A Lorentzian Lipschitz, Gromov-Hausdorff notion of distance

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Abstract

This paper is the first of three in which I study the moduli space of isometry classes of (compact) globally hyperbolic spacetimes (with boundary). I introduce a notion of Gromov-Hausdorff distance which makes this moduli space into a metric space. Further properties of this metric space are studied in the next papers. The importance of the work can be situated in fields such as cosmology, quantum gravity and - for the mathematicians - global Lorentzian geometry.

1 Introduction

The aim of this paper is to make first steps in the construction of a convergence theory for partially ordered spaces equipped with a Lorentz distance, which we shall refer to as Lorentz spaces. Typical examples of such spaces are Lorentz manifolds, which constitute the geometrical playground for general relativity. The field of application should in the end be quantum gravity and in particular the path integral formulation thereof. In this application, the purpose is twofold: on one hand a convergence theory will serve as a tool for taking a continuum limit, on the other hand it will provide a mechanism to control which geometrical objects to sum over and which not. It is my hope that in a later stage, we shall be able to link this rather abstract control theory with statistical Lorentzian geometry, in order to be able to make this passage to the former application.
At the moment, the main background-independent attempts to quantize general relativity are canonical quantum gravity and the resulting spin foam models, which are structural extensions of the causal sets introduced by Rafael Sorkin. The main difficulty in all these approaches consists in making precise what it means for such a Lorentz space (in fact, causal sets are not Lorentz spaces in the above sense since the Lorentzian distance is not accounted for) to be close to a Lorentz manifold. Other difficult questions with respect to these objects concern a good definition of dimensionality and relativistic scale. In these three papers we shall present an *abstract* solution to all these problems in the way

Gromov did for locally compact metric spaces. The Lorentzian analogue is not just a copy of the 23 year old Gromov theory [8] [10]: some intermediate results have to be stated differently and the proofs are considerably more difficult.
In section two, we generalize the Lipschitz notion of distance between two compact metric spaces to a notion of distance between two globally hyperbolic compact Lorentz manifolds with spacelike boundary. It turns out that we can only measure a distance between conformally equivalent structures, and in this sense this chapter is only a warm-up. The most important result is a Lorentzian analogue for the Ascoli-Arzela theorem which guarantees convergence to isometry. The most important lesson, however, is that we have found a class of mappings that is rich enough to compare conformally equivalent structures and which is poor enough to keep a good control over. In section three we introduce a Lorentzian notion of Gromov - Hausdorff distance d_{GH}. I will give some examples to show that the Lorentzian theory is rather different from the Riemannian one.
Previous attempts in the literature to construct a metric on the modulo space of isometry classes of *Lorentzian* spacetimes can be found in [11], [4] , [5] , [6]. But all these attempts failed since one could only prove that one had obtained a *pseudo* distance. Moreover, I do not agree with the philosophy behind them, since in all these papers (including mine) the canonical volume measure has been used. In particular, this means that I take the point of view that the construction of a *statistical* Lorentzian convergence theory should follow a *geometrical* Lorentzian convergence theory and not the other way around.
The readers not familiar with the following notions and results concerning causality are, if not mentioned otherwise referred to the bible of general relativity, [2].

2 A Lipschitz distance

Our aim is to define a Lorentzian analogue of the classical Riemannian Lipschitz distance between (locally) compact (pointed) metric spaces. Let (X, d_X) and (Y, d_Y) two compact metric spaces and $f : X \to Y$ be a bi-Lipschitz mapping, i.e., there exist numbers $0 < \alpha < \beta$ such that

$$\alpha d_X(x, y) \leq d_Y(f(x), f(y)) \leq \beta d_X(x, y) \quad \forall x, y \in X$$

The minimal such β is the *dilatation* $\mathrm{dil}(f)$ of f and the maximal such α the *co-dilatation* of f (or the inverse of the dilatation of f^{-1} if f^{-1} exists). The Lipschitz distance $d_L(X, Y)$ between X and Y is the infimum over all bi-Lipschitz homeomorphisms of the expression:

$$|\ln(\mathrm{dil}(f))| + \left|\ln(\mathrm{dil}(f^{-1}))\right|$$

The key result is that $d_L(X, Y) = 0$ iff (X, d_X) is isometric to (Y, d_Y), which is a direct consequence of the Ascoli-Arzela theorem [1].

Theorem 1 ***(Ascoli-Arzela)*** *Let X and Y be second countable, locally compact spaces, moreover (Y, d_Y) is assumed to be metrically complete. Assume that the sequence $\{f_n\}$ of functions $f_n : X \to Y$ is equicontinuous such that the sets $\bigcup_n \{f_n(x)\}$ are bounded with respect to d_Y for every $x \in X$. Then there exists a continuous function $f : X \to Y$ and a subsequence of $\{f_n\}$ which converges uniformly on compact sets in X to f.*

Let us now make some analogy and discrepancy with the Lorentzian case. For now we restrict to spacetimes, i.e., pairs $(\mathcal{M}, g)$ where $\mathcal{M}$ is a C^∞, paracompact, Hausdorff manifold and g is a Lorentzian metric on it, such that $(\mathcal{M}, g)$ is time orientable. Abstract Lorentzian spaces will be defined later on in analogy with Seifert and Busemann. We will make now a convention in terminology which is not standard in the literature, but is somehow necessary to keep the discussion clear.

Definition 1 *Let X be a topological space with a partial order $\prec$. A Lorentzian distance is a function $d : X \times X \to \mathbb{R}^+ \cup \{\infty\}$ which satisfies*

- *$d(x, x) = 0$*
- *$d(x, y) > 0$ implies $d(y, x) = 0$ (antisymmetry)*
- *if $x \prec y \prec z$ then $d(x, z) \geq d(x, y) + d(y, z)$ (reverse triangle inequality)*

It is well known that every chronological spacetime (with partial order $\ll$) determines a canonical Lorentzian distance by defining $d_g(x, y)$ as the supremum over all lengths of future oriented causal curves from x to y if such curves exist and zero otherwise. One has that d_g is continuous and finite if $(\mathcal{M}, g)$ is globally hyperbolic and, vice versa, that if d_g is continuous then $(\mathcal{M}, g)$ is causally continuous. More equivalences between properties of d_g and causality restrictions can be found in [9]. We shall only be interested in globally hyperbolic spacetimes since the continuity of d_g is a desirable property if one wants to work out a comparison theory according to Lipschitz. Note immediately that a compact globally hyperbolic spacetime does not exist unless we consider manifolds with a boundary. We assume the boundary is spacelike and that $\mathcal{M}$ and $\mathcal{N}$ are locally extendible across their boundary (all the results in this paper are also valid when an extra timelike boundary is allowed, it is up to the reader to fill in the details in the proofs). Remark first that every point of the boundary is contained in a neighborhood $\mathcal{U}$ which is diffeomorphic to a hypercube in $\mathbb{R}^n$, where exactly one side hyperplane, say $t = 0$, belongs to $\mathcal{U}$ and no other points of any other side hyperplane belong to $\mathcal{U}$. By local extendibility I mean that there exists an isometric embedding of $(\mathcal{U}, g_{|\mathcal{U}})$ in a open spacetime $(\mathcal{V}, g_{|\mathcal{V}})$ such that the image of $\mathcal{U}$ has compact closure in $\mathcal{V}$. We stress that this does not correspond to the usual notion of *causal* local extendibility to which we come back later on.
First we have to contemplate which mappings between two spacetimes need to be considered for comparison. To this purpose, let $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$ denote globally hyperbolic spacetimes. A mapping $f : \mathcal{M} \to \mathcal{N}$ is said to be timelike

Lipschitz if and only if it has bounded timelike dilatation tdil(f), i.e. there exists a (smallest) number β such that

$$d_h(f(x), f(y)) \leq \beta d_g(x,y), \quad \forall x, y \in \mathcal{M}.$$

The above construction for the Riemannian case suggests that we consider timelike bi-Lipschitz homeomorphisms. However, a slight generalization of a classical result for homotheties teaches us that a surjective timelike bi-Lipschitz map is automatically a homeomophism (see Appendix A). Indeed, a result of Hawking, Mc Carthy and King [3], proves that such mapping is a C^∞ conformal diffeomorphism. One might be concerned that such maps are too restrictive in the sense that they only allow conformally equivalent spacetimes to be compared but as mentioned before, this section is meant as a warm-up to get used to the techniques needed for the next section. The next logical step is formulating and proving a Lorentzian version of a suitably modified Ascoli-Arzela theorem.

Theorem 2 ***(Lorentzian Ascoli-Arzela)*** *Let $f_n : \mathcal{M} \to \mathcal{N}$ be onto bi-Lipschitz mappings such that $\bigcup_n \{f_n(x)\}$ and $\bigcup_n \{f_n^{-1}(y)\}$ are precompact[1] in $\mathcal{N}$ respectively $\mathcal{M}$ for all $x \in \mathcal{M}$ and $y \in \mathcal{N}$. Moreover, let $(c_n)_{n\in\mathbb{N}}$ be a descending sequence ($c_n < 1$) converging to zero such that tdil$(f_n) \leq 1 + c_n$ and tdil$(f_n^{-1}) \leq \frac{1}{1-c_n}$; then there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ and a isometry f such that f_{n_k} converges to f pointwise.*

We recall that $x \prec y$ means that x is in the causal past of y and $x \ll y$ indicates that y is in the chronological future of x. Note also that we did not specify that $\ll$ is a partial order relation induced by a metric tensor g since this would unnecessarily complicate the notation. It should be clear from the context by which metric the particular partial order relation is defined. We shall also not denote the distinction between d_g and d_h.

Proof

Let $\mathcal{C}$ be a countable dense subset of $\mathcal{M}$. By a diagonalization argument one obtains a subsequence $\{f_{n_k}\}$ such that $f_{n_k}(p) \stackrel{k\to\infty}{\to} f(p) \quad \forall p \in \mathcal{C}$. Let r be any interior point of $\mathcal{M}$ which is not in $\mathcal{C}$, we show now that the definition of f can be extended to r such that $\lim_{k\to\infty} f_{n_k}(r) = f(r)$. Let $\mathcal{U}$ be a causally convex normal neighborhood of r and choose a point $p_1 \in \mathcal{C} \cap I^-(r) \cap \mathcal{U}$ close enough to r. Let γ be the unique timelike geodesic from p_1 through r and define $\tilde{p}_i, \tilde{q}_i \in \gamma$ by $d(\tilde{p}_i, r) = \frac{d(p_1,r)}{i}$ and $d(r, \tilde{q}_i) = \frac{d(p_1,r)}{i}$. Hence $d(\tilde{p}_i, \tilde{p}_{i+j}) = \frac{jd(p_1,r)}{i(i+j)} = d(\tilde{q}_{i+j}, \tilde{q}_i)$ and $d(\tilde{p}_i, \tilde{q}_i) = \frac{2d(p_1,r)}{i}$. Define now sequences $p_i, q_i \in \mathcal{C}$ such that $\tilde{p}_i \ll p_i \ll \tilde{p}_{i+1}$, $\tilde{q}_{i+1} \ll q_i \ll \tilde{q}_i$ with the exception that $p_1 = \tilde{p}_1$. We shall now prove the following claims:

- the sequence $(f(p_i))_i$ is contained in the compact set $A(f(\tilde{p}_1), f(q_1))$[2] and has exactly one accumulation point $f_\uparrow(r)$ which turns out to be a limit point.

[1] A subset A of a topological space X is precompact iff the closure of A is compact.

[2] $A(p,q) = \{r | p \leq r \leq q\}$.

- $f_\uparrow(r)$ is independent of the choice of $(p_i)_{i\in\mathbb{N}}$

The first claim is an easy consequence of the observation that for all i one has that :

$$\begin{aligned} d(f(\tilde{p}_1), f(p_i)) &= \lim_{k\to\infty} d(f_{n_k}(\tilde{p}_1), f_{n_k}(p_i)) \\ &= d(\tilde{p}_1, p_i), \end{aligned}$$

where we used the continuity of d in the target space and the property of the convergence of the timelike dilatation and co-dilatation of the mappings f_n. The above also proves that $d(f(p_i), f(p_{i+j}) = d(p_i, p_{i+j})$ and hence $f(p_i) \ll f(p_{i+j})$. This in turn implies that any accumulation point of the sequence $(f(p_i))_i$ must lie to the future of all $f(p_i)$, hence it is a limit point which must be unique.
The second claim follows from the observation that if $\hat{p}_i$ is another such sequence with corresponding $\hat{f}_\uparrow(r)$ then one has that

$$p_i \ll \hat{p}_{i+1} \ll p_{i+2} \ll \hat{p}_{i+3} \ll \ldots \ll r.$$

Hence

$$0 < d(f(\hat{p}_{i+1}), f_\uparrow(r)) \leq d(f(p_i), f_\uparrow(r)) - d(f(p_i), f(\hat{p}_{i+1}))$$

However, the first term on the rhs. converges to zero for $i \to \infty$ and the second term is estimated by $d(f(p_i), f(\hat{p}_{i+1})) \leq \frac{2d(\tilde{p}_1, r)}{i(i+2)}$. Hence $\hat{f}_\uparrow(r) \in E^-(f_\uparrow(r))$. The reverse is proven similary and this concludes the second claim.

The same result is of course also true for p replaced by q, and we denote the corresponding accumulation point by $f_\downarrow(r)$. Note that $d(f(\tilde{p}_1), f_\uparrow(r)) = d(\tilde{p}_1, r)$, $d(f_\downarrow(r), f(q_1)) = d(r, q_1)$ and $d(f(\tilde{p}_1), f(q_1)) = d(\tilde{p}_1, q_1)$, which all follow from continuity of d and the present properties of f. $f_\uparrow(r) = f_\downarrow(r)$ follows from the observation that changing q_1 by a point in the future of q_1, so that we can come arbitrary close to $\tilde{q}_1$, does not change the point $f_\downarrow(r)$. For the same reasons as before such a sequence of points q_1 will define a sequence $f(q_1)$ which converges to a point, say, $f_\uparrow(\tilde{q}_1)$ in the future of all points $f(q_1)$. Hence due to continuity we have that $d(f_\downarrow(r), f_\uparrow(\tilde{q}_1)) = d(r, \tilde{q}_1)$ and $d(f(\tilde{p}_1), f_\uparrow(\tilde{q}_1)) = d(\tilde{p}_1, \tilde{q}_1)$. But this implies that $d(f_\downarrow(r), f_\uparrow(r)) = 0$ and more strongly $f_\uparrow(r) = f_\downarrow(r)$ otherwise by "rounding off the edges" we could find a timelike curve with length larger than $d(f(\tilde{p}_1), f_\uparrow(\tilde{q}_1))$, which is a contradiction. It is now easy to see that $f_{n_k}(r)$ converges to $f(r)$, since for every i we can find a k_0 such that for all $k \geq k_0$ one has that $f(p_i) \ll f_{n_k}(p_{i+1}) \ll f(r) \ll f_{n_k}(q_{i+1}) \ll f(q_i)$, which implies (because of the properties of f_{n_k}) that $f(p_i) \ll f_{n_k}(r) \ll f(q_i)$. This concludes the proof when r is an interior point, since the open Alexandrov sets $int(A(f(p_i), f(q_i)))$ form a basis for the topology around $f(r)$. The case when r is a past boundary point is rather different, since then we cannot squeeze the point r anymore in an Alexandrov set (the case of a future boudary point is identical). Obviously, $f_{n_k}(r)$ belongs to the past boundary of $\mathcal{N}$. Let γ be the unique geodesic segment orthogonal to the past boundary in r, and choose the sequences $(\tilde{q}_i)_{i\in\mathbb{N}}$ and $(q_i)_{i\in\mathbb{N}}$ as before. Then we can find a subsequence $f_{n_{k_l}}$ such that $f_{n_{k_l}}(r) \stackrel{l\to\infty}{\to} f(r)$, where $f(r)$ belongs to the past boundary and $f_{n_{k_l}}(\gamma_{|[r,\tilde{q}_1]}) \to f(\gamma_{|[r,\tilde{q}_1]})$ in the C^0 topology of curves. It is easy to see that $f(\gamma_{|[r,\tilde{q}_1]})$ is the unique geodesic segment in $\mathcal{N}$ orthogonal to the past boundary in $f(r)$. But in this case, we have that

$$f(q_i) \gg f_{n_k}(q_{i+1}) \gg f_{n_k}(r),$$

and since the $I^-(f(q_i))$ form a basis for the topology around $f(r)$, we have that $\lim_{k\to\infty} f_{n_k}(r) = f(r)$, which concludes the proof. It is not difficult to see that f is continuous by construction. As a matter of fact, we should still prove that f is onto. Performing the same construction for $f^{-1}_{n_k}$ we find (by eventually taking a subsequence) a limit mapping f^{-1}. We now show that $f^{-1}\circ f = id_{\mathcal{M}}$, $f\circ f^{-1} = id_{\mathcal{N}}$. We shall prove the former, the proof of the latter is identical. Suppose there exists an interior point x such that $\lim_{k\to\infty} f^{-1}_{n_k}\circ f(x) \neq x$, then there exist points $p_1, p_2, p_3, q_1, q_2, q_3$ such that

$$p_1 \ll p_2 \ll p_3 \ll f^{-1}\circ f(x) \ll q_3 \ll q_2 \ll q_1$$

and $x \notin A(p_1, q_1)$. Then for k big enough:

- $p_3 \ll f^{-1}_{n_k}\circ f(x) \ll q_3$
- $f_{n_k}(p_1) \ll f(p_2) \ll f_{n_k}(p_3)$
- $f_{n_k}(q_3) \ll f(q_2) \ll f_{n_k}(q_1)$,

hence

$$f_{n_k}(p_1) \ll f(p_2) \ll f(x) \ll f(q_2) \ll f_{n_k}(q_1),$$

but $f_{n_k}(x) \notin A(f_{n_k}(p_1), f_{n_k}(q_1))$, which implies that $f(x)$ cannot lie between $f(p_2)$ and $f(q_2)$, which is a contradiction. Hence $f^{-1}\circ f$ equals the identity on the interior of $\mathcal{M}$, and therefore it equals the identity everywhere since it is continuous. The conclusion that f is an isometry follows from the discussion in appendix A. $\square$

Remark first that in the proof of the theorem we needed the requirement that $\bigcup_n \left\{f_n^{-1}(y)\right\}$ is precompact in $\mathcal{M}$ for all $y \in \mathcal{N}$ only to guarantee the surjectivity and hence the smoothness of f. Sensible questions are the following:

- Is f not surjective a priori? If not give a counterexample.
- Is the convergence uniform on compact sets with respect to some Riemannian metric $\tilde{d}$ on $\mathcal{N}$? In either case, is the family of mappings $\{f_n\}$ equicontinuous with respect to $\tilde{d}$?
- Give a counterexample to the conclusion of Ascoli-Arzela in case the spacetimes are not globally hyperbolic, but, say, causally continuous. One might

expect that such counterexample exists since we made use of *all* properties of global hyperbolicity, that is the compactness of the Alexandrov sets to guarantee the convergence of the sequence $(f(p_i))_{i\in\mathbb{N}}$ and the continuity of d.

- Can one extend the theorem to the case where the timelike dilatation and co-dilatation of the mappings are only bounded and not necesserily convergent to 1?

I shall only examine the last question. Remark first that the proof made crucial use of the fact that one has convergence to isometry. The key argument was that the continuous timelike extension of f maps 3 points on a distance maximizing geodesic in $\mathcal{M}$ to 3 points on a distance maximizing geodesic in $\mathcal{N}$. This argument will clearly not be valid anymore when the limit mapping (if it exists) is not an isometry. One could however invoke earlier the construction of f^{-1} in the proof which is not a priori preferable considering the above questions. However, such a strategy leads towards the following stronger result:

Theorem 3 *Let $\alpha < 1 < \beta$, $\;f_n : \mathcal{M} \to \mathcal{N}$ be as in Theorem 2 with the difference that tdil(f_n) $\leq \beta$ and tdil(f_n^{-1}) $\leq \frac{1}{\alpha}$. Then there exists a subsequence f_{n_k} and an f such that f_{n_k} converges pointwise to f. Moreover one has that tdil(f) $\leq \beta$ and tdil(f^{-1}) $\leq \frac{1}{\alpha}$.*

Proof
Let $\mathcal{C}$ and $\mathcal{D}$ be countable dense subsets in $\mathcal{M}$ and $\mathcal{N}$ respectively. By a diagonalization argument we find a subsequence f_{n_k} such that $f_{n_k}(p)$ converges to $f(p)$ and $f^{-1}_{n_k}(q)$ converges to $f^{-1}(q)$ for all $p \in \mathcal{C}$ and $q \in \mathcal{D}$ respectively. Suppose r is an interior point and let γ, $(p_i)_{i\in\mathbb{N}}$ and $(\tilde{p})_{i\in\mathbb{N}}$ be as before. Take $q \in \mathcal{D}$ arbitrarily close in the chronological future of $f_\uparrow(r)$, we have then that

$$\begin{aligned} d(r, f^{-1}(q)) &= \lim_{i\to\infty} d(p_i, f^{-1}(q)) \\ &= \lim_{i\to\infty}\lim_{k\to\infty} d(p_i, f^{-1}_{n_k}(q)) \\ &\geq \frac{1}{\beta}\lim_{i\to\infty}\lim_{k\to\infty} d(f_{n_k}(p_i), q) \\ &\geq \frac{d(f_\uparrow(r), q)}{\beta}. \end{aligned}$$

Hence $r \ll f^{-1}(q)$. Take now $q_1, q_2 \in \mathcal{D}$ such that $f_\uparrow(r) \ll q_1 \ll q_2$ with q_2 arbitrarily close to $f_\uparrow(r)$. Choose $i > 0$, then for k sufficiently large one has

$$r \ll f^{-1}_{n_k}(q_1) \ll f^{-1}(q_2)$$

and

$$f(p_i) \ll f_{n_k}(p_{i+1}) \ll f_\uparrow(r).$$

Hence

$$f(p_i) \ll f_{n_k}(p_{i+1}) \ll f_{n_k}(r) \ll q_1,$$

which proves $\lim_{k\to\infty} f_{n_k}(r) = f_{\uparrow}(r)$.
Let r be a point of the "past" boundary (the future situation is dealt with identically). Let γ be a distance maximizing geodesic with past endpoint r and let $(\tilde{q}_i)_{i\in\mathbb{N}}, (q_i)_{i\in\mathbb{N}}$ be sequences of points as before where now the "futuremost" point $\tilde{q}_1$ is sufficiently close to r and q_1 can be chosen equal to $\tilde{q}_1$. Without loss of generality, we can assume that $\tilde{q}_1 \ll q \in \mathcal{D}$ such that $J^-(q)$ is compact. For k sufficiently large we have that

$$f_{n_k}(q) \gg f(q_1) \gg f(q_2) \gg \ldots$$

Since f_{n_k} is continuous $J^-(f_{n_k}(q)) = f_{n_k}(J^-(q))$ is compact, therefore the sequence $(f(p_i))_{i\in\mathbb{N}}$ has an accumulation point $f_{\downarrow}(r)$, which is as usual also a limit point. Suppose $f_{\downarrow}(r)$ is not on the past boundary, then we can find a point $p \in \mathcal{D}$ such that $p \ll f_{\downarrow}(r)$. The calculation above shows that $f^{-1}(p) \ll r$, which is impossible. Hence $f_{\downarrow}(r)$ belongs to the past boundary. Since the past light cones $I^-(f(p_i))$ constitute a local basis for the topology around $f_{\downarrow}(r)$, the result follows.
The other conclusions of the theorem are obvious. $\square$

Having this theorem in the pocket, the theorem which guarantees convergence to isometry follows immediately.

Theorem 4 *Let $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$ be compact globally hyperbolic spacetimes with boundary, then $d_L((\mathcal{M}, g), (\mathcal{N}, h)) = 0$ iff $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$ are isometric.*

The notion of Lipschitz distance however is too severe and does not give rise to a rich comparison theory since there is too much geometric control. A result of Defrise-Carter [12] shows that every Lie algebra $\mathcal{L}$[3] of *local* conformal diffeomorphisms of four dimensional Lorentz manifolds are all, with two exceptions, essentially isometries. By this, I mean that for every spacetime $(\mathcal{M}, g)$ not conformally equivalent to Minkowski or a plane-wave spacetime with parallel rays, there exists a conformal factor Ω such that $\mathcal{L}$ constitutes a r dimensional Lie algebra of isometries for the spacetime $(\mathcal{M}, \Omega g)$, with $r \leq 10$. In Minkowski spacetime, there is a 15-dimensional group of proper conformal transformations[4] and in the latter only a 6 or 7 dimensional group or homotheties[5]. Hence, there are "not many" infinitesimal conformal diffeomorphisms, and there are even fewer which can be integrated. Note that the result of Defrise-Carter does not mention anything about *discrete* conformal diffeomorphisms. However, the results of this section are still very important, since:

- we shall be forced to generalize this Lipschitz theory to *abstract* globally hyperbolic Lorentzian spaces, which will be done in the next paper.
- the proofs give a hunch how to prove convergence to isometry in case the family of mappings gets enlarged, such as will happen in the next section.

[3]The assumption in the paper of Defrise-Carter that the group needs to be finite dimensional, is not necessary.
[4]Generators consist of the 10 Poincare transformations, 1 dilatation and 4 accelerations.
[5]5 respectively 6 generators form an isometry group, and 1 generator forms a dilatation.

In a complete Riemannian manifold which is not locally flat, Kobayashi and Numizu have proven that there are no homotheties which are not isometries. As stated before, this is not true in the Lorentzian case as the next plane wave spacetime shows [9].

Example Consider $\mathbb{R}^3$ with the metric $ds^2 = e^{xz}dxdy + dz^2$. $(\mathbb{R}^3, ds^2)$ is not flat and the mappings $\phi_t(x, y, z) = (e^t x, e^{-3t}y, e^{-t}z)$ are proper homothecies with factor e^{-2t}.

Moreover, we will see that for every compact globally hyperbolic spacetime $(\mathcal{M}, g)$ there exists a Riemannian metric $d_{\mathcal{M}}$ such that all d_g isometries are $d_{\mathcal{M}}$ isometries. Suppose $(\mathcal{M}, g)$ is not a compact piece cut out of Minkowski or a plane-wave spacetime (with parallel rays), then there exists a Riemannian metric $\tilde{d}_{\mathcal{M}}$ such that "most" (apart from eventual discrete conformal diffeomorphisms) g-conformal diffeomorphisms are $\tilde{d}_{\mathcal{M}}$ isometries[6].

3 A Gromov-Hausdorff distance

As in the previous chapter we recall the notion of Gromov-Hausdorff distance in the Riemannian case. For this purpose define the Hausdorff distance d_H between subsets U, V of a metric space (X, d_X) as

$$d_H(U, V) = \inf\{\epsilon | U \subset B(V, \epsilon), V \subset B(U, \epsilon)\}$$

where $B(U, \epsilon) = \{x \in X | \exists a \in U : d_X(x, a) < \epsilon\}$. Gromov had around 1980 the following idea [8] : consider two compact metric spaces (X, d_X) and (Y, d_Y), define a metric d on the disjoint union $X \cup Y$ to be *admissible* iff the restrictions of d to X and Y equal d_X and d_Y respectively. Then

$$d_{GH}((X, d_X), (Y, d_Y)) = \inf\{d_H(X, Y) | \text{all admissible metrics on} X \cup Y\}.$$

In other words the Gromov-Hausdorff distance between two metric spaces is the infimum over all Hausdorff distances in $X \cup Y$ with respect to metrics which extend the given metrics on X and Y. Suppose d is an admissible metric on $X \cup Y$; then there exist mappings $f : X \to Y$, $g : Y \to X$ such that $d(x, f(x)) \leq d_H(X, Y)$ and $d(y, g(y)) \leq d_H(X, Y)$ for all $x \in X$, $y \in Y$ respectively. The triangle inequality and the properties of d imply that :

$$\begin{aligned} |d_Y(f(x_1), f(x_2)) - d_X(x_1, x_2)| &\leq 2d_H(X, Y) \quad (1) \\ |d_X(g(y_1), g(y_2)) - d_Y(y_1, y_2)| &\leq 2d_H(X, Y) \quad (2) \\ d_X(x, g \circ f(x)) &\leq 2d_H(X, Y) \quad (3) \\ d_Y(y, f \circ g(y)) &\leq 2d_H(X, Y) \quad (4) \end{aligned}$$

Observe that the last two inequalities imply that in the limit for $d_H(X, Y)$ to zero, f becomes invertible. But for compact metric spaces, invertibility also

[6]We know there exists a global conformal factor Ω such that essentially all g conformal diffeomorphisms are Ωg isometries, hence the claim follows.

follows from the observation that in the limit for $d_H(X,Y)$ to zero, f and g become distance-preserving maps. Hence $g \circ f$ and $f \circ g$ are distance-preserving maps on X and Y respectively. The compactness assumption then implies that they are both bijections and, as a consequence, so are f and g. We shall first prove a similar result in the Lorentzian case.

Theorem 5 *Let $f : \mathcal{M} \to \mathcal{M}$ be continuous and Lorentzian distance preserving on the interior of $\mathcal{M}$; then f maps the interior onto itself.*

Proof Remark that an interior point is mapped by a distance-preserving map to an interior point. Suppose p is an interior point not in $f(\mathcal{M})$, then there exists a neighborhood $\mathcal{U}$ of p for which $f(\mathcal{M}) \cap \mathcal{U} = \emptyset$. For suppose not, then we can find a sequence $r_n \overset{n\to\infty}{\to} r$ such that $f(r_n) \overset{n\to\infty}{\to} p$. Hence r is not an interior point and without loss of generality we can assume it belongs to the future boundary. But then $f(\mathcal{M}) \cap I^+(p) = \emptyset$, otherwise there would exist an interior point to the future of all r_n, which is impossible.
Hence we may assume that there exist points $r \ll p \ll s$ such that $f(\mathcal{M}) \cap I^+(r) \cap I^-(s) = \emptyset$ and $d_g(r,p) = d_g(p,s) > 0$. Since $f^k(p) \notin I^+(r) \cap I^-(s)$ for all k, we get that $f^k(p) \notin I^+(f^l(r)) \cap I^-(f^l(s))$ for all $k \geq l$. By taking a subsequence if necessary, we can assume that $f^n(p) \overset{n\to\infty}{\to} \tilde{p}$, $f^n(r) \overset{n\to\infty}{\to} \tilde{r}$, $f^n(s) \overset{n\to\infty}{\to} \tilde{s}$. Hence $\tilde{r} \ll \tilde{p} \ll \tilde{s}$, but this is impossible since this implies that for n big enough $\tilde{p} \in I^+(f^n(r)) \cap I^-(f^n(s))$. $\square$

Let us now make the following definition.

Definition 2 *(**Lorentzian Gromov-Hausdorff**) We call $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$ ϵ-close iff there exist mappings $\psi : \mathcal{M} \to \mathcal{N}$, $\zeta : \mathcal{N} \to \mathcal{M}$ such that*

$$|d_h(\psi(p_1), \psi(p_2)) - d_g(p_1, p_2)| \leq \epsilon \quad \forall p_1, p_2 \in \mathcal{M} \tag{5}$$

$$|d_g(\zeta(q_1), \zeta(q_2)) - d_h(q_1, q_2)| \leq \epsilon \quad \forall q_1, q_2 \in \mathcal{N}. \tag{6}$$

The Gromov-Hausdorff distance $d_{GH}((\mathcal{M}, g), (\mathcal{N}, h))$ is defined as the infimum over all ϵ such that $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$ are ϵ-close.

Suppose we are given sequences $(\psi_n)_{n\in\mathbb{N}}, (\zeta_n)_{n\in\mathbb{N}}$ of -possibly discontinuous - maps which make $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$ $\frac{1}{n}$ close. Then, because of the previous theorem, any limit mapping is necessarily an isometry.

Theorem 6 *$d_{GH}((\mathcal{M}, g), (\mathcal{N}, h)) = 0$ iff $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$ are isometric.*

Proof
Let $\mathcal{C}$ and $\mathcal{D}$ be countable dense subsets of $\mathcal{M}$ respectively $\mathcal{N}$, and take subsequences $(\psi_{n_k})_{k\in\mathbb{N}}$ and $(\zeta_{n_k})_{k\in\mathbb{N}}$ such that

- $\psi_{n_k}(p) \overset{k\to\infty}{\to} \psi(p)$ for all $p \in \mathcal{C}$
- $\zeta_{n_k}(q) \overset{k\to\infty}{\to} \zeta(q)$ for all $q \in \mathcal{D}$

Obviously $d_h(\psi(p), \psi(\tilde{p})) = d_g(p, \tilde{p})$ for all $p, \tilde{p} \in \mathcal{C}$ and $d_g(\zeta(q), \zeta(\tilde{q})) = d_h(q, \tilde{q})$ for all $q, \tilde{q} \in \mathcal{D}$, which is an easy consequence of the global hyperbolicity and the limiting properties of the sequences $(\psi_{n_k})_{k\in\mathbb{N}}$ and $(\zeta_{n_k})_{k\in\mathbb{N}}$.
We shall now prove that the limit map ψ exists and is distance-preserving. Let r be an interior point of $\mathcal{M}$ and take sequences $(\tilde{p}_i)_{i\in\mathbb{N}}$, $(\tilde{q}_i)_{i\in\mathbb{N}}$, $(p_i)_{i\in\mathbb{N}}$ and $(q_i)_{i\in\mathbb{N}}$ in $\mathcal{M}$ as before. In exactly the same way as in the proof of theorem 2, we obtain that $\psi_{\uparrow}(r) = \psi_{\downarrow}(r)$. Also $\psi(r) = \lim_{k\to\infty} \psi_{n_k}(r)$ since for arbitrary i we can find a k_0 such that $\forall k \geq k_0$

- $\frac{1}{k} < \min\{d(p_{i+1}, r), d(r, q_{i+1})\}$
- $\psi(p_i) \ll \psi_{n_k}(p_{i+1}) \ll \psi_{n_k}(q_{i+1}) \ll \psi(q_i)$

hence
$$\psi(p_i) \ll \psi_{n_k}(p_{i+1}) \ll \psi_{n_k}(r) \ll \psi_{n_k}(q_{i+1}) \ll \psi(q_i)$$
which proves the case. From this it is easy to prove that ψ is continuous on the interior points.
In exactly the same way one constructs a continuous limit map ζ on the interior of $\mathcal{N}$.
The previous theorem now shows that ψ and ζ are distance preserving homeomorphisms from the interior of $\mathcal{M}$ to $\mathcal{N}$ and from the interior of $\mathcal{N}$ to $\mathcal{M}$ respectively. Using this, it is not difficult to show that one can continuously extend ψ to the boundary so that $\lim_{k\to\infty} \psi_{n_k}(r) = \psi(r)$ for every boundary point r. Hence the result follows.

Furthermore, it is obvious that d_{GH} is symmetric and satisfies the triangle inequality. We will now discuss some properties of d_{GH}. Let us start with an obvious one which is similar to the Riemannian case.

Theorem 7 $d_{GH}((\mathcal{M}, g), (\mathcal{N}, h)) \leq \max\{tdiam(\mathcal{M}), tdiam(\mathcal{N})\}$ *where* $tdiam(\mathcal{M})$ *denotes the timelike diameter, ie.*
$$tdiam(\mathcal{M}) = \max_{p,\tilde{p}\in\mathcal{M}} d_g(p, \tilde{p}).$$

We shall now give an example that might feel strange in the beginning for people used to Riemannian geometry, although the result itself is what one should expect from Lorentzian geometry.

Example
As mentioned before, a need will present itself for abstraction of the concept of Lorentzian manifold. Therefore it is not hard to imagine that a Riemannian manifold is a Lorentzian space where every point is null-connected with itself and not causally related to any other point (imagine that the Riemannian manifold serves as a spacelike Cauchy surface in a globally hyperbolic spacetime). The previous theorem shows then that *any* two Riemannian manifolds are a distance zero apart since their timelike diameters are zero. This is very much different from the usual Riemannian theory *but* in a purely Lorentzian theory

this result is obvious from the fact that the causal distance does not provide us with any information whatsoever. Hence the manifold would be unobservable, so how could one compare two things which cannot be observed? This result shows that, if one wants the moduli space to be a complete metric space, the timelike diameter needs to be controlled, i.e., bounded away from zero as the next example shows.
Consider cylinders $C_T = S^1 \times [0,T]$ with Lorentz metric $ds^2 = -dt^2 + d\theta^2$. A Gromov-Hausdorff limit for $T \to 0$ is S^1, but it could equally well be any other Riemannian manifold. $\square$

Now we shall show that we can construct a metric $d_{\mathcal{M}}$ such that every d_g isometry is a $d_{\mathcal{M}}$ isometry. In order to achieve this, I construct a uniformity (using the causal structure alone) from which a metric can be constructed by choice of a particular algorithm. This is in contrast to the usual extra assumption of a preferred class of observers in the major part of the literature. Such a preferred class of observers is for example given if the energy momentum tensor satisfies the type I weak energy condition [2], i.e., determines a preferred timelike eigenvectorfield. However, our approach is purely geometrical and matter is not assumed to determine geometry through the Einstein equations. This is moreover the only sensible strategy if

- one wants to construct a theory of *vacuum* quantum gravity
- one considers spacetime *not* to be a manifold. What would the analogue be of the Einstein-Hilbert action on something like a causal set [14] or a spin network [13], anyway?

We shall find the metric by constructing a uniformity. Readers not familiar with this topological concept are referred to Appendix C. For the sake of clarity we introduce the following definition.

Definition 3 *Denote by $T_{-1}\mathcal{M}$ the bundle of all timelike vectors of norm -1. Let p be an interior point of $\mathcal{M}$. Then we introduce the following concept :*

- $M_-(p) = \{v \in T_{-1}(\mathcal{M})(p) | v$ *is past oriented and the geodesic determined by v maximizes the distance of p to the past boundary*$\}$

$M_+(p)$ is defined dually. Define now the following open sets

$$\mathcal{O}_\epsilon(p) = \{r | \exists v \in M_-(p), w \in M_+(p) : \exp_p(\epsilon v) \ll r \ll \exp_p(\epsilon w)\}$$

if the timelike distances from the past boundary to p and from p to the future boundary are greater than or equal to ϵ. If, for example, the former is not the case then $\mathcal{O}_\epsilon(p)$ is simply defined as

$$\mathcal{O}_\epsilon(p) = \{r | \exists w \in M_+(p) : r \ll \exp_p(\epsilon w)\},$$

and the dual if the latter is not satisfied. This definition extends in a trivial way to both boundaries.

It is obvious to see that $M_{\pm}(p)$ is compact for every $p \in \mathcal{M}$, moreover we have the following result.

Theorem 8 *The mappings $p \to M_{\pm}(p)$ are continuous in the compact open topology on the space of all closed subsets of $T_{-1}\mathcal{M}$ in every interior point $p \in \mathcal{M}$.*

Proof
We shall only prove the + case, the other one being similar. Suppose there exists an interior point $p \in \mathcal{M}$, a neighborhood U of $M_+(p)$ in $T_{-1}\mathcal{M}$ and sequences $p_n \to p$, $v_n \in M_+(p_n)$ such that $v_n \notin U$ for all $n \in \mathbb{N}$. The sequence of distance maximizing geodesics $s \to \exp_{p_n}(sv_n)$ from p_n to the future boundary has a subsequence that converges to a distance-maximizing geodesic from p to the future boundary. Hence the set of unit directions v_{n_k} of this subsequence has an accumulation point $v \in M_+(p)$, which is a contradiction. $\square$

This theorem has as consequence that $p \to \mathcal{O}_\epsilon(p)$ is continuous for all p except those which are exactly a distance ϵ apart from the past and future boundary. This leads to the following theorem.

Theorem 9 *For any ϵ there exists a $\delta > 0$ such that $q \in \mathcal{O}_\delta(p)$ implies that $p \in \mathcal{O}_\epsilon(q)$.*

Proof
Choose any p of $\mathcal{M}$ and suppose there does not exist a $\delta > 0$ such that $q \in \mathcal{O}_\delta(p)$ implies that $p \in \mathcal{O}_\epsilon(q)$. Then there exists a sequence $q_n \to p$ such that $p \notin \mathcal{O}_\epsilon(q_n)$, but then $p \notin \mathcal{O}_\epsilon(p)$. This is obviously true because of the preceding remark when p is a distance different from ϵ apart from both boundaries. When p is a distance ϵ apart from, say, the past boundary, then the result is also true since the obvious continuous extension $q \to \tilde{\mathcal{O}}_\epsilon(q)$ of $q \to \mathcal{O}_\epsilon(q)$ restricted to points which are a distance greater or equal than ϵ apart from the past boundary to points which are not is contained in the latter. By this we mean that $\tilde{\mathcal{O}}_\epsilon(q) \subset \mathcal{O}_\epsilon(q)$ for q a distance smaller than ϵ apart from the past boundary. Choose γ smaller than δ and the distance of p to both boundaries if p is an interior point (if p is a boundary point then γ must be smaller than ϵ and the distance to the other boundary), if we show that p has a neighborhood $\mathcal{U}$ such that $\tilde{p} \in \mathcal{U}$ and $\tilde{q} \in \mathcal{O}_\gamma(\tilde{p})$ imply that $\tilde{p} \in \mathcal{O}_\epsilon(\tilde{q})$ then we are done since $\mathcal{M}$ is compact. Suppose we find sequences $p_n \to p$ and $q_n \in \mathcal{O}_\gamma(p_n)$ and $p_n \notin \underline{\mathcal{O}_\epsilon(q_n)}$. Then by passing to a subsequence if necessary, one has that $q_n \to q \in \overline{\mathcal{O}_\gamma(p)} \subset \mathcal{O}_\delta(p)$ and $p \notin \mathcal{O}_\epsilon(q)$, which is a contradiction. The former is true since $q \to \mathcal{O}_\gamma(q)$ is continuous in p. The latter is certainly true when ϵ is different from the distance of q to both boundaries, if not then the continuous extension argument concludes the proof. $\square$

We now finish the proof that the open sets $\mathcal{O}_\epsilon(p)$ define a uniform neighborhood system by proving a generalization of the triangle inequality.

Theorem 10 *For any ϵ there exists a $\delta > 0$ such that $r \in \mathcal{O}_\delta(q)$ and $q \in \mathcal{O}_\delta(p)$ imply that $r \in \mathcal{O}_\epsilon(p)$.*

Proof If p is any point and $\epsilon > 0$, it is easy to see that there exists a $\delta > 0$ which satisfies the above conditions. For suppose not, then there exist sequences $p_n \in \mathcal{O}_{1/n}(p)$ and $q_n \in \mathcal{O}_{1/n}(p_n)$ but $q_n \notin \mathcal{O}_\epsilon(p)$. Choose n_0 sufficiently large such that $\frac{1}{n_0}$ is smaller than ϵ and the distances from p to both boundaries if p is not a boundary point (and the distance to the other boundary if p is a boundary point), then $q \to \mathcal{O}_{\frac{1}{n_0}}(q)$ is continuous in a neighborhood of p. Moreover, let $\mathcal{U}$ be a neighborhood of p such that $r \in \mathcal{U}$ implies that $\mathcal{O}_{\frac{1}{n_0}}(r) \subset \mathcal{O}_\epsilon(p)$, then $n > n_0$ implies that $\mathcal{O}_{1/n}(r) \subset \mathcal{O}_\epsilon(p)$, which is a contradiction. We finish the proof in the same way as in the previous theorem. Let p be an interior point and choose $\gamma < \delta$ small enough such that $q \in \mathcal{O}_{2\gamma}(p)$ implies that the distance of q to both boundaries is greater than γ. Suppose that we can find sequences $p_n \to p$, $q_n \in \mathcal{O}_\gamma(p_n)$, $r_n \in \mathcal{O}_\gamma(q_n)$ such that $r_n \notin \mathcal{O}_\epsilon(p_n)$. By if necessary passing to subsequences, we can assume that $q_n \to q \in \overline{\mathcal{O}_\gamma(p)}$ and $r_n \to r \in \overline{\mathcal{O}_\gamma(q)}$, but $r \notin \mathcal{O}_\epsilon(p)$, which is a contradiction. The case when p is a boundary point is left to the reader. $\square$

Hence we have proven that the $\mathcal{O}_\epsilon(p)$ define a Hausdorff[7] uniform neighborhood system, and as such theorem 12 tells us that we can construct a metric $d_\mathcal{M}$. Let us say a bit more about the metrics $d_\mathcal{M}$, $d_\mathcal{N}$. Suppose ψ is an isometry from (M, g) onto $(\mathcal{N}, h)$, then $\psi(\mathcal{O}_\epsilon(p)) = \mathcal{O}_\epsilon(\psi(p))$ for all $p \in \mathcal{M}$ and $\epsilon > 0$. Hence $d_\mathcal{N}(\psi(p), \psi(\tilde{p})) = d_\mathcal{M}(p, \tilde{p})$ for all $p, \tilde{p} \in \mathcal{M}$, which means that every Lorentzian isometry is a Riemannian isometry, which is intuitively a very nonsurprising result! The reverse however is not necesserily true. Notice also that $d_\mathcal{M}(p, \tilde{p}) \leq \mathrm{tdiam}(\mathcal{M})$ for all $p, \tilde{p} \in \mathcal{M}$.

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5 Appendix A

The next theorem is a *slight* generalization of the result in [9].

Theorem 11 *If $(\mathcal{M}, g)$ is strongly causal, then every onto map $f : \mathcal{M} \to \mathcal{N}$ with finite, strictly positive timelike dilatation and co-dilatation is a homeomorphism*

Proof

[7] Hausdorff since $\bigcap_{\epsilon>0} \mathcal{O}_\epsilon(p) = \{p\}$ for all $p \in \mathcal{M}$.

Observe first that for all $p, q \in \mathcal{M}$ one has that $d(f(p), f(q)) > 0$ if and only if $d(p, q) > 0$. Hence $f(I^{\pm}(x)) = I^{\pm}(f(x))$ (since f is onto) and $f(I^{+}(p) \cap I^{-}(q)) = I^{+}(f(p)) \cap I^{-}(f(q))$. Since $(\mathcal{M}, g)$ is strongly causal, the Alexandrov topology coincides with the manifold topology, hence f is an open mapping. f is also injective, since if $p \neq q$ and $f(p) = f(q)$, we arrive to the following contradiction. Let $\mathcal{U}$ be a locally convex neighborhood of p which does not contain q and satisfies the condition that every causal curve intersects $\mathcal{U}$ exactly once. Take then $r \ll p \ll s$ with $r, s \in \mathcal{U}$ then $I^{+}(r) \cap I^{-}(s) \subset \mathcal{U}$. A fortiori $f(r) \ll f(p) = f(q) \ll f(s)$ which is a contradiction since $q \notin I^{+}(r) \cap I^{-}(s)$. We are done if we prove that f^{-1} is open. For this it is sufficient to prove that $(\mathcal{N}, h)$ is strongly causal. Suppose that strong causality is not satisfied at $f(p)$. First choose a locally convex neighborhood $\mathcal{U}$ of $f(p)$ such that $(\mathcal{U}, h_{|\mathcal{U}})$ is globally hyperbolic. Let $\mathcal{W}$ be a neighborhood of $f(p)$ of compact closure in $\mathcal{U}$. If strong causality is not satisfied at $f(p)$ then there exist points $q_n \ll f(p) \ll r_n$ in $\mathcal{W}$ such that $q_n, r_n \overset{n\to\infty}{\to} f(p)$ and causal curves λ_n from p_n to q_n which leave $\mathcal{U}$. Denote by z_n the first intersection with $\partial\mathcal{W}$ of λ_n. Then there exists a subsequence z_{n_k} such that $z_{n_k} \overset{k\to\infty}{\to} z$. Obviously $p = f^{-1}(z)$ otherwhise the continuity of f^{-1} would contradict the strong causality of $(\mathcal{M}, g)$. But on the other hand $p = f^{-1}(z)$ contradicts the injectivity of f. $\square$

We show now that f takes null geodesics to null geodesics. Take a small enough convex normal neighborhood $\mathcal{U}$ of p which no causal curve intersects more than once and such that $(\mathcal{U}, g_{|\mathcal{U}})$ is globally hyperbolic. Moreover we assume that the closure of $f(\mathcal{U})$ belongs to a convex normal neighborhood $\mathcal{V}$ of $f(p)$ which no causal curve intersects more than once with $(\mathcal{V}, h_{|\mathcal{V}})$ globally hyperbolic. Let $\alpha(q, r)$ be a null geodesic in $\mathcal{U}$ and take sequences $q_n \to q$, $r_n \to r$ with $q_n \ll r_n$ for all n. f takes timelike geodesics $\alpha(q_n, r_n)$ with length $d(q_n, r_n)$ to timelike curves $\gamma(f(q_n), f(r_n))$ with length at most $\beta d(q_n, r_n)$; moreover $f(q_n) \to f(q)$ and $f(r_n) \to f(r)$. The geodesics $\alpha(q_n, r_n)$ converge to the null geodesic $\alpha(q, r)$. Because of the global hyperbolicity of $(\mathcal{V}, h_{|\mathcal{V}})$ a subsequence of the timelike curves $\gamma(f(q_n), f(r_n))$ converges to a causal curve from q to r. This causal curve need to be an unbroken null geodesic $\alpha(f(q), f(r))$ since $d(f(q), f(r)) = 0$. In fact it is easy to see that the whole sequence $\gamma(f(q_n), f(r_n))$ converges in the C^0 topology of curves to $\alpha(f(q), f(r))$, which concludes the proof.

It is easy to check that if $(\mathcal{M}, g)$ is a strongly causal spacetime with spacelike boundary, then the above results are still valid, ie. the homeomorphism extends to the boundary. A well known result of Hawking, King and McCarthy [3]- which is the Lorentzian equivalent of an earlier theorem by Palais- states that every homeomorphism which maps null geodesics to null geodesics must be a conformal diffeomorphism.

6 Appendix B

In this appendix we sketch how to locally extend a conformal diffeomorphism across the boundary. Let r be a boundary point of $\mathcal{M}$ and choose a neighborhood $\mathcal{U}$ of r diffeomorphic (under ψ) to the open hypercube union one side hyperplane

H. Let $(\mathcal{V}, g_{\mathcal{V}})$ be a local extension in $\mathbb{R}^n$ of $(H, \psi_* g)$. Choose $(W, \phi_* h)$ to be a similar hypercube neighborhood of $f(r)$ and interpret f as a conformal mapping of $(H, \psi_* g)$ to $(W, \phi_* h)$ with conformal factor Ω. Obviously f can be C^∞ locally extended over the boundary around $\psi(r)$, as can Ω such that Ω has almost vanishing normal first derivative [8]. Then we can construct an extension of $(W, \phi_* h)$ by defining the extension $\widetilde{\phi_* h}$ of $\phi_* h$ around $\phi(f(p))$ as $\widetilde{\phi_* h} = \frac{(f \circ \psi)_* g}{\Omega}$. Now the real question is the following: let z be a conformal C^∞ diffeomorphism from $(H, \psi_* g)$ to $(W, \phi_* h)$; does there exist an extension $\tilde{z}$ from $(\mathcal{V}, g_{\mathcal{V}})$ to $(f(\mathcal{V}), \widetilde{\phi_* h})$? Clearly, if such a *local* extension extension exists, it must be unique. This can be seen as follows. Take $x \in \mathcal{V} \cap H^c$ close enough to the boundary $t = 0 \leftrightarrow \Sigma$ such that $E^+(x) \cap \Sigma$ is diffeomorphic to the 2 - sphere and such that the null geodesics can be pushed over the boundary a bit (ie. there are no cut points in a neighborhood of Σ wrt. $\psi_* g$ for x sufficiently close to it). Denote by $T_{\text{null}}\Sigma$ the bundle of null vectors over Σ, hence x determines a unique (discontinuous) section ρ_x with support in $E^+(x) \cap \Sigma$ such that

$$\exp(-\rho_x(y)) = x \quad \forall y \in E^+(x) \cap \Sigma.$$

The push forward under z of the section ρ_x determines uniquely the point $z(x)$ as the first past intersection point of the null rays defined by $z_* \rho_x$ - if it exists. Now, since $f^{-1} \circ z$ is a conformal diffeomorphism if only if z is, it is sufficient to prove the existence of the unique conformal extension of the former. Clearly, in two dimensions, such intersection point exists and therefore also the conformal extension. However, I have no argument for now which proves the result in dimension greater than 2.

7 Appendix C

Let (X, d) be a topological space where d is a (pseudo) distance and denote by τ the corresponding locally compact topology. It is an elementary fact that the open balls $B_{1/n}(p)$ with radius $1/n : n \in \mathbb{N}_0$ around p define a countable basis for τ in p. In this chapter I, J will denote index sets. A (X, τ) cover C is defined as follows:

$$C = \{A_i | A_i \in \tau, i \in I\}$$

such that

$$\bigcup_{i \in I} A_i = X.$$

If $C = \{A_i | A_i \in \tau, i \in I\}, D = \{B_j | B_j \in \tau, j \in J\}$ are (X, τ) covers then we say that C is finer than or is a refinement of D, $C < D$ if and only if

$$\forall i \in I \quad \exists j \in J : A_i \subset B_j.$$

Next we define a few operations on the set of covers $C(X, \tau)$:

Operations on covers

[8]remember that we have chosen our coordinates in such a fashion that the boundary corresponds to a part of the hypersurface $t = 0$.

- Let C, D be as before,

$$C \wedge D = \{A_i \cap B_j | A_i, B_j \in \tau \quad i \in I, j \in J\}$$

  $C \wedge D$ is obviously a cover, moreover the doublet $C(X, \tau), \wedge$ is a commutative semigroup.

- For $A \subset X$ the star of A with respect to C is defined as follows:

$$St(A, C) = \cup_{A_i \in C : A \cap A_i \neq \emptyset} A_i$$

- The star of C, C^* is then defined as:

$$C^* = \{St(A_i, C) | A_i \in C\}$$

  Remark that $C < C^* < C^{**} \ldots$ and that if I is finite then there exists a $n \in \mathbb{N}$ such that after n star operations C has become the trivial cover.

Using the topological basis of open balls, we can define elementary covers $C_n \quad n \in \mathbb{N}_0$ as follows:

$$C_n = \{B_{1/n}(p) | p \in X\}$$

These elementary covers now define a subset U of $C(X, \tau)$:

$$U = \{C \in C(X, \tau) | \exists C_n : C_n < C\}$$

The set U satisfies the following obvious properties:

1. If $C \in U$ and $C < D$ then $D \in U$
2. If $C, D \in U$ then $C \wedge D \in U$
3. If $C \in U$ then $\exists D \in U : D^* < C$

From now on we take the above properties as a **definition** for a **uniformity**:

Definition 4 *Let X be a set, a cover C is defined as:*

$$C = \{A_i | A_i \subset X, i \in I\}$$

such that

$$\bigcup_{i \in I} A_i = X$$

*A collection of covers U is called a **uniformity** for X if and only if*

1. *If $C \in U$ and $C < D$ then $D \in U$*
2. *If $C, D \in U$ then $C \wedge D \in U$*
3. *If $C \in U$ then $\exists D \in U : D^* < C$*

where all definitions of $<, \wedge$ and * are independent of τ.

It has been proven that any uniformity can be generated by a family of pseudodistances [7]. This indicates a uniformity defines a topology. For our applications we need a different ingredient.

Definition 5 *Let I be a directed net, and suppose $B_i(x) \subset X$ satisfy the following properties:*

1. $x \in B_i(x) \quad \forall x \in X, i \in I$
2. *If* $i \leq j$ *then* $B_i(x) \subset B_j(x) \quad \forall x \in X$
3. $\forall i \in I, \exists j \in I$ *such that* $\forall y \in B_j(x) : x \in B_i(y)$
4. $\forall i \in I, \exists j \in I$ *such that if* $z \in B_j(y), y \in B_j(x)$ *then* $z \in B_i(x)$.

then we call the family of all $B_i(x)$ ***a uniform neighborhood system****.*

Now it has been proven that if $\{B_i(x)|x \in X, i \in I\}$ is a uniform neighborhood system then the family of covers:

$$C_i = \{B_i(x)|x \in X\}$$

$i \in I$ is a basis for a uniformity on X. On the other hand every uniformity can be constructed from a uniform neighborhood system.
The topology τ_U defined by a uniformity U, the **uniform topology**, is constructed as follows:

$$O(x) \in \tau_U \Longleftrightarrow \exists C \in U : St(x, C) \subset O(x)$$

so $\{St(x, C)|x \in X, C \in U\}$ defines a basis for the topology. The topology is Hausdorff if and only if $\bigcap_{O(x) \in \tau_U} O(x) = \{x\}$ but it is not difficult to see that this is equivalent with:

$$\bigcap_{i \in I} B_i(x) = \{x\}$$

where $\{B_i(x)|i \in I, x \in X\}$ is the uniform neighborhood system which generates U. We state a few facts about **quotient uniformities**. **Terminology**

- Let (X, U) and (Y, V) be uniform spaces, a map $f : X \rightarrow Y$ is **uniformly continuous** if and only if

  $$\forall C \in V : f^{-1}(C) \in U$$

  where for $C = \{A_i|i \in I\}$, $f^{-1}(C) = \{f^{-1}(A_i)|i \in I\}$.

- A uniformity $\tilde{U}$ on X is finer than U if and only if every cover in U belongs to $\tilde{U}$.

- Let $\pi : X \rightarrow \tilde{X}$ be a surjective map and (X, U) a uniform space, the **quotient uniformity** $\tilde{U}$ on $\tilde{X}$ is the finest uniformity which makes π uniformly continuous.

Notice that the existence of a quotient uniformity is guaranteed by the lemma of Zorn, the uniqueness is immediate. The obvious question now is if $\tau_{\tilde{U}}$ is equal to the quotient topology of τ_U. The answer is in general no, but under some special circumstances it works.

Definition 6 *A uniform neighborhood system $\{B_i(x)|x \in X, i \in I\}$ is **compatible** with an equivalence relation on X if and only if*

$$\forall i \in I, x' \sim x \text{ and } y \in B_i(x) \quad \exists y' \sim y : y' \in B_i(x')$$

As envisaged, compatibility implies that $\tau_{\tilde{U}}$ is equal to the quotient topology of τ_U.

Theorem 12 *If U is generated by $\{B_i(x)|i \in I, x \in X\}$ which is compatible with $\sim$ which is for example defined by a surjective map, then the quotient uniformity $\tilde{U}$ on $\tilde{X} = X/\sim$ is generated by the uniform neighborhood system defined by:*

$$\tilde{B}_i(\tilde{x}) = \{\tilde{y}|\exists x \in \tilde{x} \text{ and } y \in \tilde{y} : y \in B_i(x)\}$$

$\forall \tilde{x} \in \tilde{X}, i \in I$. Moreover $\tau_{\tilde{U}}$ is equal to the quotient topology of τ_U and a basis of neighborhoods of $\tilde{x} \in \tilde{X}$ is $\{\tilde{B}_i(\tilde{x})|i \in I\}$

As mentioned, every uniformity can be generated by a family of pseudodistances. In the case that the uniformity is generated by a countable uniform neighborhood system, the topology is defined by one pseudodistance, which is a distance when the uniformity is Hausdorff. Suppose $C_n = \{B_n(x)|x \in X\}$, $n \in \mathbb{N}$, is a countable basis for a uniformity U, then we can find a subsequence $(n_k)_k$ such that:

$$\forall k, \quad w \in B_{n_k}(z), z \in B_{n_k}(y), y \in B_{n_k}(x) \Rightarrow w \in B_{n_{k-1}}(x)$$

Assume C_n is such a basis.

Theorem 13 *Let C_n be a countable basis of U, then with*

$$\rho(x, y) = \inf_{\{n \geq 0, y \in B_n(x)\}} 2^{-n}$$

the function

$$d(x, y) = \inf_{K \in \mathbb{N}, x_k} \sum_{k=1}^{K} \frac{1}{2}(\rho(x_{k-1}, x_k) + \rho(x_k, x_{k-1}))$$

is a pseudodistance which generates U. $\{x_0, \ldots, x_K\}$ with $x_0 = x, x_K = y$ is a path in X. If U is Hausdorff then d is a distance.

Note that the function d depends on the choice of basis C_n and is therefore not canonical.

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