Journal of Mathematics Research; Vol. 4, No. 6; 2012
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Stochastic Dependence Modelling Using Conditional Elliptical Processes

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Received: November 13, 2012 Accepted: November 27, 2012 Online Published: November 29, 2012

doi:10.5539/jmr.v4n6p130 URL: http://dx.doi.org/10.5539/jmr.v4n6p130

Abstract

The class of elliptical structures includes a vast field of known symmetric distributions and copulas. This article investigates the properties of the copula underlying a stochastic elliptical process in conditional context. Specifically we characterize the conditional high dimensional copulas of the elliptical process both in non-spatial framework and for space-varying models. A spatial conditional measure is constructed to model the joint dependence of these copulas and distributions with applications to the three most known elliptical structures.

Keywords: elliptical process, Gaussian distribution, t-copula, conditional copulas, extremal copulas

2010 Mathematical Subject Classification: 60G15, 62G32, 91B72

1. Introduction

Modelling multivariate dependence, finance analysts, macroeconomists and econometricians often know a great deal about marginal distributions of individual variables but little about their joint behaviour. Consider, for example, the analysis of a portfolio consisting of a vector of stocks, one may be interested in the behaviour of the portfolio when the values of each component portfolio X_1, ..., X_n falls short of a certain high threshold. In particular, in bivariate case when data allow asymptotic interpolation, the coefficients of tail or extremal dependence quantify the magnitude of the occurence that one component be large, assuming that the other component is also extremely large at its tail.

In multivariate analysis when the marginal distributions are known with certainty, the copula function enables to capture and to piece together the joint distribution via Sklar's Theorem (Sklar, 1959). Therefore, every n-dimensional continuous distribution H can be canonically parameterized by its univariate marginal $H_1, ..., H_n$ using a copula C defined on the unit cube $[0, 1]^n$, such as

$$C(u_1, ..., u_n) = H\left(H_1^{-1}(u_1), ..., H_n^{-1}(u_n)\right) \text{ for all } (u_1, ..., u_n) \in [0, 1]^n\,; \tag{1}$$

where H_i^{-1} is the quantile function of H_i; that is, $H_i^{-1}(u) = \inf\{x \in \mathbb{R}, H_i(x) \geq u\}$. Standard references for copulas analysis are Joe (1997) or Nelsen (2007) which provide detailed and readable introductions to copulas and their statistical and mathematical foundations while Bouyé et al. (2000) or Cherubini et al. (2004) deal with applications of copulas to different levels of financial issues and derivatives pricing. Differentiating the formula (1) shows that the density function of the copula is equal to the ratio of the joint density h of H to the product of marginal densities h_i such as, for all $(u_1, ..., u_n) \in [0, 1]^n$,

$$c(u_1, ..., u_n) = \frac{\partial^n C(u_1, ..., u_n)}{\partial u_1 ... \partial u_m} = \frac{h\left[H_1^{-1}(u_1), ..., H_n^{-1}(u_n)\right]}{h_1\left[H_1^{-1}(u_1)\right] \times ... \times h_n\left[H_n^{-1}(u_n)\right]}. \tag{2}$$

Elliptical copulas form a very important class of copulas that has been receiving attention in financial applications this last years. From a practical point of view elliptical copulas and distributions are attractive particularly while modelling of financial data (see Tangho, 2007). For example, the modern theory of portfolio risk management relies Gaussian distributions hypothesis and it quintessence is that portfolio diversification effect depends essentially on covariance matrix (Embrechts et al., 2001).

Elliptical copulas are generally defined as the copulas of elliptical distributions. In particular, if Ψ_Σ is a multivariate elliptical distribution with common marginal Ψ and with dispersion matrix Σ, then the elliptical copula is derived from the relation (1) by

$$C_{\Sigma_\theta}(u_1, ..., u_n) = \Psi_\Sigma\left(\Psi_1^{-1}(u_1), ..., \Psi_n^{-1}(u_n)\right) = \frac{1}{\sqrt{\det\Sigma}}\exp\left\{-\tfrac{1}{2}\tilde{U}^T\left(\Sigma^{-1}\right)\tilde{U}\right\}; \quad (3)$$

where $\tilde{U} = \left(\Psi^{-1}(u_1), ..., \Psi^{-1}(u_n)\right)$ is the quantile-vector of marginal distribution. Despite they cannot be expressed by an explicit form, elliptical structures allow instead different degrees of correlation and provide a rich source of distributions with many tractable properties of the multivariate normal distributions. Moreover, they inherit to multivariate extreme dependence and a lot of Gaussian properties. Two most known families of elliptical structures (distributions and copulas) are the Gaussian family and the Student-t family.

The main contribution of this article is to investigate the properties of a conditional copula of a stochastic elliptical process both in non-spatial context and in space varying field. We model a conditional time dependent measure with applications to the three usual elliptical families of copulas.

2. Preliminaries

In this section we collect important definitions and properties on conditional extremal copulas and stochastic elliptical modelling, that turn out to be necessary for our approach. We refer the reader to Joe (1997) or to Nelsen (2007) for a general introduction to multivariate copulas theory and to Cambanis et al. (1981) or to Embrechts et al. (2004) for stochastic elliptical analysis and its applications to different degrees of financial issues.

2.1 Results on Stochastic Elliptical Analysis

Let S_n be the unit n-dimensional hypersphere given, for some arbitrary norm $\|\cdot\|$ in $\mathbb{R}^n$ by

$$S_n = \{t = (t_1, ..., t_n) \in \mathbb{R}^n; \|t\| = 1\} \subset \mathbb{R}^n.$$

Further, let Ω_{n-1} denote the restriction of S_n to the unit cube $[0,1]^{n-1}$of $\mathbb{R}^{n-1}$ for the 1-norm, that is:

$$\Omega_{n-1} = \left\{(t_1, ..., t_{n-1}) \in [0,1]^{n-1}; \sum_1^{n-1} t_i \leq 1\right\}.$$

A complete definiton of elliptical distributions provides to be usefull for a better understanding of our results.

Definition 1 (Cambanis et al., 1981) Let $X = (X_1, ..., X_n)$ be an n-dimensional random vector. X is said to be elliptically distributed (or simply elliptical) if for some vector $\mu \in \mathbb{R}^n$, some $n \times n$ positive definite symmetric matrix Σ and some function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}$, the characteristic function $\varphi_{X-\mu}$ of $X - \mu$ is the form $\varphi_{X-\mu}(t) = \phi\left(t^T\Sigma t\right)$ and we writte $X \rightsquigarrow E_n(\mu, \Sigma, \phi)$.

More specifically, the density function φ of an n-dimensional elliptical random vector X with expectation $\mu = (\mu_1, ..., \mu_n) \in \mathbb{R}^n$ and with dispersion matrix Σ, is written (if it exists) as

$$\varphi(x) = \frac{c_n}{\sqrt{\det(\Sigma)}} g_n\left\{\frac{1}{2}(x-\mu)^t\Sigma^{-1}(x-\mu)\right\} \text{ for all } x = (x_1, ..., x) \in \mathbb{R}^n; \quad (4)$$

for some function g_n called the density generator, c_n being a normalizing constant such as

$$c_n = \frac{\Gamma\left(\frac{n}{2}\right)}{(2\pi)^{\frac{n}{2}}}\left[\int_0^{+\infty} x^{\frac{n}{2}-1} g_n(x)\,dx\right]^{-1}; \quad (5)$$

verifying the constraint $\int_0^{+\infty} x^{\frac{n}{2}-1} g_n(x)\,dx < +\infty$. The form of the generator gives commonly the multivariate normal family or the Student-t family. The particular case where $n = 1$ provides the class of symmetric distributions. The following result provides a stochastic representation of elliptical random vectors when the dispersion matrix has full rank.

Theorem 2 (Frahm et al., 2003) *A random vector* $X \backsim E_n(\mu, \Sigma, \phi)$ *with* $r(\Sigma) = k$ *if and only if*

$$X \overset{d}{=} \mu + RAU^{(k)}; \quad (6)$$

where $U^{(k)}$ is a k-dimensional random vector uniformly distributed on S^{k-1}, R is a non-negative random variable being stochastically independent of $U^{(k)}$, where $\mu \in \mathbb{R}^n$. Further, $A \in \mathbb{R}^{n\times k}$ with $r(A) = k$ and satisfies the Cholesky decomposition $\Sigma = A^T A$.

2.2 Results on Multivariate Extreme Values Modelling

Copulas form a natural way to construct multivariate distributions with uniform margins. Particularly a 2-dimensional copula can be simply defined as follows.

Definition 3 A bivariate function: $C : [0, 1]^2 \longrightarrow [0, 1]$ is an 2-copula if, for all $(u, v) \in [0, 1]^2$:

i) $C(u, 0) = C(0, v) = 0$

ii) $C(u, 1) = u$; $C(1, v) = v$

iii) C is 2-increasing i.e. $C(u_2, v_2) - C(u_2, v1) - C(u_1, v_2) + C(u_1, v_1) \geq 0$ for all $(u_1, v_1), (u_2, v_2) \in [0, 1]^2$ such as $u_1 \leq u_2$ and $v_1 \leq v_2$ (positiveness of the volume of any rectangle in $\mathbb{R}^2$).

Particularly, if an n-dimensional copula C satisfies the following max-stability property

$$C(u_1, ..., u_n) = C\left(u_1^{1/k}, ..., u_n^{1/k}\right)^k \text{ for all } (u_1, ..., u_n) \in [0, 1]^n \text{ and } k > 0, \tag{7}$$

then, C is an extremal copula. For such a copula the associated distribution H via the canonical parameterization (1) is a multivariate extreme values (MEV) distribution (see Joe, 2007). In addition, the distribution H or its random vector X have to satisfy the regularly varying property: there exist an index $\alpha > 0$ and a random vector S distributed on S_n such that, for any $t > 0$ and any Borel-set $B \subset S_{n-1}$

$$\frac{P\left(\|X\| > tx, \frac{X}{\|X\|} \in B\right)}{P(\|X\| > x)} \underset{x\longrightarrow+\infty}{\overset{v}{\longrightarrow}} P(S \in B)\, t^{-\alpha}; \tag{8}$$

where υ symbolizes vague convergence. That is written as $X \in RV(\alpha)$ see (Resnick, 1987).

Otherwise, the univariate marginal distribution of any MEV distribution is the real parametric model H_ξ given by

$$H_{\xi_i}(x_i) = \begin{cases} \exp\left\{-\left[1 + \xi_i\left(\frac{x_i-\mu_i}{\sigma_i}\right)\right]^{\frac{-1}{\xi_i}}\right\} \text{ if } \xi_i \neq 0 \\ \exp\left\{-\exp\left(-\frac{x_i-\mu_i}{\sigma_i}\right)\right\} \text{ if } \xi_i = 0 \end{cases} \quad \text{for } 1 \leq i \leq n. \tag{9}$$

defined on the domain $D_{\xi_i} = \{x_i \in \mathbb{R}; \sigma_i + \xi_i (x_i - \mu_i) > 0\}$ where $\{\mu_i \in \mathbb{R}\}$, $\{\sigma_i > 0\}$ and $\{\xi_i \in \mathbb{R}\}$ are respectively the parameters of location, scale and shape of the margin law X_i.

Furthermore, any given distribution F belongs to the maximum domain of attraction of H_ξ, denoted $F \in MDA\left(H_\xi\right)$, if there exist normalising sequences $\{\sigma_{n,i} > 0\}$ and $\{\mu_{n,i} \in \mathbb{R}\}$ such that

$$\lim_{n\to+\infty} F_i^n (\sigma_{n,i} x_i + \mu_{n,i}) = H_{\xi_i}(x_i) \text{ for all } 1 \leq i \leq n. \tag{10}$$

In classical MEV analysis however, there exists no parametric family summarizing all types of asymptotic behaviours like H_ξ of (7) in univariate case. Nonetheless, number of structures have been constructed to model the joint dependences. One of these structures is the extremal dependence function V, defined on the unit simplex Ω_n by

$$V(x_1, ..., x_n) = \int_{\Omega_n} \max\left(\frac{w_1}{x_1}, ..., \frac{w_n}{x_n}\right) dH(w_1, ..., w_n);$$

where H is a finite non-negative measure of probability, arbitrary except for the moments constraint $\int_{\Omega_n} w_i dH(w_1, ..., w_n) = 1$ for each $1 \leq i \leq n$, see (Joe, 1997) or (Resnick, 1987). An extension of V to S_{n-1}, referred to as Pickands dependence function and has been developed, related to V, by

$$V(x_1, ..., x_n) = \textstyle\sum_1^n x_i A(t_1, ..., t_{n-1}) \text{ where } t_i = \frac{x_i}{\sum_1^n x_i}. \tag{11}$$

Then, for all $u_i \in [0, 1]$; $\tilde{u}_i = \log u_i$; $1 \leq i \leq n$, the extremal copula C is described via V and A by

$$C(u) = \exp\{-V(-(\tilde{u}_1)^{-1}, ..., -(\tilde{u}_n)^{-1})\} = \exp\left\{-\left[\sum_{i=1}^n \tilde{u}_i A\left(\frac{\tilde{u}_1}{\sum_{i=1}^n \tilde{u}_i}, ..., \frac{\tilde{u}_{n-1}}{\sum_{i=1}^n \tilde{u}_i}\right)\right]\right\}. \tag{12}$$

3. A Characterization of Conditional Elliptical Copulas

In this section $X = (X_1; ...; X_n)$ denotes an elliptical random vector with joint distribution function $H = (H_1; ...; H_n)$ with copula C. We are interested to investigate properties of a conditional copula given a conditioning set W.

Proposition 4 *Let $X = (X_1 ..., X_n)$ be an elliptical vector with distribution H and dispersion matrix Σ. Assume that there exists a real number $\alpha > 0$ such that $X \in RV(\alpha)$. Then there exists a convex conditional measure D_Σ mapping $\left(\bar{\mathbb{R}}\right)^n \times \left(\bar{\mathbb{R}}\right)^n$ to $[0, 1]$ such that, for all vectors $x, u \in \left(\bar{\mathbb{R}}\right)^n$ with $x_i \in [0, u_i]$, the conditional elliptical distribution is given by*

$$H_C((x_1, ...x_n) / (u_1, ..., u_n)) = \exp\{-D_{\Sigma_t}(x, u)\} \tag{13}$$

where $(x, u) = ((x_1, u_1); ..., (x_n, u_n)) \in \left(\bar{\mathbb{R}}\right)^n \times \left(\bar{\mathbb{R}}\right)^n$.

Proving Proposition 4 requires the following result given in (Frahm et al., 2003).

Theorem 5 (Frahm et al., 2003) *Let $X \overset{d}{=} \mu + RAU^{(k)} \sim E_n(\mu, \Sigma, \varphi)$ where $\Sigma = A^T A$ is positive definite. Further, let Ψ_Σ be the generating distribution function of X. Then $\Psi_\Sigma \in MDA(H_\xi)$ if and only if X is regularly varying with tail index $\alpha = \frac{1}{\xi} > 0$.*

Proof. (of Proposition 4) Let's extend to higher dimensional case the conditional copula considered by Patton (2006). For all realization $x = (x_1; ...; x_n) \in \left(\bar{\mathbb{R}}\right)^n$ and $u = (u_1; ...; u_n) \in \left(\bar{\mathbb{R}}\right)^n$ of the process X such as, for all $x_i \in [0, u_i]$, then the conditional distribution H_C of (13) is given by:

$$H_C((x)|u) = P\left(\bigcap_{i=1}^{n}(H_i(X_i) \le x_i)\middle|\bigcap_{i=1}^{n}(H_i(X_i) \le u_i)\right).$$

Moreover, since U_i are the probability integral transformation, $U_i \sim H_i(X_i)$, it follows that

$$H_C((x)|u) = P\left(\bigcap_{i=1}^{n}(U_i \le x_i)\middle|\bigcap_{i=1}^{n}(U_i \le u_i)\right) = \frac{P\left(\bigcap_{i=1}^{n}(U_i \le x_i); \bigcap_{i=1}^{n}(U_i \le u_i)\right)}{P\left(\bigcap_{i=1}^{n}(U_i \le u_i)\right)}.$$

Then, since $x_i \in [0, u_i]$ and every copula C is the distribution function of uniform vector $(U_1, ..., U_n)$ even $U_i \sim H_i(X_i)$ regardless of the original distribution H, it results that

$$H_C((x_t)|u_t) = \frac{P(U_1 \le x_1, ..., U_n \le x_n)}{P(U_1 \le u_1, ..., U_n \le u_n)} = \frac{C(x_1; ...; x_n)}{C(u_1; ...; u_n)}. \tag{14}$$

Otherwise, by assumption there exists an index $\alpha > 0$ such that $X \in RV(\alpha)$. Therefore, from Theorem 5 it yields that $H \in MDA\left(H_\xi\right)$. That means both that every margin H_i of H satisfies the relation (10) and particularly H is a MEV distribution. Therefore, the copula C is an extremal copula since associated to a MEV distribution by formula (1).

Let V_Σ denote the convex extremal dependence function of C. Then, it follows from (12) that,

$$C(u_1, ..., u_n) = \exp\{-V_\Sigma(-(\bar{u}_1)^{-1}, ..., (-\tilde{u}_n)^{-1})\} \text{ for all } (u_1, ..., u_n) \in [0, 1]^n;$$

It follows, from the relation (14) that

$$H_C((x_t)|u_t) = \exp\{-V_\Sigma(-(\tilde{x}_1)^{-1}, ..., (-\tilde{x}_n)^{-1}) + V_\Sigma(-(\bar{u}_1)^{-1}, ..., (-\tilde{u}_n)^{-1})\}$$

Finally, it follows a convex measure D_Σ, mapping $[(\mathbb{R} \cup \{\pm\infty\})^n]^2$ to $[0, 1]$ such as

$$D_\Sigma(x, u) = V_\Sigma(-(\tilde{x}_{11})^{-1}, ..., (-\tilde{x}_n)^{-1}) - V_\Sigma(-(\bar{u}_1)^{-1}, ..., (-\tilde{u}_n)^{-1}) \tag{15}$$

for all $(x_t, u_t) \in \left[(\mathbb{R} \cup \{\pm\infty\})^n\right]^2$. So, (13) is proved as disserted. □

4. Conditional Copulas in Elliptical Spatial Framework

In this section we consider a spatial elliptical process $\{Y_t = (Y_{t,i}), 1 \le i \le n; t \in T\}$ observed on a set of locations $D = \{x, ..., x_n\}$ where the date t of the realizations $y_{t,i} = y_t(x_i)$ is assumed to be the same (see, e.g., Smith &

Stephenson, 2009). Let $H_t = (H_{t,1}; ...; H_{t,n})$ and C_t be respectively the joint distribution function and the copula associated to the process. Hence, it follows from relation (1) that

$$H_t(y_{t,1}, ..; y_{t,n}) = P(Y_t(x_1) \leq y_{t,1}, ..., Y_t(x_n) \leq y_{t,n}) = C_t(H_{t,i}(y_{t,i}); ..; H_{t,n}(y_{t,n})).$$

Dealing with bivariate copulas Fantazzini (2004) defined a conditional copula concept given a conditioning set W. Aiming to extend this concept both to high dimensional case and to time-varying framework, let's have to deal with areal data on D_t (see Ribatet, 2010). Indeed, such a kind of geostatistical dataset allows to be partitioned into a finite number of zones $D_1; ...; D_k$ as Fereira et al. (2011) partitioned their random vector into two blocks while modelling conditional tail dependence. Then, it follows that

$$D = \bigcup_{i=1}^{k} D_i \text{ with } k < n \text{ where } D_i \cap D_j \neq \emptyset \text{ if } i \neq j.$$

Let's consider zones W_i as the conditioning subsets, that is $W_i = D_i$. The following result gives the spatial high dimensional conditional distributions where, for simplicity purpose, the conditioning zone W_i includes a single site w_t.

Proposition 6 *Let H_{t,w_t} denotes the joint distribution of $(\tilde{Y}_{t,n-1}, W_t), t \in T$ with $\tilde{Y}_{t,n-1} = (Y_{t,1}, ..., Y_{t,n-1})$, then the conditional spatial distribution of $(\tilde{Y}_{t,n-1}, W_t)$ is given, for all $\tilde{y}_t \in (\bar{\mathbb{R}})^{n-1}$ by*

$$H_{t,w_t}(\tilde{y}_t/w_t) = f_w^{-1}(w_t)\frac{\partial H_{t,w_t}(y_{t,1}; ...; y_{t,n-1}, w_t)}{\partial w_t}; \tag{16}$$

where f_w is the spatial density of the law of W_t. Moreover, the following properties are satisfied

1) $H_{t,w_t}(y_{t,1}; ..., -\infty; ...; y_{t,n-1}, w_t) = 0$ for all $\tilde{y}_t \in (\bar{\mathbb{R}})^{n-1}$.

2) $H(\infty, ..., \infty/w_t) = 1$ for all $\tilde{y}_t \in (\bar{\mathbb{R}})^{n-1}$.

3) *For all $\tilde{y}_t^{(1)} = \left(y_{t,1}^{(1)}, ..., y_{t,n-1}^{(1)}\right) \in (\bar{\mathbb{R}})^{n-1}$ and $\tilde{y}_t^{(2)} = \left(y_{t,1}^{(2)}, ..., y_{t,n-1}^{(2)}\right) \in (\bar{\mathbb{R}})^{n-1}$ such as $y_{t,j}^{(1)} \leq y_{t,j}^{(2)}$ then*

$$\sum_{(i_1,...,i_n)\in\{1,2\}^n} (-1)^{\sum_{j=1}^{n} i_j} H_{n-1,w_t}\left(y_{t,1}^{(i_1)}, ..., y_{t,n-1}^{(i_{n-1})}, w_t\right) \geq 0. \tag{17}$$

Proof. By extending the proposition 1.3 in Patton (2002) both to space-varying case and to higher dimensional framework, we obtain the three properties above since the cumulative distribution $H_{t,n-1}$ must satisfy the positiveness of the volume of any hyperrectangle of $(\bar{\mathbb{R}})^n$. □

The following result provides a key property of conditional space-varying copula.

Theorem 7 *For a given conditioning set W_i, t he conditional copula C_{W_i} of $Y_t|W_i$ where $\{Y_{t,i}|W_i \sim H_{t,i}; 1 \leq i \leq k\}$ exists and it coincides with the joint distribution function of $U_i \equiv H_{t,i}(Y_t|W_i); 1 \leq i \leq k$ given W_i. Moreover, if the elliptical process Y_t is regularly varying, then C_{W_i} is a spatial extremal copula.*

Proof. It suffice to prove that the spatial conditional copula C_W satisfies the n-dimensional properties i), ii) and iii) of Definition 3. By using simultaneously Definition 5 and Proposition 6, it follows that, for all $(u_{t,1}, ...u_{t,i-1}, u_{t,i+1}, u_{t,n}) \in [0,1]^{n-1}$

$$\begin{aligned} & C_{W_i}(u_{t,1}, ...u_{t,i-1}, 0, u_{t,i+1} ..., u_{t,n}/w_t) \\ = \; & C_t(H_{t,1}(y_{t,1}|w_t), ..., H_{t,i-1}(y_{t,i-1}|w_t), H_{t,i}(-\infty|w_t), H_{t,1+1}(y_{t,i+1}|w_t), ..., H_{t,n}(y_{t,n}|w_t)/w_t) \end{aligned}$$

Then,

$$C_{W_i}(u_{t,1}, ...u_{t,i-1}, 0, u_{t,i+1} ..., u_{t,n}/w_t) = H_{t,w_t}(y_{t,1}; ..., -\infty; ...; x_{t,n-1}, w_t) = 0$$

Hence, the copula C_{W_i} is grounded, that is, it satisfies the generalization of the property i).

Moreover, notice that for every margin $Y_{t,i}$ the generalized inverse function $H_{i,t}^{-1}$ exists since an elliptical law is continuous. Then, for all $u_{t,i} \in [0,1]$ with $1 \leq i \leq n$,

$$\begin{aligned} C_{W_i}(1, .., 1, u_{t,i}, 1, ..., 1/w_t) &= C(H_{t,1}(+\infty|w_t), ..., H_{t,i-1}(+\infty|w_t), u_{t,i}, H_{t,i+1}(+\infty|w_t), ..., H_{t,n}(+\infty|w_t)/w_t) \\ &= H_{t,w_t}\left(+\infty; .. + \infty, H_{t,i}^{-1}(u_{t,i})\right), +\infty, ... + \infty) \\ &= \lim_{y\to+\infty} H_{t,w_t}\left(y; ..x, H_{t,i}^{-1}(u_{t,i})\right), y, ..., x) = H_{i,t}\left[H_{i,t}^{-1}(u_i)\right] = u_i \end{aligned}$$

Hence, the margins of the copula C_{W_i} are uniform (the generalization of the property ii).

Otherwise, for proving iii) notice that there is no loss of generality by restricting it to 2-dimensional case since a systematic extension of the latter yields the high dimensional settings.

Then, for all $t \in T$, consider $(u_{t,1}, v_{t,1})\,;(u_{t,2}, v_{t,2}) \in [0,1]^2$ such that $u_{t,1} \le u_{t,2}$ and $v_{t,1} \le v_{t,2}$,

$$\begin{aligned} C_{W_i}(u_{t,2}, v_{t,2}/w_t) - C_{W_i}(u_{t,1}, v_{t,2}/w_t) - C_{W_i}(u_{t,2}, v_{t,1}/w_t) - C_{Wi}(u_{t,1}, u_{t,1}/w_t) &= \\ H_{t,w_t}(y_{t,2}, z_{t,2}/w_t) - H_{t,w_t}(y_{t,1}, z_{t,2}/w_t) - H_{t,w_t}(y_{t,2}, z_{t,1}/w_t) - H_{t,w_t}(y_{t,1}, z_{t,1}) &\ge 0 \end{aligned}$$

The positiveness here is provided by that H_{t,w_t} satisfies the relation (17). Hence, we conclude that C_{W_i} is a n-dimensional spatial copula.

Furthermore, the process Y_t is assumed to be regularly varying, so from Barro et al. (2012), the spatial distribution H_t lies also in the MDA of a spatial MEV distribution. Particularly, the copula C_t is a spatial extremal copulas and therefore satisfies the max-stability property (7) which equivalently written gives

$$C_t(u_{t,1}^k, ..., u_{t,n}^k) = C_t^k(u_{t,1}, ..., u_{t,n}) \text{ for all } (u_{t,1}, ..., u_{t,n}) \in [0,1]^n\,; t \in T \text{ and } k > 0.$$

Then, whatever the varible $Y_{t,i}$ the integral transformation $H_{t,i}(Y_{t,i}) \sim U_i$ regardless $H_{t,i}$.

Then, for all $(u_{t,1}, ..., u_{t,n}) \in [0,1]^n$ and $k > 0$.

$$\begin{aligned} C_{W_i}(u_{t,1}^k, ..., u_{t,n}^k/w) &= P\left(H_{t,1}(Y_{t,1}, |W_i) \le u_{t,1}^k, ..., H_{i,t}(Y_t|W_i) \le u_{t,n}^k\right) \\ &= P\left(Y_t|W_i) \le H^{-1}\left(u_{t,1}^k\right), ..., Y_t|W_i) \le H^{-1}\left(u_{t,1}^k\right)\right) \\ &= P\left(U_i \le H^{-1}\left(u_{t,1}^k\right), ..., U_i \le H^{-1}\left(u_{t,1}^k\right)\right) = C_t(u_{t,1}^k, ..., u_{t,n}^k/w_t) \end{aligned}$$

Therefore, for all $(u_{t,1}, ..., u_{t,n}) \in [0,1]^n$ and $k > 0$.

$$C_{Wi}(u_1^k, ..., u_n^k/w) = C_t(u_1^k, ..., u_n^k) = C_t^k(u_1, ..., u_n) = C_t^k(u_1, ..., u_n/w)$$

So, the copula C_{Wi} is max-stable and so, is a spatial extremal copula. □

Remark 8 Notice that the conditioning realization of process W_t have to be the same for each of marginal distributions and the copula.

5. Applications to Bivariate Usual Elliptical Families

In geostatistical analysis the separating distance $h_{ij} = \left\|x_i - x_j\right\|$ between pairs of localities x_i and x_j is provides to be very important while modelling spatial variability. Let h denote the mean value of the separating distance between the sites.

5.1 A Spatial Conditional Measure for Bivariate Gaussian Process

Gaussian process is one of the most used elliptical processes even in the spatial context.

Corollary 9 *For an elliptical vector X, if the density generator is given by $g(u) = e^{\frac{-u}{2}}$ then, X is the multivariate normal process*

$$X \overset{d}{=} \mu + \sqrt{\chi_k^2} A U^{(k)} \sim N_n(\mu, \Sigma) \text{ with } R \sim \sqrt{\chi_k^2}.$$

Moreover, in bivariate case with correlation ρ, the spatial conditional measure is given for $t \in \Omega_2$ by

$$D_\rho(t) = \frac{t}{1+t}\left[\frac{1}{t}\Phi\left(\frac{2-\rho^2 \log t}{2\rho}\right) - \Phi\left(-\frac{2+\rho^2 \log t}{2\rho}\right)\right]; \tag{18}$$

where Φ is the standard univariate Gaussian distribution.

Proof. By replacing the density generator in (4) by $g(u) = e^{\frac{-u}{2}}$ and by calculating c_n in (5), it follows that the density function of the process yields the density of the multivariate normal family (see Zinoviy et al., 2003), that is

$$\varphi(x_1, ..., x_n) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det \Sigma}} \exp\left\{-\frac{1}{2}(x-\mu)^t \Sigma^{-1}(x-\mu)\right\}.$$

Particularly, the bivariate normal copula with correlation ρ is derived from the relation (1) by

$$C_\rho(u_1, u_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \exp\left\{-\frac{\left(r^2 - 2\rho rs + s^2\right)}{2(1-\rho^2)}\right\} drds,$$

where Φ^{-1} is the quantile function of the standard normal distribution.

Furthermore, in a bivariate study, Hüsler and Réïss (see Joe 1997) showed that the corresponding Pickands dependence function is given by

$$A_\rho(t) = t\Phi\left[\frac{1}{\rho} + \frac{\rho}{2}\log\left(\frac{t}{1-t}\right)\right] + (1-t)\Phi\left(\frac{1}{\rho} - \frac{\rho}{2}\log\left(\frac{t}{1-t}\right)\right);$$

where ρ is the correlation parameter.

Otherwise, for two-dimensional case, the dependence measure is obtained by using simultaneously the relation (12) and (16), by

$$D(t) = A\left(\tfrac{1}{1+t}\right) - \tfrac{t}{1+t}. \tag{19}$$

Finaly and spatially, we obtain easily (18). □

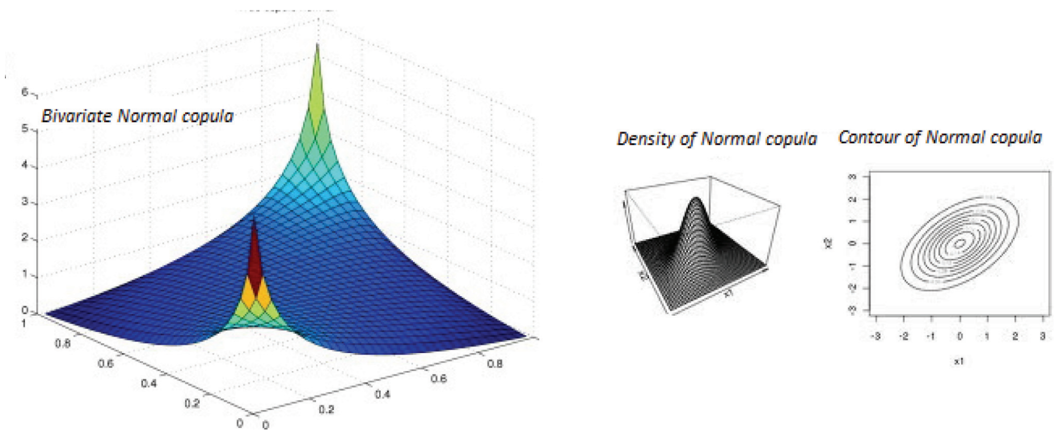

Figure 1. Graphic of bivariate normal copula, density and contour

5.2 A Spatial Conditional Measure for Bivariate Student t-Process

While modelling spatial storm profiles (Ribatet et al., 2010) noticed that the Gaussian structures do not model appropriately than the t-family.

Corollary 10 *If the density generator is given by* $g_n(t) = \left(1 + \frac{t}{\nu}\right)^{-\frac{1+\nu}{2}}$*, then, the elliptical vector X is the multivariate Student t-process with the number* ν *of degrees of freedom*

$$X \overset{d}{=} \mu + \sqrt{nT_{n,\nu}}AU^{(k)} \sim T_{\Sigma,n}(\mu, \Sigma, \nu) \text{ with } R \sim \sqrt{nT_{n,\nu}}.$$

Then, for the bivariate case with correlation ρ*, the spatial conditional measure* D_ρ *is given by*

$$(1+t)D_\rho(t) = T_{n,\nu}\left[t^{\frac{-1}{\nu}}\left(\frac{1-\rho(h)}{1+\nu}\right)^{\frac{-1}{2}}\right] + t\left(T_{n,\nu}\left[t^{\frac{1}{\nu}}\left(\frac{1-\rho(h)}{1+\nu}\right)^{\frac{-1}{2}}\right] - 1\right) \tag{20}$$

where $T_{n,\nu}$ *is the standard t-random normal distribution.*

Proof. By replacing the density generator in (4) by $g_n(t) = \left(1 + \frac{t}{\nu}\right)^{-\frac{1+\nu}{2}}$ and by calculating c_n in (5), it follows that the density function of the process is the traditional function of the t-density

$$\varphi(x_1, ..., x_n) = \frac{\Gamma\left(\frac{n+\nu}{2}\right)}{(\pi)^{\frac{n}{2}}\Gamma\left(\frac{\nu}{2}\right)\sqrt{\det\Sigma}}\left(1 + \frac{(x-\mu)^t\Sigma^{-1}(x-\mu)}{\nu}\right)^{-\frac{\nu+n}{2}};$$

Particularly, for the bivariate t-copula with correlation ρ Ribatet et al. (2010) showed that the multivariate corresponding Pickands dependence function is given by,

$$A_{\theta(h)}(t) = tT_{\nu+1}\left[\frac{\{t/1-t\}^{1/\nu}}{\left\{\frac{1-\rho(h)}{\nu+1}\right\}^{\frac{1}{2}}}\right] + (1-t)\,T_{\nu+1}\left[\frac{\{(1-t)/t\}^{1/\nu}}{\left\{\frac{1-\rho(h)}{\nu+1}\right\}^{\frac{1}{2}}}\right] \tag{21}$$

Then, by replacing the relation (21) in the formula (19), it follows (20) as disserted. □

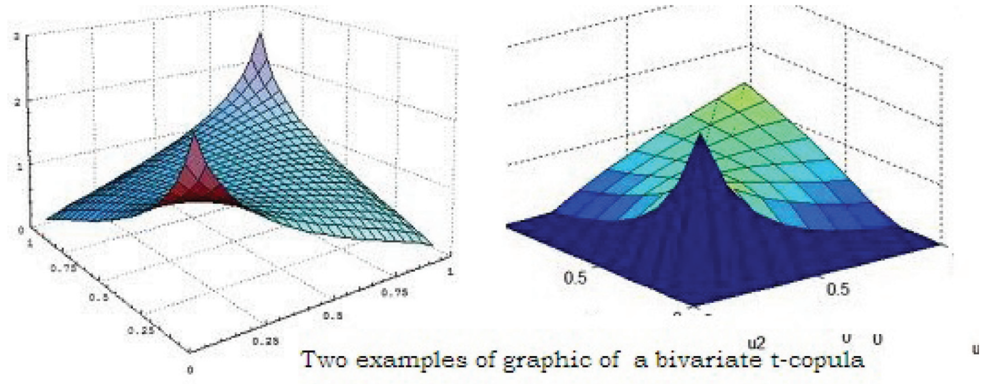

Figure 2. Two examples of graphic of a bivariate t-copula

5.3 A Spatial Conditional Measure for Bivariate Logistic Process

In MEV analysis, many high dimensional models derive from a symmetrical or non symmetrical extension of logistic model.

Corollary 11 *If the density generator is given by* $g(t) = \frac{e^{-t}}{(1+e^{-t})^2}$*, then, the elliptical vector* X *is the multivariate logistic. In particular, for the bivariate case with common parameter* θ*, the spatial conditional measure* D_θ *is given by*

$$D_\theta(t) = D_\theta(t) = \frac{t}{1+t}\left[\left(1+t^{-\theta}\right)^{\frac{1}{\theta}} - 1\right] \text{ with } \theta > 0. \tag{22}$$

Proof. By replacing the density generator in (4) by $g(t) = \frac{e^{-t}}{(1+e^{-t})^2}$ it follows that the density fonction of the process is logistical density

$$\varphi(x_1, ..., x_n) = \frac{c_n}{\sqrt{\det \Sigma}} \frac{\exp\left[\frac{-1}{2}(x-\mu)^t \Sigma^{-1}(x-\mu)\right]}{\left(1+\exp\left[\frac{-1}{2}(x-\mu)^t \Sigma^{-1}(x-\mu)\right]\right)^2};$$

where the normalizing c_n calculated in (5) gives $c_n = (2\pi)^{-n/2}\left[\sum(-1)^{j-1} j^{1-n/2}\right]^{-1}$ (see Zinoviy et al., 2003). The multivariate Pickands dependence function is

$$A_\theta(t_1, ...t_{m-1}) = \left[\sum_{i=1}^{m-1} t_i^\theta + \left(1-\sum_{i=1}^{m-1} t_i\right)^\theta\right]^{\frac{1}{\theta}} ;(t_1, ...t_{m-1}) \in S_m \tag{23}$$

Particularly in bivariate case, we easily obtain the formula (22) by replacing the relation (23) in (19). □

6. Conclusion and Discussion

In this study, we have investigated about properties of multivariate copulas associated to stochastic elliptical processes. Specifically, we have constructed a dependence measure to piece together the dependence of stochastic elliptical margins in a non-spatial case. Then, the time parametric conditional copula has also been shown to be a spatial extremal copula under the assumption of regularly varying of the process. Applications have been made by calculating the main expressions of the spatial measure for the three main families of elliptical family.

These results differ from the previous characterizations of elliptical copulas because they focus both on spatial and non-spatial analysis and also with extension to high dimensional settings. They characterize each of the three most known models of elliptical families by making the relation between the generator, the pickands measure and the spatial measure.

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