Fractional Integro-Differential Equations of Mixed Type with Solution Operator and Optimal Controls

W.Wittayakiattilerd & A.Chonwerayuth (Corresponding author)

Department of Mathematics Faculty of Science Chulalongkorn University

Payathai Road, District Pathumwan, Bangkok, Thailand

E-mail: waichi_mum@hotmail.com, anusorn.c@chula.ac.th

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Abstract

Local and global existence and uniqueness of mild solution for the fractional integro-differential equations of mixed type with delay are proved by using a family of solution operators and the contraction mapping principle on Banach space. The Bolza optimal control problem of a corresponding controlled system is solved. The Gronwall lemma with singular and time lag is derived to be tool for obtaining a priori estimate. In addition, the application to the fractional nonlinear heat equation is shown.

Keywords: Banach fixed point theorem, Solution operator, Gronwall lemma, Fractional calculus, Delay system, Integro-differential equations

1. Introduction

In this paper, we consider fractional integro-differential equations of mixed type with delay;

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + f(t, x(t), Gx(t), Sx(t)) + B(t)u(t), & t \in I \\ x(t) = \varphi(t), & t \in [-r, 0], \end{cases} \tag{1}$$

on infinite dimensional Banach space X, where $I = [0, T]$, $0 < \alpha \leq 1$, D_t^α denote the fractional derivative in the sense of Riemann-Liouville, $f : I \times X \times X \times X \to X$ and $\varphi \in C([-r, 0], X)$ are given, A is a linear operator corresponding to a solution operator $\{T_\alpha(t)\}_{t\geq 0}$ in the Banach space X and G, S are nonlinear integral operators given by

$$Gx(t) = \int_{-r}^{t} k(t, s)g(s, x(s))ds, \quad Sx(t) = \int_0^T h(t, s)q(s, x(s))ds. \tag{2}$$

Many research groups have studied and reported on integro-differential systems and fractional differential systems. These reports include the proof of the existence and uniqueness of a classical solution of an integro-differential equation by Chonwerayuth and a portion of work on the nonlinear impulsive integro-differential equations of mixed type by Wei.W. Furthermore, in 2009, Gisele M.Mophou proved existence and uniqueness of mild solution to impulsive fractional differential equations .

The scope of our work is to extend some results of these reports starting with preliminaries, some necessary definitions and theorems for proving main results such as description of fractional calculus and some different generalized Gronwall lemmas are introduced. The proof of the existence and uniqueness of solution for system (1) without control is then shown in Section 3. Moreover, the optimal control for system (1) via the Bolza cost functional is solved and reported in Section 4,. In the last section, we apply our result to fractional nonlinear heat equation.

2. Preliminaries

Let X be a Banach space and $I = [0, T]$, some important definitions and theorems those are used in this work are given as follows.

Definition 2.1. Let $f : \mathfrak{R} \to X$ be a continuous(but not necessarily differentiable) function and let $h > 0$ denote a constant discretization span. The fractional difference of order α ($\alpha \in \mathfrak{R}^+$) of f is defined by the expression

$$\Delta^\alpha f(t) \equiv \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f[t + (\alpha - k)h], \quad \binom{\alpha}{k} = \frac{\Gamma(\alpha + 1)}{k!\Gamma(\alpha - k + 1)}$$

and its fractional derivative of order α is

$$D^\alpha f(t) \equiv \lim_{h\to 0} \frac{\Delta^\alpha f(t)}{h^\alpha}. \tag{3}$$

Definition 2.2. Assume that the function in the definition 2.1. has a Laplace 's transform. Then its fractional derivative of order α is defined by the following expression

$$D_t^\alpha f(t) \equiv \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f(s) ds. \tag{4}$$

where $0 < \alpha < 1$, and the fractional integral of order $\alpha > 0$ is defined by

$$I^\alpha f(t) \equiv \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds. \tag{5}$$

These expression are called *Riemann-Liouville definition*, in particular, let $f,\ u,\ v \in C(\Re, X)$ and w be a real value function, we obtain some properties for $0 < \alpha \leq 1$

$$D_t^\alpha [u(t)v(t)] = u(t)D_t^\alpha v(t) + v(t)D_t^\alpha u(t) \tag{6}$$

$$D_t^\alpha f(w(t)) = \frac{df(w)}{dw}.D_t^\alpha w(t) = D_w^\alpha f(w)(\frac{dw}{dt})^\alpha, \tag{7}$$

see more detail in Jumarie G.

Let X and Y be two Banach spaces, $L(X, Y)$ denote the space of bounded linear operators from X to Y. Particularly $L(X) = L(X, X)$ whose norm is denoted by $\|\cdot\|_{L(X)}$. Suppose that $r > 0$. Let $C([-r, a], X)$ be the Banach spaces of continuous functions from $[-r, a]$ to X with the usual supremum norm $\|\cdot\|_{C([-r,a],X)}$. If $a = 0$, we denote this space simply by C and its norm by $\|\cdot\|_C$. Throughout this paper, we let φ be a given continuous function, denote

$$B = \{x \in C([-r, T], X) \mid x(t) = \varphi(t) \quad \text{for} \quad -r \leq t \leq 0\} \tag{8}$$

whose moving norm is defined by $\|x_t\|_B = \sup_{-r\leq s\leq t} \|x(s)\|$. From this moving norm, we generalize Gronwall lemma with time delay as follow.

Lemma 2.3. Suppose $x \in C([-r, T], X)$ satisfied the following inequality

$$\begin{cases} \|x(t)\| \leq a + \int_0^t b(s)(t-s)^{\beta-1}\|x(s)\|ds + \int_0^t c(s)(t-s)^{\beta-1}\|x_s\|_B ds; \quad t \in I, \\ x(t) = \varphi(t); \quad -r \leq t \leq 0 \end{cases} \tag{9}$$

where $0 < \beta \leq 1$, $a \geq 0$, $b(s)$ and $c(s)$ are non-negative continuous functions. Then

$$\|x(t)\| \leq [\|\varphi\|_C + a]e^{\frac{bt^\beta}{\beta}}, \quad t \in I \quad \text{where} \quad b = \sup_{s\in I}[b(s) + c(s)].$$

Using lemma 2.3, we devise the following new generalized Gronwall lemma which is very important for our work.

Lemma 2.4. Suppose $x \in C([-r, T], X)$ satisfies the following inequality

$$\begin{cases} \|x(t)\| \leq a + b \int_0^t (t-s)^{\beta-1}\|x(s)\|ds + c \int_0^t (t-s)^{\beta-1}\|x_s\|_B ds \\ \qquad + e \int_0^t (t-s)^{\beta-1}\|x(s)\|^\gamma ds, \quad t \in I \\ x(t) = \varphi(t); \quad t \in [-r, 0] \end{cases} \tag{10}$$

where $0 < \gamma < 1$, $a, b, c, e \geq 0$ are constants. Then $\|x(t)\| \leq [\|\varphi\|_C + a + \frac{eT^\beta}{\beta}]e^{\frac{(b+c+e)t^\beta}{\beta}}$, $t \in [0, T]$.

Proof. Note that $\|x(s)\| \leq \sup_{-r\leq\tau\leq s} \|x(\tau)\| = \|x_s\|_B$, for $s \in I$ and $\|x_t\|_B$ is increasing function, then one can show that $\int_0^t (t-s)^{\beta-1}\|x_s\|_B ds$ is monotonously increasing and there is a $t_0 \in [0, T]$ such that $\|x_t\|_B \leq 1$ for all $t \in [0, t_0]$ and $\|x_t\|_B > 1$ for all $t \in (t_0, T]$. Then, by (10),

$$\begin{aligned} \|x(t)\| &\leq a + b \int_0^t (t-s)^{\beta-1}\|x(s)\|ds + c \int_0^t (t-s)^{\beta-1}\|x_s\|_B ds + e \int_0^t (t-s)^{\beta-1}\|x_s\|^\gamma ds \\ &\leq a + b \int_0^t (t-s)^{\beta-1}\|x(s)\|ds + c \int_0^t (t-s)^{\beta-1}\|x_s\|_B ds + e \int_0^{t_0} (t_0-s)^{\beta-1}\|x_s\|_B^\gamma ds \\ &+ e \int_{t_0}^t (t-s)^{\beta-1}\|x_s\|_B^\gamma ds \\ &\leq a + \frac{et_0^\beta}{\beta} + b \int_0^t (t-s)^{\beta-1}\|x(s)\|ds + (c+e) \int_0^t (t-s)^{\beta-1}\|x_s\|_B ds. \end{aligned}$$

Apply lemma 2.3 to obtain that $\|x(t)\| \leq [\|\varphi\|_C + a + \frac{et_0^\beta}{\beta}]e^{\frac{(b+c+e)t^\beta}{\beta}}$, $t \in [t_0, T]$. Therefore, we conclude that $\|x(t)\| \leq [\|\varphi\|_C + a + \frac{eT^\beta}{\beta}]e^{\frac{(b+c+e)t^\beta}{\beta}}$, $t \in [0, T]$. □

The notion of solution operator plays a basic role in this study. We now consider a closed linear operator A densely defined in a Banach Space X and give a definition for the solution operator following.

Definition 2.5. Let $A : X \to X$. For each $\alpha \in (0, 1]$, a family of bounded linear operators $\{T_\alpha(t)\}_{t\geq 0}$ on X is called a solution operator corresponding to A if it satisfies the following conditions;

1. $T_\alpha(t)$ is strongly continuous for $t \geq 0$ and $T_\alpha(0) = I$;
2. $T_\alpha(t)x \in D(A)$ for all $x \in D(A)$ and $D_t^\alpha T_\alpha(t)x = AT_\alpha(t)x = T_\alpha(t)Ax$.

3. Existence of Solutions to Fractional Integro-differential equations of mixed type

Consider the nonlinear fractional system (1),

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + f(t, x(t), Gx(t), Sx(t)), & t \in I \\ x(t) = \varphi(t), & t \in [-r, 0], \end{cases}$$

where $A : D(A) \to X$ be an operator corresponding to a solution operator $\{T_\alpha(t)\}_{t\geq 0}$ satisfying $\|T_\alpha(t)\|_{L(X)} \leq Me^{\omega t}$ for some $M \geq 1, \omega > 0$ for all $t \geq 0$, $f : I\times X\times X\times X \to X$ and $\varphi \in C([-r, T], X)$ are given functions satisfies following conditions;

(HF1) $f : I \times X \times X \times X \to X$ is uniformly continuous in t and locally Lipschitz in x, ξ, η that for every $\tau > 0$ and $\rho > 0$, there is a constant $a_f = a_f(\rho, \tau)$ such that

$$\|f(t, x_1, \xi_1, \eta_1) - f(t, x_2, \xi_2, \eta_2)\| \leq a_f[\|x_1 - x_2\| + \|\xi_1 - \xi_2\| + \|\eta_1 - \eta_2\|]$$

provided $\|x_1\|, \|x_2\|, \|\xi_1\|, \|\xi_2\|, \|\eta_1\|, \|\eta_2\| \leq \rho$ and $t \in [0, \tau]$.

(HF2) There exists $c \geq 0$ such that $\|f(t, x, \xi, \eta)\| \leq c(1 + \|x\| + \|\xi\| + \|\eta\|)$ for all $x, \xi, \eta \in X$ and $t \in I$.

First of all, we study the properties of integral operators;

$$Gx(t) = \int_{-r}^{t} k(t, s)g(s, x(s))ds, \quad Sx(t) = \int_0^T h(t, s)q(s, x(s))ds.$$

We introduce the following assumptions (HG) and (HS);

(HG1) $g : [-r, T] \times X \to X$ is measurable in t on $[-r, T]$ and locally Lipschitz in x, i.e., let $\rho > 0$, there exists a constant $L_g = L_g(\rho)$ such that

$$\|g(t, x_1) - g(t, x_2)\| \leq L_g\|x_1 - x_2\| \quad \text{provided} \quad \|x_1\|, \ \|x_2\| \leq \rho.$$

(HG2) There exists a constant a_g such that

$$\|g(t, x)\| \leq a_g(1 + \|x\|), \quad \text{for all } \ t \in [-r, T], \ x \in X.$$

(HG3) $k \in C([-r, T]^2, \mathfrak{R})$.

(HS1) $q : I \times X \to X$ is measurable in t on I and locally Lipschitz in x, i.e., let $\rho > 0$, there exists a constant $L_q = L_q(\rho)$ such that

$$\|q(t, x_1) - q(t, x_2)\| \leq L_q\|x_1 - x_2\| \quad \text{provided} \quad \|x_1\|, \ \|x_2\| \leq \rho.$$

(HS2) There exists a constant a_q and $\gamma \in (0, 1)$ such that

$$\|q(t, x)\| \leq a_q(1 + \|x\|^\gamma), \quad \text{for all } \ t \in I, \ x \in X. \tag{11}$$

(HS3) $h \in C(I^2, \mathfrak{R})$.

Using moving norm $\|\cdot\|_B$ one can verify that integral operator G and S have the following properties.

Lemma 3.1. Under the assumption (HG), the operator G has the following properties;

(1) $G : C([-r, T], X) \to C([-r, T], X)$.

(2) For $\rho > 0$, if $x_1, x_2 \in C([-r,T],X)$ and $\|x_1\|, \|x_2\| \le \rho$, then

$$\|Gx_1(t) - Gx_2(t)\| \le L_g\|k\|(T+r)\|(x_1)_t - (x_2)_t\|_B, \quad \text{for all} \ \ t \in [-r,T].$$

(3) For $x \in C([-r,T],X)$, we have $\|Gx(t)\| \le a_g(T+r)\|k\|(1+\|x_t\|_B)$, for all $t \in [-r,T]$.

Proof. (1) Let $x \in C([-r,T],X)$ and $t \in [-r,T]$. Given $\epsilon > 0$. Since $k \in C([-r,T]^2, \Re)$, there exist $\delta = \delta(\epsilon) > 0$ such that if $|t-a| < \delta$, then $|k(t,s) - k(a,s)| < \epsilon$ for all $a, s \in [-r,T]$. Let $0 < \tau < \delta$. Then

$$\begin{aligned}\|Gx(t+\tau) - Gx(t)\| =& \| \int_{-r}^{t+\tau} k(t+\tau,s)g(s,x(s))ds - \int_{-r}^{t} k(t,s)g(s,x(s))ds\| \\ \le& \int_{-r}^{t} |k(t+\tau,s) - k(t,s)|\|g(s,x(s))\|ds + \int_{t}^{t+\tau} |k(t+\tau,s)|\|g(s,x(s))\|ds \\ \le& (T+r)\epsilon a_g(1+\|x_t\|_B) + \delta(\epsilon)\|k\|(1+\|x_t\|_B).\end{aligned}$$

Since ϵ is arbitrary, $Gx \in C([-r,T],X)$.

(2) Given $\rho > 0$ and $x_1, x_2 \in C([-r,T],X)$ such that $\|x_1\|, \|x_2\| \le \rho$. Then

$$\begin{aligned}\|Gx_1(t) - Gx_2(t)\| =& \| \int_{-r}^{t} k(t,s)g(s,x_1(s))ds - \int_{-r}^{t} k(t,s)g(s,x_2(s))ds\| \\ \le& \int_{-r}^{t} |k(t,s)|\|g(s,x_1(s)) - g(s,x_2(s))\|ds \le a_g\|k\|(T+r)L_g\|(x_1)_t - (x_2)_t\|_B.\end{aligned}$$

(3) Let $x \in C([-r,T],X)$. Then $\|Gx(t)\| \le \int_{-r}^{t} |k(t,s)|\|g(s,x(s))\|ds \le \|k\|(T+r)(1+\|x_t\|_B)$, for all $t \in [-r,T]$.

□

We can similarly obtain the following lemma.

Lemma 3.2. Under the assumption (HS), the operator S has the following properties;

(1) $S : C(I,X) \to C(I,X)$.

(2) For $\rho > 0$, if $x_1, x_2 \in C(I,X)$ and $\|x_1\|, \|x_2\| \le \rho$, then

$$\|Sx_1(t) - Sx_2(t)\| \le L_q\|h\|T\|x_1 - x_2\|_{C(I,X)}, \quad \text{for all} \ \ t \in I$$

(3) For $x \in C(I,X)$, we have $\|Sx(t)\| \le a_qT\|h\|(1+\|x\|^{\gamma}_{C(I,X)})$, for all $t \in I$.

Proof. The proof is similar to the proof of the lemma 3.1. □

Recall fractional integro-differential equations of mixed type (1), let $0 < \alpha \le 1$. By using (6) and (7), if x is a solution of (1), then the $X-$value function $w(s) = T_\alpha(t-s)x(s)$ is $\alpha-$differentiable for $0 < s < t$ and

$$D^\alpha_s w(s) = T_\alpha(t-s)D^\alpha_s x(s) - AT_\alpha(t-s)x(s) = T_\alpha(t-s)f(s,x(s),Gx(s),Sx(s)). \tag{12}$$

Since f is integrable, the right hand side of (12) is integrable in the sense of Bochner and apply $w(0) = T_\alpha(t)\varphi(0)$ yields,

$$x(t) = T_\alpha(t)\varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s,x(s),Gx(s),Sx(s))ds, \ t \in I.$$

Definition 2.1. Let $x \in C([-r,t_0],X)$. If there exists a $t_0 > 0$ such that

$$\begin{cases} x(t) = T_\alpha(t)\varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s,x(s),Gx(s),Sx(s))ds, & t \in [0,t_0] \\ x(t) = \varphi(t), \quad t \in [-r,0] \end{cases} \tag{13}$$

then the system (1) is called mildly solvable on $[-r,t_0]$ and this x is called a mild solution on $[-r,t_0]$.

Lemma 3.4. (An a priori bound) If $x \in C([-r,T],X)$ is any solution of system (1) then x has an a priori bound, i.e., there exist a constant $\rho > 0$, if x is solution of (1) on $[-r,T]$ then $\|x(t)\| \le \rho$, for all $t \in [-r,T]$.

Proof. Let $x \in C([-r,T],X)$. For $t \in [0,T]$, we use $(HF2)$, lemma 3.1 and lemma 3.2, there exists a constant $\tilde{L}$ such that

$$\|f(s,x(s),Gx(s),Sx(s))\| \le \tilde{L}(1+\|x(s)\|+\|x_s\|_B+\|x(s)\|^\gamma),\quad s\in[0,T] \tag{14}$$

and

$$\begin{aligned}\|x(t)\| &\le Me^{\omega T}\|\varphi\|_C+\frac{Me^{\omega T}\tilde{L}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}(1+\|x(s)\|+\|x_s\|_B+\|x(s)\|^\gamma)ds.\\ &\le Me^{\omega T}\|\varphi\|_C+\frac{Me^{\omega T}\tilde{L}T^\alpha}{\alpha\Gamma(\alpha)}+\frac{Me^{\omega T}\tilde{L}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}(\|x(s)\|+\|x_s\|_B)ds\\ &+\frac{Me^{\omega T}\tilde{L}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|x(s)\|^\gamma ds.\end{aligned}$$

By lemma 2.4, there exists a constant $\rho>0$ such that $\|x(t)\|\le\rho$, for $t\in I$. □

The existence and uniqueness of mild solution of (1) is then proved by constructed an operator F and proved that it is a strictly contraction by the following lemmas.

For each $\tau>0$, $C^\tau\equiv C([-r,\tau],X)$ with the usual supremum norm and for $\lambda>0$, we set

$$S(\lambda,\tau)=\{y\in C^\tau|\ \max_{0\le t\le\tau}\|y(t)-\varphi(0)\|\le\lambda\ \text{ and }\ y(t)=\varphi(t),\ \ t\in[-r,0]\}.$$

Then $S(\lambda,\tau)$ is a nonempty closed convex subset of C^τ. Define $F:S(\lambda,\tau)\to C^\tau$ by

$$\begin{cases}Fy(t)=T_\alpha(t)\varphi(0)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)f(s,y(s),Gy(s),Sy(s))ds, & t\in[0,\tau]\\ Fy(t)=\varphi(t),\ \ t\in[-r,0].\end{cases} \tag{15}$$

Then the map F is bounded. Indeed, by using (14), we obtain that

$$\|Fy(t)\|\le Me^{\omega T}\|\varphi\|_C+\frac{Me^{\omega T}\tilde{L}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}(1+\|y(s)\|+\|y_s\|_B+\|y(s)\|^\gamma)ds.$$

Since $y\in C^\tau$, there is a constant $N>0$ such that $1+\|y(s)\|+\|y_s\|_B+\|y(s)\|^\gamma\le N$, so

$$\|Fy(t)\|\le Me^{\omega T}\|\varphi\|_C+\frac{Me^{\omega T}\tilde{L}NT^\alpha}{\alpha\Gamma(\alpha)}<\infty.$$

Moreover, the properties of the map F are listed as following.

Lemma 3.5. The operator F is well-defined on $S(\lambda,\tau)$ for each $\tau>0$. Moreover, there exists $\tau_0>0$ such that F maps $S(\lambda,\tau_0)$ into itself, i.e., $F(S(\lambda,\tau_0))\subseteq S(\lambda,\tau_0)$.

Proof. For $\lambda>0$ and $\tau>0$, let $\{y_n\}$ be a sequence in $S(\lambda,\tau)$ and $y\in S(\lambda,\tau)$ such that $y_n\to y$. By condition $(HF1)$, lemma 3.1 and lemma 3.2, there exists a Lipschitz constant $\widetilde{L}(\rho,\tau)>0$ such that

$$\|f(s,y_n(s),Gy_n(s),Sy_n(s))-f(s,y(s),Gy(s),Sy(s))\|\le\widetilde{L}(\rho,\tau)[\|y_n(s)-y(s)\|+\|(y_n)_s-y_s\|_B].$$

for all $s\in[0,\tau]$. Then, for $t\in[0,\tau]$

$$\begin{aligned}\|Fy_n(t)-Fy(t)\|&\le\frac{Me^{\omega T}\widetilde{L}(\rho,\tau)}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[\|y_n(s)-y(s)\|+\|(y_n)_s-y_s\|_B]ds\\ &\le\frac{Me^{\omega T}\widetilde{L}(\rho,\tau)T^\alpha}{\alpha\Gamma(\alpha)}[\|y_n-y\|_{C(I,X)}+\|(y_n)_t-y_t\|_B].\end{aligned}$$

Since $\|(y_n)_t-y_t\|_B=\sup_{0\le s\le t}\|y_n(s)-y(s)\|\le\|y_n-y\|_{C(I,X)}\to0$ as $n\to+\infty$, $\|Fy_n-Fy\|\to0$ as $n\to+\infty$. This implies that the map F is well-defined. We next show that there is a τ_0 such that F map $S(\lambda,\tau_0)$ into itself, i.e., $F(S(\lambda,\tau_0))\subseteq S(\lambda,\tau_0)$. For each $y\in S(\lambda,\tau)$, by assumptions $(HF1)$, $(HF2)$, lemma 3.1 and lemma 3.2, there exist κ, $L(\lambda,\tau)>0$ such that

$$\|f(0,y(0),Gy(0),Sy(0))\|\le\kappa(1+\|\varphi\|_C)$$

and for all $s\in[0,\tau]$

$$\begin{aligned}\|f(s,y(s),Gy(s),Sy(s)))-f(0,y(0),Gy(0),Sy(0)))\|&\le L(\lambda,\tau)[\|y(s)-\varphi(0)\|+\|y_s-y_0\|_B]\\ &\le2\lambda L(\lambda,\tau).\end{aligned}$$

We obtain,

$$\begin{aligned}\|Fy(t)-\varphi(0)\| &\le \|T_\alpha(t)\varphi(0)-\varphi(0)\| + \frac{Me^{\omega\tau}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(0,y(0),Gy(0),Sy(0))\|ds\\ &+ \frac{Me^{\omega\tau}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,y(s),Gy(s),Sy(s))-f(0,y(0),Gy(0),Sy(0))\|ds\\ &\le \max_{0\le t\le\tau}\|T_\alpha(t)\varphi(0)-\varphi(0)\| + \frac{Me^{\omega\tau}[\kappa(1+\|\varphi\|_C)+2\lambda L(\lambda,\tau)]\tau^\alpha}{\alpha\Gamma(\alpha)} \le \lambda q(\tau)\end{aligned}$$

where $q(\tau)=\frac{1}{\lambda}[\max\limits_{0\le t\le\tau}\|T_\alpha(t)\varphi(0)-\varphi(0)\| + \dfrac{Me^{\omega\tau}[\kappa(1+\|\varphi\|_C)+2\lambda L(\lambda,\tau)]\tau^\alpha}{\alpha\Gamma(\alpha)}]$.

Since $q(\tau)\to 0$ as $\tau\to 0^+$, a suitable τ_0 can be found such that $0<q(\tau_0)<1$, so we conclude that the F maps $S(\lambda,\tau_0)$ into itself. □

Theorem 3.6. Suppose (HF), (HS), (HG) holds and A is a corresponding generator to a solution operator $\{T_\alpha(t)\}_{t\ge0}$ with exponentially bound. Then there exists a $\tau_0>0$ such that the system (1) is mildly solvable on $[-r,\tau_0]$ and the mild solution is unique.

Proof. For $\tau>0$, set $S(1,\tau)=\{y\in C^\tau \mid \max\limits_{0\le t\le\tau}\|y(t)-\varphi(0)\|\le 1,\ \ y(t)=\varphi(t),\ t\in[-r,0]\}$. Then $S(1,\tau)$ is the nonempty closed convex set. Define the operator $F: S(1,\tau)\to C^\tau$ by (15). Then, by lemma 3.5, the operator F is well-defined on $S(1,\tau)$ and there exists a τ_0 such that F maps $S(1,\tau_0)$ into itself. We now only show that F is a strictly contraction on $S(1,\tau_0)$. Given $\rho>0$, let $y_1,y_2\in S(1,\tau_0)$ such that $\|y_1\|,\|y_2\|\le\rho$. By $(HF1)$, lemma 3.1, lemma 3.2 and lemma 3.5, for $0\le s\le\tau\le\tau_0$, there exists $b(\rho,\tau)>0$ such that

$$\begin{aligned}&\|f(s,y_1(s),Gy_1(s),Sy_1(s))-f(s,y_2(s),Gy_2(s),Sy_2(s))\|\\ &\qquad\le b(\rho,\tau)[\|y_1(s)-y_2(s)\|+\|(y_1)_s-(y_2)_s\|_B]\le 2b(\rho,\tau)\|y_1-y_2\|_{C([-r,\tau_0],X)}.\end{aligned}$$

Then

$$\|Fy_1(t)-Fy_2(t)\|\le \frac{2Me^{\omega\tau}b(\rho,\tau)\tau^\alpha}{\alpha\Gamma(\alpha)}\|y_1-y_2\|_{C([-r,\tau_0],X)} = p(\tau)\|y_1-y_2\|_{C([-r,\tau_0],X)}$$

where $p(\tau)=\frac{2Me^{\omega\tau}b(\rho,\tau)\tau^\alpha}{\alpha\Gamma(\alpha)}$ for all $t\in[0,\tau]$. Since $p(\tau)\to0$ as $\tau\to0^+$, a suitable $\bar\tau_0\le\tau_0$ can be found such $0<p(\bar\tau_0)<1$, so we conclude that the map F is strictly contraction. By the contraction mapping on Banach space, F has a unique fixed point $x\in S(1,\tau_0)$ such that $Fx(t)=x(t)$, i.e.,

$$\begin{cases} x(t)=T_\alpha(t)\varphi(0)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)f(s,x(s),Gx(s),Sx(s))ds, & t\in[0,\tau_0]\\ x(t)=\varphi(t), \quad t\in[-r,0]. \end{cases}$$

In other word, we say that $x(t)$ is the unique mild solution of system (1) on $[-r,\tau_0]$. □

We break the main system (1) for a moment and consider the initial value problem,

$$\begin{cases} D_t^\alpha x(t)=Ax(t)+f(t,x(t),Gx(t),Sx(t)), & t\ge t_0\\ x(t_0)=x_0, \end{cases} \tag{16}$$

where A is an operator corresponding to the solution operator $\{T_\alpha(t)\}_{t\ge0}$ and $f:[t_0,T]\times X\times X\times X\to X$ is continuous in t on $[t_0,T]$ and uniformly Lipschitz continuous on X. We have the following results.

Definition 3.7. A continuous solution x of the integral equation,

$$x(t)=T_\alpha(t-t_0)x_0+\frac{1}{\Gamma(\alpha)}\int_{t_0}^t(t-s)^{\alpha-1}T_\alpha(t-s)f(s,x(s),Gx(s),Sx(s))ds, \quad t\in[t_0,T] \tag{17}$$

will be called a mild solution of the system (16).

Theorem 3.8. Under the assumptions $(HF2)$, (HG) and (HS), if $f:[t_0,T]\times X\times X\times X\to X$ is continuous in t on $[t_0,T]$ and uniformly Lipschitz continuous (with constant L) on X then for every $x_0\in X$ the system (16) has a unique mild solution $x\in C([t_0,T],X)$. Moreover, the map $x_0\to x$ is Lipschitz from X into $C([t_0,T],X)$.

Proof. For a given $x_0 \in X$, we define a mapping $F : C([t_0, T], X) \to C([t_0, T], X)$ by

$$Fx(t) = T_\alpha(t - t_0)x_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha-1} T_\alpha(t - s) f(s, x(s), Gx(s), Sx(s))ds, \quad t \in [t_0, T]. \tag{18}$$

Then F is well-defined and bounded. For each $x, y \in C([t_0, T], X)$, it follows readily from the definition of F, lemma 3.1 and lemma 3.2 that

$$\|Fx(t) - Fy(t)\| \le M_\alpha L(t - t_0)\|x - y\|_{C([t_0,T],X)} \tag{19}$$

where M_α is a bound of $\frac{1}{\alpha\Gamma(\alpha)}\|T_\alpha(t)\|$ on $[t_0, T]$. Using (18), (19) and induction on n it follows that

$$\|F^n x(t) - F^n y(t)\| \le \frac{(M_\alpha L(t - t_0)^\alpha)^n}{n!}\|x - y\|_{C([t_0,T],X)} \tag{20}$$

whence

$$\|F^n x - F^n y\| \le \frac{(M_\alpha L T^\alpha)^n}{n!}\|x - y\|_{C([t_0,T],X)}. \tag{21}$$

For n large enough $\frac{(M_\alpha LT^\alpha)^n}{n!} < 1$ and by a well-known extension of the contraction principle, F has a unique fixed point x in $C([t_0, T], X)$. This fixed point is desired mild solution of (16).

The uniqueness of x and the Lipschitz condition of the map $x_0 \to x$ are consequences of the following argument. Let y be a mild solution of (16) on $[t_0, T]$ with the initial value y_0. Then,

$$\begin{aligned}\|x(t) - y(t)\| &\le \|T_\alpha(t - t_0)x_0 - T_\alpha(t - t_0)y_0\| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha-1}\|T_\alpha(t - s)\|\|f(s, x(s), Gx(s), Sx(s)) - f(s, y(s), Gy(s), Sy(s))\|ds \\ &\le \alpha\Gamma(\alpha)M_\alpha\|x_0 - y_0\| + M_\alpha L \int_{t_0}^{t} (t - s)^{\alpha-1}[\|x(s) - y(s)\| + \|x_s + y_s\|_B]ds\end{aligned}$$

which implies, by lemma 2.3, that

$$\|x(t) - y(t)\|_{C([t_0,T],X)} \le \alpha\Gamma(\alpha)M_\alpha e^{M_\alpha L(T - t_0)^\alpha}\|x_0 - y_0\|$$

and therefore

$$\|x - y\| \le \alpha\Gamma(\alpha)M_\alpha e^{M_\alpha L(T - t_0)^\alpha}\|x_0 - y_0\|$$

which yields both the uniqueness of x and the Lipschitz continuity of the map $x_0 \to x$. □

From the result of theorem 3.8, if f is uniformly Lipschitz, then we have the existence and uniqueness of a global mild solution for system (1). However, if we assume that f satisfies only local Lipschitz in x and uniformly continuous in t on bounded intervals, then we have the following local version of theorem 3.8.

Theorem 3.9. Assume the assumptions of theorem 3.6 are holding. Then for every $x_0 \in X$, there is a $t_{max} \le \infty$ such that the initial value problem

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + f(t, x(t), Gx(t), Sx(t)), & t > 0 \\ x(0) = x_0 \end{cases} \tag{22}$$

has a unique mild solution x on $[0, t_{max})$. Moreover, if $t_{max} < \infty$, then $\lim_{t \to t_{max}} \|x(t)\| = \infty$.

Proof. We start by showing that for every $\tau_0 \ge 0$ and $x_0 \in X$, and there exists a $\delta = \delta(\tau_0, \|x_0\|)$ such that the system (16) has a unique mild solution x on an interval $[\tau_0, \tau_0 + \delta]$ whose length δ is define by,

$$\delta(\tau_0, \|x_0\|) = \min\{1, [\frac{\|x_0\|\alpha\Gamma(\alpha)}{\rho(\tau_0)L(\rho(\tau_0), \tau_0 + 1) + N(\tau_0)}]^{1/\alpha}\} \tag{23}$$

where $L(c, t)$ is the local Lipschitz constant of f following from $(HF1)$, lemma 3.1 and lemma 3.2, $M(\tau_0) = \sup\{\|T_\alpha(t)\| \mid 0 \le t \le \tau_0 + 1\}$, $\rho(\tau_0) = 2\|x_0\|M(\tau_0)$ and $N(\tau_0) = \max\{\|f(t, 0, G0(t), S0(t))\| \mid 0 \le t \le \tau_0 + 1\}$. Indeed, Let $\tau_1 = \tau_0 + \delta$ where δ

is given by (23). Define a map F by (18) maps the ball of radius $\rho(\tau_0)$ centered at 0 of $C([\tau_0, \tau_1], X)$ into itself as a result from the following estimation,

$$\begin{aligned}\|Fx(t)\| &\le M(\tau_0)\|x_0\| + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\|T_\alpha(t-s)\|(\|f(s,x(s),Gx(s),Sx(s))\\ &\quad - f(s,0,G0(s),S0(s))\| + \|f(s,0,G0(s),S0(s))\|)ds\\ &\le M(\tau_0)\|x_0\| + \frac{M(\tau_0)\rho(\tau_0)L(\rho(\tau_0),\tau_0+1)}{\alpha\Gamma(\alpha)}(t-\tau_0)^\alpha + \frac{M(\tau_0)N(\tau_0)}{\alpha\Gamma(\alpha)}(t-\tau_0)^\alpha\\ &\le 2M(\tau_0)\|x_0\| = \rho(\tau_0), \quad \text{for all } t\in[\tau_0,\tau_1]\end{aligned}$$

where the last inequality is a consequence from the definition of τ_1. In this ball, F satisfies a uniform Lipschitz condition with constant $L = L(\rho(\tau_0), \tau_0 + 1)$ and thus in the proof of theorem 3.8, it possesses a unique fixed point x in the ball. This fixed point is the desired solution of (16) on the interval $[\tau_0, \tau_1]$.

From what we have just proved, it shows that if x is a mild solution of (22) on the interval $[0, \tau]$, it can be extended to the interval $[0, \tau + \delta]$ with $\delta > 0$ by defining on $[\tau, \tau + \delta]$, $x(t) = w(t)$ where $w(t)$ is the solution of the integral equation,

$$w(t) = T_\alpha(t-\tau)x(\tau) + \frac{1}{\Gamma(\alpha)}\int_\tau^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s,w(s),Gw(s),Sw(s))ds, \quad t\in[\tau,\tau+\delta].$$

Moreover, δ depends only on $\|x(\tau)\|$, $\rho(\tau)$ and $N(\tau)$.

Let $[0, t_{max})$ be the maximum interval of existence of mild solution x for (22). If $t_{max} < \infty$, then $\lim\limits_{t\to t_{max}} \|x(t)\| = +\infty$, indeed, if it is false, then there exists a sequence $\{t_n\}$ and $C > 0$ such that $t_n \to t_{max}$ and $\|x(t_n)\| \le C$ for all n, this implies that for each t_n near enough to t_{max}, x define on $[0, t_n]$ can be extended to $[0, t_n + \delta]$ where $\delta > 0$ is independent of t_n, hence x can be extend beyond t_{max}, this contradicts the definition of t_{max}. So if $t_{max} < \infty$, then $\lim\limits_{t\to t_{max}} \|x(t)\| = +\infty$.

To prove the uniqueness of the local mild solution of (22) we note that if y is a mild solution of (22), then on every closed interval $[0, \tau_0]$ on which both x and y exist, they coincide by the uniqueness argument given in the end of the proof of theorem 3.8. Therefore, both x and y have the same t_{max} and on $[0, t_{max})$, $x = y$. □

Theorem 3.10. If the assumptions of theorem 3.6 are holding, then the system (1) has a unique mild solution on $[-r, T]$.

Proof. Let $[-r, t_{max})$ be the maximum interval of existence of mild solution x for (1). If $t_{max} > T$, there is nothing to prove. If $t_{max} < T$, by theorem 3.9, then $\lim\limits_{t\to t_{max}} \|x(t)\| = +\infty$, contradicts with an a priori bound of solution. So the system (1) has a unique mild solution on $[-r, T]$. □

4. Existence of Optimal Controls

In this section, the existence of optimal controls of system governed by the fractional integro-differential equation (1) will be discussed.

Suppose that A is a linear operator corresponding to a solution operator $\{T_\alpha(t)\}_{t\ge0}$ and Y is another separable reflexive Banach space from which the controls u take the values. Let $U_{ad} = L_q(I, Y)$, $1 < q < \infty$ denoting the admissible controls set. Consider the following controlled system;

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + f(t,x(t),Gx(t),Sx(t)) + B(t)u(t), & t\in I\\ x(t) = \varphi(t) \quad t\in[-r,0]. \end{cases} \tag{24}$$

(HB) Suppose that $B \in L(I, L(L_q(I, Y), L_p(I, X)))$ where $1 < q < \infty$ and $p > 1/\alpha$. Then $B(\cdot)u \in L_p(I, X)$ for all $u \in U_{ad}$ and we give the definition of mild solution with respect to a control in U_{ad}.

Definition 4.1. Let $x \in C([-r, T], X)$ and $u \in U_{ad}$. If x is a solution of,

$$\begin{cases} x(t) = T_\alpha(t)\varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)[f(s,x(s),Gx(s),Sx(s)) + B(s)u(s)]ds, \ t\in I\\ x(t) = \varphi(t), \quad t\in[-r,0] \end{cases}$$

then x is said to be a mild solution with respect to (w.r.t.) u on $[-r, T]$.

Theorem 4.2. Under assumptions (HF), (HG), (HS), (HB) and A is a linear operator corresponding to a solution operator $\{T_\alpha(t)\}_{t\ge0}$ with exponentially bound. Then for every $u \in U_{ad}$, the system (24) has a unique mild solution w.r.t. u on $[-r, T]$.

Proof. Let $u \in U_{ad}$, define $\widetilde{f}(t, x(t)) = f(t, x(t), Gx(t), Sx(t)) + B(t)u(t)$ for all $x \in X$. Use the fact that $B(\cdot)u \in L_p(I, X)$ for all $u \in U_{ad}$ and use assumption (HF), lemma 3.1 and lemma 3.2, we obtain that $\widetilde{f}$ satisfies the assumption (HF). By theorem 3.10, the system (24) has a unique mild solution w.r.t. u on $[-r, T]$. □

We consider the Bolza problem (P_0): Find $(x^0, u^0) \in X \times U_{ad}$ such that

$$J(x^0, u^0) \le J(x^u, u), \quad \text{for all } u \in U_{ad} \tag{25}$$

where $J(x^u, u) = \int_0^T l(t, x^u(t), x_t^u, u(t))dt + \Phi(x^u(T))$, for short, denoting by $J(u)$ and x^u denote the mild solution of the system (24) corresponding to the control $u \in U_{ad}$.

We impose some assumptions for l, say (HL);

1) $l : I \times X \times X \times Y \to (-\infty, \infty]$ is Borel measurable and $\Phi : X \to \mathfrak{R}$ is continuous and nonnegative.

2) $l(t, \cdot, \cdot, \cdot)$ is sequentially lower semicontinuous on $X \times X \times Y$ for a.e. $t \in I$.

3) $l(t, x, y_t, \cdot)$ is convex on Y for each $x, y_t \in X$ and for a.e. $t \in I$.

4) There are $a, b \ge 0$, $c > 0$ and $\eta \in L_1(I, \mathfrak{R})$ such that $l(t, x, y_t, u) \ge \eta(t) + a\|x\| + b\|y_t\|_B + c\|u\|_Y^q$, for all $t \in I$ and all $x, y_t \in X$, $u \in U_{ad}$

A pair (x^u, u) is said to be feasible if it satisfies equation (24).

Theorem 4.3. Suppose the assumption (HL) and the assumptions of theorem 4.2 hold. Then problem (P_0) admits at least one optimal pair.

Proof. If $inf\{J(u)|u \in U_{ad}\} = +\infty$ there is nothing to prove. So we assume that $inf\{J(u)|u \in U_{ad}\} = m < +\infty$. By $(HL4)$, there exist $a, b \ge 0, c > 0$ and $\eta \in L_1(I, \mathfrak{R})$ such that $l(t, x^u, x_t^u, u) \ge \eta(t) + a\|x^u\| + b\|x_t^u\|_B + c\|u\|_Y^q$ for all feasible pair (x^u, u). Since Φ is nonnegative, we have

$$\begin{aligned} J(u) &= \int_0^T l(t, x^u(t), x_t^u, u(t))dt + \Phi(x^u(T)) \\ &\ge \int_0^T \eta(t)dt + a\int_0^T \|x^u(t)\|dt + b\int_0^T \|x_t^u\|_B dt + c\int_0^T \|u(t)\|_Y^q dt + \Phi(x^u(T)) \ge -\xi > -\infty \end{aligned}$$

for some $\xi > 0$, for all $u \in U_{ad}$. Hence $m \ge -\xi > -\infty$. By definition of minimum, there exists a minimizing sequence $\{u_n\}$ of J , that is $\lim_{n\to\infty} J(u_n) = m$ and

$$J(u_n) \ge \int_0^T \eta(t)dt + a\int_0^T \|x^{u_n}(t)\|dt + b\int_0^T \|x_t^{u_n}\|_B dt + c\int_0^T \|u_n(t)\|_Y^q dt + \Phi(x^{u_n}(T)).$$

So there exist $N_0 > 0$ and $\tilde{m} > 0$ such that $m \ge J(u_n) \ge -\tilde{m} + c\int_0^T \|u(t)\|_Y^q dt$ for all $n \ge N_0$, hence $\|u_n\|_{L_q(I,Y)}^q \le \frac{\tilde{m}+m}{c}$. This implies that u_n is contained in a bounded subset of the reflexive Banach space $L_q(I, Y)$. So u_n has a convergence subsequence relabeled as u_n and $u_n \to u^0$ for some $u^0 \in U_{ad} = L_q(I, Y)$. Let $x_n \subseteq C([-r, T], X)$ be the corresponding sequence of solutions for the integral equation;

$$\begin{cases} x_n(t) = T_\alpha(t)\varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)[f(s, x_n(s), Gx_n(s), Sx_n(s)) + B(s)u_n(s)]ds, & t \in I, \\ x_n(t) = \varphi(t) \quad t \in [-r, 0]. \end{cases}$$

From the a priori estimate, there exists a constant $\rho > 0$ such that

$$\|x_n\|_{C([-r,T],X)} \le \rho \quad \text{for all } n = 0, 1, 2, ...$$

where x^0 denote the solution corresponding to u^0, that is

$$\begin{cases} x^0(t) = T_\alpha(t)\varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)[f(s, x^0(s), Gx^0(s), Sx^0(s)) + B(s)u^0(s)]ds, & t \in I, \\ x^0(t) = \varphi(t), \quad t \in [-r, 0] \end{cases}$$

By (HF), (HG), (HS), (HL), lemma 3.1 and lemma 3.2, for every $t \in I$ there is a constant $a(\rho)$ such that

$$\begin{aligned}
\|x_n(t) - x^0(t)\| &\le \frac{Me^{\omega T}a(\rho)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[\|x_n(s) - x^0(s)\| + \|(x_n)_t - (x^0)_t\|_B]ds \\
&+ \frac{Me^{\omega T}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|B(s)u_n(s) - B(s)u^0(s)\|ds \\
&\le \frac{Me^{\omega T}a(\rho)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[\|x_n(s) - x^0(s)\| + \|(x_n)_t - (x^0)_t\|_B]ds \\
&+ \frac{Me^{\omega T}}{\Gamma(\alpha)}[\frac{(p-1)T^{(\alpha p-1)/(p-1)}}{\alpha p - 1}]^{\frac{p-1}{p}}\|B(\cdot)u_n - B(\cdot)u^0\|_{L_p(I,X)}.
\end{aligned}$$

By using lemma 2.3, we found that $\|x_n(t) - x^0(t)\| \le \tilde{M}\|B(\cdot)u_n - B(\cdot)u^0\|_{L_p(I,X)}$ where $\tilde{M}$ is a constant, is independent of u, n and t. Since B is strongly continuous, we have $\|B(\cdot)u_n - B(\cdot)u^0\|_{L_p(I,X)} \to 0$. This implies that $\|x_n - x^0\| \to 0$ in $C([-r,T],X)$. We know that $l(t, x_n(t), (x_n)_t, u_n(t))$ and Φ are nonnegative and by using $(HL2)$, $(HL3)$ and Fatou's Theorem,

$$\begin{aligned}
m = \varliminf_{n\to\infty} J(u_n) &= \varliminf_{n\to\infty}\int_0^T l(t, x_n(t), (x_n)_t, u_n(t))dt + \varliminf_{n\to\infty}\Phi(x_n(T)) \\
&\ge \int_0^T \varliminf_{n\to\infty} l(t, x_n(t), (x_n)_t, u_n(t))dt + \Phi(\varliminf_{n\to\infty} x_n(T)) \\
&= \int_0^T l(t, x^0(t), (x_n^0)_t, u^0(t))dt + \Phi(x^0(T)) = J(u^0).
\end{aligned}$$

This show that $J(u^0) = m$, i.e., $J(u^0) \le J(u)$ for all $u \in U_{ad}$. □

5. Application to Fractional Nonlinear Heat Equation

Consider the nonlinear heat equation control;

$$\begin{cases}
\frac{\partial^\alpha y(x,t)}{\partial t^\alpha} = \Delta y(x,t) + f_1(x,t,y(x,t)) + \int_{-r}^t k(t-s)g(x,s,y(x,s))ds \\
+\int_0^T h(t-s)q(x,s,y(x,s))ds + \int_\Omega B(x,\xi)u(\xi,t)d\xi, \quad (x,t) \in \bar{\Omega}\times I \\
y(x,t) = 0, \quad (x,t) \in \partial\Omega \times I \text{ and } y(x,0) = y_0(x), \quad x \in \bar{\Omega} \\
y(x,t) = \varphi(x,t), \quad (x,t) \in \bar{\Omega}\times[-r,0],
\end{cases} \tag{26}$$

where Ω is a bounded domain of $\Re^N$, $u \in L_q(\Omega \times I)$ $(1 < q < \infty)$, $k \in C([-r,T]^2, \Re)$, $h \in C(I^2, \Re)$ and $B : \bar{\Omega}\times\bar{\Omega} \to \Re$ and $\varphi : \bar{\Omega}\times[-r,0] \to \Re$ are continuous. Suppose that $f : \bar{\Omega}\times I\times\Re \to \Re$, $g : \bar{\Omega}\times[-r,T]\times\Re \to \Re$, $q : \bar{\Omega}\times I\times\Re \to \Re$, and for each $\rho > 0$ there are $L_1, L_2, L_3 > 0$ such that

$$|f(x,t,\xi) - f(x,s,\tilde{\xi})| \le L_1(|t-s| + |\xi - \tilde{\xi}|), \tag{A1}$$

$$|g(x,t,\xi) - g(x,s,\tilde{\xi})| \le L_2(|t-s| + |\xi - \tilde{\xi}|), \tag{A2}$$

$$|q(x,t,\xi) - q(x,s,\tilde{\xi})| \le L_3(|t-s| + |\xi - \tilde{\xi}|), \tag{A3}$$

provided $\|\xi\|$, $\|\tilde{\xi}\| \le \rho$ and s, $t \in I$. If we interpret $y(x,t)$ as temperature at the point $x \in \Omega$ at time t, then the initial condition $y(x,0)$ means that the temperature at the initial time $t = 0$ is prescribed. Condition $y(x,t) = 0$, $(x,t) \in \partial\Omega \times I$ means that the temperature on the boundary $\partial\Omega$ is equal to zero at any time. The function f describes an external heat sources. In this system, f and u are given. We then introduce the integral $Gy(x,t) = \int_{-r}^t k(t-s)g(x,s,y(x,s))ds$ and $Sy(x,t) = \int_0^T h(t-s)q(x,s,y(x,s))ds$, which directly impact to the system. Moreover, the system is controlled by controlling u via the sensor mapping $\int_\Omega B(x,\xi)u(\xi,t)d\xi$. Let $U_{ad} = L_q(\Omega \times I)$ be the admissible control set. We will solve the optimal problem (P_0) via the cost functional;

$$J(u) = \int_0^T\int_\Omega |y(\xi,t)|^2 d\xi dt + \int_0^T\int_\Omega\int_{-r}^0 |y(\xi,t+s)|^2 ds d\xi dt + \int_0^T\int_\Omega |u(\xi,t)|^2 d\xi dt + \Phi(y(x,T)).$$

where $\Phi : X \to \Re$ is continuous and nonnegative. Let $X = L_p(\Omega)$ $(p > 1/\alpha)$. For $t \in [-r,T]$, define $y(t) : \Omega \to X$ by

$$y(t)(x) = y(x,t) \quad \text{for all } x \in \Omega,$$

and define

$$\frac{\partial^\alpha y(x,t)}{\partial t^\alpha} = \lim_{h\to 0}\frac{\Delta_h^\alpha y(t)(x)}{h^\alpha}, \quad \text{for all } y \in X, \text{ and } \Delta_h^\alpha y(t)(x) = \sum_{k=0}^\infty (-1)^k\binom{\alpha}{k}y[t + (\alpha - k)h](x).$$

We define

$$f(t, y(t), Gy(t), Sy(t))(x) = f_1(x, t, y(x, t)) + Gy(t)(x) + S(t)(x), \tag{27}$$

$$B(t)u(t)(x) = \int_\Omega B(x, \xi)u(\xi, t)d\xi, \tag{28}$$

where

$$Gy(t)(x) = \int_{-r}^{t} k(t-s)g(x, s, y(x, s))ds, \quad Sy(t)(x) = \int_0^T h(t-s)q(x, s, y(x, s))ds.$$

Define an operator $A : X \to X$ as $Ay = \Delta y$ for all $y \in D(A)$, Δ denote the Laplacian operator on $\mathfrak{R}^N$ where $D(A)$ consists of all $C^2(\bar{\Omega})$ function vanishing on $\partial\Omega$.

Lemma 5.1. The operator $Ay = \Delta y$ is a linear operator corresponding to a solution operator $\{T_\alpha(t)\}_{t\geq 0}$ on X.

Proof. Consider the general heat equation of fractional order $0 < \alpha \leq 1$,

$$D_t^\alpha u = Au, \quad u(0, x) = f(x). \tag{29}$$

Applying the Fourier transformation, we obtain

$$D_t^\alpha \hat{u} = -|\xi|^2 \hat{u}, \quad \hat{u}(0, \xi) = \hat{f}(\xi). \tag{30}$$

By solving (30),

$$\hat{u}(\xi, t) = E_\alpha(-t^\alpha|\xi|^2)\hat{f}(\xi). \tag{31}$$

Take the inverse Fourier formula, the solution of (29) is,

$$u(t, x) = E_\alpha(t^\alpha A)f(x) = (2\pi)^{-n/2} \int_{\mathfrak{R}^n} E_\alpha(-t^\alpha|\xi|^2)\hat{f}(\xi)e^{ix\xi}d\xi \tag{32}$$

where $E_\alpha(t)$ is denoted by the Mittag-Leffler function. Set $T_\alpha(t) = E_\alpha(t^\alpha A)$. Then $T_\alpha(t)$ satisfies the conditions of definition 2.5. Therefore $A = \Delta$ is a linear operator corresponding to a solution operator $\{T_\alpha(t)\}_{t\geq 0}$ on X. □

Then by lemma 5.1 and all above, the system (26) can transform to the abstract problem as followed;

$$\begin{cases} D_t^\alpha y(t) = Ay(t) + f(t, y(t), Ky(t)) + Gy(t) + B(t)u(t), & t \in I \\ y(t) = \varphi(t), & t \in [-r, 0]. \end{cases} \tag{33}$$

Theorem 5.2. Suppose conditions $(A1)$, $(A2)$ and $(A3)$ hold. Then the control problem (P_0) for system(26) has a solution, that is there exists an admissible state-control pair (u^0, y^0) such

$$J(u^0, y^0) \leq J(u, y) \text{ for all } u \in U_{ad}.$$

Proof. We solve the control problem (P_0) for system(26) via the Cauchy abstract form (33). By using the conditions $(A1)$, $(A2)$, $(A3)$ and the cost functional J, it satisfies all the assumptions given in theorem 4.3 and theorem 3.6. Then the control problem (P_0) for system(26) has a solution. □

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