Journal of Mathematics Research; Vol. 6, No. 1; 2014
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

A Minimal Pair of Turing Degrees

Patrizio Cintioli[1]

[1] Dipartimento di Matematica e Informatica, Università di Camerino, Camerino, Italy

Correspondence: Patrizio Cintioli, Dipartimento di Matematica e Informatica, Via Madonna delle Carceri, 9, Camerino (MC) 62032, Italy. E-mail: patrizio.cintioli@unicam.it

Received: November 1, 2013 Accepted: December 24, 2013 Online Published: January 21, 2014

doi:10.5539/jmr.v6n1p76 URL: http://dx.doi.org/10.5539/jmr.v6n1p76

Abstract

Let $P(A)$ be the following property, where A is any infinite set of natural numbers:

$$(\forall X)[X \subseteq A \wedge |A - X| = \infty \Rightarrow A \not\leq_m X].$$

Let $(\mathbf{R}, \leq)$ be the partial ordering of all the r.e. Turing degrees. We propose the study of the order theoretic properties of the substructure $(\mathbf{S}_m, \leq_{\mathbf{S}_m})$, where $\mathbf{S}_m =_{\mathrm{dfn}} \{\mathbf{a} \in \mathbf{R}$: $\mathbf{a}$ contains an infinite set A such that P(A) is true$\}$, and $\leq_{\mathbf{S}_m}$ is the restriction of $\leq$ to $\mathbf{S}_m$. In this paper we start by studying the existence of minimal pairs in $\mathbf{S}_m$.

Keywords: computably enumerable sets, low_1 Turing degrees, minimal-pair method

1. Introduction

For every infinite set A let us formulate the following property $P(A)$:

$$(\forall X)[X \subseteq A \wedge |A - X| = \infty \Rightarrow A \not\leq_m X], \tag{1}$$

where $\leq_m$ denotes the many-one reducibility. Sets with the property (1) are known as m-introimmune sets. This terminology was introduced in (Cintioli & Silvestri, 2003) to denote those sets that fail to be m-introreducible in a strong way, namely, those infinite sets that are not many-one reducible to any of their co-infinite subsets. In particular, every infinite set A satisfying property (1) does not contain subsets of higher many-one degree. The study of sets with no subsets of higher Turing degree began with Soare (1969) and Cohen (unpublished), and continued with Jockusch (1973) and Simpson (1978). This study was then reconsidered for some strong reducibilities in (Cintioli & Silvestri, 2003) and (Ambos-Spies, 2003).

Let $(\mathbf{R}, \leq)$ be the partially ordered structure of all the r.e. Turing-degrees. Let us define $\mathbf{S}_m =_{\mathrm{dfn}} \{\mathbf{a} \in \mathbf{R} : \mathbf{a}$ contains an infinite set which has the property (1)$\}$. In this paper we propose the study of the partially ordered substructure $(\mathbf{S}_m, \leq_{\mathbf{S}_m})$, where $\leq_{\mathbf{S}_m}$ denotes the order $\leq$ restricted to $\mathbf{S}_m$. We know that $\mathbf{0}' \in \mathbf{S}_m$ (Cintioli, 2005), that $\mathbf{S}_m$ contains a low_1 Turing degree (Cintioli, 2011), and that $\mathbf{0} \notin \mathbf{S}_m$ (Cintioli & Silvestri, 2003). What other properties hold for $(\mathbf{S}_m, \leq_{\mathbf{S}_m})$? In particular, what order theoretic properties hold for $(\mathbf{S}_m, \leq_{\mathbf{S}_m})$? For example:

1) is $(\mathbf{S}_m, \leq_{\mathbf{S}_m})$ dense?

2) is $(\mathbf{S}_m, \leq_{\mathbf{S}_m})$ an upper/lower semi-lattice?

3) does $(\mathbf{S}_m, \leq_{\mathbf{S}_m})$ has minimum element? or minimal elements?

and so on. This problematic collapses if it is true that $\mathbf{S}_m = \mathbf{R} - \{\mathbf{0}\}$: essentially all the order theoretic properties true in $(\mathbf{R}, \leq)$ will be inherited by $(\mathbf{S}_m, \leq_{\mathbf{S}_m})$ but at most some exception, due to the fact that $\mathbf{0} \notin \mathbf{S}_m$. However, this study could be extended to substructures $(\mathbf{S}_r, \leq_{\mathbf{S}_r})$ for other strong reducibilities $\leq_r$ with $\leq_r \neq \leq_T$, where $\mathbf{S}_r = \{\mathbf{a} \in \mathbf{R} : \mathbf{a}$ contains an infinite set which has the property (1) with $\leq_r$ in place of $\leq_m\}$. The reason why $\leq_r$ must be different from $\leq_T$ is that we know by the results contained in (Jockusch,1973) and (Simpson,1978) that $\mathbf{S}_T = \emptyset$. We do not know if $S_r \neq \emptyset$ for strong reducibilities $\leq_r \neq \leq_m$.

In this paper we start by studying the existence of minimal pairs in $\mathbf{S}_m$. The existence of minimal pairs in $\mathbf{S}_m$ follows by known results: such pairs are constituted of two high_1 Turing degrees. The class of high_1 Turing degrees is an important class of Turing degrees below $\mathbf{0}'$. Another important class of Turing degrees below $\mathbf{0}'$ is that one of low_1

Turing degrees. We recall that a Turing degree $\mathbf{a} < \mathbf{0}'$ is low$_1$ if $\mathbf{a}' = \mathbf{0}'$. In this paper we provide a minimal pair in $\mathbf{S}_m$ constituted of two low$_1$ Turing degrees.

2. Notations

Throughout the paper we will use some concepts of Computability Theory without define them. We refer to Odifreddi (1999), Rogers (1967) and Soare (1987) for a full exposition on the subject.

Letter N denotes the set of natural numbers. Given two sets $A, B \subseteq N$, $A - B$ denotes the set theoretic difference of A and B, and $|A|$ denotes the cardinality of A. We fix an acceptable numbering $\{\varphi_e\}_{e\geq 0}$ of all the Turing computable unary functions. $\{W_e\}_{e\geq 0}$ is the corresponding enumeration of all the recursively enumerable (r.e.) sets. For every $e, s \in N$, $W_{e,s}$ is the finite approximation of W_e obtained by performing s steps in the enumeration of W_e. For every $e \in N$ and every $X \subseteq N$, $\varphi_e^X : N \to N$ denotes the unary partial function computable by the e-th oracle Turing machine with the aid of the oracle X. For every $e, s, x \in N$ and for every oracle X, $\varphi_{e,s}^X(x)$ denotes $\varphi_e^X(x)$ if the e-th oracle Turing machines with oracle X on input x halts in $t \leq s$ steps, and in this case we write $\varphi_{e,s}^X(x)\downarrow$; otherwise, we say that $\varphi_{e,s}^X(x)$ is undefined. We assume here that if $\varphi_{e,s}^X(x)\downarrow$ then $x, e < s$ and the elements asked to the oracle X in the computation of $\varphi_{e,s}^X(x)$ are less than s. Finally, given two sets $A, B \subseteq N$, A is many-one reducible to B, in short $A \leq_m B$, if there exist a recursive function $f : N \to N$ such that for every $x \in N$, $x \in A \Leftrightarrow f(x) \in B$. In this case we say that f m-reduces A to B.

3. Main Result

We know that $(\mathbf{S}_m, \leq_{\mathbf{S}_m})$ has the maximum element $\mathbf{0}'$ because every cohesive co-r.e. set, actually every cohesive set, satisfies the property (1) (Lemma 3.2 below), and we know of the existence of maximal sets that are Σ_1^0-complete w.r.t. the Turing reducibility $\leq_T$ (Yates, 1965). For what concerns the width of $\mathbf{S}_m$ we can say at the moment that

$$\mathbf{S}_m \supset \{\mathbf{a} \in \mathbf{R} : \mathbf{a} \text{ is high}_1\}, \tag{2}$$

because every high$_1$ r.e. Turing degree contains sets which have the property (1), for example the co-maximal sets of the high$_1$ r.e. Turing degree. The inclusion (2) is proper, because there are low$_1$ r.e. Turing degrees in $\mathbf{S}_m$ (Cintioli, 2011), and the present paper.

In this paper we propose the study of the order theoretic properties of the substructure $(\mathbf{S}_m, \leq_{\mathbf{S}_m})$. To begin with, we study here the existence of minimal pairs in $\mathbf{S}_m$. We recall that two nonrecursive Turing degrees $\mathbf{a}$ and $\mathbf{b}$ form a minimal pair if for every Turing degree $\mathbf{c}$, $\mathbf{c} \leq \mathbf{a}$ and $\mathbf{c} \leq \mathbf{b}$ implies $\mathbf{c} = \mathbf{0}$. By the existence of minimal pairs in $\mathbf{S}_m$ we derive two consequences about the structure $(\mathbf{S}_m, \leq_{\mathbf{S}_m})$:

(i) there is not minimum, because $\mathbf{0}$ is not in $\mathbf{S}_m$,

(ii) $(\mathbf{S}_m, \leq_{\mathbf{S}_m})$ is not a lower semi-lattice, because such minimal pair does not have greater lower bound in $\mathbf{S}_m$.

We must say that we got (ii) in an unusual way, and even in some sense opposite to the usual way of proving that a certain partially ordered structure of r-degrees is not a lower semi-lattice. In fact, if the structure has minimum, then two r-degrees forming a minimal pair are nontrivial examples of elements having greatest lower bound, and such elements typically do not certify that the structure is not a lower semi-lattice.

Actually, we observe that the existence of minimal pairs in $\mathbf{S}_m$ is a consequence of known results, mainly of a result contained in Lachlan (1966). Exactly, we are referring to the following theorem:

Theorem 3.1 (Lachlan, 1966, cf. Theorem 2, p. 545) *There exist maximal r.e. sets A, B, the greatest lower bound of whose degrees is* $\mathbf{0}$.

The Turing degrees of the two sets of Theorem 3.1 form a minimal pair in $\mathbf{S}_m$, because every cohesive set possesses property (1), as stated in the following lemma.

Lemma 3.2 (Cintioli, 2005) *Let C be a cohesive set. Then, for every $X \subseteq C$ with $|C - X| = \infty$, $C \nleq_m X$.*

The Turing degrees of the two sets A and B of Theorem 3.1 are high$_1$, because of the maximality of A and B (Martin, 1966). We now prove the existence of a minimal pair in $\mathbf{S}_m$ constituted of two low$_1$ Turing degrees, and this is our main contribution of this paper.

Theorem 3.3 *There are two low$_1$ Turing degrees in $\mathbf{S}_m$ which form a minimal pair.*

Proof. By the *minimal-pair method* we will construct two co-immune low$_1$ r.e. sets A and B with both $\overline{A}$ and $\overline{B}$ satisfying property (1), and such that their Turing degrees form a minimal pair. Namely:

- $P(\overline{A})$: for every subset X of $\overline{A}$ with $|\overline{A} - X| = \infty$, $\overline{A} \nleq_m X$,
- $P(\overline{B})$: for every subset Y of $\overline{B}$ with $|\overline{B} - Y| = \infty$, $\overline{B} \nleq_m Y$,
- for every set C, if $C \leq_T A$ and $C \leq_T B$ then C is recursive.

The two sets A and B will be constructed by infinitely many stages. At every stage $s \geq 0$ we will define two finite sets A_s and B_s, with $A_0 = B_0 = \emptyset$, $A_s \subseteq A_{s+1}$ and $B_s \subseteq B_{s+1}$. Our final sets will be $A = \bigcup_{s\geq 0} A_s$ and $B = \bigcup_{s\geq 0} B_s$. We now describe our strategy.

3.1 Strategy

The construction of the two sets A and B is perfectly symmetric. It suffices to meet the following requirements, for every $e, i, j \in N$:

- P_{9e}: $|W_e| = \infty \Rightarrow W_e \nsubseteq \overline{A}$, (immunity of $\overline{A}$, together with requirement N_{9e+2})
- P_{9e+1}: $|W_e| = \infty \Rightarrow W_e \nsubseteq \overline{B}$, (immunity of $\overline{B}$, together with requirement N_{9e+3})
- N_{9e+2}: $(\exists x)[x \geq e \wedge x \in \overline{A}]$, (co-infinity of A)
- N_{9e+3}: $(\exists x)[x \geq e \wedge x \in \overline{B}]$, (co-infinity of B)
- N_{9e+4}: $[(\exists^\infty s)(\varphi^{A_s}_{e,s}(e)$ is defined$)] \Rightarrow \varphi^A_e(e)$ is defined (lowness of A),
- N_{9e+5}: $[(\exists^\infty s)(\varphi^{B_s}_{e,s}(e)$ is defined$)] \Rightarrow \varphi^B_e(e)$ is defined (lowness of B),
- R_{9e+6}: $(\forall X \subseteq \overline{A})[|\overline{A} - X| = \infty \Rightarrow \varphi_e$ does not m-reduce $\overline{A}$ to $X]$ ($\overline{A}$ does not contain subsets of higher m-degree),
- R_{9e+7}: $(\forall Y \subseteq \overline{B})[|\overline{B} - Y| = \infty \Rightarrow \varphi_e$ does not m-reduce $\overline{B}$ to $Y]$ ($\overline{B}$ does not contain subsets of higher m-degree),
- $N_{9\langle i,j\rangle+8}$: $(\varphi^A_i = \varphi^B_j = C) \Rightarrow C$ is recursive, (minimality).

The definitions of the requirements N_{9e+4} and N_{9e+5} are justified by the following known lemma.

Lemma 3.4 *If $D = \bigcup_{t\geq 0} D_t$ is a recursively enumerable set and for every $n \in N$*

$$[(\exists^\infty t)\varphi^{D_t}_{n,t}(n)\downarrow] \Rightarrow \varphi^D_n(n) \text{ is defined},$$

then D is low$_1$.

Furthermore, by an argument of Posner, to satisfy all the requirements $\{N_{9\langle i,j\rangle+8}\}_{i,j\geq 0}$ it suffices to satisfy all the requirements $\{N'_{9e+8}\}_{e\geq 0}$ formulated in a simpler way. Precisely:

Lemma 3.5 *For every $e \geq 0$, let N'_{9e+8} be the requirement*

$$(\varphi^A_e = \varphi^B_e = C) \Rightarrow C \text{ is recursive.}$$

If all the requirements $P_{9e}, P_{9e+1}, N_{9e+2}, N_{9e+3}$ and N'_{9e+8} are met for every $e \geq 0$, then all the requirements $N_{9\langle i,j\rangle+8}$ are met for every $i, j \geq 0$.

Proof. From the hypothesis it follows that both A and B are not recursive, and this implies that $A \neq B$. Without loss of generality assume that $A \nsubseteq B$, and let $x_0 \in A - B$. For the sake of contradiction, assume that there are i_0 and j_0 such that

$$\varphi^A_{i_0} = \varphi^B_{j_0} = C \text{ with } C \text{ not recursive.} \tag{3}$$

Let $e = e(x_0, i_0, j_0)$ be such that for every oracle X and every $x \in N$

$$\varphi^X_e(x) = \begin{cases} \varphi^X_{i_o}(x) & \text{if } x_0 \in X, \\ \varphi^X_{j_o}(x) & \text{if } x_0 \notin X. \end{cases}$$

Then clearly $\varphi^A_e = \varphi^A_{i_0}$ and $\varphi^B_e = \varphi^B_{j_0}$. It follows from (3) that $\varphi^A_e = \varphi^B_e = C$ with C not recursive, contrary to the hypothesis that N'_{9e+8} is met. □

By the above lemma we can replace each requirement $N_{9\langle i,j\rangle+8}$ with the requirement

- N_{9e+8}: $(\varphi^A_e = \varphi^B_e = C) \Rightarrow C$ is recursive.

Requirements P_{9e} and P_{9e+1} are *positive*, because to satisfy them we will enumerate numbers into A and B. Requirements N_{9e+2}, N_{9e+3}, N_{9e+4}, N_{9e+5} and N_{9e+8} are *negative*, because to satisfy them we will keep numbers out of A and B. Requirements R_{9e+6} and R_{9e+7} are *positive* and *negative*, because to satisfy them we will keep some numbers out of A and B, and we will enumerate some others numbers into A and B.

3.2 Actions to Fulfill the Requirements

In this subsection we give a description of the actions to fulfill all the requirements. These actions will be formally stated in the subsequent subsections *Requirements requiring attention* and *Active requirements*. We will use a restraint function $r : N \times N \to N$ with $r(n, 0) = -1$ for every $n \in N$. To fulfill requirements P_{9e} and P_{9e+1} we do not need to restrains elements, being both P_{9e} and P_{9e+1} positive, thus we conventionally set $r(9e, s) = r(9e+1, s) = -1$ for every $e, s \in N$. We said above that the construction of the two sets A and B is symmetric. Therefore, except for N_{9e+8}, we describe the actions to fulfil requirements relative to one set, say A.

The actions to fulfill P_{9e}, N_{9e+2} and N_{9e+4} are the usual.

- For P_{9e}, we wait for an opportune stage $s + 1 \geq 9e$, if it exists, such that
  - $W_{e,s} \cap A_s = \emptyset$, and
  - there is a number $x \in W_{e,s}$ not restrained by requirements of higher priority than P_{9e}.

Then, we enumerate x into A_{s+1}. If W_e is infinite, then eventually $W_e \nsubseteq \overline{A}$.

- For N_{9e+2} we wait for an opportune stage $s + 1 \geq 9e + 2$ such that no number $x \geq e$ is restrained yet, that is $r(9e + 2, s) = -1$, and we set $r(9e + 2, s + 1) =$ the minimum number $x \in \overline{A}_s$ with $x \geq e$. If x will not enumerated into A later, then such x will certificate that N_{9e+2} is met.

- For N_{9e+4}, we wait for an opportune stage $s+1 \geq 9e+4$, if it exists, such that $\varphi^{A_s}_{e,s}(e)$ is defined and the elements used in the computation of $\varphi^{A_s}_{e,s}(e)$ are not restrained yet, that is $r(9e+4, s) = -1$. Then it suffices to set the restraint function $r(9e+4, s+1) = s$, by remembering that the computation of $\varphi^{A_s}_{e,s}(e)\downarrow$ uses only elements less than s. This guarantees that $\varphi^{A}_{e}(e)$ is defined, if no numbers $x \leq s$ will be enumerated into A after stage $s + 1$.

For what concerns requirement R_{9e+6}, we describe first the strategy to meet it. Strategy: *to prevent that φ_e m-reduces $\overline{A}$ to some its subset X with $|\overline{A} - X| = \infty$ it is enough to have a number y such that*

$$y \in \overline{A} \text{ and } \varphi_e(y) \notin \overline{A}. \tag{4}$$

Therefore, we do the following actions: we wait for an opportune stage $s+1 \geq 9e+6$ such that for some $y \leq s+1$

$$\varphi_{e,s}(y)\downarrow \neq y \text{ and } y \in \overline{A}_s, \tag{5}$$

with $\varphi_{e,s}(y)$ not restrained by requirements of higher priority than R_{9e+6}. Then we force φ_e to be wrong by enumerating $\varphi_{e,s}(y)$ into A_{s+1} and keeping y out of A by setting $r(9e + 6, s + 1) = y$. If φ_e is a *potential* m-reduction of $\overline{A}$ to some $X \subseteq \overline{A}$ with $|\overline{A} - X| = \infty$, then the certainty that s and y of (5) exists is a consequence of the immunity of $\overline{A}$, as stated in the following lemma.

Lemma 3.6 *Let C be any immune set and let X be a subset of C with $|C - X| = \infty$. Let us suppose that $C \leq_m X$ via a recursive function f. Then, the set $\{f(x)\colon f(x) \neq x \wedge x \in C\}$ is infinite.*

Proof. By hypothesis $C \leq_m X$ via f, in particular for every $x \in C - X$ is $f(x) \neq x$. Hence, it suffices to prove that $f(C - X)$ is infinite. Observe that

$$C - X \subseteq f^{-1}(f(C - X)) \subseteq C \tag{6}$$

with $C - X$ infinite by hypothesis. So $f(C - X)$ cannot be finite, otherwise $f^{-1}(f(C - X))$ would be an infinite r.e. subset of C, contrary to the hypothesis that C is immune. □

For the requirement N_{9e+8} we employ the *minimal-pair method*, which is based on the particular definition of the restraint function, in our case the definition of $r(9e + 8, s)$, for every $e, s \geq 0$. For a full exposition of the minimal-pair method see for example either (Odifreddi, 1999, p. 543 et seq.), or (Soare, 1987, p. 152 et seq.). At every stage s of the construction of the two sets A and B we define the *length-agreement function* $l(e, s)$, for every $e \geq 0$, in the following way:

$$l(e, s) =_{\text{dfn}} \max\{x : (\forall y < x)[\varphi^{A_s}_{e,s}(y)\downarrow = \varphi^{B_s}_{e,s}(y)\downarrow]\}. \tag{7}$$

Furthermore, at every stage s we define the restraint function $r(9e + 8, s)$ by induction on $e \geq 0$.

Definition 3.7 For every stage $s \geq 0$, let us define

$$r(9 \cdot 0 + 8, s) = \begin{cases} 0, & \text{if } s \text{ is a 0-expansionary stage,} \\ \max\{t : t < s \text{ and } t \text{ is a 0-expansionary stage}\}, & \text{otherwise,} \end{cases}$$

where a stage s is 0-*expansionary* if either $s = 0$ or

$$l(0, s) > \max\{l(0, t) : t < s\}.$$

Given $r(9e + 8, s)$, define $r(9(e + 1) + 8, s)$ as the maximum of:

1) $r(9e + 8, s)$,

2) $\{t : t < s \text{ and } r(9e + 8, t) < r(9e + 8, s)\}$,

3) $\{t : t < s, r(9e + 8, t) = r(9e + 8, s)$ and stage t is $(e + 1)$-expansionary, if s is not an $(e + 1)$-expansionary stage$\}$,

where a stage s is $(e + 1)$-expansionary if either $s = 0$ or

$$(\forall t < s)[r(9e + 8, t) = r(9e + 8, s) \Rightarrow l(e + 1, t) < l(e + 1, s)].$$

It is possible to prove that for every $e \geq 0$

$$\liminf_{s\to\infty} r(9e + 8, s) < \infty. \tag{8}$$

From Definition 3.7 clearly it follows that for every $e, s \geq 0$

$$r(9e + 8, s) \geq \max_{i<e}\{r(9i + 8, s)\}. \tag{9}$$

Therefore, for every $e \geq 0$

$$\liminf_{s\to\infty} (\max_{i\leq e}\{r(9i + 8, s)\}) < \infty, \tag{10}$$

and (10) will allows us to prove that all the P-requirements and R-requirements are met. Moreover, the minimal-pair method guarantees that to satisfy each requirement N_{9e+8} two conditions are sufficient:

C1: every P-requirement and R-requirement requires attentions at most finitely often, and

C2: at every stage $s > 0$ at most one side of the equality $\varphi^{A_s}_{e,s}(y) \downarrow = \varphi^{B_s}_{e,s}(y) \downarrow$ in the definition (7) of the length-agreement function change its values.

We will construct our sets A and B in such a way that both C1 and C2 above are satisfied.

3.3 Requirements Requiring Attention

As a direct consequence of the descriptions contained in the previous subsection we make the following formal definitions. We say that:

- Requirement P_{9e} requires attention at stage $s + 1 \geq 9e$ if

$$W_{e,s} \cap A_s = \emptyset \text{ and there is } x \in W_{e,s} \text{ with } x > \max_{i<9e}\{r(i, s)\}. \tag{11}$$

- Requirement P_{9e+1} requires attention at stage $s + 1 \geq 9e + 1$ if the condition (11) holds with B and $9e + 1$ respectively in place of A and $9e$.
- Requirement N_{9e+2} requires attention at stage $s + 1 \geq 9e + 2$ if $r(9e + 2, s) = -1$.
- Requirement N_{9e+3} requires attention at stage $s + 1 \geq 9e + 3$ if $r(9e + 3, s) = -1$.
- Requirement N_{9e+4} requires attention at stage $s + 1 \geq 9e + 4$ if $\varphi^{A_s}_{e,s}(e) \downarrow$ and $r(9e + 4, s) = -1$.
- Requirement N_{9e+5} requires attention at stage $s + 1 \geq 9e + 5$ if $\varphi^{B_s}_{e,s}(e) \downarrow$ and $r(9e + 5, s + 1) = -1$.
- Requirement R_{9e+6} requires attention at stage $s + 1 \geq 9e + 6$ via $x \leq s + 1$ if $r(9e + 6, s) = -1$ and

$$[x \in \overline{A}_s \wedge \varphi_{e,s}(x) \downarrow \neq x \wedge \varphi_{e,s}(x) > \max_{i<9e+6}\{r(i, s)\}. \tag{12}$$

• Likewise, requirement R_{9e+7} requires attention at stage $s+1 \geq 9e+7$ via $x \leq s+1$ if $r(9e+7, s) = -1$ and condition (12) holds with $\overline{B}$ and $9e+7$ respectively in place of $\overline{A}$ and $9e+6$.

3.4 Active Requirements

From now on, the letter $\mathcal{R}$ will denote any one of the requirements. At every stage s and for every $n \leq s$, $n \neq 9e+8$ for every $e \geq 0$, we define $\mathcal{R}_n$ *active* if it requires attention at stage s and it has the highest priority, that is $n = \min\{m : \mathcal{R}_m$ requires attention at stage $s\}$. In particular, at every stage s at most one P-requirement or R-requirement is active, and this feature allows to satisfy condition C2 above.

3.5 Injured Requirements

The negative requirement N_{9e+2} is injured at stage $s+1$ if the number $x = r(9e+2, s)$ is enumerated into A_{s+1}. In this case we set $r(9e+2, s+1) = -1$. We set $r(9e+2, s+1) = r(9e+2, s)$ otherwise. Likewise for the negative requirement N_{9e+3}.

The negative requirement N_{9e+4} is injured at stage $s+1$ if a number $x \leq r(9e+4)$ is enumerated into A_{s+1}. In this case we set $r(9e+4, s+1) = -1$. Otherwise we set $r(9e+4, s+1) = r(9e+4, s)$. Likewise for the negative requirement N_{9e+5}.

The positive and negative requirement R_{9e+6} is injured at stage $s+1$ if the number $x = r(9e+6, s)$ is enumerated into A_{s+1}. In this case we set $r(9e+6, s+1) = -1$. Otherwise we set $r(9e+6, s+1) = r(9e+6, s)$. Likewise for the negative requirement R_{9e+7}.

The negative requirement N_{9e+8} is injured at stage $s+1$ if a number $x < r(9e+8, s)$ is enumerated into either A_{s+1} or B_{s+1}.

3.6 Construction of A and B

Stage 0. Set $A_0 = B_0 = \emptyset$, and set $r(n, 0) = -1$ for every $n \geq 0$.

Stage $s+1$. Let A_s and B_s be the sets constructed up to the stage s. See if there is some requirement requiring attention. If not, then do nothing (Note 1). Otherwise, let $\mathcal{R}_n$ be the active requirement.

• If $\mathcal{R}_n = P_{9e}$ for some e, then let x be the minimum for which P_{9e} requires attention. Set $A_{s+1} = A_s \cup \{x\}$ and $B_{s+1} = B_s$.

• If $\mathcal{R}_n = P_{9e+1}$ for some e, then let x be the minimum for which P_{9e+1} requires attention. Set set $B_{s+1} = B_s \cup \{x\}$ and $A_{s+1} = A_s$.

• If $\mathcal{R}_n = N_{9e+2}$ for some e, then let x be the minimum number in $\overline{A}_s$ such that $x \geq e$. Set $r(9e+2, s+1) = x$, $A_{s+1} = A_s$ and $B_{s+1} = B_s$.

• If $\mathcal{R}_n = N_{9e+3}$ for some e, then let x be the minimum number in $\overline{B}_s$ such that $x \geq e$. Set $r(9e+3, s+1) = x$, $A_{s+1} = A_s$ and $B_{s+1} = B_s$.

• If $\mathcal{R}_n = N_{9e+4}$ for some e, then set $r(9e+4, s+1) = s$. Set $A_{s+1} = A_s$ and $B_{s+1} = B_s$.

• If $\mathcal{R}_n = N_{9e+5}$ for some e, then set $r(9e+5, s+1) = s$. Set $A_{s+1} = A_s$ and $B_{s+1} = B_s$.

• If $\mathcal{R}_n = R_{9e+6}$ for some e, then let $x \leq s+1$ be the minimum number via the which R_{9e+6} requires attention. Set $A_{s+1} = A_s \cup \{\varphi_{e,s}(x)\}$ and set $r(9e+6, s+1) = x$. Set $B_{s+1} = B_s$.

• If $\mathcal{R}_n = R_{9e+7}$ for some e, then let $x \leq s+1$ be the minimum number via the which R_{9e+7} requires attention. Set $B_{s+1} = B_s \cup \{\varphi_{e,s}(x)\}$ and set $r(9e+7, s+1) = x$. Set $A_{s+1} = A_s$.

End construction of A and B.

Let us pose $A = \bigcup_{s \geq 0} A_s$ and $B = \bigcup_{s \geq 0} B_s$. We have to prove that every requirement is satisfied. We prove first that each requirement $\mathcal{R}_{9e+i}$ with $e \geq 0$ and $i \leq 7$ requires attention at most finitely often. From now on, for every set X and every $n \in N$, $X|n$ denotes the set $X \cap \{0, 1, \ldots, n\}$.

Lemma 3.8 *For every $e \geq 0$ and $i \leq 7$, requirement $\mathcal{R}_{9e+i}$ requires attention at most finitely often. In particular, for every $e \geq 0$ and $i \leq 7$, $\lim_{s \to \infty} r(9e+i, s) < \infty$.*

Proof. The proof is by complete induction on $n = 9e+i$, for every $e \geq 0$ and $i \leq 7$. Given n, let s_0 be the minimum stage after which no requirement of higher priority than $\mathcal{R}_n$ requires attention. We consider only the cases $n = 9e$, $n = 9e+2$, $n = 9e+4$ and $n = 9e+6$, because the other cases are similar.

• Case $\mathcal{R}_n = P_{9e}$. We conventionally defined $r(9e, s) = -1$ for every s, and trivially $\lim_{s\to\infty} r(9e, s) = -1$. If $W_{e,s_0} \cap A_{s_0} \neq \emptyset$, then it will be $W_{e,t} \cap A_t \neq \emptyset$ for every $t \geq s_0$ and P_{9e} will not require attention after s_0. So, let us suppose that $W_{e,s_0} \cap A_{s_0} = \emptyset$. Let $s \geq s_0$ be the minimun stage such that P_{9e} requires attention. By hypothesis P_{9e} is active at stage s, so a number $x \in W_{e,s}$ is enumerated into A_s, which implies that for every $t \geq s$ will be $W_{e,t} \cap A_t \neq \emptyset$ and P_{9e} will not require attention anymore after s.

• Case $\mathcal{R}_n = N_{9e+2}$. Let us suppose that N_{9e+2} requires attention at stage $s + 1 \geq s_0$. Then it becomes active and we set $r(9e + 2, s + 1) = x$ for the minimum $x \in \overline{A}_s$ with $x \geq e$. By hypothesis after stage $s + 1 \geq s_0$ N_{9e+2} will not be injured, thus for every $t \geq s + 1$ will be $r(9e + 2, t) = r(9e + 2, s + 1) = x > -1$, and N_{9e+2} will not require attention after $s + 1$.

• Case $\mathcal{R}_n = N_{9e+4}$. Let us suppose that N_{9e+4} requires attention at some stage $s + 1 \geq s_0$. Then $\varphi^{A_s}_{e,s}(e)\downarrow$, N_{9e+4} is active and we set $r(9e + 4, s + 1) = s$. The computation of $\varphi^{A_s}_{e,s}(e)$ will not be destroyed anymore after $s + 1$, because $A_s|s = A|s$, and $r(9e + 4, t) = r(9e + 4, s + 1)$ for every $t \geq s + 1$.

• Case $\mathcal{R}_n = R_{9e+6}$. Let us suppose that R_{9e+6} requires attention at stage $s + 1 \geq s_0$. Then, there is a number $x \in \overline{A}_s$ with $\varphi_{e,s}(x) \downarrow\neq x$. At the end of this stage the value of $r(9e + 6, s + 1)$ is x. By hypothesis after stage $s + 1 \geq s_0$ R_{9e+6} will not be injured anymore, hence for every $t \geq s + 1$ will be $r(9e + 6, t) = r(9e + 6, s + 1) = x$ and $x \notin A_t$. □

By (10) and Lemma 3.8 just proved it follows

Corollary 3.9 *For every $n \in N$, $\liminf_{s\to\infty}(\max_{i\leq n}\{r(i, s)\})$ is finite.*

We prove now that every requirement is met.

Lemma 3.10 *Every requirement is met.*

Proof. Given $n \in N$, let s_0 be the minimum stage such that for every $s \geq s_0$ no requirement $\mathcal{R}_m$ with $m < n$ is active at stage s. We prove only the cases $\mathcal{R}_n = P_{9e}$, $\mathcal{R}_n = N_{9e+2}$, $\mathcal{R}_n = N_{9e+4}$, $\mathcal{R}_n = R_{9e+6}$ and $\mathcal{R}_n = N_{9e+8}$, because the other cases are similar.

• $\mathcal{R}_n = P_{9e}$. Let W_e be infinite. If $W_{e,s_0} \cap A_{s_0} \neq \emptyset$ then for all $t \geq s_0$ is $W_{e,t} \cap A_t \neq \emptyset$ and P_{9e} is met. If $W_{e,s_0} \cap A_{s_0} = \emptyset$, then let

$$k_{9e} = \lim_{s\to\infty}\inf(\max_{i<9e}\{r(i, s)\}). \tag{13}$$

By Corollary 3.9 such k_{9e} exists and is finite , and in particular there are infinitely many stages $s' \geq s_0$ with $\max_{i<9e}\{r(i, s')\} = k_{9e}$. Let $s_1 \geq s_0$ be the minimum stage such that there is a number $x \leq s_1$ with

$$x \in W_{e,s_1} \text{ and } x > k_{9e}. \tag{14}$$

If $W_{e,s_1} \cap A_{s_1} = \emptyset$, then P_{9e} become active at stage $s_1 + 1$ and the minimum x of (14) is enumerated into A_{s_1+1}. Hence $W_{e,t} \cap A_t \neq \emptyset$ for every $t \geq s_1 + 1$ and P_{9e} is met.

• $\mathcal{R}_n = N_{9e+2}$. If $r(9e + 2, s_0) = x \geq e$, then by hypothesis no requirement of higher priority than N_{9e+2} will be active after s_0, so $x \in \overline{A}$ and N_{9e+2} is met. If $r(9e + 2, s_0) = -1$, then at the stage $s_0 + 1$ requirement N_{9e+2} is active, hence $r(9e + 2, s_0 + 1) = x' \geq e$ for the minimum such $x' \in \overline{A}_{s_0}$. By hypothesis no requirement of higher priority than N_{9e+2} will be active after $s_0 + 1$: x' will be permanently in $\overline{A}$ and N_{9e+2} is met.

• $\mathcal{R}_n = N_{9e+4}$. Let us suppose that there are infinitely many stages s such that $\varphi^{A_s}_{e,s}(e)$ is defined. Let us assume first that $r(9e + 4, s_0) = -1$. Then, there is a minimum stage $s' + 1 \geq s_0$ such that $\varphi^{A_{s'}}_{e,s'}(e)$ is defined. At stage $s' + 1$ N_{9e+4} requires attention and by hypothesis it becomes active. At this stage $s' + 1$ we set $r(9e + 4, s' + 1) = s'$ and N_{9e+4} will not be injured after stage $s' + 1$, in particular

$$r(9e + 4, t) = r(9e + 4, s' + 1) = s'$$

for every $t \geq s' + 1$. Furthermore $A_{s'} = A|s'$, so

$$\varphi^{A_{s'}}_{e,s'}(e)\downarrow= \varphi^{A|s'}_{e,s'}(e)\downarrow= \varphi^{A}_{e,s'}(e)\downarrow= \varphi^{A}_{e}(e),$$

that is $\varphi^A_e(e)$ is defined, which implies that N_{9e+4} is met.

If $r(9e + 4, s_0) = s'' > -1$, then N_{9e+4} has been active at $s'' + 1 < s_0$ and it was not injured after $s'' + 1$; for the same argument above, N_{9e+4} is met.

• $\mathcal{R}_n = R_{9e+6}$. For the sake of contradiction, let us suppose that R_{9e+6} is not met. So, there is a subset X of $\overline{A}$ with $|\overline{A} - X| = \infty$ and $\overline{A} \leq_m X$ via φ_e. Then necessarily it has to be $r(9e + 6, s_0) = -1$. In fact, if it were $r(9e + 6, s_0) = x > -1$ for some x, then at some stage $s' + 1 < s_0$ the number $\varphi_{e,s'}(x)$ has been enumerated into $A_{s'+1}$. Since by hypothesis R_{9e+6} will not be injured anymore after s_0, it would be $x \in \overline{A}$ and $\varphi_e(x) \notin \overline{A}$, contrary to the assumption that φ_e m-reduces $\overline{A}$ to X. Let

$$k_{9e+6} = \lim_{s\to\infty} \inf (\max_{i<9e+6} \{r(i, s)\}).$$

Requirements P_{9e} and N_{9e+2} are met for every $e \geq 0$, hence the set $\overline{A}$ is immune. By Lemma (3.6) the set $\{\varphi_e(x) : \varphi_e(x) \neq x \wedge x \in \overline{A}\}$ is infinite. Let $s' + 1 \geq s_0$ be the minimum stage such that

(i) $\max_{i<9e+6}\{r(i, s')\} = k_{9e+6}$,

(ii) there exists $x \leq s' + 1$ with $x \in \overline{A}_{s'}$, $\varphi_{e,s'}(x) \downarrow\neq x$ and $\varphi_{e,s'}(x) > k_{9e+6}$.

Let x' be the minimum x satisfying (ii). This means that R_{9e+6} requires attention at stage $s' + 1$ via x' and R_{9e+6} has the highest priority by hypothesis, that is R_{9e+6} is active. At stage $s' + 1$ the number $\varphi_{e,s'}(x')$ is enumerated into $A_{s'+1}$ and $r(9e + 6, s' + 1) = x'$. After stage $s' + 1$ R_{9e+6} will not be injured anymore, therefore $x' \in \overline{A}$ and $\varphi_e(x') \notin \overline{A}$, contrary to the assumption that $\overline{A} \leq_m X$ via φ_e.

• $\mathcal{R}_n = N_{9e+8}$. Let us suppose that

$$\varphi_e^A = \varphi_e^B = C. \tag{15}$$

We have to prove that C is recursive, and this is possible by the known technique developed in the minimal-pair method. If $e > 0$, then let

$$k_{9e+8} = \lim_{s\to\infty} \inf r(9(e - 1) + 8, s) \tag{16}$$

and let

$$S = \{s : r(9(e - 1) + 8, s) = k_{9e+8}\}. \tag{17}$$

If $e = 0$ then let $k_{9e+8} = 0$ and $S = N$. In any case S is recursive. Given x, decide "$x \in C$" by looking for a stage $s' \geq s_0$ such that $s' \in S$, $l(e, s') > x$ and s' is e-expansionary. In S there are infinitely many e-expansionary stages by (15). Then

$$C(x) = \varphi_{e,s'}^{A_{s'}}(x), \tag{18}$$

where (18) follows by the following claim.

Claim 3.11 *Let $s_1 = s' < s_2 < s_3 < \cdots$ be all the infinite e-expansionary stages in S from s' onwards. Let $y = \varphi_{e,s_1}^{A_{s_1}}(x) = \varphi_{e,s_1}^{B_{s_1}}(x)$. Then, for every $n \geq 1$ and for every stage t with $s_1 \leq t \leq s_n$ at least one of 1) and 2) below occur:*

1) $\varphi_{e,t}^{A_t}(x) = y$,

2) $\varphi_{e,t}^{B_t}(x) = y$.

Claim 3.11 can be proved by the definition of $r(9e + 8, s)$ and assuming the following two conditions (see either Soare, 1987, Lemma 3, p. 155 et seq., or Odifreddi, 1999, p. 545):

• requirement N_{9e+8} is not injured after s', and

• at every stage $s + 1$ at most one of the two sets A_s and B_s changes.

These two conditions are satisfied respectively by the choice of $s' \geq s_0$ and by the fact that at every stage $s + 1$ at most one of either P-requirement or R-requirement is active; it follows that N_{9e+8} is met. □

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Notes

Note 1. That is, set $A_{s+1} = A_s$, $B_{s+1} = B_s$, and $r(n, s+1) = r(n, s)$ for every $n \neq 9e + 8$, $e \geq 0$.