A brief note on stochastic homogenization in 1D

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Abstract

In this brief note, we discuss the homogenization of a 1D elliptic operator with a random coefficient function. As is generally well-known, under assumptions of stationarity and ergodicity, there is a closed form expression for the homogenized coefficient in 1D. We provide a simple derivation of this result and also discuss the connections to the general theoretical framework in [4]. We also discuss the method of periodization in the 1D case and provide some numerical examples.

Contents

1 Introduction

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. Recall that a measurable mapping $T : \Omega \mapsto \Omega$ is called measure preserving if $\mu\left(T^{-1}(E)\right) = \mu(E)$ for all $E \in \mathcal{F}$. A family of mappings, $T = \{T_x\}_{x \in \mathbf{R}}$, on Ω is called an 1-dimensional dynamical system if $T_x : \Omega \mapsto \Omega$ satisfies the following group property:

$$\begin{cases} T_{x+y} = T_x T_y \\ T_0 = I \end{cases}$$

Moreover, in the context of homogenization, we require the following additional properties:

- T_x is measure preserving for all $x \in \mathbf{R}$
- For any measurable function f on Ω, the composition $f(T_x\omega) \equiv (f \circ T_x)(\omega)$ is $Leb \otimes \mu$ measurable, where Leb is the Lebesgue measure on $\mathbf{R}$.

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Last revised: July 12, 2013

Suppose $a : \Omega \mapsto \mathbf{R}$ satisfies the following:

1. a is μ-measurable.
2. There exists positive constants α and β such that

$$\alpha \leq a(\omega) \leq \beta \quad \text{almost everywhere in } \Omega.$$

Realizations of a with respect to T are given by $a\big(T_x(\omega)\big)$. We use the following notation for realizations of a,

$$a^T(x,\omega) = a\big(T_x(\omega)\big).$$

We consider the following problem,

$$\begin{cases} -\big(a^T(x,\omega)u'(x,\omega)\big)' = f(x) \quad \text{in} \quad \mathcal{O} = (0,1) \\ u(0,\omega) = u(1,\omega) = 0 \end{cases} \tag{1.1}$$

for $\omega \in \Omega$.

The goal is to show that for almost all $\omega \in \Omega$, the problem (1.1) admits homogenization with the homogenized coefficient given by

$$\bar{a} = \frac{1}{\mathrm{E}\left\{\frac{1}{a}\right\}}.$$

Here $\mathrm{E}\{a\}$ denotes the expectation,

$$\mathrm{E}\{a\} := \int_\Omega a(\omega)\, d\mu(\omega).$$

The proof of the above result relies on several major results one of which is Birkoff's Ergodic Theorem. For convenience, we recall Birkhoff's Theorem in the next section.

2 Birkhoff Ergodic Theorem

Since we are working in 1D, the results in this section are specialized to the 1D case. Let us first recall the notion of the mean value of a function [4].

Definition 2.1 (Mean Value). *Let $f \in L^1_{loc}(\mathbf{R})$ and suppose the limit, $\lim_{\epsilon\to 0} \frac{1}{|K|}\int_K f(\frac{\boldsymbol{x}}{\epsilon})\, d\boldsymbol{x}$ exists for any Lebesgue measurable subset $K \subset \mathbf{R}$ independently of K. Then, we say that f has a mean-value f^* given by*

$$f^* = \lim_{\epsilon\to 0} \frac{1}{|K|}\int_K f(\frac{\boldsymbol{x}}{\epsilon})\, d\boldsymbol{x}.$$

The following result, due to Birkhoff, is one of the fundamental results in Ergodic Theory[1].

Theorem 2.2. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, and suppose $T = \{T_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\mathbf{R}}$ is a measure-preserving dynamical system on Ω. Let $f \in L^p(\Omega)$, where $p \geq 1$. Then for almost all $\omega \in \Omega$ the realization $f^T(\boldsymbol{x},\omega)$ has a mean value $F^*(\omega)$ in the following sense: denote $f^T_\epsilon(\boldsymbol{x},\omega) = f^T(\boldsymbol{x}/\epsilon,\omega)$, then*

$$f^T_\epsilon(\cdot,\omega) \rightharpoonup F^*(\omega) \quad \text{in } L^p_{loc}(\mathbf{R}).$$

[1] See [3] or [5] for references on Ergodic Theory.

Moreover, $F^* = F^*(\omega)$ is an invariant function; that is,

$$F^*\big(T_{\boldsymbol{x}}(\omega)\big) = F^*(\omega) \quad \forall x \in \mathbf{R},\ \mu\text{-almost everywhere}\ . \tag{2.1}$$

Also,

$$\mathrm{E}\,\{f\} = \int_\Omega F^*(\omega)\, d\mu(\omega). \tag{2.2}$$

We briefly mention the notion of an ergodic dynamical system here [3]. First, let us recall what an invariant function is. A measurable function f on Ω is called an invariant function (with respect to the dynamical system T) if,

$$f\big(T_x(\omega)\big) = f(\omega), \quad \mu\text{-a.e}, \forall\, x \in \mathbf{R}.$$

A 1-dimensional dynamical system $T = \{T_x\}_{x\in\mathbf{R}}$ is ergodic if every invariant function is a constant almost everywhere.

Corollary 2.3. *Let f and T be as in Theorem 2.2; moreover, suppose the dynamical system T is ergodic. Then, the mean value F^* is a constant and is given by*

$$F^* = \mathrm{E}\,\{f\}\,. \tag{2.3}$$

Proof. Using ergodicity of T, (2.1) implies $F^* = const$ almost everywhere. Then, (2.2) gives,

$$\mathrm{E}\,\{f\} = \int_\Omega F^*\, d\mu = F^*. \tag{2.4}$$

□

3 Proof of Convergence

Recall the problem (1.1), and suppose a is as before. Moreover, suppose T is an ergodic dynamical system. By Birkhoff Ergodic Theorem (along with Corollary 2.3), we know that there is a set $E \in \mathcal{F}$, with $\mu(E) = 1$ such that for all $\omega \in E$,

$$\frac{1}{a_\epsilon^T(\cdot,\omega)} \rightharpoonup \mathrm{E}\left\{\frac{1}{a}\right\} \quad \text{in } L^2(\mathcal{O}). \tag{3.1}$$

From now on, let $\omega \in E$ be fixed but arbitrary. Consider the problem,

$$\begin{cases} -\big(a_\epsilon^T(x,\omega)u_\epsilon'(x,\omega)\big)' = f(x) \quad \text{in} \quad \mathcal{O} = (0,1) \\ u_\epsilon(0,\omega) = 0 = u_\epsilon(1,\omega) \end{cases} \tag{3.2}$$

Using standard arguments (we show that $\{u_\epsilon(\cdot,\omega)\}_{\epsilon>0}$ is bounded in $H_0^1(\mathcal{O})$, etc.)

$$u_\epsilon(\cdot,\omega) \rightharpoonup \bar{u} \quad \text{in } H_0^1(\mathcal{O}). \tag{3.3}$$

At this point, however, it is not clear whether $\bar{u}$ is deterministic. From (3.3) we immediately get that,

$$u_\epsilon'(\cdot,\omega) \rightharpoonup \bar{u}' \quad \text{in } L^2(\mathcal{O}). \tag{3.4}$$

Next, let $\sigma_\epsilon(x,\omega) = a_\epsilon^T(x,\omega)u_\epsilon'(x,\omega)$ and note that again by standard arguments, $\sigma_\epsilon(\cdot,\omega) \rightharpoonup \bar{\sigma}$ in $H^1(\Omega)$. Now, by compact embedding of $H^1(\mathcal{O})$ into $L^2(\mathcal{O})$ we have,

$$\sigma_\epsilon(\cdot,\omega) \to \bar{\sigma} \quad \text{in } L^2(\mathcal{O}). \tag{3.5}$$

Now, consider the following obvious equality,

$$u'_\epsilon(\cdot,\omega) = \frac{a^T_\epsilon(\cdot,\omega)}{a^T_\epsilon(\cdot,\omega)} u'_\epsilon(\cdot,\omega) = \sigma_\epsilon(\cdot,\omega)\frac{1}{a_\epsilon(\cdot,\omega)}. \tag{3.6}$$

By (3.1) and (3.5) we have

$$\sigma_\epsilon(\cdot,\omega)\frac{1}{a_\epsilon(\cdot,\omega)} \rightharpoonup \bar{\sigma}\frac{1}{\bar{a}} \quad \text{in } L^2(\mathcal{O}), \tag{3.7}$$

where

$$\bar{a} = \frac{1}{\mathrm{E}\left\{\frac{1}{a}\right\}}. \tag{3.8}$$

Then, (3.4) and (3.7) give

$$\bar{\sigma} = \bar{a}\bar{u}'. \tag{3.9}$$

Moreover, we know $-\sigma'_\epsilon(\cdot,\omega) = f$ for all ϵ and therefore, $-\bar{\sigma}' = f$; this latter conclusion along with (3.9) gives,

$$-(\bar{a}\bar{u}')' = f.$$

Hence, we have shown that for almost all $\omega \in \Omega$, the problem

$$\begin{cases} -\big(a^T(x,\omega)u'(x,\omega)\big)' = f(x) \quad \text{in} \quad \mathcal{O} = (0,1) \\ u(0,\omega) = u(1,\omega) = 0 \end{cases}$$

admits homogenization, with the homogenized coefficient given by (3.8).

4 Connections to Classical Theory

The homogenization result shown in the previous section relies heavily on the fact the we are working in a 1D setting. In higher dimension, the analysis becomes significantly more complicated. However, the homogenization theory for the general n-dimensional case is fully developed [4]. Our goal in this section is to verify that, in the special case of 1D, the homogenization results such as those in [4] give the same expression for the homogenized coefficient $\bar{a}$ as in (3.8).

The homogenization results in [4] state that (here we specialize everything to the 1D case): The homogenized coefficient $\bar{a}$ is characterized as follows.

For $\xi \in \mathbf{R}$,

$$\bar{a}\xi = \int_\Omega a(\omega)(\xi + v_\xi(\omega))\, d\mu(\omega), \tag{4.1}$$

where $v_\xi \in \boldsymbol{\mathcal{V}}^2_{\text{pot}}(\Omega, T)$ (let T be as before), is solution to

$$a(\xi + v_\xi) \in \boldsymbol{L}^2_{\text{sol}}(\Omega, T). \tag{4.2}$$

Now, in the above equations we let $\xi = 1$. Denote,

$$q(\omega) = a(\omega)(1 + v_1(\omega)), \tag{4.3}$$

and note that by (4.2), for almost all ω, $q\big(T_x(\omega)\big)$ is a constant (depending on ω). Therefore, we get that for almost all $\omega \in \Omega$,

$$q\big(T_x(\omega)\big) = q(\omega), \quad \forall x \in \mathbf{R}.$$

Consequently, by ergodicity of the dynamical system T, we have

$$q(\omega) = const =: \bar{c}, \quad \mu\text{-a.e.} \tag{4.4}$$

Using (4.3) and (4.4) we solve for v_1

$$v_1(\omega) = \frac{\bar{c}}{a(\omega)} - 1.$$

Moreover, using using $\int_\Omega v_1(\omega)\, d\mu(\omega) = 0$, we have

$$\bar{c} = \frac{1}{\int_\Omega \frac{1}{a(\omega)}\, d\mu(\omega)}. \tag{4.5}$$

Then, (4.1) gives (recall we let $\xi = 1$),

$$\begin{aligned}
\bar{a} &= \int_\Omega a(\omega)(1 + v_1(\omega))\, d\mu(\omega) \\
&= \int_\Omega \bar{c}\, d\mu(\omega) \\
&= \bar{c} \\
&= \frac{1}{\int_\Omega \frac{1}{a(\omega)}\, d\mu(\omega)},
\end{aligned}$$

where the last equality follows from (4.5). Hence, we see that the homogenization theory of random media as developed in [4], when specialized to the case of 1D, gives the same conclusion as that we saw in the previous section.

5 Numerical Computations

While (3.8) gives a closed form expression for the homogenized coefficient $\bar{a}$, in practice one cannot compute this quantity, because it involves integrating in an abstract probability space. Of course it is possible to do Monte-Carlo like simulations to approximate the expectation in (3.8) by ensemble averages; however, our approach in this section will be different. We will proceed in the framework of periodization [2] and consider specialized convergence results in our 1D context. The developments in this section follow that of [1] closely.

Denote $S^\rho = (0, \rho)$ and let $a : \Omega \mapsto \mathbf{R}$ be as in the previous section. Define $a^\rho_{per}(x, \omega)$ as follows,

$$a^\rho_{per}(x, \omega) = a\big(T_{x \bmod S^\rho}(\omega)\big) = a^T(x \bmod S^\rho, \omega). \tag{5.1}$$

One thing to note immediately is that for $x \in S^1$,

$$a^\rho_{per}(\rho x, \omega) = a^T(\rho x \bmod S^\rho) = a^T(\rho x, \omega). \tag{5.2}$$

Now, for each ρ and for a fixed ω, we can consider the (periodized) problem,

$$\begin{cases} -\big(a^\rho_{per}(x, \omega) u'_\rho\big)' = f(x) \quad \text{in} \quad \mathcal{O} = (0, 1) \\ u_\rho(0, \omega) = 0 = u_\rho(1, \omega) \end{cases} \tag{5.3}$$

We know from homogenization of periodic operators that this problem admits homogenization with homogenized coefficient given by

$$\bar{a}^\rho(\omega) = \frac{1}{\frac{1}{\rho}\int_0^\rho \frac{1}{a^\rho_{per}(x, \omega)}} \tag{5.4}$$

The following theorem [1] provides a convergence result for method periodization in 1D:

Theorem 5.1. *For almost all* $\omega \in \Omega$, $\bar{a}^{\rho}(\omega) \to \bar{a}$, *as* $\rho \to \infty$. *with* $\bar{a}$ *as in* (3.8).

Proof. First note that using the change of variable $x \mapsto x/\rho$ in (5.4) we get

$$\bar{a}^{\rho}(\omega) = \frac{1}{\int_0^1 \frac{1}{a_{per}^{\rho}(\rho x, \omega)}} = \frac{1}{\int_0^1 \frac{1}{a^T(\rho x, \omega)}}, \tag{5.5}$$

where the second equality follows from (5.2). Now, as we saw in the previous section, we can invoke Birkhoff's Ergodic Theorem to conclude that for almost all $\omega \in \Omega$, as $\rho \to \infty$,

$$\frac{1}{a^T(\rho x, \omega)} \rightharpoonup \mathrm{E}\left\{\frac{1}{a}\right\} \quad \text{in } L^2(\mathcal{O}). \tag{5.6}$$

Therefore,

$$\int_0^1 \frac{1}{a^T(\rho x, \omega)} \to \mathrm{E}\left\{\frac{1}{a}\right\}, \quad \text{as } \rho \to \infty. \tag{5.7}$$

But then it follows from (5.5) that for almost all $\omega \in \Omega$,

$$\lim_{\rho \to \infty} \bar{a}^{\rho}(\omega) = \lim_{\rho \to \infty} \frac{1}{\int_0^1 \frac{1}{a^T(\rho x, \omega)}} = \frac{1}{\mathrm{E}\left\{\frac{1}{a}\right\}},$$

which completes the proof. $\square$

5.1 Numerical experiments and results

Let us begin by considering a simple numerical study. Suppose we are working with a one dimensional rod whose conductivity is a random function which in each interval $[n-1, n)$, $n \in \mathbf{Z}$, takes the value 1 or 2 with probability of $\frac{1}{2}$. Figure 1 depicts the conductivity function for a sample realization of such a rod. The function a used to

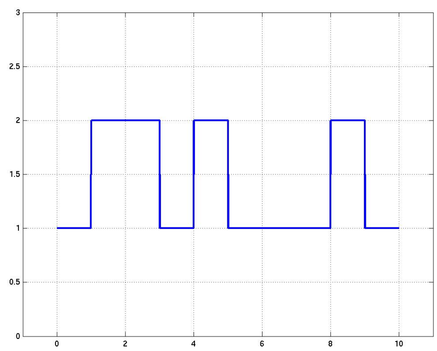

Figure 1: Conductivity function for a rod with random structure.

model such a medium, will take values of 1 or 2 with equal probability of $\frac{1}{2}$. Without delving too much into the technical details, we know from the results of the previous sections that the effective conductivity $\bar{a}$ will be

$$\bar{a} = \left(\mathrm{E}\left\{\frac{1}{a}\right\}\right)^{-1} = \frac{4}{3}.$$

To get some numerical insight on the periodization process, we compute $\bar{a}^\rho(\omega_j)$ for $\rho = 2^i$, $i = 6, \ldots 9$ and for realizations ω_j, $j = 1, \ldots, N$. We used number of realizations $N = 10000$ in our simulations. The histograms in Figure 2 show the improvement of the approximation $\bar{a}^\rho$ to $\bar{a}$ as ρ gets larger. We note that the distribution of $\bar{a}^\rho$ gets more and more centered around the value of $\frac{4}{3}$. We also look at the arithmetic mean the harmonic mean:

$$M_a(\rho) = \frac{1}{N}\sum_{j=1}^{N} \bar{a}^\rho(\omega_j), \qquad M_h(\rho) = \frac{N}{\sum_{j=1}^{N} \frac{1}{\bar{a}^\rho(\omega_j)}},$$

for different values of ρ listed in Table 1. Also, we record the sample standard deviation for each choice of ρ in Table 1.

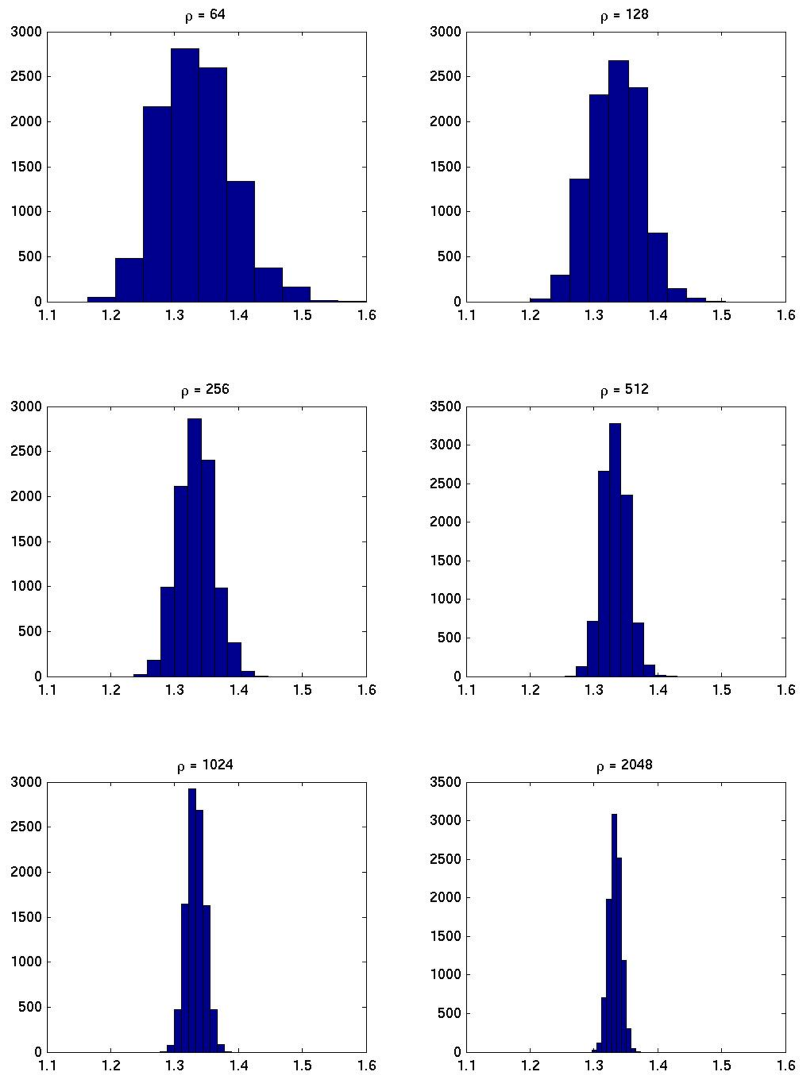

Figure 2: Distribution of $\bar{a}^\rho$ for different ρ

ρ	$M_a(\rho)$	$M_h(\rho)$	Sample Std. Deviation
64	1.335595	1.333235	0.056361
128	1.334577	1.333399	0.039705
256	1.333984	1.333402	0.027889
512	1.333653	1.333355	0.019961
1024	1.333453	1.333307	0.013974
2048	1.333131	1.333059	0.009797

Table 1: Sample averages for different ρ

Remark 5.2. *The histograms in Figure 2 suggest that for large values of ρ the distribution of ρ approaches that of a normal distribution. As shown in detail in [1],this is in fact the case; that is, asymptotically, distribution of $\bar{a}^\rho$ approaches that of a normal distribution. See [1] for details and further numerical examples.*

References

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