g(g) = \frac{|C_g|}{|G|} \sum_{i=1}^N \overline{\chi_{\rho_i}(g)} \chi_{\rho_i}(g)$, which proves the second formula. $\square$

4 Projection Operators

We introduce the projection operators onto the isotypic components of the decomposition of the vector space of any representation. Let (E, ρ) be a representation of G and let $\rho = \oplus_{i=1}^N m_i \rho_i$ be the decomposition of ρ into isotypic components. The support of the isotypic component $m_i \rho_i$, is $m_i E_i = E_i \oplus \cdots \oplus E_i$ (m_i terms). We denote this vector subspace of E by V_i. We shall write

$$V_i = m_i E_i = \overset{m_i}{\underset{j=1}{\oplus}} E_{i,j},$$

where each $E_{i,j}$, $1 \leq j \leq m_i$, is equal to E_i. We thus have $E = \oplus_{i=1}^N V_i$.

Theorem 4.1. *For each i, $1 \leq i \leq N$, we set*

$$P_i = \frac{\dim \rho_i}{|G|} \sum_{g \in G} \overline{\chi_i(g)} \rho(g).$$

Then

(*i*) *P_i is the projection of E onto V_i under the decomposition $E = \oplus_{i=1}^N V_i$.*
(*ii*) *$P_i P_j = \delta_{ij} P_i$, for $1 \leq i \leq N$, $1 \leq j \leq N$.*
(*iii*) *If ρ is unitary, then P_i is Hermitian, that is, ${}^t\overline{P_i} = P_i$.*

Proof. (i) Let us choose i_0, $1 \leq i_0 \leq N$, and show that $P_{i_0}\big|_{V_{i_0}} = Id_{V_{i_0}}$, while if $i \neq i_0$, then $P_{i_0}\big|_{V_i} = 0$. Let $x = \sum_{i=1}^N x_i$, where $x_i \in V_i$, and let $x_i = \sum_{j=1}^{m_i} x_{i,j}$, where $x_{i,j} \in E_{i,j}$, whence $x = \sum_{i=1}^N \sum_{j=1}^{m_i} x_{i,j}$. Then

$$\begin{aligned} P_{i_0}(x) &= \frac{\dim \rho_{i_0}}{|G|} \sum_{i=1}^N \sum_{j=1}^{m_i} \sum_{g \in G} \overline{\chi_{i_0}(g)} \rho(g) x_{i,j} \\ &= \frac{\dim \rho_{i_0}}{|G|} \sum_{i=1}^N \sum_{j=1}^{m_i} \left(\sum_{g \in G} \overline{\chi_{i_0}(g)} \rho_i(g) \right) x_{i,j}. \end{aligned}$$

Because χ_{i_0} is a class function and ρ_i is irreducible, we may apply Lemma 3.5, and we obtain

$$\sum_{g \in G} \overline{\chi_{i_0}(g)} \rho_i(g) = \rho_{i, \overline{\chi_{i_0}}} = \frac{|G|}{\dim \rho_i} (\chi_{i_0} \mid \chi_i) \mathrm{Id}_{E_i} = \frac{|G|}{\dim \rho_{i_0}} \delta_{i i_0} \mathrm{Id}_{E_{i_0}},$$

which finally leads to

$$P_{i_0}(x) = \sum_{i=1}^{N} \sum_{j=1}^{m_i} \delta_{i_0 i} x_{i,j} = \sum_{j=1}^{m_{i_0}} x_{i_0,j} = x_{i_0}.$$

(ii) The equations $P_i P_j = 0$ if $i \neq j$ and $P_i^2 = P_i$ follow from (i).
(iii) If ρ is unitary, then

$$\begin{aligned}\frac{|G|}{\dim \rho_i} {}^t\overline{P_i} &= \sum_{g \in G} \chi_i(g) {}^t\overline{\rho(g)} = \sum_{g \in G} \chi_i(g) \rho(g^{-1}) \\ &= \sum_{g \in G} \chi_i(g^{-1}) \rho(g) = \sum_{g \in G} \overline{\chi_i(g)} \rho_i(g),\end{aligned}$$

which is equal to $\frac{|G|}{\dim \rho_i} P_i$, which proves (iii). □

The decomposition $E = \oplus_{i=1}^N V_i$ is unique up to order. On the other hand, the decomposition $V_i = \oplus_{j=1}^{m_i} E_{i,j}$ is not always unique. For example, if $\rho = \mathrm{Id}_E$, then ρ can be written in an infinite number of ways as a direct sum of one-dimensional representations.

5 Induced Representations

Induction is an operation that associates to a representation of a subgroup H of a group G a representation of the group G itself.

5.1 Definition

Let G be a finite group and H a subgroup. Let (F, π) be a representation of H. We define the vector space

$$E = \{\varphi : G \to F \mid \forall h \in H, \varphi(gh) = \pi(h^{-1})\varphi(g)\}, \tag{5.1}$$

and a representation $\rho = \pi^{\uparrow G}$ of G on E by

$$\forall \varphi \in E, \quad (\rho(g_0)\varphi)(g) = \varphi(g_0^{-1} g), \tag{5.2}$$

for every $g_0 \in G$ and for every $g \in G$. We can see that $\rho(g_0)\varphi$ lies in E because

$$(\rho(g_0)\varphi)(gh) = \varphi(g_0^{-1} gh) = \pi(h^{-1})\varphi(g_0^{-1} g) = \pi(h^{-1})\left((\rho(g_0)\varphi)(g)\right),$$

and on the other hand, we see that $g \mapsto \rho(g)$ is a group morphism of G into $\mathrm{GL}(E)$.

Definition 5.1. *The representation $\rho = \pi^{\uparrow G}$ of G on E is called the representation of G* induced *from the representation π of the subgroup H of G.*

For example, if $H = \{e\}$ and if π is the trivial representation of H on $\mathbb{C}$, then the vector space E is equal to $\mathbb{C}[G]$ and the representation of G induced from π is the regular representation of G.

5.2 Geometric Interpretation

We can interpret the vector space E as the space of sections of a "vector bundle." We consider the Cartesian product $G \times F$ and we introduce the equivalence relation

$$(g, x) \sim (gh, \pi(h^{-1})x), \ \forall h \in H. \tag{5.3}$$

Let $G \times_\pi F$ be the quotient of $G \times F$ by this equivalence relation, and let

$$q : G \times_\pi F \to G/H$$

be the projection that sends the equivalence class of (g, x) to gH. Note that this is well defined, because if $(g', x') \sim (g, x)$, then $g' = gh$, for some $h \in H$. The inverse image under the projection q of any point in G/H is isomorphic to the vector space F. We call $G \times_\pi F$ a *vector bundle* over G/H with *fiber* F.

A *section* of the projection $q : G \times_\pi F \to G/H$ (or of the vector bundle $G \times_\pi F$) is, by definition, a map ψ from G/H to $G \times_\pi F$ such that $q \circ \psi = \mathrm{Id}_{G/H}$.

Proposition 5.2. *The support E of the induced representation $\pi^{\uparrow G}$ is the vector space of sections of the projection $q : G \times_\pi F \to G/H$.*

Proof. To $\varphi \in E$ and $g \in G$ we associate the equivalence class of $(g, \varphi(g))$. The result depends only on the class of g modulo H. In fact, if $g' = gh$, with $h \in H$, we obtain the equivalence class of $(gh, \varphi(gh))$, which is equal to the equivalence class of $(g, \pi(h)\varphi(gh)) = (g, \varphi(g))$, since $\varphi \in E$. Thus one defines a section of $q : G \times_\pi F \to G/H$. On the other hand, to any given section of q we may associate an element of E by considering the second component of the equivalence class associated to an element of G/H. Since this construction is the inverse of the previous one, we have thus obtained an isomorphism of the space E of the induced representation onto the vector space of sections of the vector bundle $G \times_\pi F$. $\square$

The notion of an induced representation can be defined more generally than just for finite groups, and has many applications in mathematics and physics.

References

The representations of finite groups are the subject of Serre's book (1997), of which Part I is an exposition of fundamental results. Finite groups are studied in Sternberg (1994), Simon (1996), Artin (1991) and Ledermann–Weir (1996). Also see the first chapters of Fulton–Harris (1991), which are followed by chapters on Lie algebra representations, or the textbook by James and Liebeck (1993, 2001), which stresses arithmetic. All of these works discuss induced representations. For applications to physics, see Ludwig–Falter (1996), Tung (1985), or Blaizot–Tolédano (1997).

Tensor products of vector spaces are introduced in Exercise 2.5 below. (Also see Exercise 2.7.) For supplementary material on the tensor, exterior, and symmetric algebras of a vector space, see Greub (1967), Warner (1983), Sternberg (1994), or Knapp (2002).

The theory of characters was created by Frobenius in a series of articles published, starting in 1896, in the *Sitzungsberichte* of the Berlin Academy. These articles, reprinted in Frobenius (1968), contain beautiful character tables, p. 345, for a subgroup of $\mathfrak{S}_{12}$ of order $12 \cdot 11 \cdot 10 \cdot 9 \cdot 8$ with 15 irreducible representations, and, on the folding page between p. 346 and p. 347, for a subgroup of $\mathfrak{S}_{24}$ of order $24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 48$ with 26 irreducible representations. One can find a historical and mathematical analysis of this theory in the book by Curtis (1999). Also see Hawkins (2000) and Rossmann (2002).

Heinrich Maschke (1858–1908) published the theorem that bears his name in 1899, as a preliminary result in an attempt to prove a property of finite groups of matrices with complex coefficients.

For and in-depth study of induced representations, including those of Lie groups, see, e.g., Gurarie (1992), which includes applications to physics.

Exercises

Exercise 2.1 *The symmetric group* $\mathfrak{S}_3$.

We write c for the cyclic permutation (123) and t for the transposition (23). Show that $\{c, t\}$ generates $\mathfrak{S}_3$, and that $tc = c^2t, ct = tc^2$. Find the conjugacy classes of the group $\mathfrak{S}_3$.

Exercise 2.2 *Representations of* $\mathfrak{S}_3$.

(a) Find the one-dimensional representations of the group $\mathfrak{S}_3$.

(b) Let e_1, e_2, e_3 be the canonical basis of $\mathbb{C}^3$. For $g \in \mathfrak{S}_3$, set $\sigma(g)e_i = e_{g(i)}$. Show that this defines a three-dimensional representation σ of $\mathfrak{S}_3$ and that $V = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1 + z_2 + z_3 = 0\}$ is invariant under σ. This representation is called the *permutation representation* of the symmetric group.

We denote by ρ the restriction to V of the representation σ.

(c) Show that there is a basis (u_1, u_2) of V such that $\rho(t)u_1 = u_2$, $\rho(t)u_2 = u_1$, $\rho(c)u_1 = ju_1$, $\rho(c)u_2 = j^2u_2$, where $j^2 + j + 1 = 0$. Is the representation ρ irreducible?

(d) Find the character table of $\mathfrak{S}_3$.

(e) What is the geometric interpretation of $\mathfrak{S}_3$ as a group of symmetries? What is the geometric interpretation of the representation ρ?

Exercise 2.3 *The symmetric group* $\mathfrak{S}_4$.

Find the conjugacy classes and character table of the symmetric group $\mathfrak{S}_4$.

Exercise 2.4 *The alternate group* $\mathfrak{A}_4$.

Find the character table of $\mathfrak{A}_4$. Which representations of $\mathfrak{A}_4$ are the restriction of a representation of $\mathfrak{S}_4$? Which representations of $\mathfrak{S}_4$ have an irreducible restriction to $\mathfrak{A}_4$? Which have a reducible restriction?

Exercise 2.5 *Tensor products of vector spaces and of representations.*

We denote the dual of a vector space E by E^*, and the duality pairing by $\langle\ ,\ \rangle$.

If E and F are finite-dimensional vector spaces over $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), one can define the tensor product $E \otimes F$ as the vector space of bilinear maps of $E^* \times F^*$ into the scalar field $\mathbb{K}$. For $x \in E$, $y \in F$, we define the element $x \otimes y \in E \otimes F$ by

$$(x \otimes y)(\xi, \eta) = \langle \xi, x\rangle\langle \eta, y\rangle,$$

for every $\xi \in E^*$, $\eta \in F^*$.

(a) Let $(e_1, \dots, e_n)$ be a basis of E and let $(f_1, \dots, f_p)$ be a basis of F. Show that $(e_i \otimes f_j)_{1 \le i \le n, 1 \le j \le p}$ is a basis of $E \otimes F$.

(b) An element of $E \otimes E$ is called a *contravariant tensor* (or simply a *tensor*) of order 2 on E. Every contravariant tensor of order 2 on E can be written $T = \sum_{i,j=1}^{n} T^{ij} e_i \otimes e_j$, where the T^{ij} are scalars, called the *components* of T in the basis (e_i). What are the components of T after a change of basis?

(c) We can associate to $\xi \otimes y \in E^* \otimes F$ the linear map u of E into F defined by $u(x) = \langle \xi, x\rangle y$, for $x \in E$. Show that this defines an isomorphism of $E^* \otimes F$ onto the vector space of linear maps of E into F, $\mathcal{L}(E, F)$.

(d) Show that if $u : E \to E$ and $v : F \to F$ are linear maps, then there is a unique endomorphism $u \otimes v$ of $E \otimes F$ satisfying $(u \otimes v)(x \otimes y) = u(x) \otimes v(y)$ for each $x \in E$, $y \in F$. In $E \otimes F$, we choose the basis

$$(e_1 \otimes f_1, e_1 \otimes f_2, \dots, e_1 \otimes f_p, e_2 \otimes f_1, e_2 \otimes f_2, \dots, e_2 \otimes f_p, \dots, e_n \otimes f_1, \dots, e_n \otimes f_p).$$

Write the matrix of $u \otimes v$, where u (respectively, v) is an endomorphism of E (respectively, F) with matrix $A = (a_{ij})$ (respectively, $B = (b_{ij})$) in the chosen bases.

(e) If (E_1, ρ_1) and (E_2, ρ_2) are representations of a group G, we set, for $g \in G$,

$$(\rho_1 \otimes \rho_2)(g) = \rho_1(g) \otimes \rho_2(g).$$

Show that this defines a representation $\rho_1 \otimes \rho_2$ of G on $E_1 \otimes E_2$. What can one say about the character of $\rho_1 \otimes \rho_2$? If ρ_1 and ρ_2 are irreducible, is the representation $\rho_1 \otimes \rho_2$ irreducible?

Exercise 2.6 *The dual representation.*

Let (E, π) be a representation of a group G. For $g \in G$, $\xi \in E^*$, $x \in E$, we set $\langle \pi^*(g)(\xi), x \rangle = \langle \xi, \pi(g^{-1})(x) \rangle$. (As in Excercise 2.5, E^* is the dual of E, and $\langle\ ,\ \rangle$ is the duality pairing.)

(a) Show that this defines a representation π^* of G on E^*. The representation π^* is called the *dual* (or *contragredient*) of π.

(b) Show that if (E, π) and (F, ρ) are representations of a group G, then $g \cdot u = \rho(g) \circ u \circ \pi(g^{-1})$, for $u \in \mathcal{L}(E, F)$ and $g \in G$, defines a representation of G on $\mathcal{L}(E, F)$, equivalent to $\pi^* \otimes \rho$.

Exercise 2.7 *Exterior and symmetric powers.*

Let E be a finite-dimensional vector space, with basis $(e_1, \ldots, e_n)$. We denote by $\bigwedge^2 E$ (respectively, S^2E) the vector subspace of $E \otimes E$ generated by $e_i \otimes e_j - e_j \otimes e_i$, $1 \le i < j \le n$ (respectively, $e_i \otimes e_j + e_j \otimes e_i$, $1 \le i \le j \le n$). These definitions are independent of the choice of basis and $E \otimes E = \bigwedge^2 E \oplus S^2 E$. The space $\bigwedge^2 E$ is the *exterior* (or *antisymmetric*) *power* of degree 2 of E, and the space S^2E is the *symmetric power* of degree 2 of E.

(a) If (E, ρ) is a representation of a group G, then $\bigwedge^2 E$ and S^2E are invariant under $\rho \otimes \rho$. We denote the restriction of $\rho \otimes \rho$ to $\bigwedge^2 E$ (respectively, S^2E) by $\bigwedge^2 \rho$ (respectively, $S^2\rho$). Suppose that G is finite. Show that the characters of these representations satisfy, for each $g \in G$,

$$\chi_{\wedge^2\rho}(g) = \frac{1}{2}\Big((\chi_\rho(g))^2 - \chi_\rho(g^2)\Big), \quad \chi_{S^2\rho}(g) = \frac{1}{2}\Big((\chi_\rho(g))^2 + \chi_\rho(g^2)\Big).$$

(b) If ρ is the two-dimensional irreducible representation of $\mathfrak{S}_3$, find $\chi_{\wedge^2\rho}$ and $\chi_{S^2\rho}$. Decompose $\rho \otimes \rho$ into a direct sum of irreducible representations.

Exercise 2.8 *Equivalence of the left and right regular representations.*

Show that the left and right regular representations of a finite group are equivalent.

Exercise 2.9 *Representations of abelian and cyclic groups.*

(a) Show that every irreducible representation of a finite group is one-dimensional if and only if the group is abelian.
(b) Find all the inequivalent irreducible representations of the cyclic group of order n.

Exercise 2.10 *An application of the orthogonality relations.*

Let ρ_i and ρ_j be irreducible representations of a finite group G. Let $\chi_i = \chi_{\rho_i}$ and $\chi_j = \chi_{\rho_j}$. Show that for every $h \in G$,

$$\frac{1}{|G|} \sum_{g \in G} \chi_i(g)\chi_j(g^{-1}h) = \frac{1}{\dim\, \rho_i} \chi_i(h)\delta_{ij}.$$

Exercise 2.11 *Regular representation of $\mathfrak{S}_3$.*

Decompose the regular representation of $\mathfrak{S}_3$ into a direct sum of irreducible representations.

Find a basis of each one-dimensional invariant subspace and a projection onto the support of the representation 2ρ, where ρ is the irreducible representation of dimension 2.

Exercise 2.12 *Real and complexified representations.*

Let E be a vector space over $\mathbb{R}$, of dimension n. A morphism of a finite group G into $\mathrm{GL}(E)$ is called a *real representation* of G, of (real) dimension n.

We consider $E^{\mathbb{C}} = E \oplus iE = E \otimes \mathbb{C}$, a vector space over $\mathbb{C}$, of complex dimension n, called the *complexification* of E.

(a) Show that every real representation of G on E can be extended uniquely to a (complex) representation of G on $E^{\mathbb{C}}$. This representation is called the *complexification* of the real representation.
(b) Let the symmetric group $\mathfrak{S}_3$ act on $\mathbb{R}^2$ by rotation through angles of $2k\pi/3$ and reflection. Show that the complexification of this representation is equivalent to the irreducible representation of $\mathfrak{S}_3$ on $\mathbb{C}^2$.
(c) Let the cyclic group of order 3 act on $\mathbb{R}^2$ by rotations through angles of $2k\pi/3$. Is this real representation irreducible?
(d) Are all irreducible real representations of abelian groups one-dimensional?

Exercise 2.13 *Representations of the dihedral group.*

(a) Show that if H is an abelian subgroup of order p of a finite group G of order n, then every irreducible representation of G is of dimension $\leq n/p$.
(b) Conclude that for every $n \geq 3$, every irreducible representation of the dihedral group $D_{(n)}$ is one- or two-dimensional.

Exercise 2.14 *Peter–Weyl theorem for finite groups.*

Let $\rho^1, \rho^2, \ldots, \rho^N$ be unitary representations of a finite group G, chosen from each equivalence class of irreducible representations.

Show that the matrix coefficients of the representations ρ^k, $k = 1, \ldots, N$, in orthonormal bases form an orthogonal basis of $L^2(G)$. Conclude that every function $f \in L^2(G)$ has a "Fourier series"

$$f = \sum_{k=1}^{N} \sum_{i,j=1}^{\dim \rho_k} d_k (\rho_{ij}^k | f) \rho_{ij}^k,$$

where the d_k are integers.

Exercise 2.15 *Representation of* $\mathrm{GL}(2, \mathbb{C})$ *on the polynomials of degree* 2.

Let G be be a group and let ρ be a representation of G on $V = \mathbb{C}^n$. Let $P^{(k)}(V)$ be the vector space of complex polynomials on V that are homogeneous of degree k.

(a) For $f \in P^{(k)}(V)$, we set $\rho^{(k)}(g)(f) = f \circ \rho(g^{-1})$. Show that this defines a representation $\rho^{(k)}$ of G on $P^{(k)}(V)$.
(b) Compare $\rho^{(1)}$ and the dual representation of ρ.
(c) Suppose that $G = \mathrm{GL}(2, \mathbb{C})$, $V = \mathbb{C}^2$, and ρ is the fundamental representation. Let $k = 2$. To the polynomial $f \in P^{(2)}(\mathbb{C}^2)$ defined by $f(x, y) = ax^2 + 2bxy + cy^2$ we associate the vector $v_f = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{C}^3$. Let $\tilde{\rho}$ denote the representation of $\mathrm{GL}(2, \mathbb{C})$ on $\mathbb{C}^3$ defined by $\rho^{(2)}$ and the isomorphism above. Find the dual of $\tilde{\rho}$.

Exercise 2.16 *Convolution.*

Let G be be a finite group and let $\mathbb{C}[G]$ be the *group algebra*, of G that is, the vector space $\mathcal{F}(G)$ with multiplication defined by $\epsilon_g \epsilon_{g'} = \epsilon_{gg'}$, for g and $g' \in G$, and extended by linearity.

(a) Show that the product of two functions $f_1, f_2 \in \mathbb{C}[G]$ is the *convolution product* $(f_1 * f_2)(g) = \sum_{h \in G} f_1(h) f_2(h^{-1}g)$.
(b) Let ρ be a representation of G and suppose $f \in \mathbb{C}[G]$. Set $\rho_f = \sum_{g \in G} f(g)\rho(g)$. Show that $\rho_{f_1 * f_2} = \rho_{f_1} \circ \rho_{f_2}$.
(c) Show that $f \in \mathbb{C}[G]$ is a class function if and only if f is in the center of the algebra $\mathbb{C}[G]$ equipped with convolution (that is, f commutes in the sense of convolution with every function on G).

Exercise 2.17 *On the map $f \mapsto \rho_f$.*

For every representation (E, ρ) of G and each function f on G, consider the endomorphism ρ_f of E defined by

$$\rho_f = \sum_{g \in G} f(g)\rho(g).$$

(a) Let R be the regular representation of G. Consider $R_f(\epsilon_g)$, for $g \in G$. Show that $R_f(\epsilon_e) = f$. Is the map $f \in \mathbb{C}[G] \mapsto R_f \in \mathrm{End}(\mathbb{C}[G])$ injective?
(b) Let ρ_i and ρ_j be irreducible representations of G and let χ_i (respectively, χ_j) be the character of ρ_i (respectively, ρ_j). Find ρ_f for $\rho = \rho_j$ and $f = \overline{\chi_i}$.

Exercise 2.18 *Tensor products of representations.*

Let ρ be the irreducible representation of dimension 2 of the symmetric group $\mathfrak{S}_3$. We set $\rho = \rho^{\otimes 1}$, and by induction we define for every integer $k \geq 2$,

$$\rho^{\otimes k} = \rho^{\otimes (k-1)} \otimes \rho.$$

(a) For each positive integer k, decompose $\rho^{\otimes k}$ into a direct sum of irreducible representations.
(b) Let $\mathfrak{A}_3 \subset \mathfrak{S}_3$ denote the alternate group. For each positive integer k, decompose the restriction of $\rho^{\otimes k}$ to $\mathfrak{A}_3$ into a direct sum of irreducible representations.

Issai Schur, born in 1875 in Mohilev, Belorussia, professor in Bonn and then in Berlin, member of the Prussian Academy of Sciences, lost his university position in 1935. Forced to emigrate to Palestine, he died in Tel Aviv in 1941. Schur is, with Frobenius, a founder of representation theory.

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Chapter 3

Representations of Compact Groups

In this chapter, we shall extend to compact topological groups many of the properties proved in the case of finite groups. Some properties will be stated without proof.

1 Compact Groups

Recall that a topological group is a group equipped with the structure of a Hausdorff topological space (for example, a subset of a normed vector space) such that multiplication and inversion are continuous maps. A topological space is locally compact if each point has a compact neighborhood. A topological group that is a compact (respectively, locally compact) space is called a *compact group* (respectively, *locally compact group*).

If E is a Banach space (i.e., a complete normed vector space) over the real or complex field, we denote by $\mathcal{L}(E,E)$ the vector space of continuous linear maps of E into E (also called *endomorphisms* of E or *continuous linear operators* or *bounded operators* on E). We equip it with the standard norm on linear maps defined, for each continuous, linear $u : E \to E$, by

$$\|u\| = \sup_{\|x\| \leq 1} \|u(x)\|.$$

For each Banach space E we denote by $\mathrm{GL}(E) \subset \mathcal{L}(E,E)$ the group of isomorphisms of E, that is, bijective and bicontinuous endomorphisms of E. In order to show that the inverse of a continuous linear operator is continuous, it suffices to show that the operator is bijective. We consider $\mathrm{GL}(E)$ to be a topological subspace of the normd vector space $\mathcal{L}(E,E)$.

The unit ball in a normed vector space is compact if and only if the space is finite-dimensional. Thus each *closed and bounded* subset of $\mathrm{GL}(E)$, where E is a finite-dimensional vector space, is compact. For example $\mathrm{U}(n) \subset \mathrm{GL}(n,\mathbb{C})$ and $\mathrm{O}(n) \subset \mathrm{GL}(n,\mathbb{R})$ are compact. Similarly, $\mathrm{SU}(n)$ and $\mathrm{SO}(n)$ are compact. The abelian group $\mathbb{R}$ with its usual metric is locally compact but not compact. The groups $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{GL}(n,\mathbb{R})$, and $\mathrm{SL}(n,\mathbb{R})$ are locally compact but not compact.

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2 Haar Measure

On a finite group G, we know that for each function $f \in \mathcal{F}(G)$ and for every $g \in G$,

$$\sum_{h\in G} f(h) = \sum_{h\in G} f(gh) = \sum_{h\in G} f(hg).$$

If we denote by l_g (respectively, r_g) the left (respectively, right) multiplication by $g \in G$, we have, by definition, $f(gh) = (f \circ l_g)(h)$ and $f(hg) = (f \circ r_g)(h)$. Consequently, the averaging operation

$$M : f \mapsto M(f) = \frac{1}{|G|} \sum_{g\in G} f(g)$$

satisfies the following conditions:

- M is a positive linear form on $\mathcal{F}(\mathcal{G})$, that is, M takes nonnegative values on real nonnegative functions;
- M is both left- and right-invariant, that is,

$$\forall g \in G,\ \ M(f \circ l_g) = M(f \circ r_g) = M(f);$$

- $M(1) = 1$.

On any compact group there is a measure, the *Haar measure*, that has analogous properties. More generally, on a locally compact group, there are measures that are either left-invariant or right-invariant but in general, not both.

Theorem 2.1. *Let G be a locally compact group.*

(*i*) *There exists a positive measure on G, finite on compact sets and not identically zero, that is left-invariant, i.e., for each integrable function f and for every $h \in G$,*

$$\int_G f(hg)\mathrm{d}\mu(g) = \int_G f(g)\mathrm{d}\mu(g).$$

Such a measure is unique up to multiplication by a positive real number. If f is a continuous, nonnegative function, then $\int_G f(g)\mathrm{d}\mu(g) = 0$, implies $f = 0$.

(*ii*) *If G is compact, there is a unique left-invariant measure μ such that $\int_G \mathrm{d}\mu(g) = 1$.*

(*iii*) *On a compact group, every left-invariant measure is right-invariant.*

Proof. (i) The proof of this result is beyond the scope of this book. See, e.g., Loomis (1953).

(ii) If μ_0 is a left-invariant measure on a compact group G and if $\int_G \mathrm{d}\mu_0(g) = m$, we set $\mu = \frac{1}{m}\mu_0$, and μ is clearly the unique left-invariant measure such that $\int_G \mathrm{d}\mu(g) = 1$.

(iii) Let μ be a left-invariant measure on a locally compact group G. For any continuous f with compact support, we set $\mu(f) = \int_G f(g)\mathrm{d}\mu(g)$. Let $h \in G$ and consider $\mu_h(f) = \int_G f(gh)\mathrm{d}\mu(g)$, that is, $\mu_h(f) = \mu(f \circ r_h)$. Then

$$\forall k \in G, \mu_h(f \circ l_k) = \int_G f(kgh)\mathrm{d}\mu(g) = \int_G f(gh)\mathrm{d}\mu(g) = \mu_h(f),$$

and hence, by the uniqueness of left-invariant measures up to a factor, there is a positive real number $\Delta(h)$ depending on h satisfying

$$\mu_h(f) = \Delta(h)\mu(f).$$

If G is compact, the constant function 1 is integrable. Therefore we obtain $\mu_h(1) = \mu(1) = \Delta(h)\mu(1)$. Hence $\Delta = 1$, and μ is thus also right-invariant, i.e.,

$$\int_G f(gh)\mathrm{d}\mu(g) = \int_G f(g)\mathrm{d}\mu(g),$$

for every $h \in G$. □

Definition 2.2. *On a compact group, the unique left- and right-invariant measure of total mass* 1 *is called the Haar measure.*

On a locally compact group G, the function $\Delta : h \in G \mapsto \Delta(h) \in \mathbb{R}$ is called the *modular function* of G. It satisfies

$$\Delta(hh') = \Delta(h)\Delta(h'),$$

since $\Delta(hh')\mu(f) = \mu_{hh'}(f) = \mu(f \circ r_{hh'}) = \mu(f \circ r_{h'} \circ r_h) = \Delta(h)\mu(f \circ r_{h'}) = \Delta(h)\Delta(h')\mu(f)$. We say that a locally compact group G is *unimodular* if $\Delta = 1$. The preceding theorem says that if G is compact, then G is unimodular.

One often writes $\int f(g)\mathrm{d}g$ instead of $\int f(g)\mathrm{d}\mu(g)$. Thus, if G is compact, then for each measurable function f,

$$\forall h \in G, \int_G f(g)\mathrm{d}g = \int_G f(hg)\mathrm{d}g = \int_G f(gh)\mathrm{d}g,$$

and we require that μ satisfy the normalization condition $\int_G \mathrm{d}g = 1$.

Examples.

- On an abelian group, each left-invariant measure is obviously also right-invariant.
- In the case of the locally compact abelian group $\mathbb{R}$, every invariant measure is proportional to the Lebesgue measure.
- If $G = \mathrm{U}(1) = S^1 = \{e^{i\theta} | \theta \in \mathbb{R}/2\pi\mathbb{Z}\} = \{z \in \mathbb{C} \mid |z| = 1\}$, then

$$\mathrm{d}g = \frac{\mathrm{d}\theta}{2\pi} \quad \text{or} \quad \mathrm{d}g = \frac{\mathrm{d}z}{2i\pi z}.$$

- For the groups SU(2) and SO(3), see Exercises 5.4, 5.5, and 5.6.

3 Representations of Topological Groups and Schur's Lemma

All the Hilbert spaces we consider are over the field of *complex numbers*, and we assume each Hilbert space to be *separable*, that is, we assume that it has a countable Hilbert space basis.

3.1 General Facts

Definition 3.1. *Let G be a topological group. A* continuous representation (*or simply a representation*) *of G is defined to be a Hilbert space E and a morphism group $\rho : G \to \mathrm{GL}(E)$ such that for every $x \in E$,*

$$g \in G \mapsto \rho(g)x \in E$$

is a continuous mapping.

A sufficient condition for the continuity of the map $g \mapsto \rho(g)x$, for every $x \in E$, is that

$$\lim_{g \to e} \|\rho(g) - \mathrm{Id}_E\| = 0,$$

that is, ρ should be continuous as a map of G into $\mathrm{GL}(E)$ with the topology induced by the norm of $\mathcal{L}(E, E)$. If E is finite-dimensional, this sufficient condition is also necessary.

The dimension, finite or infinite, of E is called the *dimension* of ρ.

The *trivial representation* of G on a vector space E is defined by $\rho(g) = \mathrm{Id}_E$, for every $g \in G$.

Let E be a complex Hilbert space. If $u \in \mathcal{L}(E, E)$, the *adjoint* u^* of u is defined by

$$\forall x, y \in E, (ux \mid y) = (x \mid u^* y),$$

and an element $u \in \mathrm{GL}(E)$ is a *unitary operator* if $uu^* = u^*u = \mathrm{Id}_E$. The group of unitary operators of E is denoted by $\mathrm{U}(E)$. In finite dimensions and in an orthonormal basis, a unitary operator is represented by a unitary matrix.

A representation ρ of G on E is called *unitary* if E is a complex Hilbert space and if for every $g \in G$, $\rho(g)$ is a unitary operator. Then for every $g \in G$, $x, y \in E$, we have $(\rho(g)x \mid \rho(g)y) = (x \mid y)$, and in particular, $\|\rho(g)x\| = \|x\|$.

Remark. In the same way, one can define representations on real Hilbert spaces. In this case one speaks of *orthogonal representations*.

3.2 Coefficients of a Representation

Let E^* be the dual of E. For $x \in E$ and $\xi \in E^*$, we set

$$\rho_{x\xi}(g) = \langle \xi, \rho(g)x \rangle,$$

where $\langle\ ,\ \rangle$ denotes the duality pairing. In finite dimensions, given a basis (e_i) of E, and the dual basis (e_i^*) in E^*, we recover the matrix coefficients ρ_{ij} defined in Chapter 2, because

$$\rho_{e_j e_i^*}(g) = \langle e_i^*, \rho(g)e_j\rangle = \rho_{ij}(g).$$

One can also consider, for $x, y \in E$,

$$\varphi^\rho_{xy}(g) = (x \mid \rho(g)y).$$

If ρ is unitary, then

$$\varphi^\rho_{xy}(g^{-1}) = \overline{\varphi^\rho_{yx}(g)}.$$

In finite dimensions, for each basis (e_i) of E, we consider the coefficients

$$\varphi^\rho_{ij}(g) = (e_i \mid \rho(g)e_j).$$

If the basis (e_i) is orthonormal, then

$$\varphi^\rho_{ij} = \rho_{ij}.$$

3.3 Intertwining Operators

Definition 3.2. *Let (E_1, ρ_1) and (E_2, ρ_2) be unitary representations of G. An* intertwining operator *for ρ_1 and ρ_2 is defined to be any* continuous *linear mapping $T : E_1 \to E_2$ such that for every $g \in G$, $\rho_2(g) \circ T = T \circ \rho_1(g)$.*

In particular, if (E, ρ) is a unitary representation of G, a continuous linear mapping $T : E \to E$ intertwines ρ with itself if and only if it commutes with ρ.

Two representations (E_1, ρ_1) and (E_2, ρ_2) are *equivalent* if there is a bijective (and thus bicontinuous) intertwining operator T for ρ_1 and ρ_2. Then E_1 is isomorphic to E_2 and $\rho_2(g) = T \circ \rho_1(g) \circ T^{-1}$. Thus one defines the *equivalence class* of a representation.

Definition 3.3. *A representation (E, ρ) of G is called* irreducible *if $E \neq \{0\}$ and if no* closed *nontrivial vector subspace of E is invariant under ρ.*

For a closed vector subspace F invariant under ρ, the restriction of ρ to F is called a *subrepresentation* of ρ.

Proposition 3.4. *If (E, ρ) is a unitary representation and if $F \subset E$ is a closed vector subspace invariant under ρ, then the orthogonal complement $F^\perp$ of F is also a closed vector subspace invariant under ρ, and the orthogonal projections P_F on F and $P_{F^\perp}$ on $F^\perp$ commute with ρ.*

Proof. We know that for every vector subspace F of E, the orthogonal complement $F^\perp$ of F is closed. By hypothesis, for every $x \in F$, $\rho(g)x \in F$. Let $y \in F^\perp$. Then for every $x \in F$, $(x \mid y) = 0$. Hence for every $x \in F$, $(x \mid \rho(g)y) = (\rho(g)^{-1}x \mid y) = 0$, and so $\rho(g)y \in F^\perp$. Furthermore,

$$\rho(g)y = \rho(g)(P_F y + P_{F^\perp} y) = \rho(g)P_F y + \rho(g)P_{F^\perp} y.$$

The first term is $P_F(\rho(g)y)$ and the second is $P_{F^\perp}(\rho(g)y)$. □

3.4 Operations on Representations

One can take the *Hilbert direct sum* $(\hat{\oplus} E_i, \hat{\oplus} \rho_i)$ of a countable family of unitary representations (E_i, ρ_i) of a topological group G. In the vector space of the representation, each E_i is a closed subspace, the E_i are pairwise orthogonal, and the direct sum is dense.

A representation is called *completely reducible* (or *semisimple*) if it is the Hilbert direct sum of irreducible representations.

If ρ is an irreducible representation, we use the notation $m\rho$ to denote the *isotypic component* of type ρ of a completely reducible representation, where m can be an integer or ∞.

We define the tensor product $(E_1 \otimes E_2, \rho_1 \otimes \rho_2)$ of two representations (E_1, ρ_1) and (E_2, ρ_2) of G by setting, for $g \in G$, $x_1 \in E_1$, and $x_2 \in E_2$,

$$((\rho_1 \otimes \rho_2)(g))\,(x_1 \otimes x_2) = \rho_1(g)x_1 \otimes \rho_2(g)x_2.$$

If ρ_1 and ρ_2 are infinite-dimensional representations, we consider $E_1 \hat{\otimes} E_2$, the tensor product completed under the norm associated to the scalar product defined by

$$(x_1 \otimes x_2 \mid x'_1 \otimes x'_2) = (x_1 \mid x'_1)(x_2 \mid x'_2),$$

for x_i and $x'_i \in E_i$, $i = 1, 2$, which can be shown to be positive. The Hilbert space $E_1 \hat{\otimes} E_2$ is called the *Hilbert tensor product* of E_1 and E_2.

Theorem 3.5. *Every* finite-dimensional *unitary representation of a topological group is completely reducible.*

The proof is the same as in the case of finite groups using Proposition 3.4.

There exist (nonunitary) representations that are reducible but not completely reducible.

3.5 Schur's Lemma

Theorem 3.6 (Schur's Lemma). *Let G be a topological group and let (E_1, ρ_1) and (E_2, ρ_2) be irreducible unitary representations of G. Let T be a continuous linear mapping of E_1 into E_2 that intertwines ρ_1 and ρ_2. Then either $T = 0$ or T is an isomorphism (and consequently ρ_1 and ρ_2 are equivalent), and T is then unique up to a multiplicative constant.*

In summary, if $\rho_2(g) \circ T = T \circ \rho_1(g)$ for every $g \in G$, then either $\rho_1 \nsim \rho_2$ and $T = 0$, or $\rho_1 \sim \rho_2$ and T is an isomorphism. The proof, which we omit, uses the spectral decomposition of Hermitian operators. See, e.g., Gurarie(1972).

Corollary 3.7. *Let G be a topological group and let (E, ρ) be a unitary representation of G. The representation ρ is irreducible if and only if every endomorphism of E that commutes with ρ is a scalar multiple of the identity.*

Proof. If ρ is not irreducible, the projection onto a nontrivial closed invariant subspace is a nonscalar endomorphism of E that commutes with ρ. The converse is a consequence of Schur's lemma. $\square$

Corollary 3.8. *The unitary irreducible representations of an abelian group are one-dimensional.*

Proof. Let (E, ρ) be an unitary irreducible representation of an abelian group G. Let g be an element of G. Then for every $h \in G$, $\rho(g)\rho(h) = \rho(h)\rho(g)$, and thus $\rho(g)$ commutes with ρ. By Corollary 3.7 of Schur's lemma, G acts by scalar multiplication. Because the representation ρ is assumed irreducible, it must be one-dimensional. □

4 Representations of Compact Groups

In this section, we consider a topological group G that we assume to be *compact*, and as above, by "representation," we mean a continuous representation on a separable complex Hilbert space.

4.1 Complete Reducibility

Theorem 4.1. *Every representation of a compact group is unitarizable.*

Sketch of Proof. Let G be be a compact group, and let (E, ρ) be a representation of G. We set, for $x, y \in E$,

$$(x \mid y)' = \int_G (\rho(g)x \mid \rho(g)y)\mathrm{d}g,$$

where dg is the Haar measure on G. This is indeed a scalar product because if $(x|x)' = 0$, then by Theorem 2.1, $(\rho(g)x \mid \rho(g)x) = 0$ for every $g \in G$, and consequently $x = 0$. On the other hand,

$$(\rho(g)x \mid \rho(g)y)' = \int_G (\rho(hg)x \mid \rho(hg)y)\mathrm{d}h = \int_G (\rho(h)x \mid \rho(h)y)\mathrm{d}h = (x \mid y)'.$$

Thus $\rho(g)$ is unitary for $(\ \mid\ )'$. Next one shows that the norms associated to $(\ \mid\ )$ and $(\ \mid\ )'$ are equivalent, which implies that the representation remains continuous and that the representations ρ on the Hilbert spaces $(E, (\ \mid\ ))$ and $(E, (\ \mid\ )')$ are equivalent, which means that the identity mapping of E is bicontinuous and intertwines the representations. □

Combining this theorem with Theorem 3.5, we obtain the following corollary:

Corollary 4.2. *Every finite-dimensional representation of a compact group is completely reducible.*

A proof of the following result may be found in, e.g., Naimark–Štern(1982).

Theorem 4.3. *Every irreducible representation of a compact group is finite-dimensional.*

Remark. This theorem, as specified above, assumes representations to be continuous and representation spaces to be separable complex Hilbert spaces. The theorem is not true in all generality, but it is true for continuous representations taking values in certain topological vector spaces more general than Hilbert spaces.

4.2 Orthogonality Relations

We define a scalar product on the vector space of complex-valued functions on G by

$$(f_1 \mid f_2) = \int_G \overline{f_1(g)} f_2(g) \mathrm{d}g,$$

where $\mathrm{d}g$ is the haar measure. We denote by $L^2(G)$ the Hilbert space obtained by completing this pre-Hilbert space under the norm defined by the scalar product. This is the Hilbert space of *square-integrable function* on G. (Strictly speaking, this vector space consists of equivalence classes of square-integrable functions under the equivalence relation of equality almost everywhere.)

We know that the irreducible representations of G are finite-dimensional. The orthogonality relations of characters of irreducible representations of finite groups extend to the compact case.

Theorem 4.4. *Let G be be a compact group and let (E_1, ρ_1) and (E_2, ρ_2) be irreducible unitary representations of G. For every $x_1, y_1 \in E_1$ and $x_2, y_2 \in E_2$,*

$$\left(\varphi^{\rho_1}_{x_1 y_1} \mid \varphi^{\rho_2}_{x_2 y_2}\right) \begin{cases} 0 & \text{if } \rho_1 \nsim \rho_2, \\ \frac{1}{\dim E}(x_2 \mid x_1)(y_1 \mid y_2) & \text{if } E_1 = E_2 = E \text{ and } \rho_1 = \rho_2 = \rho. \end{cases}$$

Proof. Generalizing the process used in the proof of Proposition 2.9 of Chapter 2, we define, for each continuous linear mapping $u : E_1 \to E_2$, the following operator intertwining ρ_1 and ρ_2:

$$T_u = \int_G \rho_2(g) u \rho_1(g)^{-1} \mathrm{d}g.$$

For $y_1 \in E_1, y_2 \in E_2$, we consider the linear map $u_{y_1 y_2} : E_1 \to E_2$ defined by $u_{y_1 y_2}(x) = (y_1 \mid x) y_2$, for x in E_1. Using the fact that ρ_1 is unitary, we then obtain the equation $(\varphi^{\rho_1}_{x_1 y_1} \mid \varphi^{\rho_2}_{x_2 y_2}) = (x_2 \mid T_{u_{y_1 y_2}} x_1)$.

Then we apply Schur's lemma. The quantity above is zero if ρ_1 is not equivalent to ρ_2. If $E_1 = E_2 = E$ and $\rho_1 = \rho_2 = \rho$, then $T_{u_{y_1 y_2}} = \tau(y_1, y_2)\mathrm{Id}_E$, where $\tau(y_1, y_2)$ is antilinear in x_1 and linear in x_2. We calculate $\tau(y_1, y_2)$ by calculating the trace of $T_{u_{y_1 y_2}}$. This is equal to the trace of $u_{y_1 y_2}$ because for each linear mapping u, $\operatorname{Tr} T_u = \int_G \operatorname{Tr}(\rho(g) \circ u \circ \rho(g^{-1}))\mathrm{d}g = \int_G \operatorname{Tr} u \, \mathrm{d}g = \operatorname{Tr} u$. Because $\operatorname{Tr}(u_{y_1 y_2}) = (y_1 \mid y_2)$, we obtain the desired result. □

In particular, if ρ_1 and ρ_2 are inequivalent, then in every orthonormal basis,

$$\left(\varphi^{\rho_1}_{ij} \mid \varphi^{\rho_2}_{k\ell}\right) = 0, \tag{4.1}$$

and if $\rho_1 = \rho_2 = \rho$, then

$$\left(\varphi^{\rho}_{ij} \mid \varphi^{\rho}_{k\ell}\right) = \frac{1}{\dim E} \delta_{ik} \delta_{j\ell} \ . \tag{4.2}$$

We denote by $\widehat{G}$ the set of equivalence classes of irreducible representations of a compact group G. When $L^2(G)$ is separable—which is usually true in the cases one meets in practice—the orthogonality relations above imply that $\widehat{G}$ is countable.

By (4.1) and (4.2) the matrix coefficients in orthonormal bases of inequivalent irreducible unitary representations of G form an orthogonal set in $L^2(G)$. We now state without proof the Peter–Weyl theorem, which states that these matrix coefficients form an orthogonal basis of $L^2(G)$ in the Hilbert space sense. For a proof, see, e.g., Simon (1996) or Gurarie(1992).

Theorem 4.5 (Peter–Weyl Theorem for Compact Groups). *Every function $f \in L^2(G)$ has a "Fourier series"*

$$f = \sum_{\alpha \in \widehat{G}} \sum_{i,j=1}^{\dim \rho^\alpha} c_{ij}^\alpha \rho_{ij}^\alpha \tag{4.3}$$

converging in the L^2 sense, where the ρ^α are unitary representatives of the classes of inequivalent irreducible representations of G, the ρ_{ij}^α are their matrix coefficients in orthonormal bases, and

$$c_{ij}^\alpha = (\dim \rho^\alpha)(\rho_{ij}^\alpha \mid f) = (\dim \rho^\alpha) \int_G f(g)\overline{\rho_{ij}^\alpha(g)}\mathrm{d}g. \tag{4.4}$$

Example. The irreducible representations of the abelian compact group $G = \mathbb{R}/2\pi\mathbb{Z}$ are the exponentials $e_k : x \mapsto e^{ikx}$ for $k \in \mathbb{Z}$. Thus in this case $\widehat{G} = \mathbb{Z}$.

We thus see that the expansion of functions of $L^2(G)$ as series, where G is a compact group, given by the Peter–Weyl theorem is a generalization of the Fourier series expansion of 2π-periodic functions square-integrable on $[0, 2\pi]$. If $G = \mathbb{R}/2\pi\mathbb{Z}$, then for every $f \in L^2(G)$,

$$f = \sum_{k \in \mathbb{Z}} c_k e_k, \tag{4.5}$$

where

$$c_k = (e_k \mid f) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)e^{-ik\theta}\mathrm{d}\theta. \tag{4.6}$$

In this case, each representation is one-dimensional, and its unique matrix coefficient is the character of the representation. The formulas (4.3) and (4.4) are thus seen to be generalizations of the classic formulas (4.5) and (4.6) of Fourier analysis.

If (E, ρ) is a finite-dimensional representation of G, we define the *character* χ_ρ of ρ by

$$\forall g \in G,\ \chi_\rho(g) = \mathrm{Tr}\ (\rho(g)).$$

The complex-valued function thus defined on G is continuous. It is a class function that depends only on the equivalence class of the representation ρ.

Theorem 4.6 (Orthogonality Relations). *Let ρ_1 and ρ_2 be irreducible representations of G. Then*

$$(\chi_{\rho_1} \mid \chi_{\rho_2}) = \begin{cases} 0 & \text{if } \rho_1 \nsim \rho_2, \\ 1 & \text{if } \rho_1 \sim \rho_2. \end{cases}$$

Proof. With the help of Theorem 4.1, these relations are a consequence of the preceding formulas (4.1) and (4.2). □

A representation ρ is irreducible if and only if $(\chi_\rho \mid \chi_\rho) = 1$.

If ρ is a representation of G, we can decompose it as a Hilbert sum of irreducible representations ρ_i, where $i \in \widehat{G}$. We shall write

$$\rho = \widehat{\bigoplus_{i \in \widehat{G}}} m_i \rho_i,$$

where

$$m_i = (\chi_{\rho_i} \mid \chi_\rho).$$

We may have $m_i = \infty$.

Theorem 4.1 of Chapter 2 concerning the projection operators generalizes as follows. A projection on the isotypic component $m_i \rho_i$ is

$$P_i = \dim \rho_i \int_G \overline{\chi_i(g)} \rho(g) \mathrm{d}g.$$

In general, many results from the representation theory of finite groups extend to the case of compact groups, replacing averaging over the group with the Haar integral.

We shall determine the irreducible representations of the compact groups SU(2) and SO(3) in Chapter 6.

The following section summarizes the principal properties of the representations of compact groups, compared to those of finite groups.

5 Summary of Chapter 3

We summarize the properties of finite groups that are true also for *compact* groups. The representations are assumed continuous on separable *complex* Hilbert spaces.

Representations of Finite G	**Representations of Compact G**
Every representation is unitarizable	*true*
Every finite-dimensional representation is completely reducible	*true*
Every irreducible representation is finite-dimensional	*true*
There is a finite number of equivalence classes of irreducible representations = number of conjugacy classes of the group = dimension of the vector space of class functions	*false*
The irreducible characters form an orthonormal basis of the vector space of class functions	*true* *as a Hilbert space basis*

The matrix coefficients in orthonormal bases of unitary irreducible representations form an orthogonal basis of the vector space $L^2(G)$	*true* *as an orthogonal set*
Every representation has a decomposition into a direct sum $\rho = \oplus_{i=1}^{N} m_i \rho_i$ $m_i = (\chi_i \mid \chi_\rho)$	*into a direct Hilbert sum* $\rho = \oplus_{i \in \widehat{G}} m_i \rho_i$ $m_i = (\chi_i \mid \chi_\rho)$
Decomposition of the regular representation $\sum_{i=1}^{N} (\dim \rho_i)^2 = \lvert G \rvert$	*false*
Each irreducible representation is contained in the regular representation a number of times equal to its dimension	*true*
Projection onto an isotypic component $P_i = \frac{\dim \rho_i}{\lvert G \rvert} \sum_{g \in G} \overline{\chi_i(g)} \rho(g)$	$P_i = \dim \rho_i \int_G \overline{\chi_i(g)} \rho(g) \mathrm{d}g$

References

One will find a study of representations of compact groups in Barut–Raçzka (1977, 1986), Naimark–Štern (1982), Sternberg (1994), Bump (2004), and many other textbooks.

For a proof of the bicontinuity of continuous bijective operators, as well as other results in analysis, see Rudin (1991) or Brezis (1983).

The existence of a left-invariant measure on a locally compact group is proved in the book by Weil (1965), in Loomis (1953), and in Section IV.5 of Hewitt–Ross (1963). The proof was first given in 1932 in a communication to the Hungarian Academy by Alfréd Haar (1885–1933), following work of Adolf Hurwitz (1859–1919) at the end of the nineteenth century. The complete works of Haar, *Összegyűjtött Munkái/Gesammelte Arbeiten*, published in Budapest in 1959, reproduce the original article "A folytonos csoportok elméletéről," *Mat. Term. Ért.*, **49** (1932), 287–307, followed by its translation into German by the author, "Der Massbegriff in der Theorie der kontinuierlichen Gruppen," *Annals of Mathematics* (1933), 147–169. The uniqueness of the Haar measure was proved by von Neumann in 1934.

See Gurarie (1972) for a thorough introductory text, including a proof of Schur's lemma (Theorem 3.6), and Naimark–Štern (1982) for a comprehensive study of irreducible representations on Hilbert spaces, including a proof of Theorem 4.3. For a generalization to locally convex topological vector spaces, see Kirillov (1976).

The Peter–Weyl theorem is proved in most of the books mentioned. It was published in 1926 by Hermann Weyl and one of his students, Fritz Peter (1899–1949), who became headmaster of a secondary school.

Exercises

Exercise 3.1 *Continuity condition for a representation*

Let E be a Hilbert space and let ρ be a morphism of the locally compact group G into $\mathrm{GL}(E)$, the group of bijective continuous linear operators from E to E. (One knows that such an operator is bicontinuous.) We consider the following conditions:

(i) ρ is a continuous mapping of G into $\mathrm{GL}(E)$ with the usual norm.
(ii) $\lim_{g \to e} \|\rho(g) - id_E\| = 0$.
(iii) The mapping $(g, x) \in G \times E \mapsto \rho(g)(x) \in E$ is continuous.
(iv) For each $x \in E$, the mapping $g \in G \mapsto \rho(g)(x) \in E$ is continuous (in which case ρ is called a *representation* of G on E).

(a) Show that conditions (i) and (ii) are equivalent.
(b) Show that (i) implies (iii).
(c) Show that the conditions (iii) and (iv) are equivalent. [We know by the Banach–Steinhaus theorem (uniform boundedness theorem) that if K is a compact subset of G and if $\sup_{g \in K} \|\rho(g)(x)\| < \infty$, for every $x \in E$, then $\sup_{g \in K} \|\rho(g)\| < \infty$.]
(d) Show that if E is finite-dimensional, these conditions are equivalent.

Exercise 3.2 *Dual of a unitary representation.*

Show that the dual representation of a unitary representation of a compact group is equivalent to the conjugate representation.

Exercise 3.3 *Regular representation of a compact group.*

Let G be a compact group, and let $E = L^2(G)$ be the Hilbert space of square-integrable functions on G with respect to the Haar measure. We set, for every $f \in L^2(G), g, h \in G$.

$$(R(g)(f))(h) = f(g^{-1}h).$$

Show that R is a unitary representation of G on $L^2(G)$.

Exercise 3.4 *Equivalence of representations.*

Show that two equivalent unitary representations of a topological group are unitarily equivalent. [Hint: Introduce the adjoint T^* of the intertwining operator T and the positive square root of the positive self-adjoint operator TT^*.]

Exercise 3.5 *An application of Schur's lemma.*

Let (E, ρ) be an irreducible representation of a compact group G. Show that if two scalar products on E are invariant under the representation ρ, they are proportional.

Exercise 3.6 *A not completely reducible representation.*

Let G be be the group of complex matrices $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$ such that $|a| = 1$, with the usual topology.

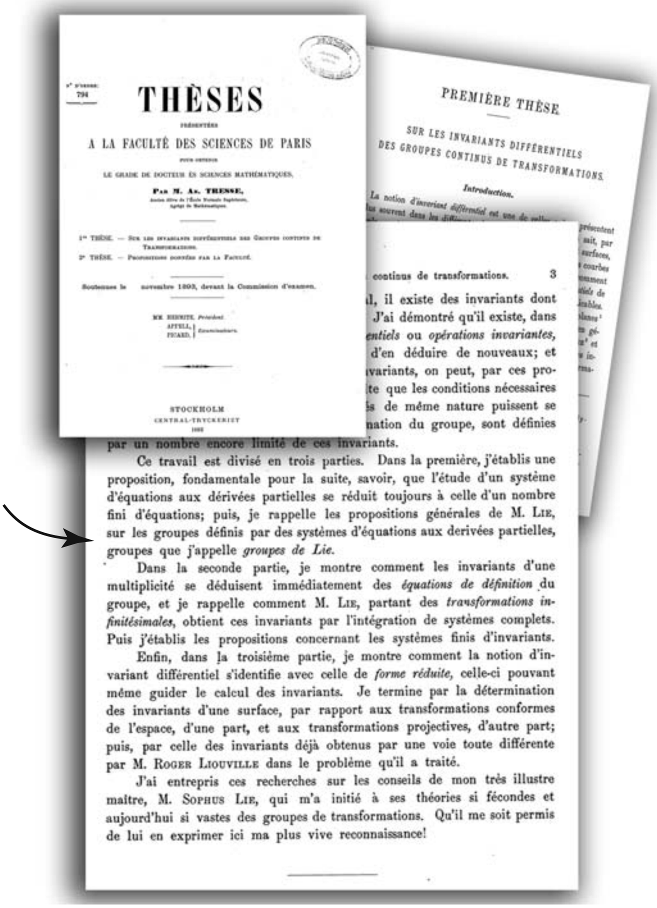
(a) Is this group compact?
(b) Show that the fundamental representation of G on $\mathbb{C}^2$ is reducible, but not completely reducible.
(c) Find the endomorphisms of $\mathbb{C}^2$ that commute with the fundamental representation of G.

Exercise 3.7 *Haar measure.*

(a) Find the Haar measure on $\mathrm{O}(2)$.
(b) Find the left-invariant and the right-invariant measures on the group of affine transformations of $\mathbb{R}$.

Exercise 3.8 *Representations of the direct product of two compact groups.*

Let (E_1, ρ_1) and (E_2, ρ_2) be finite-dimensional representations of compact groups G_1 and G_2, respectively. Show that we can define a representation $\rho = \rho_1 \times \rho_2$ of $G_1 \times G_2$ on $E_1 \otimes E_2$ by $\rho(g_1, g_2)(v_1 \otimes v_2) = \rho_1(g_1)v_1 \otimes \rho_2(g_2)v_2$, for $g_1 \in G_1, g_2 \in G_2, v_1 \in E_1, v_2 \in E_2$. Show that each irreducible representation of $G_1 \times G_2$ is of the form $(E_1 \otimes E_2, \rho_1 \times \rho_2)$, where (E_i, ρ_i) is an irreducible representation of G_i, $i = 1, 2$.



THÈSES

PRÉSENTÉES

A LA FACULTÉ DES SCIENCES DE PARIS

POUR OBTENIR

LE GRADE DE DOCTEUR ÈS SCIENCES MATHÉMATIQUES,

PAR M. AR. TRESSE,

STOCKHOLM

CENTRAL-TRYCKERIET

PREMIÈRE THÈSE.

SUR LES INVARIANTS DIFFÉRENTIELS DES GROUPES CONTINUS DE TRANSFORMATIONS.

Introduction.

continus de transformations. 3

l, il existe des invariants dont
J'ai démontré qu'il existe, dans
entiels ou *opérations invariantes,*
d'en déduire de nouveaux; et
ıvariants, on peut, par ces pro-
ite que les conditions nécessaires
s de même nature puissent se
nation du groupe, sont définies
par un nombre encore limité de ces invariants.

Ce travail est divisé en trois parties. Dans la première, j'établis une proposition, fondamentale pour la suite, savoir, que l'étude d'un système d'équations aux dérivées partielles se réduit toujours à celle d'un nombre fini d'équations; puis, je rappelle les propositions générales de M. LIE, sur les groupes définis par des systèmes d'équations aux derivées partielles, groupes que j'appelle *groupes de Lie.*

Dans la seconde partie, je montre comment les invariants d'une multiplicité se déduisent immédiatement des *équations de définition* du groupe, et je rappelle comment M. LIE, partant des *transformations infinitésimales,* obtient ces invariants par l'intégration de systèmes complets. Puis j'établis les propositions concernant les systèmes finis d'invariants.

Enfin, dans la troisième partie, je montre comment la notion d'invariant différentiel s'identifie avec celle de *forme réduite,* celle-ci pouvant même guider le calcul des invariants. Je termine par la détermination des invariants d'une surface, par rapport aux transformations conformes de l'espace, d'une part, et aux transformations projectives, d'autre part; puis, par celle des invariants déjà obtenus par une voie toute différente par M. ROGER LIOUVILLE dans le problème qu'il a traité.

J'ai entrepris ces recherches sur les conseils de mon très illustre maître, M. SOPHUS LIE, qui m'a initié à ses théories si fécondes et aujourd'hui si vastes des groupes de transformations. Qu'il me soit permis de lui en exprimer ici ma plus vive reconnaissance!

Arthur Tresse (1868–1958), a student of Lie, submitted his thesis in 1893 in Paris. It was published in Acta Mathematica, *18 (1893), pp. 1–88. In the introduction, page 3, he gives the groups studied by Lie the name “groupes de Lie”*: “then I recall the general propositions of Mr. Lie concerning the groups that are defined by systems of partial differential equations, groups which I shall call *Lie groups*.”

Chapter 4

Lie Groups and Lie Algebras

We restrict ourselves to the study of *linear Lie groups*, that is, to closed subgroups of $\mathrm{GL}(n, \mathbb{R})$, for an integer n, in other words, to groups of real matrices. We adopt the convention, introduced in Chapter 1, of calling such a group simply a *Lie group*. We shall show that to each Lie group there corresponds a Lie algebra. For ease of exposition, we start by defining Lie algebras, and return later to the study of groups.

1 Lie Algebras

1.1 Definition and Examples

We can define Lie algebras over any field. We restrict ourselves to the real or complex case. So let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition 1.1. *A* Lie algebra $\mathfrak{a}$ *over* $\mathbb{K}$ *is a finite- or infinite-dimensional* $\mathbb{K}$*-vector space with a* $\mathbb{K}$*-bilinear, antisymmetric operation* $[\ ,\ ]$ *satisfying the Jacobi identity*

$$\forall X, Y, Z \in \mathfrak{a},\ \ [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0. \tag{1.1}$$

The bilinear operation $[\ ,\ ]$ *is called the* Lie bracket (*or simply* bracket).

Examples.

- The vector space $\mathbb{R}^3$ with the cross product is a real Lie algebra of dimension 3.
- Let E be a $\mathbb{K}$-vector space. We define the commutator of two $\mathbb{K}$-linear operators u and v on E by

$$[u, v] = u \circ v - v \circ u. \tag{1.2}$$

The verification of the Jacobi identity is immediate, because

$$u \circ (v \circ w - w \circ v) - (v \circ w - w \circ v) \circ u + v \circ (w \circ u - u \circ w) - (w \circ u - u \circ w) \circ v \\ + w \circ (u \circ v - v \circ u) - (u \circ v - v \circ u) \circ w = 0.$$

P.Y. Kosmann-Schwarzbach, *Groups and Symmetries*, Universitext,
DOI 10.1007/978-0-387-78866-1_4, © Springer Science+Business Media, LLC 2010

Thus the commutator defines a Lie algebra structure on the vector space of $\mathbb{K}$-linear operators on E.

In particular, the vector space of $n \times n$ matrices with coefficients in the field $\mathbb{K}$, with the commutator

$$[X, Y] = XY - YX, \tag{1.3}$$

is a Lie algebra over $\mathbb{K}$ of dimension n^2, denoted by $\mathfrak{gl}(n, \mathbb{K})$.

Definition 1.2. *A* Lie subalgebra *of a Lie algebra is a vector subspace closed under the bracket.*

Thus each $\mathbb{K}$-vector subspace of $\mathfrak{gl}(n, \mathbb{K})$ closed under the commutator is a Lie algebra on $\mathbb{K}$.

Examples.

- The vector space of real (respectively, complex) matrices of trace zero is a real (respectively, complex) Lie algebra.
- The vector space of complex anti-Hermitian matrices is a real Lie algebra.
- The vector space of real antisymmetric matrices is a real Lie algebra.
- More generally, if $p + q = n$, we denote by J_{pq} the diagonal matrix with p diagonal elements 1 followed by q diagonal elements -1, and we set

$$\mathfrak{so}(p, q) = \{X \in \mathfrak{gl}(n, \mathbb{R}) \mid X J_{pq} + J_{pq}\, {}^tX = 0\}.$$

This is a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{R})$. In fact, for X and $Y \in \mathfrak{so}(p, q)$,

$$J_{pq}({}^tY\, {}^tX - {}^tX\, {}^tY) = -Y J_{pq}\, {}^tX + X J_{pq}\, {}^tY = -(XY - YX) J_{pq}.$$

Remark. Neither the vector space of symmetric matrices nor that of Hermitian matrices is stable under the commutator.

A Lie algebra is called *abelian* if the bracket is identically zero.

An *ideal* of $\mathfrak{a}$ is a Lie subalgebra $\mathfrak{b}$ of $\mathfrak{a}$ such that $[\mathfrak{a}, \mathfrak{b}] \subset \mathfrak{b}$, i.e., for every element X in $\mathfrak{a}$ and for every element Y in $\mathfrak{b}$, $[X, Y]$ is in $\mathfrak{b}$. In any Lie algebra $\mathfrak{a}$, the ideals $\{0\}$ and $\mathfrak{a}$ itself are called the *trivial ideals*.

The *center* of a Lie algebra $\mathfrak{a}$ is the abelian ideal of $\mathfrak{a}$,

$$\{X \in \mathfrak{a} \mid \forall Y \in \mathfrak{a},\ [X, Y] = 0\}.$$

A Lie algebra is called *simple* if it has no nontrivial ideals and is not of dimension 0 or 1. It is called *semisimple* if it has no nonzero abelian ideals.

If E is a normed vector space over $\mathbb{K}$, we consider the vector space of continuous $\mathbb{K}$-linear maps of E into E, with the standard norm of linear maps satisfying the inequality

$$\|u \circ v\| \leq \|u\| \|v\|. \tag{1.4}$$

This vector space with the commutator (1.2) is a Lie subalgebra of the Lie algebra of all linear operators on E. We denote it by $\mathfrak{gl}(E)$.

If E is finite-dimensional, every linear map on E is continuous; therefore in this case $\mathfrak{gl}(E)$ is the Lie algebra of all linear operators on E. In particular, $\mathfrak{gl}(\mathbb{K}^n)$ can be identified with the Lie algebra $\mathfrak{gl}(n, \mathbb{K})$ of $n \times n$ matrices with coefficients in $\mathbb{K}$.

1.2 Morphisms

Let $\mathfrak{a}$ and $\mathfrak{b}$ be Lie algebras over $\mathbb{K}$. A *morphism* of Lie algebras of $\mathfrak{a}$ into $\mathfrak{b}$ is a $\mathbb{K}$-linear map φ of $\mathfrak{a}$ into $\mathfrak{b}$ such that

$$\forall X, Y \in \mathfrak{a}, [\varphi X, \varphi Y]_{\mathfrak{b}} = \varphi([X, Y]_{\mathfrak{a}}).$$

An *isomorphism* of $\mathfrak{a}$ onto $\mathfrak{b}$ is a bijective morphism. An *automorphism* of $\mathfrak{a}$ is an isomorphism of $\mathfrak{a}$ onto $\mathfrak{a}$. We write $\mathrm{Aut}(\mathfrak{a})$ for the group of automorphisms of $\mathfrak{a}$.

For all of the following, we shall consider *finite-dimensional* Lie algebras.

1.3 Commutation Relations and Structure Constants

Let $\mathfrak{a}$ be a Lie algebra of finite dimension d, and let $(e_i),\ i = 1, \ldots, d$, be a basis of the vector space $\mathfrak{a}$. The *commutation relations* of $\mathfrak{a}$ in the basis (e_i) are the Lie brackets $[e_i, e_j], 1 \leq i < j \leq d$. The constants, real or complex numbers C^i_{jk} such that

$$[e_i, e_j] = \sum_{k=1}^{d} C^k_{ij} e_k,$$

are called *structure constants*. If two finite-dimensional Lie algebras have the same commutation relations, they are isomorphic. More precisely, if there exist in the Lie algebras $\mathfrak{a}$ and $\mathfrak{b}$ of the same finite dimension bases (e_i) and (f_i) such that the structure constants are the same, then the linear isomorphism of $\mathfrak{a}$ onto $\mathfrak{b}$ defined by $e_i \mapsto f_i$ is an isomorphism of Lie algebras.

1.4 Real Forms

On the one hand, each complex vector space E can be considered as a vector space over $\mathbb{R}$, denoted by $E^{\mathbb{R}}$, of twice the dimension. On the other hand, each real vector space E can be complexified by setting $E^{\mathbb{C}} = E \oplus iE = E \otimes \mathbb{C}$ and by setting $i(X + iY) = -Y + iX$, for every X and $Y \in E$. Note that if E is a complex (respectively, real) vector space and $E \neq \{0\}$, then $(E^{\mathbb{R}})^{\mathbb{C}} \neq E$ (respectively, $(E^{\mathbb{C}})^{\mathbb{R}} \neq E$), as one can easily see by calculating the dimensions.

If $\mathfrak{a}$ is a complex Lie algebra, we denote by $\mathfrak{a}^{\mathbb{R}}$ the Lie algebra over $\mathbb{R}$ (of twice the dimension) obtained by "realification." On the other hand, if $\mathfrak{a}$ is a real Lie algebra, the corresponding complexified vector space can be given the structure of a complex Lie algebra by setting $[X, iY] = i[X, Y]$ and $[iX, iY] = -[X, Y]$, for every X and $Y \in \mathfrak{a}$. We denote by $\mathfrak{a}^{\mathbb{C}}$ the complex Lie algebra thus obtained, called the *complexification* of the real Lie algebra $\mathfrak{a}$.

A *real form* of a complex Lie algebra $\mathfrak{g}$ is a real Lie algebra $\mathfrak{a}$ such that $\mathfrak{a}^{\mathbb{C}} = \mathfrak{g}$. For example, $(\mathfrak{gl}(n, \mathbb{R}))^{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{C})$. In general, a complex Lie algebra has several nonisomorphic real forms.

1.5 Representations of Lie Algebras

Here we define Lie algebra representations, restricting ourselves to the case of finite-dimensional representations. We shall see in Section 7.2 the relationship between representations of groups and representations of Lie algebras.

Definition 1.3. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$. A* representation *of $\mathfrak{g}$ is a $\mathbb{K}$-linear mapping R of $\mathfrak{g}$ into $\mathfrak{gl}(E)$, where E is a finite-dimensional complex vector space satisfying*

$$\forall X, Y \in \mathfrak{g},\ [R(X), R(Y)] = R([X, Y]).$$

A representation of a complex Lie algebra $\mathfrak{g}$ on a complex vector space E is thus a morphism of complex Lie algebras of $\mathfrak{g}$ into $\mathfrak{gl}(E)$. The dimension of E is called the *dimension* of the representation.

A representation of a real Lie algebra $\mathfrak{g}$ on a complex vector space E is a morphism of real Lie algebras of $\mathfrak{g}$ into $\mathfrak{gl}(E)^{\mathbb{R}}$. We show below (Proposition 1.4) that such a morphism has a unique extension to a morphism of complex Lie algebras of $\mathfrak{g}^{\mathbb{C}}$ into $\mathfrak{gl}(E)$. Later, we study numerous examples. Here is a very simple one. The matrices

$$\eta_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \eta_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

form a basis of a real Lie algebra $\mathfrak{g} = \mathfrak{so}(3)$, of dimension 3, whose commutation relations are

$$[\eta_k, \eta_\ell] = \eta_m$$

for each cyclic permutation k, ℓ, m of $1, 2, 3$. To the elements η_1, η_2, η_3 of $\mathfrak{g}$ we associate the complex 2×2 matrices

$$\xi_1 = \frac{i}{2}\sigma_1, \quad \xi_2 = -\frac{i}{2}\sigma_2, \quad \xi_3 = \frac{i}{2}\sigma_3,$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

are the Pauli matrices. Because

$$[\xi_k, \xi_\ell] = \xi_m,$$

we see immediately that the $\mathbb{R}$-linear map of $\mathfrak{so}(3)$ into $\mathfrak{gl}(2, \mathbb{C})^{\mathbb{R}}$ defined by $\eta_k \mapsto \xi_k$ is a representation of $\mathfrak{g}$ on $\mathbb{C}^2$. Furthermore, extending this map by $\mathbb{C}$-linearity defines a morphism of complex Lie algebras of $\mathfrak{so}(3)^{\mathbb{C}}$ into $\mathfrak{gl}(2, \mathbb{C})$.

Remark. We can also define the *real representations* of Lie algebras over $\mathbb{K}$, which are $\mathbb{R}$-linear maps into $\mathfrak{gl}(E)$, where E is a real vector space.

A representation (E, R) of $\mathfrak{g}$ is called *irreducible* if $E \neq \{0\}$ and there are no nontrivial subspaces of E invariant under R, i.e., the only vector subspaces of E invariant under R are $\{0\}$ and E itself.

Representations (E_1, R_1) and (E_2, R_2) of a Lie algebra $\mathfrak{g}$ are *equivalent* if there is an isomorphism T of E_1 onto E_2 such that

$$\forall X \in \mathfrak{g},\ R_2(X) \circ T = T \circ R_1(X).$$

Proposition 1.4. *Let $\mathfrak{g}^{\mathbb{C}}$ be the complexification of a real Lie algebra $\mathfrak{g}$. Every representation of $\mathfrak{g}$ can be extended uniquely to a representation of $\mathfrak{g}^{\mathbb{C}}$. There is a bijective correspondence between irreducible representations of $\mathfrak{g}$ and of $\mathfrak{g}^{\mathbb{C}}$.*

Proof. Let (E, R) be a representation of the Lie algebra $\mathfrak{g}$ on a complex vector space E. We set

$$\forall X, Y \in \mathfrak{g},\ R(X + iY) = R(X) + iR(Y) \in \mathfrak{gl}(E).$$

A calculation using the equation $R([X, X']) = R(X) \circ R(X') - R(X') \circ R(X)$, for X and $X' \in \mathfrak{g}$, shows that

$$R([X + iY, X' + iY']) = R(X + iY) \circ R(X' + iY') - R(X' + iY') \circ R(X + iY).$$

Thus the extension of R is a representation of $\mathfrak{g}^{\mathbb{C}}$. The uniqueness is obvious. On the other hand, from a representation of $\mathfrak{g}^{\mathbb{C}}$ one can define a representation of $\mathfrak{g}$ by restriction. Thus it is clear that there is a bijective correspondence between representations of $\mathfrak{g}$ and representations of $\mathfrak{g}^{\mathbb{C}}$. Furthermore, if a representation of $\mathfrak{g}$ is irreducible, the corresponding representation of $\mathfrak{g}^{\mathbb{C}}$ is irreducible as well. □

In summary, the equivalence relation on the representations of $\mathfrak{g}$ corresponds to the equivalence relation on the representations of $\mathfrak{g}^{\mathbb{C}}$. Thus the equivalence classes of irreducible representations of $\mathfrak{g}$ and of $\mathfrak{g}^{\mathbb{C}}$ are in bijective correspondence.

2 Review of the Exponential Map

Let $X \in \mathfrak{gl}(n, \mathbb{K})$. We set

$$\exp X = e^X = \sum_{p=0}^{\infty} \frac{X^p}{p!}. \tag{2.1}$$

For each compact set $K \subset \mathfrak{gl}(n, \mathbb{K})$ there exists a positive real number M such that $\|X\| \leq M$ for all $X \in K$. Therefore when we equip $\mathfrak{gl}(n, \mathbb{K})$ with the standard norm of linear maps, which satisfies the inequality (1.4), the series (2.1) is convergent and uniformly convergent on each compact set.

We have

$$\exp 0 = I, \tag{2.2}$$

$$\forall s, t \in \mathbb{R},\ \exp(sX)\exp(tX) = \exp(s + t)X. \tag{2.3}$$

Furthermore, if $XY = YX$ (but in this case only!), then $\exp(X + Y) = \exp X \exp Y$.

We deduce from (2.3) that for every X, $\exp X$ is invertible with inverse $\exp(-X)$,

$$(\exp X)^{-1} = \exp(-X).$$

Thus the exponential map is a map from $\mathfrak{gl}(n,\mathbb{K})$ into the group of invertible matrices $\mathrm{GL}(n,\mathbb{K}) \subset \mathfrak{gl}(n,\mathbb{K})$.

Lemma 2.1. *The map* $t \in \mathbb{R} \mapsto \exp(tX) \in \mathrm{GL}(n,\mathbb{K})$ *is differentiable and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp(tX) = X\ \exp(tX) = \exp(tX)X. \tag{2.4}$$

Proof. The term-by-term differentiation of the exponential series for $\exp(tX)$ is legitimate and gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{p=0}^{\infty}\frac{X^p}{p!}t^p = \sum_{p=1}^{\infty}X^p\frac{t^{p-1}}{(p-1)!} = X\sum_{p=0}^{\infty}X^p\frac{t^p}{p!} = X\ \exp(tX) = \exp(tX)\ X,$$

for every $t \in \mathbb{R}$. □

In particular,

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp(tX)\big|_{t=0} = X. \tag{2.5}$$

Proposition 2.2. *The exponential map of* $\mathfrak{gl}(n,\mathbb{K})$ *into* $\mathrm{GL}(n,\mathbb{K}) \subset \mathfrak{gl}(n,\mathbb{K})$ *is smooth, and its differential at the origin is the identity map of* $\mathfrak{gl}(n,\mathbb{K})$ *into itself.*

Proof. We have

$$\exp X = I + X + X\sum_{p=2}^{\infty}\frac{X^{p-1}}{p!},$$

whence $\|\exp X - \exp 0 - X\| \leq \|X\|\ \varepsilon(X)$, where

$$\varepsilon(X) = \left\|\sum_{p=2}^{\infty}\frac{X^{p-1}}{p!}\right\| \leq \|X\|\left\|\sum_{q=0}^{\infty}\frac{X^q}{(q+2)!}\right\|.$$

This last series converges, so $\lim_{\|X\|\to 0}\|\varepsilon(X)\| = 0$, which shows that exp is differentiable at 0, with differential $\mathrm{Id}_{\mathfrak{gl}(n,\mathbb{K})}$.

One can show, that the map exp is differentiable at each point and is of class C^∞ (see Exercise 4.6), and we shall use this result. □

By the inverse function theorem, we obtain the following result.

Corollary 2.3. *There exist an open neighborhood* U_0 *of* 0 *in* $\mathfrak{gl}(n,\mathbb{K})$ *and an open neighborhood* V_0 *of* I *in* $\mathrm{GL}(n,\mathbb{K})$ *such that* exp *is a diffeomorphism of* U_0 *onto* V_0.

We call the inverse diffeomorphism, defined on a neighborhood of $I \in \mathrm{GL}(n, \mathbb{K})$, the *logarithm*. We denote it by log (or sometimes ln). Then,

$$\forall X \in \mathfrak{gl}(n, \mathbb{K}) \text{ such that } \|X\| < 1,\ \log(I + X) = \sum_{p=0}^{\infty} (-1)^p \frac{X^{p+1}}{p+1}.$$

Proposition 2.4. *For every $X, Y \in \mathfrak{gl}(n, \mathbb{K})$,*

$$\exp(tX)\ \exp(tY) = \exp\left(t(X+Y) + \frac{t^2}{2}[X, Y] + \mathrm{O}(t^3)\right). \tag{2.6}$$

Proof. For t near 0, on the one hand,

$$\begin{aligned}\exp(tX)\ \exp(tY) &= \left(I + tX + \frac{t^2}{2}X^2 + \mathrm{O}(t^3)\right)\left(I + tY + \frac{t^2}{2}Y^2 + \mathrm{O}(t^3)\right)\\ &= I + t(X+Y) + \frac{t^2}{2}\left(X^2 + 2XY + Y^2\right) + \mathrm{O}(t^3).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\exp\left(t(X+Y) + \frac{t^2}{2}(XY - YX) + \mathrm{O}(t^3)\right)\\ &\quad = I + t(X+Y) + \frac{t^2}{2}(XY - YX) + \frac{t^2}{2}(X^2 + XY + YX + Y^2) + \mathrm{O}(t^3).\end{aligned}$$

The two expressions are thus equal. □

One could expand further and consider the coefficients of the terms of order t^3, t^4, etc. The result is known as the Baker–Campbell–Hausdorff formula.

Corollary 2.5. *The exponential of a sum can be expressed as*

$$\exp(X+Y) = \lim_{k \to +\infty}\left(\exp\frac{X}{k}\ \exp\frac{Y}{k}\right)^k. \tag{2.7}$$

Proof. We have, for every positive integer k,

$$\begin{aligned}\left(\exp\frac{X}{k}\ \exp\frac{Y}{k}\right)^k &= \left(\exp\left(\frac{1}{k}(X+Y) + \mathrm{O}\left(\frac{1}{k^2}\right)\right)\right)^k\\ &= \exp\left(X + Y + \mathrm{O}\left(\frac{1}{k}\right)\right),\end{aligned}$$

so the result follows by the continuity of the exponential map. □

Proposition 2.6. *For each $X \in \mathfrak{gl}(n, \mathbb{K})$,*

$$\exp(\mathrm{Tr}X) = \det(\exp X). \tag{2.8}$$

In particular, the exponential of a traceless matrix is a matrix of determinant 1.

Proof. Over $\mathbb{C}$, one can triangularize the matrix X. If X is similar to a triangular matrix T, then $\exp X$ is similar to $\exp T$. If the diagonal coefficients of T are $a_1, \ldots, a_n$, then the diagonal coefficients of the matrix $\exp T$ are $e^{a_1}, \ldots, e^{a_n}$. Hence

$$\det(\exp X) = \prod_{i=1}^{n} e^{a_i} = e^{\sum_{i=1}^{n} a_i} = e^{\mathrm{Tr} X}.$$

Over $\mathbb{R}$, the equation is valid for real numbers. □

Corollary 2.7. *We have*

$$\frac{\mathrm{d}}{\mathrm{d}t} \det(\exp(tX))\big|_{t=0} = \mathrm{Tr} X.$$

Proof. In formula (2.8), we replace X by tX and differentiate. □

3 One-Parameter Subgroups of $\mathrm{GL}(n, \mathbb{K})$

We shall show that in the case of one-parameter subgroups of the group of invertible matrices, the hypothesis of continuity implies differentiability.

Definition 3.1. *A* one-parameter subgroup *of a topological group G is a* continuous *map $f : \mathbb{R} \to G$ such that*

$$\forall t \in \mathbb{R}, \forall s \in \mathbb{R},\ f(t+s) = f(t)f(s).$$

We may also simply write *one-parameter group*. The definition implies that $f(0) = e$.

Lemma 3.2. *Let $f : \mathbb{R} \to \mathrm{GL}(n, \mathbb{K})$ be a one-parameter subgroup of* $\mathrm{GL}(n, \mathbb{K})$*. Then f is differentiable.*

Proof. Let $a \in \mathbb{R}$, $a > 0$. The map f is continuous, and thus $\int_0^a f(t)\mathrm{d}t$ exists. For each $s \in \mathbb{R}$,

$$\int_0^a f(t+s)\mathrm{d}t = f(s) \int_0^a f(t)\mathrm{d}t = \int_s^{s+a} f(t)\mathrm{d}t.$$

This quantity is differentiable with respect to s. To prove that f is differentiable, it is sufficient to show that there is a real number $a > 0$ such that $\int_0^a f(t)\mathrm{d}t$ is invertible. Now, because f is continuous at 0 and $f(0) = I$,

$$\exists a > 0,\ |t| \leq a,\ \|f(t) - I\| \leq \frac{1}{2},$$

whence we deduce that

$$\left\| \frac{1}{a} \left(\int_0^a f(t)\mathrm{d}t \right) - I \right\| = \left\| \frac{1}{a} \int_0^a (f(t) - I)\mathrm{d}t \right\| \leq \frac{1}{2} < 1,$$

and thus $\int_0^a f(t)\mathrm{d}t$ is invertible. □

Proposition 3.3. *Let $f : \mathbb{R} \to \mathrm{GL}(n,\mathbb{K})$ be a one-parameter subgroup of $\mathrm{GL}(n,\mathbb{K})$. There exists a unique $X \in \mathfrak{gl}(n,\mathbb{K})$ such that for each $t \in \mathbb{R}$,*

$$f(t) = \exp(tX). \tag{3.1}$$

Proof. The uniqueness is clear, since f is differentiable by the preceding lemma, and hence the desired matrix X must satisfy the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t)\Big|_{t=0} = X. \tag{3.2}$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t) = \lim_{s\to 0} \frac{f(t+s) - f(t)}{s} = \lim_{s\to 0} f(t) \frac{f(s) - f(0)}{s} = f(t)\ X.$$

By Lemma 2.1, we see that $f(t)$ and $\exp(tX)$ are solutions of the same differential equation with the same initial condition $f(0) = I$. The proposition follows. □

Definition 3.4. *The element X of $\mathfrak{gl}(n,\mathbb{K})$ satisfying* (3.1) *is called the* infinitesimal generator *of the one-parameter group $t \mapsto f(t)$.*

We have

$$X = f'(0)$$

and, more generally,

$$\forall t \in \mathbb{R},\ (f(t))^{-1} f'(t) = X. \tag{3.3}$$

Example. The rotations around the axes $0x$, $0y$, $0z$, respectively, constitute the one-parameter groups

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}, \quad \begin{pmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{pmatrix}, \quad \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

of the form $\exp(t\eta_i), i = 1, 2, 3$, where the infinitesimal generators are the matrices η_1, η_2, η_3 defined in Section 1.5.

Important Remark. In physics, the name "infinitesimal generator" of a one-parameter group $f(t)$ is given to the matrix A satisfying

$$f(t) = \exp(-itA).$$

Thus $A = iX$. In particular, if the one-parameter group consists of unitary transformations of a finite-dimensional Hilbert space, then the operator A is Hermitian (self-adjoint), and the operator X is anti-Hermitian. In infinite dimensions, the problem is that the operator A is not necessarily bounded.

4 Lie Groups

Definition 4.1. *A linear Lie group is a closed subgroup of a group* $\mathrm{GL}(n, \mathbb{K})$.

We can identify $\mathrm{GL}(m, \mathbb{C})$ with a closed subgroup of $\mathrm{GL}(2m, \mathbb{R})$. In fact, each complex $m \times m$ matrix A can be written $A = B + iC$, where B and C are real $m \times m$ matrices and the $\mathbb{R}$-linear map $A \mapsto \left(\begin{smallmatrix} B & -C \\ C & B \end{smallmatrix}\right)$ of $\mathrm{GL}(m, \mathbb{C})$ onto a closed subgroup of $\mathrm{GL}(2m, \mathbb{R})$ is a morphism of groups. *A linear Lie group is thus a closed subgroup of* $\mathrm{GL}(n, \mathbb{R})$, for a certain n. In Chapter 1 we have given an abstract definition of real Lie groups as smooth manifolds equipped with a group structure compatible with their differentiable structure, and we have stated that each linear Lie group is a real Lie group. In the following, we study only linear Lie groups, and *by convention*, we call them simply *Lie groups*. There exist Lie groups that are not linear Lie groups, such as the universal cover of $\mathrm{SL}(2, \mathbb{R})$. Most of the properties proved below extend to the general case.

In Chapter 1, we saw several examples of Lie groups: the real and complex special linear groups in dimension n, the orthogonal groups, and the unitary groups, related by

$$\begin{array}{ccc} \mathrm{SO}(n) \subset \mathrm{O}(n) \subset \mathrm{GL}(n, \mathbb{R}) & & \overbrace{\mathrm{SU}(n) \subset \mathrm{U}(n)}^{\text{real Lie groups}} \subset \mathrm{GL}(n, \mathbb{C}) \\ \cap & & \cap \\ \mathrm{SL}(n, \mathbb{R}) & & \mathrm{SL}(n, \mathbb{C}) \end{array}$$

For each positive integer n, the additive group $\mathbb{R}^n$ is isomorphic to the group of $(n+1) \times (n+1)$ matrices of the form $\left(\begin{smallmatrix} I & x \\ 0 & 1 \end{smallmatrix}\right)$, where $x \in \mathbb{R}^n$, or to the group of matrices $\begin{pmatrix} e^{x_1} & & \\ & \ddots & \\ & & e^{x_n} \end{pmatrix}$, and is thus a Lie group.

The group of affine transformations of $\mathbb{R}^n$ is a Lie group.

The symplectic groups $\mathrm{Sp}(n, \mathbb{C})$ and $\mathrm{Sp}(n)$ are defined by

$$\mathrm{Sp}(n, \mathbb{C}) = \{A \in \mathrm{SL}(2n, \mathbb{C}) \mid AJ\,{}^tA = J\},$$
$$\mathrm{Sp}(n) = \mathrm{Sp}(n, \mathbb{C}) \cap \mathrm{U}(2n),$$

where $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$, and are Lie groups.

5 The Lie Algebra of a Lie Group

Let G be be a Lie group, i.e., a closed subgroup of $\mathrm{GL}(n, \mathbb{R})$. We consider

$$\mathfrak{g} = \{X = \gamma'(0) \mid \gamma \;:\; \mathcal{I} \to G, \;\; \text{of class } C^1, \; \gamma(0) = I\},$$

where $\mathcal{I}$ is an open interval of $\mathbb{R}$ containing 0. This is the set of vectors tangent to parametrized curves of class C^1 in G passing through I for the value 0 of the parameter. We shall show that $\mathfrak{g}$ is not only a real vector space, but also a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{R})$.

Theorem 5.1. *Let G be a Lie group and let $\mathfrak{g}$ be defined as above.*

(*i*) $\mathfrak{g}$ *is a vector subspace of* $\mathfrak{gl}(n,\mathbb{R})$.
(*ii*) $X \in \mathfrak{g}$ *if and only if for every* $t \in \mathbb{R}$, $\exp(tX) \in G$.
(*iii*) *If* $X \in \mathfrak{g}$ *and if* $g \in G$*, then* $gXg^{-1} \in \mathfrak{g}$.
(*iv*) $\mathfrak{g}$ *is closed under the matrix commutator.*

Proof. (i) Let $\gamma : \mathcal{I} \to G$ be of class C^1 and such that $\gamma(0) = I$. Set $\gamma'(0) = X$. For each $\lambda \in \mathbb{R}$, the parametrized curve $t \mapsto \gamma(\lambda t)$ has tangent vector λX at 0. Thus $\mathfrak{g}$ is closed under multiplication by real scalars. Let $\gamma_i : \mathcal{I} \to G$, $i = 1, 2$, be of class C^1 and such that $\gamma_i(0) = I$. Let $X_1 = \gamma_1'(0), X_2 = \gamma_2'(0)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma_1(t)\gamma_2(t))\Big|_{t=0} = X_1 + X_2.$$

Thus $\mathfrak{g}$ is closed under addition.

(ii) It is clear that if $\exp(tX) \in G$, then $X = \frac{\mathrm{d}}{\mathrm{d}t}\exp(tX)\big|_{t=0} \in \mathfrak{g}$. Conversely, if $X \in \mathfrak{g}$, then by hypothesis, $X = \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)\big|_{t=0}$, with $\gamma(t) \in G$. By the Taylor expansion, for any positive integer k,

$$\gamma\left(\frac{t}{k}\right) = I + \frac{t}{k}X + \mathrm{O}\left(\frac{1}{k^2}\right) = \exp\left(\frac{t}{k}X + \mathrm{O}\left(\frac{1}{k^2}\right)\right).$$

We can deduce that

$$\left(\gamma\left(\frac{t}{k}\right)\right)^k = \exp\left(tX + \mathrm{O}\left(\frac{1}{k}\right)\right),$$

and hence

$$\lim_{k\to+\infty}\left(\gamma\left(\frac{t}{k}\right)\right)^k = \exp(tX).$$

This equation implies that $\exp(tX) \in G$ because $\gamma(\frac{t}{k}) \in G$, and G is a closed subgroup.

(iii) If $X \in \mathfrak{g}$ and $g \in G$, then $gXg^{-1} = \frac{\mathrm{d}}{\mathrm{d}t}\exp(t(gXg^{-1}))\big|_{t=0}$, and for every t, the matrix $\exp(t(gXg^{-1})) = g(\exp(tX))g^{-1}$ belongs to G, so $gXg^{-1} \in \mathfrak{g}$.

(iv) Let X and $Y \in \mathfrak{g}$. Then, $\exp(tX) \in G$ and $\exp(tX)Y\exp(-tX) \in \mathfrak{g}$, by (iii). In the vector space $\mathfrak{g}$, we consider

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp(tX)Y\exp(-tX)\right)\big|_{t=0}.$$

This element of $\mathfrak{g}$ is $XY - YX = [X,Y]$. □

Definition 5.2. *The Lie algebra* $\mathfrak{g}$*, the tangent space to* G *at* I*, is called* the Lie algebra of the Lie group G.

Property (ii) of Theorem 5.1 is the fundamental characterization of the Lie algebra of a Lie group.

The dimension of the real vector space $\mathfrak{g}$ is called the *dimension* of the Lie group G. This is the number of independent parameters necessary to parametrize the points of the group. (This is also the dimension of the Lie group as a real manifold.)

If the group G is a discrete subgroup of $\mathrm{GL}(n,\mathbb{R})$, a continuous curve in G is necessarily constant, and the Lie algebra of the group is thus reduced to $\{0\}$, and the dimension of the group is 0.

Examples. The Lie algebras of $\mathrm{GL}(n,\mathbb{K}), \mathrm{SL}(n,\mathbb{K}), \mathrm{O}(n), \mathrm{SO}(n), \mathrm{U}(n), \mathrm{SU}(n)$ are denoted respectively by $\mathfrak{gl}(n,\mathbb{K}), \mathfrak{sl}(n,\mathbb{K}), \mathfrak{o}(n), \mathfrak{so}(n), \mathfrak{u}(n), \mathfrak{su}(n)$. The notation $\mathfrak{gl}(n,\mathbb{K})$ is consistent with our previous notation: it is the vector space of all the $n \times n$ matrices with coefficients in $\mathbb{K}$ equipped with the commutator.

Proposition 5.3. *The Lie algebras of* $\mathrm{SL}(n,\mathbb{K}), \mathrm{O}(n), \mathrm{SO}(n), \mathrm{U}(n),$ *and* $\mathrm{SU}(n)$ *are*

- $\mathfrak{sl}(n,\mathbb{K}) = \{X \in \mathfrak{gl}(n,\mathbb{K}) \mid \mathrm{Tr}\ X = 0\}$ *is the Lie algebra of traceless* $n \times n$ *matrices with coefficients in* $\mathbb{K}$*, and*

$$\dim_{\mathbb{R}} \mathfrak{sl}(n,\mathbb{R}) = n^2 - 1, \quad \dim_{\mathbb{C}} \mathfrak{sl}(n,\mathbb{C}) = n^2 - 1, \quad \dim_{\mathbb{R}} \mathfrak{sl}(n,\mathbb{C}) = 2(n^2 - 1).$$

- $\mathfrak{o}(n) = \mathfrak{so}(n) = \{X \in \mathfrak{gl}(n,\mathbb{R}) \mid X +^t X = 0\}$ *is the Lie algebra of antisymmetric real* $n \times n$ *matrices and* $\dim \mathfrak{so}(n) = n(n-1)/2$.
- $\mathfrak{u}(n) = \{X \in \mathfrak{gl}(n,\mathbb{C}) \mid X +^t \overline{X} = 0\}$ *is the Lie algebra of anti-Hermitian complex* $n \times n$ *matrices and* $\dim \mathfrak{u}(n) = n^2$.
- $\mathfrak{su}(n) = \{X \in \mathfrak{gl}(n,\mathbb{C}) \mid X +^t \overline{X} = 0, \mathrm{Tr} X = 0\}$ *is the Lie algebra of traceless anti-Hermitian complex* $n \times n$ *matrices and* $\dim_{\mathbb{R}} \ \mathfrak{su}(\mathrm{n}) = \mathrm{n}^2 - 1$.

Proof. In each case we express the condition for $\exp(tX)$ to belong to the indicated Lie group, for every $t \in \mathbb{R}$. □

In certain, but not all, cases, where the Lie group G is a subgroup of $\mathrm{GL}(n,\mathbb{C})$, its Lie algebra has the structure of a complex vector space. This is the case for $\mathfrak{sl}(n,\mathbb{C})$, but not for $\mathfrak{u}(n)$.

Theorem 5.4. *The application* exp *is a diffeomorphism of a neighborhood of* 0 *in* $\mathfrak{g}$ *onto a neighborhood of* I *in* G.

Proof. Let U_0 be a neighborhood of 0 in $\mathfrak{gl}(n,\mathbb{R})$ and let V_0 be a neighborhood of I in $\mathrm{GL}(n,\mathbb{R})$ such that exp is a diffeomorphism of U_0 onto V_0, as in Corollary 2.3. We consider a supplementary vector subspace $\mathfrak{g}'$ of $\mathfrak{g}$ in $\mathfrak{gl}(n,\mathbb{R})$.

We shall first show by contradiction that there exists a neighborhood $U_0' \subset U_0$ of 0 in $\mathfrak{g}'$ such that the two conditions $X \in U_0'$ and $\exp X \in G$ imply $X = 0$. Suppose then that in each neighborhood of 0 in $\mathfrak{g}'$ there is an $X \neq 0$ such that $\exp X \in G$. In particular, for every n, there is an $X_n \in \frac{1}{n}(U_0 \cap \mathfrak{g}'), X_n \neq 0$, such that $\exp X_n \in G$. Thus we construct a sequence (X_n) in $\mathfrak{g}'$ that tends to 0 as n tends to infinity. We set $Y_n = X_n/\|X_n\|$, which implies $\|Y_n\| = 1$. Because the unit sphere is compact, there is a subsequence of (Y_n) that converges in the vector space $\mathfrak{g}'$ to a limit Y of norm 1. For $t \in \mathbb{R}$, we consider the integral part $p_n(t)$ of $t/\|X_n\|$ defined by

$$\frac{t}{\|X_n\|} = p_n(t) + u_n(t), \quad p_n(t) \in \mathbb{Z}, \quad u_n(t) \in [0,1].$$

Then

$$\exp\left(\frac{t}{\|X_n\|}X_n\right) = (\exp X_n)^{p_n(t)} \exp(u_n(t)X_n).$$

The sequence $\exp(u_n(t)X_n)$ tends to I, since the sequence $(u_n(t))$ is bounded and the sequence (X_n) tends to 0, and $(\exp X_n)^{p_n(t)}$ is in G since G is a group. Therefore the sequence $\exp\left(\frac{t}{\|X_n\|}X_n\right)$ tends to an element in G. Because the exponential map is continuous, the limit of $\exp\left(\frac{t}{\|X_n\|}X_n\right)$ is $\exp(tY)$, which is therefore in G. By Theorem 5.1 (ii), this in turn implies that Y is in $\mathfrak{g}$, which contradicts the hypothesis $\mathfrak{g} \cap \mathfrak{g}' = \{0\}$.

Now let $\alpha : \mathfrak{g} \oplus \mathfrak{g}' \to \mathrm{GL}(n, \mathbb{R})$ be the map defined by

$$\alpha(X, X') = \exp X \exp X'.$$

The differential at 0 of α is the identity, and thus α is a local diffeomorphism of a product $U \times U'$ of neighborhoods of 0 in $\mathfrak{g}$ and $\mathfrak{g}'$ respectively onto a neighborhood W of I in $\mathrm{GL}(n, \mathbb{R})$. We can assume that $U' \subset U'_0$. We show that $W \cap G = \exp U$. It is clear that $\exp U \subset W \cap G$. On the other hand, if $g \in W \cap G$, then $g = \exp X \exp X'$, where $X \in U \subset \mathfrak{g}$, $X' \in U'_0 \subset \mathfrak{g}'$. Then we have $X' = 0$, since $X' \in U'_0$ and $\exp X' = (\exp X)^{-1} g \in G$. Thus $W \cap G \subset \exp U$. □

6 The Connected Component of the Identity

Definition 6.1. *The* connected component *of a point $g \in G$ is the largest connected set contained in G and containing g. The* connected component of the identity *is often called* the connected component *of G for short. We shall denote it by G_0.*

The connected component of a point $g \in G$ is the union of the connected sets contained in G and containing g. Clearly, if G is connected, $G = G_0$ and G has only one component.

Proposition 6.2. *Each connected component of G is open and closed in G.*

Proof. Each connected component is closed because the closure of a connected set is connected. It is also open. In fact, each point of G has a connected neighborhood: for I, the image of a ball centered at the origin under the exponential map; for an arbitrary element of the group, the translate of such a neighborhood. For h in the connected component of $g \in G$, let V be a connected neighborhood of h. Then V is contained in the connected component of g, and thus the connected component of g is a neighborhood of h. □

Recall that the only nonempty open subset of a connected set is the set itself. It follows that G is the disjoint union of its connected components.

Proposition 6.3. *The connected component of the identity of G is a subgroup of G. Any open and closed subgroup of G contains the connected component of the identity of G.*

Proof. If $g \in G_0$, the left translate of G_0 by g^{-1} is diffeomorphic to G_0, and thus is connected, and it contains I, so it is contained in G_0. Thus, G_0 is a subset containing I and such that for every g and $g' \in G_0$, $g^{-1}g'$ is in G_0, i.e., G_0 is a subgroup.

Let H be an open and closed subgroup of G. Then $H \cap G_0$ is open and closed. Since it contains I, it is nonempty. Therefore $H \cap G_0 = G_0$. □

Example. The connected component of $\mathrm{O}(n)$ is $\mathrm{SO}(n)$. For $n = 3$, this fact will be proved in Section 2.1 of Chapter 5.

Proposition 6.4. *Every neighborhood of the identity in G generates a subgroup of G containing the connected component of the identity. In particular, if the Lie group G is connected, every neighborhood of the identity generates G.*

Proof. Let U be an open neighborhood of I in G, and let H be the subgroup generated by U. Because the translations are diffeomorphisms, for every $g \in H$, the subset Ug of H is open, and it is contained in H because H is a subgroup of G. Thus H is open. On the other hand, each open subgroup H of any topological group is also closed. In fact, the entire group is the union of left (or right) cosets modulo H and each class is open. Since H is the complement of the union of cosets other than that of the identity, and since this union of open sets is open, we deduce that H is closed. Thus the subgroup H is open and closed. We deduce that it contains the connected component of the identity. □

Theorem 6.5. *Let G be be a Lie group and let $\mathfrak{g}$ be its Lie algebra. The subgroup of G generated by* $\exp \mathfrak{g}$ *is the connected component of the identity. In particular, if G is connected, each element of G is the product of a finite number of exponentials.*

Proof. By Theorem 5.4, the image of the exponential map is a neighborhood of the identity. Thus the subgroup that it generates contains G_0, by Proposition 6.4. Because $\exp \mathfrak{g}$ is connected, it is contained in G_0 and so is the subgroup that it generates, since G_0 is a subgroup of G, by Proposition 6.3. Therefore $\exp \mathfrak{g}$ generates G_0. □

Every $g \in G_0$ is thus a finite product $\exp X_1 \cdots \exp X_k,\ X_i \in \mathfrak{g}$.

7 Morphisms of Lie Groups and of Lie Algebras

We defined in Section 1.2 the algebraic notion of a morphism of Lie algebras. Now we define the morphisms of Lie groups and consider the close relationship between these two definitions.

7.1 Differential of a Lie Group Morphism

Definition 7.1. *Let G and G' be Lie groups. A* morphism *of Lie groups, or Lie group morphism, $f : G \to G'$ is a continuous morphism of groups of G into G'. Most authors use the term* homomorphism*.*

Theorem 7.2. *Let f be a Lie group morphism of G into G', and let $\mathfrak{g}$ and $\mathfrak{g}'$ be the Lie algebras of G and G', respectively.*

(*i*) *For each $X \in \mathfrak{g}$, the map $t \in \mathbb{R} \mapsto f(\exp(tX)) \in G'$ is a one-parameter subgroup of G'.*

(*ii*) *Let*

$$\varphi(X) = \frac{\mathrm{d}}{\mathrm{d}t} f(\exp(tX))\big|_{t=0}. \tag{7.1}$$

Then for every $t \in \mathbb{R}$,

$$f(\exp(tX)) = \exp\left(t\varphi(X)\right), \tag{7.2}$$

and in particular,

$$f(\exp X) = \exp(\varphi(X)). \tag{7.3}$$

(*iii*) *The map $\varphi : X \mapsto \varphi(X)$ is a morphism of Lie algebras from $\mathfrak{g}$ into $\mathfrak{g}'$.*

Proof. (i) The map $t \mapsto f(\exp(tX))$ is continuous and also satisfies the group property

$$f\left(\exp((t+s)X)\right) = f(\exp(tX)\ \exp(sX)) = f(\exp(tX))\ f(\exp(sX)).$$

It is therefore a one-parameter subgroup of G', and by Lemma 3.2, we can differentiate the map $t \mapsto f(\exp(tX))$.

(ii) Because $\varphi(X) = \frac{\mathrm{d}}{\mathrm{d}t} f(\exp(tX))\big|_{t=0}$ is the infinitesimal generator of the one-parameter group $t \mapsto f(\exp(tX))$, we obtain (7.2) and in particular, (7.3).

$$f(\exp X) = \exp(\varphi(X)). \tag{7.4}$$

(iii) Let us show that φ is $\mathbb{R}$-linear. Let s be a real number. Then $\varphi(sX) = s\varphi(X)$. In fact, $t \mapsto f(\exp(t(sX)))$ is a one-parameter subgroup with infinitesimal generator

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\exp(tsX))\big|_{t=0} = s\frac{\mathrm{d}}{\mathrm{d}u} f(\exp(uX))\big|_{u=0} = s\varphi(X).$$

Now this infinitesimal generator is $\varphi(sX)$ by definition. Thus $\varphi(sX) = s\varphi(X)$.

Let us show that φ is additive. Let X and Y be elements of $\mathfrak{g}$. On the one hand, the map $t \mapsto f(\exp(t(X+Y)))$ is a one-parameter subgroup of G', with infinitesimal generator $\varphi(X+Y)$. On the other hand, using Corollary 2.5 and the continuity of f, we obtain

$$\begin{aligned}
\exp(t\varphi(X+Y)) = f\left(\exp(t(X+Y))\right) &= \lim_{k\to+\infty} f\left(\left(\exp\left(\frac{t}{k}X\right)\exp\left(\frac{t}{k}Y\right)\right)^k\right) \\
&= \lim_{k\to+\infty}\left(f\left(\exp\left(\frac{t}{k}X\right)\right)\ f\left(\exp\left(\frac{t}{k}Y\right)\right)\right)^k \\
&= \lim_{k\to+\infty}\left(\exp\left(\frac{t}{k}\varphi(X)\right)\ \exp\left(\frac{t}{k}\varphi(Y)\right)\right)^k,
\end{aligned}$$

because $\frac{1}{k}\varphi(X)$ is the infinitesimal generator of $t \mapsto f(\exp(\frac{t}{k}X))$. Thus

$$\exp(t\varphi(X+Y)) = \exp\left(t(\varphi(X)+\varphi(Y))\right).$$

Hence

$$\varphi(X+Y) = \varphi(X)+\varphi(Y).$$

It remains to show that φ preserves the commutator. Let us show first that

$$\forall g \in G,\ \forall X \in \mathfrak{g},\ \varphi(gXg^{-1}) = f(g)\varphi(X)(f(g))^{-1}.$$

In fact,

$$\begin{aligned}\exp(t\varphi(gXg^{-1})) &= f(\exp(g(tX)g^{-1})) = f(g\exp(tX)g^{-1}) \\ &= f(g)f(\exp(tX))f(g^{-1}) = f(g)\exp(t\varphi(X))\ (f(g))^{-1} \\ &= \exp(tf(g)\varphi(X)f(g^{-1}))\end{aligned}$$

whence for every $Y \in \mathfrak{g}$ and for every $t \in \mathbb{R}$,

$$\varphi(\exp(tY)\ X\ \exp(-tY)) = f(\exp(tY))\varphi(X)f(\exp(-tY)).$$

Differentiating at 0, since the map φ is linear, we obtain

$$\varphi(YX - XY) = \varphi(Y)\varphi(X) - \varphi(X)\varphi(Y),$$

which is the desired equation. □

Definition 7.3. *The morphism of Lie algebras φ defined by* (7.1) *is called the* differential *of the Lie group morphism f, and is denoted by Df.*

Thus, by definition,

$$(Df)(X) = \frac{\mathrm{d}}{\mathrm{d}t} f(\exp(tX))\Big|_{t=0}. \tag{7.5}$$

Remark. A Lie group, more generally a topological space, is called *simply connected* if each continuous loop can be deformed continuously to a point. If the Lie group G is connected and simply connected, then for every morphism of real Lie algebras φ of $\mathfrak{g}$ into $\mathfrak{g}'$, there is conversely a unique Lie group morphism of G into G' whose differential is φ. For a proof of this important theorem, see, e.g., Rossmann (2002) or Hall (2003).

7.2 Differential of a Lie Group Representation

One may define and study the infinite-dimensional representations of Lie groups. In particular, if the Lie group is compact, the results of Chapter 3 concerning the representations on separable complex Hilbert spaces apply. But we shall nonetheless limit ourselves here to the case of finite-dimensional representations. If E is a finite-dimensional complex vector space, then $\mathrm{GL}(E)$ is a (real) Lie group, with Lie algebra $\mathfrak{gl}(E)^{\mathbb{R}}$.

Definition 7.4. *A* representation *of a Lie group G is a Lie group morphism from G into* $\mathrm{GL}(E)$*, where E is a finite-dimensional complex vector space.*

Let ρ be a representation of a Lie group G on a finite-dimensional complex vector space E. We denote by $D\rho$ the differential of the Lie group morphism $\rho : G \to \mathrm{GL}(E)$. This is a morphism of Lie algebras of $\mathfrak{g}$ into $\mathfrak{gl}(E)^{\mathbb{R}}$ (and in particular into $\mathfrak{gl}(n, \mathbb{C})^{\mathbb{R}}$ if $E = \mathbb{C}^n$). Thus $D\rho$ is a representation of the Lie algebra $\mathfrak{g}$ on E in the sense of Definition 1.3. We have, by definition, for each $X \in \mathfrak{g}$,

$$(D\rho)(X) = \frac{\mathrm{d}}{\mathrm{d}t}\rho(\exp(tX))\Big|_{t=0}. \tag{7.6}$$

In other words, $\rho(\exp(tX))$ is a one-parameter subgroup of $\mathrm{GL}(E)$ with infinitesimal generator $(D\rho)(X)$, and we then have, for every $t \in \mathbb{R}$,

$$\rho(\exp(tX)) = \exp(t(D\rho)(X)). \tag{7.7}$$

The following definition is thus justified.

Definition 7.5. *Let (E, ρ) be be a representation of a Lie group G. The* differential *of the representation ρ is the Lie algebra representation $D\rho \ : \mathfrak{g} \to \mathfrak{gl}(E)$ defined by* (7.5)*.*

Theorem 7.6. *Let (E, ρ) be a representation of a Lie group G.*

(*i*) *If $F \subset E$ is invariant under ρ, then F is invariant under $D\rho$.*
(*ii*) *If $D\rho$ is irreducible, then ρ is irreducible.*
(*iii*) *If $(E, (\ \mid\ ))$ is a finite-dimensional complex Hilbert space and if (E, ρ) is unitary, then the Lie algebra representation $(E, D\rho)$ is anti-Hermitian.*
(*iv*) *Let (E_1, ρ_1) and (E_2, ρ_2) be representaions of G. If $\rho_1 \sim \rho_2$, then $D\rho_1 \sim D\rho_2$.*

The converses are true if G is connected.

Proof. (i) If for every $X \in \mathfrak{g}$ and for every $t \in \mathbb{R}$, F is invariant under $\rho(\exp(tX))$, then F is invariant under $(D\rho)(X)$.

(ii) is an immediate consequence of (i).

(iii) Suppose ρ is unitary. Then for every $X \in \mathfrak{g}$ and every $t \in \mathbb{R}$,

$$(\rho(\exp(tX))u \mid \rho(\exp(tX))v) = (u \mid v),$$

for every u and $v \in E$. Differentiating with respect to t at $t = 0$, we obtain

$$(D\rho(X)u \mid v) + (u \mid D\rho(X)v) = 0, \tag{7.8}$$

which says that $D\rho$ is anti-Hermitian.

(iv) If there exists a bijective linear map $T : E_1 \to E_2$ satisfying

$$\forall g \in G,\ \rho_2(g) \circ T = \ T \circ \rho_1(g),$$

then by setting $g = \exp(tX)$ and differentiating with respect to t, we obtain the desired equation

$$\forall X \in \mathfrak{g},\ (D\rho_2)(X) \circ T = T \circ (D\rho_1)(X).$$

The converses are true if G is connected because, in this case, every element of G is the product of a finite number of exponentials, by Theorem 6.5. We prove, for example, the converse of (iii). Suppose $D\rho$ is anti-Hermitian. Using the equation $\frac{\mathrm{d}}{\mathrm{d}t}\rho(\exp tX) = D\rho(X) \circ \rho(\exp tX)$, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\rho(\exp tX)u \mid \rho(\exp tX)v)\\ &\quad = (D\rho(X)\rho(\exp tX)u \mid \rho(\exp tX)v) + (\rho(\exp tX)u \mid D\rho(X)\rho(\exp tX)v).\end{aligned}$$

This quantity is zero for every $t \in \mathbb{R}$ by hypothesis. Thus the value of the scalar product $(\rho(\exp tX)u \mid \rho(\exp tX)v)$ remains constant and equal to the value $(u \mid v)$ it takes at $t = 0$. Thus ρ is unitary. □

If (E_1, ρ_1) and (E_2, ρ_2) are representations of G, then

$$D(\rho_1 \oplus \rho_2) = D\rho_1 \oplus D\rho_2$$

and

$$D(\rho_1 \otimes \rho_2) = D\rho_1 \otimes \mathrm{Id}_{E_2} + \mathrm{Id}_{E_1} \otimes D\rho_2, \tag{7.9}$$

that is, explicitly, for every $X \in \mathfrak{g}$ and for every $v_1 \in E_1$, $v_2 \in E_2$,

$$D(\rho_1 \otimes \rho_2)(X)(v_1 \otimes v_2) = (D\rho_1)(X)v_1 \otimes v_2 + v_1 \otimes (D\rho_2)(X)v_2. \tag{7.10}$$

This equation is proved by taking the derivative at $t = 0$ of the equation defining $\rho_1 \otimes \rho_2$,

$$(\rho_1 \otimes \rho_2)(g)(v_1 \otimes v_2) = \rho_1(g)v_1 \otimes \rho_2(g)v_2,$$

for $g \in G$ and $v_1 \in E_1$, $v_2 \in E_2$, where $g = \exp(tX)$.

7.3 The Adjoint Representation

Let G be be a Lie group with Lie algebra $\mathfrak{g}$. We consider the conjugation action of G on itself. For each $g \in G$,

$$\mathcal{C}_g : h \in G \mapsto ghg^{-1} \in G$$

is an automorphism of the Lie group G that leaves the identity element of G invariant. The differential of $\mathcal{C}_g$ at the identity is a linear mapping of $\mathfrak{g}$ into $\mathfrak{g}$, called the *adjoint action* of g and denoted by Ad_g. From the equation $\mathcal{C}_{gg'} = \mathcal{C}_g \circ \mathcal{C}_{g'}$, for g and $g' \in G$, we see that

$$\mathrm{Ad}_{gg'} = \mathrm{Ad}_g \circ \mathrm{Ad}_{g'}.$$

Furthermore, because Ad_g is the identity of $\mathfrak{g}$ when $g = I$, we deduce that $\mathrm{Ad} : g \in G \mapsto \mathrm{Ad}_g \in \mathrm{GL}(\mathfrak{g})$ is a representation of G on $\mathfrak{g}$.

Definition 7.7. *The representation* Ad *of G on $\mathfrak{g}$ is called the* adjoint representation of the Lie group G.

To the representation Ad of the Lie group G on $\mathfrak{g}$ there corresponds a representation $D(\mathrm{Ad})$ of the Lie algebra $\mathfrak{g}$ on itself, called the *adjoint representation of the Lie algebra* $\mathfrak{g}$, and denoted by ad. The following equation is thus true by definition, for every $t \in \mathbb{R}$ and each $X \in \mathfrak{g}$:

$$\mathrm{Ad}_{\exp(tX)} = \exp(t\mathrm{ad}_X). \tag{7.11}$$

Proposition 7.8. (*i*) *Let A be an invertible matrix belonging to the Lie group G and let X be a matrix belonging to the Lie algebra $\mathfrak{g}$ of G. Then*

$$\mathrm{Ad}_A(X) = AXA^{-1}.$$

(*ii*) *Let X and $Y \in \mathfrak{g}$. Then*

$$\mathrm{ad}_X(Y) = [X, Y].$$

(*iii*) *Let X and $Y \in \mathfrak{g}$. Then*

$$\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y]. \tag{7.12}$$

Proof. (i) By definition, for $B \in G$, we have $\mathcal{C}_A(B) = ABA^{-1}$, and thus

$$\mathrm{Ad}_A(X) = \frac{\mathrm{d}}{\mathrm{d}t} A \exp(tX) A^{-1}\big|_{t=0} = AXA^{-1}.$$

(ii) Furthermore,

$$\begin{aligned}\mathrm{ad}_X(Y) &= \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ad}_{\exp(tX)}(Y)\big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\exp(tX)Y\exp(-tX)\big|_{t=0} \\ &= XY - YX = [X, Y].\end{aligned}$$

(iii) Equation (7.12) expresses the fact that ad is a Lie algebra representation. It is a direct consequence of the Jacobi identity. □

References

Most of the results of this chapter extend beyond the case of linear Lie groups to the case of Lie groups in general. Rossmann (2002) first treats the linear Lie groups and then introduces the general notion of Lie groups. In this spirit, one could also consult the books of Hall (2003) and of Baker (2002), while Sagle and Walde (1973), Postnikov (1986), Marsden and Ratiu (1999), Sattinger and Weaver (1993), Kirillov (2008) treat the general case. See Gilmore (1974) for many examples and applications to physics.

There are numerous more advanced textbooks, such as Bump (2004) and Godement (2004). The most classical book still available is Chevalley (1946). For the study of the differential geometry of Lie groups as manifolds, see for example Warner (1983) or Duistermaat–Kolk (2000); Warner's book also contains manifold cohomology theory, while Duistermaat and Kolk treat the representation theory of compact Lie groups. There one will find a proof of the Baker–Campbell–Hausdorff formula. The work of Naimark and Štern (1982) provides an exposition of the theory of Lie groups and of their representations. For a very complete treatment, see Bourbaki (1989, 2002, 2005).

For the theorem on the integration of a morphism of Lie algebras to a morphism of Lie groups in the simply connected case mentioned in Section 7.1, see Hall (2003), Naimark–Štern (1982), or Rossmann (2002).

Exercises

Exercise 4.1 *The exponential mapping and Lie brackets.*

Let X and Y be real or complex $n \times n$ matrices. Show that for $t, s \in \mathbb{R}$, $k \in \mathbb{N}^*$,

(a) $\exp(tX)\exp(tY)\exp(-tX) = \exp\left(tY + t^2[X,Y] + \mathrm{O}(t^3)\right)$.

(b) $\exp(tX)\exp(tY)\exp(-tX)\exp(-tY) = \exp\left(t^2[X,Y] + \mathrm{O}(t^3)\right)$.

(c) $\exp t[X,Y] = \lim_{k\to+\infty} \left(\exp\left(t\frac{X}{k}\right)\exp\left(\frac{Y}{k}\right)\exp\left(-t\frac{X}{k}\right)\exp\left(-\frac{Y}{k}\right)\right)^{k^2}$.

(d) $[X,Y] = \frac{\partial^2}{\partial t \partial s}\left(\exp(tX)\exp(sY)\exp(-tX)\exp(-sY)\right)\big|_{t=s=0}$.

Exercise 4.2 *Infinite-dimensional Lie algebra: vector fields on* $\mathbb{R}^n$.

Consider the vector space of first-order linear differential operators on $\mathbb{R}^n$, of the form $D = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}$, where each X^i is a smooth function on $\mathbb{R}^n$. Show that the commutator gives this vector space the structure of an infinite-dimensional Lie algebra over $\mathbb{R}$.

Exercise 4.3 *Infinite-dimensional Lie algebra: vector fields on the circle* (*the Virasoro algebra*).

Consider the vector space E of smooth maps from the circle $S^1 = \{\mathrm{e}^{i\theta} \mid 0 \le \theta < 2\pi\}$ into $\mathbb{C}$ and set, for $f \in E$, $X_f = f(\theta)\,\mathrm{d}/\mathrm{d}\theta$.

(a) Let $f, g \in E$. Show that there exists $h \in E$ such that $[X_f, X_g] = X_h$. Conclude that E is a Lie algebra, whose Lie bracket we denote as usual by $[\ ,\ ]$. Calculate $[\mathrm{e}^{ni\theta}, \mathrm{e}^{mi\theta}]$, for $n, m \in \mathbb{Z}$. This Lie algebra is called the *Witt algebra*.

(b) Set

$$c(f,g) = \int_0^{2\pi} f(\theta)\frac{\mathrm{d}^3 g}{\mathrm{d}\theta^3}\mathrm{d}\theta.$$

Let $\widetilde{E} = E \oplus \mathbb{C}$, and let k be a complex constant. For every (f,λ) and $(g,\mu) \in \widetilde{E}$, set

$$[(f,\lambda),(g,\mu)]_k = ([f,g], k\,c(f,g)).$$

Show that $(\widetilde{E}, [\ ,\ ]_k)$ is a Lie algebra. Calculate $[(\mathrm{e}^{ni\theta},\lambda),(\mathrm{e}^{mi\theta},\mu)]_k$, for $n, m \in \mathbb{Z}$. This Lie algebra is called the *Virasoro algebra*.

Exercise 4.4 *Example of a nonsurjective exponential map.*

(a) Show that the exponential map of $\mathfrak{sl}(2,\mathbb{C})$ into $\mathrm{SL}(2,\mathbb{C})$ is not surjective.
(b) Is the exponential map always injective?

Exercise 4.5 *The Heisenberg group.*

(a) Let

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Calculate $[H,X]$, $[H,Y]$ and $[X,Y]$.
Calculate $\exp(tX)$, $\exp(tY)$ and $\exp(tH)$, for $t \in \mathbb{R}$.
The real Lie algebra $\mathfrak{g}$ generated by X, Y, H is called the *Heisenberg algebra*, and the group G of real matrices of the form $\left(\begin{smallmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{smallmatrix}\right)$ is called the *Heisenberg group.*
Show that the exponential map is a bijection of the Heisenberg algebra onto the Heisenberg group.

(b) What is the center of $\mathfrak{g}$? of G?
(c) More generally, we consider, for every positive integer n, the Lie algebra $\mathfrak{g}$ generated by $Q_k, P_k, H,\ k = 1, \ldots, n$, with the commutation relations

$$[Q_k, H] = [P_k, H] = [Q_k, Q_\ell] = [P_k, P_\ell] = 0,$$
$$[Q_k, P_\ell] = \alpha \delta_{k\ell} H,$$

where α is a nonzero scalar. (The equations $[Q_k, P_\ell] = i\hbar\delta_{k\ell}$ are the *Heisenberg relations*.) Show that this Lie algebra has no finite-dimensional irreducible representation such that $\rho(H) \neq 0$, and that the irreducible finite-dimensional representations such that $\rho(H) = 0$ are one-dimensional.

Exercise 4.6 *Differential of the exponential map.*

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Show that $\mathrm{d}_X(\exp)$, the differential at the point $X \in \mathfrak{g}$ of the map $\exp : \mathfrak{g} \to G$, satisfies

$$\exp(-X)\mathrm{d}_X(\exp) = \frac{1 - \exp(-\mathrm{ad}_X)}{\mathrm{ad}_X},$$

where the right-hand side denotes the sum of a series. [Hint: Consider, for X and Y in $\mathfrak{g}$, $A_s(X,Y) = \exp(-sX)\frac{\partial}{\partial t}\exp(s(X+tY))\big|_{t=0}$ and calculate $\frac{\partial}{\partial s}A_s(X,Y)$.]

Exercise 4.7 *Killing form.*

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The *Killing form* of $\mathfrak{g}$ is the $\mathbb{K}$-bilinear form on $\mathfrak{g}$ defined by

$$\forall X, Y \in \mathfrak{g}, \quad K(X,Y) = \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y).$$

(a) Show that for every $X, Y, Z \in \mathfrak{g}$,

$$K([X,Y],Z) + K(Y,[X,Z]) = 0.$$

(b) Let G be a Lie group with Lie algebra $\mathfrak{g}$. Show that K is an *invariant* bilinear form, that is, for every $g \in G$,

$$K(\mathrm{Ad}_g X, \mathrm{Ad}_g Y) = K(X, Y).$$

(c) Find the Killing form and determine its signature for each of the Lie algebras $\mathfrak{sl}(2, \mathbb{C})$, $\mathfrak{sl}(2, \mathbb{R})$, and $\mathfrak{so}(3)$.

Exercise 4.8 *Dimension of the symplectic groups.*

Let $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Calculate the dimension of the Lie algebra

$$\mathfrak{sp}(n, \mathbb{C}) = \{X \in \mathfrak{sl}(2n, \mathbb{C}) \mid XJ + J^t X = 0\}.$$

What are the dimensions of $\mathrm{Sp}(n, \mathbb{C})$ and of $\mathrm{Sp}(n) = \mathrm{Sp}(n, \mathbb{C}) \cap \mathrm{U}(2n)$?

Exercise 4.9 *Derivations and the adjoint representation.*

A *derivation* of a Lie algebra $\mathfrak{g}$ is a linear mapping of $\mathfrak{g}$ into $\mathfrak{g}$ satisfying, for every $X, Y \in \mathfrak{g}$,

$$D[X, Y] = [DX, Y] + [X, DY]\ .$$

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be a Lie algebra. Show that if ρ is a representation of G on $\mathfrak{h}$ by automorphisms, then $D\rho$ is a representation of $\mathfrak{g}$ on $\mathfrak{h}$ by derivations. What can one say about the representations Ad and ad?

Exercise 4.10 *Coadjoint representation.*

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{g}^*$ be the dual vector space of $\mathfrak{g}$. By definition, the *coadjoint representation* of G on $\mathfrak{g}^*$ is the dual of the adjoint representation. The *coadjoint orbit* of an element ξ of $\mathfrak{g}^*$ is its orbit under the coadjoint action of G.

Find the coadjoint orbits in the case $G = \mathrm{SO}(3)$.

Exercise 4.11 *Real forms.*

Show that $\mathfrak{so}(3), \mathfrak{su}(2), \mathfrak{so}(2,1), \mathfrak{sl}(2, \mathbb{R})$ are real forms of $\mathfrak{sl}(2, \mathbb{C})$. Are they isomorphic as real Lie algebras?

Exercise 4.12 *The connected component of the identity of a Lie group.*

Let G be a Lie group and let G_0 be the connected component of the identity of G.

(a) Show that G_0 is a normal subgroup of G.
(b) Show that the Lie groups G and G_0 have the same Lie algebra.

Exercise 4.13 *Connected components.*

(a) Show that $\mathrm{GL}(n, \mathbb{C})$, for $n \geq 2$, and $\mathrm{U}(n)$, for $n \geq 1$, are connected. Find the connected component of the identity of $\mathrm{GL}(n, \mathbb{R})$.
(b) Study the connected component of the identity of the Lorentz group. How many connected components does the Lorentz group have?

Exercise 4.14 *Ideals of Lie algebras and normal subgroups.*

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and let H be a closed connected subgroup of G. Show that H is normal if and only if the Lie algebra $\mathfrak{h}$ of H is an ideal of $\mathfrak{g}$.

Exercise 4.15 *Invariant measure on Lie groups.*

Suppose that the elements of a Lie group G of dimension p are parametrized by $x = (x_1, x_2, \ldots, x_p)$. We denote by $g^{-1}\frac{\mathrm{d}g}{\mathrm{d}x}$ the matrix whose columns are the components of the vectors $g^{-1}(x)\frac{\partial g}{\partial x_k}(x), k = 1, 2, \ldots, p$, in a basis of $\mathfrak{g}$. Show that $|\det(g^{-1}\frac{\mathrm{d}g}{\mathrm{d}x})|$ is a left-invariant volume element on G.

Élie Cartan, who was born in Dolomieu (*Isère*) *in 1869 and died in Paris in 1951, is famous for his work on Lie groups and algebras and his fundamental contributions to differential geometry. Here he is on vacation in 1931, with two of his sons, his daughter, Hélène, and a nephew. On the far left is his son Henri* (*1904–2008*)*, one of the founders of the Bourbaki group and a great mathematician of the twentieth century.*

(*Private collection*)

Chapter 5

Lie Groups SU(2) and SO(3)

1 The Lie Algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$

We know that $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ are real Lie algebras of dimension 3. We shall show that they are isomorphic by showing that there are bases of each in which the commutation relations are the same.

1.1 Bases of $\mathfrak{su}(2)$

The Lie algebra $\mathfrak{su}(2) = \left\{X \in \mathfrak{gl}(2,\mathbb{C}) \mid {}^t\overline{X} + X = 0, \mathrm{Tr}X = 0\right\}$ is the real vector space of dimension 3 of traceless anti-Hermitian 2×2 complex matrices.

(a) The three linearly independent matrices

$$\xi_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \xi_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \xi_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

form a basis over $\mathbb{R}$ of $\mathfrak{su}(2)$ and satisfy the commutation relations

$$[\xi_k, \xi_\ell] = \xi_m,$$

where k, ℓ, m is a cyclic permutation of $1, 2, 3$.

(b) In physics one considers the Pauli matrices which are Hermitian matrices

$$\sigma_1 = -2i\xi_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = 2i\xi_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = -2i\xi_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They satisfy the commutation relations

$$[\sigma_k, \sigma_\ell] = 2i\sigma_m.$$

One also sometimes uses the matrices

$$\widetilde{\sigma}_1 = \frac{1}{2}\sigma_1 = -i\xi_1, \quad \widetilde{\sigma}_2 = \frac{1}{2}\sigma_2 = i\xi_2, \quad \widetilde{\sigma}_3 = \frac{1}{2}\sigma_3 = -i\xi_3,$$

which satisfy the commutation relations

$$[\widetilde{\sigma}_k, \widetilde{\sigma}_\ell] = i\widetilde{\sigma}_m.$$

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(c) We also introduce the matrices

$$J_k = i\xi_k,$$

which satisfy the commutation relations,

$$[J_k, J_\ell] = iJ_m.$$

We also consider the matrices

$$\begin{aligned}
&J_3 = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \\
&J_+ = J_1 + iJ_2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad J_- = J_1 - iJ_2 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

We have

$$\begin{aligned}
&J_3 = i\xi_3 = -\widetilde{\sigma}_3 = -\frac{1}{2}\sigma_3, \\
&J_\pm = J_1 \pm iJ_2 = i\xi_1 \mp \xi_2 = -\widetilde{\sigma}_1 \pm i\widetilde{\sigma}_2 = \frac{1}{2}(-\sigma_1 \pm i\sigma_2),
\end{aligned}$$

and so

$$\xi_3 = -iJ_3, \quad \xi_1 = -\frac{i}{2}(J_+ + J_-), \quad \xi_2 = -\frac{1}{2}(J_+ - J_-).$$

The matrices J_3, J_+, J_- satisfy the commutation relations,

$$[J_+, J_-] = 2J_3, \quad [J_3, J_\pm] = \pm J_\pm.$$

(d) Finally, we introduce the basis of $\mathfrak{su}(2)$,

$$\mathcal{I} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{K} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

We see immediately that

$$\begin{aligned}
&\mathcal{I}^2 = \mathcal{J}^2 = \mathcal{K}^2 = -I, \\
&\mathcal{I}\mathcal{J} = \mathcal{K} = -\mathcal{J}\mathcal{I}, \quad \mathcal{J}\mathcal{K} = \mathcal{I} = -\mathcal{K}\mathcal{J}, \quad \mathcal{K}\mathcal{I} = \mathcal{J} = -\mathcal{I}\mathcal{K},
\end{aligned}$$

and the commutation relations of $\mathfrak{su}(2)$ can thus be written

$$[\mathcal{I}, \mathcal{J}] = 2\mathcal{K}, \quad [\mathcal{J}, \mathcal{K}] = 2\mathcal{I}, \quad [\mathcal{K}, \mathcal{I}] = 2\mathcal{J}.$$

We have

$$\begin{aligned}
\mathcal{I} &= 2\xi_1 = 2i\widetilde{\sigma}_1 = i\sigma_1 = -2iJ_1, \\
\mathcal{J} &= 2\xi_2 = -2i\widetilde{\sigma}_2 = -i\sigma_2 = -2iJ_2, \\
\mathcal{K} &= 2\xi_3 = 2i\widetilde{\sigma}_3 = i\sigma_3 = -2iJ_3.
\end{aligned}$$

A matrix X of $\mathfrak{su}(2)$ can be written $X = x_1\mathcal{I} + x_2\mathcal{J} + x_3\mathcal{K}$, i.e.,

$$X = \begin{pmatrix} ix_3 & -x_2 + ix_1 \\ x_2 + ix_1 & -ix_3 \end{pmatrix},$$

where x_1, x_2, x_3 are real numbers. We have then

$$\det X = \|\overrightarrow{X}\|^2, \tag{1.1}$$

where $\|\overrightarrow{X}\|$ is the Euclidean norm of the vector $\overrightarrow{X}$ with components (x_1, x_2, x_3).

Interpretation in terms of quaternions. Let $\mathbb{H}$ be the associative algebra of the quaternions, a vector space of dimension 4, with basis $(1, i, j, k)$, with the multiplication defined by

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

The map $i \mapsto \mathcal{I}$, $j \mapsto \mathcal{J}$, $k \mapsto \mathcal{K}$ defines a bijection of the vector space $\mathfrak{su}(2)$ onto the vector subspace of $\mathbb{H}$ generated by i, j, and k, called the vector space of *pure quaternions*, and denoted by $\mathbb{H}_0$. The vector subspace $\mathbb{H}_0$ is not a subalgebra of the associative algebra $\mathbb{H}$, but $\mathbb{H}_0$ is a Lie algebra under the commutator, and the bijection above is an isomorphism of Lie algebras.

Remark. All the triples of matrices considered above form bases over $\mathbb{C}$ of $(\mathfrak{su}(2))^{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$. While the ξ_k and the matrices $\mathcal{I}, \mathcal{J}, \mathcal{K}$ belong to $\mathfrak{su}(2)$, the Pauli matrices σ_k and the matrices J_k belong to $i\mathfrak{su}(2)$.

1.2 Bases of $\mathfrak{so}(3)$

The Lie algebra $\mathfrak{so}(3)$ is the vector space of antisymmetric real 3×3 matrices. We have already seen that the matrices

$$\eta_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \eta_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

form a basis of this Lie algebra, with the commutation relations

$$[\eta_k, \eta_\ell] = \eta_m.$$

It is clear that $\eta_k \mapsto e_k$, where (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$, defines an isomorphism of the Lie algebra $\mathfrak{so}(3)$ onto the Lie algebra $(\mathbb{R}^3, \wedge)$.

On the other hand,

$$\mathfrak{su}(2) \approx \mathfrak{so}(3).$$

More precisely, an isomorphism of Lie algebras is realized by the $\mathbb{R}$-linear map defined by $\xi_k \mapsto \eta_k$. From the fact that $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ are isomorphic, we deduce that $(\mathfrak{so}(3))^{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$.

We may also consider the matrices

$$\widehat{J}_k = i\eta_k.$$

In the preceding isomorphism, $\widetilde{\sigma}_1 \mapsto -\widehat{J}_1$, $\widetilde{\sigma}_2 \mapsto \widehat{J}_2$, $\widetilde{\sigma}_3 \mapsto -\widehat{J}_3$.

1.3 Bases of $\mathfrak{sl}(2, \mathbb{C})$

Other than the bases already described above, the most frequently used basis is

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

which satisfies the commutation relations

$$[H, X_\pm] = \pm 2X_\pm, \quad [X_+, X_-] = H.$$

We have

$$H = -2J_3 = -2i\xi_3 = 2\widetilde{\sigma}_3 = \sigma_3,$$
$$X_\pm = -J_\mp = -J_1 \pm iJ_2 = -(i\xi_1 \pm \xi_2) = \widetilde{\sigma}_1 \pm i\widetilde{\sigma}_2 = \frac{1}{2}(\sigma_1 \pm i\sigma_2),$$

and thus

$$J_3 = -\frac{1}{2}H, \quad J_\pm = -X_\mp,$$

and

$$\mathcal{I} = i(X_+ + X_-), \quad \mathcal{J} = -(X_+ - X_-), \quad \mathcal{K} = iH.$$

2 The Covering Morphism of SU(2) onto SO(3)

We shall see that although the Lie algebras of the Lie groups SO(3) and SU(2) are isomorphic, the groups themselves are not. The one, SU(2), is connected and simply connected, but the other, SO(3), is connected but not simply connected, and there is a surjective group morphismn of the first onto the second, whose kernel consists of the elements I and $-I$ of SU(2).

2.1 The Lie Group SO(3)

Every orthogonal transformation of $\mathbb{R}^3$ of determinant $+1$ leaves a unit vector a of $\mathbb{R}^3$ invariant. This is then a rotation through an angle $t \in \mathbb{R}/2\pi\mathbb{Z}$, denoted by $\mathrm{Rot}(a, t)$. Thus an element of SO(3) is determined by $a \in \mathbb{R}^3$, $\|a\| = 1$, and $t \in [0, 2\pi]$. The image of an element $x \in \mathbb{R}^3$ by $\mathrm{Rot}(a, t)$ is

$$\mathrm{Rot}(a, t)(x) = x + (1 - \cos t) a \wedge (a \wedge x) + \sin t \; a \wedge x. \tag{2.1}$$

For the proof, we observe first that if x is orthogonal to a, then

$$\mathrm{Rot}(a, t)(x) = \cos t \; x + \sin t \; a \wedge x.$$

Next we decompose any x as $\lambda a + \mu b$, where b is orthogonal to a, and we exploit the linearity of $\mathrm{Rot}(a, t)$. Hence

$$\mathrm{Rot}(a, t)(x) = \cos t \; x + (1 - \cos t)(x \mid a)a + \sin t \; a \wedge x.$$

Finally, we use the triple product formula, $u \wedge (v \wedge w) = (u \mid w)v - (u \mid v)w$, which gives $(x \mid a)a = x + a \wedge (a \wedge x)$, which proves (2.1).

Proposition 2.1. *Two rotations are conjugate in* SO(3) *if and only if they have equal or opposite angles.*

Proof. It is clear that for $t \neq 0$, the pair (a', t') defines the same rotation as the pair (a, t) if and only if $(a', t') = (a, t)$ or $(a', t') = (-a, -t)$. We shall now prove that rotations $\mathrm{Rot}(a, t)$ and $\mathrm{Rot}(a', t')$ are conjugate in SO(3) if and only if $t' = \pm t$. In fact, if a rotation R leaves a invariant, then for each $g \in \mathrm{SO}(3)$, the rotation gRg^{-1} leaves ga invariant. More precisely, it follows from the preceding formulas that

$$\forall\, g \in \mathrm{SO}(3), \quad g\ \mathrm{Rot}(a, t)\ g^{-1} = \mathrm{Rot}(ga, t), \tag{2.2}$$

for $x \in \mathbb{R}^3$. Therefore, if $R' = \mathrm{Rot}(a', t')$ is a rotation conjugate to $R = \mathrm{Rot}(a, t)$ by an element $g \in \mathrm{SO}(3)$, then $\mathrm{Rot}(a', t') = \mathrm{Rot}(ga, t)$, and therefore R and R' have equal or opposite angles. Conversely, each rotation $\mathrm{Rot}(a', t)$ is conjugate to $\mathrm{Rot}(a, t)$ by an element $g \in \mathrm{SO}(3)$ such that $a' = ga$, and each rotation $\mathrm{Rot}(a', -t)$ is conjugate to $\mathrm{Rot}(a, t)$ by an element $g \in \mathrm{SO}(3)$ such that $-a' = ga$. □

Surjectivity of the exponential map of $\mathfrak{so}(3)$ *onto* SO(3). Let η_k be the infinitesimal generators of the one-parameter groups of rotations around the axes e_k, $k = 1, 2, 3$. By definition,

$$\exp(t\eta_k) = \mathrm{Rot}(e_k, t).$$

Let a be any unit vector in $\mathbb{R}^3$. We choose $k = 1, 2$, or 3, and let g be an element of SO(3) such that $a = g(e_k)$. Then

$$\mathrm{Rot}(a, t) = g\ \mathrm{Rot}(e_k, t)g^{-1} = g\ \ \exp(t\eta_k)g^{-1} = \exp(t\, g\, \eta_k\, g^{-1}).$$

By Theorem 5.1 (iii) of Chapter 4, the element $g\eta_k g^{-1}$ belongs to $\mathfrak{so}(3)$. We have thus shown that the exponential map is surjective from $\mathfrak{so}(3)$ onto SO(3). We deduce that SO(3) is arc-connected, hence connected, and that it is the connected component of the identity of O(3). We remark that the exponential map is clearly not injective from $\mathfrak{so}(3)$ into SO(3) because for $X = \eta_k$ ($k = 1, 2$ or 3) and ℓ integer, $\exp((t + 2\ell\pi)X) = \exp(tX)$.

Identification of $\mathfrak{so}(3)$ *with* $\mathbb{R}^3$. Let $a \mapsto M_a$ be the $\mathbb{R}$-linear map from $\mathbb{R}^3$ onto $\mathfrak{so}(3)$ defined by $e_k \mapsto \eta_k$. The equation $M_{e_k}(x) = \eta_k(x) = e_k \wedge x$ implies by linearity that $M_a(x) = a \wedge x$, for every $x \in \mathbb{R}^3$, and for $g \in \mathrm{SO}(3)$, $M_{ga}(x) = ga \wedge x = g(a \wedge g^{-1}x) = gM_a g^{-1}(x)$. Thus

$$\forall g \in \mathrm{SO}(3),\ \forall a \in \mathbb{R}^3,\ M_{ga} = Ad_g M_a.$$

Proposition 2.2. *For each* $a \in \mathbb{R}^3$ *of unit norm and for every* $t \in \mathbb{R}$,

$$\exp(tM_a) = \mathrm{Rot}(a, t). \tag{2.3}$$

Proof. The two sides are one-parameter subgroups of SO(3). By (2.1),

$$\forall x \in \mathbb{R}^3,\ \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Rot}(a, t)(x)\big|_{t=0} = a \wedge x,$$

while

$$\forall x \in \mathbb{R}^3, \ \frac{\mathrm{d}}{\mathrm{d}t} \exp(tM_a)(x)\big|_{t=0} = M_a(x),$$

which proves (2.3). □

Remark. Despite the differences of notation, formula (2.3) is the same as a formula to be found in physics books, e.g., (1.48) in Blaizot–Tolédano (1997),

$$\mathcal{D}(\overrightarrow{u}, \alpha) = \mathrm{e}^{-i\alpha \overrightarrow{u} \cdot \overrightarrow{J}},$$

where $\overrightarrow{J}$ denotes the triple of matrices $i\eta_1, i\eta_2, i\eta_3$, which we have denoted by $\widehat{J}_1, \widehat{J}_2, \widehat{J}_3$ (but which are denoted by J_1, J_2, J_3 in that book). This equation says that the rotation determined by the unit vector $\overrightarrow{u} = (u_1, u_2, u_3)$ and angle α is the exponential of the matrix $-i\alpha(u_1\widehat{J}_1 + u_2\widehat{J}_2 + u_3\widehat{J}_3)$. Since this matrix is equal to $\alpha(u_1\eta_1 + u_2\eta_2 + u_3\eta_3)$, this equation is in fact the same as (2.3). In Tung (1985), this equation is (7.2-8), written

$$R_n(\psi) = \mathrm{e}^{-i\psi J_k n^k},$$

where n is a unit vector with components n^1, n^2, n^3, and ψ is the angle of the rotation.

2.2 The Lie Group SU(2)

The Lie group $\mathrm{SU}(2) = \{A \in \mathrm{GL}(2, \mathbb{C}) \mid A\,{}^t\overline{A} = I, \ \det A = 1\}$ is diffeomorphic to the sphere $\mathrm{S}^3 \subset \mathbb{R}^4$ because

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mid a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}.$$

In fact, for a matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, the equations $A\,{}^t\overline{A} = I$ and $\det A = 1$ imply that $|a|^2 + |b|^2 = 1, \ |c|^2 + |d|^2 = 1, \ ad - bc = 1, \ a\bar{c} + b\bar{d} = 0$, whence $ad\bar{c} + bd\bar{d} = 0$, whence $c = -\bar{b}$, and similarly, $d = \bar{a}$. The elements of the Lie group SU(2) thus depend on three independent real parameters.

Like the sphere S^3, the group SU(2) is compact, connected, and simply connected.

Every unitary matrix is diagonalizable by means of a unitary matrix. Because the eigenvalues of a unitary matrix are of modulus 1, for every $A \in \mathrm{SU}(2)$, there are a real number t and a special unitary matrix g such that

$$A = g \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} g^{-1}.$$

Furthermore $\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}$ and $\begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{pmatrix}$ are conjugate via $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ in SU(2).

Surjectivity of the exponential map of $\mathfrak{su}(2)$ *on* SU(2). Let $A = g\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} g^{-1}$ be a matrix of SU(2). From the equation

$$\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} = \exp(t\mathcal{K}),$$

we deduce that

$$A = \exp(tg\mathcal{K}g^{-1}),$$

with $g\mathcal{K}g^{-1} \in \mathfrak{su}(2)$. Thus the exponential map is surjective from $\mathfrak{su}(2)$ onto SU(2).

Identification of $\mathfrak{su}(2)$ *with* $\mathbb{R}^3$. Let $a \mapsto X_a$ be the $\mathbb{R}$-linear map from $\mathbb{R}^3$ onto $\mathfrak{su}(2)$ defined by

$$e_1 \mapsto \mathcal{I}, \quad e_2 \mapsto \mathcal{J}, \quad e_3 \mapsto \mathcal{K}.$$

Then (see formula (1.1))

$$\det X_a = \|a\|^2,$$

and in addition,

$$(X_a)^2 = -(\det X_a)I.$$

We deduce the following lemma:

Lemma 2.3. *For each* $X \in \mathfrak{su}(2)$, $X^2 = -(\det X)I$.

Proposition 2.4. *For each* $X \in \mathfrak{su}(2)$ *such that* $\det X = 1$, *and for every* $t \in \mathbb{R}$,

$$\exp(tX) = \cos t \ I + \sin t \ X.$$

Proof. It is clear that the two sides have X for derivative at $t = 0$. It thus suffices to show that if $\det X = 1$, the map $t \mapsto \cos t \ I + \sin t \ X$ is a one-parameter subgroup of $\mathrm{GL}(2, \mathbb{C})$. For t and $s \in \mathbb{R}$,

$$\begin{aligned}(\cos t \ I + \sin t \ X)&(\cos s \ I + \sin s \ X) \\ &= \cos t \cos s \ I + \sin t \sin s \ X^2 + (\sin t \cos s + \cos t \sin s)X.\end{aligned}$$

Thus by the lemma, if $\det X = 1$, then

$$(\cos t \ I + \sin t \ X)(\cos s \ I + \sin s \ X) = \cos(t+s)I + \sin(t+s)X.$$

One can also use the definition of the exponential as a series of matrices and the preceding lemma. □

As a consequence of Proposition 2.4, we see that each matrix of SU(2) can be written $\alpha_0 I + \alpha_1\mathcal{I} + \alpha_2\mathcal{J} + \alpha_3\mathcal{K}$, where the vector $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ of $\mathbb{R}^4$ has norm 1. Thus SU(2) is identified with the group of quaternions of unit norm.

2.3 Projection of SU(2) onto SO(3)

We consider the adjoint representation of SU(2) on $\mathfrak{su}(2)$,

$$\mathrm{Ad} : \mathrm{SU}(2) \to \mathrm{GL}(\mathfrak{su}(2)).$$

When we identify $\mathfrak{su}(2)$ with $\mathbb{R}^3$ by means of the isomorphism $a \mapsto X_a$, the map Ad is identified with a map

$$\varphi : \mathrm{SU}(2) \to \mathrm{GL}(3, \mathbb{R}).$$

This is the map that to $g \in \mathrm{SU}(2)$ associates the matrix of Ad_g in the basis $(\mathcal{I}, \mathcal{J}, \mathcal{K})$ of $\mathfrak{su}(2)$.

For each $g \in \mathrm{SU}(2), \mathrm{Ad}_g : \mathfrak{su}(2) \to \mathfrak{su}(2)$ is the map $X \mapsto gXg^{-1}$ that preserves determinants. By (1.1), in the identification of $\mathfrak{su}(2)$ with $\mathbb{R}^3$, the determinant of a matrix corresponds to the square of the Euclidean norm of the vector; hence for every $g \in \mathrm{SU}(2)$, $\varphi(g)$ preserves the norm, and thus $\varphi(\mathrm{SU}(2)) \subset \mathrm{O}(3)$. In fact, since φ is continuous and SU(2) is connected, $\varphi(\mathrm{SU}(2)) \subset \mathrm{SO}(3)$. We know that the map $\varphi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ is a morphism of groups. We shall show that the morphism φ is surjective from SU(2) onto SO(3), and we shall determine its kernel.

Proposition 2.5. (*i*) *In the basis* $(\mathcal{I}, \mathcal{J}, \mathcal{K})$ *of* $\mathfrak{su}(2)$, *the matrix of* ad_{ξ_k} *is* η_k, *for* $k = 1, 2, 3$.

(*ii*) *For each* $t \in \mathbb{R}$,

$$\varphi(\exp(t\xi_k)) = \exp(t\eta_k).$$

(*iii*) *For each* $a \in \mathbb{R}^3$ *of unit norm, and for every* $t \in \mathbb{R}$,

$$\varphi(\exp(tX_a)) = \mathrm{Rot}(a, 2t).$$

(*iv*) *The map* φ *is surjective.*

Proof. (i) It suffices to write $\mathrm{ad}_{\xi_k}(\xi_\ell) = [\xi_k, \xi_\ell] = \xi_m$, and we immediately obtain the matrix of ad_{ξ_k} in the basis (ξ_1, ξ_2, ξ_3), which is also its matrix in the basis $(\mathcal{I}, \mathcal{J}, \mathcal{K})$.

(ii) is the consequence of (i).

(iii) Let the components of $a \in \mathbb{R}^3$ be a_1, a_2, a_3. Then, by definition, $X_a = a_1\mathcal{I} + a_2\mathcal{J} + a_3\mathcal{K} = 2(a_1\xi_1 + a_2\xi_2 + a_3\xi_3)$. Thus the matrix of ad_{X_a} in the basis $(\mathcal{I}, \mathcal{J}, \mathcal{K})$ is $2(a_1\eta_1 + a_2\eta_2 + a_3\eta_3) = 2M_a$. Therefore the matrix of $\mathrm{Ad}_{\exp(tX_a)}$ in this basis is $\exp(2tM_a)$. If a is of unit norm, then it is $\mathrm{Rot}(a, 2t)$.

(iv) is the consequence of (iii). □

The kernel of Ad is $\{g \in \mathrm{SU}(2) \mid \mathrm{Ad}_g = \mathrm{Id}_{\mathfrak{su}(2)}\}$. Let $g = \left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix}\right)$ be such that $g\mathcal{K} = \mathcal{K}g,\ g\mathcal{I} = \mathcal{I}g,\ g\mathcal{J} = \mathcal{J}g$. One sees that $b = 0$ and a is real. Because $\det g = 1$, we have $g = I$ or $g = -I$. Thus $\mathrm{Ker}\varphi = \{I, -I\}$. The morphism φ is thus not injective, since its kernel contains two elements.

Because φ is continuous and surjective, and since SU(2) is simply connected, we can state the following result:

Proposition 2.6. *The group* SU(2) *is the double-sheeted universal cover of the group* SO(3).

References

The books of Rossmann (2002) and of Hall (2003), among many others, study the group of rotations and its universal cover SU(2). Already in 1847, in a paper published in the *Proceedings of the Irish Royal Academy*, William Rowan Hamilton (1805–1865) applied the theory of quaternions to the problem of the composition of rotations in space.

In this chapter and the next we introduce notation used in physics for which see Basdevant–Dalibard (2005), Ludwig–Falter (1996), Tung (1985), Edmonds (1974), Blaizot–Tolédano (1997), or Rougé (2005).

Exercises

Exercise 5.1 *Conjugate rotations*

We use the notation of Section 2.1. Let x be a vector in $\mathbb{R}^3$ and let $g \mapsto g(t)$ be a one-parameter subgroup of SO(3), with infinitesimal generator M_x.

(a) Show that the rotations $g(t)$ leave the vector x of $\mathbb{R}^3$ invariant.
(b) For $g_0 \in \mathrm{SO}(3)$, show that $t \mapsto g_0 g(t) g_0^{-1}$ is a one-parameter subgroup of SO(3) with infinitesimal generator $M_{g_0 x}$.

Exercise 5.2 *Eigenvalues* -1 *and* 1 *of* Ad_g.

(a) Show that the endomorphism $\mathrm{Ad}_g + \mathrm{Id}_{\mathfrak{su}(2)}$ of $\mathfrak{su}(2)$ is invertible for every $g \in \mathrm{SU}(2)$ except for a set of measure zero. Find this set of measure zero.
(b) Find the dimension of the kernel of $\mathrm{Ad}_g - \mathrm{Id}_{\mathfrak{su}(2)}$ as a function of $g \in \mathrm{SU}(2)$.

Exercise 5.3 *An isomorphism of Lie algebras.*

Show that the Lie algebra $\mathfrak{su}(2) \times \mathfrak{su}(2)$ is isomorphic to the Lie algebra $\mathfrak{so}(4)$.

Exercise 5.4 *Haar measure on the group* SU(2).

We parametrize SU(2) off a set of measure zero by setting, for an element $g \in \mathrm{SU}(2)$, $g = \left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix}\right)$, $a = a_1 + ia_2$ and $b = b_1 + ib_2$, and

$$a_1 = \cos\theta_3, \ a_2 = \sin\theta_3 \cos\theta_2, \ b_1 = \sin\theta_3 \sin\theta_2 \cos\theta_1, \ b_2 = \sin\theta_3 \sin\theta_2 \sin\theta_1,$$

for $\theta_1 \in (0, 2\pi)$, $\theta_2 \in (0, \pi)$, $\theta_3 \in (0, \pi)$.

(a) Show that the Haar integral on SU(2) can be written

$$I(f) = \frac{1}{2\pi^2} \int_{\mathcal{O}} f(\theta_1, \theta_2, \theta_3) \sin^2\theta_3 \sin\theta_2 \, \mathrm{d}\theta_1 \, \mathrm{d}\theta_2 \, \mathrm{d}\theta_3,$$

where $\mathcal{O} = (0, 2\pi) \times (0, \pi) \times (0, \pi)$.
(b) Set $\theta_3 = t$. Show that the Haar integral of a class function f is

$$\frac{2}{\pi} \int_0^\pi f(t) \sin^2 t \, \mathrm{d}t.$$

Exercise 5.5 *Euler angles and Haar measure on the group* SO(3).

(a) Show that each rotation $g \in \mathrm{SO}(3)$ can be written

$$g = \exp(\alpha\eta_3)\exp(\beta\eta_1)\exp(\gamma\eta_3)$$

with $0 \leq \alpha < 2\pi$, $0 \leq \beta \leq \pi$, $0 \leq \gamma < 2\pi$, uniquely if $\beta \neq 0$ and $\beta \neq \pi$. The angles α, β, γ are called the *Euler angles* of the rotation.

(b) Show that the Haar integral of SO(3) can be written

$$I(f) = \frac{1}{8\pi^2}\int_{\Omega_0} f(\alpha, \beta, \gamma)\,\sin\beta\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma,$$

where $\Omega_0 = (0, 2\pi) \times (0, \pi) \times (0, 2\pi)$.

Exercise 5.6 *Euler angles and Haar measure on the group* SU(2).
Let $\Omega = (0, 2\pi) \times (0, \pi) \times (-2\pi, 2\pi)$.

(a) Show that for every pair of complex numbers a and b such that $|a|^2 + |b|^2 = 1$, $\mathcal{I}m(a) \neq 0$, and $\mathcal{R}e(b) \neq 0$, there is a unique triple $(\phi, \theta, \psi) \in \Omega$ such that $a = \cos\frac{\theta}{2} e^{i\phi+\psi/2}$, $b = i\sin\frac{\theta}{2} e^{i\phi-\psi/2}$. Conclude that off a set of measure zero of SU(2), each matrix $g = \left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix}\right) \in \mathrm{SU}(2)$ can be written in one and only one way as

$$g = \exp(\phi\xi_3)\exp(\theta\xi_1)\exp(\psi\xi_3),$$

with $(\phi, \theta, \psi) \in \Omega$. Show that the image of $g(\phi, \theta, \psi)$ under the morphism φ of SO(3) onto SU(2) is the rotation with Euler angles (ϕ, θ, ψ).

(b) Show that the Haar integral on SU(2) can be written

$$I(f) = \frac{1}{16\pi^2}\int_{\Omega} f(\phi, \theta, \psi)\,\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}\psi.$$

Chapter 6

Representations of SU(2) and SO(3)

To study the representations of the Lie group SU(2), we first study those of its Lie algebra $\mathfrak{su}(2)$, by studying the representations of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. In fact, by Proposition 1.4 of Chapter 4, if $\mathfrak{g}^{\mathbb{C}}$ is the complexification of a real Lie algebra $\mathfrak{g}$, there is a bijective correspondence between irreducible representations of $\mathfrak{g}$ and of $\mathfrak{g}^{\mathbb{C}}$. In order to determine the irreducible representations of $\mathfrak{su}(2)$, we shall thus study those of $\mathfrak{sl}(2, \mathbb{C})$. We recall that by our convention, a *representation* is a representation on a finite-dimensional complex vector space.

1 Irreducible Representations of $\mathfrak{sl}(2, \mathbb{C})$

1.1 The Representations D^j

We consider $\mathfrak{sl}(2, \mathbb{C})$ equipped with the basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_+ = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

in which the commutation relations can be written

$$\begin{aligned} [H, X_\pm] &= \pm 2X_\pm, \\ [X_+, X_-] &= H. \end{aligned} \tag{1.1}$$

Let (E, R) be an irreducible finite-dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$. The operator $R(H)$ has at least one eigenvalue λ and an eigenvector v for this eigenvalue that by definition satisfies $v \neq 0$ and

$$R(H)v = \lambda v.$$

By the commutation relations (1.1),

$$R(H)R(X_+)v = (R(X_+)R(H) + 2R(X_+))v = (\lambda + 2)R(X_+)v \tag{1.2}$$

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and

$$R(H)R(X_-)v = (R(X_-)R(H) - 2R(X_-))v = (\lambda - 2)R(X_-)v. \tag{1.3}$$

Because there can be no more than a finite number of distinct eigenvalues of $R(H)$, there are an eigenvalue λ_0 and an associated eigenvector v_0 such that $R(H)v_0 = \lambda_0 v_0$, and furthermore,

$$R(X_+)v_0 = 0. \tag{1.4}$$

Then we set, for $k \in \mathbb{N}$,

$$v_k = R(X_-)^k v_0.$$

By iterating equation (1.3), we obtain

$$R(H)v_k = (\lambda_0 - 2k)v_k. \tag{1.5}$$

The nonzero vectors v_k are thus eigenvectors of $R(H)$ with distinct eigenvalues.

We shall show, by induction on k that for any positive integer k,

$$R(X_+)v_k = k(\lambda_0 - k + 1)v_{k-1}.$$

The equation is satisfied for $k = 1$ because by (1.1) and (1.4),

$$R(X_+)v_1 = R(X_+)R(X_-)v_0 = R(H)v_0 = \lambda_0 v_0.$$

Assume that the equation holds for k. Using the hypothesis $v_{k+1} = R(X_-)v_k$ and the second equation of (1.1), we obtain

$$\begin{aligned} R(X_+)v_{k+1} &= R(X_+)R(X_-)v_k = (R(X_-)R(X_+) + R(H))v_k \\ &= R(X_-)k(\lambda_0 - k + 1)v_{k-1} + (\lambda_0 - 2k)v_k \\ &= (k(\lambda_0 - k + 1) + \lambda_0 - 2k)v_k = (k+1)(\lambda_0 - k)v_k, \end{aligned}$$

which proves the equation.

Since the nonzero vectors v_k are linearly independent and the vector space E was assumed finite-dimensional, there is an integer n such that

$$v_0 \neq 0, \quad v_1 \neq 0, \quad \ldots, \quad v_n \neq 0, \quad v_{n+1} = 0\ .$$

Hence, writing $R(X_+)v_{n+1} = 0$, we have

$$\lambda_0 = n.$$

One can then verify that for every $k \in \mathbb{N}$,

$$[R(X_+), R(X_-)]v_k = R(H)v_k, \quad [R(H), R(X_\pm)]v_k = \pm 2R(X_\pm)v_k.$$

The vectors, $v_0, v_1, \ldots, v_n$ generate a subspace of E invariant under R, and since the desired representation is irreducible, they generate E, which is thus of dimension $n+1$. In summary, we have found a representation $(E^{(n)}, R^{(n)})$ of dimension $n+1$ of $\mathfrak{sl}(2,\mathbb{C})$ and a basis $v_0, v_1, \ldots, v_n$ of $E^{(n)}$ such that

$$\begin{cases} R^{(n)}(H)v_k \ = (n-2k)v_k, \\ R^{(n)}(X_-)v_k = v_{k+1}, \\ R^{(n)}(X_+)v_k = k(n-k+1)v_{k-1}, \end{cases} \tag{1.6}$$

for $0 \leq k \leq n$, and $v_{-1} = 0$, $v_{n+1} = 0$. Let us show that the representation $(E^{(n)}, R^{(n)})$ is irreducible. We suppose that u is a nonzero vector in a vector subspace F of $E^{(n)}$, invariant under $R^{(n)}$. Then

$$u = \sum_{k=0}^{n} u_k v_k, \ u_k \in \mathbb{C}.$$

Suppose $u_{k_0} \neq 0, u_{k_0+1} = \cdots = u_n = 0$, where $0 < k_0 \leq n$. Then $(R^{(n)}(X_+))^{k_0} u$ is proportional to v_0 with a nonzero coefficient. Thus v_0 belongs to F, and consequently so do the v_k, for $k = 0, \ldots, n$. Thus each nontrivial invariant subspace coincides with the entire space. On the other hand, the proof above showed that every irreducible finite-dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$ is of the form $(E^{(n)}, R^{(n)})$, for a certain $n \in \mathbb{N}$.

Proposition 1.1. *The irreducible finite-dimensional representations of* $\mathfrak{sl}(2,\mathbb{C})$ *are the* $(E^{(n)}, R^{(n)})$, $n \in \mathbb{N}$.

Often, particularly in physics texts, one writes simply Hv_k, X_-v_k, X_+v_k, where the name of the representation is understood. Physicists set $n = 2j$, with j integer or half-integer, and they prefer the basis J_3, J_+, J_- of $\mathfrak{sl}(2,\mathbb{C})$. They use (E^j, D^j) to denote the representation $(E^{(2j)}, R^{(2j)})$ and they use the basis of $E^j = E^{(2j)}$ indexed by the number m, $-j \leq m \leq j$ (m is integer if j is integer, half-integer if j is half-integer), denoted by $|j,m\rangle$ and called the *standard basis of the representation* D^j, defined by

$$|j,m\rangle = (-1)^{j+m} \sqrt{\frac{(j-m)\,!}{(j+m)\,!}}\ v_{j+m}. \tag{1.7}$$

The notation comes from the “bras” and “kets” formalism of Dirac in quantum mechanics. In the basis $|j,m\rangle$,

$$D^j \begin{cases} J_3|j,m\rangle \ = m|j,m\rangle, \\ J_+|j,m\rangle = \sqrt{(j-m)(j+m+1)}|j,m+1\rangle, \\ J_-|j,m\rangle = \sqrt{(j+m)(j-m+1)}|j,m-1\rangle. \end{cases} \tag{1.8}$$

Hence

$$J_\pm|j,m\rangle = \sqrt{j(j+1) - m(m\pm 1)}|j,m\pm 1\rangle. \tag{1.9}$$

1.2 The Casimir Operator

In each representation space (E,R) of $\mathfrak{sl}(2,\mathbb{C})$, we can consider the operator $(R(J_1))^2+(R(J_2))^2+(R(J_3))^2$, which we write in shorthand:

$$J^2 = J_1^2 + J_2^2 + J_3^2. \tag{1.10}$$

Because $J_+J_- = (J_1+iJ_2)(J_1-iJ_2) = J_1^2+J_2^2-i[J_1,J_2] = J_1^2+J_2^2+J_3$, we have

$$J^2 = J_+J_- + J_3(J_3-I) = J_-J_+ + J_3(J_3+I).$$

It is clear that if $R=D^j$, each vector $|j,m\rangle$ is an eigenvector of J^2. More precisely, we find from (1.8) and (1.9) that

$$J^2|j,m\rangle = j(j+1)|j,m\rangle.$$

Thus we see that, in the representation D^j, not only is each vector $|j,m\rangle$ an eigenvector of J^2, but in fact J^2 acts as the multiple of the identity $j(j+1)I$. The operator J^2 is called the *Casimir operator* of the representation D^j. It commutes with each operator of the representation D^j.

1.3 Hermitian Nature of the Operators J_3 and J^2

We shall show that the operators J_3 and J^2 in the representation (E^j,D^j) of $\mathfrak{sl}(2,\mathbb{C})$ are Hermitian with respect to a well-chosen scalar product on E^j. We define the scalar product on E^j by requiring that $|j,m\rangle$ be an orthonormal basis. We denote by X^* the adjoint of a matrix X, i.e., $X^* = {}^t\overline{X}$. Then

Proposition 1.2.

$$(J_+)^* = J_-, \quad (J_-)^* = J_+, \quad (J_3)^* = J_3, \quad (J^2)^* = J^2.$$

Proof. We use the basis

$$(|j,-j\rangle,\ |j,-j+1\rangle,\ \dots,\ |j,j-1\rangle,\ |j,j\rangle),$$

in the given order.

- The matrix of J_3 in the basis above is $\begin{pmatrix} -j & & \\ 0 & \ddots & 0 \\ & & j \end{pmatrix}$; thus J_3 is real and symmetric, thus Hermitian (self-adjoint).
- The matrix of J_+ is

$$\begin{pmatrix}
0 & 0 & 0 & & & & \\
\sqrt{2j} & 0 & 0 & & & & \\
0 & \sqrt{2(2j-1)} & 0 & & & & \\
 & & \ddots & \ddots & & & \\
 & & & \ddots & 0 & 0 & 0 \\
 & & & & \sqrt{2(2j-1)} & 0 & 0 \\
 & & & & 0 & \sqrt{2j} & 0
\end{pmatrix}.$$

- The matrix of J_- is

$$\begin{pmatrix} 0 & \sqrt{2j} & 0 & & & & \\ 0 & 0 & \sqrt{2(2j-1)} & & & & \\ 0 & 0 & 0 & & & & \\ & & \ddots & \ddots & & & \\ & & & \ddots & 0 & \sqrt{2(2j-1)} & 0 \\ & & & & 0 & 0 & \sqrt{2j} \\ & & & & 0 & 0 & 0 \end{pmatrix}.$$

We have $(J_-)_{m+1,m} = \sqrt{j(j+1) - m(m+1)} = (J_+)_{m,m+1}$. Thus the matrices of J_+ and J_- are real and are transposes of one another. Furthermore,

$$(J^2)^* = J_-^* J_+^* + J_3(J_3 - I) = J_+ J_- + J_3(J_3 - I) = J^2.$$

which proves the proposition. □

In quantum physics, the Hermitian operators J_3 and J^2 are observables that correspond, respectively, to the component along the z-axis of the *angular momentum* and the square of its norm.

By restricting the representation D^j to the real form $\mathfrak{su}(2)$ of $\mathfrak{sl}(2,\mathbb{C})$, we obtain a representation of $\mathfrak{su}(2)$, which we shall denote by the same symbol, D^j.

Because $J_3 = i\xi_3$, we deduce from Proposition 1.2 that

$$D^j(\xi_3)^* = -D^j(\xi_3).$$

Because $J_\pm = i\xi_1 \mp \xi_2$, we have

$$\xi_1 = -\frac{i}{2}(J_+ + J_-), \quad \xi_2 = \frac{1}{2}(J_- - J_+),$$

whence

$$D^j(\xi_1)^* = -D^j(\xi_1), \quad D^j(\xi_2)^* = -D^j(\xi_2).$$

As a consequence, for every $X \in \mathfrak{su}(2)$, $D^j(X)^* = -D^j(X)$. We thus have proved the following proposition.

Proposition 1.3. *The operators of the representation D^j of $\mathfrak{su}(2)$ are anti-Hermitian for the scalar product on E^j defined by the condition that $|j,m\rangle$ be an orthonormal basis.*

Thus, denoting this scalar product by $(\ \mid\ )$, we have

$$(D^j(X)x_1 \mid x_2) = -(x_1 \mid D^j(X)x_2)$$

for every $X \in \mathfrak{su}(2)$ and for every $x_1, x_2 \in E^j$. The matrix of $D^j(X)$ in the basis $|j,m\rangle$ is anti-Hermitian.

2 Representations of SU(2)

2.1 The Representations $\mathcal{D}^j$

We shall study representations $\mathcal{D}^j$ of SU(2), show that they have as differentials $D\mathcal{D}^j$ the representations D^j of $\mathfrak{su}(2)$ studied above, and show that these are the only irreducible representations of SU(2).

The group $\mathrm{SL}(2,\mathbb{C})$ acts on $\mathbb{C}^2$ by the fundamental representation, so that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} az_1 + bz_2 \\ cz_1 + dz_2 \end{pmatrix}.$$

As we have seen in Section 3.5 of Chapter 2, it is natural to make $\mathrm{SL}(2,\mathbb{C})$ act on the space of functions from $\mathbb{C}^2$ to the complex numbers by

$$\rho(g)f = f \circ g^{-1},$$

for each function f on $\mathbb{C}^2$. Thus one defines a representation of $\mathrm{SL}(2,\mathbb{C})$. (Here "representation" means only that $\rho(gg') = \rho(g) \circ \rho(g')$ if g and $g' \in \mathrm{SL}(2,\mathbb{C})$.)

If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ has determinant 1, then $g^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$; thus explicitly,

$$(\rho(g)f)(z_1, z_2) = f(dz_1 - bz_2, -cz_1 + az_2). \tag{2.1}$$

We shall study the representation ρ restricted to SU(2), but it is certainly not irreducible, and we shall find vector subspaces on which SU(2) acts irreducibly.

For each $g \in \mathrm{SU}(2)$, $g^{-1} = {}^t\bar{g}$, we then have $\rho(g)f = f \circ {}^t\bar{g}$, and if $g = \left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix}\right)$,

$$(\rho(g)f)(z_1, z_2) = f(\bar{a}z_1 - bz_2, \bar{b}z_1 + az_2). \tag{2.2}$$

Let V^j be the vector space of homogeneous polynomials with complex coefficients in two variables, (z_1, z_2), of degree $2j$, where $j \in \frac{1}{2}\mathbb{N}$. This complex vector space is of dimension $2j+1$. A basis is

$$z_2^{2j},\ z_1 z_2^{2j-1},\ \ldots,\ z_1^{j+m} z_2^{j-m},\ \ldots,\ z_1^{2j},\ -j \le m \le j.$$

For example, these bases are

$$\begin{aligned} &\text{if } j = \frac{1}{2}, \quad z_2,\ z_1, \\ &\text{if } j = 1, \quad z_2^2,\ z_1 z_2,\ z_1^2, \\ &\text{if } j = \frac{3}{2}, \quad z_2^3,\ z_1 z_2^2,\ z_1^2 z_2,\ z_1^3. \end{aligned}$$

Clearly, V^j is invariant under the representation ρ of SU(2). Now we shall study the representation $\mathcal{D}^j$ of SU(2) on V^j obtained by restriction.

We introduce the notation

$$f_m^j(z_1, z_2) = z_1^{j+m} z_2^{j-m}.$$

Action of diagonal matrices in SU(2). In the representation $\mathcal{D}^j$, the diagonal matrices of SU(2) act in a simple way on V^j. We set

$$g_t = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} = \exp(t\mathcal{K}) = \exp(2t\xi_3), \quad \text{where} \quad \xi_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Then

$$(\mathcal{D}^j(g_t)f_m^j)(z_1, z_2) = f_m^j(e^{-it}z_1, e^{it}z_2) = e^{-2imt}f_m^j(z_1, z_2),$$

and so

$$\mathcal{D}^j(g_t)f_m^j = e^{-2imt}f_m^j.$$

Each f_m^j is thus an eigenvector of $\mathcal{D}^j(g_t)$ for the eigenvalue e^{-2imt}.
Calculation of the differential of $\mathcal{D}^j$. Let $X = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathfrak{sl}(2, \mathbb{C})$ and

$$g(t) = \exp(tX) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}.$$

Then $g(0) = I$ and $g'(0) = X$, and thus $a(0) = d(0) = 1$, $b(0) = c(0) = 0$, and

$$a'(0) = \alpha, \quad b'(0) = \beta, \quad c'(0) = \gamma, \quad d'(0) = \delta.$$

Thus, for every polynomial f in two variables,

$$\begin{aligned}((D\rho)(-X)f)(z_1, z_2) &= \frac{\mathrm{d}}{\mathrm{d}t}(\rho(g(t)^{-1})f)(z_1, z_2)\big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ g(t))(z_1, z_2)\big|_{t=0} \\ &= (\alpha z_1 + \beta z_2)\partial_1 f(z_1, z_2) + (\gamma z_1 + \delta z_2)\partial_2 f(z_1, z_2).\end{aligned}$$

Thus

$$\begin{aligned}((D\rho)(\xi_3)f)(z_1, z_2) &= -\frac{i}{2}(z_1\partial_1 f - z_2\partial_2 f), \\ ((D\rho)(\xi_1)f)(z_1, z_2) &= -\frac{i}{2}(z_2\partial_1 f + z_1\partial_2 f), \\ ((D\rho)(\xi_2)f)(z_1, z_2) &= \frac{1}{2}(z_2\partial_1 f - z_1\partial_2 f).\end{aligned}$$

Therefore we obtain

$$\begin{aligned}(D\rho)(J_3) &= i(D\rho)(\xi_3) = \frac{1}{2}(z_1\partial_1 - z_2\partial_2), \\ (D\rho)(J_+) &= i(D\rho)(\xi_1) - (D\rho)(\xi_2) = z_1\partial_2, \\ (D\rho)(J_-) &= i(D\rho)(\xi_1) + (D\rho)(\xi_2) = z_2\partial_1.\end{aligned}$$

We shall study the action $D\rho$ restricted to V^j. It suffices to determine the value of the operators $(D\mathcal{D}^j)(\xi_k)$, for $k = 1, 2, 3$, or $(D\mathcal{D}^j)(J_3), (D\mathcal{D}^j)(J_+)$, $(D\mathcal{D}^j)(J_-)$, on the basis vectors f_m^j, $-j \leq m \leq j$, of V^j. It is clear that

$$(D\mathcal{D}^j)(\xi_3)f_m^j = -imf_m^j.$$

Using

$$\partial_1 f^j_m = (j+m) z_1^{j+m-1} z_2^{j-m}, \quad \partial_2 f^j_m = (j-m) z_1^{j+m} z_2^{j-m-1},$$

we obtain

$$\begin{aligned}(D\mathcal{D}^j)(\xi_1) f^j_m &= -\frac{i}{2}((j+m) f^j_{m-1} + (j-m) f^j_{m+1}),\\ (D\mathcal{D}^j)(\xi_2) f^j_m &= \frac{1}{2}((j+m) f^j_{m-1} - (j-m) f^j_{m+1}),\end{aligned}$$

and the simple formulas

$$\begin{aligned}(D\mathcal{D}^j)(J_3) f^j_m &= m f^j_m,\\ (D\mathcal{D}^j)(J_+) f^j_m &= (j-m) f^j_{m+1},\\ (D\mathcal{D}^j)(J_-) f^j_m &= (j+m) f^j_{m-1}.\end{aligned}$$

We set

$$|j,m\rangle = \frac{1}{\sqrt{(j-m)!(j+m)!}} f^j_m.$$

We have then

$$D\mathcal{D}^j \begin{cases} J_3 |j,m\rangle = m|j,m\rangle,\\ J_+ |j,m\rangle = \sqrt{(j-m)(j+m+1)}\,|j,m+1\rangle,\\ J_- |j,m\rangle = \sqrt{(j+m)(j-m+1)}\,|j,m-1\rangle.\end{cases}$$

We deduce immediately from these formulas that the differential $D\mathcal{D}^j$ of the representation $(V^j, \mathcal{D}^j)$ of SU(2) is equivalent to the representation D^j of the Lie algebra $\mathfrak{su}(2)$, because they are the restrictions of the same representation of $\mathfrak{sl}(2,\mathbb{C})$. We thus have proved the following proposition.

Proposition 2.1. *The differential of the representation $\mathcal{D}^j$ of* SU(2) *can be identified with the representation D^j of $\mathfrak{su}(2)$.*

From now on, we shall identify the support V^j of $D\mathcal{D}^j$ with the support E^j of D^j.

Proposition 2.2. *For each $j \in \frac{1}{2}\mathbb{N}$, the representation $(V^j, \mathcal{D}^j)$ of* SU(2) *is unitary for the scalar product on V^j defined by requiring that $|j,m\rangle$ be an orthonormal basis.*

Proof. In light of Theorem 7.6 of Chapter 4, this result is the consequence of Propositions 1.3 and 2.1, and of the fact that SU(2) is connected. □

For each $g \in \mathrm{SU}(2)$, the matrix of $\mathcal{D}^j(g)$ in the basis $|j,m\rangle$ is thus unitary.

Proposition 2.3. *For each j, integer or half-integer, $(V^j, \mathcal{D}^j)$ is an irreducible representation of* SU(2) *of dimension $2j+1$.*

Proof. By Theorem 7.6 of Chapter 4, the proposition follows from the fact that the differential of $\mathcal{D}^j$ is an irreducible representation. (If V is a vector subspace of V^j invariant under $\mathcal{D}^j$, it is invariant under $D\mathcal{D}^j$, and thus V is trivial.) $\square$

Corollary 2.4. *Every irreducible representation of* SU(2) *is equivalent to one of the representations* $(V^j, \mathcal{D}^j)$, $j \in \frac{1}{2}\mathbb{N}$.

Proof. Because the group SU(2) is compact, we know, by Theorem 4.3 of Chapter 3, that every irreducible representation of SU(2) is finite-dimensional. We have seen that each finite-dimensional irreducible representation of $\mathfrak{su}(2)$ is one of the D^j, $j \in \frac{1}{2}\mathbb{N}$. Because SU(2) is connected, the result follows. $\square$

2.2 Characters of the Representations $\mathcal{D}^j$

By definition, for $j \in \frac{1}{2}\mathbb{N}$, $\chi_j(t) = \chi_{\mathcal{D}^j}(t) = \sum_{m=-j}^{j} e^{2imt}$. Therefore, for every $t \in (0, \pi)$,

$$\chi_j(t) = \frac{\sin(2j+1)t}{\sin t}.$$

We have $\chi_j(0) = 2j+1$, which is the dimension of the representation, and $\chi_j(\pi) = (-1)^{2j}(2j+1)$.

Character of $\mathcal{D}^{j_1} \otimes \mathcal{D}^{j_2}$. One can express $\chi_j(t)$ as a function of $X = e^{2it}$, for $j \in \frac{1}{2}\mathbb{N}$. If $t \neq 0$ and $t \neq \pi$,

$$\chi_j(t) = \sum_{m=-j}^{j} X^m = \frac{X^{j+1} - X^{-j}}{X-1}.$$

Let j_1 and $j_2 \in \frac{1}{2}\mathbb{N}$. We have

$$\begin{aligned}\chi_{j_1}(t)\chi_{j_2}(t) &= \left(\sum_{m=-j_1}^{j_1} X^m\right)\left(\frac{X^{j_2+1} - X^{-j_2}}{X-1}\right) \\ &= \frac{1}{X-1}\left(\sum_{m=-j_1}^{j_1} X^{m+j_2+1} - \sum_{m=-j_1}^{j_1} X^{m-j_2}\right).\end{aligned}$$

We set $m = j_1 - k$ in the first sum and $m = k - j_1$ in the second. We obtain

$$\chi_{j_1}(t)\chi_{j_2}(t) = \frac{1}{X-1}\left(\sum_{k=0}^{2j_1} X^{j_1+j_2+1-k} - \sum_{k=0}^{2j_1} X^{-j_1-j_2+k}\right).$$

Suppose $j_1 \leq j_2$. Then $0 \leq k \leq 2j_1$ implies $j_1 + j_2 - k \geq 0$, and thus

$$\chi_{j_1}(t)\chi_{j_2}(t) = \sum_{k=0}^{2j_1} \chi_{j_1+j_2-k}(t) = \chi_{j_1+j_2}(t) + \chi_{j_1+j_2-1}(t) + \cdots + \chi_{j_2-j_1}(t).$$

For $t = 0$, the left-hand side is $\chi_{j_1}(t)\chi_{j_2}(t) = (2j_1+1)(2j_2+1)$. On the right-hand side, there are $2j_1+1$ terms whose sum is

$$\begin{aligned}\sum_{k=0}^{2j_1} \chi_{j_2-j_1+k}(0) &= \sum_{k=0}^{2j_1} (2(j_1+j_2)+1-2k) \\ &= (2j_1+1)(2(j_1+j_2)+1) - 2j_1(2j_1+1) \\ &= (2j_1+1)(2j_2+1).\end{aligned}$$

The formula thus holds in this case. The proof for $t = \pi$ is analogous. We have thus proved the formula

$$\chi_{j_1}\chi_{j_2} = \chi_{|j_2-j_1|} + \chi_{|j_2-j_1|+1} + \cdots + \chi_{j_1+j_2}. \tag{2.3}$$

From (2.3) and from equation (2.2) of Chapter 2, we deduce the decomposition of the tensor product $\mathcal{D}^{j_1} \otimes \mathcal{D}^{j_2}$ into a direct sum of irreducible representations, called the *Clebsch–Gordan formula*:

$$\mathcal{D}^{j_1} \otimes \mathcal{D}^{j_2} = \mathcal{D}^{|j_2-j_1|} \oplus \mathcal{D}^{|j_2-j_1|+1} \oplus \cdots \oplus \mathcal{D}^{j_1+j_2}. \tag{2.4}$$

3 Representations of SO(3)

We recall the existence of the surjective morphism φ from SU(2) onto SO(3), with kernel $\{I, -I\}$, studied in Section 2.3 of Chapter 5. If (E, ρ) is a representation of SU(2), then ρ factors through the projection φ if and only if

$$\rho(-I) = \rho(I) = \mathrm{Id}_E. \tag{3.5}$$

If ρ factors through φ as $\sigma \circ \varphi$, where $\sigma : \mathrm{SO}(3) \to \mathrm{GL}(E)$, then σ is a representation of SO(3), and ρ is irreducible if and only if σ is irreducible. The representation $\mathcal{D}^j$ of SU(2) satisfies the condition (3.5) if and only if j is integer. In fact $-I = g_\pi$ and $\mathcal{D}^j(g_\pi)f_m^j = e^{-2im\pi}f_m^j$, where the scalar factor is 1 if and only if m is integer, thus if and only if j is integer. Thus we see that the representations obtained by factorization of $\mathcal{D}^0$, $\mathcal{D}^1$, $\mathcal{D}^2$, ..., $\mathcal{D}^j$, ..., where j is integer, are irreducible representations of SO(3). We denote these representations of SO(3) by $\underline{\mathcal{D}}^j$ or even $\mathcal{D}^j$, by abuse of notation. Because the Lie group SO(3) is connected, we can conclude the following:

Theorem 3.1. *Every irreducible representation of* SO(3) *is equivalent to one of the representations* $(V^j, \mathcal{D}^j)$, $j \in \mathbb{N}$.

References

One can find the results concerning the representations of SU(2) and of SO(3) in many sources, including Fulton–Harris (1991), Sternberg (1994), Talman (1969), and Gilmore (1974), since they are fundamental in mathematics and in physics.

Also see the beginning of the book by Knapp (2002). They are used in the theory of angular momentum in quantum mechanics; see Basdevant–Dalibard (2005), Tung (1985), Edmonds (1974), or Rougé (2005).

Alfred Clebsch (1833–1872) and Paul Gordan (1837–1912) developed the theory of invariants. The "Clebsch–Gordan formula" is a modern group-theoretic interpretation of the finite series expansion for the invariants of binary forms (polynomials in two variables), which they derived independently in 1872 and 1871, respectively.

Hendrik Casimir (1909–2000) was led to the definition of the operator that bears his name in the early 1930s after defending his thesis on the quantum mechanics of the rigid body and finding inspiration in Weyl (1931).

Exercises

Exercise 6.1 *Equivalent representations.*

We set, for $g \in \mathrm{SU}(2)$ and for each complex-valued function f on $\mathbb{C}^2$,

$$\tilde{\rho}(g)f = f \circ \bar{g}^{-1}.$$

Show that $\tilde{\rho} : \mathrm{SU}(2) \to \mathrm{GL}(V^j)$ is a representation of SU(2) equivalent to $\mathcal{D}^j$.

Exercise 6.2 *The representations $\mathcal{D}^{\frac{1}{2}}$ and $\mathcal{D}^1$.*

(a) Write the matrices of J_3, J_+, and J_- in the basis f^j_m for the representations $D^{\frac{1}{2}}$ and D^1. Same question for ξ_1, ξ_2, ξ_3.

(b) Is the representation $\mathcal{D}^{\frac{1}{2}}$ equivalent to the fundamental representation of SU(2) on $\mathbb{C}^2$? Is the representation $\underline{\mathcal{D}}^1$ equivalent to the complexification of the fundamental representation of SO(3) on $\mathbb{R}^3$?

(c) Decompose $\mathcal{D}^{1/2} \otimes \mathcal{D}^{1/2}$, $\mathcal{D}^1 \otimes \mathcal{D}^1$, $\mathcal{D}^{1/2} \otimes \mathcal{D}^1$, and $\mathcal{D}^{1/2} \otimes \mathcal{D}^{1/2} \otimes \mathcal{D}^{1/2}$ into direct sums of irreducible representations.

Exercise 6.3 *Differential operators on polynomials.*

(a) Using the Euler identity for the homogeneous polynomials of degree $2j$, show that

$$(D\mathcal{D}^j)(J_3)f = (j - z_1\partial_1)f = (z_2\partial_2 + j)f.$$

(b) Express $J_+J_- + J_-J_+$, $(J_3)^2$ and J^2 as differential operators of order less than or equal to 2 on the homogeneous polynomials of two variables of degree $2j$.

Exercise 6.4 *Clebsch–Gordan coefficients.*

In the tensor product $V^{j_1} \otimes V^{j_2}$, we consider the two orthonormal bases

$$|j_1, m_1\rangle \otimes |j_2, m_2\rangle, \quad -j_1 \le m_1 \le j_1, \quad -j_2 \le m_2 \le j_2,$$

and

$$|J, M\rangle, \quad |j_1 - j_2| \le J \le j_1 + j_2, \quad -J \le M \le J.$$

The *Clebsch–Gordan coefficients* are the coefficients $C(J, M, j_1, m_1, j_2, m_2)$ of the change-of-basis matrix

$$|J, M\rangle = \sum_{m_1, m_2} C(J, M, j_1, m_1, j_2, m_2)\, |j_1, m_1\rangle \otimes |j_2, m_2\rangle \ .$$

Calculate the Clebsch–Gordan coefficients in the case of $V^{1/2} \otimes V^{1/2}$ and of $V^{1/2} \otimes V^1$.

Exercise 6.5 *Restriction to* SO(2) *of the representations* $\underline{\mathcal{D}}^j$*.*

(a) Find the irreducible representations of the group SO(2).
(b) Find the character of the representation $\underline{\mathcal{D}}^j$ of SO(3), for $j \in \mathbb{N}$.
(c) Decompose the restriction to SO(2) of the representation $\underline{\mathcal{D}}^j$ of SO(3), for $j \in \mathbb{N}$, into a direct sum of irreducible representations.
(d) Identify $\underline{\mathcal{D}}^1$ with the complexification of the fundamental representation of SO(3), and find a basis of $\mathbb{C}^3$ adapted to the decomposition. of the restriction of $\underline{\mathcal{D}}^1$ to SO(2). Is the restriction to SO(2) of the fundamental representation of SO(3) reducible?

Exercise 6.6 *Matrix coefficients of* $\mathcal{D}^j$*.*
We set, for $\theta \in [0, \pi]$,

$$d^j(\theta) = \mathcal{D}^j(\exp(\theta\xi_1)).$$

Calculate the matrix coefficients d^1_{00}, d^1_{10}, and $d^1_{-1,0}$ in the orthonormal basis $|j, m\rangle$.

Exercise 6.7 *Orthogonality of characters.*

(a) Find the value of the integral

$$\int_0^{2\pi} \frac{\sin(j_1 + \frac{1}{2})t \ \sin(j_2 + \frac{1}{2})t \ \sin(j + \frac{1}{2})t}{\sin \frac{t}{2}}\, \mathrm{d}t$$

in terms of the values of j_1, j_2, $j \in \mathbb{N}$.
(b) Same question for j_1, j_2, $j \in \frac{1}{2}\mathbb{N}$.

Chapter 7

Spherical Harmonics

Spherical harmonics play an important role in electrodynamics and in quantum mechanics. We shall show how they appear in the representation theory of the rotation group SO(3). Each irreducible representation of SO(3) can be realized in a finite-dimensional Hilbert space of functions on the sphere, the restrictions of harmonic homogeneous polynomials of a given degree, and this representation is unitary. We shall determine an orthonormal basis of this space that is transformed in a simple way under the action of the group of rotations: the elements of such a basis are simultaneous eigenfunctions for the operators J_3 and J^2, defined below. The functions thus defined on the sphere are called spherical harmonics.

1 Review of $L^2(S^2)$

We denote by S^2 the unit sphere in $\mathbb{R}^3$,

$$S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1)^2 + (x_2)^2 + (x_3)^2 = 1\}.$$

We introduce spherical cordiates (r, θ, ϕ) on $\mathbb{R}^3$, $r \geq 0$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$, such that

$$x_1 = r \sin\theta \cos\phi,\ x_2 = r \sin\theta \sin\phi,\ x_3 = r \cos\theta.$$

The angle ϕ is the longitude and θ is the colatitude. On $\mathbb{R}^3$ minus the axis $\mathrm{O}x_3$ ($r > 0$ and $0 < \theta < \pi$), the passage from Cartesian coordinates to spherical coordinates is smooth. The volume element of $\mathbb{R}^3$ is then $r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi$, and the surface area element on the unit sphere is

$$\mathrm{d}\mu = \frac{1}{4\pi} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi.$$

We denote by $L^2(S^2)$ the separable Hilbert space of (equivalence classes of) complex functions on S^2 that are square-integrable for the scalar product

$$(f_1 \mid f_2) = \frac{1}{4\pi} \int_{S^2} \overline{f_1(\theta, \phi)} f_2(\theta, \phi) \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi.$$

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In spherical coordinates, the *Laplacian* $\Delta = (\partial/\partial x_1)^2 + (\partial/\partial x_2)^2 + (\partial/\partial x_3)^2$ can be written

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\,\Delta_{S^2},$$

where Δ_{S^2} is the *spherical Laplacian*,

$$\Delta_{S^2} = \frac{\partial^2}{\partial \theta^2} + \cot\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial \phi^2}.$$

2 Harmonic Polynomials

2.1 Representations of Groups on Function Spaces

We shall make use of the important fact, stated in Section 3.1 of Chapter 2, that each group of transformations of a space can be represented on the vector space of complex-valued functions on that space. If f is a function on $\mathbb{R}^3$, and if $g \in \mathrm{SO}(3)$, we set, for $x \in \mathbb{R}^3$,

$$(g \cdot f)(x) = f(g^{-1}x),$$

and this defines a representation of SO(3) on the vector space of functions on $\mathbb{R}^3$. (This turns out to be an infinite-dimensional representation. Here we talk of representation from the algebraic point of view only, without insisting on continuity.) We shall denote this representation by σ, defined by

$$\sigma(g)f = g \cdot f.$$

By abuse of language, we denote its restriction to certain subspaces of functions also by σ.

We introduce the harmonic polynomials and show that, by restricting σ to spaces of harmonic polynomials, we obtain all the irreducible representations of SO(3), which were determined in Chapter 6.

2.2 Spaces of Harmonic Polynomials

Definition 2.1. *We call a function f of class C^2 harmonic on $\mathbb{R}^3$ if*

$$\Delta f = 0.$$

For any nonnegative integer ℓ we denote by $P^{(\ell)}$ the vector space of homogeneous polynomials of degree ℓ with complex coefficients on $\mathbb{R}^3$. We then consider the vector subspace of $P^{(\ell)}$ consisting of harmonic polynomials, i.e., polynomials with vanishing Laplacian, which we denote by $H^{(\ell)}$.

Lemma 2.2. *The vector space $H^{(\ell)}$ is of dimension $2\ell + 1$.*

Proof. First we calculate the dimension of the vector space $P^{(\ell)}$. A homogeneous polynomial of degree ℓ on $\mathbb{R}^3$ is determined by $\ell+1$ homogeneous polynomials in two variables, of respective degrees $0, 1, \ldots, \ell$. Because the vector space of homogeneous polynomials in two variables of degree k is of dimension $k+1$, we obtain

$$\dim P^{(\ell)} = 1 + 2 + \cdots + (\ell+1) = \frac{(\ell+1)(\ell+2)}{2}.$$

We observe next that the operator Δ sends $P^{(\ell)}$ into $P^{(\ell-2)}$. Let us show that the linear map $\Delta : P^{(\ell)} \to P^{(\ell-2)}$ is surjective. First we observe that for every $q_3 \in \mathbb{N}$, $x_3^{q_3}$ belongs to the image of Δ, because

$$\Delta(x_3^{q_3+2}) = (q_3+2)(q_3+1)x_3^{q_3}.$$

Similarly, one can easily see that $x_1 x_3^{q_3}$ and $x_2 x_3^{q_3}$ are in $\operatorname{Im}\Delta$. The formula

$$\begin{aligned}\Delta(x_1^{q_1} x_2^{q_2} x_3^{q_3}) = {} & q_1(q_1-1)x_1^{q_1-2}x_2^{q_2}x_3^{q_3} + q_2(q_2-1)x_1^{q_1}x_2^{q_2-2}x_3^{q_3} \\ & + q_3(q_3-1)x_1^{q_1}x_2^{q_2}x_3^{q_3-2},\end{aligned}$$

which holds for every $q_1, q_2, q_3 \in \mathbb{N}$, shows that if the property $x_1^{q_1}x_2^{q_2}x_3^{q_3} \in \operatorname{Im}\Delta$ is true for $q_1+q_2 = q-2$, it is true for $q_1+q_2 = q$. Since this property is true for $q = 0$ and for $q = 1$, the surjectivity of the linear map $\Delta : P^{(\ell)} \to P^{(\ell-2)}$ is thus proved by induction on q. Consequently, $\dim H^{(\ell)} = \dim P^{(\ell)} - \dim P^{(\ell-2)} = 2\ell+1$. □

2.3 Representations of SO(3) on Spaces of Harmonic Polynomials

Let us prove that SO(3) acts on $H^{(\ell)}$. It is clear that if $P \in P^{(\ell)}$ and if $g \in \mathrm{SO}(3)$, then $\sigma(g)P = g \cdot P$ is still a homogeneous polynomial of degree ℓ.

Proposition 2.3. *The subspace $H^{(\ell)}$ of $P^{(\ell)}$ is invariant under σ.*

Proof. Let f be a function of three variables of class C^2 and let g be in SO(3). We denote by (A_{ij}), $i,j = 1,2,3$, the matrix of g, and by y_i the components of $y = g(x)$, for $x \in \mathbb{R}^3$. We have

$$\frac{\partial}{\partial x_i}(f \circ g)(x) = \sum_{j=1}^{3} A_{ji} \frac{\partial f}{\partial y_j}(y).$$

Hence

$$(\Delta(f \circ g))(x) = \sum_{i,j,k=1}^{3} A_{ji} A_{ki} \frac{\partial^2 f}{\partial y_j \partial y_k}(y).$$

Because (A_{ij}) is an orthogonal matrix, we obtain

$$(\Delta(f \circ g))(x) = \sum_{j=1}^{3} \frac{\partial^2 f}{\partial y_j^2}(y) = (\Delta f)(g(x)),$$

that is, $\Delta(f \circ g) = (\Delta f) \circ g$. Consequently, if P is a harmonic polynomial, then for every $g \in \mathrm{SO}(3)$, the polynomial $\sigma(g)P = g \cdot P = P \circ g^{-1}$ is also harmonic. □

On the other hand, for a given homogeneous polynomial P, the coefficients of the polynomials $g \cdot P$ depend continuously on the coefficients of the matrix $g \in \mathrm{SO}(3)$. Thus we can state the following.

Proposition 2.4. *By restricting σ to $H^{(\ell)}$, we obtain, for each $\ell \in \mathbb{N}$, a representation $(H^{(\ell)}, \sigma^\ell)$ of* SO(3) *of dimension $2\ell + 1$.*

We shall show that the representation σ^ℓ of SO(3) on $H^{(\ell)}$ is equivalent to the representation $\mathcal{D}^\ell$ defined in Section 3 of Chapter 6. For this, we shall use the following lemma.

Lemma 2.5. *For $\ell \in \frac{1}{2}\mathbb{N}$, let (V, ρ) be a representation of dimension $2\ell + 1$ of* SU(2)*. If $e^{2i\ell\theta}$ or $e^{-2i\ell\theta}$ is an eigenvalue of $\rho(g_\theta)$, where $g_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$, then ρ is equivalent to $\mathcal{D}^\ell$.*

Proof. We can decompose the given representation into a direct sum of irreducible representations,

$$\rho = \bigoplus_{j=0}^{j_0} m_j \mathcal{D}^j,$$

and since $\dim V = 2\ell + 1$, it follows that $j_0 \leq \ell$. If we had $j_0 < \ell$, all the eigenvalues of $\rho(g_\theta)$ would be of the form $e^{2im\theta}$, with $|m| < \ell$. Thus, one of the $\mathcal{D}^j$ in the direct sum is equal to $\mathcal{D}^\ell$. The condition on the dimension of the representation then implies that $\rho = \mathcal{D}^\ell$. □

We immediately deduce from this lemma that a $(2\ell+1)$-dimensional representation (V, σ) of SO(3), where $\ell \in \mathbb{N}$, is equivalent to $\mathcal{D}^\ell$ if and only if $e^{2i\ell\theta}$ or $e^{-2i\ell\theta}$ is an eigenvalue of $\sigma(\varphi(g_\theta))$. Here φ is the morphism of SU(2) onto SO(3) defined in Section 2.3 of Chapter 5.

Proposition 2.6. *The representation σ^ℓ of* SO(3) *on $H^{(\ell)}$ is equivalent to the representation $\mathcal{D}^\ell$.*

Proof. The polynomial $p_\ell = (x_1 + ix_2)^\ell$ belongs to $P^{(\ell)}$. We can easily see that it is harmonic. Furthermore, in the representation σ^ℓ of SO(3) on $H^{(\ell)}$, it is an eigenfunction of $\varphi(g_\theta)$ for the eigenvalue $e^{-2i\ell\theta}$, because

$$\varphi(g_\theta) = \varphi(\exp 2\theta\xi_3) = \mathrm{Rot}(e_3, 2\theta) = \begin{pmatrix} \cos 2\theta & -\sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and consequently,

$$(\varphi(g_\theta)) \cdot p_\ell = e^{-2i\ell\theta} p_\ell.$$

The lemma then gives the desired result. □

Proposition 2.7. *For each $\ell \geq 2$,*

$$P^{(\ell)} = H^{(\ell)} \oplus r^2 P^{(\ell-2)}.$$

Proof. The sum of the dimensions of the subspaces $H^{(\ell)}$ and $r^2 P^{(\ell-2)}$ of $P^{(\ell)}$ is equal to the dimension of $P^{(\ell)}$. Let us show that their intersection is trivial. If $P \in P^{(\ell)}$, then with the help of the Euler identity $x_1 \frac{\partial P}{\partial x_1} + x_2 \frac{\partial P}{\partial x_2} + x_3 \frac{\partial P}{\partial x_3} = \ell P$, we establish, for every integer $k \geq 0$, the formula

$$\Delta(r^{2k} P) = 2k(2\ell + 2k + 1) r^{2k-2} P + r^{2k} \Delta P.$$

For $P \in H^{(\ell)}$, let k be the largest integer such that there exists a polynomial $Q \in P^{(\ell-2k)}$ satisfying $P = r^{2k} Q$. We thus have $0 = 2k(2\ell - 2k + 1) r^{2k-2} Q + r^{2k} \Delta Q$. We must have $k = 0$, since if not, Q would be divisible by r^2, which contradicts the hypothesis on k. □

We deduce from this proposition that

$$P^{(\ell)} = H^{(\ell)} \oplus r^2 H^{(\ell-2)} \oplus \cdots , \tag{2.1}$$

where the last term is $r^\ell H^{(0)}$ if ℓ is even, and $r^{\ell-1} H^{(1)}$ if ℓ is odd.

3 Definition of Spherical Harmonics

A homogeneous polynomial on $\mathbb{R}^3$ is entirely determined by its restriction to the unit sphere S^2.

Definition 3.1. *The functions on the sphere obtained by restriction of harmonic homogeneous polynomials are called* spherical harmonics.

For each nonnegative integer ℓ, the spherical harmonics of degree ℓ form a vector space $\widetilde{H}^{(\ell)}$ of dimension $2\ell + 1$ that is isomorphic to $H^{(\ell)}$ and contained in the space of smooth functions on the sphere, itself contained in $L^2(S^2)$.

We observe first that by equation (2.1), the space of restrictions to the sphere of homogeneous polynomials of degree ℓ can be written

$$\widetilde{P}^{(\ell)} = \widetilde{H}^{(\ell)} \oplus \widetilde{H}^{(\ell-2)} \oplus \cdots , \tag{3.1}$$

where the last term is $\widetilde{H}^{(0)}$ if ℓ is even, and $\widetilde{H}^{(1)}$ if ℓ is odd.

3.1 Representations of SO(3) on Spaces of Spherical Harmonics

For each $\ell \in \mathbb{N}$, by the identification of $\widetilde{H}^{(\ell)}$ with $H^{(\ell)}$, we obtain a representation, also denoted by σ^ℓ, of SO(3) on the space of spherical harmonics of degree ℓ. These representations are *unitary*, as one can easily see using the rotational invariance of the measure on S^2. In fact, for all functions f_1 and $f_2 \in L^2(S^2)$, and each $g \in \mathrm{SO}(3)$, setting $\sigma(g) f = f \circ g^{-1}$ yields

$$(\sigma(g) f_1 | \sigma(g) f_2) = \int_{S^2} \overline{f_1(g^{-1}x)} f_2(g^{-1}x) \mathrm{d}\mu(x) = \int_{S^2} \overline{f_1(x)} f_2(x) \mathrm{d}\mu(x) = (f_1 | f_2).$$

We have already seen (see Chapter 6, Section 2.1) that when a Lie group is represented on a function space, the differential of the representation associated to each element of the Lie algebra of the group is a differential operator on the functions (when these are differentiable). By Theorem 7.6 of Chapter 4, if the representation of the group is *unitary*, the representation of the Lie algebra is *anti-Hermitian.*

Let η_1, η_2, η_3 be the basis of the Lie algebra $\mathfrak{so}(3)$ of SO(3), introduced in Section 1.2 of Chapter 5. We see easily, by calculating $\frac{\mathrm{d}}{\mathrm{d}t} f(\exp(-t\eta_k)x)\big|_{t=0}$ for each $f \in C^\infty(\mathbb{R}^3)$, $x \in \mathbb{R}^3$ with components x_1, x_2, x_3, and $k = 1, 2, 3$, that the differential $D\sigma$ of the representation σ of SO(3) on $C^\infty(\mathbb{R}^3)$ is such that

$$\begin{aligned}
(D\sigma)\eta_1 &= x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3}, \\
(D\sigma)\eta_2 &= x_1 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1}, \\
(D\sigma)\eta_3 &= x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}.
\end{aligned}$$

By restriction, these formulas define the action of $\mathfrak{so}(3)$ on $P^{(\ell)}$ and $H^{(\ell)}$, and thus on $\widetilde{H}^{(\ell)}$. In spherical coordinates,

$$\begin{aligned}
(D\sigma)\eta_1 &= \sin\phi \frac{\partial}{\partial \theta} + \cos\phi \cot\theta \frac{\partial}{\partial \phi}, \\
(D\sigma)\eta_2 &= -\cos\phi \frac{\partial}{\partial \theta} + \sin\phi \cot\theta \frac{\partial}{\partial \phi}, \\
(D\sigma)\eta_3 &= -\frac{\partial}{\partial \phi}.
\end{aligned}$$

Each of these operators is *anti-Hermitian* for the scalar product of $L^2(S^2)$, since the operators associated to elements of the group SO(3) by σ are unitary, and this fact can also be proved by a direct calculation. We introduce the operators

$$\begin{aligned}
J_3 &= -i \frac{\partial}{\partial \phi}, \\
J_1 &= i \left(\sin\phi \frac{\partial}{\partial \theta} + \cos\phi \cot\theta \frac{\partial}{\partial \phi} \right), \\
J_2 &= i \left(-\cos\phi \frac{\partial}{\partial \theta} + \sin\phi \cot\theta \frac{\partial}{\partial \phi} \right),
\end{aligned}$$

which are thus *Hermitian.* We introduce also $J_\pm = J_1 \pm iJ_2$,

$$J_\pm = e^{\pm i\phi} \left(\pm \frac{\partial}{\partial \theta} + i \cot\theta \frac{\partial}{\partial \phi} \right).$$

While the operator J_3 is Hermitian, the operators J_+ and J_- are adjoint to one another.

Remark. The notation is consistent with that of Chapter 5. The Hermitian operators J_k are associated to the matrices $J_k = i\xi_k$ or $\widehat{J}_k = i\eta_k$, while the operators $J_\pm$ are associated to $J_\pm = J_1 \pm iJ_2 = i\xi_1 \mp \xi_2$ or $\widehat{J}_\pm = \widehat{J}_1 \pm i\widehat{J}_2 = i\eta_1 \mp \eta_2$. But to be prefectly rigorous, the operators we denote by J_k are by definition $i(D\sigma)\eta_k$ rather than $(D\sigma)(i\eta_k)$, since the $i\eta_k$ do not belong to $\mathfrak{so}(3)$.

3.2 The Casimir Operator

Now we introduce the Casimir operator of the representation in question (see Section 1.2 of Chapter 6),

$$J^2 = J_1^2 + J_2^2 + J_3^2 = J_+J_- + J_3^2 - J_3,$$

with the notation as above. The operator J^2 is *Hermitian*, since J_3 as well as J_+J_- are Hermitian.

A simple calculation shows that J^2 is equal to the negative of the spherical Laplacian,

$$J^2 = -\Delta_{S^2}, \tag{3.2}$$

that is,

$$\sum_{k=1}^{3}((D\sigma)\eta_k)^2 = \Delta_{S^2}.$$

3.3 Eigenfunctions of the Casimir Operator

We can write for $P \in P^{(\ell)}$,

$$P(r,\theta,\phi) = r^\ell Y(\theta,\phi).$$

The condition $\Delta P = 0$ is equivalent to

$$\Delta_{S^2} Y = -\ell(\ell+1)Y.$$

Therefore, the numbers $-\ell(\ell+1)$, for nonnegative integer ℓ, are eigenvalues of the operator Δ_{S^2}, of multiplicity $2\ell+1$, since the corresponding eigenspace is $\widetilde{H}^{(\ell)}$. Thus,

$$\Delta_{S^2}\big|_{\widetilde{H}^{(\ell)}} = -\ell(\ell+1)\mathrm{Id}_{\widetilde{H}^{(\ell)}}.$$

Theorem 3.2. *The space $L^2(S^2)$ is the Hilbert direct sum of the $\widetilde{H}^{(\ell)}$, for $\ell \in \mathbb{N}$.*

Proof. We show first that the $\widetilde{H}^{(\ell)}$ are pairwise orthogonal in $L^2(S^2)$. This is the consequence of the fact that these are the eigenspaces of the operator Δ_{S^2}, which is Hermitian, as we have just shown.

Because the space of continuous functions on S^2 is dense in $L^2(S^2)$, it suffices to show that each continuous function on S^2 is a uniform limit of sums of elements of the $\widetilde{H}^{(\ell)}$. By the Weierstrass theorem, we can approximate any continuous function on S^2 by the restriction of polynomials on $\mathbb{R}^3$. We then decompose the polynomials into sums of homogeneous polynomials, and in view of the decomposition (3.1), the proof is then complete. □

We see that the space of square-integrable functions on the sphere decomposes into a Hilbert sum of eigenspaces of the spherical Laplacian, which are the spaces of spherical harmonics. Thus we can conclude as follows.

Corollary 3.3. *The spectrum of the Casimir operator operating on the functions of class C^2 on the sphere is discrete, its eigenvalues are the integers $\ell(\ell+1)$, $\ell \in \mathbb{N}$, and the associated eigenfunctions are the spherical harmonics of degree ℓ.*

Remark. The observables of quantum mechanics are Hermitian operators on the Hilbert space of states. As we oserved in Chapter 6, the Hermitian operators J_k, $k = 1, 2, 3$, and J^2 have a physical interpretation: up to a factor of $\hbar$ ($\hbar = h/2\pi$, where h is Planck's constant), J_1, J_2, J_3 are the observables associated to the components of the *angular momentum* of a particle, and up to the factor $\hbar^2$, J^2 is the observable associated to the square of the norm of the angular momentum. Thus the numbers $\hbar^2\ell(\ell+1)$, for nonnegative integers ℓ, are the eigenvalues of the square of the norm of the angular momentum.

3.4 Bases of Spaces of Spherical Harmonics

More particularly, we use the name *spherical harmonics* for the elements of an orthonormal basis Y^ℓ_m, $-\ell \le m \le \ell$, of $\widetilde{H}^{(\ell)} \subset L^2(S^2)$, for each nonnegative integer ℓ. Some authors seek to realize the orthonormalization condition according to the normalized scalar product on $L^2(S^2)$ introduced in Section 1, while others use the unnormalized scalar product defined by

$$\langle f_1 \mid f_2 \rangle = \int_0^\pi \int_0^{2\pi} \overline{f_1(\theta, \phi)} f_2(\theta, \phi) \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi.$$

We define the following functions, for $\ell \ge 0$. If $0 \le m \le \ell$, we set

$$Y^\ell_m(\theta, \phi) = \gamma^\ell_m Z^\ell_m(\theta) e^{im\phi},$$

where

$$Z^\ell_m(\theta) = \sin^m\theta \, Q^\ell_m(\cos\theta), \quad Q^\ell_m(x) = \frac{\mathrm{d}^{\ell+m}}{\mathrm{d}x^{\ell+m}}(1-x^2)^\ell,$$

and γ^ℓ_m is the real number

$$\gamma^\ell_m = \frac{(-1)^{\ell+m}}{2^\ell \ell!} \sqrt{\frac{2\ell+1}{4\pi}} \sqrt{\frac{(\ell-m)!}{(\ell+m)!}}.$$

If $-\ell \le m < 0$, we set

$$Y^\ell_m = (-1)^m \overline{Y^\ell_{-m}}.$$

Thus we define, for each $\ell \ge 0$, a family Y^ℓ_m, $-\ell \le m \le \ell$, of $2\ell+1$ functions on the sphere. We shall show that these functions are spherical harmonics in the preceding sense, and that they form an orthonormal basis of the space $\widetilde{H}^{(\ell)}$ for the unnormalized scalar product $\langle \ \mid \ \rangle$.

On the one hand, the functions Y^ℓ_m are eigenvectors of the operator $J_3 = -i\frac{\partial}{\partial\phi}$ with eigenvalue m,

$$J_3 Y^\ell_m = m Y^\ell_m. \tag{3.3}$$

On the other hand, the functions Y_m^ℓ satisfy the equations

$$J_+ Y_m^\ell = \sqrt{(\ell - m)(\ell + m + 1)}\, Y_{m+1}^\ell \tag{3.4}$$

and

$$J_- Y_m^\ell = \sqrt{(\ell + m)(\ell - m + 1)}\, Y_{m-1}^\ell, \tag{3.5}$$

which we can easily verify by distinguishing the cases $m \geq 0$ and $m < 0$. Thus we see that the functions Y_m^ℓ are eigenvectors of $J^2 = -\Delta_{S^2}$ with eigenvalue $\ell(\ell+1)$,

$$J^2 Y_m^\ell = \ell(\ell+1) Y_m^\ell, \tag{3.6}$$

and we note that the eigenvalue does not depend on m. Each function Y_m^ℓ is thus seen to be a spherical harmonic. Because the operator J_3 is Hermitian, the functions Y_m^ℓ are pairwise orthogonal. These $2\ell+1$ functions of $\widetilde{H}^{(\ell)}$ thus form an orthogonal basis.

The functions Y_m^ℓ correspond to the "kets" $|\ell, m\rangle$ introduced in Section 1.1 of Chapter 6, equation (1.7). More precisely, the formulas (3.3), (3.4), and (3.5) show that one defines an isomorphism of $(\widetilde{H}^{(\ell)}, \sigma^\ell)$ onto $(V^\ell, \mathcal{D}^\ell)$ by mapping the basis Y_m^ℓ to the basis $|\ell, m\rangle$.

Remark. We know that J^2 commutes with the operators J_k, $k = 1, 2, 3$. Because the operators J^2 and J_3 commute, we can diagonalize them simultaneously. This is what we have done by our choice of the basis (Y_m^ℓ).

We would like to show that for the scalar product $\langle \quad | \quad \rangle$, this basis is orthonormal. We know already that it is orthogonal. We show next that for fixed ℓ, all the basis vectors Y_m^ℓ of $\widetilde{H}^{(\ell)}$ have the same norm, using the equalities (3.4) and (3.5) and the fact that J_+ and J_- are adjoints of one another. For $-\ell + 1 \leq m \leq \ell$,

$$\langle J_+ Y_{m-1}^\ell \mid Y_m^\ell \rangle = \sqrt{(\ell + m)(\ell - m + 1)}\langle Y_m^\ell \mid Y_m^\ell \rangle,$$

while

$$\langle Y_{m-1}^\ell \mid J_- Y_m^\ell \rangle = \sqrt{(\ell + m)(\ell - m + 1)}\langle Y_{m-1}^\ell \mid Y_{m-1}^\ell \rangle.$$

Because these two quantites are equal, we conclude, by induction on m, that all the vectors Y_m^ℓ have the same norm. It suffices then to calculate $\langle Y_\ell^\ell \mid Y_\ell^\ell \rangle$. The polynomial Q_ℓ^ℓ is the constant $(-1)^\ell (2\ell)!$, and consequently,

$$Y_\ell^\ell(\theta, \phi) = (-1)^\ell \, \gamma_\ell^\ell \, (2\ell)! \, \sin^\ell \theta \, e^{i\ell\phi}.$$

We then have

$$\langle Y_\ell^\ell \mid Y_\ell^\ell \rangle = \gamma_\ell \mathcal{I}_\ell,$$

where

$$\gamma_\ell = 2\pi(\gamma_\ell^\ell (2\ell)!)^2$$

and

$$\mathcal{I}_\ell = \int_0^\pi \sin^{2\ell+1} \theta \, \mathrm{d}\theta.$$

We have $\mathcal{I}_0 = 2$ and $\gamma_0 = 1/2$, whence $\langle Y_0^0 \mid Y_0^0 \rangle = 1$. Integration by parts shows that $\mathcal{I}_{\ell+1} = \mathcal{I}_\ell - \frac{1}{2\ell+2}\mathcal{I}_{\ell+1}$, that is, $(2\ell+3)\mathcal{I}_{\ell+1} = (2\ell+2)\mathcal{I}_\ell$. Because $(2\ell+2)\gamma_{\ell+1} = (2\ell+3)\gamma_\ell$, we can deduce that

$$\langle Y_\ell^\ell \mid Y_\ell^\ell \rangle = 1,$$

for every $\ell \in \mathbb{N}$. By Theorem 3.2, we can thus state the following theorem.

Theorem 3.4. *The spherical harmonics Y_m^ℓ, $\ell \in \mathbb{N}$, $-\ell \leq m \leq \ell$, form a Hilbert basis of $L^2(S^2)$ equipped with the unnormalized scalar product.*

In other words, each function belonging to $L^2(S^2)$ has a series expansion in spherical harmonics, convergent in the sense of the norm of $L^2(S^2)$,

$$f = \sum_{\ell=0}^{\infty} \sum_{-\ell \leq m \leq \ell} f_m^\ell Y_m^\ell = f_0^0 Y_0^0 + f_1^1 Y_1^1 + f_0^1 Y_0^1 + f_{-1}^1 Y_{-1}^1 + \cdots ,$$

where the coefficients of the expansion are given by the unnormalized scalar products of functions on the sphere,

$$f_m^\ell = \langle Y_m^\ell \mid f \rangle.$$

Remark. The *Legendre polynomials* are defined, for $\ell \in \mathbb{N}$, by

$$P_\ell(x) = \frac{(-1)^\ell}{2^\ell \ell!} \frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}(1-x^2)^\ell,$$

and the *Legendre functions* are defined, for $m \in \mathbb{N}$ and for $x \in [-1, 1]$, by

$$P_{\ell,m}(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{\mathrm{d}^m}{\mathrm{d}x^m} P_\ell(x) = \frac{(-1)^{\ell+m}}{2^\ell \ell!} (1-x^2)^{\frac{m}{2}} \frac{\mathrm{d}^{\ell+m}}{\mathrm{d}x^{\ell+m}} (1-x^2)^\ell.$$

In general one expresses the spherical harmonics Y_m^ℓ by means of the Legendre functions $P_{\ell,m}$. We see that for $m \geq 0$,

$$Y_m^\ell(\theta, \phi) = \gamma_{\ell,m} P_{\ell,m}(\cos\theta) e^{im\phi},$$

where

$$\gamma_{\ell,m} = \sqrt{\frac{2\ell+1}{4\pi}} \sqrt{\frac{(\ell-m)!}{(\ell+m)!}}.$$

The functions denoted by $Z_m^\ell(\theta)$ above differ from the Legendre functions $P_{\ell,m}(\cos\theta)$ only by a numerical factor,

$$Z_m^\ell(\theta) = (-1)^{\ell+m}\ 2^\ell\ \ell!\ P_{\ell,m}(\cos\theta),$$

and the constants γ_m^ℓ and $\gamma_{\ell,m}$ are related by

$$\gamma_m^\ell = \frac{(-1)^{\ell+m}}{2^\ell \ell!} \gamma_{\ell,m}.$$

This close link between spherical harmonics and Legendre functions allows us to deduce the properties of Legendre functions from those of spherical harmonics that were proved using the representation theory of groups. This method generalizes to other groups to which there correspond other *special functions*.

We can obtain the orthonormality property of the basis functions (Y_m^ℓ) by expressing them as functions of the matrix coefficients of the unitary representations σ^ℓ and using the orthogonality relations proved in Theorem 4.4 of Chapter 3 (see Problem 9).

3.5 Explicit Formulas

In spectroscopy, the levels $\ell = 0, 1, 2, 3, 4, 5$ are denoted by the letters s, p, d, f, g, h. Here are the explicit formulas for the levels s, p, and d:

s	$Y_0^0 = \sqrt{\frac{1}{4\pi}}$
p_0	$Y_0^1 = \sqrt{\frac{3}{4\pi}} \cos\theta$
$p_\pm$	$Y_{\pm 1}^1 = \mp\sqrt{\frac{3}{8\pi}} \sin\theta \ e^{\pm i\phi}$
d_0	$Y_0^2 = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$
$d_{\pm 1}$	$Y_{\pm 1}^2 = \mp\sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta \ e^{\pm i\phi}$
$d_{\pm 2}$	$Y_{\pm 2}^2 = \sqrt{\frac{15}{32\pi}} \sin^2\theta \ e^{\pm 2i\phi}$

We also calculate for the level f,

$$Y_0^3 = \sqrt{\frac{7}{16\pi}} \cos\theta(5\cos^2\theta - 3),$$

and so forth.

We also introduce, for $m > 0$, real linear combinations of spherical harmonics,

$$Y_{m,c}^\ell = (-1)^m \sqrt{2}\gamma_m^\ell Z_m^\ell(\theta) \cos m\phi$$

and

$$Y_{m,s}^\ell = (-1)^m \sqrt{2}\gamma_m^\ell Z_m^\ell(\theta) \sin m\phi.$$

When one restricts a representation $\mathcal{D}^\ell$ to a finite subgroup of SO(3), such as a dihedral group, one obtains a representation, in general not irreducible, of this subgroup. With linear combinations of spherical harmonics Y_m^ℓ, one can construct bases for the spaces of irreducible representations of finite subgroups of SO(3).

The theory of spherical harmonics has vast generalizations in which the groups considered are compact Lie groups more general than SO(3). What generalizes the spheres are the homogeneous spaces, which are manifolds that are quotients of Lie groups by closed subgroups.

References

Sternberg (1994) treats the theory of spherical harmonics, and one can find a survey of some of their properties in Basdevant–Dalibard (2005) and Edmonds (1974). See also Singer (2005). For a more complete theory, one might consult Vilenkin (1968), Vilenkin–Klimyk (1991), Talman (1969), or Dieudonné (1980). The book by Dym and McKean (1985) deals with spherical harmonics in terms of Fourier analysis, and so does that of Axler, Bourdon, and Ramey(2001) on a more advanced level. The comprehensive textbook of Gurarie (1992) relates representation theory and harmonic analysis.

Exercises

Exercise 7.1 *Representations of* SO(2).

Let (r, θ) be polar coordinates on $\mathbb{R}^2$ minus the origin. We recall that the Laplacian in polar coordinates can be written

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

(a) Show that $(\sigma(g)f)(x) = f(g^{-1}(x))$, $x \in \mathbb{R}^2$, defines a unitary representation σ of SO(2) on $L^2(\mathbb{R}^2)$.

Denote by $\mathcal{H}^{(\ell)}$ the vector space of harmonic homogeneous polynomials on $\mathbb{R}^2$ of degree ℓ, where $\ell \in \mathbb{N}$.

(b) Find a basis of $\mathcal{H}^{(\ell)}$.

(c) Show that $\mathcal{H}^{(\ell)}$ is invariant under σ. Is the restriction of σ to $\mathcal{H}^{(\ell)}$ irreducible for every $\ell \geq 0$? If not, find its decomposition into a direct sum of irreducible representations of SO(2).

Exercise 7.2 *Orthogonality relations for the Legendre functions.*

Deduce from the properties of spherical harmonics the orthogonality relations for the Legendre functions,

$$\int_{-1}^{1} P_{\ell,m}(x) P_{\ell',m}(x) \mathrm{d}x = 0 \quad \text{if} \quad \ell \neq \ell',$$

and calculate the value of the integral for $\ell = \ell'$.

Exercise 7.3 *Addition theorem for spherical harmonics.*

Let $d^\ell_{mn}(\theta)$, $-\ell \le m \le \ell$, $-\ell \le n \le \ell$, $\theta \in [0, \pi]$, be the matrix coefficients of the rotation $\exp(\theta\eta_1)$ in the representation $\mathcal{D}^\ell$ in the orthonormal basis $|\ell, m\rangle$. Then

$$d^\ell_{00}(\theta) = P_\ell(\cos\theta),$$

where P_ℓ is the ℓth Legendre polynomial, and the coefficients $d^\ell_{m0}(\theta)$ are related to the spherical harmonics $Y^\ell_m(\theta, \phi)$ by

$$Y^\ell_m(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}(-i)^m d^\ell_{m0}(\theta) e^{im\phi}.$$

[These formulas are proved in Problem 9. The factor $(-i)^m$ appears here instead of the factor i^m because $\varphi^\ell_m = (-1)^{\ell-m}|\ell m\rangle$ implies $\rho^\ell_{m0} = (-1)^m d^\ell_{m0}$.]

(a) Verify these formulas when $\ell = 1$ using the result of Exercise 6.6.
(b) Prove the equation

$$\frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} \overline{Y^\ell_m(\theta,\phi)}\, Y^\ell_m(\theta',\phi) = P_\ell(\cos(\theta-\theta')).$$

This equation, called the *addition theorem* for spherical harmonics, has important applications in quantum mechanics.

Exercise 7.4 *Recurrence relation.*

Show that there are constants $\alpha(\ell, m)$ and $\beta(\ell, m)$ such that

$$\cos\theta\, Y^\ell_m = \alpha(\ell,m) Y^{\ell+1}_m + \beta(\ell,m) Y^{\ell-1}_m\ .$$

Murray Gell-Mann, born in New York in 1929, Nobel Prize in physics in 1969, member of the Los Alamos National Laboratory and professor at the University of New Mexico, proposed the theory of the eightfold way and quarks in 1961, at about the same time as Ne'eman.

(*American Institute of Physics, Emilio Segrè Visual Archives*)

Yuval Ne'eman (1925–2006), founder and director of the physics and astronomy department of Tel Aviv University, discovered the SU(3) *symmetry of subatomic particles in 1961, at about the same time as Gell-Mann.*

(*American Institute of Physics, Emilio Segrè Visual Archives*)

Chapter 8

Representations of SU(3) and Quarks

1 Review of $\mathfrak{sl}(n,\mathbb{C})$, Representations of $\mathfrak{sl}(3,\mathbb{C})$ and SU(3)

1.1 Review of $\mathfrak{sl}(n,\mathbb{C})$

In $\mathfrak{sl}(n,\mathbb{C})$, we denote by $\mathfrak{h}$ the abelian Lie subalgebra of the traceless diagonal matrices. We note that $\mathfrak{h}$ is maximal among the abelian Lie subalgebras of $\mathfrak{sl}(n,\mathbb{C})$. In fact, if a Lie subalgebra of $\mathfrak{sl}(n,\mathbb{C})$ contains $\mathfrak{h}$ and a nondiagonal matrix, it is nonabelian, since no nondiagonal matrix commutes with every traceless diagonal matrix. The Lie subalgebra $\mathfrak{h}$ of $\mathfrak{sl}(n,\mathbb{C})$ is called a *Cartan subalgebra* of $\mathfrak{sl}(n,\mathbb{C})$.

We denote by E_{ij}, $1 \le i \le n$, $1 \le j \le n$, the matrix having 1 on the intersection of the ith line and the jth column and having 0 everywhere else. We set $H_i = E_{ii} - E_{i+1,i+1}$. Recall that a basis of $\mathfrak{sl}(n,\mathbb{C})$ is formed by the E_{ij}, $1 \le i \le n$, $1 \le j \le n$, $i \ne j$, and the H_i, $i = 1,\ldots,n-1$, where the matrices H_i, $i = 1,\ldots,n-1$, are a basis of $\mathfrak{h}$.

We have $E_{ij}E_{kl} = \delta_{jk}E_{il}$, whence the commutation relations of these matrices,

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj}\,, \tag{1.1}$$

$$[H, E_{ij}] = (\lambda_i - \lambda_j)(H)E_{ij}\,, \tag{1.2}$$

where $\lambda_i : H \mapsto \lambda_i(H)$ denotes the linear map of $\mathfrak{h}$ into $\mathbb{C}$ that associates to each traceless diagonal matrix the element on the ith line.

1.2 The Case of $\mathfrak{sl}(3,\mathbb{C})$

In the case of $\mathfrak{sl}(3,\mathbb{C})$, the Cartan subalgebra $\mathfrak{h}$ is of dimension 2 and has as a basis

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

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We can take as a basis of the vector space of strictly upper triangular matrices,

$$E_1 = E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; E_2 = E_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; E_3 = E_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and as a basis of the vector space of strictly lower triangular matrices,

$$F_1 = {}^tE_1 = E_{21}, \quad F_2 = {}^tE_2 = E_{32}, \quad F_3 = {}^tE_3 = E_{31}.$$

We set

$$H_3 = H_1 + H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The commutation relations can be written

$$\begin{gathered} [H_1, H_2] = 0, \quad [E_i, F_i] = H_i, \quad i = 1, 2, 3, \\ [E_1, E_2] = E_3, \quad [E_1, E_3] = 0, \quad [E_2, E_3] = 0, \\ [F_1, F_2] = -F_3, \quad [F_1, F_3] = 0, \quad [F_2, F_3] = 0, \end{gathered}$$

and

$$\begin{aligned} &[E_1, F_2] = 0, \quad [E_2, F_1] = 0, \\ &[E_1, F_3] = -F_2, \quad [E_3, F_1] = -E_2, \\ &[E_2, F_3] = F_1, \quad [E_3, F_2] = E_1. \end{aligned}$$

The remaining commutation relations will interest us more particularly in what follows. Some of them say that E_1, E_2, and E_3 are common eigenvectors of the endomorphisms ad_{H_1} and ad_{H_2} of $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$,

$$\begin{aligned} &[H_1, E_1] = 2E_1, \quad [H_2, E_1] = -E_1, \\ &[H_1, E_2] = -E_2, \quad [H_2, E_2] = 2E_2, \\ &[H_1, E_3] = E_3, \quad [H_2, E_3] = E_3. \end{aligned}$$

The others say that, similarly, F_1, F_2, and F_3 are common eigenvectors of the endomorphisms ad_{H_1} and ad_{H_2},

$$\begin{aligned} &[H_1, F_1] = -2F_1, \quad [H_2, F_1] = F_1, \\ &[H_1, F_2] = F_2, \quad [H_2, F_2] = -2F_2, \\ &[H_1, F_3] = -F_3, \quad [H_2, F_3] = -F_3. \end{aligned}$$

We let $\alpha_1(H_1)$ (respectively, $\alpha_1(H_2)$) denote the eigenvalue of ad_{H_1} (respectively, ad_{H_2}) corresponding to the eigenvector E_1. By the preceding formulas,

$$\alpha_1(H_1) = 2, \quad \alpha_1(H_2) = -1.$$

In fact, each element of the Cartan subalgebra $\mathfrak{h}$ can be written as a linear combination of H_1 and H_2, and we thus define a linear form α_1 on $\mathfrak{h}$.

The values of this linear form on the basis vectors (H_1, H_2), are $(2, -1)$. These are also the components of this linear form in the basis dual to the basis (H_1, H_2), by the definition of the notion of dual basis. We introduce similarly the notation $\alpha_2(H)$ for the eigenvalues of $H \in \mathfrak{h}$ corresponding to the eigenvector E_2,

$$\alpha_2(H_1) = -1, \quad \alpha_2(H_2) = 2.$$

Finally, let $\alpha_3(H)$ be the eigenvalue of $H \in \mathfrak{h}$ corresponding to the eigenvector E_3,

$$\alpha_3(H_1) = 1, \quad \alpha_3(H_2) = 1.$$

We clearly have

$$\alpha_3 = \alpha_1 + \alpha_2.$$

By definition,

$$[H_i, E_j] = \alpha_j(H_i)E_j, \quad 1 \le i \le 3, \quad 1 \le j \le 3,$$

and we also see that

$$[H_i, F_j] = -\alpha_j(H_i)F_j, \quad 1 \le i \le 3, \quad 1 \le j \le 3.$$

The linear forms α_1, α_2, α_3, $-\alpha_1$, $-\alpha_2$, $-\alpha_3$ on $\mathfrak{h}$ play an important role in representation theory. They are called the *roots* of $\mathfrak{sl}(3,\mathbb{C})$.

Above we have introduced the linear form on $\mathfrak{h}$, denoted by λ_i, that associates to each matrix in $\mathfrak{h}$ the element on the ith line. We see immediately, directly or from (1.2), that

$$\alpha_1 = \lambda_1 - \lambda_2, \quad \alpha_2 = \lambda_2 - \lambda_3, \quad \alpha_3 = \lambda_1 - \lambda_3.$$

1.3 The Bases (I_3, Y) and (I_3, T_8) of $\mathfrak{h}$

In the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}(3,\mathbb{C})$ we introduce a new basis (I_3, Y) defined by

$$I_3 = \frac{1}{2}H_1, \quad Y = \frac{1}{3}(H_1 + 2H_2) = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

We introduce also

$$T_8 = \frac{\sqrt{3}}{2}Y = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

and we consider the basis (I_3, T_8) of $\mathfrak{h} \subset \mathfrak{sl}(3,\mathbb{C})$ as well.

Remark. In many physics books, I_3 is denoted by T_3. The quantity $Q = \frac{1}{2}Y + I_3$ is introduced as well. The eigenvalues of the operators I_3, Y, and Q of the symmetry algebra in a representation correspond to the third component of the *isospin*, to the *hypercharge*, and to the *electric charge*, respectively.

1.4 Representations of $\mathfrak{sl}(3,\mathbb{C})$ and of SU(3)

There is a bijection between the irreducible representations of $\mathfrak{sl}(3,\mathbb{C})$ and those of $\mathfrak{su}(3)$, by complexification and, conversely, by restriction. Furthermore, the irreducible representations of the Lie algebra $\mathfrak{su}(3)$ are in bijective correspondence with those of the Lie group SU(3), since this group is connected and simply connected.

2 The Adjoint Representation and Roots

In what follows we are interested in the representations of the Lie group SU(3). By the preceding remarks, it suffices to study the representations of the Lie algebra $\mathfrak{sl}(3,\mathbb{C})$, which is of complex dimension 8. We make use of the notation of Section 1.1. The commutation relations of $\mathfrak{sl}(3,\mathbb{C})$ imply that if $H = \begin{pmatrix} \lambda_1(H) & 0 & 0 \\ 0 & \lambda_2(H) & 0 \\ 0 & 0 & \lambda_3(H) \end{pmatrix}$, then $[H, E_{ij}] = (\lambda_i(H) - \lambda_j(H))E_{ij}$. The roots of $\mathfrak{sl}(3,\mathbb{C})$ are thus given by the following table:

root	relations		basis $(I_3$,Y)	basis(I_3, T_8)
α_1	$[I_3, E_1] = E_1$	[Y,E_1] $= 0$	(1,0)	(1,0)
α_2	$[I_3, E_2] = -\frac{1}{2}E_2$	$[Y, E_2] = E_2$	$(-\frac{1}{2}, 1)$	$(-\frac{1}{2}, \frac{\sqrt{3}}{2})$
α_3	$[I_3, E_3] = \frac{1}{2}E_3$	$[Y, E_3] = E_3$	$(\frac{1}{2}, 1)$	$(\frac{1}{2}, \frac{\sqrt{3}}{2})$
$\alpha_1^{'}$	$[I_3, F_1] = -F_1$	[Y,F_1] $= 0$	$(-1,0)$	$(-1,0)$
$\alpha_2^{'}$	$[I_3, F_2] = \frac{1}{2}F_2$	$[Y, F_2] = -F_2$	$(\frac{1}{2}, -1)$	$(\frac{1}{2}, -\frac{\sqrt{3}}{2})$
$\alpha_3^{'}$	$[I_3, F_3] = -\frac{1}{2}F_3$	$[Y, F_3] = -F_3$	$(-\frac{1}{2}, -1)$	$(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$

The following equalities are obvious:

$$\alpha_3 = \alpha_1 + \alpha_2, \quad \alpha_1' = -\alpha_1, \quad \alpha_2' = -\alpha_2, \quad \alpha_3' = -\alpha_3.$$

The first can also be seen as a consequence of the Jacobi identity, because

$$\begin{aligned}[H_i, E_3] &= [H_i, [E_1, E_2]] = [[H_i, E_1], E_2] + [E_1, [H_i, E_2]] \\ &= \alpha_1(H_i)[E_1, E_2] + \alpha_2(H_i)[E_1, E_2] = (\alpha_1 + \alpha_2)(H_i)E_3.\end{aligned}$$

We recall also that $\alpha_1 = \lambda_1 - \lambda_2$ and $\alpha_2 = \lambda_2 - \lambda_3$.

Now we draw the roots of $\mathfrak{sl}(3,\mathbb{C})$ in the plane relative to the axes I_3 and T_8 (Figure 1). We see that these six roots form a regular hexagon, which would not have been the case had we used the basis (I_3, Y). Technically, 0 is not called a root, but for every $H \in \mathfrak{h}$, 0 is a double eigenvalue of the endomorphism ad_H. This is why we draw the point 0 as a double point (a point surrounded by a little circle). The adjoint representation is of dimension 8, and we shall denote it in the rest of this chapter by 8. In the following sections, we shall talk more generally of *weights* of representations of the algebra $\mathfrak{sl}(3,\mathbb{C})$ and we shall see that *the roots are the weights of the adjoint representation.*

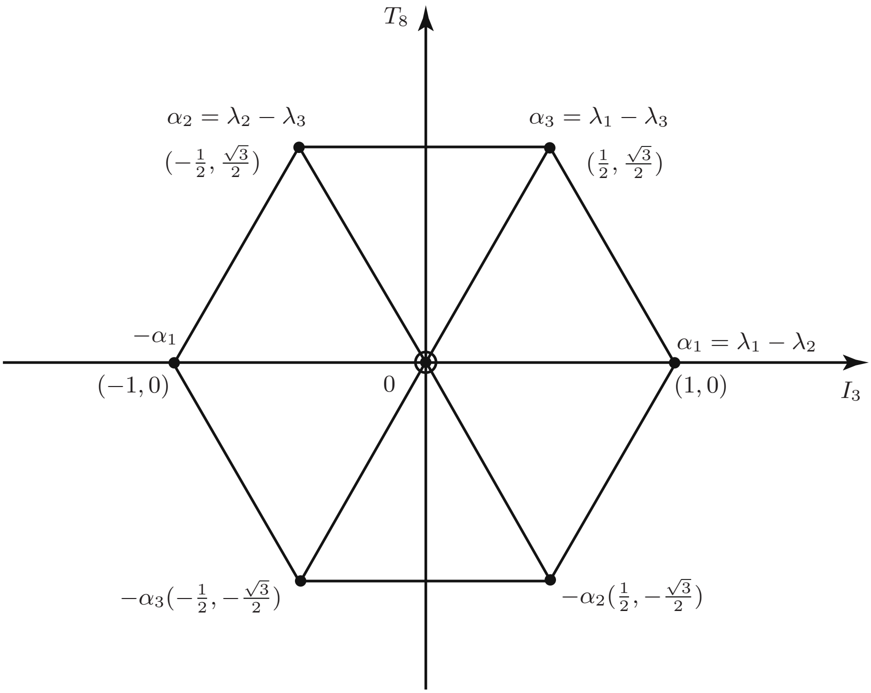

Figure 1. The representation **8**.

3 The Fundamental Representation and Its Dual

3.1 The Fundamental Representation

Now we consider the fundamental representation. We have

$$H_1 e_1 = e_1, \quad H_2 e_1 = 0,$$
$$H_1 e_2 = -e_2, \quad H_2 e_2 = e_2,$$
$$H_1 e_3 = 0, \quad H_2 e_3 = -e_3.$$

By definition, the *weights* of this representation are the linear forms λ_1, λ_2, and λ_3 on $\mathfrak{h}$ satisfying, for $H \in \mathfrak{h}$ and $i = 1, 2, 3$,

$$He_i = \lambda_i(H)e_i.$$

Thus in the basis (H_1, H_2) the components of the three weights are

$$\lambda_1 = (1, 0), \quad \lambda_2 = (-1, 1), \quad \lambda_3 = (0, -1).$$

Hence we have the following table:

weight	relations		basis (I_3,Y)	basis(I_3, T_8)	quark
λ_1	$I_3 e_1 = \frac{1}{2} e_1$	$Y e_1 = \frac{1}{3} e_1$	$(\frac{1}{2}, \frac{1}{3})$	$(\frac{1}{2}, \frac{1}{2\sqrt{3}})$	u
λ_2	$I_3 e_2 = -\frac{1}{2} e_2$	$Y e_2 = \frac{1}{3} e_2$	$(-\frac{1}{2}, \frac{1}{3})$	$(-\frac{1}{2}, \frac{1}{2\sqrt{3}})$	d
λ_3	$I_3 e_3 = 0$	$Y e_3 = -\frac{2}{3} e_3$	$(0, -\frac{2}{3})$	$(0, \frac{1}{3\sqrt{3}})$	s

Thus there are three linearly independent vectors e_1, e_2, e_3 that are common eigenvectors of all elements H of $\mathfrak{h}$. We denote them by $u,\ d,\ s$, which designate the *quarks*,

$$u = \text{up},\ \ d = \text{down},\ \ s = \text{strange}.$$

(See more information on the quarks in Sections 6 and 7 below.)

In the basis (I_3, T_8), the weights $\lambda_1 = w_u, \lambda_2 = w_d, \lambda_3 = w_s$ of the fundamental representation form an equilateral triangle. We have

$$\lambda_1 + \lambda_2 + \lambda_3 = 0\ .$$

In Figure 2 we draw the weight diagram of the fundamental representation of $\mathfrak{sl}(3, \mathbb{C})$, which we shall call **3** from now on.

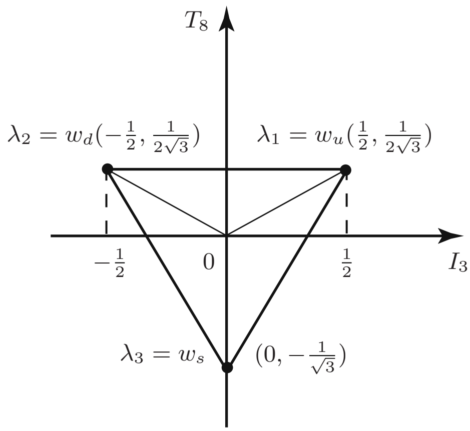

Figure 2. The representation **3**.

3.2 The Dual of the Fundamental Representation

We pass from the fundamental representation to its dual by replacing the matrices $X \in \mathfrak{sl}(3, \mathbb{C})$ by $X' = -{}^tX$, and so $H'_j e_i = -\lambda_i(H_j)e_i$. The equations

$$\begin{aligned} &H'_1 e_1 = -e_1, \quad H'_2 e_1 = 0, \\ &H'_1 e_2 = e_2, \quad H'_2 e_2 = -e_2, \\ &H'_1 e_3 = 0, \quad H'_2 e_3 = e_3, \end{aligned}$$

show that the components of the three weights $\lambda'_1, \lambda'_2, \lambda'_3$ of this representation in the basis (H_1, H_2) are

$$\lambda'_1 = (-1, 0), \quad \lambda'_2 = (1, -1), \quad \lambda'_3 = (0, 1).$$

One obtains thus the following table for the weights of the dual representation of the fundamental representation:

weight	relations		basis (I_3, Y)	basis(I_3, T_8)	antiquarks
λ_1'	$I_3 e_1 = -\frac{1}{2}e_1$	$Ye_1 = -\frac{1}{3}e_1$	$(-\frac{1}{2}, -\frac{1}{3})$	$(-\frac{1}{2}, -\frac{1}{2\sqrt{3}})$	$\bar{u}$
λ_2'	$I_3 e_2 = \frac{1}{2}e_2$	$Ye_2 = -\frac{1}{3}e_2$	$(\frac{1}{2}, -\frac{1}{3})$	$(\frac{1}{2}, -\frac{1}{2\sqrt{3}})$	$\bar{d}$
λ_3'	$I_3 e_3 = 0$	$Ye_3 = \frac{2}{3}e_3$	$(0, \frac{2}{3})$	$(0, \frac{1}{\sqrt{3}})$	$\bar{s}$

The weight vectors $\overline{u}, \overline{d}$, and $\overline{s}$ are called the *antiquarks*.

In Figure 3 we draw the weight diagram of the representation dual to the fundamental representation of $\mathfrak{sl}(3, \mathbb{C})$, which we shall call $\overline{\mathbf{3}}$ from now on.

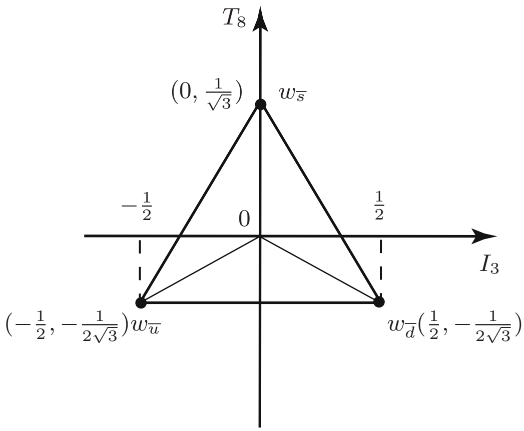

Figure 3. The representation $\overline{\mathbf{3}}$.

4 Highest Weight of a Finite-Dimensional Representation

4.1 Highest Weight

A *weight* of a representation (E, ρ) is a linear form ω on $\mathfrak{h}$ such that there is a nonzero vector $v \in E$ satisfying $\rho(H)v = \omega(H)v$ for every $H \in \mathfrak{h}$.

We shall see that in each of the three representations described above, one of the weights plays a particular role. It will be called the *highest weight* of the representation. An eigenvector corresponding to the highest weight, which is defined up to multiplication by a nonzero scalar, is called the *highest-weight vector.*

Example 1. In $\mathbf{3}$, the highest weight is $\lambda_1 = w_u$, with highest-weight vector $e_1 = u$. We verify the following properties:

(i) $H_1 e_1 = e_1,\ H_2 e_1 = 0$ $\quad$ (e_1 is an eigenvector for each $H \in \mathfrak{h}$),
(ii) $E_i e_1 = 0, \quad i = 1, 2, 3.$

Furthermore, we obtain e_2 and e_3 by the action of F_1, F_2, and F_3 on e_1. More precisely,

$$F_1 e_1 = e_2, \quad F_2 e_1 = 0, \quad F_3 e_1 = e_3.$$

Example 2. In $\overline{\mathbf{3}}$, the highest weight is $\lambda'_3 = w_{\overline{s}}$, with highest-weight vector $e_3 = \overline{s}$, that is,

(*i*) $H'_1 e_3 = 0, \quad H'_2 e_3 = e_3 \quad$ (e_3 is an eigenvector of each $H \in \mathfrak{h}$),
(*ii*) $E'_i e_3 = -F_i e_3 = 0, \quad i = 1, 2, 3.$

As in the preceding example, we obtain scalar multiples of e_1 and e_2 by the action of F'_1, F'_2, and F'_3 on e_3. More precisely,

$$F'_1 e_3 = -E_1 e_3 = 0, \quad F'_2 e_3 = -E_2 e_3 = -e_2, \quad F'_3 e_3 = -E_3 e_3 = -e_1.$$

Example 3. In $\mathbf{8}$, the highest weight is α_3, with highest-weight vector E_3. We have

(*i*) $[H_1, E_3] = E_3, \quad [H_2, E_3] = E_3 \quad$ (E_3 is an eigenvector of each $H \in \mathfrak{h}$),
(*ii*) $[E_i, E_3] = 0, \quad i = 1, 2, 3.$

Here we obtain scalar multiples of $E_1, E_2, F_1, F_2, F_3, H_1, H_2$ by the action of F_1, F_2, and F_3 on E_3. More precisely,

$$\begin{aligned} &[F_1, E_3] = E_2, \quad [F_2, E_3] = -E_1, \quad [F_3, E_3] = -H_3, \\ &[F_2, [F_1, E_3]] = -H_2, \quad [F_3, [F_1, E_3]] = -F_1, \\ &[F_1, [F_2, E_3]] = -H_1, \quad [F_3, [F_2, E_3]] = F_2, \\ &[F_3, [F_3, E_3]] = -2F_3. \end{aligned}$$

In summary, we have seen that for the representations ρ studied so far, there is a *highest weight* $w_{\max}$ and a vector $e_{\max}$ called *highest-weight vector*, defined up to multiplication by a nonzero scalar, satisfying the following properties:

(*i*) for every $H \in \mathfrak{h}$, the vector $e_{\max}$ is an eigenvector of $\rho(H)$ for the eigenvalue $w_{\max}(H)$,
(*ii*) for $i = 1, 2, 3$,
$$\rho(E_i)(e_{\max}) = 0,$$
(*iii*) the images of the vector $e_{\max}$ by repeated application of the operators $\rho(F_i)$, for $i = 1, 2, 3$, generate the entire space of the representation.

4.2 Weights as Linear Combinations of the $\boldsymbol{\lambda_i}$

We have set

$$H = \begin{pmatrix} \lambda_1(H) & 0 & 0 \\ 0 & \lambda_2(H) & 0 \\ 0 & 0 & \lambda_3(H) \end{pmatrix},$$

and thus defined linear forms λ_1, λ_2, and λ_3 on $\mathfrak{h}$. We shall see that each weight of the representations above can be written

$$w = m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3$$

(which means that $w(H) = m_1\lambda_1(H) + m_2\lambda_2(H) + m_3\lambda_3(H)$, for every $H \in \mathfrak{h}$), where the m_i are integers. We can denote a weight by a triple (m_1, m_2, m_3).

Such a triple is not unique. In fact, $\lambda_1 + \lambda_2 + \lambda_3 = 0$, and consequently we can always reduce to the case in which one of the m_i is zero. For example, we can write

$$w = (m_1 - m_3)\lambda_1 + (m_2 - m_3)\lambda_2 + m_3(\lambda_1 + \lambda_2 + \lambda_3) = m_1'\lambda_1 + m_2'\lambda_2 \ .$$

We can also reduce to the case in which the m_i are all three positive or zero, as we shall show in the following examples.

Example 1. The representation **3**. In **3**, the weights are

$$w_u = \lambda_1, \ w_d = \lambda_2, \ w_s = \lambda_3.$$

We see that

$$\begin{aligned} w_u &= \lambda_1 = \lambda_1 + 0\lambda_2 + 0\lambda_3 \to (1,0,0), \\ w_d &= \lambda_2 = 0\lambda_1 + 1\lambda_2 + 0\lambda_3 \to (0,1,0), \\ w_s &= \lambda_3 = 0\lambda_1 + 0\lambda_2 + 1\lambda_3 \to (0,0,1). \end{aligned}$$

Example 2. The representation $\overline{\mathbf{3}}$. In $\overline{\mathbf{3}}$, the weights are

$$w_{\overline{u}} = -\lambda_1, \ w_{\overline{d}} = -\lambda_2, \ w_{\overline{s}} = -\lambda_3.$$

We then have

$$\begin{aligned} w_{\overline{u}} &= -\lambda_1 = \lambda_2 + \lambda_3 \to (0,1,1), \\ w_{\overline{d}} &= -\lambda_2 = \lambda_1 + \lambda_3 \to (1,0,1), \\ w_{\overline{s}} &= -\lambda_3 = \lambda_1 + \lambda_2 \to (1,1,0). \end{aligned}$$

Example 3. The representation **8**. In the representation **8**, the weights are the roots $\alpha_i, -\alpha_i$, $i = 1,2,3$. We see that

$$\begin{aligned} \alpha_1 &= \lambda_1 - \lambda_2 \to (2,0,1), & \alpha_1' &= \lambda_2 - \lambda_1 \to (0,2,1), \\ \alpha_2 &= \lambda_2 - \lambda_3 \to (1,2,0), & \alpha_2' &= \lambda_3 - \lambda_2 \to (1,0,2), \\ \alpha_3 &= \lambda_1 - \lambda_3 \to (2,1,0), & \alpha_3' &= \lambda_3 - \lambda_1 \to (0,1,2). \end{aligned}$$

In these three examples, we see that the highest weight is the one for which the triple (m_1, m_2, m_3) is the largest in lexicographic order.

4.3 Finite-Dimensional Representations and Weights

For each finite-dimensional irreducible representation ρ of $\mathfrak{sl}(3,\mathbb{C})$, there are a highest weight w and a highest-weight vector v, defined up to multiplication by a nonzero scalar, satisfying, for every $H \in \mathfrak{h}$,

$$\rho(H)v = w(H)v, \quad \rho(E_i)v = 0, \quad i = 1,2,3.$$

Furthermore, the action of compositions of powers of $\rho(F_i)$, $i = 1,2,3$, on v generates the entire space of the irreducible representation of $\mathfrak{sl}(3,\mathbb{C})$.

These facts are a special case of general theorems on the representations of semisimple Lie algebras, for which see the references at the end of this chapter.

From the calculation

$$\rho(H)\rho(F_i)v = \rho(F_i)\rho(H)v + \rho([H, F_i])v = w(H)\rho(F_i)v - \alpha_i(H)\rho(F_i)v,$$

we obtain

$$\rho(H)\rho(F_i)v = (w - \alpha_i)(H)\rho(F_i)v, \tag{4.1}$$

and it follows that each vector $\rho(F_1)^{k_1}\rho(F_2)^{k_2}\rho(F_3)^{k_3}v$, for nonnegative integers k_i, $i = 1, 2, 3$, is a weight vector corresponding to the weight $w - k_1\alpha_1 - k_2\alpha_2 - k_3\alpha_3$. The weights thus obtained are part of the translate of the root diagram that contains the point w. The following is a useful rule:

Rule. In the root diagram of $\mathfrak{sl}(3, \mathbb{C})$, the weights of an irreducible representation of $\mathfrak{sl}(3, \mathbb{C})$ lie on concentric hexagons that may degenerate into triangles. The multiplicity of each weight is 1 on the edge, it increases by 1 on each concentric hexagon approaching the center, and it is constant on the triangles.

For a proof of this rule, see Fulton–Harris (1991).

4.4 Another Example: The Representation 6

We consider the representation of highest weight $w = 2\lambda_1$. The other weights will be constructed by adding the negative integer multiples of $\lambda_1 - \lambda_2$, $\lambda_2 - \lambda_3$, and $\lambda_1 - \lambda_3$ such that the weights obtained are not higher than the highest weight. Let v be a highest-weight vector. By definition,

$$H_1 v = 2v, \ H_2 v = 0, \ E_1 v = E_2 v = E_3 v = 0.$$

Knowing that $Hv = w(H)v$, we obtain, using equation (4.1),

$$H(F_i v) = (w - \alpha_i)(H)(F_i v),$$

whence

$$\begin{aligned} H(F_1 v) &= (w - (\lambda_1 - \lambda_2))(H)(F_1 v) \\ &= (\lambda_1 + \lambda_2)(H)(F_1 v) = -\lambda_3(H)(F_1 v), \\ H(F_3 v) &= (w - (\lambda_1 - \lambda_3))(H)(F_3 v) \\ &= (\lambda_1 + \lambda_3)(H)(F_3 v) = -\lambda_2(H)(F_3 v). \end{aligned}$$

By repeated action of F_1, F_2, F_3, we obtain Figure 4. This representation is of dimension 6, and we shall denote it by **6**.

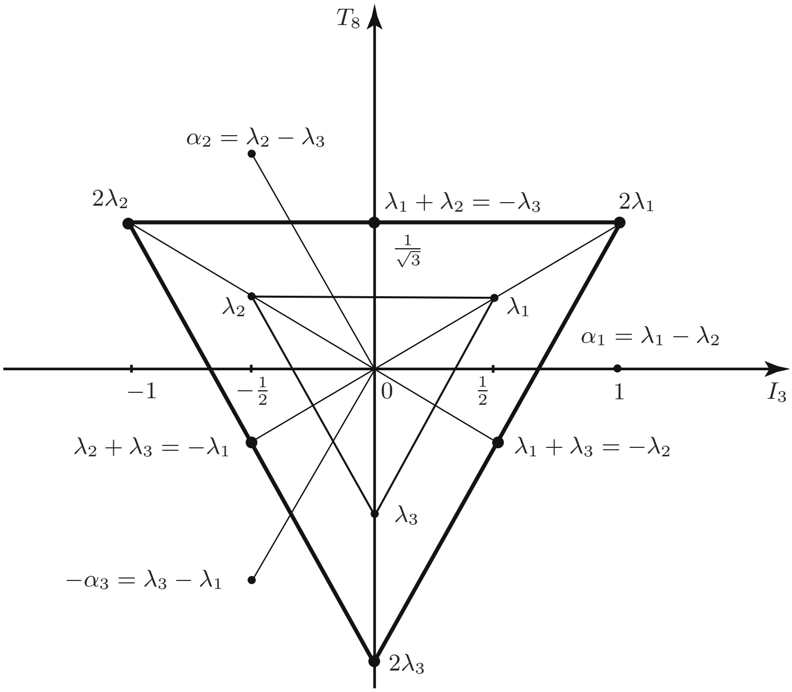

Figure 4. The representation **6**.

4.5 One More Example: The Representation 10

We consider the representation of highest weight $w = 3\lambda_1$. Below is the list of weights of this representation. Because of the applications to particle physics, which we shall discuss briefly in Section 6, we have given the names of the elementary particles associated to each weight, as well as the value of the component of each weight on the element $Q = \frac{1}{2}Y + I_3$ that corresponds to the charge of each particle:

weight w	$w(I_3)$	$w(Y)$	$w(T_8)$	$w(Q)$	particle
$3\ \lambda_1$	$\frac{3}{2}$	1	$\frac{\sqrt{3}}{2}$	2	Δ^{++}
$2\lambda_1 + \lambda_2$	$\frac{1}{2}$	1	$\frac{\sqrt{3}}{2}$	1	Δ^{+}
$\lambda_2 - \lambda_3 = \lambda_1 + 2\lambda_2$	$-\frac{1}{2}$	1	$\frac{\sqrt{3}}{2}$	0	Δ^{0}
$3\lambda_2$	$-\frac{3}{2}$	1	$\frac{\sqrt{3}}{2}$	-1	Δ^{-}
$\lambda_2 - \lambda_1 = 2\lambda_2 + \lambda_3$	-1	0	0	-1	Σ^{*-}
$\lambda_3 - \lambda_1 = \lambda_2 + 2\lambda_3$	$-\frac{1}{2}$	-1	$-\frac{\sqrt{3}}{2}$	-1	Ξ^{*-}
$3\lambda_3 = 2\lambda_3 - \lambda_1 - \lambda_2$	0	-2	$-\sqrt{3}$	-1	Ω^{-}
$\lambda_3 - \lambda_2 = 2\lambda_3 + \lambda_1$	$\frac{1}{2}$	-1	$-\frac{\sqrt{3}}{2}$	0	Ξ^{*0}
$\lambda_3 + 2\lambda_1 = \lambda_1 - \lambda_2$	1	0	0	1	Σ^{*+}
0	0	0	0	0	Σ^{*0}

In Figure 5 we draw the weight diagram of this representation, which is of dimension 10, and which we denote by the symbol **10**. To each weight corresponds a unique weight vector up to scalar multiplication. This representation is irreducible, since one can show by examining the weight diagram that it is not the direct sum of representations of smaller dimension. In fact, we can give a complete list of these irreducible representations of small dimension:

- the trivial representation, denoted by **1**,
- the fundamental representation **3**,
- the dual $\overline{\mathbf{3}}$ of the fundamental representation **3**,
- the representation **6**,
- the dual $\overline{\mathbf{6}}$ of the representation **6**,
- the adjoint representation **8**.

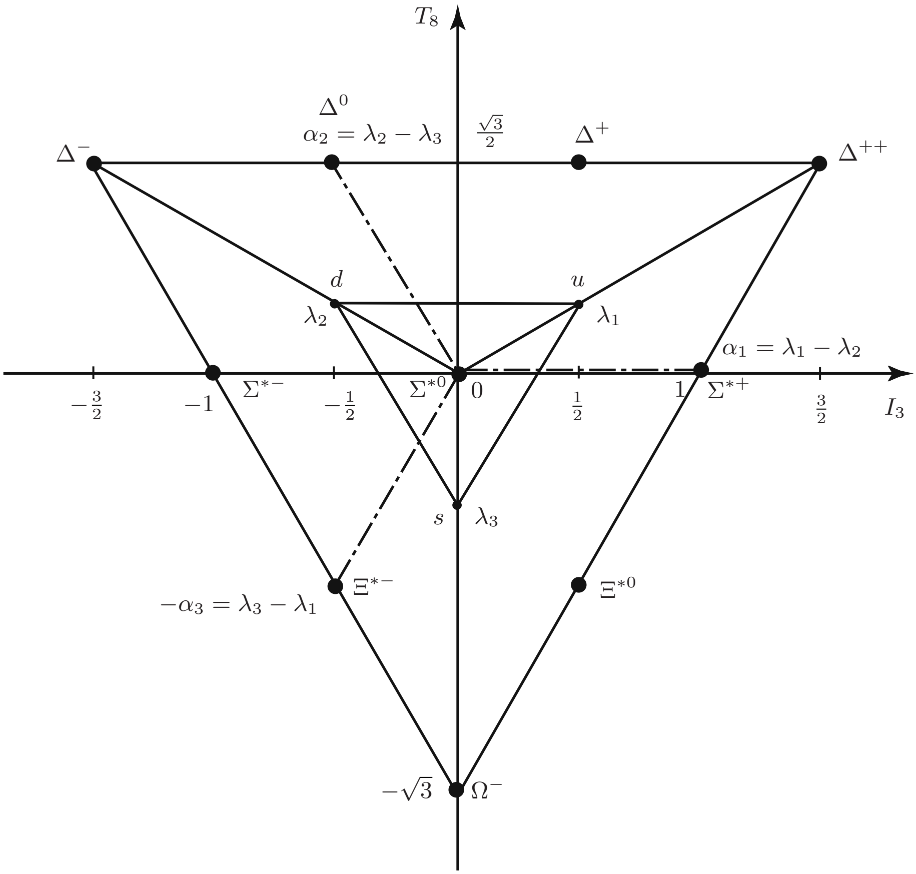

Figure 5. The representation **10**.

5 Tensor Products of Representations

Let (E_1, ρ_1) and (E_2, ρ_2) be representations of the same group G. We recall that by definition, $\rho_1 \otimes \rho_2$ is the representation of G on $E_1 \otimes E_2$ defined by

$$(\rho_1 \otimes \rho_2)(g)(v_1 \otimes v_2) = \rho_1(g)v_1 \otimes \rho_2(g)v_2,$$

for $g \in G$ and $v_1 \in E_1$, $v_2 \in E_2$. Then the differential of this representation satisfies equation (7.9) of Chapter 4,

$$D(\rho_1 \otimes \rho_2)(X)(v_1 \otimes v_2) = (D\rho_1)(X)v_1 \otimes v_2 + v_1 \otimes (D\rho_2)(X)v_2,$$

for every $X \in \mathfrak{g}$. Abusing language, we write

$$D(\rho_1 \otimes \rho_2) = D\rho_1 \otimes D\rho_2.$$

The following result is almost obvious given the formula above, but it is very important, and is in fact valid in general.

Proposition 5.1. *If v_1 is a weight vector of a representation R_1 of $\mathfrak{sl}(3,\mathbb{C})$ for a weight w_1, and if v_2 is a weight vector of a representation R_2 of $\mathfrak{sl}(3,\mathbb{C})$ for a weight w_2, then $v_1 \otimes v_2$ is a weight vector of the representation $R_1 \otimes R_2$ for the weight $w_1 + w_2$.*

Example 1: $\mathbf{3} \otimes \overline{\mathbf{3}}$. We shall treat the example of the representation $\mathbf{3} \otimes \overline{\mathbf{3}}$ graphically (see Figure 6). We see that

$$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}.$$

The weight 0 is of multiplicity 3, and the other weights are of multiplicity 1.

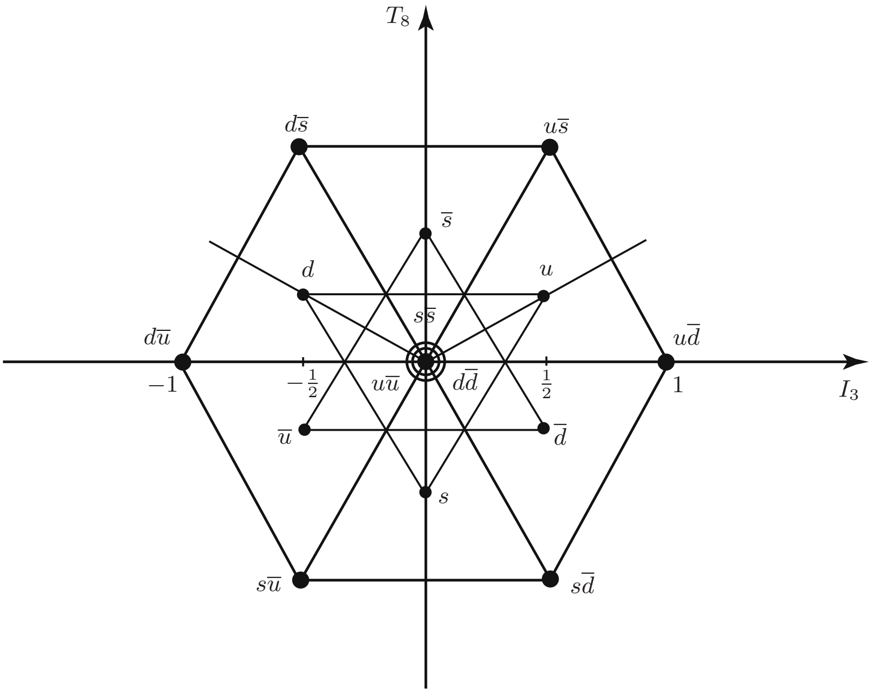

Figure 6. $\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$.

Example 2: $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$. In this tensor product, there are three weights of multiplicity 1, six weights of multiplicity 3, and the weight 0 is of multiplicity 6 (see Figure 7). We obtain

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}.$$

The table below gives the components of the weights with the corresponding weight vectors:

weight vectors						I_3	Y
					uuu	$\frac{3}{2}$	1
					ddd	$-\frac{3}{2}$	1
					sss	0	-2
			uud	udu	duu	$\frac{1}{2}$	1
			uus	usu	suu	1	0
			ddu	dud	udd	$-\frac{1}{2}$	1
			dds	dsd	sdd	-1	0
			ssu	sus	uss	$\frac{1}{2}$	-1
			ssd	sds	dss	$-\frac{1}{2}$	-1
uds	dus	sud	usd	dsu	sdu	0	0

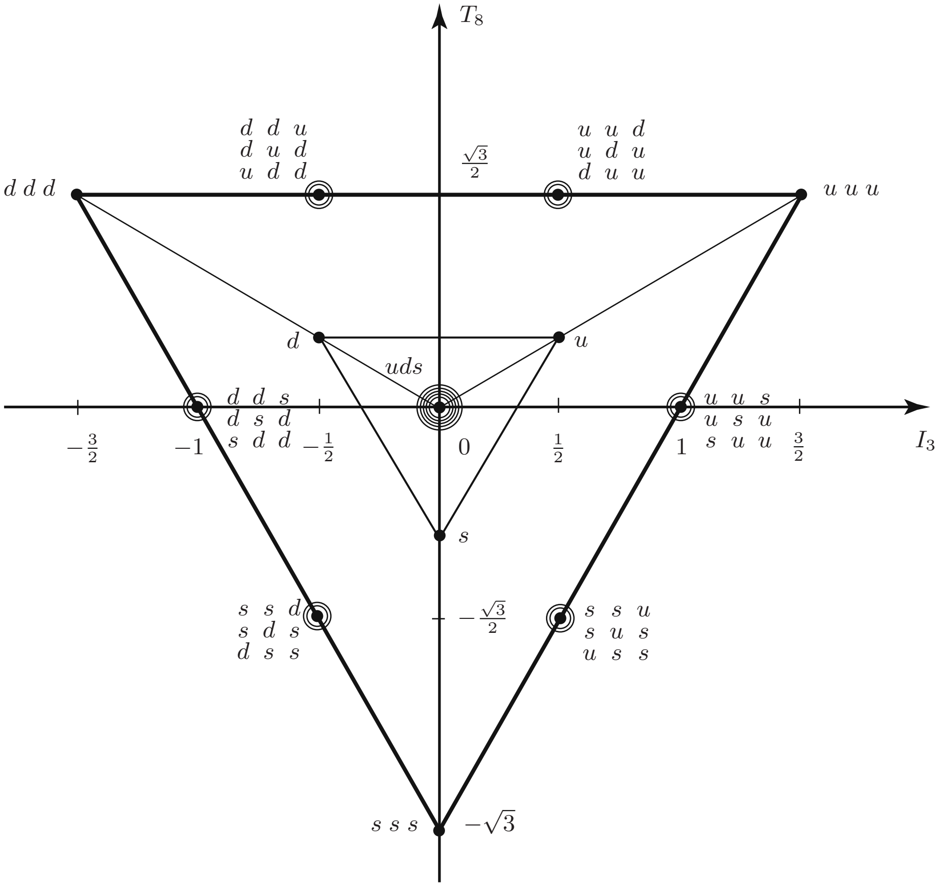

Figure 7. $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}$.

6 The Eightfold Way

Up until 1955, scientists had observed seven baryons, and an eighth was observed in 1958, the Ξ^0. On the other hand, there were seven known mesons. This set of observations led Murray Gell-Mann and Yuval Ne'eman, independently, during the winter of 1960–1961, to propose the symmetry group SU(3) for the classification of the elementary particles. The eight baryons were grouped in the octet of the adjoint representation, and the seven known mesons formed part of such an octet. The publication of Ne'eman's results took place in 1961, while the internal report of the California Institute of Technology by Gell-Mann was not published as an article in *Physical Review* until 1962. The particle η completing the octet of mesons and whose existence was predicted by this theory was discovered in 1962.

The basic idea is that each elementary particle "is" a weight vector of the representation. To each weight vector corresponds a weight that is the collection of the eigenvalues of the operators associated to the chosen basis vectors in the Cartan subalgebra. Thus each weight is defined by two numbers, the eigenvalues of H_1 and H_2, or of I_3 and Y, or of I_3 and T_8, according to the basis chosen. These two numbers are the *quantum numbers*, which characterize the particle when the weight is simple:

- the quantum number corresponding to I_3 is the third component of the *isospin*;
- the quantum number corresponding to Y is the *hypercharge*.

Then the charge is $Q = \frac{1}{2}Y + I_3$, and furthermore, $Y = B + S$, where B is the baryon number and S is the *strangeness*, which is zero for the proton and the neutron, and for the three π mesons.

The *baryon* and *meson octets* were obtained as the representation **8** of SU(3), whence the name of *eightfold way* given to the theory, echoing the "noble eightfold path" of Buddhism.

Other extremely short-lived elementary particles called *resonances* were observed in 1961, the four Δ and the three Σ^*. These corresponded to seven of the ten points of the weight diagram of the representation **10** of SU(3). The theory predicted the existence of two particles of hypercharge -1, the Ξ^*, which were discovered in 1962, as well as the particle corresponding to the tenth point, which was called Ω^-, and was discovered later, in 1964. This discovery was a great success for the SU(3) theory.

We reproduce below tables of baryons and of mesons, the corresponding diagrams (Figures 8a and 9a), and the diagram of baryon resonances (Figure 10a), all three adapted from Sternberg (1994), as well as the same diagrams (Figures 8b, 9b, and 10b) adapted from Blaizot–Tolédano (1997). We recognize immediately the weight diagram of the representation **8** for the baryons and the mesons, and that of the representation **10** for the baryon resonances. In these tables, J denotes the spin and I the isospin of the particle.

6.1 Baryons ($B = 1$)

	S	I	I_3	Q	J	Mass (MeV/c^2)	Half-life (s)
Ξ^-	-2	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{1}{2}$	1321	1.6×10^{-10}
Ξ^0	-2	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1315	2.9×10^{-10}
Σ^-	-1	1	-1	-1	$\frac{1}{2}$	1197	1.5×10^{-10}
Σ^0	-1	1	0	0	$\frac{1}{2}$	1193	7.4×10^{-20}
Σ^+	-1	1	1	1	$\frac{1}{2}$	1189	0.8×10^{-10}
Λ^0	-1	0	0	0	$\frac{1}{2}$	1116	2.6×10^{-10}
n	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	940	0.9×10^3
p	0	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	938	stable

The eight observed baryons belong to the weight diagram of the representation **8** as shown in Figures 8a, b.

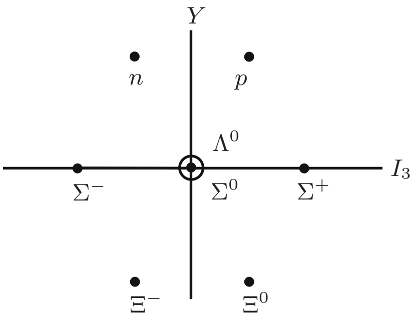

Figure 8a. The baryon octet.

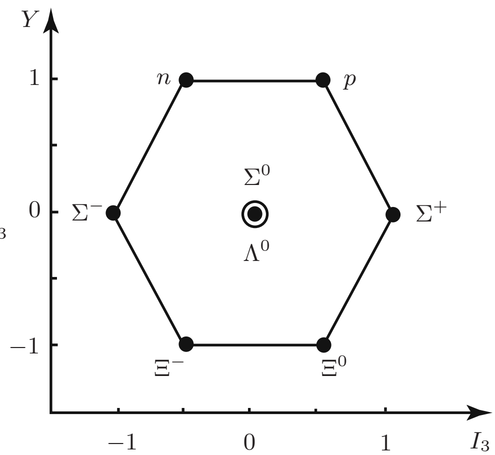

Figure 8b. The baryon octet.

6.2 Mesons ($B = 0$)

	S	I	I_3	Q	J	Mass (MeV/c^2)	Half-life (s)	
K^+	1	$\frac{1}{2}$	$\frac{1}{2}$	1	0	494	1.2×10^{-8}	
K^0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	498	$K_S = \frac{1}{\sqrt{2}}(K^0 + \overline{K}^0)$	0.9×10^{-10}
$\overline{K}^0$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	498	$K_L = \frac{1}{\sqrt{2}}(K^0 - \overline{K}^0)$	5.2×10^{-8}
K^-	-1	$\frac{1}{2}$	$-\frac{1}{2}$	-1	0	494	1.2×10^{-8}	
π^+	0	1	1	1	0	140	2.6×10^{-8}	
π^0	0	1	0	0	0	135	8.4×10^{-17}	
π^-	0	1	-1	-1	0	140	2.6×10^{-8}	

For the meson η, $S = 0$, $I = 0$, $I_3 = 0$, $Q = 0$, $J = 0$. Its mass is 548 MeV/c^2.

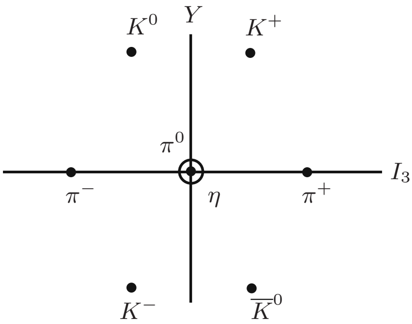

Figure 9a.
The meson octet.

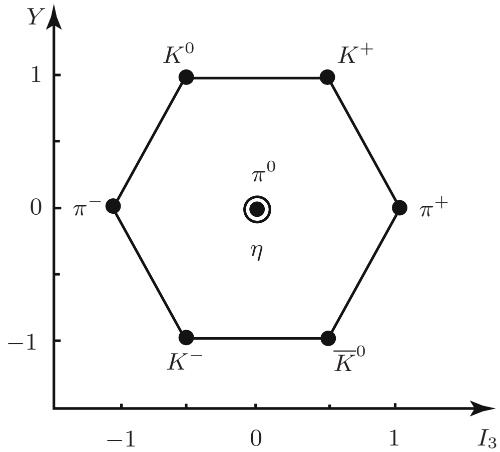

Figure 9b. The meson octet.

6.3 Baryon Resonances

The observed and predicted baryon resonances belong to the weight diagram of the representation **10** as shown in Figures 10a, b.

The baryon resonances have a very short half-life, of order 10^{-23} s for $\Delta^{++}, \Delta^{+}, \Delta^{0}$, and Δ^{-}.

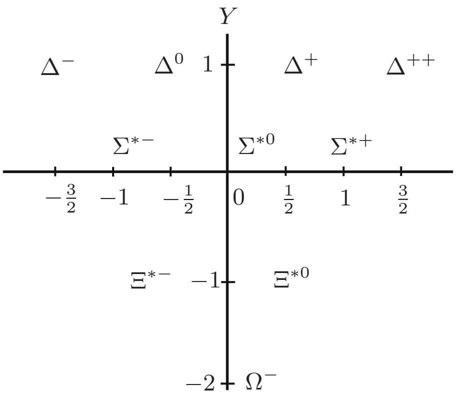

Figure 10a.
The baryon resonances.

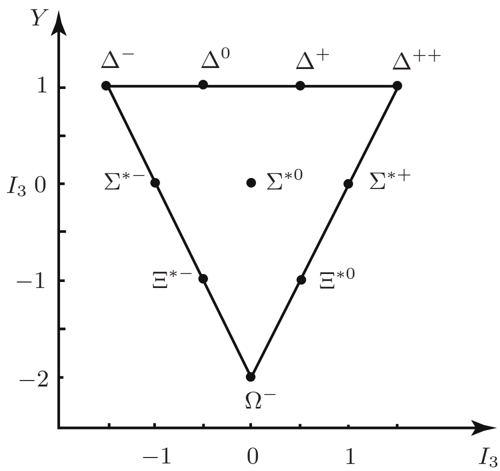

Figure 10b. The baryon resonances.

7 Quarks and Antiquarks

However, the representations $\mathbf{3}$ and $\overline{\mathbf{3}}$ did not correspond to any observed particle. The quark model allows us to consider the hadrons (baryons and mesons) as "states composed of quarks and antiquarks." Mathematically, this means the following:

- The *quarks* are the particles corresponding to the representation $\mathbf{3}$ of SU(3): the three quarks, u (*up*), d (*down*), and s (*strange*), are the weight vectors of this representation:

$\mathbf{3}$	w	$w(I_3)$	$w(Y)$	$w(T_8)$	$w(Q)$
u	$w_u = \lambda_1$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2\sqrt{3}}$	$\frac{2}{3}$
d	$w_d = \lambda_2$	$-\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2\sqrt{3}}$	$-\frac{1}{3}$
s	$w_s = \lambda_3 = -\lambda_1 - \lambda_2$	0	$-\frac{2}{3}$	$-\frac{1}{\sqrt{3}}$	$-\frac{1}{3}$

- The *antiquarks* are the particles corresponding to the representation $\overline{\mathbf{3}}$ of SU(3): the three antiquarks, $\overline{u}$, $\overline{d}$, and $\overline{s}$, are the weight vectors of this representation:

$\overline{\mathbf{3}}$	w	$w(I_3)$	$w(Y)$	$w(T_8)$	$w(Q)$
$\overline{u}$	$w_{\overline{u}} = \lambda_2 + \lambda_3 = -\lambda_1$	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{2}{3}$
$\overline{d}$	$w_{\overline{d}} = \lambda_1 + \lambda_3 = -\lambda_2$	$\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{2\sqrt{3}}$	$\frac{1}{3}$
$\overline{s}$	$w_{\overline{s}} = \lambda_1 + \lambda_2 = -\lambda_3$	0	$\frac{2}{3}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{3}$

- One obtains the meson octet in the decomposition of the tensor product $\mathbf{3} \otimes \overline{\mathbf{3}}$, more precisely, $$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1},$$

whence "a meson is composed of a quark–antiquark pair."

- We obtain the baryon octet and decuplet in the decomposition of the tensor product $$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1},$$

whence "a baryon is composed of three quarks."

These decompositions were proved graphically in Section 5 above.

To avoid difficulties due to fractional charges and an incompatibility with the Pauli exclusion principle, an additional SU(3) symmetry was in fact introduced later, corresponding to a new quantum number, the *color*, and leading to the theory of *quantum chromodynamics*.

References

Roots, Dynkin diagrams, and weights of the representations of Lie algebras are the subject of the book by Humphreys (1978). There one can find the classification of semisimple complex Lie algebras, established by Killing and Cartan at the end of the nineteenth century. (See Hawkins (2000) for a historical account.) Also see Serre (1987) and Knapp (2002). The book of Sattinger and Weaver (1993) contains applications to particle physics. The case of $\mathfrak{sl}(3,\mathbb{C})$ is treated in detail in Hall (2003). The study of the representations of low-dimensional Lie algebras precedes the general theory in Fulton–Harris (1991). See in particular p. 184 for the rule stated in Section 4.3.

For the proof of the simple connectedness of SU(3), see for example Rossmann (2002) or Hall (2003).

The tables of baryons and mesons in Section 6 are taken from K. Hagiwara et al., “Review of Particle Properties,” *Physical Review* D 66, 010001 (2002). See http://pdg.lbl.gov for more recent and much more detailed experimental results. Weight diagrams of the representations of SU(3) can be found, not only in Sternberg (1994) and Blaizot–Tolédano (1997), but also in Fulton–Harris (1991), Ludwig–Falter (1996), Tung (1985), Greiner–Müller (1989), Rougé (2005). For information on the theoretical and experimental discoveries in the years 1950–1962, see Sternberg (1994) and Rougé (2005). The original articles are reproduced in the collection edited by Gell-Mann and Ne’eman (1964). Gell-Mann first called the quarks “the leptons ν, e^-, and μ^-.” Subsequently, in 1963, he introduced the term *quark*, taken from *Finnegans Wake* by James Joyce: “Three quarks for Muster Mark!”

Exercises

Exercise 8.1 *Orthonormal basis of* $\mathfrak{h}$

(a) Show that the Killing form of $\mathfrak{sl}(3,\mathbb{C})$ is $K(X,Y) = 6\ \mathrm{Tr}XY$.
(b) Equip the Cartan subalgebra $\mathfrak{h}$ with the restriction of the Killing form of $\mathfrak{sl}(3,\mathbb{C})$. Show that (I_3, Y) and (I_3, T_8) are orthogonal bases of $\mathfrak{h}$. Find an orthonormal basis of $\mathfrak{h}$.

Exercise 8.2 *The representation* $\mathbf{3}\otimes\mathbf{3}$.

Decompose the representation $\mathbf{3}\otimes\mathbf{3}$ of SU(3) into a direct sum of irreducible representations with the help of the weight diagram.

Exercise 8.3 *The representation* **10**.

Prove that the representation **10** of SU(3) is irreducible.

Exercise 8.4 *On the representation* $\mathbf{3}\otimes\overline{\mathbf{3}}$.

Prove that the adjoint representation of $\mathfrak{sl}(3,\mathbb{C})$ is equivalent to the restriction of $\mathbf{3}\otimes\overline{\mathbf{3}}$ to the smallest invariant subspace containing the vector $u\otimes\overline{u}$.

Exercise 8.5 *Restrictions to* $\mathfrak{sl}(2,\mathbb{C})$.

The subalgebra $\mathfrak{g}$ of $\mathfrak{sl}(3,\mathbb{C})$ generated by E_1, F_1, H_1 is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. Decompose the restriction to $\mathfrak{g}$ of the adjoint representation of $\mathfrak{sl}(3,\mathbb{C})$ into a direct sum of irreducible representations of $\mathfrak{g}$. Same question for the representations **6** and **10**.

Exercise 8.6 *Representations of dimension* 15.

(a) Show that there are four inequivalent irreducible representations of dimension 15 of $\mathfrak{sl}(3,\mathbb{C})$, denoted by $\mathbf{15}$, $\overline{\mathbf{15}}$, $\mathbf{15}'$, $\overline{\mathbf{15}}'$, and draw their weight diagrams in the basis (I_3, T_8).

(b) Show that $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3} = 3\,\mathbf{3}\oplus 2\,\overline{\mathbf{6}}\oplus 3\,\mathbf{15}\oplus\mathbf{15}'$.

Eugene Wigner was born in Budapest in 1902 and died in Princeton, NJ, in 1995. He introduced the theory of groups into quantum mechanics as early as 1927 and received the Nobel Prize for physics in 1963 for his discovery of the fundamental symmetry principles in the theory of elementary particles.

(*American Institute of Physics, Emilio Segrè Visual Archives*)

Problems and Solutions

1 Restriction of a Representation to a Finite Group

Let $\mathrm{Rot}(a, \theta)$ be the rotation of the space $\mathbb{R}^3$ through the angle θ leaving invariant the unit vector a. Consider the rotations, for $p = 0, 1, 2$,

$$g_p = \mathrm{Rot}\left(e_3, \frac{2p\pi}{3}\right) \quad \text{and} \quad h_p = \mathrm{Rot}(g_p e_1, \pi),$$

where (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$. Set

$$\Gamma = \{g_0, g_1, g_2, h_0, h_1, h_2\}.$$

1. Prove the equalities

$$g_1 h_0 = h_0 g_2, \quad g_2 h_0 = h_0 g_1, \quad h_1 = h_0 g_1, \quad h_2 = h_0 g_2.$$

Show that Γ is a subgroup of the group of rotations $\mathrm{SO}(3)$ isomorphic to the symmetric group on three elements, $\mathfrak{S}_3$.

2. Find the character table of the group Γ.

For $j \in \mathbb{N}$, we denote by $\underline{\mathcal{D}}^j$ the representation of $\mathrm{SO}(3)$ obtained by passing to the quotient in the representation $\mathcal{D}^j$ of $\mathrm{SU}(2)$.

3. Show that the value of the character $\underline{\chi}_j$ of $\underline{\mathcal{D}}^j$ on rotation through an angle θ, $\theta \in (0, 2\pi]$, is

$$\underline{\chi}_j(\theta) = \frac{\sin(2j+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}}.$$

What is $\underline{\chi}_j(0)$?

4. Find, for each $j \in \mathbb{N}$, the decomposition of the restriction to Γ of the representation $\underline{\mathcal{D}}^j$ into a direct sum of irreducible representations of Γ. [Hint: Consider the cases $j = 3k$, $j = 3k+1$, $j = 3k+2$, where $k \in \mathbb{N}$.]

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Solutions

1. It is clear that g_0 is the identity and that

$$g_2 = (g_1)^2, \quad (g_p)^3 = g_0, \quad (h_p)^2 = g_0, \quad p = 0, 1, 2.$$

We have $h_0 = \exp \pi\eta_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $g_1 = \exp \frac{2\pi}{3}\eta_3 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $g_2 = \exp \frac{4\pi}{3}\eta_3 = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$. By calculating the products of the matrices, we see that $g_1h_0 = h_0g_2 = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$. We deduce that $g_2h_0g_2 = g_2g_1h_0 = g_0h_0 = h_0$, whence $g_2h_0 = h_0g_1$.

The rotation h_1 leaves g_1e_1 invariant. The same is true of h_0g_1 because $h_0g_1(g_1e_1) = h_0(g_2e_1) = g_1(h_0e_1) = g_1e_1$. On the other hand, h_1 and h_0g_1 both square to the identity; the rotations h_1 and h_0g_1 are thus the same rotation around g_1e_1 and through an angle π. Similarly, the rotation h_0g_2 leaves g_2e_1 invariant, and squares to the identity; thus it is equal to h_2. The set Γ is thus

$$\Gamma = \{g_0, g_1, g_2, h_0, h_0g_1, h_0g_2\}.$$

The symmetric group on three elements can be written

$$\mathfrak{S}_3 = \{e, c, c^2, t, tc, tc^2\}.$$

It is generated by c and t with the relations $c^3 = e$, $t^2 = e$, $ct = tc^2$. We define a bijection α of Γ onto $\mathfrak{S}_3$ by setting $\alpha(g_p) = c^p$ and $\alpha(h_p) = tc^p$. The equations shown above imply that Γ is a group, and that α is an isomorphism of groups.

2. We denote by $\underline{1}$ the trivial one-dimensional representation, by ϵ the signature representation, and by ρ_0 the irreducible representation of dimension 2 of $\mathfrak{S}_3$. The character table of Γ is then as follows:

	(1)	(2)	(3)
Γ	g_0	g_1	h_0
$\chi_{\underline{1}}$	1	1	1
χ_ϵ	1	1	- 1
χ_{ρ_0}	2	-1	0

3. The conjugacy classes of SO(3) are parametrized by $\theta \in [0, \pi]$. By definition, $\underline{\mathcal{D}}^j(\mathrm{Rot}(e_3, \theta)) = \mathcal{D}^j(A)$, where $A \in \mathrm{SU}(2)$ is such that $\varphi(A) = \mathrm{Rot}(e_3, \theta)$, where φ is the covering morphism of SU(2) onto SO(3). Because $\varphi(\pm\exp(\theta\xi_3)) = \exp(\theta\eta_3)$, we have

$$\underline{\mathcal{D}}^j(\mathrm{Rot}(e_3, \theta)) = \mathcal{D}^j \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix}.$$

Thus $\underline{\chi}_j(\theta) = \chi_j(\theta/2)$, where $\chi_j(t)$ is the value of the character of $\mathcal{D}^j$ on the matrix $\left(\begin{smallmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{smallmatrix}\right)$, that is, $\sin(2j+1)t/\sin t$. Consequently,

$$\underline{\chi}_j(\theta) = \frac{\sin(2j+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}}, \quad 0 \le \theta \le \pi.$$

4. We thus have the following characters:

Γ	(1) g_0	(2) g_1	(3) h_0
$\underline{\chi}_{3k}$	6k+1	1	$(-1)^k$
$\underline{\chi}_{3k+1}$	6k+3	0	$(-1)^{k+1}$
$\underline{\chi}_{3k+2}$	6k+5	-1	$(-1)^k$

The restriction to the subgroup Γ of the representation $\underline{\mathcal{D}}^j$ can be written

$$\underline{\mathcal{D}}^j\big|_\Gamma = m_{\underline{1}}\underline{1} \oplus m_\epsilon \epsilon \oplus m_{\rho_0}\rho_0,$$

where for $\rho = \underline{1}, \epsilon, \rho_0$,

$$m_\rho = (\underline{\chi}_j \mid \chi_\rho) = \frac{1}{6}\Big((2j+1)\chi_\rho(g_0) + 2\underline{\chi}_j(g_1)\chi_\rho(g_1) + 3\underline{\chi}_j(h_0)\chi_\rho(h_0)\Big).$$

Let $j = 3k + r$, where k is an integer and $r = 0, 1$, or 2. The calculation of the coefficients $m_{\underline{1}}$, m_ϵ, and m_{ρ_0} gives us the following decomposition. If j is even (k and r have the same parity), then

$$\underline{\mathcal{D}}^j\big|_\Gamma = (k+1)\underline{1} \oplus k\epsilon \oplus (2k+r)\rho_0.$$

If j is odd (k and r have opposite parity), then

$$\underline{\mathcal{D}}^j\big|_\Gamma = k\underline{1} \oplus (k+1)\epsilon \oplus (2k+r)\rho_0.$$

In each case we verify that the direct sum is of dimension $2j + 1$.

2 The Group O(2)

Let O(2) be the group of rotations and symmetries of the plane preserving the origin, generated by the matrices

$$r_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \ 0 \le \theta < 2\pi, \quad \text{and} \quad s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

1. Find the conjugacy classes of O(2).
2. (a) Show that each irreducible representation of O(2) is of dimension ≤ 2.
   [Hint: Consider the restriction of the representation to the subgroup SO(2).]
   (b) Find the representations of dimension 1 of O(2).
   (c) Show that the two-dimensional irreducible representations of O(2), up to equivalence, are such that

$$\pi_n(r_\theta) = \begin{pmatrix} e^{in\theta} & 0 \\ 0 & e^{-in\theta} \end{pmatrix},$$

for some integer $n \geq 1$. Find the character χ_{π_n} of π_n.

3. Let Ad be the adjoint representation of O(2) in its Lie algebra. Show that it is equivalent to one of the representations studied in Question 2.

4. Decompose $\pi_n \otimes \pi_m$, for $n \geq 1$ and $m \geq 1$, into a direct sum of irreducible representations.

5. (a) Show that $r_{\frac{2\pi}{3}}$ and s generate a subgroup G of O(2) isomorphic to the symmetric group $\mathfrak{S}_3$ on three elements.
   (b) Find, for every integer $n \geq 1$, the decomposition of the restriction σ_n of π_n to the subgroup G as a direct sum of irreducible representations. Give a geometric interpretation of the result.

Solutions

1. The equations $s^2 = I$ and $sr_\theta s^{-1} = r_{-\theta}$ imply $r_\theta s = sr_{-\theta}$, whence $r_\theta sr_{-\theta} = sr_{-2\theta}$. Thus the conjugacy class of s is $\{sr_\theta \mid \theta \in [0, 2\pi)\}$. On the other hand, r_θ and $r_{\theta'}$ are conjugate if and only if $\theta' = \pm\theta$. Thus the other conjugacy classes of O(2) are $\{I\}$, $\{-I\}$, and $\{r_\theta, r_{-\theta}\}$, for $0 < \theta < \pi$.

2. (a) Let ρ be an irreducible representation of O(2) on $\mathbb{C}^d$. Because SO(2) is abelian, the restriction of ρ to SO(2) is a direct sum of representations of dimension 1. Let $x \in \mathbb{C}^d$ be a vector generating one of the lines invariant under the restriction of ρ. The vector space generated by x and $\rho(s)x$ is invariant under O(2), since $\rho(r_\theta)\rho(s)x = \rho(s)\rho(r_{-\theta})x$ is collinear with $\rho(s)x$. Because the representation is assumed irreducible, this is the whole space, and thus $d \leq 2$. The representation is of dimension 1 if and only if x and $\rho(s)x$ are collinear.

(b) Let $\rho : \mathrm{O}(2) \to \mathbb{C}$ be a representation of dimension 1 of O(2). One has either $\rho(s) = 1$ or $\rho(s) = -1$. Because $\rho(sr_\theta) = \rho(s)$, we must have $\rho(r_\theta) = 1$, for every $\theta \in [0, 2\pi)$. Thus, either $\rho = \chi_1$ (the *trivial representation*) or $\rho = \chi_1'$ (the *determinant representation*).

(c) Let $\rho : \mathrm{O}(2) \to \mathrm{GL}(\mathbb{C}^2)$ be a representation of dimension 2 of O(2). There exists $x \in \mathbb{C}^2$ such that x and $\rho(s)x$ are linearly independent (otherwise the representation is of dimension 1). In the basis $(x, \rho(s)x)$ of $\mathbb{C}^2$, the matrix of $\rho(s)$ is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Because the restriction of ρ to SO(2) is a direct sum of representations of dimension 1, there are integers n and m such that the matrix of $\rho(r_\theta)$ in the basis $(x, \rho(s)x)$ of $\mathbb{C}^2$ is $\begin{pmatrix} e^{in\theta} & 0 \\ 0 & e^{im\theta} \end{pmatrix}$. The equation $r_\theta s = sr_{-\theta}$ implies $m = -n$. We verify that for every integer n, $\pi_n(s) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\pi_n(r_\theta) = \begin{pmatrix} e^{in\theta} & 0 \\ 0 & e^{-in\theta} \end{pmatrix}$ defines a two-dimensional representation of O(2). The representations π_n and π_{-n} are equivalent.

The only lines in $\mathbb{C}^2$ invariant under $\pi_n(s)$ are $\mathbb{C}(e_1 + e_2)$ and $\mathbb{C}(e_1 - e_2)$. These are invariant under $\pi_n(r_\theta)$ for every θ if and only if $n = 0$. We have $\pi_0 = \chi_1 \oplus \chi_1'$. For $n \geq 1$, π_n is irreducible. The character χ_{π_n} of π_n is such that

$$\chi_{\pi_n}(r_\theta) = 2\cos n\theta, \quad \chi_{\pi_n}(s) = 0.$$

3. The Lie algebra of O(2) is $\mathfrak{o}(2) = \mathfrak{so}(2)$, a real one-dimensional abelian Lie algebra. We can choose the basis $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ in $\mathfrak{o}(2)$ to identify $\mathfrak{o}(2)$ with $\mathbb{R}$. Thus $\mathrm{Ad} : \mathrm{O}(2) \to \mathrm{GL}(\mathfrak{o}(2))$ is a representation of O(2) of dimension 1. It must be equivalent to either χ_1 or χ_1'. We have $\mathrm{Ad}_s : X \in \mathfrak{o}(2) \mapsto sXs^{-1} \in \mathfrak{o}(2)$ and

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

thus $\mathrm{Ad}_s = -1$. Hence $\mathrm{Ad} \sim \chi_1'$, the determinant representation.

4. Let $n \geq 1$ and $m \geq 1$ be integers. We have

$$\chi_{\pi_n \otimes \pi_m} = \chi_{\pi_n} \chi_{\pi_m}.$$

This character is zero on s and satisfies

$$(\chi_{\pi_n} \chi_{\pi_m})(r_\theta) = 4\cos n\theta \cos m\theta = 2(\cos(n+m)\theta + \cos(|n-m|\theta)).$$

If $n \neq m$, we find that

$$\chi_{\pi_n} \otimes \chi_{\pi_m} = \chi_{\pi_{n+m}} + \chi_{\pi_{|n-m|}},$$

whence

$$\pi_n \otimes \pi_m = \pi_{n+m} \oplus \pi_{|n-m|}.$$

If $n = m$, then $(\chi_{\pi_n} \chi_{\pi_n})(r_\theta) = \chi_{\pi_{2n}}(r_\theta) + 2$. The two-dimensional representation π_0 whose character takes the value 2 on r_θ and 0 on s is reducible, with $\pi_0 = \chi_1 \oplus \chi_1'$. We then have

$$\pi_n \otimes \pi_n = \pi_{2n} \oplus \chi_1 \oplus \chi_1'.$$

5. (a) We set $c = r_{\frac{2\pi}{3}}$. We verify that c and s satisfy the relations $c^3 = I$, $s^2 = I$, and $c^2 s = sc$. The subgroup G of O(2) generated by c and s is the group of six elements

$$\{e, c, c^2, s, sc, sc^2\}$$

isomorphic to $\mathfrak{S}_3$.

(b) The representation $\sigma_n = \pi_n\big|_G$ decomposes as a direct sum of representations of $G \approx \mathfrak{S}_3$. We have

$$\chi_{\sigma_n}(c) = 2\cos\frac{2n\pi}{3} = \begin{cases} 2 & \text{if } n \equiv 0 \pmod 3, \\ -1 & \text{if } n \not\equiv 0 \pmod 3. \end{cases}$$

On the other hand, the restriction of χ_1 to $\mathfrak{S}_3$ is the trivial representation $\underline{1}$ while the restriction of χ_1' is the signature representation ϵ. The restriction σ_1 of π_1 is the two-dimensional irreducible representation of $\mathfrak{S}_3$. One obtains the following character table:

	e	c	s
$\underline{1}$	1	1	1
ϵ	1	1	-1
σ_1	2	-1	0
$n \equiv 0 \pmod 3)$ σ_n	2	2	0
$n \not\equiv 0 \pmod 3)$ σ_n	2	-1	0

Hence, for $k \in \mathbb{N}, \sigma_{3k} = \underline{1} \oplus \epsilon, \sigma_{3k+1} = \sigma_{3k+2} = \sigma_1$.

Geometric interpretation. If $n \equiv 0 \pmod 3$, the rotations $c = r_{\frac{2\pi}{3}}$ and $c^2 = r_{\frac{4\pi}{3}}$ act by π_n as the identity on the plane $\mathbb{C}^2$. On the other hand, $\pi_n(s)$ always leaves invariant the line D_1 generated by the vector $e_1 + e_2$ and the line D_2 generated by the vector $e_1 - e_2$. More precisely $\pi_n(s)(e_1 + e_2) = e_1 + e_2$ and $\pi_n(s)(e_1 - e_2) = -(e_1 - e_2)$. Because these two lines are invariant under σ_n, we have

$$\sigma_n\,|_{D_1} = \underline{1} \quad \text{and} \quad \sigma_n\,|_{D_2} = \epsilon.$$

If $n \not\equiv 0 \pmod 3$, then $\sigma_n(c) = \pi_n(r_{\frac{2\pi}{3}})$ leaves none of the lines in $\mathbb{C}^2$ invariant. Thus σ_n is a two-dimensional irreducible representation of $\mathfrak{S}_3$. This is the complexification of the representation of $\mathfrak{S}_3$ in $\mathbb{R}^2$ by the symmetries of an equilateral triangle centered at the origin: rotations through the angles $2\pi/3$ and $4\pi/3$ and symmetries with respect to the three altitudes.

3 Representations of the Dihedral and Quaternion Groups

Let n be an integer, $n \geq 2$. We denote by C_n the cyclic group of order n and by $D_{(n)}$ the dihedral group of order $2n$. We know that

$$C_n = \{I, r, r^2, \ldots, r^p, \ldots, r^{n-1}\},$$
$$D_{(n)} = \{I, r, r^2, \ldots, r^{n-1}, s, sr, sr^2, \ldots, sr^{n-1}\},$$

where $r^n = I, s^2 = I$, and $srs = r^{-1}$.

Part I

For $0 \leq k \leq n-1$, we denote by χ_k the character of C_n such that

$$\chi_k(r) = e^{\frac{2i\pi k}{n}}.$$

1. What are the inequivalent irreducible representations of C_n?

2. Find the conjugacy classes of $D_{(n)}$. [Hint: Distinguish the case of n odd from the case of n even.]

3. Find the one-dimensional representations of $D_{(n)}$.

For each k, $0 \leq k \leq n-1$, we consider the induced representation of the representation χ_k from C_n to $D_{(n)}$, which we denote by ρ_k.

4. (a) Show that the representation ρ_k is of dimension 2. Choose a basis of the vector space of this representation and write the matrix of $\rho_k(r)$ and the matrix of $\rho_k(s)$ in this basis.

   (b) Find the character of ρ_k.

5. (a) Show that ρ_k and ρ_{n-k} are equivalent. Is the representation ρ_0 irreducible? Show that if n is even, with $n = 2m$, the representation ρ_m is not irreducible.

   (b) Find the inequivalent irreducible representations of $D_{(n)}$.

Part II

In the group O(2), each element can be written r_θ or $s_0 r_\theta$, $0 \leq \theta < 2\pi$, with

$$r_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{and} \quad s_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We set

$$\widehat{r_\theta} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \widehat{s_0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

6. (a) What is the geometric interpretation of $\widehat{r_\theta}$ and of $\widehat{s_0}$?

   (b) Show that there is a unique injective group morphism $\psi : \mathrm{O}(2) \to \mathrm{SO}(3)$ such that
   $$\psi(r_\theta) = \widehat{r_\theta} \quad \text{and} \quad \psi(s_0) = \widehat{s_0}.$$

Let φ be the projection (covering morphism) from SU(2) onto SO(3). We introduce the matrices $\mathcal{I} = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$, $\mathcal{J} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and $\mathcal{K} = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$.

7. (a) What is the inverse image under φ of $\widehat{r_\theta}$? of $\widehat{s_0}$?

   (b) Show that $D_{(n)}$ can be identified with a subgroup of O(2) generated by a rotation and s_0. What is the order of the group $\Gamma_n = \varphi^{-1}(\psi(D_{(n)}))$? What are the elements of this group?

8. Show that Γ_2 is the quaternion group,
$$\{\pm I, \pm\mathcal{I}, \pm\mathcal{J}, \pm\mathcal{K}\}.$$

9. Create the character table of the group Γ_2.

Let $\mathcal{R}^{(N)}$, where N is a nonnegative integer, be the representation of Γ_2 obtained as the restriction to Γ_2 of the irreducible representation of dimension $N+1$ of SU(2).

10. (a) Find the character of $\mathcal{R}^{(N)}$.
 (b) Find the decomposition of $\mathcal{R}^{(N)}$ as a direct sum of irreducible representations of Γ_2.
 (c) Same question for each of the representations $\rho \otimes \mathcal{R}^{(1)}$, where ρ is an irreducible representation of Γ_2.

Solutions

Part I

1. Because C_n is an abelian group, all the irreducible representations of C_n are one-dimensional. The group C_n thus has $|C_n| = n$ inequivalent irreducible representations: these are the characters χ_k, $0 \leq k \leq n-1$, such that for $0 \leq p \leq n-1$, $\chi_k(r^p) = e^{\frac{2i\pi kp}{n}}$.

2. From $srs = r^{-1}$ we deduce the equation $sr = r^{-1}s$, whence for every integer p,

$$sr^p s = r^{-p}.$$

(i) *Conjugacy classes of rotations.* We deduce from the preceding formula that r^p and r^{n-p} are conjugate in $D_{(n)}$. For $r^{p'}$ to be conjugate to r^p, it is necessary and sufficient that there be an integer q such that $r^{p'} = sr^q r^p sr^q$, which is equivalent to $r^{p'} = sr^p s$, or to $p + p' \equiv 0 \pmod{n}$.

- If n is odd, $n = 2m+1$, there are $m+1 = \frac{n+1}{2}$ conjugacy classes of rotations,
$$\{1\}, \{r, r^{n-1}\}, \dots, \{r^m, r^{m+1}\}.$$
- If n is even, $n = 2m$, there are $m = \frac{n}{2}+1$ conjugacy classes of rotations,
$$\{1\}, \{r, r^{n-1}\}, \dots, \{r^{m-1}, r^{m+1}\}, \{r^m\}.$$

(ii) *Conjugacy classes of symmetries.* For $sr^{p'}$ to be conjugate to sr^p, it is necessary and sufficient that there be an integer q such that $sr^{p'} = r^q sr^p r^{-q}$ or $sr^{p'} = sr^q sr^p r^{-q} s$. The first condition gives $p' - p + 2q \equiv 0 \pmod{n}$, and the second condition gives $p' + p - 2q \equiv 0 \pmod{n}$.

- If n is odd, such a choice of q is always possible. Thus there is only one conjugacy class of symmetries, $\{s, sr, \dots, sr^{n-1}\}$.
- If n is even, there are two conjugacy classes of symmetries, $\{s, sr^2, \dots, sr^{n-2}\}$ and $\{sr, sr^3, \dots, sr^{n-1}\}$, each having $\frac{n}{2}$ elements.

To summarize, if n is odd, there are $\frac{n+3}{2}$ conjugacy classes,

$$\begin{array}{ccccc} (1) & (2) & & (2) & (n) \\ 1 & r & \dots & r^{\frac{n-1}{2}} & s \end{array}$$

and if n is even, there are $\frac{n}{2}+3$ conjugacy classes,

$$\begin{array}{ccccccc} (1) & (2) & & (2) & (1) & (\frac{n}{2}) & (\frac{n}{2}) \\ 1 & r & \dots & r^{\frac{n}{2}-1} & r^{\frac{n}{2}} & s & sr \end{array}$$

In both cases, we verify that there are $2n$ elements in the group $D_{(n)}$.

3. Let $\pi : D_{(n)} \to \mathbb{C}^*$ be a one-dimensional representation of $D_{(n)}$. Let $a = \pi(r)$ and $b = \pi(s)$. It must be true that

$$a^n = 1, \quad b^2 = 1, \quad bab = a^{-1}.$$

Hence $a = \pm 1$ and $b = \pm 1$.

If n is odd, we must have $a = +1$; there are two one-dimensional representations: the trivial representation π_1 and the representation π_1' defined by

$$\pi_1'(r) = 1, \quad \pi_1'(s) = -1.$$

If n is even, there are four one-dimensional representations: π_1, π_1', and π_2, π_2' defined by

$$\begin{aligned} \pi_2(r) &= -1, & \pi_2(s) &= 1, \\ \pi_2'(r) &= -1, & \pi_2'(s) &= -1, \end{aligned}$$

whence $\pi_2(r^p) = \pi_2'(r^p) = (-1)^p,\ \pi_2(sr^p) = (-1)^p,\ \pi_2'(sr^p) = (-1)^{p+1}$.

4. (a) By definition, the space E_k of the representation ρ_k is the vector space of maps f of $D_{(n)}$ into $\mathbb{C}$ such that

$$\forall g \in D_{(n)}, \forall h \in C_n,\ \ f(gh) = \chi_k(h^{-1})f(g),$$

i.e., $f(gr^p) = e^{-\frac{2i\pi kp}{n}} f(g)$, for $0 \leq p \leq n-1$. Therefore

$$f(r^p) = e^{-\frac{2i\pi kp}{n}} f(I)$$

and

$$f(sr^p) = e^{-\frac{2i\pi kp}{n}} f(s),$$

and the function f is entirely determined by $f(I)$ and $f(s)$, which are arbitrary complex numbers. Thus the representation ρ_k is of dimension2.

Let f_1 be the element of E_k such that $f_1(I) = 1, f_1(s) = 0$ and let f_2 be the element of E_k such that $f_2(I) = 0$, $f_2(s) = 1$. Then (f_1, f_2) is a basis of E_k.

By definition, for $f \in E_k$ and $g_0, g \in D_{(n)}$,

$$(\rho_k(g_0)f)(g) = f(g_0^{-1}g),$$

whence

$$\begin{aligned} (\rho_k(r)f)(I) &= f(r^{-1}) = \chi_k(r)f(I) = e^{\frac{2i\pi k}{n}} f(I), \\ (\rho_k(r)f)(s) &= f(r^{-1}s) = f(sr) = \chi_k(r^{-1})f(s) = e^{-\frac{2i\pi k}{n}} f(s). \end{aligned}$$

Thus $\rho_k(r)f_1 = e^{\frac{2i\pi k}{n}} f_1$ and $\rho_k(r)f_2 = e^{-\frac{2i\pi k}{n}} f_2$, whence the matrix of $\rho_k(r)$ in the basis (f_1, f_2) of E_k is given by

$$\rho_k(r) = \begin{pmatrix} e^{\frac{2i\pi k}{n}} & 0 \\ 0 & e^{-\frac{2i\pi k}{n}} \end{pmatrix}.$$

Similarly, $(\rho_k(s)f)(I) = f(s)$ and $(\rho_k(s)f)(s) = f(I)$, whence $\rho_k(s)f_1 = f_2$ and $\rho_k(s)f_2 = f_1$. Hence we have

$$\rho_k(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(b) We have

$$\rho_k(r^p) = \begin{pmatrix} e^{\frac{2i\pi kp}{n}} & 0 \\ 0 & e^{-\frac{2i\pi kp}{n}} \end{pmatrix}$$

and

$$\rho_k(sr^p) = \begin{pmatrix} 0 & e^{-\frac{2i\pi kp}{n}} \\ e^{\frac{2i\pi kp}{n}} & 0 \end{pmatrix}.$$

The character of ρ_k is thus given by

$$\begin{cases} \mathrm{Tr}(\rho_k(r^p)) = 2\cos\frac{2kp\pi}{n}, & 0 \leq p \leq n-1 \;, \\ \mathrm{Tr}(\rho_k(sr^p)) = 0. \end{cases}$$

We verify that it takes the same value on each conjugacy class of $D_{(n)}$.

5. (a) The representations ρ_k and ρ_{n-k} are inequivalent since they have same character.

The representation ρ_0 is not irreducible. In fact, $\rho_0 = \pi_1 \oplus \pi_1'$.

If n is even, $n = 2m$, the character of ρ_m takes the values $2(-1)^p$ at r^p and 0 at sr^p. We have $\chi_{\rho_m} = \chi_{\pi_2} + \chi_{\pi_2'}$, whence $\rho_m = \pi_2 \oplus \pi_2'$.

(b) (i) Case of odd n, $n = 2m+1$. The representations $\rho_0, \rho_1, \ldots, \rho_{m-1}, \rho_m$ are inequivalent, since the values of their characters at r are different.

Each of the representations ρ_k, $1 \leq k \leq m$, is irreducible. In fact, if $1 \leq k \leq m$, the only vector subspaces of E_k invariant under $\rho_k(r)$ are the lines $\mathbb{C}f_1$ and $\mathbb{C}f_2$, and these are not invariant under $\rho_k(s)$. [One can also show that the modulus of the character of each of these representations is equal to 1.]

The representations $\pi_1, \pi_1', \rho_1, \ldots, \rho_m$ are $2 + m = \frac{n+3}{2}$ inequivalent irreducible representations. This is the number of conjugacy classes of $D_{(n)}$ for n odd. Thus we have determined all the irreducible representations of $D_{(n)}$. [Another argument: the sum of the squares of the dimensions of these representations is $(\dim \pi_1)^2 + (\dim \pi_1')^2 + \sum_{k=1}^{m}(\dim \rho_k)^2 = 2 + 4m = 2n$, which is the order of $D_{(n)}$.]

(ii) Case of even n, $n = 2m$. The representations $\rho_1, \ldots, \rho_{m-1}$ are inequivalent and irreducible (see the odd case).

The representations $\pi_1, \pi_1', \pi_2, \pi_2', \rho_1, \ldots, \rho_{m-1}$ are $4 + m - 1 = \frac{n}{2} + 3$ inequivalent irreducible representations, and this number is the number of conjugacy classes of $D_{(n)}$ for even n. We then have the complete list. [Another argument: the sum of the squares of the dimensions of these representations is $4 + 4(m-1) = 2n$, which is the order of $D_{(n)}$.]

In both cases, the irreducible representations of $D_{(n)}$ are one- or two-dimensional.

Part II

6. (a) The transformation $\widehat{r_\theta}$ of $\mathbb{R}^3$ is the rotation around the axis Oz and through an angle θ, $\mathrm{Rot}(e_3, \theta)$, and $\widehat{s_0}$ is the symmetry with respect to the axis Ox, $\mathrm{Rot}(e_1, \pi)$.

(b) We see immediately that $\widehat{r_\theta}$ and $\widehat{s_0}$ belong to SO(3). We set

$$\psi(s_0 r_\theta) = \widehat{s_0}\widehat{r_\theta}.$$

Thus we define an injective map from O(2) into SO(3). It is clear that $\widehat{r_\theta}\widehat{r_{\theta'}} = \widehat{r_\theta r_{\theta'}} = \widehat{r_{\theta+\theta'}}, \quad (\widehat{s_0})^2 = (\widehat{s_0^2}) = I$. The matrix product $\widehat{s_0}\widehat{r_\theta}\widehat{s_0}$ is equal to $\widehat{r_{-\theta}} = (\widehat{r_\theta})^{-1}$. We deduce that ψ is a group morphism. In fact,

$$\psi(r_\theta(s_0 r_{\theta'})) = \psi(s_0 r_{\theta'-\theta}) = \widehat{s_0}\widehat{r_{\theta'-\theta}},$$

while

$$\psi(r_\theta)\psi(s_0 r_{\theta'}) = \widehat{r_\theta}\widehat{s_0}\widehat{r_{\theta'}} = \widehat{s_0}\widehat{r_{-\theta}}\widehat{r_{\theta'}} = \widehat{s_0}\widehat{r_{\theta'-\theta}}.$$

Similarly,

$$\psi((s_0 r_\theta)(s_0 r_{\theta'})) = \psi(r_{\theta'-\theta}) = \widehat{r_{\theta'-\theta}},$$

while

$$\psi(s_0 r_\theta)\psi(s_0 r_{\theta'}) = \widehat{s_0}\widehat{r_\theta}\widehat{s_0}\widehat{r_{\theta'}} = \widehat{r_{\theta'-\theta}}.$$

Geometrically, we associate to the rotation through an angle θ in the plane xOy the rotation of $\mathbb{R}^3$ with the same angle and with axis Oz, and to the symmetry s_0 of the plane xOy that exchanges the axes, the symmetry of $\mathbb{R}^3$ with respect to the axis x. (While in $\mathbb{R}^2$ a symmetry with respect to a line has determinant -1, a symmetry with respect to a line in $\mathbb{R}^3$ is a rotation through an angle π, with determinant $+1$.)

7. (a) We know that on the one hand, $\varphi(\exp \frac{\theta}{2}\mathcal{K}) = \mathrm{Rot}\,(e_3, \theta) = \widehat{r_\theta}$, and on the other hand, $\varphi(\exp \frac{\pi}{2}\mathcal{I}) = \mathrm{Rot}(e_1, \pi) = \widehat{s_0}$. We then have

$$\varphi^{-1}(\widehat{r_\theta}) = \left\{\exp\frac{\theta}{2}\mathcal{K}, -\exp\frac{\theta}{2}\mathcal{K}\right\},$$
$$\varphi^{-1}(\widehat{s_0}) = \left\{\exp\frac{\pi}{2}\mathcal{I}, -\exp\frac{\pi}{2}\mathcal{I}\right\}.$$

We have $\exp t\mathcal{I} = \cos t I + \sin t\mathcal{I}$, whence $\exp\frac{\pi}{2}\mathcal{I} = \mathcal{I}$. Thus

$$\varphi^{-1}(\widehat{s_0}) = \{\mathcal{I}, -\mathcal{I}\}.$$

Similarly, $\exp t\mathcal{K} = \cos t I + \sin t\mathcal{K}$, whence

$$\varphi^{-1}(\widehat{r_\theta}) = \left\{\cos\frac{\theta}{2}I + \sin\frac{\theta}{2}\mathcal{K}, -\cos\frac{\theta}{2}I - \sin\frac{\theta}{2}\mathcal{K}\right\}.$$

(b) The group $D_{(n)}$ can be identified with the subgroup of O(2) generated by the rotation $r_{\frac{2\pi}{n}}$ and the symmetry s_0.

The group $\psi(D_{(n)})$ is of the same order as $D_{(n)}$, i.e., $2n$. The group $\Gamma_n = \varphi^{-1}(\psi(D_{(n)}))$ is of order $4n$, since for each element g of $\psi(D_{(n)})$, $\varphi^{-1}(g)$ contains two opposite elements of SU(2).

The group $\psi(D_{(n)})$ is generated by $\widehat{r_{\frac{2\pi}{n}}}$ and $\widehat{s_0}$. It consists of transformations of $\mathbb{R}^3$, $\widehat{r_{\frac{2p\pi}{n}}}$ and $\widehat{s_0 r_{\frac{2p\pi}{n}}}$, for $0 \le p \le n-1$. The group $\Gamma_n = \varphi^{-1}(\psi(D_{(n)}))$ is thus generated by the matrices of SU(2), $\pm(\cos\frac{\pi}{n}I + \sin\frac{\pi}{n}\mathcal{K})$, $\mathcal{I}$, and $-\mathcal{I}$. It consists of the matrices

$$\cos\frac{p\pi}{n}I + \sin\frac{p\pi}{n}\mathcal{K}, \quad -\cos\frac{p\pi}{n}I - \sin\frac{p\pi}{n}\mathcal{K},$$

and the matrices

$$\cos\frac{p\pi}{n}\mathcal{I} - \sin\frac{p\pi}{n}\mathcal{J}, \quad -\cos\frac{p\pi}{n}\mathcal{I} + \sin\frac{p\pi}{n}\mathcal{J},$$

for $0 \le p \le n-1$.

8. If $n = 2$, the group $\Gamma_2 = \varphi^{-1}(\psi(D_{(2)}))$ consists of the matrices

$$I,\ -I,\ \mathcal{I},\ -\mathcal{I},\ \mathcal{J},\ -\mathcal{J},\ \mathcal{K},\ -\mathcal{K}.$$

Therefore it is the quaternion group. We can study the case $n = 2$ directly. The group $D_{(2)}$ is the group $\{I, r, s, sr\}$, with $r^2 = s^2 = I,\ rs = sr$. It can be realized in $\mathbb{R}^2$ as the group

$$\left\{ I, r_\pi = -I,\ s_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ s_0 r_\pi = -s_0 \right\},$$

and in $\mathbb{R}^3$ as the group

$$\left\{ I, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}.$$

We have $\varphi^{-1}(I) = \{I, -I\}$. (The letter I sometimes denotes the identity on $\mathbb{R}^2$, sometimes the identity on $\mathbb{R}^3$ and sometimes the identity on $\mathbb{C}^2$.) On the other hand,

$$\begin{aligned} \varphi^{-1}(\widehat{r_\pi}) &= \{\mathcal{K}, -\mathcal{K}\}, \\ \varphi^{-1}(\widehat{s_0}) &= \{\mathcal{I}, -\mathcal{I}\}, \\ \varphi^{-1}(\widehat{s_0 r_\pi}) &= \{\mathcal{J}, -\mathcal{J}\}, \quad \text{because } \mathcal{K}\mathcal{I} = \mathcal{J}. \end{aligned}$$

Thus we obtain the list of elements of the group Γ_2.

9. We note that $\mathcal{I}^{-1} = -\mathcal{I},\ \mathcal{J}^{-1} = -\mathcal{J},\ \mathcal{K}^{-1} = -\mathcal{K}$.

The conjugacy classes of Γ_2 are

$$\{I\},\ \{-I\},\ \{\mathcal{I}, \mathcal{I}^{-1}\},\ \{\mathcal{J}, \mathcal{J}^{-1}\},\ \{\mathcal{K}, \mathcal{K}^{-1}\}.$$

In fact, $\mathcal{J}\mathcal{I}\mathcal{J}^{-1} = -\mathcal{J}\mathcal{I}\mathcal{J} = -\mathcal{J}\mathcal{K} = -\mathcal{I}$, and $\mathcal{K}\mathcal{I}\mathcal{K}^{-1} = -\mathcal{K}\mathcal{I}\mathcal{K} = -\mathcal{J}\mathcal{K} = -\mathcal{I}$. These calculations show that $\mathcal{I}, \mathcal{J}, \mathcal{K}$ are pairwise not conjugate.

There are four one-dimensional representations of Γ_2. To show this, let a and b be the values of a one-dimensional representation χ of Γ_2 at $\mathcal{I}$ and $\mathcal{K}$ respectively. Then

$$a^2b^2 = 1 \quad \text{and} \quad a^2 = 1,$$

since

$$\mathcal{I}^2\mathcal{K}^2 = I \quad \text{and} \quad \mathcal{K}^{-1}\mathcal{I}\mathcal{K}\mathcal{I} = \mathcal{K}^{-1}\mathcal{I}\mathcal{J} = I.$$

Thus $a = \pm 1$ and $b = \pm 1$. These choices determine the value of χ at $\mathcal{J} = \mathcal{K}\mathcal{I}$ and at $-I = \mathcal{I}^2$.

Because this group has five conjugacy classes, there is another irreducible representation, α, of dimension $d \geq 2$. We must have $4 + d^2 = |\Gamma_2|$. Because the order of Γ_2 is 8, necessarily $d = 2$. We complete the character table using the orthogonality of columns. The representation α is in fact the fundamental representation of $\Gamma_2 \subset \mathrm{SU}(2)$. One obtains the table

	(1) I	(1) $-I$	(2) $\mathcal{I}$	(2) $\mathcal{J}$	(2) $\mathcal{K}$
α_0	1	1	1	1	1
α_1	1	1	1	-1	-1
α_2	1	1	-1	1	-1
α_3	1	1	-1	-1	1
α	2	-2	0	0	0

(We have used the same letter to denote the representation and its character.)

10. (a) Let $\chi^{(N)}$ be the character of the irreducible representation of dimension $N+1$ of $\mathrm{SU}(2)$. We have $\chi^{(N)}(I) = N + 1$.

We know that if $g \in \mathrm{SU}(2)$ is conjugate to $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$, $0 < \theta < \pi$, then

$$\chi^{(N)}(g) = \sum_{m=-\frac{N}{2}}^{\frac{N}{2}} e^{2im\theta} = \frac{\sin(N+1)\theta}{\sin\theta}.$$

Since $\mathcal{I}, \mathcal{J}$, and $\mathcal{K}$ are conjugate in $\mathrm{SU}(2)$ to the matrix $\begin{pmatrix} e^{i\frac{\pi}{2}} & 0 \\ 0 & e^{-i\frac{\pi}{2}} \end{pmatrix}$, we have

$$\chi^{(N)}(\mathcal{I}) = \chi^{(N)}(\mathcal{J}) = \chi^{(N)}(\mathcal{K}) = \sin(N+1)\frac{\pi}{2} = \cos N\frac{\pi}{2}.$$

Thus we obtain

$$\chi^{(N)}(\mathcal{I}) = 0, \ if \ N \text{ is odd},$$
$$\chi^{(N)}(\mathcal{I}) = 1, \ if \ N \equiv 0 \pmod 4,$$
$$\chi^{(N)}(\mathcal{I}) = -1, \ if \ N \equiv 2 \pmod 4.$$

We have $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} e^{i\pi} & 0 \\ 0 & e^{-i\pi} \end{pmatrix}$. If N is even, $N = 2N'$,

$$\chi^{(N)}(-I) = \sum_{m=-N'}^{N'} e^{2im\pi} = 2N' + 1 = N + 1,$$

while if N is odd, $N = 2N' + 1$, the index m takes half-integer values, and thus each of the $N+1$ terms of the sum is equal to $e^{-i\pi} = -1$, and we obtain

$$\chi^{(N)}(-I) = -(N+1).$$

Hence

$$\chi^{(N)}(-I) = (-1)^N (N+1).$$

(b) In order to find the decomposition of $\mathcal{R}^{(N)}$ as a direct sum of irreducible representations, we calculate $(\alpha_i \mid \chi^{(N)}), \quad 0 \le i \le 3$, and $(\alpha \mid \chi^{(N)})$. We have

$$\begin{aligned}
(\alpha_0 \mid \chi^{(N)}) &= \frac{1}{8}(\chi^{(N)}(I) + \chi^{(N)}(-I) + 6\chi^{(N)}(\mathcal{I})),\\
(\alpha_i \mid \chi^{(N)}) &= \frac{1}{8}(\chi^{(N)}(I) + \chi^{(N)}(-I) - 2\chi^{(N)}(\mathcal{I})), \quad 1 \le i \le 3,\\
(\alpha \mid \chi^{(N)}) &= \frac{1}{4}(\chi^{(N)}(I) - \chi^{(N)}(-I)).
\end{aligned}$$

If N is odd, then $(\alpha_i \mid \chi^{(N)}) = 0$, for $0 \le i \le 3$, and $(\alpha \mid \chi^{(N)}) = \frac{1}{2}(N+1)$, whence

$$\mathcal{R}^{(N)} = \frac{N+1}{2}\alpha.$$

In particular, $\mathcal{R}^{(1)} = \alpha$.

If N is even and $N \equiv 0 \pmod 4$, then $(\alpha_0 \mid \chi^{(N)}) = \frac{N}{4} + 1$, $(\alpha_i \mid \chi^{(N)}) = \frac{N}{4}$ for $1 \le i \le 3$, and $(\alpha \mid \chi^{(N)}) = 0$, whence

$$\mathcal{R}^{(N)} = \left(\frac{N}{4} + 1\right)\alpha_0 \oplus \frac{N}{4}(\alpha_1 \oplus \alpha_2 \oplus \alpha_3).$$

If N is even and $N \equiv 2 \pmod 4$, then $(\alpha_0 \mid \chi^{(N)}) = \frac{1}{8}(2(N+1) - 6) = \frac{N-2}{4}$, $(\alpha_i \mid \chi^{(N)}) = \frac{1}{8}(2(N+1)+2) = \frac{N+2}{4}$ for $1 \le i \le 3$, and $(\alpha \mid \chi^{(N)}) = 0$, whence

$$\mathcal{R}^{(N)} = \frac{N-2}{4}\alpha_0 \oplus \frac{N+2}{4}(\alpha_1 \oplus \alpha_2 \oplus \alpha_3).$$

We verify that in each case, the dimension of $\mathcal{R}^{(N)}$ is $N+1$.

(c) We have $\mathcal{R}^{(1)} = \alpha$. It is clear that $\alpha_0 \otimes \alpha = \alpha$. By calculating the scalar products of the characters, we obtain $\alpha_i \otimes \alpha = \alpha$, for $0 \le i \le 3$, and

$$\alpha \otimes \alpha = \alpha_0 \oplus \alpha_1 \oplus \alpha_2 \oplus \alpha_3.$$

The preceding results can be generalized to the subgroups $\varphi^{-1}(\psi(D_{(n)}))$, $n > 2$, and more generally, to the finite subgroups of SU(2). These either are of the form $\varphi^{-1}(G)$, for a finite subgroup G of SO(3), or are cyclic groups of odd order. For each finite subgroup of SU(2), one can decompose each tensor product of an irreducible representation and the restriction of the two-dimensional representation of SU(2) into a direct sum of irreducible representations. One can show that the matrix of coefficients appearing in these decompositions can be calculated from a matrix, called the Cartan matrix, associated to a simple Lie algebra. The correspondence between finite subgroups of SU(2) and simple Lie algebras is called the *McKay correspondence*. See B. Kostant, "The McKay correspondence, the Coxeter element and representation theory," in *Élie Cartan et les mathématiques d'aujourd'hui. The Mathematical Heritage of Élie Cartan*, Astérisque, hors série, Société Mathématique de France, 1985, pp. 209–255, and "The Coxeter element and the branching law for the finite subgroups of SU(2)," in *The Coxeter Legacy. Reflections and Projections*, C. Davis and E. W. Ellers, eds., American Mathematical Society, Providence, RI, 2006, pp. 63–70.

4 Representations of SU(2) and of $\mathfrak{S}_3$

We denote by σ the fundamental representation of SU(2) in $\mathbb{C}^2$, and by (u_0, u_1) the canonical basis of $\mathbb{C}^2$.

Let $u_{k\ell m} = u_k \otimes u_\ell \otimes u_m$, where $k, \ell, m = 0$ or 1, be the vectors of the canonical basis of the vector space $E = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$.

1. Show that σ is equivalent to $\mathcal{D}^{\frac{1}{2}}$. Find an isomorphism of $\mathbb{C}^2$ onto the support $V^{\frac{1}{2}}$ of the representation $\mathcal{D}^{\frac{1}{2}}$ that intertwines σ and $\mathcal{D}^{\frac{1}{2}}$.

2. Decompose the representation $\rho = \sigma \otimes \sigma \otimes \sigma$ of SU(2) in E into a direct sum of irreducible representations.

3. Let $D\rho$ be the differential of the representation ρ. Set

$$\xi_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathfrak{su}(2).$$

Is each $u_{k\ell m}$ an eigenvector of $(D\rho)(\xi_3)$? Find the eigenvalues of $(D\rho)(\xi_3)$ and the multiplicity of each one.

4. Let $\mathfrak{S}_3$ be the symmetric group on three elements.

   (a) Give the list of inequivalent irreducible representations and the character table of $\mathfrak{S}_3$.

   (b) We define a representation π of $\mathfrak{S}_3$ in E by

$$\pi(g)(u_{k\ell m}) = u_{g^{-1}(k)g^{-1}(\ell)g^{-1}(m)},$$

for $g \in \mathfrak{S}_3$. Find the character of π and decompose π as a direct sum of irreducible representations.

5. Find a basis of each of the isotypic components of (E, π).

6. Show that each isotypic component of (E, π) is the support of a representation of SU(2) and find the representation. Find a basis of E adapted to the decomposition of the representation ρ of SU(2) into irreducible representations.

Solutions

1. Because σ is two-dimensional, in order to show that σ is equivalent to $\mathcal{D}^{\frac{1}{2}}$, it suffices to show that σ is irreducible. In fact, the eigenspaces of $g_t = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}$ are $\mathbb{C}u_0$ and $\mathbb{C}u_1$, and these lines are not invariant under $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SU}(2)$.

By definition, if $g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \mathrm{SU}(2)$, then

$$\sigma(g)u_0 = au_0 - \bar{b}u_1, \quad \sigma(g)u_1 = bu_0 + \bar{a}u_1,$$

while

$$(\mathcal{D}^{\frac{1}{2}}(g)f)(z_1, z_2) = f(\bar{a}z_1 - bz_2, \bar{b}z_1 + az_2)$$

and $f^{\frac{1}{2}}_{\frac{1}{2}}(z_1, z_2) = z_1,\ f^{\frac{1}{2}}_{-\frac{1}{2}}(z_1, z_2) = z_2$. Thus

$$\mathcal{D}^{\frac{1}{2}}(g)f^{\frac{1}{2}}_{\frac{1}{2}} = \bar{a}f^{\frac{1}{2}}_{\frac{1}{2}} - bf^{\frac{1}{2}}_{-\frac{1}{2}}, \quad \mathcal{D}^{\frac{1}{2}}(g)f^{\frac{1}{2}}_{-\frac{1}{2}} = \bar{b}f^{\frac{1}{2}}_{\frac{1}{2}} + af^{\frac{1}{2}}_{-\frac{1}{2}}.$$

The isomorphism T of $\mathbb{C}^2$ onto $V^{\frac{1}{2}}$ defined by

$$T(u_0) = f^{\frac{1}{2}}_{-\frac{1}{2}}, \quad T(u_1) = -f^{\frac{1}{2}}_{\frac{1}{2}}$$

intertwines σ and $\mathcal{D}^{\frac{1}{2}}$. This can be easily verified by calculating $\mathcal{D}^{\frac{1}{2}}(g)T(u_k)$ and $T(\sigma(g)u_k)$, for $k = 0$ and 1.

2. Because the representation σ is equivalent to $\mathcal{D}^{\frac{1}{2}}$, $\rho = \sigma \otimes \sigma \otimes \sigma$ is equivalent to $\mathcal{D}^{\frac{1}{2}} \otimes \mathcal{D}^{\frac{1}{2}} \otimes \mathcal{D}^{\frac{1}{2}}$. We can thus apply the Clebsch–Gordan formula, which yields $\mathcal{D}^{\frac{1}{2}} \otimes \mathcal{D}^{\frac{1}{2}} = \mathcal{D}^0 \oplus \mathcal{D}^1$, $\mathcal{D}^0$ being the trivial one-dimensional representation. Thus

$$\rho = (\mathcal{D}^0 \oplus \mathcal{D}^1) \otimes \mathcal{D}^{\frac{1}{2}} = \mathcal{D}^{\frac{1}{2}} \oplus (\mathcal{D}^1 \otimes \mathcal{D}^{\frac{1}{2}}).$$

Because $\mathcal{D}^1 \otimes \mathcal{D}^{\frac{1}{2}} = \mathcal{D}^{\frac{1}{2}} \oplus \mathcal{D}^{\frac{3}{2}}$, we obtain

$$\rho = 2\mathcal{D}^{\frac{1}{2}} \oplus \mathcal{D}^{\frac{3}{2}}.$$

3. Because $\rho = \sigma \otimes \sigma \otimes \sigma$, for $X \in \mathfrak{su}(2)$, we have

$$\begin{aligned}(D\rho)(X)u_{k\ell m} = {} & (D\sigma)(X)u_k \otimes u_\ell \otimes u_m \\ & + u_k \otimes (D\sigma)(X)u_\ell \otimes u_m \\ & + u_k \otimes u_\ell \otimes (D\sigma)(X)u_m.\end{aligned}$$

If $X = \xi_3$, we have $(D\sigma)(\xi_3)u_k = \xi_3 u_k = (-1)^k \frac{i}{2} u_k$, whence

$$(D\rho)(\xi_3)u_{k\ell m} = ((-1)^k + (-1)^\ell + (-1)^m)\frac{i}{2}u_{k\ell m}.$$

Thus each $u_{k\ell m}$ is an eigenvector of $(D\rho)(\xi_3)$.

The eigenvalues are

$$\frac{3i}{2}, \quad -\frac{3i}{2}, \quad \frac{i}{2}, \quad -\frac{i}{2},$$

with the eigenvectors u_{000} for $3i/2$; u_{111} for $-3i/2$; u_{001}, u_{010}, and u_{100} for $i/2$; u_{011}, u_{101}, and u_{110} for $-i/2$. Thus $3i/2$ and $-3i/2$ are simple, while $i/2$ and $-i/2$ have multiplicity 3.

4. (a) We know that $\mathfrak{S}_3$ has two irreducible one-dimensional representations, the trivial representation $\underline{1}$ and the signature representation ϵ, and a two-dimensional irreducible representation, α. The conjugacy class of a cyclic permutation c contains two elements (the two cyclic permutations different from the identity), and the conjugacy class of a transposition t contains three elements (the three transpositions). Thus the character table is

	(1) e	(2) c	(3) t
$\chi_{\underline{1}}$	1	1	1
χ_ϵ	1	1	-1
χ_α	2	-1	0

(b). The group $\mathfrak{S}_3$ acts on ordered triples (k, ℓ, m) by permutations. The trace $\chi_\pi(g)$ of $\pi(g)$ is the number of triples that are invariant under the action of g^{-1}. We have $\chi_\pi(e) = 8$. If g is a cyclic permutation, only u_{000} and u_{111} are invariant, and thus $\chi_\pi(c) = 2$. If t is a transposition, then $\chi_\pi(t) = 4$. In fact, $t = (12)$ leaves $u_{000}, u_{111}, u_{001}, u_{110}$ invariant.

We find the decomposition of $\pi = m_{\underline{1}}\underline{1} \oplus m_\epsilon \epsilon \oplus m_\alpha \alpha$ by calculating, by means of the character table,

$$m_{\underline{1}} = (\chi_{\underline{1}}|\chi_\pi) = 4, \quad m_\epsilon = (\chi_\epsilon|\chi_\pi) = 0, \quad m_\alpha = (\chi_\alpha|\chi_\pi) = 2.$$

Thus

$$\pi = 4\ \underline{1} \oplus 2\ \alpha.$$

5. We have seen that $E = 4E^{\underline{1}} \oplus 2E^\alpha$. We denote by $V^{\underline{1}} = 4E^{\underline{1}}$ and $V^\alpha = 2E^\alpha$ the isotypic components of E. Thus $E = V^{\underline{1}} \oplus V^\alpha$.

The projection of E onto $V^{\underline{1}}$ is defined by

$$P_{\underline{1}} u_{k\ell m} = \frac{1}{6} \sum_{g \in \mathfrak{S}_3} \pi(g) u_{k\ell m} = \frac{1}{6} \sum_{g \in \mathfrak{S}_3} u_{g(k)g(\ell)g(m)}.$$

We obtain

$$\begin{aligned}
&P_{\underline{1}} u_{000} = u_{000}, \quad P_{\underline{1}} u_{111} = u_{111}\ , \\
&P_{\underline{1}} u_{001} = P_{\underline{1}} u_{010} = P_{\underline{1}} u_{100} = \frac{1}{3}(u_{001} + u_{010} + u_{100}) = u\ , \\
&P_{\underline{1}} u_{011} = P_{\underline{1}} u_{101} = P_{\underline{1}} u_{110} = \frac{1}{3}(u_{011} + u_{101} + u_{110}) = u'.
\end{aligned}$$

A basis of $V^{\underline{1}} = 4E^{\underline{1}}$ is thus $(u_{000}, u_{111}, u, u')$.

For the projection onto V^α, we obtain

$$P_\alpha u_{k\ell m} = \frac{1}{3}(2u_{k\ell m} - u_{\ell mk} - u_{mk\ell}).$$

Thus $P_\alpha u_{000} = P_\alpha u_{111} = 0$ and

$$P_\alpha u_{001} = \frac{1}{3}(2u_{001} - u_{100} - u_{010}) = v, \quad P_\alpha u_{100} = \frac{1}{3}(2u_{100} - u_{010} - u_{001}) = v',$$
$$P_\alpha u_{110} = \frac{1}{3}(2u_{110} - u_{011} - u_{101}) = w, \quad P_\alpha u_{011} = \frac{1}{3}(2u_{011} - u_{101} - u_{110}) = w',$$

and

$$P_\alpha u_{010} = -P_\alpha(u_{001} + u_{100}),$$
$$P_\alpha u_{101} = -P_\alpha(u_{110} + u_{011}).$$

A basis of $V^\alpha = E^\alpha \oplus E^\alpha$ is thus given by the four vectors v, v', w, w'.

We can also obtain this result by observing that V^α is the orthogonal complement of $V^{\underline{1}}$ for the scalar product on E such that the vectors u_{klm} form an orthonormal basis.

Set $j = e^{\frac{2i\pi}{3}}$, so that $j^2+j+1 = 0$. We see that vectors $x =$ (j+2)v + (2j+1)v' $= u_{001} + ju_{100} + j^2u_{010}$ and $y = (j-1)v - (j+2)v' = u_{010} + ju_{001} + j^2u_{100}$, on the one hand, and vectors $x' = (j+2)w + (2j+1)w' = u_{110} + ju_{011} + j^2u_{101}$ and $y' = (j-1)w - (j+2)w' = u_{101} + ju_{110} + j^2u_{011}$, on the other hand, form a basis of a vector space E^α, the support of the representation α of $\mathfrak{S}_3$.

6. Let us show that the subspace $V^{\underline{1}}$ is closed under the representation $D\rho$ of $\mathfrak{su}(2)$, and thus under the representation ρ of SU(2), which is connected. For convenience we denote $(D\rho)(X)u_{klm}$ by $X \cdot u_{klm}$, for $X \in \mathfrak{su}(2)$. The four vectors of the basis given above are eigenvectors of $(D\rho)(\xi_3)$,

$$\xi_3 \cdot u_{000} = \frac{3i}{2}u_{000}, \quad \xi_3 \cdot u_{111} = -\frac{3i}{2}u_{111}, \quad \xi_3 \cdot u = \frac{i}{2}u, \quad \xi_3 \cdot u' = -\frac{i}{2}u'.$$

On the other hand,

$$(D\sigma)(\xi_1)u_0 = \frac{i}{2}u_1, \quad (D\sigma)(\xi_1)u_1 = \frac{i}{2}u_0,$$
$$(D\sigma)(\xi_2)u_0 = \frac{1}{2}u_1, \quad (D\sigma)(\xi_2)u_1 = -\frac{1}{2}u_0.$$

Thus,

$$\xi_1 \cdot u_{000} = \frac{3i}{2}u, \quad \xi_1 \cdot u_{111} = \frac{3i}{2}u', \quad \xi_1 \cdot u = \frac{i}{2}(u_{000} + 2u') \in V^{\underline{1}},$$

and similarly, $\xi_1 \cdot u' = \frac{i}{2}(u_{111} + 2u) \in V^{\underline{1}}$. Furthermore,

$$\xi_2 \cdot u_{000} = \frac{3}{2}u, \quad \xi_2 \cdot u_{111} = -\frac{3}{2}u', \quad \xi_2 \cdot u = -\frac{1}{2}(u_{000} - 2u') \in V^{\underline{1}},$$

and similarly, $\xi_2 \cdot u' = \frac{1}{2}(u_{111} - 2u) \in V^{\underline{1}}$.

The restriction of $D\rho$ to $V^{\underline{1}}$ is a representation of dimension 4 of $\mathfrak{su}(2)$ in which ξ_3 has eigenvalue $3i/2$. We know that this property implies that $\rho|_{V^{\underline{1}}}$ is equivalent to $\mathcal{D}^{\frac{3}{2}}$.

We verify that $T : V^{\underline{1}} \to V^{\frac{3}{2}} = E^{\frac{3}{2}}$, defined by $T(u_{000}) = -f^{\frac{3}{2}}_{-\frac{3}{2}}$, $T(u_{111}) = f^{\frac{3}{2}}_{\frac{3}{2}}$, $Tu = f^{\frac{3}{2}}_{-\frac{1}{2}}$, $Tu' = -f^{\frac{3}{2}}_{\frac{1}{2}}$, intertwines $D\rho\big|_{V^{\underline{1}}}$ and $D^{\frac{3}{2}}$.

Next we consider the space $V^\alpha = 2E^\alpha$. We have

$$\xi_3 \cdot v = \frac{i}{2}v, \quad \xi_3 \cdot v' = \frac{i}{2}v', \quad \xi_3 \cdot w = -\frac{i}{2}w, \quad \xi_3 \cdot w' = -\frac{i}{2}w'.$$

The vector subspace generated by v and w is the support of a representation $\mathcal{D}^{\frac{1}{2}}$. In fact, it is closed under $D\rho$. We have $\xi_1 \cdot v = -\frac{i}{2}w$ and $\xi_1 \cdot w = -\frac{i}{2}v$, because $\xi_1^2 = -\frac{1}{4}I$. Furthermore, $\xi_2 \cdot v = -\frac{1}{2}w$ and $\xi_2 \cdot w = \frac{1}{2}v$.

We verify that if we set $Tv = f^{\frac{1}{2}}_{-\frac{1}{2}}$ and $Tw = f^{\frac{1}{2}}_{\frac{1}{2}}$, the isomorphism T intertwines the restriction of $D\rho$ to the vector space generated by v and w with $D^{\frac{1}{2}}$.

Analogous equations for v' and w' show that the vector subspace generated by v' and w' is also the support of are presentation $\mathcal{D}^{\frac{1}{2}}$. In conclusion, the two vector subspaces generated by v and w on the one hand, and by v' and w' on the other hand, are each the support of a representation $\mathcal{D}^{\frac{1}{2}}$.

Using the decomposition of E under the action of $\mathfrak{S}_3$, we have obtained a basis of E adapted to the decomposition of $\rho = \sigma \otimes \sigma \otimes \sigma$.

We see that the isotypic components in the decompositions of the representation ρ of SU(2) and of the representation π of $\mathfrak{S}_3$ correspond to one another: $V^{\underline{1}} = 4E^{\underline{1}} \sim E^{\frac{3}{2}}$ and $V^\alpha = 2E^\alpha \sim 2E^{\frac{1}{2}}$. We have proved

$$E = (\dim E^\alpha)E^{\frac{1}{2}} \oplus (\dim E^{\underline{1}})E^{\frac{3}{2}} = (\dim E^{\frac{1}{2}})E^\alpha \oplus (\dim E^{\frac{3}{2}})E^{\underline{1}}.$$

The fact that $\rho = \oplus_{k\in I}\, p_k\rho_k = \oplus_{k\in I}\, m_k\pi_k$, where the ρ_k are irreducible representations of SU(2) and the π_k irreducible representations of $\mathfrak{S}_3$, with $p_k = \dim \pi_k$ and $m_k = \dim \rho_k$, is a particular case of a general property, the *Schur–Weyl duality* between representations of $\mathrm{GL}(V)$ and of $\mathfrak{S}_r$ on $V^{\otimes r}$, for a given vector space V and a positive integer r (here $V = \mathbb{C}^2$ and $r = 3$). See Goodman–Wallach (1998).

5 Pseudo-unitary and Pseudo-orthogonal Groups

We consider the Lie group

$$\mathrm{SU}(1,1) = \{A \in \mathrm{SL}(2,\mathbb{C}) \mid AJ\,{}^t\bar{A} = J\}, \quad \text{where} \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

1. Show that $A \in \mathrm{SU}(1,1)$ if and only if $A = \left(\begin{smallmatrix} a & b \\ \bar{b} & \bar{a} \end{smallmatrix}\right)$, where a and b are complex numbers such that $|a|^2 - |b|^2 = 1$.

Is the Lie group $\mathrm{SU}(1,1)$ compact?

2. For $g \in \mathrm{SL}(2,\mathbb{R})$, we set $\mu(g) = \tau g \tau^{-1}$, where $\tau = \left(\begin{smallmatrix} 1 & i \\ i & 1 \end{smallmatrix}\right)$. What is the image of μ? Show that the Lie groups $\mathrm{SU}(1,1)$ and $\mathrm{SL}(2,\mathbb{R})$ are isomorphic.

3. Let $\xi_1 = \frac{1}{2}\left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$, $\xi_2 = \frac{1}{2}\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $\xi_3 = \frac{1}{2}\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$. Show that the matrices $M_1 = -i\xi_1, M_2 = -i\xi_2, M_3 = \xi_3$ form a basis of the Lie algebra $\mathfrak{su}(1,1)$ of $\mathrm{SU}(1,1)$. Write the commutation relations of $\mathfrak{su}(1,1)$ in this basis.

On $\mathbb{R}^3$ we consider the quadratic form defined by $Q(x) = x_1^2 + x_2^2 - x_3^2$, where x_1, x_2, x_3 denote the components of $x \in \mathbb{R}^3$in the canonical basis (e_1, e_2, e_3).

4. Define an isomorphism α from $\mathbb{R}^3$ onto $\mathfrak{su}(1,1)$ by setting

$$\alpha(e_k) = M_k, \ k = 1, 2, 3.$$

(a) For each $x \in \mathbb{R}^3$, calculate the determinant of $\alpha(x)$ and compare it to $Q(x)$.

(b) Denote by K the Killing form of $\mathfrak{su}(1,1)$. Show that for every $x \in \mathbb{R}^3$,

$$K(\alpha(x), \alpha(x)) = 2Q(x).$$

Consider the Lie group

$$\mathrm{O}(2,1) = \{A \in \mathrm{GL}(3,\mathbb{R}) \mid AJ_{21}\,{}^tA = J_{21}\}, \quad \text{where} \quad J_{21} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Let $\mathrm{SO}(2,1) = \mathrm{O}(2,1) \cap \mathrm{SL}(3,\mathbb{R})$, and let $\mathrm{SO}_0(2,1)$ be the connected component of the identity of $\mathrm{SO}(2,1)$.

5. Let $A \in \mathrm{GL}(3,\mathbb{R})$. Show that $A \in \mathrm{O}(2,1)$ if and only if for every $x \in \mathbb{R}^3$,

$$Q(Ax) = Q(x).$$

6. Find a basis of the Lie algebra $\mathfrak{so}(2,1)$ of $\mathrm{SO}(2,1)$. Write the commutation relations of $\mathfrak{so}(2,1)$ in this basis.

What can one say about the Lie algebra of $\mathrm{O}(2,1)$?

7. Are the Lie algebras $\mathfrak{su}(1,1)$ and $\mathfrak{so}(2,1)$ isomorphic? Are the Lie algebras $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{so}(2,1)$ isomorphic?

8. Consider the adjoint representation Ad of the group $\mathrm{SU}(1,1)$ on its Lie algebra. To $g \in \mathrm{SU}(1,1)$ we associate the matrix $\psi(g)$ of Ad_g in the basis (M_1, M_2, M_3) of $\mathfrak{su}(1,1)$. It is known that $\mathrm{SU}(1,1)$ is connected.

(a) Show that for every $g \in \mathrm{SU}(1,1)$, $\psi(g) \in \mathrm{O}(2,1)$.

(b) Show that ψ is a Lie group morphism from $\mathrm{SU}(1,1)$ to $\mathrm{SO}_0(2,1)$. What is the kernel of ψ?

(c) Find the differential of ψ. Show that ψ is surjective from $\mathrm{SU}(1,1)$ onto $\mathrm{SO}_0(2,1)$.

Solutions

1. The condition $AJ\,{}^t\bar{A} = J$ can be written

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which is equivalent to $a\bar{a} - b\bar{b} = 1$, $c\bar{c} - d\bar{d} = -1$, $a\bar{c} - b\bar{d} = 0$, $\bar{a}c - \bar{b}d = 0$. We deduce that $ac\bar{c} - bc\bar{d} = 0$. Because $ad - bc = 1$, we have $ac\bar{c} - ad\bar{d} + \bar{d} = 0$, whence $d = \bar{a}$ and $c = \bar{b}$. Thus $A = \left(\begin{smallmatrix} a & b \\ \bar{b} & \bar{a} \end{smallmatrix}\right)$, and the condition $\det A = 1$ is equivalent to $|a|^2 - |b|^2 = 1$.

To show that the Lie group $\mathrm{SU}(1,1)$ is not compact, it suffices to show that the set $\{(a,b) \in \mathbb{C}^2 \mid\ |a|^2 - |b|^2 = 1\}$ is not bounded; it contains, for example, all pairs of positive reals of the form $(\sqrt{x^2+1}, x)$.

2. If $\tau = \left(\begin{smallmatrix} 1 & i \\ i & 1 \end{smallmatrix}\right)$, then $\tau^{-1} = \frac{1}{2}\left(\begin{smallmatrix} 1 & -i \\ -i & 1 \end{smallmatrix}\right)$. Let $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{R})$. Then $\mu(g)$ is the matrix

$$\tau g \tau^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \alpha+\delta-i(\beta-\gamma) & \beta+\gamma-i(\alpha-\delta) \\ \beta+\gamma+i(\alpha-\delta) & \alpha+\delta+i(\beta-\gamma) \end{pmatrix}.$$

We verify that $d = \bar{a},\ c = \bar{b}$, and $|a|^2 - |b|^2 = \alpha\delta - \beta\gamma = 1$ because $g \in \mathrm{SL}(2,\mathbb{R})$. Furthermore, $\det \mu(g) = \det g = 1$. Thus the image of μ is contained in $\mathrm{SU}(1,1)$.

On the other hand, if $h = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SU}(1,1)$, we have

$$\tau^{-1} h \tau = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a+d+i(b-c) & b+c+i(a-d) \\ b+c-i(a-d) & a+d-i(b-c) \end{pmatrix}.$$

We see that under the hypothesis $d = \bar{a}$ and $c = \bar{b}$, the numbers α, β, γ, and δ are real. Furthermore, because $h \in \mathrm{SU}(1,1)$,

$$\det(\tau^{-1} h \tau) = \det h = 1.$$

Thus μ is a bijection of $\mathrm{SL}(2,\mathbb{R})$ onto $\mathrm{SU}(1,1)$. Furthermore, μ is a group morphism since $\mu(gg') = \mu(g)\mu(g')$ for every g and $g' \in \mathrm{SL}(2,\mathbb{R})$. Finally, the coefficients of $\mu(g)$ are linear functions and thus are continuous in the coefficients of g. Thus μ is a Lie group isomorphism of $\mathrm{SL}(2,\mathbb{R})$ onto $\mathrm{SU}(1,1)$.

3. Let M be a matrix of $\mathfrak{gl}(2,\mathbb{C})$. By differentiating both sides of the equation

$$\exp(sM) J \exp(s\,{}^t\overline{M}) = J$$

with respect to $s \in \mathbb{R}$, we obtain

$$\exp(sM)(MJ + J\,{}^t\overline{M}) \exp(s\,{}^t\overline{M}) = 0.$$

By evaluating at $s = 0$, we see that in order for $\exp(sM)$ to belong to $\mathrm{SU}(1,1)$ for every $s \in \mathbb{R}$, we must have $MJ + J\,{}^t\overline{M} = 0$. And conversely, if the condition $MJ + J\,{}^t\overline{M} = 0$ is satisfied, the matrix $\exp(sM) J \exp(s\,{}^t\overline{M})$ is constant and equal to its initial value, which is J. On the other hand, the condition $\mathrm{Tr}M = 0$ is equivalent to $\mathrm{Det}(\exp sM) = 1$, for every s. Hence the Lie algebra of $\mathrm{SU}(1,1)$ is

$$\mathfrak{su}(1,1) = \{M \in \mathfrak{sl}(2,\mathbb{C}) \mid MJ + J\,{}^t\overline{M} = 0\}.$$

The matrices of $\mathfrak{su}(1,1)$ are of the form $\left(\begin{smallmatrix} u & v \\ \bar{v} & -u \end{smallmatrix}\right)$, where u is pure imaginary. A basis of $\mathfrak{su}(1,1)$ is thus

$$\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -i\xi_1 = M_1, \quad \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = -i\xi_2 = M_2, \quad \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \xi_3 = M_3.$$

The commutation relations are

$$[M_1, M_2] = -M_3, \quad [M_2, M_3] = M_1, \quad [M_3, M_1] = M_2.$$

4. (a) If $x = \sum\limits_{k=1}^{3} x_k e_k$, then $\alpha(x) = \sum\limits_{k=1}^{3} x_k M_k = \frac{1}{2}\left(\begin{smallmatrix} ix_3 & x_1+ix_2 \\ x_1-ix_2 & -ix_3 \end{smallmatrix}\right)$, whose determinant is $\frac{1}{4}(-(x_1^2+x_2^2)+x_3^2) = -\frac{1}{4}Q(x)$.

(b) By definition, $K(X,Y) = \mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_Y)$. We have, in the basis (M_k),

$$\mathrm{ad}_{M_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \mathrm{ad}_{M_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}_{M_3} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Set $X = \alpha(x) = \sum_{k=1}^{3} x_k M_k$ and set $Y = \alpha(y) = \sum_{k=1}^{3} y_k M_k$. Then

$$\mathrm{ad}_X = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ x_2 & -x_1 & 0 \end{pmatrix},$$

whence $\mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_Y) = 2(x_1y_1 + x_2y_2 - x_3y_3)$. Thus $K(\alpha(x), \alpha(x)) = 2Q(x)$.

5. The group $\mathrm{O}(2,1)$ is the group of transformations of $\mathbb{R}^3$ preserving the quadratic form Q of signature (2,1), and therefore not positive definite. In fact, if X is the column vector of components x_1, x_2, x_3, then

$$Q(x) = {}^tXJ_{21}X,$$

and if $A \in \mathrm{GL}(3,\mathbb{R})$,

$$Q(Ax) = {}^tX\,{}^tA\ J_{21}\ A\ X,$$

then $Q(Ax) = Q(x)$ for every x is equivalent to ${}^tAJ_{21}A = J_{21}$. Now, because $(J_{21})^2 = I$, the condition ${}^tA \in \mathrm{O}(2,1)$ is equivalent to $A \in \mathrm{O}(2,1)$. In fact, ${}^tAJ_{21}A = J_{21}$ implies $AJ_{21}\,{}^tAJ_{21}A = A$, whence $AJ_{21}\,{}^tA = J_{21}$.

6. Using the same argument as in Question 3, we see that if $M \in \mathfrak{gl}(3,\mathbb{R})$, the condition $\exp(sM)J_{21}\exp(s\,{}^tM) = J_{21}$, for every $s \in \mathbb{R}$, is equivalent to the condition $MJ_{21} + J_{21}\,{}^tM = 0$. The matrices M of $\mathfrak{gl}(3,\mathbb{R})$ satisfying the condition $MJ_{21} + J_{21}\,{}^tM = 0$ are of the form $\left(\begin{smallmatrix} 0 & u & v \\ -u & 0 & w \\ v & w & 0 \end{smallmatrix}\right)$, where $u, v, w \in \mathbb{R}$. Thus $\mathrm{Tr}M = 0$. The Lie algebra of $\mathrm{SO}(2,1)$ is thus

$$\mathfrak{so}(2,1) = \{M \in \mathfrak{sl}(3,\mathbb{R}) \mid MJ_{21} + J_{21}\,{}^tM = 0\}.$$

The matrices

$$\zeta_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \zeta_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \zeta_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

form a basis of $\mathfrak{so}(2,1)$.

The commutation relations are

$$[\zeta_1, \zeta_2] = -\zeta_3, \quad [\zeta_2, \zeta_3] = \zeta_1, \quad [\zeta_3, \zeta_1] = \zeta_2.$$

It follows from the remark above on the vanishing of the trace that the Lie algebra of $\mathrm{O}(2,1)$ is equal to $\mathfrak{so}(2,1)$.

7. The linear mapping of $\mathfrak{su}(1,1)$ on $\mathfrak{so}(2,1)$ defined by $M_k \mapsto \zeta_k$, for $k = 1,2,3$, is an isomorphism of Lie algebras, since in these bases the commutation relations are the same in the two Lie algebras.

On the other hand, one can deduce from Question 2 above that the Lie algebras of $\mathrm{SU}(1,1)$, and $\mathrm{SL}(2,\mathbb{R})$ are isomorphic. Thus the Lie algebras $\mathfrak{so}(2,1), \mathfrak{su}(1,1)$, and $\mathfrak{sl}(2,\mathbb{R})$ are isomorphic.

Observe that because $\mathrm{Tr}((\mathrm{ad}\xi_k)^2) = -2$, the Killing form of $\mathfrak{su}(2)$ is negative definite. The Lie algebras $\mathfrak{su}(1,1)$ and $\mathfrak{su}(2)$ are not isomorphic, since their Killing forms do not have the same signature.

8. (a) By definition $\alpha \circ \psi(g) = \mathrm{Ad}_g \circ \alpha$, for every $g \in \mathrm{SU}(1,1)$. The linear mapping Ad_g preserves the determinants, and hence $\psi(g)$ preserves Q, because $\det \alpha(x) = -\frac{1}{4}Q(x)$, for every $x \in \mathbb{R}^3$. Thus $\psi(g) \in \mathrm{O}(2,1)$.

(b) Because $\mathrm{SU}(1,1)$ is connected, the image of ψ is contained in the connected component of the identity of $\mathrm{O}(2,1)$, which is the component $\mathrm{SO}_0(2,1)$ of its subgroup $\mathrm{SO}(2,1)$.

It is clear that ψ is a Lie group morphism. The kernel of ψ is the subgroup of $\mathrm{SU}(1,1)$,

$$\{g \in \mathrm{SU}(1,1) \mid \mathrm{Ad}_g = \mathrm{Id}_{\mathfrak{su}(1,1)}\} = \{g \in \mathrm{SU}(1,1) \mid \forall M \in \mathfrak{su}(1,1), gM = Mg\}.$$

A calculation shows that $g = \left(\begin{smallmatrix} a & b \\ \bar{b} & \bar{a} \end{smallmatrix}\right)$ satisfies this condition if and only if $b = 0$, $a = \bar{a}$, whence $a = \pm 1$ and $\mathrm{Ker}\ \psi = \{I, -I\}$.

(c) The differential of ψ is the morphism of Lie algebras that associates to $X \in \mathfrak{su}(1,1)$ the matrix of ad_X in the basis (M_k). This morphism is the isomorphism from $\mathfrak{su}(1,1)$ onto $\mathfrak{so}(2,1)$ constructed in Question 7 above, because the matrix of ad_{M_k} is ζ_k, for $k = 1,2,3$, as we have seen in the proof of Questions 4(b) and 6.

On the other hand, each element of $\mathrm{SO}_0(2,1)$ is a product of exponentials. Thus to show that ψ is surjective, it suffices to show that for every $M \in \mathfrak{so}(2,1)$, there is an $N \in \mathfrak{su}(1,1)$ such that $\psi(\exp N) = \exp M$. Because $\psi(\exp N) = \exp((D\psi)N)$, and because $D\psi$ is an isomorphism, it suffices to take $N = (D\psi)^{-1}(M)$.

The morphism ψ is a double cover of $\mathrm{SO}_0(2,1)$. But $\mathrm{SU}(1,1)$ is not simply connected. The universal cover of $\mathrm{SO}_0(2,1)$ is an example of a Lie group that is not a linear Lie group.

We can define the analogue of Euler angles for $\mathrm{SU}(1,1)$ and express the Haar integral in this parametrization. Since the Lie group $\mathrm{SU}(1,1)$ is not compact, it has infinite-dimensional irreducible unitary representations. One can realize the representations of $\mathrm{SU}(1,1)$ in spaces of homogeneous functions of a complex variable, study the special functions associated to the matrix coefficients of these representations, etc. See Vilenkin (1968) or Vilenkin–Klimyk (1991).

In physics, the noncompact Lie group $\mathrm{SO}_0(3,2)$ is called the *anti–de Sitter group*, while $\mathrm{SO}_0(4,1)$ is called the *de Sitter group*. The importance of the anti–de Sitter group is due to the fact that the Poincaré group can be obtained from it by contraction. The group $\mathrm{O}(3,2)$ is the symmetry group of a one-sheeted hyperboloid in $\mathbb{R}^5$, called the *anti–de Sitter space-time*, AdS_5.

The case of $\mathrm{O}(2,1)$ corresponding to the two-dimensional space-time AdS_3 has been studied intensively. See for example the anti–de Sitter version of Hamiltonian mechanics of J.-P. Gazeau, "On two analytic elementary systems in quantum mechanics," in *Géométrie complexe*, F. Norguet, S. Ofman, and J.-J. Szczeciniarz, eds., Hermann, Paris, 1996, pp. 175–199. We quote from an article by S. Detournay, D. Orlando, P. Petropoulos, and P. Spindel, "Three-dimensional black holes from deformed anti-de Sitter," *JHEP* **07** (2005) 072, hep-th/0504231: "Three-dimensional anti–de Sitter space provides a good laboratory for studying many aspects of gravity and strings, including black-holes physics." The geometry of the anti–de Sitter space-time as well as the representations of the groups $\mathrm{SU}(1,1)$ and $\mathrm{SL}(2,\mathbb{R})$ play an essential role in these theories. In the preprint list http://fr.arxiv.org/find/hep-th, the number of papers containing "anti–de Sitter" or "AdS" in the title was 77 in 2005, 100 in 2006, 123 in 2007, and 176 in 2008.

6 Irreducible Representations of SU(2) × SU(2)

Let $\mathbb{H}$ be the associative algebra of quaternions, a real vector space of dimension 4 with basis $(1,i,j,k)$ and multiplication defined by

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

For $q = \alpha_0 1 + \alpha_1 i + \alpha_2 j + \alpha_3 k$, we set $\bar{q} = \alpha_0 1 - \alpha_1 i - \alpha_2 j - \alpha_3 k$ and $\|q\|^2 = q\bar{q}$. It is known that $\|q\|^2 = \sum_{p=0}^{3} (\alpha_p)^2$ and that for all q, $q' \in \mathbb{H}$,

$$\|qq'\| = \|q\|\|q'\|.$$

Denote by $\mathbb{S}$ the group of unit quaternions

$$\mathbb{S} = \{q \in \mathbb{H} \mid \|q\| = 1\}.$$

Consider the real vector space $\mathcal{H}$ generated by the matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{I} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{K} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Observe that $\mathrm{SU}(2)$ and $\mathfrak{su}(2)$ are contained in $\mathcal{H}$.

Part I

1. We write $M \mapsto q_M$ for the isomorphism of real vector spaces from $\mathcal{H}$ onto $\mathbb{H}$ defined by $I \mapsto 1$, $\mathcal{I} \mapsto i$, $\mathcal{J} \mapsto j$, $\mathcal{K} \mapsto k$.

   (a) Show that the restriction to $\mathrm{SU}(2)$ of this map is an isomorphism from the group $\mathrm{SU}(2)$ onto the group $\mathbb{S}$.

   (b) Show that for every $X \in \mathfrak{su}(2)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(q_{\exp(tX)})\Big|_{t=0} = q_X.$$

2. Let $(A, B) \in \mathrm{SU}(2) \times \mathrm{SU}(2)$. We consider the map

$$\sigma(A, B) : \mathbb{H} \to \mathbb{H}$$

defined by

$$\sigma(A, B)q = q_A \ q \ (q_B)^{-1}, \quad \forall q \in \mathbb{H}.$$

We identify $\mathbb{H}$ and $\mathbb{R}^4$ by means of the $\mathbb{R}$-linear map defined by $1 \mapsto e_0$, $i \mapsto e_1$, $j \mapsto e_2$, $k \mapsto e_3$, where (e_0, e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^4$.

   (a) Show that $\sigma(A, B)$ can be identified with an element of $\mathrm{O}(4)$.

   (b) Show that σ is a group morphism from $\mathrm{SU}(2) \times \mathrm{SU}(2)$ into $\mathrm{SO}(4)$.

   (c) Find the kernel of the morphism σ.

3. (a) Find the differential $D\sigma : \mathfrak{su}(2) \oplus \mathfrak{su}(2) \to \mathfrak{so}(4)$ of σ.

   (b) Show that $D\sigma$ is an isomorphism of Lie algebras.

   (c) Show that the image of σ is $\mathrm{SO}(4)$.

Part II

Let (E_1, ρ_1) and (E_2, ρ_2) be finite-dimensional representations of compact Lie groups G_1 and G_2, respectively. We define a representation ρ of $G_1 \times G_2$ in $E_1 \otimes E_2$ by

$$\rho(g_1, g_2)(v_1 \otimes v_2) = \rho_1(g_1)v_1 \otimes \rho_2(g_2)v_2,$$

for $g_1 \in G_1$, $g_2 \in G_2$, $v_1 \in E_1$, $v_2 \in E_2$. We denote this representation by $\rho = \rho_1 \times \rho_2$.

We state without proof that each irreducible representation of $G_1 \times G_2$ is of the form $(E_1 \otimes E_2, \rho_1 \times \rho_2)$, where (E_i, ρ_i) is an irreducible representation of G_i, $i = 1, 2$.

4. Find the irreducible representations of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and those of $\mathrm{SO}(4)$.

5. Let $\mathfrak{g}_1$ (respectively, $\mathfrak{g}_2$) be the Lie algebra of G_1 (respectively, G_2). Let R be the differential of $\rho = \rho_1 \times \rho_2$. Evaluate $R(X_1, X_2)(v_1 \otimes v_2)$, for $X_1 \in \mathfrak{g}_1$, $X_2 \in \mathfrak{g}_2$, $v_1 \in E_1$, $v_2 \in E_2$, by means of $D\rho_1$ and $D\rho_2$. Denote by $D\rho_1 \times D\rho_2$ such a representation of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ or of its complexification $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$.

6. Let $j_1, j_2 \in \frac{1}{2}\mathbb{N}$. Consider, with the notation of Question 5, the representation $D^{(j_1 j_2)} = D^{j_1} \times D^{j_2}$ of $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ on $V^{j_1} \otimes V^{j_2}$. Set

$$J_1 = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J_2 = -\frac{i}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J_3 = -\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $|j, m\rangle$ be the usual basis of the space V^j. Choose a basis of $V^{j_1} \otimes V^{j_2}$ and write the matrices of $D^{(j_1 j_2)}(X, 0)$ and of $D^{(j_1 j_2)}(0, X)$ in this basis,

(a) for $X = J_1,\ J_2,\ J_3$, when $j_1 = j_2 = \frac{1}{2}$,
(b) for $X = J_3$, when $j_1 = 1$ and $j_2 = \frac{1}{2}$,
(c) for $X = J_3$, when $j_1 = \frac{1}{2}$ and $j_2 = 1$.

7. Are the representations $D^{(j_1 j_2)}$ and $D^{(j_2 j_1)}$ equivalent? Why?

8. For $X \in \mathfrak{sl}(2,\mathbb{C})$, set $D_1^{(j_1 j_2)}(X^2) = D^{(j_1 j_2)}(X, 0) \circ D^{(j_1 j_2)}(X, 0)$ and $D_2^{(j_1 j_2)}(X^2) = D^{(j_1 j_2)}(0, X) \circ D^{(j_1 j_2)}(0, X)$. Set $C_i^{(j_1 j_2)} = \sum_{\ell=1}^{3} D_i^{(j_1 j_2)}(J_\ell^2)$, $i = 1,\ 2$. Evaluate the operators $C_1^{(j_1 j_2)}$ and $C_2^{(j_1 j_2)}$.

9. We continue to use the notation of the preceding questions. We want to evaluate the multiplicity of the negative-energy bound states of a hydrogen atom. Let e, m, and $\hbar$ be constants. We define the Rydberg constant

$$R_H = \frac{me^4}{2\hbar^2}.$$

Let R_ℓ and l_ℓ, $\ell = 1, 2, 3$, be elements of $\mathfrak{sl}(2,\mathbb{C})$. We set $R^2 = \sum_{\ell=1}^{3} D_1^{(jj)}(R_\ell^2)$ and $l^2 = \sum_{\ell=1}^{3} D_2^{(jj)}(l_\ell^2)$. Assume that

$$R^2 = e^4 + 2E\frac{l^2 + \hbar^2}{m},$$

and that

$$-\frac{m}{2E}R^2 + l^2 = 2\hbar^2(C_1^{(jj)} + C_2^{(jj)}).$$

Find the value of the energy E as a function of the constant R_H and the integer $n = 2j + 1$.

Solutions

Part I

1. (a) The map $M \mapsto q_M$ respects multiplication, since

$$\mathcal{I}^2 = \mathcal{J}^2 = \mathcal{K}^2 = -I,\ \mathcal{I}\mathcal{J} = -\mathcal{J}\mathcal{I} = \mathcal{K},$$
$$\mathcal{J}\mathcal{K} = -\mathcal{K}\mathcal{J} = \mathcal{I},\ \mathcal{K}\mathcal{I} = -\mathcal{I}\mathcal{K} = \mathcal{J}.$$

The restriction of this map is thus a group morphism. The image of SU(2) is $\mathbb{S}$, since if $A \in \mathrm{SU}(2)$, $A = \alpha_0 I + \alpha_1 \mathcal{I} + \alpha_2 \mathcal{J} + \alpha_3 \mathcal{K}$ with $\sum_{p=0}^{3} (\alpha_p)^2 = 1$; hence $q_A \in \mathbb{S}$, and conversely, each quaternion of norm 1 is the image of a matrix A belonging to SU(2). Thus $A \in \mathrm{SU}(2) \mapsto q_A \in \mathbb{S}$ is a group isomorphism.

(b) The vector space $\mathcal{H}$ is in fact a subalgebra of the associative algebra of complex 2×2 matrices (with matrix multiplication). If $X \in \mathfrak{su}(2)$, then $X^k/k! \in \mathcal{H}$ for each $k \in \mathbb{N}$, and one has, in $\mathbb{H}$,

$$q_{\exp(tX)} = q_{I\,+\,tX\,+\,\mathrm{O}(t^2)} = 1 + t q_X + \mathrm{O}(t^2).$$

Hence $\frac{\mathrm{d}}{\mathrm{d}t} q_{\exp(tX)}\big|_{t=0} = q_X$.

One can also use the equation $\exp(tX) = \cos t\ I + \sin t\ X$, which holds for $X \in \mathfrak{su}(2)$ of determinant 1, whence for every $X \in \mathfrak{su}(2)$,

$$\exp(tX) = \cos(t\|X\|)I + \sin(t\|X\|)\frac{X}{\|X\|},$$

whence $q_{\exp(tX)} = \cos(t\|X\|)1 + \frac{1}{\|X\|}\sin(t\|X\|)q_X$, and therefore $\frac{\mathrm{d}}{\mathrm{d}t} q_{\exp(tX)}\big|_{t=0} = q_X$.

2. (a) In the identification of $\mathbb{H}$ with $\mathbb{R}^4$, the quaternion norm can be identified with the Euclidean norm. The map $\sigma(A, B)$ preserves the quaternion norm because

$$\|q_A q (q_B)^{-1}\| = \|q_A\|\|q\|\|(q_B)^{-1}\|,$$

which is equal to $\|q\|$, since $\|q_A\| = \|q_B\| = \|(q_B)^{-1}\| = 1$, for $A, B \in \mathrm{SU}(2)$. Thus σ can be identified with an element of O(4).

(b) Let (A, B) and (A', B') be elements of SU(2) × SU(2). Then

$$\sigma(A, B) \circ \sigma(A', B')\ q = \sigma(A, B)(q_{A'} q (q_{B'})^{-1}) = q_A q_{A'} q (q_{B'})^{-1} (q_B)^{-1}.$$

Now $A \mapsto q_A$ is a group morphism, and hence

$$q_A q_{A'} = q_{AA'} \quad \text{and} \quad (q_{B'})^{-1}(q_B)^{-1} = (q_B q_{B'})^{-1} = (q_{BB'})^{-1}.$$

Hence

$$\sigma(A, B) \circ \sigma(A', B') = \sigma(AA', BB').$$

The image of SU(2) × SU(2) is contained in SO(4) because SU(2) × SU(2) is connected.

(c) The kernel of σ is the set of ordered pairs (A, B) such that for every $q \in \mathbb{H}$, $q_A q = q q_B$. Taking $q = 1$, we see that $q_A = q_B$.We set

$$q_A = \alpha_0 1 + \alpha_1 i + \alpha_2 j + \alpha_3 k.$$

The condition $q_A i = i q_A$ implies

$$\alpha_0 i - \alpha_1 - \alpha_2 k + \alpha_3 j = \alpha_0 i - \alpha_1 + \alpha_2 k - \alpha_3 j,$$

whence

$$\alpha_2 = \alpha_3 = 0.$$

The condition $q_A j = j q_B$ implies $\alpha_1 = 0$. One thus obtains $q_A = q_B = \alpha_0 1$. Because $\|q_A\| = 1$, the solutions are $q_A = q_B = 1$ and $q_A = q_B = -1$. The kernel of σ is thus the set

$$\{(I, I), (-I, -I)\} \subset \mathrm{SU}(2) \times \mathrm{SU}(2).$$

3. (a) We have

$$\begin{aligned}(D\sigma)(X,Y)q &= \frac{\mathrm{d}}{\mathrm{d}t}\sigma(\exp(tX), \exp(tY))q\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(q_{\exp(tX)}\ q\ q_{\exp(-tY)})\Big|_{t=0} \\ &= q_X q - q q_Y,\end{aligned}$$

by the equation $\frac{\mathrm{d}}{\mathrm{d}t} q_{\exp(tX)}\big|_{t=0} = q_X$.

(b) We know that $D\sigma : \mathfrak{su}(2) \oplus \mathfrak{su}(2) \to \mathfrak{so}(4)$ and that $D\sigma$ is a morphism of Lie algebras.

The kernel of $D\sigma$ is the set of pairs $(X, Y) \in \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ such that $q_X q = q q_Y$, for every $q \in \mathbb{H}$. This implies $q_X = q_Y = \alpha_0 1$. Because X and Y belong to $\mathfrak{su}(2)$, q_X and q_Y are zero. Hence $\mathrm{Ker} D\sigma = (0, 0)$. Because $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and $\mathfrak{so}(4)$ have the same dimension, 6, this injective morphism is an isomorphism.

(c) Every element of SO(4) can be written $\exp Z_1 \cdots \exp Z_k$, where $Z_\ell \in \mathfrak{so}(4)$. Because $D\sigma$ is bijective, $Z_\ell = D\sigma(X_\ell, Y_\ell)$ for some ordered pair $(X_\ell, Y_\ell) \in \mathfrak{su}(2)$. So $\exp Z_\ell = \exp D\sigma(X_\ell, Y_\ell) = \sigma(\exp X_\ell, \exp Y_\ell)$; thus $\exp Z_\ell \in \mathrm{Im}\sigma$. We deduce that $\exp Z_1 \cdots \exp Z_k \in \mathrm{Im}\sigma$, and thus $\mathrm{Im}\sigma = \mathrm{SO}(4)$.

Part II

4. The irreducible representations of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ are $\mathcal{D}^{j_1} \times \mathcal{D}^{j_2}$, $(j_1, j_2) \in \frac{1}{2}\mathbb{N} \times \frac{1}{2}\mathbb{N}$. We set

$$\mathcal{D}^{(j_1 j_2)} = \mathcal{D}^{j_1} \times \mathcal{D}^{j_2}.$$

The irreducible representations of SO(4) are the $\mathcal{D}^{(j_1 j_2)}$ that can be factored through the projection σ of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ onto SO(4), that is, that satisfy $\mathcal{D}^{(j_1 j_2)}(-I, -I) = \mathrm{Id}_{\mathrm{V}^{j_1} \otimes \mathrm{V}^{j_2}}$. Now,

$$\begin{aligned}\mathcal{D}^{(j_1 j_2)}(-I, -I)(v_1 \otimes v_2) &= \mathcal{D}^{j_1}(-I)v_1 \otimes \mathcal{D}^{j_2}(-I)v_2 \\ &= (-1)^{j_1 + j_2} v_1 \otimes v_2.\end{aligned}$$

Therefore the irreducible representations of SO(4) are $\mathcal{D}^{(j_1 j_2)}$, $(j_1, j_2) \in \frac{1}{2}\mathbb{N} \times \frac{1}{2}\mathbb{N}$, such that $j_1 + j_2 \in \mathbb{N}$.

The irreducible representations of SU(2) × SU(2) and their dimensions are thus

$$
\begin{array}{cccccccc}
\mathcal{D}^{(00)} & \mathcal{D}^{(0\frac{1}{2})} & \mathcal{D}^{(01)} & \mathcal{D}^{(0\frac{3}{2})} & \mathcal{D}^{(02)} & \cdots & \mathcal{D}^{(0j_2)} & \cdots \\
1 & 2 & 3 & 4 & 5 & & 2j_2+1 & \\
\mathcal{D}^{(\frac{1}{2}0)} & \mathcal{D}^{(\frac{1}{2}\frac{1}{2})} & \mathcal{D}^{(\frac{1}{2}1)} & \mathcal{D}^{(\frac{1}{2}\frac{3}{2})} & \mathcal{D}^{(\frac{1}{2}2)} & \cdots & \mathcal{D}^{(\frac{1}{2}j_2)} & \cdots \\
2 & 4 & 6 & 8 & 10 & & 2(2j_2+1) & \cdots \\
\mathcal{D}^{(10)} & \mathcal{D}^{(1\frac{1}{2})} & \mathcal{D}^{(11)} & \mathcal{D}^{(1\frac{3}{2})} & \mathcal{D}^{(12)} & \cdots & \mathcal{D}^{(1j_2)} & \cdots \\
3 & 6 & 9 & 12 & 15 & & 3(2j_2+1) & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mathcal{D}^{(j_1 0)} & \mathcal{D}^{(j_1\frac{1}{2})} & \mathcal{D}^{(j_1 1)} & \mathcal{D}^{(j_1\frac{3}{2})} & \mathcal{D}^{(j_1 2)} & \cdots & \mathcal{D}^{(j_1 j_2)} & \cdots \\
2j_1+1 & 2(2j_1+1) & 3(2j_1+1) & 4(2j_1+1) & 5(2j_1+1) & & (2j_1+1)(2j_2+1) & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
$$

The irreducible representations of SO(4) are

$$
\begin{array}{cccccc}
\mathcal{D}^{(00)} & \mathcal{D}^{(01)} & \mathcal{D}^{(02)} & \cdots & \mathcal{D}^{(0j_2)} & j_2 \text{ integer} \\
\mathcal{D}^{(\frac{1}{2}\frac{1}{2})} & \mathcal{D}^{(\frac{1}{2}\frac{3}{2})} & & \cdots & \mathcal{D}^{(\frac{1}{2}j_2)} & j_2 \text{ half-integer} \\
\mathcal{D}^{(10)} & \mathcal{D}^{(11)} & & \cdots & \mathcal{D}^{(1j_2)} & j_2 \text{ integer} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \mathcal{D}^{(j_1j_2)} & j_1 \text{ and } j_2 \text{ integers or } j_1 \text{ and } j_2 \text{ half-integers.}
\end{array}
$$

5. By definition, we have

$$
\begin{aligned}
D(\rho_1 \times \rho_2)(X_1, X_2)(v_1 \otimes v_2) &= \frac{\mathrm{d}}{\mathrm{d}t}\rho_1(\exp(tX_1))v_1 \otimes \rho_2(\exp(tX_2))v_2\Big|_{t=0} \\
&= D\rho_1(X_1)v_1 \otimes v_2 + v_1 \otimes D\rho_2(X_2)v_2.
\end{aligned}
$$

6. In V^j, we use the basis $(|j,-j\rangle,\ |j,-j+-1\rangle, \ldots, |j,j-1\rangle,\ |j,j\rangle)$ in this order. The matrix of $D^j(J_3)$ in this basis is

$$
\begin{pmatrix}
-j & 0 & & 0 \\
0 & -j+1 & & 0 \\
 & & \ddots & \\
0 & 0 & & j
\end{pmatrix}.
$$

On the other hand, $J_\pm = J_1 \pm iJ_2$, whence $J_1 = \frac{1}{2}(J_+ + J_-)$ and $J_2 = -\frac{i}{2}(J_+ - J_-)$, whence

$$
D^j(J_1) = \frac{1}{2}
\begin{pmatrix}
0 & \sqrt{2j} & 0 & \cdot \\
\sqrt{2j} & 0 & \sqrt{2(2j-1)} & \cdot \\
0 & \sqrt{2(2j-1)} & 0 & \cdot \\
\cdots & \cdots & \cdots & 0
\end{pmatrix},
$$

$$D^j(J_2) = \frac{1}{2}\begin{pmatrix} 0 & i\sqrt{2j} & 0 & \cdot \\ -i\sqrt{2j} & 0 & i\sqrt{2(2j-1)} & \cdot \\ 0 & -i\sqrt{2(2j-1)} & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

In the basis $(|\frac{1}{2}, -\frac{1}{2}\rangle,\ |\frac{1}{2}, \frac{1}{2}\rangle)$ of $V^{\frac{1}{2}}$,

$$D^{\frac{1}{2}}(J_3) = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = J_3,$$

$$D^{\frac{1}{2}}(J_1) = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -J_1, \quad D^{\frac{1}{2}}(J_2) = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = -J_2.$$

In the basis $(|1, -1\rangle,\ |1, 0\rangle,\ |1, 1\rangle)$ of V^1,

$$D^1(J_3) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$D^1(J_1) = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}, \quad D^1(J_2) = \frac{1}{2}\begin{pmatrix} 0 & i\sqrt{2} & 0 \\ -i\sqrt{2} & 0 & i\sqrt{2} \\ 0 & -i\sqrt{2} & 0 \end{pmatrix}.$$

By definition,

$$D^{(j_1,j_2)}(X, 0) = D^{j_1}(X) \otimes \mathrm{Id}_{V^{j_2}}, \quad D^{(j_1,j_2)}(0, X) = \mathrm{Id}_{V^{j_1}} \otimes D^{j_2}(X).$$

We set, for fixed j_1 and j_2, $|j_1, m_1\rangle \otimes |j_2, m_2\rangle = |m_1\,; m_2\rangle$, and we use the basis

$$(|-j_1; -j_2\rangle, |-j_1; -j_2+1\rangle, \dots, |-j_1; j_2\rangle, |-j_1+1; -j_2\rangle, \dots,$$
$$|-j_1+1; j_2\rangle, \dots, |j_1\ ; j_2\rangle).$$

(a) Let $j_1 = j_2 = \frac{1}{2}$. Then

$$D^{(\frac{1}{2}\frac{1}{2})}(J_3, 0) = \frac{1}{2}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad D^{(\frac{1}{2}\frac{1}{2})}(0, J_3) = \frac{1}{2}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$D^{(\frac{1}{2}\frac{1}{2})}(J_1, 0) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad D^{(\frac{1}{2}\frac{1}{2})}(0, J_1) = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and

$$D^{(\frac{1}{2}\frac{1}{2})}(J_2,0) = \frac{1}{2}\begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad D^{(\frac{1}{2}\frac{1}{2})}(0,J_2) = \frac{1}{2}\begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}.$$

(b) Set $j_1 = 1$ and $j_2 = \frac{1}{2}$. Then

$$D^{(1\frac{1}{2})}(J_3,0) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$D^{(1\frac{1}{2})}(0,J_3) = \frac{1}{2}\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(c) Let $j_1 = \frac{1}{2}$ and $j_2 = 1$. Then

$$D^{(\frac{1}{2}1)}(J_3,0) = \frac{1}{2}\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$D^{(\frac{1}{2}1)}(0,J_3) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

7. If $j_1 > j_2$, then j_1 is an eigenvalue of $D^{(j_1 j_2)}(J_3,0)$ and not of $D^{(j_2 j_1)}(J_3,0)$. Thus the representation $D^{(j_1 j_2)}$ is equivalent to $D^{(j_2 j_1)}$ if and only if $j_1 = j_2$.

8. We have

$$\sum_{\ell=1}^{3} D_1^{(j_1 j_2)}(J_\ell^2)(v_1 \otimes v_2) = \sum_{\ell=1}^{3} D^{j_1}(J_\ell^2)v_1 \otimes v_2 = j_1(j_1+1)v_1 \otimes v_2$$

for $v_1 \in V^{j_1}$ and $v_2 \in V^{j_2}$, and a similar result for $\sum_{\ell=1}^{3} D_2^{(j_1 j_2)}(J_\ell^2)$. Hence

$$C_i^{(j_1 j_2)} = j_i(j_i+1)\mathrm{Id}_{V^{j_1}\otimes V^{j_2}}, \quad i = 1, 2.$$

9. From $R^2 = e^4 + 2E\frac{l^2+\hbar^2}{m}$, we find that

$$-\frac{m}{2E}R^2 + l^2 + \hbar^2 = -\frac{me^4}{2E},$$

whence, by replacement from $-\frac{m}{2E}R^2 + l^2 = 2\hbar^2(C_1^{(jj)} + C_2^{(jj)})$,

$$\hbar^2(2(C_1^{(jj)} + C_2^{jj}) + 1) = -\frac{me^4}{2E}.$$

Because $C_1^{(jj)} = C_2^{(jj)} = j(j+1)$, and since we set $\frac{me^4}{2\hbar^2} = R_H$, we obtain

$$E = -\frac{R_H}{(2j+1)^2},$$

or

$$E = -\frac{R_H}{n^2}.$$

The degeneracy of the energy level $E = -R_H/n^2$ of the hydrogen atom is the dimension $(2j+1)^2 = n^2$ of the corresponding eigenspace of the Hamiltonian operator that is the support of the irreducible representation $D^{(jj)}$ of SO(4). This degeneracy, which has been observed experimentally, can therefore be explained by the fact that the full symmetry group of the hydrogen atom is the group SO(4), a symmetry group larger than the obvious symmetry group SO(3). This is what is called the *hidden symmetry of the hydrogen atom*. See, for example, Singer (2005) or Gilmore (2008). The operator (R_1, R_2, R_3) corresponds to the Lenz–Pauli vector $\overrightarrow{R}$ in classical mechanics. The fact that it involves a representation with $j_1 = j_2$ is a consequence of the fact that the Lenz–Pauli vector $\overrightarrow{R}$ is orthogonal to the angular momentum $\overrightarrow{\ell}$. In other works, the constant called the Rydberg constant is $R'_H = R_H/h = R_H/2\pi\hbar$. The values of these constants are $R_H = 13.61$ eV and $R'_H = 3.29 \times 10^{15}$ s^{-1}. See also Sudbery (1986) and Blaizot–Toédano (1997).

7 Projection Operators

Part I

Let $G = \mathfrak{S}_3$ be the symmetric group on three elements. We denote by e the identity, by c the cyclic permutation $\left(\begin{smallmatrix}1&2&3\\2&3&1\end{smallmatrix}\right)$ and by t the transposition $\left(\begin{smallmatrix}1&2&3\\1&3&2\end{smallmatrix}\right)$. We denote by ρ_0 the representation of $\mathfrak{S}_3$ in $\mathbb{C}^2$ such that

$$\rho_0(c) = \begin{pmatrix} j & 0 \\ 0 & j^2 \end{pmatrix}, \quad \rho_0(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where j is the zero of $x^2 + x + 1$ with positive imaginary part.

Let $E = \mathcal{F}(G) = \mathbb{C}[G]$, the vector space of functions on G taking values in $\mathbb{C}$, with basis $(\epsilon_g)_{g \in G}$. Let (E, R) be the regular representation of G.

1. Write the decomposition of R as a direct sum of irreducible representations.

We propose finding a vector subspace of E invariant under R and such that the restriction of R to this subspace is equivalent to ρ_0.

We denote by $\rho_{\alpha\beta}$ the matrix coefficients of ρ_0 $(1 \le \alpha \le 2,\ 1 \le \beta \le 2)$, and we set

$$P_{\alpha\beta} = \frac{1}{3} \sum_{g \in G} \rho_{\beta\alpha}(g^{-1}) R(g) \in \text{ End } (E).$$

2. (a) Find the matrices $\rho_0(e), \rho_0(c^2), \rho_0(tc), \rho_0(tc^2)$.
   (b) Express P_{11}, P_{22}, P_{21}, and P_{12} explicitly as linear combinations of $R(g)$, $g \in G$.
   (c) Calculate $P_{11}(\epsilon_h)$, for every $h \in G$. What is the dimension of the vector subspace $P_{11}(E)$ of E? Is it invariant under R?
   (d) Same questions as in (c) replacing P_{11} by P_{22}.
3. Choose a basis (u_1, u_1') of $P_{11}(E)$.
   (a) Calculate $P_{21}(u_1)$ and show that $P_{21}(u_1) \in P_{22}(E)$.
   (b) Let $E_{0,1}$ be the vector subspace of E generated by u_1 and $P_{21}(u_1)$. Show that $E_{0,1}$ is invariant under R and that the restriction of R to $E_{0,1}$ is equivalent to ρ_0.
   (c) What can be said about $R\big|_{E'_{0,1}}$, where $E'_{0,1}$ is the vector subspace of E generated by u_1' and $P_{21}(u_1')$?
4. Show that $P_{11} + P_{22}$ is the projection P_0 of E onto the isotypic component of type ρ_0 of E, and that $P_0(E) = E_{0,1} \oplus E'_{0,1}$.

Part II

Let G be a finite group, and let $(E^{(i)}, \rho^{(i)})$, $i = 1, \ldots, N$, be its inequivalent irreducible representations. We set $\dim E^{(i)} = d_i$ and we denote by $\rho^{(i)}_{\alpha\beta}$, for $1 \le \alpha \le d_i, 1 \le \beta \le d_i$, the matrix coefficients of the representation $\rho^{(i)}$ in a basis (e_α) of $E^{(i)}$. We recall that for all integers $\alpha, \beta, \lambda, \mu$ between 1 and d_i,

$$\sum_{g \in G} \rho^{(i)}_{\beta\alpha}(g^{-1}) \rho^{(j)}_{\lambda\mu}(g) = \begin{cases} 0 & \text{if } i \neq j, \\ \frac{|G|}{d_i} \delta_{\alpha\lambda} \delta_{\beta\mu} & \text{if } i = j. \end{cases} \tag{1}$$

Let (E, ρ) be a representation of G. We set $E = \oplus_{i=1}^{N} V^{(i)}$, with $V^{(i)} = m_i E^{(i)}$, where m_i is the multiplicity of $E^{(i)}$ in E. We fix the index $i, 1 \leq i \leq N$, and for $1 \leq \alpha \leq d_i, 1 \leq \beta \leq d_i$, we consider

$$P_{\alpha\beta}^{(i)} = \frac{d_i}{|G|} \sum_{g \in G} \rho_{\beta\alpha}^{(i)}(g^{-1})\rho(g) \in \text{ End } (E).$$

5. Compare $\sum_{\alpha=1}^{d_i} P_{\alpha\alpha}^{(i)}$ to the projection $P^{(i)}$ of E onto the isotypic component $V^{(i)}$ of E.

6. Set $1 \leq i \leq N,\ 1 \leq j \leq N$. Show that $P_{\alpha\beta}^{(i)}(V^{(j)}) = \{0\}$ if $j \neq i$, and that

$$P_{\alpha\beta}^{(i)}(e_\gamma) = \delta_{\beta\gamma} e_\alpha \quad \text{for } 1 \leq \gamma \leq d_i.$$

7. (a) Let $\alpha, \beta, \lambda, \mu$ be integers between 1 and d_i. Show that $P_{\alpha\beta}^{(i)} \circ P_{\lambda\mu}^{(j)} = 0$ if $i \neq j$, and that

$$P_{\alpha\beta}^{(i)} \circ P_{\lambda\mu}^{(i)} = \delta_{\beta\lambda} P_{\alpha\mu}^{(i)}.$$

(b) Conclude that $P_{\alpha\beta}^{(i)}$ vanishes on $P_{\gamma\gamma}^{(i)}(E)$ if $\gamma \neq \beta$, and that $P_{\alpha\beta}^{(i)}$ defines an isomorphism of $P_{\beta\beta}^{(i)}(E)$ onto $P_{\alpha\alpha}^{(i)}(E)$.

8. Show that $\rho(g) \circ P_{\alpha\beta}^{(i)} = \sum_{\gamma=1}^{d_i} \rho_{\gamma\alpha}^{(i)}(g) P_{\gamma\beta}^{(i)}$.

9. Let $x \in P_{11}^{(i)}(E)$, $x \neq 0$. For $\beta = 1, \ldots,$ set $d_i,\ x_\beta = P_{\beta 1}^{(i)}(x)$. Let $E_1^{(i)}(x)$ be the vector subspace of E generated by $(x_1, \ldots, x_{d_i})$. Show that the restriction of ρ to $E_1^{(i)}(x)$ is equivalent to $\rho^{(i)}$.

10. Construct m_i vector subspaces of E whose direct sum is $V^{(i)}$ and such that the restriction of ρ to each is equivalent to $\rho^{(i)}$.

11. Define projection operators $P_{\alpha\beta}^{(i)}$ in order to apply an analogous procedure to the decomposition of a representation of a compact group.

Solutions

Part I

1. The irreducible representations of $G = \mathfrak{S}_3$ are the trivial representation $\underline{1}$ and the signature representation ϵ, both one-dimensional, and the two-dimensional representation ρ_0. Hence

$$R = \underline{1} \oplus \epsilon \oplus 2\rho_0.$$

2. (a) By calculating matrix products, we obtain

$$\rho_0(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho_0(c) = \begin{pmatrix} j & 0 \\ 0 & j^2 \end{pmatrix}, \quad \rho_0(c^2) = \begin{pmatrix} j^2 & 0 \\ 0 & j \end{pmatrix},$$

$$\rho_0(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_0(tc) = \begin{pmatrix} 0 & j^2 \\ j & 0 \end{pmatrix}, \quad \rho_0(tc^2) = \begin{pmatrix} 0 & j \\ j^2 & 0 \end{pmatrix}.$$

We shall make use of the equations $c^3 = e$, $t^2 = e$, $ct = tc^2$, $c^2t = tc$, and $ctc = t$.

(b) We note that $c^{-1} = c^2$, $(c^2)^{-1} = c$. By definition,

$$P_{11} = \frac{1}{3}(R(e) + j^2R(c) + jR(c^2)), \quad P_{22} = \frac{1}{3}(R(e) + jR(c) + j^2R(c^2)).$$

Because t, tc, and tc^2 are equal to their inverses, and since e, c, and c^2 are diagonal, we obtain

$$P_{12} = \frac{1}{3}\sum_{g\in G}\rho_{21}(g^{-1})R(g) = \frac{1}{3}(R(t) + jR(tc) + j^2R(tc^2)),$$
$$P_{21} = \frac{1}{3}\sum_{g\in G}\rho_{12}(g^{-1})R(g) = \frac{1}{3}(R(t) + j^2R(tc) + jR(tc^2)).$$

(c) We know that $R(g)\epsilon_h = \epsilon_{gh}$, for $g, h \in G$. Thus

$$P_{11}(\epsilon_h) = \frac{1}{3}(\epsilon_h + j^2\epsilon_{ch} + j\epsilon_{c^2h}).$$

Hence

$$P_{11}(\epsilon_e) = \frac{1}{3}(\epsilon_e + j^2\epsilon_c + j\epsilon_{c^2}),$$
$$P_{11}(\epsilon_t) = \frac{1}{3}(\epsilon_t + j^2\epsilon_{ct} + j\epsilon_{c^2t}) = \frac{1}{3}(\epsilon_t + j\epsilon_{tc} + j^2\epsilon_{tc^2}).$$

We set

$$P_{11}(\epsilon_e) = u_1, \quad P_{11}(\epsilon_t) = u_1' \ ,$$

which are linearly independent. Then,

$$P_{11}(\epsilon_c) = ju_1, \quad P_{11}(\epsilon_{c^2}) = j^2u_1,$$
$$P_{11}(\epsilon_{tc}) = j^2u_1', \quad P_{11}(\epsilon_{tc^2}) = ju_1'.$$

Thus $P_{11}(E)$ is two-dimensional.

We have $R(t)u_1 = \frac{1}{3}(\epsilon_t + j^2\epsilon_{tc} + j\epsilon_{tc^2})$, which does not lie in $P_{11}(E)$. Thus $P_{11}(E)$ is not invariant under R.

(d) By (b) and the equation $R(g)\epsilon_h = \epsilon_{gh}$, we obtain

$$P_{22}(\epsilon_h) = \frac{1}{3}(\epsilon_h + j\epsilon_{ch} + j^2\epsilon_{c^2h}).$$

Hence

$$P_{22}(\epsilon_e) = \frac{1}{3}(\epsilon_e + j\epsilon_c + j^2\epsilon_{c^2}),$$
$$P_{22}(\epsilon_t) = \frac{1}{3}(\epsilon_t + j\epsilon_{ct} + j^2\epsilon_{c^2t}) = \frac{1}{3}(\epsilon_t + j^2\epsilon_{tc} + j\epsilon_{tc^2}).$$

We set

$$P_{22}(\epsilon_t) = v_1, \quad P_{22}(\epsilon_e) = v_1',$$

which are linearly independent. We have

$$P_{22}(\epsilon_c) = j^2 v_1', \quad P_{22}(\epsilon_{c^2}) = j v_1',$$
$$P_{22}(\epsilon_{tc}) = j v_1, \quad P_{22}(\epsilon_{tc^2}) = j^2 v_1.$$

Thus $P_{22}(E)$ is two-dimensional.

We have $R(t)v_1 = \frac{1}{3}(\epsilon_e + j^2\epsilon_c + j\epsilon_{c^2})$, which does not lie in $P_{22}(E)$. Thus $P_{22}(E)$ is not invariant under R.

3. (a) We calculate $P_{21}(u_1) = \frac{1}{3}(R(t)u_1 + j^2 R(tc)u_1 + jR(tc^2)u_1)$. We have

$$R(t)u_1 = \frac{1}{3}(\epsilon_t + j^2\epsilon_{tc} + j\epsilon_{tc^2}) = v_1,$$
$$R(tc)u_1 = \frac{1}{3}(\epsilon_{tc} + j^2\epsilon_{tc^2} + j\epsilon_t) = jv_1,$$
$$R(tc^2)u_1 = \frac{1}{3}(\epsilon_{tc^2} + j^2\epsilon_t + j\epsilon_{tc}) = j^2 v_1.$$

Hence $P_{21}(u_1) = v_1$, which belongs to $P_{22}(E)$. Furthermore, u_1 and v_1 are linearly independent.

(b) We have $R(c)u_1 = \frac{1}{3}(\epsilon_c + j^2\epsilon_{c^2} + j\epsilon_e) = ju_1$ and $R(t)u_1 = v_1$, and thus u_1 and $P_{21}(u_1)$ generate a two-dimensional vector subspace $E_{0,1}$ of E, invariant under R. We have $R(c)v_1 = \frac{1}{3}(\epsilon_{tc^2} + j\epsilon_{tc} + j^2\epsilon_t) = j^2 v_1$ and $R(t)v_1 = \frac{1}{3}(\epsilon_e + j\epsilon_{c^2} + j^2\epsilon_c) = u_1$. Thus the matrices of $R\big|_{E_{0,1}}$ in the basis (u_1, v_1) are

$$C = \begin{pmatrix} j & 0 \\ 0 & j^2 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus $R\big|_{E_{0,1}}$ is equivalent to ρ_0.

(c) Similarly, $R\big|_{E'_{0,1}}$ is equivalent to ρ_0.

4. By definition,

$$\begin{aligned} P_{11} + P_{22} &= \frac{1}{3}\sum_{g\in G}(\rho_{11}(g^{-1}) + \rho_{22}(g^{-1}))R(g) \\ &= \frac{1}{3}\sum_{g\in G}\overline{\chi_{\rho_0}(g)}R(g), \end{aligned}$$

where χ_{ρ_0} is the character of the representation ρ_0. We know that this endomorphism of E is the projection P_0 onto the isotypic component of type ρ_0, which here has multiplicity two. Because $E_{0,1}$ and $E'_{0,1}$ are of type ρ_0 and have trivial intersection $\{0\}$, we have

$$P_0(E) = E_{0,1} \oplus E'_{0,1}.$$

Part II

5. We know that $\sum_{\alpha=1}^{d_i} \rho^{(i)}_{\alpha\alpha}(g^{-1}) = \chi_{\rho^{(i)}}(g^{-1}) = \overline{\chi_{\rho^{(i)}}(g)}$. Thus, by the definitions,

$$\sum_{\alpha=1}^{d_i} P^{(i)}_{\alpha\alpha} = P^{(i)}.$$

6. Let (φ_λ), $\lambda = 1, \dots, d_j$, be a basis of $E^{(j)}$. Since $\rho(g)(\varphi_\mu) = \sum_{\lambda=1}^{d_j} \rho^{(j)}_{\lambda\mu}(g)\varphi_\lambda$, it follows from (1) that if $i \neq j$, then

$$P^{(i)}_{\alpha\beta}(\varphi_\mu) = \frac{d_i}{|G|} \sum_{\lambda=1}^{d_j} \left(\sum_{g \in G} \rho^{(i)}_{\beta\alpha}(g^{-1}) \rho^{(j)}_{\lambda\mu}(g) \right) \varphi_\lambda = 0.$$

Because $V^{(j)} = m_j E^{(j)}$, the operator $P^{(i)}_{\alpha\beta}$ vanishes on $V^{(j)}$.

On the other hand, using (1) for $i = j$ yields

$$P^{(i)}_{\alpha\beta}(e_\gamma) = \frac{d_i}{|G|} \sum_{\lambda=1}^{d_i} \left(\sum_{g \in G} \rho^{(i)}_{\beta\alpha}(g^{-1}) \rho^{(i)}_{\lambda\gamma}(g) \right) e_\lambda = \sum_{\lambda=1}^{d_i} \delta_{\alpha\lambda}\delta_{\beta\gamma} e_\lambda = \delta_{\beta\gamma} e_\alpha.$$

7. (a) We calculate

$$\begin{aligned} P^{(i)}_{\alpha\beta} \circ P^{(j)}_{\lambda\mu} &= \frac{d_i d_j}{|G|^2} \left(\sum_{g \in G} \rho^{(i)}_{\beta\alpha}(g^{-1}) \rho(g) \right) \circ \left(\sum_{h \in G} \rho^{(j)}_{\mu\lambda}(h^{-1}) \rho(h) \right) \\ &= \frac{d_i d_j}{|G|^2} \sum_{g \in G} \sum_{h \in G} \rho^{(i)}_{\beta\alpha}(g^{-1}) \rho^{(j)}_{\mu\lambda}(h^{-1}) \rho(gh). \end{aligned}$$

Setting $gh = k$, whence $g^{-1} = hk^{-1}$, we obtain

$$P^{(i)}_{\alpha\beta} \circ P^{(j)}_{\lambda\mu} = \frac{d_i d_j}{|G|^2} \sum_{k \in G} \left(\sum_{h \in G} \rho^{(i)}_{\beta\alpha}(hk^{-1}) \rho^{(j)}_{\mu\lambda}(h^{-1}) \right) \rho(k).$$

We use $\rho^{(i)}_{\beta\alpha}(hk^{-1}) = \sum_{\gamma=1}^{d_i} \rho^{(i)}_{\beta\gamma}(h) \rho^{(i)}_{\gamma\alpha}(k^{-1})$ along with equation (1), and we obtain

$$\begin{aligned} P^{(i)}_{\alpha\beta} \circ P^{(j)}_{\lambda\mu} &= \frac{d_i d_j}{|G|^2} \sum_{k \in G} \sum_{\gamma=1}^{d_i} \left(\sum_{h \in G} \rho^{(i)}_{\beta\gamma}(h) \rho^{(j)}_{\mu\lambda}(h^{-1}) \right) \rho^{(i)}_{\gamma\alpha}(k^{-1}) \rho(k) \\ &= \delta_{ij} \frac{d_i}{|G|} \sum_{k \in G} \sum_{\gamma=1}^{d_i} \delta_{\gamma\mu} \delta_{\beta\lambda} \rho^{(i)}_{\gamma\alpha}(k^{-1}) \rho(k) \\ &= \delta_{ij} \delta_{\beta\lambda} \frac{d_i}{|G|} \sum_{k \in G} \rho^{(i)}_{\mu\alpha}(k^{-1}) \rho(k) = \delta_{ij} \delta_{\beta\lambda} P^{(i)}_{\alpha\mu}, \end{aligned}$$

which proves the desired results.

(b) In particular, $P^{(i)}_{\alpha\alpha} \circ P^{(j)}_{\beta\beta} = 0$ if $i \neq j$, and $P^{(i)}_{\alpha\beta} \circ P^{(i)}_{\gamma\gamma} = 0$ if $\gamma \neq \beta$. Furthermore,

$$P^{(i)}_{\alpha\alpha} \circ P^{(i)}_{\beta\beta} = 0 \text{ if } \alpha \neq \beta, \quad P^{(i)}_{\alpha\alpha} \circ P^{(i)}_{\alpha\alpha} = P^{(i)}_{\alpha\alpha}.$$

Therefore, the $P^{(i)}_{\alpha\alpha}$ are projection operators onto subspaces of $V^{(i)}$ such that their images $P^{(i)}_{\alpha\alpha}(E)$ and $P^{(i)}_{\beta\beta}(E)$ have trivial intersection if $\alpha \neq \beta$.

We now show, by means of the result of part (a), that $P^{(i)}_{\alpha\beta}$ is an isomorphism of $P^{(i)}_{\beta\beta}(E)$ onto $P^{(i)}_{\alpha\alpha}(E)$. We use

$$P^{(i)}_{\alpha\beta} \circ P^{(i)}_{\beta\beta} = P^{(i)}_{\alpha\beta} = P^{(i)}_{\alpha\alpha} \circ P^{(i)}_{\alpha\beta}$$

to show that $P^{(i)}_{\alpha\beta}(P^{(i)}_{\beta\beta}(E)) \subset P^{(i)}_{\alpha\alpha}(E)$. Let us show that the restriction of $P^{(i)}_{\alpha\beta}$ to $P^{(i)}_{\beta\beta}(E)$ is bijective. It is injective, since if $u \in E$ satisfies $P^{(i)}_{\alpha\beta} P^{(i)}_{\beta\beta}(u) = 0$, then $0 = P^{(i)}_{\beta\alpha} P^{(i)}_{\alpha\beta} P^{(i)}_{\beta\beta}(u) = P^{(i)}_{\beta\beta} P^{(i)}_{\beta\beta}(u) = P^{(i)}_{\beta\beta}(u)$. It is also surjective onto $P^{(i)}_{\alpha\alpha}(E)$, because for every $u \in E$, and for every $\beta = 1, \ldots, d_i$,

$$P^{(i)}_{\alpha\alpha}(u) = P^{(i)}_{\alpha\beta} P^{(i)}_{\beta\alpha}(u) = P^{(i)}_{\alpha\beta} P^{(i)}_{\beta\beta} P^{(i)}_{\beta\alpha}(u),$$

and thus $P^{(i)}_{\alpha\alpha}(u)$ belongs to the image under $P^{(i)}_{\alpha\beta}$ of $P^{(i)}_{\beta\beta}(E)$.

8. We calculate $\rho(g) \circ P^{(i)}_{\alpha\beta} = \frac{d_i}{|G|} \sum_{h \in G} \rho^{(i)}_{\beta\alpha}(h^{-1}) \rho(g) \rho(h)$. We set $gh = k$, whence $h = g^{-1}k$ and $h^{-1} = k^{-1}g$. We obtain

$$\begin{aligned} \rho(g) \circ P^{(i)}_{\alpha\beta} &= \frac{d_i}{|G|} \sum_{k \in G} \rho^{(i)}_{\beta\alpha}(k^{-1}g) \rho(k) \\ &= \frac{d_i}{|G|} \sum_{k \in G} \sum_{\gamma=1}^{d_i} \rho^{(i)}_{\beta\gamma}(k^{-1}) \rho^{(i)}_{\gamma\alpha}(g) \rho(k) \\ &= \sum_{\gamma=1}^{d_i} \rho^{(i)}_{\gamma\alpha}(g) P^{(i)}_{\gamma\beta}. \end{aligned}$$

9. We consider the family $(x_\beta = P^{(i)}_{\beta 1}(x))$, $\beta = 1, \ldots, d_i$, where $x \in P^{(i)}_{11}(E)$ and $x \neq 0$. Let us show that $(x_1, \ldots, x_\beta, \ldots, x_{d_i})$ are linearly independent. We remark first that $x_1 = x$, since $(P^{(i)}_{11})^2 = P^{(i)}_{11}$. Assume that there are complex numbers $\lambda_1, \ldots, \lambda_{d_i}$ such that $\sum_{\beta=1}^{d_i} \lambda_\beta P^{(i)}_{\beta 1}(x) = 0$. Then for every $\alpha = 1, \ldots, d_i$,

$$\sum_{\beta=1}^{d_i} \lambda_\beta P^{(i)}_{1\alpha} P^{(i)}_{\beta 1}(x) = \sum_{\beta=1}^{d_i} \lambda_\beta \delta_{\alpha\beta} P^{(i)}_{11}(x) = \lambda_\alpha x.$$

Because $x \neq 0$, we see that $\lambda_\alpha = 0$ for each α. The vector space $E^{(i)}_1(x)$ generated by $(x_1, \ldots, x_{d_i})$ is thus of dimension d_i. We also recall that $P^{(i)}_{\beta 1}$ is an isomorphism from $P_{11}^{(i)}(E)$ onto $P_{\beta\beta}^{(i)}(E)$ and that the $(P_{\beta\beta}^{(i)}(E))$, $\beta = 1, \ldots, d_i$, form a direct sum decomposition of $V^{(i)} = m_i E^{(i)}$.

By the equation proved in Question 8,

$$\rho(g)x_\beta = \rho(g)(P_{\beta 1}^{(i)}(x)) = \sum_{\gamma=1}^{d_i} \rho_{\gamma\beta}^{(i)}(g) P_{\gamma 1}^{(i)}(x)$$
$$= \sum_{\gamma=1}^{d_i} \rho_{\gamma\beta}^{(i)}(g) x_\gamma,$$

which shows that $\rho\big|_{E_1^{(i)}(x)}$ is equivalent to $\rho^{(i)}$.

10. In order to obtain m_i vector subspaces of E whose direct sum is $V^{(i)}$ and on each of which the group acts by the representation $\rho^{(i)}$, it suffices to apply the procedure of Question 9 successively to the vectors of a basis of $P_{11}^{(i)}(E)$, as we did in Part I of this problem. In fact, we have seen that for fixed x in $P_{11}^{(i)}(E)$, the corresponding vector subspace $E_1^{(i)}(x)$ is invariant under G. If x and y are linearly independent, $E_1^{(i)}(x) \cap E_1^{(i)}(y)$ reduces to $\{0\}$ by the irreducibility of the representation $\rho^{(i)}$.

We have already noted that the $(P_{\alpha\alpha}^{(i)}(E))$, $\alpha = 1, \ldots, d_i$, form a direct sum. Since $V^{(i)} = P^{(i)}(E) = \left(\sum_{\alpha=1}^{d_i} P_{\alpha\alpha}^{(i)}\right)(E)$, we obtain

$$V^{(i)} = \bigoplus_{\alpha=1}^{d_i} P_{\alpha\alpha}^{(i)}(E).$$

Because the $P_{\alpha\alpha}^{(i)}(E), \alpha = 1, \ldots, d_i$, are isomorphic, we have $\dim P_{\alpha\alpha}^{(i)}(E) = m_i$. Thus by choosing m_i basis vectors $(x^1, \ldots, x^{m_i})$ in $P_{11}^{(i)}(E)$ and by subjecting them to the process of Question 9, we obtain m_i vector subspaces $E_1^{(i)}(x^A)$, $A = 1, \ldots, m_i$, such that $\rho_{|E_1^{(i)}(x^A)}$ is equivalent to $\rho^{(i)}$ for every A, and $V^{(i)} = \bigoplus_{A=1}^{m_i} E_1^{(i)}(x^A)$. *This is the desired decomposition.*

Clearly, Part I of this problem consisted in the direct study of a particular case of this process, where $\rho^{(i)} = \rho_0$, one has chosen the basis (u_1, u_1') of $P_{11}^{(i)}(E) = P_{11}(E)$, and one has set $E_1^{(i)}(u^1) = E_{0,1}$, $E_1^{(i)}(u_1') = E_{0,1}'$.

11. For a compact group, we use the Haar integral $\int_G f(g)\mathrm{d}g$ to define

$$P_{\alpha\beta}^{(i)} = d_i \int_G \rho_{\beta\alpha}^{(i)}(g^{-1})\rho(g)\mathrm{d}g,$$

and we proceed as in the case of a finite group.

This method is useful for determining the Clebsch–Gordan coefficients, and has many applications in quantum mechanics. See for example the study of the distribution of isotopic-spin states for a collection of free particles in Barut–Rączka (1977, 1986), p. 183.

In that book, on p. 177, the authors write, “The technique of the projection operators is extremely useful, elegant and effective in the solution of various practical problems in representation theory and quantum physics.”

8 Symmetries of Fullerene Molecules

We propose to study certain properties of the icosahedron group, which is the symmetry group of rotations of “soccer-ball molecules,” C_{60}, also called *fullerenes* or *buckyballs.*

Part I: Character table of $\mathfrak{S}_5$

We give below, without proof, the list of conjugacy classes of the symmetric group $\mathfrak{S}_5$ on five elements and the number of elements of each class. We denote by

e the class of the identity, with one element,
a the class of the permutation (12): (1,2,3,4,5) $\mapsto$ (2,1,3,4,5), with 10 elements,
α the class of (123): (1,2,3,4,5) $\mapsto$ (2,3,1,4,5), with 20 elements,
b the class of (1234): (1,2,3,4,5) $\mapsto$ (2,3,4,1,5), with 30 elements,
γ the class of (12345): (1,2,3,4,5) $\mapsto$ (2,3,4,5,1), with 24 elements,
β the class of (12)(34): (1,2,3,4,5) $\mapsto$ (2,1,4,3,5), with 15 elements,
c the class of (12)(345): (1,2,3,4,5) $\mapsto$ (2,1,4,5,3), with 20 elements.

1. The permutation representation of $\mathfrak{S}_5$ decomposes as the direct sum of the trivial representation ρ_1 and a four-dimensional representation ρ_4. Find their characters. Find the character of the signature representation ρ'_1 as well as the character of the representation ρ'_4 obtained as the tensor product of ρ_4 with ρ'_1. Verify that ρ_4 and ρ'_4 are irreducible.

2. Show that for each finite-dimensional representation (V, ρ) of a finite group G of character χ_ρ, the character of the representation $\wedge^2\rho$ satisfies

$$\chi_{\wedge^2\rho}(g) = \frac{1}{2}((\chi_\rho(g))^2 - \chi_\rho(g^2));,$$

and that $\chi_{S^2\rho} = (\chi_\rho)^2 - \chi_{\wedge^2\rho}$. (Here $\wedge$ denotes the exterior product and S the symmetric product.)

3. Calculate the characters of the representations $\wedge^2(\rho_4)$ and $S^2(\rho_4)$. [Hint: Use the equations $\alpha^2 = \alpha,\ b^2 = b,\ \gamma^2 = \gamma,\ c^2 = \alpha$.] Show that $\wedge^2(\rho_4)$ is irreducible. Show that $S^2(\rho_4)$ is a direct sum of three irreducible representations, and then show that $S^2(\rho_4) = \rho_1 \oplus \rho_4 \oplus \rho_5$, where ρ_5 is a five-dimensional irreducible representation.

4. Give the character table of $\mathfrak{S}_5$.

Part II: Character table of $\mathfrak{A}_5$

Here is the list of conjugacy classes of the alternating group $\mathfrak{A}_5$ with the number of elements of each class:

e the class of the identity, with one element,
α the class of (123), with 20 elements,
β the class of (12)(34), with 15 elements,
γ the class of (12345), with 12 elements,
γ^2 the class of (23451): $(1,2,3,4,5) \mapsto (3,4,5,1,2)$, with 12 elements.

5. Show that the representations ρ_1, ρ_4, and ρ_5 stay irreducible when restricted to $\mathfrak{A}_5$. We denote these also by ρ_1, ρ_4, ρ_5.
Show that the restriction to $\mathfrak{A}_5$ of $\wedge^2(\rho_4)$ is the direct sum of two inequivalent irreducible representations ρ_3 and ρ_3', and that each of them is three-dimensional.

6. Let $\tau = \frac{1+\sqrt{5}}{2}$ be the golden mean. Show that the values of the characters of the representations ρ_3 and ρ_3' of $\mathfrak{A}_5$ on the classes $(e, \alpha, \beta, \gamma, \gamma^2)$ are $(3, 0, -1, \tau, 1-\tau)$ and $(3, 0, -1, 1-\tau, \tau)$.

7. Find the character table of $\mathfrak{A}_5$.

Part III: Some representations of the icosahedron group

It is known that the group $\mathfrak{A}_5$ can be identified with the subgroup $\mathcal{I}$ of the group of rotations of $\mathbb{R}^3$ leaving a regular icosahedron invariant. In this identification, the elements of the conjugacy classes $(e, \alpha, \beta, \gamma, \gamma^2)$ of $\mathfrak{A}_5$ become rotations through angles $\Big(0, 2\pi/3, \pi, 2\pi/5, 4\pi/5\Big)$, respectively. We recall that $2\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{2} = \tau$.

8. Denote by φ the covering morphism of SU(2) onto SO(3). What is the image under φ of the element $g_\alpha = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}$?

For $j \in \mathbb{N}$, we denote by $\underline{\mathcal{D}^j}$ the irreducible representation of dimension $2j+1$ of SO(3) obtained from the representation $\mathcal{D}^j$ of SU(2) by passing to the quotient. Show that the value of the character $\chi_{\underline{\mathcal{D}^j}}$ of $\underline{\mathcal{D}^j}$ on a rotation through an angle $\theta \in [0, \pi]$ is

$$\chi_{\underline{\mathcal{D}^j}}(\theta) = \sum_{m=-j}^{j} e^{im\theta}.$$

For each $j \in \mathbb{N}$, we consider the restriction to the subgroup $\mathcal{I}$ of the representation $\underline{\mathcal{D}^j}$. We denote it by R^j and its character by χ^j.

9. Calculate χ^j for $j \in \mathbb{N}$. Show that for each element $g \in I, g \neq e$, of $\mathcal{I}$ and for every integer $\ell \in \mathbb{N}$,

$$\chi^{30\ell+j}(g) = \chi^j(g).$$

10. Show that the representations R^0, R^1, and R^2 of the group $\mathcal{I}$ are irreducible. Decompose the representation R^3 into a direct sum of irreducible representations, and then decompose the representations $R^{30\ell+3}$, for every integer $\ell \geq 1$.

11. Decompose the representation $R^2 \otimes R^3$ of the group $\mathcal{I}$ into a direct sum of irreducible representations.

Epilogue: Representations of $G = \varphi^{-1}(\mathcal{I}) \subset \mathrm{SU2}$

Similarly, we can study the representations of the double cover of the icosahedron group, the inverse image of the group $\mathcal{I}$ under the morphism φ. It is a subgroup G of SU(2) with 120 elements. It has nine conjugacy classes, whose representatives are g_α, with

$$\alpha = 0,\ \pi,\ \frac{\pi}{2},\ \frac{\pi}{3},\ \frac{4\pi}{3},\ \frac{\pi}{5},\ \frac{7\pi}{5},\ \frac{6\pi}{5},\ \frac{2\pi}{5}.$$

These classes have $1, 1, 30, 20, 20, 12, 12, 12, 12$ elements respectively. Can you find the character table of G? Can you show that restricted to G, the representation $\mathcal{D}^j$ of SU(2) is irreducible for $0 \leq j \leq \frac{5}{2}$? Can you find the decomposition of the restrictions to G of the representations $\mathcal{D}^j,\ j \in \frac{1}{2}\,\mathbb{N}$, into a direct sum of irreducible representations?

Solutions

Part I: Character table of $\mathfrak{S}_5$

1. Let ρ_1 be the one-dimensional trivial representation. We know that $\chi_{\rho_1}(g) = 1$, for every g. In the permutation representation, $\chi(g)$ is the number of elements that are invariant under g. Because $\chi_{\rho_1} + \chi_{\rho_4} = \chi$, we obtain the following character table:

	1 e	10 a	20 α	30 b	24 γ	15 β	20 c
ρ_1	1	1	1	1	1	1	1
ρ_4	4	2	1	0	-1	0	-1
ρ_1'	1	-1	1	-1	1	1	-1
ρ_4'	4	-2	1	0	-1	0	1

We calculate $(\chi_{\rho_4}|\chi_{\rho_4}) = (\chi_{\rho_4'}|\chi_{\rho_4'}) = \frac{1}{120}\,(16 + 40 + 20 + 24 + 20) = 1$, and thus ρ_4 and ρ_4' are irreducible.

2. The representation ρ is unitarizable, so for every $g \in G,\ \rho(g)$ is diagonizable. Let (e_i) be a basis of V such that $\rho(g)e_i = \lambda_i(g)e_i$, where $\lambda_i(g) \in \mathbb{C}$. Then

$$(\rho(g) \otimes \rho(g))(e_i \otimes e_j - e_j \otimes e_i) = \lambda_i(g)\lambda_j(g)(e_i \otimes e_j - e_j \otimes e_i),$$

whence

$$\chi_{\wedge^2\rho}(g) = \sum_{i<j} \lambda_i(g)\lambda_j(g) = \frac{1}{2}\Big(\Big(\sum_i \lambda_i(g)\Big)^2 - \sum_i (\lambda_i(g))^2\Big)$$
$$= \frac{1}{2}\left((\chi_\rho(g))^2 - \chi_\rho(g^2)\right).$$

Because $\rho \otimes \rho = \wedge^2\rho \oplus \mathrm{S}^2\rho$, we have $\chi_{\mathrm{S}^2\rho} = (\chi_\rho)^2 - \chi_{\wedge^2\rho}$. Therefore

$$\chi s^2_\rho(g) = \frac{1}{2}((\chi_\rho(g))^2 + \chi_\rho(g^2)).$$

3. The calculation yields

$$\chi_{\wedge^2(\rho_4)} = (6, 0, 0, 0, 1, -2, 0) \quad \text{and} \quad \chi_{\mathrm{S}^2(\rho_4)} = (10, 4, 1, 0, 0, 2, 1).$$

We have $(\chi_{\wedge^2(\rho_4)}|\chi_{\wedge^2(\rho_4)}) = \frac{1}{120}(36+24+60) = 1$, and thus $\wedge^2(\rho_4)$ is irreducible. On the other hand,

$$(\chi_{\mathrm{S}^2(\rho_4)}|\chi_{\mathrm{S}^2(\rho_4)}) = \frac{1}{120}(100 + 160 + 20 + 60 + 20) = 3.$$

We cannot write $3 = \sum m_i^2$, where the m_i are integers, unless three of them are equal to 1 and the others are zero. We have

$$(\chi_{\rho_1}|\chi_{\mathrm{S}^2(\rho_4)}) = 1, \quad (\chi_{\rho_4}|\chi_{\mathrm{S}^2(\rho_4)}) = 1,$$

and thus $\mathrm{S}^2(\rho_4) = \rho_1 \oplus \rho_4 \oplus \rho_5$, where $\dim\rho_5 = 5$ and $\chi_{\rho_5} = (5, 1, -1, -1, 0, 1, 1)$; thus $(\chi_{\rho_5}|\chi_{\rho_5}) = 1$, and ρ_5 is therefore irreducible.

4. We denote by $\rho_6 = \wedge^2(\rho_4)$ the six-dimensional irreducible representation of $\mathfrak{S}_5$. We obtain the character table of $\mathfrak{S}_5$:

	1 e	10 a	20 α	30 b	24 γ	15 β	20 c
ρ_1	1	1	1	1	1	1	1
ρ_1'	1	-1	1	-1	1	1	-1
ρ_4	4	2	1	0	-1	0	-1
ρ_4'	4	-2	1	0	-1	0	1
ρ_5	5	1	-1	-1	0	1	1
ρ_5'	5	-1	-1	1	0	1	-1
ρ_6	6	0	0	0	1	-2	0

We verify the orthogonality of the columns and of the rows, weighted by the number of elements of each class.

Part II: Character table of $\mathfrak{A}_5$

5. Using the preceding table, we obtain the following characters of $\mathfrak{A}_5$:

	1 e	20 α	15 β	12 γ	12 γ^2
ρ_1	1	1	1	1	1
ρ_4	4	1	0	-1	-1
ρ_5	5	-1	1	0	0
ρ_6	6	0	-2	1	1

We see that

$$(\chi_{\rho_1}|\chi_{\rho_1}) = 1, \quad (\chi_{\rho_4}|\chi_{\rho_4}) = 1, \quad (\chi_{\rho_5}|\chi_{\rho_5}) = 1, \quad (\chi_{\rho_6}|\chi_{\rho_6}) = 2.$$

Thus the representations ρ_1, ρ_4, ρ_5 are irreducible and $\wedge^2(\rho_4) = \rho_6 = \rho_3 \oplus \rho_3'$ with $(\dim \rho_3)^2 + (\dim \rho_3')^2 + 1 + 16 + 25 = 60$, and hence

$$\dim \rho_3 = \dim \rho_3' = 3.$$

6. Let $(3, A, B, C, D)$ and $(3, A', B', C', D')$ be the last two rows of the character table of $\mathfrak{A}_5$. Using the orthogonality of columns and rows, we obtain $A = A' = 0$, $B = B' = -1$, $C + D = 1$, $C' + D' = 1$, $C'^2 - D'^2 = 0$. Because $D' = -C$ is impossible, we have $D' = C$ and $C' = D$, whence $C^2 - C - 1 = 0$. We can choose $C = \tau$, and then $C' = \tau - 1$, $D = 1 - \tau$, $D' = \tau$. If we had chosen $C = 1 - \tau$, we would only have inverted the last two rows of the character table.

7. Hence we obtain the character table of $\mathfrak{A}_5$:

	1 e	20 α	15 β	12 γ	12 γ^2
ρ_1	1	1	1	1	1
ρ_4	4	1	0	-1	-1
ρ_5	5	-1	1	0	0
ρ_3	3	0	-1	τ	$1-\tau$
ρ_3'	3	0	-1	$1-\tau$	τ

Part III: Some representations of the icosahedron group

8. In the projection $\varphi : \mathrm{SU}(2) \to \mathrm{SO}(3)$, the element

$$g_\alpha = \exp(2\alpha\xi_3) = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}$$

projects to the rotation through an angle 2α:

$$\mathrm{Rot}(e_3, 2\alpha) = \exp(2\alpha\eta_3) = \begin{pmatrix} \cos 2\alpha & -\sin 2\alpha & 0 \\ \sin 2\alpha & \cos 2\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We know that $\chi_{\mathcal{D}^j}(g_\alpha) = \sum_{m=-j}^{j} e^{2im\alpha}$, whence

$$\chi_{\underline{\mathcal{D}^j}}(\mathrm{Rot}(e_3, \theta)) = \chi_{\mathcal{D}^j}(g_{\frac{\theta}{2}}) = \sum_{m=-j}^{j} e^{im\theta}.$$

9. We calculate

$$\chi^j(\theta) = \sum_{m=-j}^{j} e^{im\theta} = \frac{\sin(j+\frac{1}{2})\theta}{\sin\frac{\theta}{2}},$$

for $\theta \neq 0$. We obtain $\chi^j(0) = 2j+1$ and for $\theta = 2\pi/3,\ \pi,\ 2\pi/5,\ 4\pi/5$,

$$\chi^j\left(\frac{2\pi}{3}\right) = \begin{cases} 1 & \text{if } j \equiv 0 \,(\mathrm{mod}\ 3), \\ 0 & \text{if } j \equiv 1 \,(\mathrm{mod}\ 3), \\ -1 & \text{if } j \equiv 2 \,(\mathrm{mod}\ 3), \end{cases}$$

$$\chi^j(\pi) = \begin{cases} 1 & \text{if } j \equiv 0 \,(\mathrm{mod}\ 2), \\ -1 & \text{if } j \equiv 1 \,(\mathrm{mod}\ 2), \end{cases}$$

$$\chi^j\left(\frac{2\pi}{5}\right) = \begin{cases} 1 & \text{if } j \equiv 0 \,(\mathrm{mod}\ 5), \\ 2\cos\frac{\pi}{5} = \tau & \text{if } j \equiv 1 \,(\mathrm{mod}\ 5), \\ 0 & \text{if } j \equiv 2 \,(\mathrm{mod}\ 5), \\ -2\cos\frac{\pi}{5} = -\tau & \text{if } j \equiv 3 \,(\mathrm{mod}\ 5), \\ -1 & \text{if } j \equiv 4 \,(\mathrm{mod}\ 5), \end{cases}$$

$$\chi^j\left(\frac{4\pi}{5}\right) = \begin{cases} 1 & \text{if } j \equiv 0 \,(\mathrm{mod}\ 5), \\ 1-\tau & \text{if } j \equiv 1 \,(\mathrm{mod}\ 5), \\ 0 & \text{if } j \equiv 2 \,(\mathrm{mod}\ 5), \\ \tau - 1 & \text{if } j \equiv 3 \,(\mathrm{mod}\ 5), \\ -1 & \text{if } j \equiv 4 \,(\mathrm{mod}\ 5). \end{cases}$$

It is thus clear that for every integer $\ell \in \mathbb{N}$ and each rotation $g \in \mathcal{I}$ different from the identity,

$$\chi^{j+30\ell}(g) = \chi^j(g).$$

We can find the table of all characters χ^j, for $0 \leq j \leq 29$. We write the first rows:

		1	20	15	12	12	
	j	0	$\frac{2\pi}{3}$	π	$\frac{2\pi}{5}$	$\frac{4\pi}{5}$	
R^0	0	1	1	1	1	1	χ^0
R^1	1	3	0	-1	τ	$1-\tau$	χ^1
R^2	2	5	-1	1	0	0	χ^2
R^3	3	7	1	-1	$-\tau$	$\tau-1$	χ^3
R^4	4	9	0	1	-1	-1	χ^4
R^5	5	11	-1	-1	1	1	χ^5
R^6	6	13	1	1	τ	$1-\tau$	χ^6
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

10. We see immediately that $R^0 = \rho_1$ (the trivial representation), $R^1 = \rho_3$, and $R^2 = \rho_5$. Therefore R^0, R^1, and R^2 are irreducible. We denote by $\chi_1, \chi_3, \chi_{3'}, \chi_4, \chi_5$ the characters of $\rho_1, \rho_3, \rho_3', \rho_4, \rho_5$. We have

$$(\chi_1|\chi^3) = 0, \quad (\chi_3|\chi^3) = 0, \quad (\chi_{3'}|\chi^3) = 1, \quad (\chi_4|\chi^3) = 1, \quad (\chi_5|\chi^3) = 0.$$

Thus

$$R^3 = \rho_3' \oplus \rho_4.$$

We set, for $m = 1, 3, 3', 4, 5$,

$$(\chi_m|\chi^j) = \frac{1}{60}((2j+1)\underline{m} + A_m^j),$$

where $\underline{m} = m$ for $m = 1, 3, 4, 5$ and $\underline{m} = 3$ for $m = 3'$. Then

$$(\chi_m|\chi^{30\ell+j}) = \frac{1}{60}((2j+1+60\ell)\underline{m} + A_m^j) = \ell\underline{m} + (\chi_m|\chi^j).$$

Thus if $(\chi_m|\chi^j) = (a_1^j, a_3^j, a_{3'}^j, a_4^j, a_5^j)$, then

$$(\chi_m|\chi^{30\ell+j}) = a_1^j + \ell,\ a_3^j + 3\ell,\ a_{3'}^j + 3\ell, a_4^j + 4\ell,\ a_5^j + 5\ell.$$

In particular,

$$(\chi_m|\chi^{30\ell+3}) = \ell,\ 3\ell,\ 3\ell+1,\ 4\ell+1,\ 5\ell,$$

whence

$$R^{30\ell+3} = \ell\rho_1 \oplus 3\ell\rho_3 \oplus (3\ell+1)\rho_3' \oplus (4\ell+1)\rho_4 \oplus 5\ell\rho_5.$$

We verify that $2(30\ell+3)+1 = \ell + 9\ell + 9\ell + 3 + 16\ell + 4 + 25\ell$.

11. In order to decompose $R^2 \otimes R^3$, we can use

$$\mathcal{D}^2 \otimes \mathcal{D}^3 = \mathcal{D}^1 \oplus \mathcal{D}^2 \oplus \mathcal{D}^3 \oplus \mathcal{D}^4 \oplus \mathcal{D}^5,$$

which implies the same relationship between the $\underline{\mathcal{D}^j}$ and thus between the R^j.

The decompositions of R^4 and of R^5 are

$$R^4 = \rho_4 \oplus \rho_5, \quad R^5 = \rho_3 \oplus \rho_3' \oplus \rho_5.$$

Hence

$$R^2 \otimes R^3 = 2\rho_3 \oplus 2\rho_3' \oplus 2\rho_4 \oplus 3\rho_5.$$

We verify that $5 \times 7 = 6 + 6 + 8 + 15$.

One can also write

$$R^2 \otimes R^3 = \rho_5 \otimes (\rho_3' \oplus \rho_4) = (\rho_5 \otimes \rho_3') \oplus (\rho_5 \otimes \rho_4).$$

The character of $\rho_5 \otimes \rho_3'$ is $(15, 0, -1, 0, 0)$, and thus

$$\rho_5 \otimes \rho_3' = \rho_3 \oplus \rho_3' \oplus \rho_4 \oplus \rho_5.$$

The character of $\rho_5 \otimes \rho_4$ is $(20, -1, 0, 0, 0)$, and hence

$$\rho_5 \otimes \rho_4 = \rho_3 \oplus \rho_3' \oplus \rho_4 \oplus 2\rho_5.$$

We recover

$$R^2 \otimes R^3 = 2\rho_3 \oplus 2\rho_3' \oplus 2\rho_4 \oplus 3\rho_5.$$

Epilogue: Representations of $G = \varphi^{-1}(\mathcal{I}) \subset \mathrm{SU2}$

Each conjugacy class of G has a representative $g_\alpha,\ 0 \le \alpha < 2\pi$.

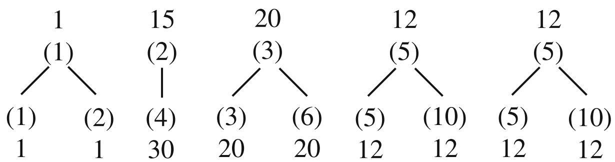
order	1	2	4	6	3	10	10	5	5
cardinality	1	1	30	20	20	12	12	12	12
α	0	π	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{4\pi}{3}$	$\frac{\pi}{5}$	$\frac{7\pi}{5}$	$\frac{6\pi}{5}$	$\frac{2\pi}{5}$

The class of $e, \theta = 0$, of order 1, yields $\alpha = 0$ (order 1) and $\alpha = \pi$ (order 2).

The class with 15 elements, $\theta = \pi$, of order 2, yields a class with 30 elements of order 4, $\alpha = \pi/2$.

The class with 20 elements, $\theta = 2\pi/3$, of order 3, yields a class with 20 elements of order 3, $\alpha = 4\pi/3$, and a class with 20 elements of order 6, $\alpha = \pi/3$.

Each class with 12 elements, $\theta = 2\pi/5$ (respectively, $\theta = 4\pi/5$), of order 5, yields a class with 12 elements of order 5, $\alpha = 6\pi/5$ (respectively, $\alpha = 2\pi/5$), and a class with 12 elements of order 10, $\alpha = \pi/5$ (respectively, $\alpha = 7\pi/5$):



We can easily write the characters of the representations $\rho_1, \rho_3, \rho_3', \rho_4, \rho_5$.

In addition, for $j = 1/2$, there are a representation ρ_2 obtained by restriction of $\mathcal{D}^{1/2}$ and a representation ρ_2'. The representations $\rho_1, \rho_3, \rho_4, \rho_5$ are the restrictions of $\mathcal{D}^0, \mathcal{D}^1, \mathcal{D}^{3/2}, \mathcal{D}^2$. Hence we obtain the character table of $G = \varphi^{-1}(\mathcal{I})$:

	order	1	2	4	6	3	10	10	5	5
	cardinality	1	1	30	20	20	12	12	12	12
		0	π	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{4\pi}{3}$	$\frac{\pi}{5}$	$\frac{7\pi}{5}$	$\frac{6\pi}{5}$	$\frac{2\pi}{5}$
$\mathcal{D}^0$	**1**	1	1	1	1	1	1	1	1	1
$\mathcal{D}^{1/2}$	**2**	2	-2	0	1	-1	τ	$1-\tau$	$-\tau$	$\tau-1$
$\mathcal{D}^1$	**3**	3	3	-1	0	0	τ	$1-\tau$	τ	$1-\tau$
$\mathcal{D}^{3/2}$	**4**	4	-4	0	-1	1	1	1	-1	-1
$\mathcal{D}^2$	**5**	5	5	1	-1	-1	0	0	0	0
$\mathcal{D}^{5/2}$	**6**	6	-6	0	0	0	-1	-1	1	1
	4′	4	4	0	1	1	-1	-1	-1	-1
	3′	3	3	-1	0	0	$1-\tau$	τ	$\tau-1$	τ
	2′	2	-2	0	1	-1	$1-\tau$	τ	$\tau-1$	$-\tau$

The first six rows are irreducible characters, as shown by the calculation of the norm squared. Since, for $j \geq \frac{1}{2}$, $\mathcal{D}^{1/2} \otimes \mathcal{D}^j = \mathcal{D}^{j-\frac{1}{2}} \oplus \mathcal{D}^{j+\frac{1}{2}}$, we obtain, for $k \geq 2$,

$$\mathbf{2} \otimes \mathbf{k} = (\mathbf{k-1}) \oplus (\mathbf{k+1}).$$

Furthermore,

$$\mathbf{2} \otimes \mathbf{2'} = \mathbf{4'}, \quad \mathbf{2} \otimes \mathbf{3'} = \mathbf{6'} = \mathbf{2'} \otimes \mathbf{3}, \quad \mathbf{2} \otimes \mathbf{4'} = \mathbf{2'} \oplus \mathbf{6}.$$

We can show that

$$\mathbf{3} \oplus \mathbf{1} = \mathbf{2}^{\otimes \mathbf{2}}, \quad \mathbf{4} \oplus 2\mathbf{2} = \mathbf{2}^{\otimes \mathbf{3}}, \quad \mathbf{5} \oplus 3\mathbf{2}^{\otimes \mathbf{2}} = \mathbf{2}^{\otimes \mathbf{4}} \otimes \mathbf{1}, \quad \mathbf{6} \oplus 4\mathbf{2}^{\otimes \mathbf{3}} = \mathbf{2}^{\otimes \mathbf{5}} \otimes 3\mathbf{2},$$

and similarly, $\mathbf{4'}, \mathbf{3'}, \mathbf{2'}$ can be written in terms of linear combinations of tensor powers of $\mathbf{2}$.

A reference for the properties proved in this problem is F. R. K. Chung, B. Kostant, S. Sternberg, "Groups and the buckyball," in *Lie Theory and Geometry*, J.-L. Brylinski, R. Brylinski, V. Guillemin, V. Kac, eds., Birkhäuser, Boston, 1994, pp. 97–126.

It is proved there that the multiplicities $a_{k,r}$ of the representations $r = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{2'}$ in the restriction to G of the representations $\mathcal{D}^{k/2}, k \in \mathbb{N}$, are the coefficients of the series expansion of a rational function, where only the numerator depends on r.

9 Matrix Coefficients and Spherical Harmonics

Let j be a nonnegative integer or half-integer. We denote by V^j the complex vector space of homogeneous polynomials in two variables (z_1, z_2) with complex coefficients, of degree $2j$.

1. To each matrix $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ belonging to $\mathrm{SL}(2,\mathbb{C})$ and each polynomial $f \in V^j$, we associate the polynomial $\rho_V^j(g)f$ defined by

$$(\rho_V^j(g)f)(z_1, z_2) = f(az_1 + cz_2, bz_1 + dz_2).$$

Show that we have thus defined a representation ρ_V^j of $\mathrm{SL}(2,\mathbb{C})$ in V^j.

We denote by P^j the complex vector space of polynomials with complex coefficients in one variable x of degree $\leq 2j$.

2. Show that the map Φ of V^j into P^j defined by

$$(\Phi(f))(x) = f(x, 1),$$

for $f \in V^j$, is an isomorphism of vector spaces.

3. For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{C})$ and $\varphi \in P^j$, we define $\rho^j(g)(\varphi) \in P^j$ by

$$(\rho^j(g)\varphi)(x) = (bx+d)^{2j} \varphi\left(\frac{ax+c}{bx+d}\right).$$

Show that we have thus defined a representation ρ^j of $\mathrm{SL}(2,\mathbb{C})$ in P^j equivalent to the representation (V^j, ρ_V^j).

4. We denote by r^j the differential of the representation ρ^j. Find the differential operator $r^j(X)$ for $X = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathfrak{sl}(2,\mathbb{C})$.

For $-j \leq m \leq j$ and $j - m$ integer, we set

$$\varphi_m^j(x) = \frac{x^{j-m}}{\sqrt{(j-m)!(j+m)!}}.$$

5. We consider the basis of $\mathfrak{sl}(2,\mathbb{C})$,

$$J_3 = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \; J_+ = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \; J_- = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

Find the images under each of the operators $r^j(J_3)$, $r^j(J_+)$, $r^j(J_-)$ of the polynomials φ_m^j, for $-j \leq m \leq j$ and $j - m$ integer.

6. Show that the representation ρ^j of $\mathrm{SL}(2,\mathbb{C})$ on P^j is irreducible.

7. Show that there is a unique scalar product on P^j invariant under the restriction to $\mathrm{SU}(2)$ of the representation ρ^j of $\mathrm{SL}(2,\mathbb{C})$ and such that the polynomials φ_m^j form an orthonormal basis of P^j. Is the matrix of $\rho^j(g)$ in the basis (φ_m^j) unitary for every $g \in \mathrm{SL}(2,\mathbb{C})$?

We write $(\ |\ )$ for the scalar product defined above. The matrix coefficients $\rho^j_{m,n}$ of the representation ρ^j in the basis (φ^j_m) are defined by

$$\rho^j_{m,n}(g) = (\varphi^j_m | \rho^j(g)\varphi^j_n),$$

for $g \in \mathrm{SL}(2,\mathbb{C})$.

8. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{C})$. Use Taylor's formula for the function $(ax+c)^{j-n}(bx+d)^{j+n}$, along with a change of variable, to show that if $a \neq 0$ and $b \neq 0$, then

$$\rho^j_{m,n}(g) = C(j,m,n)\frac{b^{n-m}}{a^{n+m}}\frac{\mathrm{d}^{j-m}}{\mathrm{d}x^{j-m}}((1-x)^{j-n}(1+x)^{j+n})\big|_{x=ad+bc},$$

where $C(j,m,n)$ is a constant to be determined.

9. (a) Let $\xi_1 = \frac{1}{2}\left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$. We set $\rho^j(\theta) = \rho^j(\exp(\theta\xi_1))$, $0 \leq \theta \leq \pi$. Find $\rho^j(\pi)$. Show that $\rho^j_{m,-n}(\theta) = \rho^j_{-m,n}(\theta)$.

(b) We assume that j and m are integers and that $m \geq 0$. We recall that the Legendre function $P_{j,m}$ is defined, for $-1 \leq x \leq 1$, by

$$P_{j,m}(x) = \frac{(-1)^{j+m}}{2^j j!}\left(1-x^2\right)^{\frac{m}{2}}\frac{\mathrm{d}^{j+m}}{\mathrm{d}x^{j+m}}\left(1-x^2\right)^j.$$

Show that $\rho^j_{m,0}(\theta) = \mathcal{P}^j_m(\cos\theta)$, where $\mathcal{P}^j_m$ is a function of one variable proportional to $P_{j,m}$, and find the coefficient of proportionality.

For $-j \leq m \leq j$, what is the relation between the function $\rho^j_{m,0}(\theta)e^{im\phi}$ and the spherical harmonic $Y^j_m(\theta,\phi)$ $(0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi)$?

10. Prove the orthonormality property of the system of spherical harmonics from the orthogonality relations for the matrix coefficients of irreducible representations of SU(2).

Solutions

1. For every matrix $g \in \mathrm{SL}(2,\mathbb{C})$ and for every $f \in V^j$, the polynomial $\rho^j_V(g)f$ is homogeneneous of degree $2j$, and its coefficients depend continuously on the coefficients of the matrix g. The definition can be written $\rho^j_V(g)f = f \circ {}^t g$; hence if g_1 and $g_2 \in \mathrm{SL}(2,\mathbb{C})$,

$$\rho^j_V(g_1g_2)f = f \circ {}^t g_2\ {}^t g_1 = \rho^j_V(g_1)(\rho^j_V(g_2)f).$$

Thus ρ^j_V is a representation of $\mathrm{SL}(2,\mathbb{C})$ on V^j.

2. It is clear that Φ is linear. The vector spaces V^j and P^j have the same dimension, $2j+1$. Furthermore, Φ is injective. In fact, if $\Phi(f) = 0$, all the coefficients of the homogeneous polynomial f are zero, and hence it is zero. Thus Φ is an isomorphism. The inverse Ψ of Φ is given by

$$(\Psi(\varphi))(z_1,z_2) = z_2^{2j}\varphi\left(\frac{z_1}{z_2}\right), \quad \varphi \in P^j.$$

3. When we identify V^j with P^j by means of the isomorphism Φ, the transformation $\rho_V^j(g)$ can be identified with the transformation $\rho^j(g)$. In fact, for $f \in V^j$,

$$\begin{aligned}\rho^j(g)(\Phi(f))(x) &= (bx+d)^{2j}(\Phi(f))\left(\frac{ax+c}{bx+d}\right) = (bx+d)^{2j} f\left(\frac{ax+c}{bx+d}, 1\right) \\ &= f(ax+c, bx+d) = (\rho_V^j(g)f)(x,1) = \Phi(\rho_V^j(g)f)(x).\end{aligned}$$

We deduce that ρ^j is a representation of $\mathrm{SL}(2,\mathbb{C})$ in P^j, equivalent to ρ_V^j. The isomorphism Φ intertwines the representations ρ_V^j and ρ^j.

4. Let $X = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathfrak{sl}(2,\mathbb{C})$. Then $\exp(tX) = \left(\begin{smallmatrix} a(t) & b(t) \\ c(t) & d(t) \end{smallmatrix}\right)$ is a matrix of $\mathrm{SL}(2,\mathbb{C})$ such that $a(0) = d(0) = 1$, $b(0) = c(0) = 0$, and $a'(0) = \alpha$, $b'(0) = \beta$, $c'(0) = \gamma$, $d'(0) = \delta$. One thus obtains, for $\varphi \in P^j$,

$$\begin{aligned}(r^j(X)\varphi)(x) &= \frac{\mathrm{d}}{\mathrm{d}t}(\rho^j(\exp tX)\varphi)(x)\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(b(t)x+d(t))^{2j}\varphi\left(\frac{a(t)x+c(t)}{b(t)x+d(t)}\right)\Bigg|_{t=0} \\ &= 2j(\beta x+\delta)\varphi(x) + (-\beta x^2 + (\alpha-\delta)x+\gamma)\frac{\mathrm{d}\varphi}{\mathrm{d}x}(x),\end{aligned}$$

that is,

$$r^j(X) = 2j(\beta x+\delta) + (-\beta x^2 + (\alpha-\delta)x+\gamma)\frac{\mathrm{d}}{\mathrm{d}x}.$$

5. In particular,

$$r^j(J_3) = j - x\frac{\mathrm{d}}{\mathrm{d}x}, \quad r^j(J_+) = -\frac{\mathrm{d}}{\mathrm{d}x}, \quad r^j(J_-) = -2jx + x^2\frac{\mathrm{d}}{\mathrm{d}x},$$

and hence

$$\begin{aligned}r^j(J_3)(\varphi_m^j) &= m\varphi_m^j, \\ r^j(J_+)(\varphi_m^j) &= -\sqrt{(j-m)(j+m+1)}\,\varphi_{m+1}^j, \\ r^j(J_-)(\varphi_m^j) &= -\sqrt{(j+m)(j-m+1)}\,\varphi_{m-1}^j.\end{aligned}$$

6. The preceding equations show that the representation r^j of $\mathfrak{sl}(2,\mathbb{C})$ is equivalent to the representation D^j, the basis $(-1)^{j-m}\varphi_m^j$ corresponding to the standard basis $|j,m\rangle$. We know that this representation is irreducible. We deduce immediately that the representation ρ^j of $\mathrm{SL}(2,\mathbb{C})$ is irreducible.

7. There exists a unique scalar product on P^j such that the basis (φ_m^j) is orthonormal. Let us show that this scalar product is invariant under $\rho^j\big|_{\mathrm{SU}(2)}$, that is, that the restriction of ρ^j to $\mathrm{SU}(2)$ is unitary for this scalar product. Because $\mathrm{SU}(2)$ is connected, it suffices to show that the matrices in the basis (φ_m^j) of the operators $r^j(\xi_k)$, $k = 1,2,3$, are anti-Hermitian, or equivalently, that the matrices of $r^j(J_+)$ and $r^j(J_-)$ are conjugate transposes of one another,

and that the matrix of $r^j(J_3)$ is Hermitian. Because the operator $r^j(J_3)$ is real and diagonal, it is Hermitian. We have

$$\begin{aligned}
(r^j(J_+)\varphi^j_m|\varphi^j_n) &= -\sqrt{(j-m)(j+m+1)}\,\delta_{m+1,n},\\
(\varphi^j_m|r^j(J_-)\varphi^j_n) &= -\sqrt{(j+n)(j-n+1)}\,\delta_{m,n-1}\\
&= -\sqrt{(j+m+1)(j-m)}\,\delta_{m+1,n},
\end{aligned}$$

which shows that the matrices of $r^j(J_+)$ and $r^j(J_-)$ are real and are transposes of one another. The operators $\rho^j(g)$ are thus unitary for $g \in \mathrm{SU}(2)$. But clearly this is not true for an arbitrary element of $\mathrm{SL}(2,\mathbb{C})$: the operators $r^j(X)$ are anti-Hermitian for $X \in \mathfrak{su}(2)$ and Hermitian for $X \in i\mathfrak{su}(2)$.

8. The coefficient of x^{j-m} in $(ax+c)^{j-n}(bx+d)^{j+n}$ is

$$\frac{1}{(j-m)!}\frac{\mathrm{d}^{j-m}}{\mathrm{d}x^{j-m}}((ax+c)^{j-n}(bx+d)^{j+n})\big|_{x=0},$$

and hence

$$\begin{aligned}
\rho^j_{m,n}(g) &= \left(\frac{x^{j-m}}{\sqrt{(j-m)!(j+m)!}}\,\middle|\,\frac{(ax+c)^{j-n}(bx+d)^{j+n}}{\sqrt{(j-n)!(j+n)!}}\right)\\
&= \frac{\sqrt{(j+m)!}}{\sqrt{(j-n)!(j+n)!(j-m)!}}\frac{\mathrm{d}^{j-m}}{\mathrm{d}x^{j-m}}((ax+c)^{j-n}(bx+d)^{j+n})\big|_{x=0}.
\end{aligned}$$

We set $\frac{y+1}{2} = a(bx+d)$. Taking the equation $ad-bc=1$ into account, we have $\frac{y-1}{2} = b(ax+c)$. We assume $a \neq 0$ and $b \neq 0$. Because the variable y is a linear function of x, we obtain

$$\rho^j_{m,n}(g) = C(j,m,n)\frac{b^{n-m}}{a^{n+m}}\frac{\mathrm{d}^{j-m}}{\mathrm{d}y^{j-m}}((1-y)^{j-n}(1+y)^{j+n})\big|_{y\,=\,ad+bc},$$

where

$$C(j,m,n) = \frac{(-1)^{j-n}}{2^{j+m}}\frac{\sqrt{(j+m)!}}{\sqrt{(j-n)!(j+n)!(j-m)!}}.$$

9. (a) We know that $\exp(\theta\xi_1) = \begin{pmatrix}\cos\frac{\theta}{2} & i\sin\frac{\theta}{2}\\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}$. In particular, $\exp(\pi\xi_1) = \begin{pmatrix}0 & i\\ i & 0\end{pmatrix}$. Now, $(\rho^j(\pi)\varphi^j_m)(x) = \frac{1}{\sqrt{(j-m)!(j+m)!}}i^{2j}x^{j+m} = i^{2j}\varphi^j_{-m}(x)$, and thus

$$\rho^j(\pi)\varphi^j_m = i^{2j}\varphi^j_{-m}, \qquad \rho^j_{m,n}(\pi) = i^{2j}\delta_{m,-n}.$$

From the equation $\rho^j(\theta)\rho^j(\pi) = \rho^j(\pi)\rho^j(\theta)$, we deduce that

$$\rho^j_{m,-n}(\theta) = \rho^j_{-m,n}(\theta).$$

In particular, $\rho^j_{m,0}(\theta) = \rho^j_{-m,0}(\theta)$.

(b) For $0 < \theta < \pi$, we calculate $\rho^j_{m,0}(\theta) = \rho^j_{-m,0}(\theta)$ by the general formula obtained above. We obtain

$$\rho^j_{m,0}(\theta) = \frac{(-1)^j}{2^j j!}\sqrt{\frac{(j-m)!}{(j+m)!}}\, i^m \sin^m\theta \frac{\mathrm{d}^{j+m}}{\mathrm{d}x^{j+m}}(1-x^2)^j\big|_{x=\cos\theta}.$$

When $0 \le \theta \le \pi$, the equation $\sin\theta = (1-\cos^2\theta)^{\frac{1}{2}}$ is valid. Consequently, for $0 < \theta < \pi$, $\rho^j_{m,0}(\theta) = \mathcal{P}^j_m(\cos\theta)$, where the function $\mathcal{P}^j_m$ is defined for $-1 \le x \le 1$ by

$$\mathcal{P}^j_m(x) = \frac{(-1)^j}{2^j j!}\sqrt{\frac{(j-m)!}{(j+m)!}}\, i^m (1-x^2)^{\frac{m}{2}} \frac{\mathrm{d}^{j+m}}{\mathrm{d}x^{j+m}}(1-x^2)^j.$$

We verify that for j and m integers, $m \ge 0$, $\mathcal{P}^j_m(1) = \delta_{m,0}$ and $\mathcal{P}^j_m(-1) = (-1)^j\delta_{m,0}$, which extends the result to the case $\theta = 0$ and $\theta = \pi$. If we use $P_{j,m}$ to denote the Legendre function, we obtain an equation valid for $0 \le \theta \le \pi$,

$$\rho^j_{m,0}(\theta) = \sqrt{\frac{(j-m)!}{(j+m)!}}\, i^{-m} P_{j,m}(\cos\theta).$$

We know that for $0 \le m \le j$, the spherical harmonics are the functions

$$Y^j_m(\theta,\Phi) = \sqrt{\frac{2j+1}{4\pi}}\sqrt{\frac{(j-m)!}{(j+m)!}}\, P_{j,m}(\cos\theta)e^{im\Phi}.$$

We deduce that in this case,

$$Y^j_m(\theta,\phi) = \sqrt{\frac{2j+1}{4\pi}}\, i^m \rho^j_{m,0}(\theta)e^{im\phi}.$$

This equation is valid for $-j \le m < 0$ as well. In fact, by definition, $Y^j_m = (-1)^m\overline{Y}^j_{-m}$; therefore; $Y^j_m = (-1)^m\sqrt{\frac{2j+1}{4\pi}}\, i^m \overline{\rho^j_{-m,0}(\theta)} e^{im\phi}$. Since $\rho^j_{-m,0}(\theta) = \rho^j_{m,0}(\theta)$ and $\overline{\rho^j_{m,0}(\theta)} = (-1)^m\rho^j_{m,0}(\theta)$, the result follows.

10. We know that a dense open set of SU(2) is parametrized by the Euler angles. For a matrix $g = \left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a}\end{smallmatrix}\right) \in \mathrm{SU}(2)$ such that $\mathcal{I}m(a) \neq 0$, $\mathcal{R}e(b) \neq 0$, there are unique parameters $\phi \in [0,2\pi]$, $\theta \in [0,\pi]$, $\psi \in [-2\pi, 2\pi]$ such that

$$g = g(\phi,\theta,\psi) = \begin{pmatrix} \cos\frac{\theta}{2}e^{i\frac{\phi+\psi}{2}} & i\sin\frac{\theta}{2}e^{i\frac{\phi-\psi}{2}} \\ i\sin\frac{\theta}{2}e^{-i\frac{\phi-\psi}{2}} & \cos\frac{\theta}{2}e^{-i\frac{\phi+\psi}{2}} \end{pmatrix} = \exp(\phi\xi_3)\exp(\theta\xi_1)\exp(\psi\xi_3).$$

In this parametrization, the Haar integral on SU(2) can be written

$$I(f) = \frac{1}{16\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_{-2\pi}^{2\pi} f(g(\phi,\theta,\psi))\sin\theta\ \mathrm{d}\phi\ \mathrm{d}\theta\ \mathrm{d}\psi.$$

Now consider the matrix coefficients of the representation ρ^j of SU(2). Because $g(\phi,0,0) = \exp(\phi\xi_3) = \begin{pmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{pmatrix}$, for each function $\varphi \in P^j$, we obtain $(\rho^j(g(\phi,0,0))\varphi)(x) = e^{-ij\phi}\varphi(e^{i\phi}x)$, whence

$$\rho^j(g(\phi,0,0))x^{j-m} = e^{-im\phi}x^{j-m}.$$

The matrix of $\rho^j(g(\phi,0,0))$ in the basis φ^j_m is thus diagonal, with elements $e^{-im\phi}$, $-j \le m \le j$. Similarly, the matrix of $\rho^j(g(0,0,\psi))$ is diagonal, with elements $e^{-im\psi}$.

We have set $\rho^j(\theta) = \rho^j(\exp(\theta\xi_1))$, i.e., $\rho^j(\theta) = \rho^j(g(0,\theta,0))$. By writing

$$\rho^j_{m,n}(g(\phi,\theta,\psi)) = \sum_{p,q} \rho^j_{m,p}(g(\phi,0,0))\rho^j_{p,q}(\theta)\rho^j_{q,n}(g(0,0,\psi)),$$

we obtain

$$\rho^j_{m,n}(g(\phi,\theta,\psi)) = e^{-i(m\phi+n\psi)}\rho^j_{m,n}(\theta).$$

In particular,

$$\rho^j_{m,0}(g(\phi,\theta,\psi)) = e^{-im\phi}\rho^j_{m,0}(\theta).$$

The orthogonality relations for the matrix coefficients of unitary irreducible representations ρ^j of SU(2) imply

$$\begin{aligned} \frac{1}{16\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_{-2\pi}^{2\pi} &\overline{\rho^j_{m,0}(g(\phi,\theta,\psi))}\rho^{j'}_{m',0}(g(\phi,\theta,\psi))\sin\theta \,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}\psi \\ &= \frac{1}{2j+1}\delta_{m,m'}\delta_{j,j'}, \end{aligned}$$

whence, because $\rho^j_{m,0}$ does not depend on ψ,

$$\frac{2j+1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \overline{\rho^j_{m,0}(g(\phi,\theta,0))}\rho^{j'}_{m',0}(g(\phi,\theta,0))\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta = \delta_{m,m'}\delta_{j,j'}.$$

The elements corresponding to $\psi = 0$ can be identified with the points of the sphere S^2: to $\exp(\phi\xi_3)\exp(\theta\xi_1)$ corresponds the point $\exp(\phi\eta_3)\exp(\theta\eta_1)e_3$, with spherical coordinates $(\theta, \phi - \frac{\pi}{2})$.

Now, for $j \in \mathbb{N}$, $-j \le m \le j$,

$$Y^j_m(\theta,\phi) = \sqrt{\frac{2j+1}{4\pi}}\, i^m e^{im\phi}\rho^j_{m,0}(\theta).$$

Furthermore, we know that $\rho^j_{m,0}(\theta)$ is either real or pure imaginary. This equation means that up to a coefficient, the spherical harmonic $Y^j_m(\theta,\phi)$ is the conjugate of the matrix coefficient $\rho^j_{m,0}(g(\phi,\theta,0))$. Taking into account the value of the modulus of this coefficient, the orthogonality relation above can be written

$$\int_0^{\pi}\int_0^{2\pi}\overline{Y_m^j(\theta,\phi)}Y_{m'}^{j'}(\theta,\phi)\sin\theta\;\mathrm{d}\theta\;\mathrm{d}\phi=\delta_{m,m'}\delta_{jj'},$$

which proves that the spherical harmonics (Y_m^j), $j\in\mathbb{N}$, $-j\leq m\leq j$, form an orthonormal set in $L^2(S^2)$ equipped with the unnormalized scalar product.

From the properties of the matrix coefficients of the irreducible representations of the group SU(2), one can prove the orthogonality properties, the addition theorems, and many other properties of the Legendre polynomials, the Legendre functions, and the spherical harmonics.

Hermann Weyl (1885–1955) wrote his doctoral dissertation in 1908 in Göttingen under the direction of David Hilbert, taught in Zurich from 1913 to 1930, succeeded Hilbert at Göttingen, and left Germany in 1933 for the United States, where he accepted a position at the newly formed Institute for Advanced Study in Princeton. His most important contributions to group theory date from the years 1920–1940.

(*Collection of Professor Konrad Jacobs, with the kind permission of the Mathematisches Forschungsinstitut Oberwolfach*)

Bibliography

The literature on finite groups, Lie algebras, Lie groups, and representation theory is immense. Here we cite the classical and recent works that seem most useful for a first approach to or for in-depth study of topics dealt with in this text. The former are marked with the symbol (*).

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