# THE BRIDGE BETWEEN THE ABSTRACT AND THE UNIMAGINABLE, A VERY ROUGH DRAFT

HARRY KIM

## 1. Introduction

1.1. **Introduction.** The purpose of this paper is to translate "Hilbert Space Operators and Quantum Mechanics" by Edward W. Packel into an undergraduate understandable paper. A common difficulty in dealing with mathematics is the abstractness and its relationship with familiar, everyday problems. The application and usefulness of mathematics is generally unfathomable, however I will elucidate a beautiful application of extremely theoretical Hilbert Spaces to Quantum Mechanics. Although quantum mechanics is not a familiar concept, the foundation of Hilbert Spaces describes the theory of quantum mechanics almost flawlessly.

## 2. Let us begin

To begin the translation of this paper, I will introduce many definitions that may be familiar concepts obfuscated by the terminology.

**Definition 2.1.** *Dense Subset*: A subset $A$ is *d*ense if $\bar{A} = X$. A subset $A$ of $X$ is dense if every $x \in X$ is the limit of a sequence of points in $A$ or is an element of $A$.

**Definition 2.2.** *Metric Space*: A *metric* is a function which defines the meaning of "distance" between two elements on a space. A *metric space* $(X, d)$ is a space $X$ with a metric $d : X \times X \to \mathbb{R}$ defined on it. In other words, if I were to ask you the distance between two points on $X$, you would get the same answer as everyone else.

A metric has 3 specific properties, $\forall x, y, z \in X$

$$d(x, y) = d(y, x)$$
$$d(x, y) \geq 0, \ d(x, y) = 0 \Leftrightarrow x = y$$
$$d(x, y) \leq d(x, z) + d(z, y)$$

e.g. The metric in hyperbolic geometry is $|dz| = \frac{2|dz|}{1-|z|^2}$

**Definition 2.3.** *Complete*: A metric space $(X,\, d)$ is complete if every Cauchy sequence in X has a limit in X. In addition, a normed vector space that is complete with respect to a metric is a Banach Space.

**Definition 2.4.** *Normed Inner Product Space*: A vector space $V$ over complex numbers where the metric $d(x, y) = ||x - y|| = \langle x - y, x - y\rangle^{1/2}$ *for all* $x, y \in V$.

Properties of the inner product is as follows, $\langle .,. \rangle : H \times H \to \mathbb{C}$, $\forall f, g, h \in H$ and $\forall \alpha, \beta \in \mathbb{C}$,

$$\langle f, g \rangle = \overline{\langle g, f \rangle} \tag{1}$$

$$\langle \alpha f + \beta g, h \rangle = \langle \alpha f, h \rangle + \langle \beta g, h \rangle \tag{2}$$

$$\langle f, f \rangle \geq 0, \text{equality iff } f \equiv 0 \tag{3}$$

The inner product of two numbers (ordinary multiplication) or vectors is equivalent to the well-known dot product or scalar product, $\langle x, y \rangle = x \cdot \bar{y}$. Suppose $f, g$ are Lebesgue integrable functions on the $[a, b]$ then

$$\langle f, g \rangle = \int_a^b f\,\bar{g},$$

if we ignore the differences between the two functions on Lebesgue measure zero. Elements of Hilbert Spaces satisfy Schwarz's (also known as Cauchy's) Inequality

$$\langle f, g \rangle \leq ||f||\,||g|| \tag{4}$$

As well as the Triangle Inequality, which can be derived from (4)

$$||f + g|| \leq ||f|| + ||g|| \tag{5}$$

With the Triangle inequality and other properties easily derived from the inner product, it can be shown that $\langle f, f \rangle = ||f||^{1/2}$ satisfies the properties of a norm and thus is indeed a metric defined by a norm.

**Theorem 2.5.** [2, I.7.6 Theorem] *Suppose the sequences $x_n \to x$ and $y_n \to y$ in a inner-product space $X$. Then $\langle x_n, y_n \rangle \to \langle x, y \rangle$. In other words, the inner product is continuous on the metric space $X \times X$.*

*Proof.*

$$\begin{aligned} \langle x_n, y_n \rangle - \langle x, y \rangle &= \langle x_n, y_n \rangle - \langle x_n, y \rangle + \langle x_n, y \rangle - \langle x, y \rangle \\ &= \langle x_n - x, y_n \rangle + \langle x, y_n - y \rangle \end{aligned}$$

By Schwarz inequality

$$|\langle x_n, y_n \rangle - \langle x, y \rangle| \leq ||x_n - x||\,||y_n|| + ||x||\,||y_n - y||$$

$y_n \to y$ so $||y_n|| \to ||y||$ and same for $x_n$. So, $||x_n - x|| \to 0$ *and* $||y_n - y|| \to= 0$. Hence, $\langle x_n, y_n \rangle \to \langle x, y \rangle$, in other words, the inner product is a continuous function. ☐

**Example 2.6.** Let

$$V = \{\{x_k\} : \{x_k\} \textit{ is a sequence of complex numbers and the series } \sum_{k=1}^{\infty} |x_k|^2 \textit{ converges in } \mathbb{R}\}.$$

Also, for every $x, y \in V$,

$$\langle x, y \rangle = \sum_{k=1}^{\infty} x_k \bar{y_k}$$

First, for all $n \in \mathbb{N}$, we use the algebraic, geometric inequality and basic properties of complex numbers and its conjugates,

$|x_n \bar{y_n}| = |x_n||y_n|$ and by the algebraic, geometric inequality, $2|x_n||y_n| \leq |x_n|^2 + |y_n|^2$.

Since this inequality is true for all $x_n$ and $y_n$,

$$\sum_{k=1}^{\infty} |x_n \bar{y_n}| \leq \frac{1}{2}[\sum_{k=1}^{\infty} |x_n|^2 + \sum_{k=1}^{\infty} |y_n|^2]$$

By the suppositions, the right hand side converges absolutely so by comparison, the left hand side converges absolutely and thus converges in $\mathbb{C}^m$. In addition, the $\langle .,. \rangle$ defined in V satisfies equations (1)-(3) due to properties of convergent series. Hence, V is a normed inner product space.

**Definition 2.7.** *Hilbert Space*: A complete normed inner product space.

**Example 2.8.** A commonly used example of a Hilbert space is the space of square Lebesgue integrable functions $L_2(\mathbb{R})$. I will use this space to exemplify many uses of Hilbert space in quantum mechanics, however it is a non-trivial proof to prove that $L_2(\mathbb{R})$ is indeed a Hilbert space.

## 3. Operators on Hilbert Spaces

A crucial link between Hilbert Spaces and Quantum Mechanics are the operators which map from one Hilbert space to another, $U : H \to H$. These operators are generally associated with observable physical quantities so we shall only consider structure preserving operators for simplicity. Such morphisms are *operators* of $H$.
A **bounded linear** operator is a linear transformation between two normed vector spaces $T : X \to Y$ (we are considering only transformations $T : H \to H$) for which $\exists M > 0$ such that $||T(h)|| \leq M||h|| \, \forall h \in H$. The norm of an operator is the minimum M so the inequality is true for all $h \in H$.
More precisely, a (linear) operator $T$ **on** Hilbert space $H$ is bounded if $\forall f, g \in H$ and $\alpha, \beta \in \mathbb{C}$, as defined in [4],

$$T : H \to H \text{ (T is defined on all of H)}$$
$$T(\alpha f + \beta g) = \alpha T(f) + \beta T(g) \text{ (T is linear)}$$
$$||T|| \equiv \sup_{||f|| \leqq 1} ||T(f)|| < \infty \text{ (T is bounded)}$$

**Example 3.1.** A couple of familiar bounded linear operators are

$$(If)(x) = f(x)$$
$$(Uf)(x) = e^{ix}(x)$$

For the rotation operator, we know that $||e^{ix} f(x)|| = ||e^{ix}|| \, ||f(x)|| \leq ||f(x)||$ so it is bounded and the identity operator is obviously bounded.

**Theorem 3.2.** [2, I.3.2 Theorem] *Let $T$ be a linear operator with domain $X$ and range $Y$ where X, Y are normed linear spaces. Then, the following statements are equivalent.*

(i) *$T$ is continuous at a point*
(ii) *$T$ is uniformly continuous on $X$*
(iii) *$T$ is bounded; i.e., there exists a number $M$ such that $\forall x \in X$*

$$||T(x)|| \leq M||x||$$

*Proof.* (*i*) *implies*(*ii*). Suppose $T$ is continuous at $x_o \in X$. Given $\epsilon > 0$, there exists a $\delta = \delta(\epsilon)$ such that

$$||T(x) - T(x_o)| < \epsilon \quad ||x - x_o|| < \epsilon$$

Let $u$ be any point in X. Then for $||u - v|| < \delta$ it follows from above and the additivity of $T$ that

$$||T(u) - T(v)|| = ||T(x_o) - T(x_o + u - v)|| < \epsilon$$

This proves (ii).

*(ii) implies (iii).* The continuity of T on X implies that there exists a $\delta > 0$ such that

$$||T(x)|| = ||T(x) - T(0)|| \leq 1 \quad ||x|| \leq \delta$$

For $x \neq 0$ in $X$, $\delta = ||\frac{\delta x}{||x||}||$, so

$$1 \geq ||T(\frac{\delta x}{||x||})|| = \delta \frac{||T(x)||}{||x||} \Rightarrow ||T(x)|| \leq \delta^{-1} ||x|| \; \forall x \in X$$

*(iii) implies (i).* The inequality $||T(x) - T(z)|| \leq M||x - z||$ *implies (i).* □

An operator $T$ is **unbounded**, as defined in [4], if,

$$T : \Omega \to H \text{ where } \Omega \text{ is a dense subset of } H$$
$$T \text{ is linear on } \Omega$$
$$||T|| \equiv \sup_{||f|| \leqq 1 \; f \in \Omega} ||T(f)|| = \infty.$$
$$T \; is \; \textbf{closed}$$

*Remark* 1. An operator is closed if the set

$$graph(T) = \{(f, Tf) \in H \times H : f \in \Omega\}$$

is a closed subspace of $H \times H$.

**Example 3.3.** Couple of important examples of unbounded operators $p$ and $q$ are

$$(p\psi)(x) = x\psi(x) \text{ (the position operator).} \tag{6}$$
$$(q\psi)(x) = -i\hbar\psi'(x) \text{ (the momentum operator).} \tag{7}$$

on suitable subsets of $L_2(\mathbb{R})$. It requires a non-trivial proof to show that these operators are dense closed and their domains dense, however to show that they are not bounded, first consider the position operator. Consider the function

$$f_k(x) = \begin{cases} 1 & : x \in [k-1, k] \\ 0 & : \quad otherwise \end{cases}$$

$||xf_k|| \nleq C||f_k||$ for all $f_k$ since $||f_k|| = 1$. For the momentum operator, there is not a constant C such that $|| - if'|| = ||f'|| \leq C||f||$ for all x if $f = 1/x$

**Definition 3.4.** *Unitary Operators*: A linear transformation $U$ that maps Hilbert spaces $H$ to $K$, $U : H \to K$, is *u*nitary if it is surjective ($U(H) = M \subseteq K$) and $||U(f)|| = ||f||, f \in H$.

**Example 3.5.** In Example 3.1 both operators are unitary operators.

**Definition 3.6.** *Unitarily Equivalent Operators*: An unitary operator U which also satisfies, for linear operators $f, g \in H_f$, $H_g$ respectively, $(U^{-1}fU)(x) = g(x)$ for any $x \in H_g$.

**Definition 3.7.** *Unitary Isomorphism*: Given Hilbert spaces $H$ and $K$, a linear mapping $U : H \to K$ is a *u*nitary isomorphism if it is bijective and preserves inner products (e.g. $\forall x, y \in X$, $\langle Ux, Uy \rangle = \langle x, y \rangle$).

**Definition 3.8.** *Self-Adjoint*: An linear operator $U$ on Hilbert space $H$ is *self-adjoint* if $\forall f, g \in domain(U)$, $\langle Uf, g\rangle = \langle f, Ug\rangle$ and $range(T - iI) = range(T + iI) = H$.

Note that for a bounded complex valued operators $U$, this implies that $U = \overline{U}^T = U^*$ (Hermition symmetric).

*Remark* 2. When an operator $U$ is bounded, then it is sufficient for self-adjointness if $\langle Ux, y\rangle = \langle x, Uy\rangle$, however it becomes a bit more complicated when the operator is unbounded. Thus the second condition is necessary a general case for all operators on $H$.

**Example 3.9.** [1, Example 5A.17C] Consider the operators $p$ and $q$ from Example 3.3. Let $H = L_2(a, b)$ where $\psi_1, \psi_2, p\psi_1, p\psi_2 \in H$, consider the difference

$$\langle p\psi_1, \psi_2\rangle - \langle \psi_1, p\psi_2\rangle = \int_a^b x\psi_1\overline{\psi_2}dt - \int_a^b \psi_1\overline{x\psi_2}dt = 0$$

since x is real on the interval. Hence, the position vector is self-adjoint.

**Example 3.10.** [1, Example 5A.17B] Now consider the momentum operator. Use the same $H = L_2(a, b)$ and a *linear manifold* (a generalization of finite vector subspaces) in $L_2(a, b)$ $M = \{\psi \in H|$ the set of points in (a,b) at which $\psi$ is not differentiable has Lebesgue measure zero, $\psi\prime \in H$, and $\psi(a) = \psi(b) = 0\}$. For $\psi_1, \psi_2 \in M$,

$$\langle q\psi_1, \psi_2\rangle - \langle \psi_1, p\psi_2\rangle = i\int_a^b \psi_1'\overline{\psi_2}dx + i\int_a^b \psi_1\overline{\psi_2'}dx = i(\psi_1\overline{\psi}_2(b) - \psi_1\overline{\psi_2}(a)) = 0.$$

Hence, the momentum vector is also self-adjoint.

## 4. The Connection with Quantum Mechanics

Now that most of the necessary machinery have been introduced, I will start to build the bridge and demonstrate the relationship between Hilbert space and quantum mechanics.
Quantum mechanics, developed in early 20th century, introduced many baffling properties such as the two-slit interference pattern which lead to the conclusion of particles having wave-like behavior as well as waves having particle-like behavior. Physicists call this the wave-particle duality. Another curious property of quantum mechanics is the probabilistic nature of tiny forms of matter. Matter, in the microscale, seemed to act completely randomly, but "macroly" it followed rules of probability, which invoked new mathematics to describe nature. A mathematician and physicist John von Neumann studied abstract generalizations of Euclidean spaces, proposed axioms for Hilbert Spaces, and used them as the mathematical foundations of quantum mechanics. A few more definitions need to be introduced from basic probability to describe quantum mechanics.

**Definition 4.1.** *Random Variable*: A function that assigns numerical values to each possible outcome of an experiment.

**Example 4.2.** Suppose our experiment is flipping a coin twice to see what the possible outcomes are. Of course the set of possible outcomes $S = \{HH, TH, HT, TT\}$ Let the random variable X represent the number of tails, which attains value from a set $P = \{0, 1, 2\}$. Thus, the random variable maps outcomes to numerical values.

*Remark* 3. Note that this random variable has discrete values 0,1,2. There are other types of random variables, continuous and mixed (both discrete and continuous).

**Definition 4.3.** *Probability Density Function*: a function $f : \mathbb{R} \to \mathbb{R}$ such that $\int_S f =$ the probability of finding the variable in the subset S $\in \mathbb{R}$.

**Example 4.4.** A Gaussian distribution is a common probability density function for continuous random variables such as salary of certain jobs in a given area code.

**Definition 4.5.** *Expectation*: The *expectation* can be interpreted as the average value of a random variable and is defined as $E_g = \int_{\mathbb{R}} xg(x)\,dx$.

*Remark* 4. The Expected Value is not necessary or usually the most probable outcome, but more the value when repetition of the experiment is taken to infinity.

**Example 4.6.** Consider the coin flipping experiment from above. The random variable is again the number of Tails so the expectation would be

$$(0 \times \frac{1}{4}) + (1 \times \frac{1}{2}) + (2 \times \frac{1}{4}) = 1.$$

*Remark* 5. The expectation for this is a possible outcome, but consider rolling a six-sided dice once; the expected value would be 3.5, which is not a possible outcome.

**Definition 4.7.** *Variance*: The *variance* can be interpreted as the deviation from the expected value $E_g$ and is defined as $D_g = \int_{\mathbb{R}} (x - E_g)^2 g(x)dx$.

*Remark* 6. Generally in physics, $(D_g)^{1/2}$ is called the *standard deviation* of the random variable from the expected value.

In quantum mechanics, the term **observables** describes quantities of a system such as energy, momentum, and position. Observables in quantum mechanics correspond to self-adjoint operators on $L^2(\mathbb{R})$ which acts on a function which is used to describe the probability of an observable (the random variable). This function is called the *wavefunction* $\psi$. The wavefunction is a function that describes the state of the system. As the state of the system changes, the outcomes of experiments done on the system changes. To go into more depth of how the wavefunction and the state of a system are related would be too much of a tangent for the purpose of this paper so we shall continue on the current path. If $T$ is a operator corresponding to some observable then,

$$E_\psi(T) \equiv \langle T\psi, \psi \rangle \tag{8}$$

The variance for an operator in quantum mechanics is defined as

$$D_\psi(T) \equiv ||(T - E_\psi(T)I)\psi||^2 = \langle (T - E_\psi(T)I)\psi, (T - E_\psi(T)I)\psi \rangle. \tag{9}$$

**Theorem 4.8.** [4, Theorem 1] *If $A$ and $B$ are self-adjoint operators on a Hilbert space $H$ and if $\psi$ is in domain($AB$) $\cap$ domain($BA$), then*

$$D_\psi(A)D_\psi(B) \geqq \frac{1}{4}|E_\psi(AB - BA)|^2$$

*Proof.*

$$
\begin{aligned}
|E_\psi(AB - BA)|^2 &= \\
|\langle (AB - BA)\psi, \psi\rangle|^2 &= \\
|\langle AB\psi, \psi\rangle - \langle BA\psi, \psi\rangle|^2 &= \\
|\langle AB\psi, \psi\rangle - \langle \psi, AB\psi\rangle|^2 &= \\
|2\Im\langle AB\psi, \psi\rangle|^2 &
\end{aligned}
$$

Note for any $a$ and $b$ in $\mathbb{R}$,

$$(AB - BA) = (A - aI)(B - bI) - (B - bI)(A - aI) =$$

letting $E_\psi A = a$ and $E_\psi B = b$, we get

$$
\begin{aligned}
\frac{1}{4}|E_\psi(AB - BA)|^2 &= \\
\frac{1}{4}|E_\psi[(A - aI)(B - bI) - (B - bI)(A - aI)] &= \\
\Im(\langle (A - aI)(B - bI)\psi, \psi\rangle)^2 &= \\
\Im(\langle (A - aI)\psi, (B - bI)\psi\rangle)^2 &\leqq \\
(Schwarz)\ ||(A - aI)\psi||^2||(B - bI)\psi||^2 &= \\
D_\psi(A)D_\psi(B) &
\end{aligned}
$$

□

The quantity $AB - BA$ from the proof is called the *commutator* of $A$ and $B$. It can be shown that the commutator of $p$ and $q$ on $L_2(\mathbb{R})$ is

$$pq - qp = -iI. \tag{10}$$

The position and momentum operators, which are associated to observables, satisfy the hypothesis of Theorem 4.8, so it can applied to $q$ and $p$ to conclude,

$$D_\psi(q)D_\psi(p) \geqq \frac{1}{4}[E_\psi(-iI)]^2 \tag{11}$$

or in more familiar terms and using the correct units of $\hbar$ as done in quantum mechanics,

$$\Delta p \Delta q \geqq \frac{\hbar}{2} \tag{12}$$

which is the famous Heisenberg Uncertainty Principle.
The commutator of Equation 10 would be useful to generalize, however the next Theorem makes it a bit more complicated.

**Theorem 4.9.** [4, Theorem 2] *There do not exist bounded operators $P$ and $Q$ on $H$ that satisfy $PQ - QP = -iI$.*

*Proof.* We can replace P with iP so the equation PQ-QP = -iI with PQ-QP = I without any loss of generality. Suppose there are bounded operators P and Q such that the equation holds. For n = 1, 2, ...

$$nQ^{n-1} = PQ^n - Q^nP \tag{13}$$

For n = 1, the equation is the initial equation PQ-QP = I which is assumed to be true and assuming it is true for n,

(14)
$$(n+1)Q^n = nQ^{(n-1)}Q + Q^n I = (PQ^n - Q^n P)Q + Q^n(PQ - QP) = PQ^{n+1} + Q^{n+1}P$$

so for n = 1, 2, ...

$$n||Q^{n-1}|| \leqq 2||P||\,||Q||\,||Q^{n-1}|| \Rightarrow n \leqq 2||P||\,||Q||\ for\ all\ n. \tag{15}$$

□

This shows that $||Q^n||$ must equal 0 for some n because P and Q are bounded operators, which implies $||Q^n|| = 0 \Rightarrow Q^n = 0 \Rightarrow Q^{n-1} = 0... \Rightarrow I = 0$ which is the contradiction so both P and Q cannot be bounded.
Although this may seem like a roadblock, there is a nice little detour that takes us in the right direction that is as follows,

**Theorem 4.10.** [4, Theorem 3] *Every self-adjoint operator T on a Hilbert space H generates a strongly continuous one-parameter group of unitary operators $e^{itT}$ on H. Conversely, every such one-parameter group is generated by a unique self-adjoint operator.*

*Proof.* The proof for this heorem is a bit above undergraduates, so like the article itself, I shall present a motivation for the theorem. If T is bounded, then using Taylor expanisons,

$$e^{it\tau} = \sum_{k=0}^{\infty} \frac{(it\tau)^k}{k!} \tag{16}$$

If T is unbounded, then the identity,

$$e^{it\tau} = \lim_{k\to\infty} (1 - \frac{it\tau}{k})^{-k} \tag{17}$$

can be generalized to obtain bounded operators. Furthermore, for a self-adjoint T and $t \in \mathbb{R}$,

$$(I - \frac{itT}{k})^{-1} \tag{18}$$

always exists and is bounded. So for every $f \in H$

$$e^{itT} f = \lim_{k\to\infty} (1 - \frac{itT}{k})^{-k} f \tag{19}$$

□

Exemplifying the converse of Theorem 4.10, consider the strongly continuous one-parameter operator, for $t \in \mathbb{R}$ and operators $U, V \in L_2(\mathbb{R})$,

$$(U(t)f)(x) = f(x+t)\ and\ (V(t)f)(x) = e^{itx} f(x).$$

With some simple manipulations, the reader can show that the generators of $U(t)$ and $V(t)$ are the position and momentum operators, respectively. $U(t) = e^{itp}$ and $V(t) = e^{itq}$.

**Example 4.11.** Consider the operator $U(t)$ from above.

$$(U'(0)f)(x) = \lim_{h\to 0} (\frac{U(h) - U(0)}{h} f)(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x).$$

Hence, $U'(0) = ip$ which implies that p is the generator of $U(t)$.

## 5. The Finale

5.1. **Conclusion.** My bridging between the abstractness of mathematics and its application in quantum mechanics began, however there is still much more to explore of the admirable advancements in quantum mechanics in the 20th century. I have shown a derivation of the Heisenberg Uncertainty Principle which is just the surface of the usefulness of mathematics and in particular Hilbert spaces. Packel does a fantastic job in his article in briefly yet thoroughly introducing quantum mechanical applications of Hilbert spaces to mathematicians and I hope I've done similar for the reader.

## References

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