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Evaluating the Partial Derivatives of Four Types of Two-Variables Functions

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Abstract This article uses the mathematical software Maple for the auxiliary tool to study the partial differential problems of four types of two-variables functions. We can obtain the infinite series forms of any order partial derivatives of these two-variables functions by using differentiation term by term theorem, and hence greatly reduce the difficulty of calculating their higher order partial derivative values. In addition, we provide some examples to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying our answers by using Maple.

Keywords: *partial derivatives, infinite series forms, differentiation term by term theorem, Maple*

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1. Introduction

The computer algebra system (CAS) has been widely employed in mathematical and scientific studies. The rapid computations and the visually appealing graphical interface of the program render creative research possible. Maple possesses significance among mathematical calculation systems and can be considered a leading tool in the CAS field. The superiority of Maple lies in its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. In addition, through the numerical and symbolic computations performed by Maple, the logic of thinking can be converted into a series of instructions. The computation results of Maple can be used to modify our previous thinking directions, thereby forming direct and constructive feedback that can aid in improving understanding of problems and cultivating research interests.

In calculus and engineering mathematics curricula, the evaluation and numerical calculation of the partial derivatives of multivariable functions are important. For example, Laplace equation, wave equation, as well as other important physical equations are involved the partial derivatives. On the other hand, calculating the q-th order partial derivative value of a multivariable function at some point, in general, needs to go through two procedures: firstly determining the q-th order partial derivative of this function, and then taking the point into the q-th order partial derivative. These two procedures will make us be faced with increasingly complex calculations when calculating higher order partial derivative values (i.e. q is large), and hence to obtain the answers by manual calculations is not easy. In this paper, we study the partial differential problem of the following four types of two-variables functions

$$f(x,y)=\left(\sqrt{a^2x^2+b^2y^2}\right)^p\times\left[\cos\left(p\tan^{-1}\frac{by}{ax}\right)\sin ax\cosh by-\sin\left(p\tan^{-1}\frac{by}{ax}\right)\cos ax\sinh by\right] \tag{1}$$

$$g(x,y)=\left(\sqrt{a^2x^2+b^2y^2}\right)^p\times\left[\cos\left(p\tan^{-1}\frac{by}{ax}\right)\cos ax\sinh by+\sin\left(p\tan^{-1}\frac{by}{ax}\right)\sin ax\cosh by\right] \tag{2}$$

$$h(x,y)=\left(\sqrt{a^2x^2+b^2y^2}\right)^p\times\left[\cos\left(p\tan^{-1}\frac{by}{ax}\right)\cos ax\cosh by+\sin\left(p\tan^{-1}\frac{by}{ax}\right)\sin ax\sinh by\right] \tag{3}$$

$$u(x,y)=\left(\sqrt{a^2x^2+b^2y^2}\right)^p\times\left[-\cos\left(p\tan^{-1}\frac{by}{ax}\right)\sin ax\sinh by+\sin\left(p\tan^{-1}\frac{by}{ax}\right)\cos ax\cosh by\right] \tag{4}$$

where a, b are real numbers, $a\neq 0$, and p is an integer. We can obtain the infinite series forms of any order partial derivatives of these four types of two-variables functions using differentiation term by term theorem; these are the major results of this study (i.e., Theorems 1-4), and hence greatly reduce the difficulty of calculating their higher order partial derivative values. The study of related partial differential problems can refer to [1-13]. In addition, we provide some examples to do calculation practically. The research methods adopted in this paper involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. Therefore, Maple provides insights and guidance regarding problem-solving methods.

2. Main Results

Firstly, we introduce some notations and formulas used in this paper.

2.1. Notations

2.1.1. Let $z = a + ib$ be a complex number, where $i = \sqrt{-1}$, a, b are real numbers. We denote a the real part of z by $\mathrm{Re}(z)$, and b the imaginary part of z by $\mathrm{Im}(z)$.

2.1.2. Suppose m, n are non-negative integers. For the two-variables function $f(x, y)$, its n-times partial derivative with respect to x, and m-times partial derivative with respect to y, forms a $m+n$-th order partial derivative, and denoted by $\dfrac{\partial^{m+n} f}{\partial y^m \partial x^n}(x, y)$.

2.1.3. Suppose r is any real number, m is any positive integer. Define $(r)_m = r(r-1)\cdots(r-m+1)$, and $(r)_0 = 1$.

2.2. Formulas

2.2.1. Euler's Formula

$e^{i\phi} = \cos\phi + i\sin\phi$, where ϕ is any real number.

2.2.2. DeMoivre's Formula

$(\cos\phi + i\sin\phi)^p = \cos p\phi + i\sin p\phi$, where p is any integer, and ϕ is any real number.

2.2.3. ([14])

$\sin(u + iv) = \sin u \cosh v + i\cos u \sinh v$, where u, v are real numbers.

2.2.4. ([14])

$\cos(u + iv) = \cos u \cosh v - i\sin u \sinh v$, where u, v are real numbers.

2.2.5. Taylor Series Expression of Complex Sine Function

$\sin z = \sum_{k=0}^{\infty} \dfrac{(-1)^k}{(2k+1)!} z^{2k+1}$, where z is any complex number.

2.2.6. Taylor Series Expression of Complex Cosine Function

$\cos z = \sum_{k=0}^{\infty} \dfrac{(-1)^k}{(2k)!} z^{2k}$, where z is any complex number.

Next, we introduce an important theorem used in this study.

2.3. Differentiation Term by Term Theorem ([15])

For all non-negative integers k, if the functions $g_k : (a, b) \to R$ satisfy the following three conditions : (i) there exists a point $x_0 \in (a, b)$ such that $\sum_{k=0}^{\infty} g_k(x_0)$ is convergent, (ii) all functions $g_k(x)$ are differentiable on open interval (a, b), (iii) $\sum_{k=0}^{\infty} \dfrac{d}{dx} g_k(x)$ is uniformly convergent on (a, b). Then $\sum_{k=0}^{\infty} g_k(x)$ is uniformly convergent and differentiable on (a, b). Moreover, its derivative $\dfrac{d}{dx} \sum_{k=0}^{\infty} g_k(x) = \sum_{k=0}^{\infty} \dfrac{d}{dx} g_k(x)$.

Before deriving the first major result in this study, we need a lemma.

2.4. Lemma

Suppose α, β are real numbers, $\alpha > 0$, and p is an integer. Then

$$(\alpha + i\beta)^p = \left(\sqrt{\alpha^2 + \beta^2}\right)^p \left[\cos\left(p\tan^{-1}\frac{\beta}{\alpha}\right) + i\sin\left(p\tan^{-1}\frac{\beta}{\alpha}\right)\right] \tag{5}$$

$$= \left(\sqrt{\alpha^2 + \beta^2}\right)^p \exp\left(ip\tan^{-1}\frac{\beta}{\alpha}\right) \tag{6}$$

Proof $(\alpha + i\beta)^p$

$$= \left[\sqrt{\alpha^2 + \beta^2}\left(\frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} + i\frac{\beta}{\sqrt{\alpha^2 + \beta^2}}\right)\right]^p$$

$$= \left(\sqrt{\alpha^2 + \beta^2}\right)^p (\cos\phi + i\sin\phi)^p$$

(where $\phi = \tan^{-1}\dfrac{\beta}{\alpha}$)

$$= \left(\sqrt{\alpha^2 + \beta^2}\right)^p \left[\cos\left(p\tan^{-1}\frac{\beta}{\alpha}\right) + i\sin\left(p\tan^{-1}\frac{\beta}{\alpha}\right)\right]$$

(By DeMoivre's formula)

$$= \left(\sqrt{\alpha^2 + \beta^2}\right)^p \exp\left(ip\tan^{-1}\frac{\beta}{\alpha}\right)$$

(By Euler's formula)

In the following, we determine the infinite series forms of any order partial derivatives of the two-variables function (1).

2.5. Theorem 1

Suppose a, b are real numbers, $a \neq 0$, p is an integer, and m, n are non-negative integers. If the domain of the two-variables function

$$f(x,y) = \left(\sqrt{a^2x^2+b^2y^2}\right)^p \times$$
$$\left[\cos\left(p\tan^{-1}\frac{by}{ax}\right)\sin ax\cosh by - \sin\left(p\tan^{-1}\frac{by}{ax}\right)\cos ax\sinh by\right]$$

is $\left\{(x,y)\in R^2 \,\middle|\, ax>0\right\}$. Then the $m+n$-th order partial derivative of $f(x,y)$,

$$\frac{\partial^{m+n}f}{\partial y^m\partial x^n}(x,y)$$
$$= a^nb^m\sum_{k=0}^{\infty}\frac{(-1)^k(2k+p+1)_{m+n}}{(2k+1)!}\left(\sqrt{a^2x^2+b^2y^2}\right)^{2k+p+1-m-n}\times \quad (7)$$
$$\cos\left[(2k+p+1-m-n)\tan^{-1}\frac{by}{ax}+\frac{m\pi}{2}\right]$$

Proof Let $z = ax+iby$, then

$$z^p\sin z$$
$$=(ax+iby)^p\sin(ax+iby)$$
$$=\left(\sqrt{a^2x^2+b^2y^2}\right)^p\begin{bmatrix}\cos\left(p\tan^{-1}\frac{by}{ax}\right)\\ +i\sin\left(p\tan^{-1}\frac{by}{ax}\right)\end{bmatrix}\times \quad (8)$$
$$(\sin ax\cosh by+i\cos ax\sinh by)$$

(By (5) and Formula 2.2.3.)
Therefore,

$$f(x,y)$$
$$=\mathrm{Re}(z^p\sin z)$$
$$=\mathrm{Re}\left[(ax+iby)^p\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}(ax+iby)^{2k+1}\right]$$

(By Formula 2.2.5.)

$$=\mathrm{Re}\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}(ax+iby)^{2k+p+1}\right] \quad (9)$$

Using differentiation term by term theorem, differentiating n-times with respect to x, and m-times with respect to y on both sides of (9), we have

$$\frac{\partial^{m+n}f}{\partial y^m\partial x^n}(x,y)$$
$$=a^nb^m\,\mathrm{Re}\left[i^m\sum_{k=0}^{\infty}\frac{(-1)^k(2k+p+1)_{m+n}}{(2k+1)!}(ax+iby)^{2k+p+1-m-n}\right]$$
$$=a^nb^m\sum_{k=0}^{\infty}\frac{(-1)^k(2k+p+1)_{m+n}}{(2k+1)!}\mathrm{Re}[i^m(ax+iby)^{2k+p+1-m-n}]$$
$$=a^nb^m\sum_{k=0}^{\infty}\frac{(-1)^k(2k+p+1)_{m+n}}{(2k+1)!}\left(\sqrt{a^2x^2+b^2y^2}\right)^{2k+p+1-m-n}\times$$
$$\cos\left[(2k+p+1-m-n)\tan^{-1}\frac{by}{ax}+\frac{m\pi}{2}\right]$$

(Using (6))

Next, we find the infinite series forms of any order partial derivatives of the two-variables function (2).

2.6. Theorem 2

If the assumptions are the same as Theorem 1. Suppose the domain of the two-variables function

$$g(x,y) = \left(\sqrt{a^2x^2+b^2y^2}\right)^p \times$$
$$\left[\cos\left(p\tan^{-1}\frac{by}{ax}\right)\cos ax\sinh by + \sin\left(p\tan^{-1}\frac{by}{ax}\right)\sin ax\cosh by\right]$$

$\left\{(x,y)\in R^2 \,\middle|\, ax>0\right\}$. Then the $m+n$-th order partial derivative of $g(x,y)$,

$$\frac{\partial^{m+n}g}{\partial y^m\partial x^n}(x,y)$$
$$= a^nb^m\sum_{k=0}^{\infty}\frac{(-1)^k(2k+p+1)_{m+n}}{(2k+1)!}\left(\sqrt{a^2x^2+b^2y^2}\right)^{2k+p+1-m-n}\times$$
$$\sin\left[(2k+p+1-m-n)\tan^{-1}\frac{by}{ax}+\frac{m\pi}{2}\right]$$
(10)

Proof

$$g(x,y)$$
$$=\mathrm{Im}(z^p\sin z)$$

(By (8))

$$=\mathrm{Im}\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}(ax+iby)^{2k+p+1}\right] \quad (11)$$

Thus, by differentiation term by term theorem, we obtain

$$\frac{\partial^{m+n}g}{\partial y^m\partial x^n}(x,y)$$
$$= a^nb^m\sum_{k=0}^{\infty}\frac{(-1)^k(2k+p+1)_{m+n}}{(2k+1)!}\left(\sqrt{a^2x^2+b^2y^2}\right)^{2k+p+1-m-n}\times$$
$$\sin\left[(2k+p+1-m-n)\tan^{-1}\frac{by}{ax}+\frac{m\pi}{2}\right]$$

In the following, we determine the infinite series forms of any order partial derivatives of the two-variables function (3).

2.7. Theorem 3

If the assumptions are the same as Theorem 1. Suppose the domain of the two-variables function

$$h(x,y) = \left(\sqrt{a^2x^2+b^2y^2}\right)^p \times$$
$$\left[\cos\left(p\tan^{-1}\frac{by}{ax}\right)\cos ax\cosh by + \sin\left(p\tan^{-1}\frac{by}{ax}\right)\sin ax\sinh by\right]$$

is $\left\{(x,y)\in R^2 \,\middle|\, ax>0\right\}$. Then the $m+n$-th order partial derivative of $h(x,y)$,

$$\frac{\partial^{m+n}h}{\partial y^m \partial x^n}(x,y)$$
$$= a^n b^m \sum_{k=0}^{\infty} \frac{(-1)^k (2k+p)_{m+n}}{(2k)!} \left(\sqrt{a^2x^2+b^2y^2}\right)^{2k+p-m-n} \times \tag{12}$$
$$\cos\left[(2k+p-m-n)\tan^{-1}\frac{by}{ax}+\frac{m\pi}{2}\right]$$

Proof Let $z = ax+iby$, then

$$z^p \cos z$$
$$= (ax+iby)^p \cos(ax+iby)$$
$$= \left(\sqrt{a^2x^2+b^2y^2}\right)^p \left[\cos\left(p\tan^{-1}\frac{by}{ax}\right)+i\sin\left(p\tan^{-1}\frac{by}{ax}\right)\right]\times$$
$$(\cos ax\cosh by - i\sin ax\sinh by) \tag{13}$$

(By (5) and Formula 2.2.4.)
Thus,

$$h(x,y)$$
$$= \mathrm{Re}(z^p \cos z)$$
$$= \mathrm{Re}\left[(ax+iby)^p \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}(ax+iby)^{2k}\right]$$

(By Formula 2.2.6.)

$$= \mathrm{Re}\left[\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}(ax+iby)^{2k+p}\right] \tag{14}$$

Also, by differentiation term by term theorem, we have

$$\frac{\partial^{m+n}h}{\partial y^m \partial x^n}(x,y)$$
$$= a^n b^m \sum_{k=0}^{\infty} \frac{(-1)^k (2k+p)_{m+n}}{(2k)!} \left(\sqrt{a^2x^2+b^2y^2}\right)^{2k+p-m-n} \times$$
$$\cos\left[(2k+p-m-n)\tan^{-1}\frac{by}{ax}+\frac{m\pi}{2}\right]$$

Finally, we obtain the infinite series forms of any order partial derivatives of the two-variables function (4).

2.8. Theorem 4

Let the assumptions be the same as Theorem 1. If the domain of the two-variables function

$$u(x,y) = \left(\sqrt{a^2x^2+b^2y^2}\right)^p \times$$
$$\left[-\cos\left(p\tan^{-1}\frac{by}{ax}\right)\sin ax\sinh by + \sin\left(p\tan^{-1}\frac{by}{ax}\right)\cos ax\cosh by\right]$$

is $\left\{(x,y)\in R^2 \middle| ax>0\right\}$. Then the m + n-th order partial derivative of u (x, y),

$$\frac{\partial^{m+n}u}{\partial y^m \partial x^n}(x,y)$$
$$= a^n b^m \sum_{k=0}^{\infty} \frac{(-1)^k (2k+p)_{m+n}}{(2k)!} \left(\sqrt{a^2x^2+b^2y^2}\right)^{2k+p-m-n} \times$$
$$\sin\left[(2k+p-m-n)\tan^{-1}\frac{by}{ax}+\frac{m\pi}{2}\right] \tag{15}$$

Proof $u(x,y)$

$$= \mathrm{Im}(z^p \cos z)$$

(By (13))

$$= \mathrm{Im}\left[\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}(ax+iby)^{2k+p}\right] \tag{16}$$

Thus, by differentiation term by term theorem,

$$\frac{\partial^{m+n}u}{\partial y^m \partial x^n}(x,y)$$
$$= a^n b^m \sum_{k=0}^{\infty} \frac{(-1)^k (2k+p)_{m+n}}{(2k)!} \left(\sqrt{a^2x^2+b^2y^2}\right)^{2k+p-m-n} \times$$
$$\sin\left[(2k+p-m-n)\tan^{-1}\frac{by}{ax}+\frac{m\pi}{2}\right]$$

3. Examples

In the following, for the partial differential problem of the four types of two-variables functions in this study, we propose four examples and use Theorems 1-4 to determine the infinite series forms of any order partial derivatives of these functions, and evaluate some of their higher order partial derivative values. In addition, we employ Maple to calculate the approximations of these higher order partial derivative values and their solutions for verifying our answers.

3.1. Example 1

Suppose the domain of the two-variables function

$$f(x,y) = \left(\sqrt{9x^2+4y^2}\right)^3 \times$$
$$\left[\cos\left(3\tan^{-1}\frac{2y}{3x}\right)\sin 3x\cosh 2y - \sin\left(3\tan^{-1}\frac{2y}{3x}\right)\cos 3x\sinh 2y\right] \tag{17}$$

is $\left\{(x,y)\in R^2 \middle| x>0\right\}$. By (7), we obtain any m + n-th order partial derivative of $f(x, y)$,

$$\frac{\partial^{m+n}f}{\partial y^m \partial x^n}(x,y)$$
$$= 3^n 2^m \sum_{k=0}^{\infty} \frac{(-1)^k (2k+4)_{m+n}}{(2k+1)!} \left(\sqrt{9x^2+4y^2}\right)^{2k+4-m-n} \times$$
$$\cos\left[(2k+4-m-n)\tan^{-1}\frac{2y}{3x}+\frac{m\pi}{2}\right] \tag{18}$$

for all $x>0$.

Thus, we can evaluate the 12-th order partial derivative value of $f(x,y)$ at $(4,-1)$,

$$\frac{\partial^{12} f}{\partial y^7 \partial x^5}(4,-1)$$
$$= 3^5 2^7 \sum_{k=0}^{\infty} \frac{(-1)^k (2k+4)_{12}}{(2k+1)!} \left(\sqrt{148}\right)^{2k-8} \sin\left[(2k-8)\tan^{-1}\frac{-1}{6}\right] \tag{19}$$

Next, we use Maple to verify the correctness of (19).

```
>f:=(x,y)->(sqrt(9*x^2+4*y^2))^3*(cos(3*arctan((2*y)/
(3*x)))*sin(3*x)*cosh(2*y)-sin(3*arctan((2*y)/(3*x)))
*cos(3*x)*sinh(2*y));
>evalf(D[1$5,2$7](f)(4,-1),22);
```

$$7.014059334657999668 \cdot 10^8$$

```
>evalf(3^5*2^7*sum((-1)^k*product(2*k+4-j,j=0..11)/(2*
k+1)!*(sqrt(148))^(2*k-8)*sin((2*k-8)*arctan(-1/6)),k=0..
infinity),22);
```

$$7.014059334657999962 \cdot 10^8$$

3.2. Example 2

If the domain of the two-variables function

$$g(x,y) = \left(\sqrt{3x^2+5y^2}\right)^7 \times \left[\begin{array}{l} \cos\left(7\tan^{-1}\frac{\sqrt{5}y}{\sqrt{3}x}\right)\cos\sqrt{3}x\sinh\sqrt{5}y \\ +\sin\left(7\tan^{-1}\frac{\sqrt{5}y}{\sqrt{3}x}\right)\sin\sqrt{3}x\cosh\sqrt{5}y \end{array}\right] \tag{20}$$

is $\left\{(x,y)\in R^2 \,\middle|\, x>0\right\}$. Using (10), we obtain

$$\frac{\partial^{m+n} g}{\partial y^m \partial x^n}(x,y)$$
$$= \sqrt{3}^n \sqrt{5}^m \sum_{k=0}^{\infty} \frac{(-1)^k (2k+8)_{m+n}}{(2k+1)!} \left(\sqrt{3x^2+5y^2}\right)^{2k+8-m-n} \times \sin\left[(2k+8-m-n)\tan^{-1}\frac{\sqrt{5}y}{\sqrt{3}x}+\frac{m\pi}{2}\right] \tag{21}$$

for all $x>0$.
Thus,

$$\frac{\partial^{14} g}{\partial y^6 \partial x^8}(3,7)$$
$$= -\sqrt{3}^8 \sqrt{5}^6 \sum_{k=0}^{\infty} \frac{(-1)^k (2k+8)_{14}}{(2k+1)!} (\sqrt{272})^{2k-6} \times \sin\left[(2k-6)\tan^{-1}\frac{7\sqrt{15}}{9}\right] \tag{22}$$

```
>g:=(x,y)->(sqrt(3*x^2+5*y^2))^7*(cos(7*arctan((sqrt(
5)*y)/(sqrt(3)*x)))*cos(sqrt(3)*x)*sinh(sqrt(5)*y)+sin(7*
arctan((sqrt(5)*y)/(sqrt(3)*x)))*sin(sqrt(3)*x)*cosh(sqrt(5
)*y));
>evalf(D[1$8,2$6](g)(3,7),26);
```

$$-1.2426756924774193402078946 \cdot 10^{20}$$

```
>evalf(-sqrt(3)^8*sqrt(5)^6*sum((-1)^k*product(2*k+8-
j,j=0..13)/(2*k+1)!*(sqrt(272))^(2*k-6)*sin((2*k-6)*
arctan(7*sqrt(15)/9)),k=0..infinity),26);
```

$$-1.2426756924774193402078943 \cdot 10^{20}$$

3.3. Example 3

Let the domain of the two-variables function

$$h(x,y) = \left(\sqrt{4x^2+y^2}\right)^9 \times \left[\cos\left(9\tan^{-1}\frac{y}{2x}\right)\cos 2x\cosh y + \sin\left(9\tan^{-1}\frac{y}{2x}\right)\sin 2x\sinh y\right] \tag{23}$$

be $\left\{(x,y)\in R^2 \,\middle|\, x>0\right\}$. By (12), we obtain

$$\frac{\partial^{m+n} h}{\partial y^m \partial x^n}(x,y)$$
$$= 2^n \sum_{k=0}^{\infty} \frac{(-1)^k (2k+9)_{m+n}}{(2k)!} \left(\sqrt{4x^2+y^2}\right)^{2k+9-m-n} \times \cos\left[(2k+9-m-n)\tan^{-1}\frac{y}{2x}+\frac{m\pi}{2}\right] \tag{24}$$

for all $x>0$.
Hence,

$$\frac{\partial^{14} h}{\partial y^8 \partial x^6}(2,-5)$$
$$= 2^6 \sum_{k=0}^{\infty} \frac{(-1)^k (2k+9)_{14}}{(2k)!} (\sqrt{41})^{2k-5} \cos\left[(2k-5)\tan^{-1}\frac{-5}{4}\right] \tag{25}$$

```
>h:=(x,y)->(sqrt(4*x^2+y^2))^9*(cos(9*arctan(y/(2*x)))
*cos(2*x)*cosh(y)+sin(9*arctan(y/(2*x)))*sin(2*x)*sinh(
y));
>evalf(D[1$6,2$8](h)(2,-5),22);
```

$$-1.114341795118739401769 \cdot 10^{14}$$

```
>evalf(2^6*sum((-1)^k*product(2*k+9-j,j=0..13)/(2*k)!*(
sqrt(41))^(2*k-5)*cos((2*k-5)*arctan(-5/4)),k=0..infinity)
,22);
```

$$-1.114341795118739401755 \cdot 10^{14}$$

3.4. Example 4

If the domain of the two-variables function

$$u(x,y) = \left(\sqrt{x^2+9y^2}\right)^{11} \times \left[-\cos\left(11\tan^{-1}\frac{3y}{x}\right)\sin x\sinh 3y + \sin\left(11\tan^{-1}\frac{3y}{x}\right)\cos x\cosh 3y\right] \tag{26}$$

is $\left\{(x,y)\in R^2 \,\middle|\, x>0\right\}$. Using (15), we have

$$\frac{\partial^{m+n} u}{\partial y^m \partial x^n}(x,y)$$
$$= 3^m \sum_{k=0}^{\infty} \frac{(-1)^k (2k+11)_{m+n}}{(2k)!} \left(\sqrt{x^2+9y^2}\right)^{2k+11-m-n} \times$$

$$\sin\left[(2k+11-m-n)\tan^{-1}\frac{3y}{x}+\frac{m\pi}{2}\right] \qquad (27)$$

for all $x > 0$.

Hence,

$$\frac{\partial^{15}u}{\partial y^8 \partial x^7}(2,9)$$
$$= 3^8 \sum_{k=0}^{\infty} \frac{(-1)^k (2k+11)_{15}}{(2k)!} (\sqrt{733})^{2k-4} \sin\left[(2k-4)\tan^{-1}\frac{27}{2}\right] \qquad (28)$$

```
>u:=(x,y)->(sqrt(x^2+9*y^2))^11*(-cos(11*arctan(3*y/x)
)*sin(x)*sinh(3*y)+sin(11*arctan(3*y/x))*cos(x)*cosh(3*
y));
>evalf(D[1$7,2$8](u)(2,9),32);
```

$$-4.4733197863947791051640802320322 \cdot 10^{32}$$

```
>evalf(3^8*sum((-1)^k*product(2*k+11-j,j=0..14)/(2*k)!
*(sqrt(733))^(2*k-4)*sin((2*k-4)*arctan(27/2)),k=0..
infinity),32);
```

$$-4.4733197863947791051640802320431 \cdot 10^{32}$$

4. Conclusion

In this paper, we provide a new technique to evaluate any order partial derivatives of four types of two-variables functions. We hope this technique can be applied to solve another partial differential problems. On the other hand, the differentiation term by term theorem plays a significant role in the theoretical inferences of this study. In fact, the applications of this theorem are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. In addition, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and solve these problems by using Maple. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

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