GEOMETRIC GENERATORS FOR BRAID-LIKE GROUPS

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ABSTRACT. We study the problem of finding generators for the fundamental group G of a space of the following sort: one removes a family of complex hyperplanes from $\mathbb{C}^n$, or complex hyperbolic space $\mathbb{C}H^n$, or the Hermitian symmetric space for $\mathrm{O}(2,n)$, and then takes the quotient by a discrete group $P\Gamma$. The classical example is the braid group, but there are many similar "braid-like" groups that arise in topology and algebraic geometry. Our main result is that if $P\Gamma$ contains reflections in the hyperplanes nearest the basepoint, and these reflections satisfy a certain property, then G is generated by the analogues of the generators of the classical braid group. We apply this to obtain generators for G in a particular intricate example in $\mathbb{C}H^{13}$. The interest in this example comes from a conjectured relationship between this braid-like group and the monster simple group M, that gives geometric meaning to the generators and relations in the Conway-Simons presentation of $(M \times M) : 2$.

1. INTRODUCTION

The braid group was described by Fox and Neuwirth [FN] as the fundamental group of $\mathbb{C}^n$, minus the hyperplanes $x_i = x_j$, modulo the action of the group generated by the reflections across them (the symmetric group S_n). The term "braid-like" in the title is meant to suggest groups that arise by this construction, generalizing the choices of $\mathbb{C}^n$ and this particular hyperplane arrangement. Artin groups [Br] and the braid groups of finite complex reflection groups [Be] are examples. The problem we address is: find generators for groups of this sort. We are mainly interested in the case that there are infinitely many hyperplanes, for example coming from hyperplane arrangements in complex hyperbolic space $\mathbb{C}H^n$. Our specific motivation is a conjecture relating the monster finite simple group to the braid-like group associated to a certain hyperplane arrangement in $\mathbb{C}H^{13}$. By our results and those of Heckman [H], this conjecture may now be within reach. We also suggest some applications to algebraic geometry.

The general setting is the following: Let X be complex Euclidean space, or complex hyperbolic space, or the Hermitian symmetric space for an orthogonal group $\mathrm{O}(2,n)$. Let $\mathcal{M}$ be a locally finite set of complex hyperplanes in X, $\mathcal{H}$ their union, and $P\Gamma \subseteq \operatorname{Isom} X$ a discrete group preserving $\mathcal{H}$. Let $a \in X - \mathcal{H}$. Then the associated "braid-like group" means the orbifold fundamental group

$$G_a := \pi_1^{\mathrm{orb}}\big((X - \mathcal{H})/P\Gamma, a\big)$$

See section 3 for the precise definition of this. In many cases, $P\Gamma$ acts freely on $X - \mathcal{H}$, so that the orbifold fundamental group is just the ordinary fundamental group.

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Our first result describes generators for $\pi_1(X-\mathcal{H},a)$. This is a subgroup of G_a, since $X-\mathcal{H}$ is an orbifold covering space of $(X-\mathcal{H})/P\Gamma$. For $H\in\mathcal{M}$ we define in section 2 a loop $\overset{\circ}{\overline{aH}}$ that travels from a to a point $c\in X-\mathcal{H}$ very near H, encircles H once, and then returns from c to a. We pronounce the notation "a loop H". See section 2 for details and a generalization (theorem 2.1) of the following result:

Theorem 1.1. *The loops $\overset{\circ}{\overline{aH}}$, with H varying over $\mathcal{M}$, generate $\pi_1(X-\mathcal{H},a)$.*

If a is generic enough then this follows easily from stratified Morse theory [GM]. But in our applications it is very important to take a non-generic, because choosing it to have large $P\Gamma$-stabilizer can greatly simplify the analysis of $\pi_1^{\mathrm{orb}}\big((X-\mathcal{H})/P\Gamma,a\big)$. So we prove theorem 1.1 with no genericity conditions on a. This lack of genericity complicates even the definition of $\overset{\circ}{\overline{aH}}$. For example, $\overset{\circ}{\overline{aH}}$ may encircle some hyperplanes other than $\mathcal{H}$, and this difficulty cannot be avoided in any natural way. One can view theorem 1.1 as a first step toward a version of stratified Morse theory for non-generic basepoints.

Next we consider generators for G_a. In our motivating examples, $P\Gamma$ is generated by complex reflections in the hyperplanes $H\in\mathcal{M}$. (A complex reflection means an isometry of finite order >1 that pointwise fixes a hyperplane, called its mirror.) So suppose $H\in\mathcal{M}$ is the mirror of some complex reflection in $P\Gamma$. Then there is a "best" such reflection R_H, characterized by the following properties: every complex reflection in $P\Gamma$ with mirror H is a power of R_H, and R_H acts on the normal bundle of H by $\exp(2\pi i/n)$, where n is the order of R_H.

For each hyperplane H, we define in section 3 an element of G_a which is the natural analogue of the standard generators for the classical braid group. Recall that the definition of $\overset{\circ}{\overline{aH}}$ referred to a point $c\in X-\mathcal{H}$ very near H, and a circle around H based at c. We define $\mu_{a,H}$ to go from a to c as before, then along the portion of this circle from c to $R_H(c)$, then along the R_H-image of the inverse of the path from a to c. This is a path in $X-\mathcal{H}$, not a loop. But R_H sends its beginning point to its end point, so we may regard it as a loop in $(X-\mathcal{H})/P\Gamma$. So we may regard $\mu_{a,H}$ as an element of G_a. (Because a may have nontrivial stabilizer, properly speaking we must take the ordered pair $(\mu_{a,H},R_H)$ rather than just $\mu_{a,H}$; see section 3 for background on the orbifold fundamental group.)

In the case of the braid group, the standard braid generators correspond to the $\mu_{a,H}$ for which H is closest possible to a. (Here we choose the basepoint a in the Weyl chamber, equidistant from its facets.) The same holds for any Artin group. Under a certain hypothesis on the mirrors nearest a, the following theorem shows that this also holds in our more general situation. It is the source of the term "geometric generators" in our title.

Theorem 1.2. *Suppose $\mathcal{C}\subseteq\mathcal{M}$ are the hyperplanes closest to a, and that the complex reflections R_C generate $P\Gamma$, where C varies over $\mathcal{C}$. Suppose that for each $H\in\mathcal{M}-\mathcal{C}$, some power of some R_C moves a closer to p, where p is the point of H closest to a. Then the $(\mu_{a,C},R_C)$ generate $G_a=\pi_1^{\mathrm{orb}}\big((X-\mathcal{H})/P\Gamma,a\big)$.*

We mentioned that our main motivation is a conjectural relation between a particular braid-like group and the monster simple group. Here are minimal details; see section 4 for more background. We take $X=\mathbb{C}H^{13}$ and $P\Gamma$ to be a particular discrete subgroup of $\operatorname{Aut}\mathbb{C}H^{13}=PU(13,1)$ generated by complex reflections of order 3. We take $\mathcal{M}$ to be the set of mirrors of the complex reflections in $P\Gamma$. It turns out that any two mirrors are $P\Gamma$-equivalent, so the image of $\mathcal{H}$ in $X/P\Gamma$ is irreducible. The positively-oriented boundary, of a small disk transverse to a generic point of this image, determines a conjugacy class in $\pi_1^{\text{orb}}\big((X-\mathcal{H})/P\Gamma\big)$. We call the elements of this conjugacy class meridians.

Conjecture 1.3 ([A3])**.** *The quotient of $\pi_1^{\text{orb}}\big((X-\mathcal{H})/P\Gamma\big)$, by the normal subgroup generated by the squares of the meridians, is the semidirect product of $M \times M$ by $\mathbb{Z}/2$, where M is the monster simple group and $\mathbb{Z}/2$ exchanges the factors in the obvious way.*

Presumably, any proof of this will require generators and relations for $\pi_1^{\text{orb}}\big((X-\mathcal{H})/P\Gamma\big)$, which is the motivation for the current paper. In [Ba1] the second author found a point $\tau \in \mathbb{C}H^{13}$ (called $\bar{\rho}$ there), such that the set $\mathcal{C}$ of mirrors closest to τ has size 26, and showed that their complex reflections generate $P\Gamma$. Because τ has nontrivial $P\Gamma$-stabilizer, the corresponding meridians are ordered pairs $(\mu_{\tau,C}, R_C)$ rather than just bare paths $\mu_{\tau,C}$. Taking τ as our basepoint, we announce the following result, which we regard as a significant step toward conjecture 1.3.

Theorem 1.4. *The 26 meridians $(\mu_{\tau,C}, R_C)$, with C varying over the 26 mirrors closest to τ, generate $\pi_1^{\text{orb}}\big((X-\mathcal{H})/P\Gamma, \tau\big)$.*

We wish this were a corollary of theorem 1.2. Unfortunately the hypothesis of theorem 1.2 about moving τ closer to the various $p \in H$ fails badly. Instead, we first prove theorem 1.5 below, which is an analogue of theorem 1.4 with a different basepoint ρ in place of τ. Then we identify the fundamental groups based at τ and ρ by means of a path from τ to ρ, and study how the generators based at τ and ρ are related under this identification. The proof of theorem 1.5 follows that of theorem 1.2, although considerable work is required. The change-of-basepoint argument is complicated and delicate, of a different character, and of more specialized interest. Therefore it will appear separately.

The main reason we prefer theorem 1.4 to theorem 1.5, i.e., we prefer the basepoint to be τ rather than ρ, is that the 26 meridians $\mu_{\tau,C}$ are closely related to the coincidences that motivated conjecture 1.3. In particular, by [Ba2], they satisfy the braid and commutation relations specified by the incidence graph of the points and lines of $P^2(\mathbb{F}_3)$. The 26 generators in the Conway-Simons presentation of $(M \times M) : 2$ in [CS] satisfy exactly the same relations. The "deflation" relations in this presentation appear to also have a good geometric interpretation in terms of the $\mu_{\tau,C}$'s. Heckman has developed ideas [H] aimed at showing that these relations account for all relations in $\pi_1^{\text{orb}}\big((X-\mathcal{H})/P\Gamma\big)$.

Another reason to prefer τ is that ρ is not actually in $\mathbb{C}H^{13}$; rather, it is an ideal point. So extra care is required when defining the meridians "based at ρ". One can proceed as follows. There turns out to be a closed horoball A centered at ρ that misses $\mathcal{H}$; we choose any basepoint a inside A. We call the mirrors that come closest to A the "Leech mirrors". The name comes from the fact that they are indexed by the elements of (a central extension of) the complex Leech lattice Λ; in particular there are infinitely many of them. If C is a Leech mirror, let $b_C \in A$ be the point of A nearest it. Then $\mu_{a,A,C}$ is defined to be the geodesic $\overline{ab_C} \subseteq A$ followed by $\mu_{b_C,C}$ followed by $R_C(\overline{b_C a})$. See figure 1 for a picture. These are meridians in the sense of conjecture 1.3, and we call them the Leech meridians. (As before, because a may have nontrivial $P\Gamma$-stabilizer, the meridian associated to C is really the ordered pair $(\mu_{a,A,C}, R_C)$ rather than just $\mu_{a,A,C}$.)

Theorem 1.5. *The Leech meridians* $(\mu_{a,A,C}, R_C)$ *generate* $\pi_1^{\mathrm{orb}}\big((X-\mathcal{H})/P\Gamma, a\big)$.

We hope that our techniques will be useful more generally. For example, they might be used to give generators for the fundamental group of the moduli space of Enriques surfaces. Briefly, this is the quotient of the Hermitian symmetric space for $\mathrm{O}(2,10)$, minus a hyperplane arrangement, by a certain discrete group. See [Na] for the original result and [A2] for a simpler description of the arrangement. The symmetric space has two orbits of 1-dimensional cusps, one of which is misses all the hyperplanes. Taking this as the base "point", the hyperplanes nearest it are analogues of the Leech mirrors. It seems reasonable to hope that the meridians associated to these mirrors generate the orbifold fundamental group.

There are many spaces in algebraic geometry with a description $(X-\mathcal{H})/P\Gamma$ of the sort we have studied. For example, the discriminant complements of many hypersurface singularities [L1][L2], the moduli spaces of del Pezzo surfaces [ACT1] [Ko1][HL], the moduli space of curves of genus four [Ko2], the moduli spaces of smooth cubic threefolds [ACT2][LS] and fourfolds [L4], and the moduli spaces of lattice-polarized K3 surfaces [Ni][D]. The orbifold fundamental groups of these spaces are "braid-like" in the sense of this paper, and we hope that our methods will be useful in understanding them.

The paper is organized as follows. In section 2 we study the fundamental group $\pi_1(X-\mathcal{H}, a)$, in particular proving theorem 1.1. The proof relies on van Kampen's theorem. In section 2 we study $\pi_1^{\mathrm{orb}}\big((X-\mathcal{H})/P\Gamma, a\big)$, in particular proving theorem 1.2. The core of that proof is lemma 3.1, which is more general than needed for theorem 1.2. The extra generality is needed for our application to $\mathbb{C}H^{13}$. Section 4 gives background on complex hyperbolic space and the particular hyperplane arrangement referred to in conjecture 1.3 and theorems 1.4–1.5. Finally, section 5 proves theorem 1.5. Most of the proof consists of tricky calculations verifying the hypothesis of theorem 1.2 that the basepoint can be moved closer to the various points $p \in H$. In a few cases this is not possible, so we have to do additional work.

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2. Loops in arrangement complements

For the rest of the paper we fix X = one of three spaces, $\mathcal{M}$ = a locally finite set of hyperplanes in X, and $\mathcal{H}$ = their union. The precise assumption on X is that it is complex affine space with its Euclidean metric, or complex hyperbolic space, or the symmetric space for $\mathrm{O}(2,n)$. To understand the general machinery in this section and the next, it is enough to think about the affine case. In our application in section 5 we specialize to the case that X is complex hyperbolic 13-space. Most of the other potential applications mentioned in the introduction would use the $\mathrm{O}(2,n)$ case. What we need about X is that it is a simply-connected complex manifold equipped with a complete positive-definite Hermitian metric of nonpositive sectional curvature, that there is a good notion of the "complex line" containing a given nontrivial geodesic segment, and that there is a good notion of a "complex hyperplane". These lines and hyperplanes should be totally geodesic. We don't know any interesting examples besides the three listed, so we haven't attempted to formulate our results in greater generality.

For $b, c \in X$ we write $\overline{bc}$ for the geodesic segment from b to c. Now suppose $b, c \notin \mathcal{H}$. It may happen that $\overline{bc}$ meets $\mathcal{H}$, so we will define a perturbation $\widehat{bc}$ of $\overline{bc}$ in the obvious way. The notation may be pronounced "b dodge c" or "b detour c". We write $\overline{bc}^{\mathbb{C}}$ for the complex line containing $\overline{bc}$. By the local finiteness of $\mathcal{M}$, $\overline{bc}^{\mathbb{C}} \cap \mathcal{H}$ is a discrete set. Consider the path got from $\overline{bc}$ by using positively oriented semicircular detours in $\overline{bc}^{\mathbb{C}}$, around the points of $\overline{bc} \cap \mathcal{H}$, in place of the corresponding segments of $\overline{bc}$. After taking the radius of these detours small enough, the construction makes sense and the resulting homotopy class in $X - \mathcal{H}$ (rel endpoints) is radius-independent. This homotopy class is what we mean by $\widehat{bc}$.

At times we will need to speak of the "restriction of $\mathcal{H}$ at p", where p is a point of X. So we write $\mathcal{M}_p$ for the set of hyperplanes in $\mathcal{M}$ that contain p, and $\mathcal{H}_p$ for their union.

Now suppose $b \in X - \mathcal{H}$ and $H \in \mathcal{M}$. We will define a homotopy class $\overset{\multimap}{bH}$ of loops in $X - \mathcal{H}$ based at b; the notation can be pronounced "b loop H". We write p for the point of H nearest b. Let U be a ball around p that is small enough that $U \cap \mathcal{H} = U \cap \mathcal{H}_p$, and let c be a point of $(U - \mathcal{H}) \cap \overline{bp}$. Consider the circular loop in $\overline{bp}^{\mathbb{C}}$ centered at p, based at c, and traveling once around p in the positive direction. It misses $\mathcal{H}$, because under the exponential map $T_pX \to X$, the elements of $\mathcal{M}_p$ correspond to some complex hyperplanes in T_pX, while $\overline{bp}^{\mathbb{C}}$ corresponds to a complex line. And the line misses the hyperplanes except at 0, because $b \notin \mathcal{H}$. Finally, $\overset{\multimap}{bH}$ means $\widehat{bc}$ followed by this circular loop, followed by reverse($\widehat{bc}$).

Remark (Caution in the non-generic case). This definition has some possibly unexpected behavior when b is not generic. For example, take $\mathcal{M}$ to be the A_2 arrangement in $X = \mathbb{C}^2$, let H be one of the three hyperplanes, and take $b \in X - \mathcal{H}$ orthogonal to H. It is easy to see that $\overset{\multimap}{bH}$ encircles *all three hyperplanes*, not just H. Furthermore, this phenomenon cannot be avoided by any procedure that respects symmetry. To explain this we note that $\pi_1(X - \mathcal{H}, b) \cong \mathbb{Z} \times F_2$ where the first factor is generated by $\overset{\multimap}{bH}$ and the second is free on $\overset{\multimap}{bH_1}$ and $\overset{\multimap}{bH_2}$, where H_1 and H_2 are the other two hyperplanes. Let f be the isometry of X that fixes b and negates H. It exchanges H_1 and H_2. So f's action on $\pi_1(X - \mathcal{H}, b)$ fixes $\overset{\multimap}{bH}$ and swaps the other two generators. It follows that the group of fixed points of f in $\pi_1(X - \mathcal{H}, b)$ is just the first factor $\mathbb{Z}$. So any symmetry-respecting definition of $\overset{\multimap}{bH}$ must give some multiple of our definition.

The main result of this section, theorem 2.1, shows that the various $\overset{\multimap}{bH}$ generate $\pi_1(X - \mathcal{H}, b)$. But for our applications to $\mathbb{C}H^{13}$ in section 5, it will be useful to formulate the fundamental group with a "fat basepoint" A in place of b. This is because we will want to choose a boundary point of $\mathbb{C}H^{13}$ as our basepoint. Strictly speaking this is not possible, since a boundary point is not a point of $\mathbb{C}H^{13}$. So we will use a closed horoball A centered at that boundary point in place of a basepoint. For purposes of understanding the current section, the reader may take A to be a point.

We add to our standing assumptions: A is a nonempty closed convex subset of X, disjoint from $\mathcal{H}$. To avoid some minor technical issues we will also assume two additional properties. First, for every $H \in \mathcal{M}$, there is a unique point of A closest to H. Second, some group of isometries of X preserving $\mathcal{M}$ and A acts cocompactly on the boundary ∂A. Under our sectional curvature assumption on X, the first property holds automatically if A is strictly convex. The second assumption is natural in our intended applications because $\mathcal{H}$ is always invariant under an arithmetic group.

Because $A - \mathcal{H} = A$ is simply connected (even contractible), the fundamental groups of $X - \mathcal{H}$ based at any two points of A are canonically identified. So we write just $\pi_1(X - \mathcal{H}, A)$. If $c \in X$ then we define $\overline{Ac}$ and $\overset{\frown}{Ac}$ as $\overline{bc}$ and $\overset{\frown}{bc}$, where b is the point of A nearest c. Similarly, if $H \in \mathcal{M}$ then we define $\overset{\circ}{\overline{AH}}$ as $\overset{\circ}{\overline{bH}}$, where b is the point of A closest to H. We sometimes write $\overline{b,c}$ and $\overset{\frown}{b,c}$ and $\overset{\circ}{\overline{b,H}}$ for $\overline{bc}$ and $\overset{\frown}{bc}$ and $\overset{\circ}{\overline{bH}}$, and similarly for $\overline{A,c}$ and $\overset{\frown}{A,c}$ and $\overset{\circ}{\overline{A,H}}$.

Theorem 2.1 (π_1 of a ball-like set minus hyperplanes)**.** *Let B_r be the open r-neighborhood of A, where $r \in (0, \infty]$. Then $\pi_1(B_r - \mathcal{H}, A)$ is generated by the $\overset{\circ}{\overline{AH}}$'s for which $d(A, H) < r$.*

The rest of the section is devoted to the proof, beginning with two lemmas.

Lemma 2.2. *Suppose X is complex Euclidean space, every $H \in \mathcal{M}$ contains the origin 0, and $c \in X - \mathcal{H}$. Write $\frac{1}{2}X$ for the open halfspace of X that contains c and is bounded by the real orthogonal complement to $\overline{c0}$. (In the trivial case $\mathcal{M} = \emptyset$ we also assume $c \neq 0$, so that $\frac{1}{2}X$ is defined.)*

1. *If c is not orthogonal to any element of $\mathcal{M}$, then $\pi_1(X - \mathcal{H}, c)$ is generated by $\pi_1(\frac{1}{2}X - \mathcal{H}, c)$.*
2. *If c is orthogonal to some $H \in \mathcal{M}$, then $\pi_1(X - \mathcal{H}, c)$ is generated by $\pi_1(\frac{1}{2}X - \mathcal{H}, c)$ together with any element of $\pi_1(X - \mathcal{H}, c)$ having linking number ± 1 with H, for example $\overset{\circ}{\overline{cH}}$.*

Proof. (2) Write H' for the translate of H containing c. Every point of $X - \mathcal{H}$ is a nonzero scalar multiple of a unique point of $H' - \mathcal{H}$. It follows that $X - \mathcal{H}$ is the product of $H' - \mathcal{H} \subseteq \frac{1}{2}X - \mathcal{H}$ with $\mathbb{C} - \{0\}$. The map $\pi_1(X - \mathcal{H}, c) \to \mathbb{Z}$ corresponding to the projection to the second factor is the linking number with H. All that remains to prove is that $\overset{\circ}{\overline{cH}}$ has linking number 1 with H. In fact more is true: essentially by definition, this loop generates the fundamental group of one of the fibers $\mathbb{C} - \{0\}$.

(1) We define H as the complex hyperplane through 0 that is orthogonal to $\overline{c0}$. We apply the previous paragraph to $\mathcal{M}' = \mathcal{M} \cup \{H\}$ and $\mathcal{H}' = \mathcal{H} \cup H$. Using $\frac{1}{2}X - \mathcal{H} = \frac{1}{2}X - \mathcal{H}'$ yields

$$(*) \qquad \pi_1(X - \mathcal{H}', c) = \pi_1(\tfrac{1}{2}X - \mathcal{H}', c) \times \langle \overset{\circ}{\overline{cH}} \rangle = \pi_1(\tfrac{1}{2}X - \mathcal{H}, c) \times \langle \overset{\circ}{\overline{cH}} \rangle.$$

Let γ be any element of $\pi_1(X - \mathcal{H}', c)$ that is freely homotopic to the boundary of a small disk transverse to H at a generic point of H. It dies under the natural map $\pi_1(X - \mathcal{H}', c) \to \pi_1(X - \mathcal{H}, c)$. Because γ has linking number ± 1 with H, the product decomposition (*) shows that every element of $\pi_1(X - \mathcal{H}', c)$ can be written as the product of a power of γ and an element of $\pi_1(\frac{1}{2}X - \mathcal{H}, c)$. It is standard that $\pi_1(X - \mathcal{H}', c) \to \pi_1(X - \mathcal{H}, c)$ is surjective. Since this map kills γ, it must send the subgroup $\pi_1(\frac{1}{2}X - \mathcal{H}, c)$ of $\pi_1(X - \mathcal{H}', c)$ surjectively to $\pi_1(X - \mathcal{H}, c)$. □

Lemma 2.3 (π_1 of a ball-like set with a bump, minus hyperplanes)**.** *Assume $p \in X$ lies at distance $r > 0$ from A and write B for the open r-neighborhood of A. Assume U is any open ball centered at p, small enough that $U \cap \mathcal{H} = U \cap \mathcal{H}_p$.*

(1) *If no $H \in \mathcal{M}_p$ is orthogonal to $\overline{Ap}$, then $\pi_1\big((B \cup U) - \mathcal{H}, A\big)$ is generated by the image of $\pi_1(B - \mathcal{H}, A)$.*
(2) *If some $H \in \mathcal{M}_p$ is orthogonal to $\overline{Ap}$, then $\pi_1\big((B \cup U) - \mathcal{H}, A\big)$ is generated by the image of $\pi_1(B - \mathcal{H}, A)$, together with any loop of the following form $\alpha\lambda\alpha^{-1}$, for example $\overset{\circ}{\overline{AH}}$. Here α is a path in $B - \mathcal{H}$ from A to a point of $(B \cap U) - \mathcal{H}$ and λ is a loop in $U - \mathcal{H}$, based at that point and having linking number ± 1 with H.*

Proof. For uniformity, in case (1) we choose a random path α in $B - \mathcal{H}$ beginning in A and ending in $(B \cap U) - \mathcal{H}$. In both cases we write c for the final endpoint of α; without loss of generality we may suppose $c \in \overline{Ap} - \{p\}$. Van Kampen's theorem shows that $\pi_1\big((B \cup U) - \mathcal{H}, c\big)$ is generated by the images of $\pi_1(B - \mathcal{H}, c)$ and $\pi_1(U - \mathcal{H}, c)$. We claim that $\pi_1(U - \mathcal{H}, c)$ is generated by the image of $\pi_1\big((B \cap U) - \mathcal{H}, c\big)$, supplemented in case (2) by λ.

Assuming this, we move the basepoint from c into A along reverse(α). This identifies $\pi_1(B - \mathcal{H}, c)$ with $\pi_1(B - \mathcal{H}, A)$, λ with $\alpha\lambda\alpha^{-1}$, and the elements of $\pi_1\big((B \cap U) - \mathcal{H}, c\big)$ with certain loops in $B - \mathcal{H}$ based in A. It follows that $\pi_1\big((B \cup U) - \mathcal{H}, A\big)$ is generated by the image of $\pi_1(B - \mathcal{H}, A)$, supplemented in case (2) by $\alpha\lambda\alpha^{-1}$. This is the statement of the theorem.

So it suffices to prove the claim. We transfer this to a problem in the tangent space T_pX by the exponential map and its inverse (written log). So we must show that $\pi_1(\log U - \log \mathcal{H}_p, \log c)$ is generated by the image of $\pi_1\big(\log(B \cap U) - \log \mathcal{H}_p, \log c\big)$, supplemented in case (2) by $\log \lambda$. The key to this is that the vertical arrows in the following commutative diagram are homotopy equivalences.

$$\begin{array}{ccc}
\log(B \cap U) - \log \mathcal{H}_p & \longrightarrow & \log U - \log \mathcal{H}_p \\
\big\downarrow & & \big\downarrow \\
\frac{1}{2}T_pX - \log \mathcal{H}_p & \longrightarrow & T_pX - \log \mathcal{H}_p
\end{array}$$

Here $\frac{1}{2}T_pX$ is as in lemma 2.2: the open halfspace containing $\log c$ and bounded by the (real) orthogonal complement of $\log(\overline{cp}) = \overline{\log c, 0}$.

The right vertical arrow is a homotopy equivalence by a standard scaling argument. The same argument works for the left one, because the boundary of $\log B$ is tangent to the boundary of $\frac{1}{2}T_pX$ at 0. It follows that any compact subset of $\frac{1}{2}T_pX$ can be sent into $\log(B \cap U)$ by multiplying it by a sufficiently small scalar. Since scaling respects $\log \mathcal{H}_p$, this establishes a weak homotopy equivalence. Homotopy equivalence then follows from Whitehead's theorem (or one could refine the scaling argument).

By these homotopy equivalences, it suffices to show that $\pi_1(T_pX - \log \mathcal{H}_p, \log c)$ is generated by the image of $\pi_1(\frac{1}{2}T_pX - \log \mathcal{H}_p, \log c)$, supplemented in case (2) by $\log \lambda$. This is just lemma 2.2, completing the proof. $\square$

Proof of theorem 2.1. Let R be the set of $r \in (0, \infty]$ for which the conclusion of the theorem holds. By the cocompactness assumption on A, the distances $d(A, H)$ are bounded away from 0, as H varies over $\mathcal{M}$. That is, for any sufficiently small r we have $B_r \cap \mathcal{H} = \emptyset$ and therefore $r \in R$. So $R \neq \emptyset$. Since $B_{r_0} = \cup_{r < r_0} B_r$ we see that $(0, r_0) \subseteq R$ implies $(0, r_0] \subseteq R$. By a connectedness argument it suffices to show that if $r \in (0, \infty)$ and $(0, r] \subseteq R$, then $(0, r + \delta) \subseteq R$ for some $\delta > 0$.

So we fix $r \in (0, \infty)$, abbreviate B_r to B, assume the conclusion of the theorem holds for B, and define S as the "sphere" ∂B. For each $p \in S$ there is an open ball U_p centered at p such that $U_p \cap \mathcal{H} = U_p \cap \mathcal{H}_p$. By the cocompactness hypothesis, there exists $\delta > 0$ such that $B_{r+\delta}$ is covered by them and B.

To prove $(0, r + \delta) \subseteq R$ we suppose given some $r' \in (r, r + \delta)$ and write B' for $B_{r'}$. Since B' is covered by B and the U_p's, every mirror that meets B' either meets B or is tangent to S. So we must prove that $\pi_1(B' - \mathcal{H}, A)$ is generated by $\pi_1(B - \mathcal{H}, A)$ and the $\overline{AH}$'s with $H \in \mathcal{M}$ tangent to S. For $p \in S$ we define V_p as $(B \cup U_p) \cap B'$. It is easy to see that $V_p - \mathcal{H} \to (B \cup U_p) - \mathcal{H}$ is a homotopy equivalence. (Retract points of $U_p - B'$ along geodesics toward p.) It follows from lemma 2.3 that $\pi_1(V_p - \mathcal{H}, A)$ is generated by $\pi_1(B - \mathcal{H}, A)$, supplemented by $\overline{AH}$ if p is the point of tangency of S with some $H \in \mathcal{M}$.

Because $B' = \cup_{p \in S} V_p$, repeatedly using van Kampen's theorem shows that $\pi_1(B' - \mathcal{H}, A)$ is generated by the $\pi_1(V_p - \mathcal{H}, A)$, finishing the proof. This use of van Kampen's theorem requires checking that every set got from the V_p's by finite unions and intersections is connected. We call a subset Y of X star-shaped (around A) if it contains A and the geodesics $\overline{Ay}$ for all $y \in Y$. Nonpositive curvature shows that $B \cup U_p$ is star-shaped. Intersecting with B' preserves star-shapedness and yields V_p. Since unions and intersections of star-shaped sets are again star-shaped, our repeated application of van Kampen's theorem is legitimate. □

3. Loops in quotients of arrangement complements

We continue using the notation X, $\mathcal{M}$ and $\mathcal{H}$ from the previous section. We also suppose a group $P\Gamma$ acts faithfully, isometrically and properly discontinuously on X, preserving $\mathcal{H}$. At this point we have no group Γ in mind; the notation $P\Gamma$ is just for compatibility with sections 4–5. Our goal is to understand the orbifold fundamental group of $(X - \mathcal{H})/P\Gamma$. We use the following definition from [L3] and [Ba2]; more general formulations exist [R][Ka].

Fixing a basepoint $a \in X - \mathcal{H}$, consider the set of pairs (γ, g) where $g \in P\Gamma$ and γ is a path in $X - \mathcal{H}$ from a to $g(a)$. We regard one such pair as equivalent to another one (γ', g') if $g = g'$ and γ and γ' are homotopic in $X - \mathcal{H}$, rel endpoints. The orbifold fundamental group $G_a := \pi_1^{\mathrm{orb}}\big((X - \mathcal{H})/P\Gamma, a\big)$ means the set of equivalence classes. The group multiplication is $(\gamma, g) \cdot (\gamma', g') = (\gamma \text{ followed by } g \circ \gamma', gg')$. Projection of (γ, g) to g defines a homomorphism $G_a \to P\Gamma$. The kernel is obviously $\pi_1(X - \mathcal{H}, a)$, yielding the exact sequence

$$1 \to \pi_1(X - \mathcal{H}, a) \to G_a \to P\Gamma \to 1 \tag{1}$$

Although we don't need it, we remark that if a has trivial $P\Gamma$ stabilizer then there is a simpler $P\Gamma$-invariant description of the orbifold fundamental group. Writing o for a's orbit, we define $G_o := \pi_1^{\mathrm{orb}}\big((X - \mathcal{H})/P\Gamma, o\big)$ as the set of $P\Gamma$-orbits on the homotopy classes (rel endpoints) of paths in $X - \mathcal{H}$ that begin and end in o. The $P\Gamma$-action is the obvious one: $g \in P\Gamma$ sends a path γ to $g \circ \gamma$. To define $\gamma\gamma'$, where $\gamma, \gamma' \in G_o$, one translates γ' so that it begins where γ ends, and then composes paths in the usual way. Well-definedness of multiplication, and the identification with the definition of G_a, uses the fact that every path starting in o has a unique translate starting at a.

A complex reflection means a finite-order isometry of X whose fixed-point set is a complex hyperplane, called its mirror. In our applications, $P\Gamma$ is generated

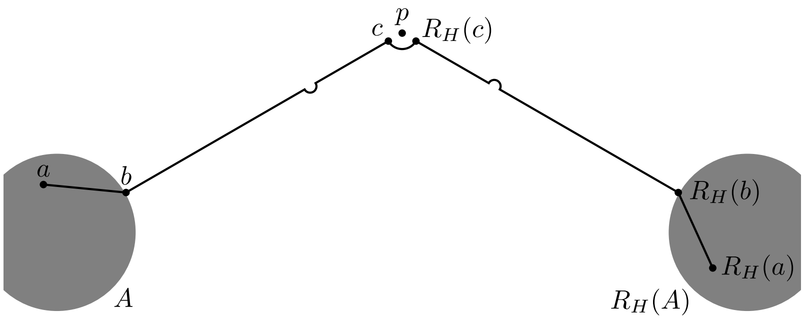

FIGURE 1. The path $\mu_{a,A,H}$ goes from left to right. Here R_H is the complex reflection of order 3, acting by counter-clockwise rotation by $2\pi/3$. The hyperplane H is not shown except for its point p closest to A. The small semicircles indicate that the path from b to c may detour around some points of $\mathcal{H}$.

by complex reflections whose mirrors are hyperplanes in $\mathcal{M}$. This leads to certain natural elements of the orbifold fundamental group: for $H \in \mathcal{M}$ we next define a loop $\mu_{a,H} \in G_a$ which is a fractional power of $\overline{aH}^{\circ}$. Write p for the point of H closest to a and n_H for the order of the cyclic group generated by the complex reflections in $P\Gamma$ with mirror H. Write R_H for the complex reflection of $P\Gamma$ with mirror H, that acts on the normal bundle to H by $\exp(2\pi i/n_H)$. (In the special case that no complex reflection in $P\Gamma$ has H as its mirror, this yields $n_H = 1$ and $R_H = 1$.)

Recall that the definition of $\overline{aH}^{\circ}$ involved a point c of $\overline{ap}$ very near p, and a circular loop in $\overline{ap}^{\mathbb{C}}$ centered at p and based at c. We define $\mu_{a,H}$ as $\widehat{ac}$ followed by the first $(1/n_H)$th of this loop (going from c to $R_H(c)$), followed by $R_H(\text{reverse}(\widehat{ac}))$. This is a path from a to $R_H(a)$, so the pair $(\mu_{a,H}, R_H)$ is an element of the orbifold fundamental group G_a. Using the definition of multiplication, the first component of $(\mu_{a,H}, R_H)^{n_H}$ is the path got by following $\mu_{a,H}$, then $R_H(\mu_{a,H})$, then $R_H^2(\mu_{a,H}), \ldots$ and finally $R_H^{n_H-1}(\mu_{a,H})$. It is easy to see that this is homotopic to $\overline{aH}^{\circ}$. So we have $(\mu_{a,H}, R_H)^{n_H} = \overline{aH}^{\circ}$.

At this point we have defined everything in the statement of theorem 1.2. But before proving it, we will adapt our construction to accomodate the "fat basepoints" of the previous section. This is necessary for our application to $\mathbb{C}H^{13}$. So we fix A as in section 2, and assume it contains our basepoint a. We will use A as the base "point" when discussing $\pi_1(X - \mathcal{H})$, and a as the basepoint when discussing $\pi_1^{\text{orb}}\big((X - \mathcal{H})/P\Gamma\big)$. In particular, the left term of (1) could also be written $\pi_1(X - \mathcal{H}, A)$. The analogue of $\mu_{a,H}$ is defined as follows, in terms of the point b of A that is closest to H. We define $\mu_{a,A,H}$ to be $\overline{ab}$ followed by $\mu_{b,H}$ followed by $R_H(\overline{ba})$. See figure 1 for a picture. Essentially the same argument as before shows that $(\mu_{a,A,H}, R_H)^{n_H} = \overline{AH}^{\circ} \in \pi_1(X - \mathcal{H}, A)$.

In applications one typically has some distinguished set of $\mu_{a,H}$'s or $\mu_{a,A,H}$'s in mind and wants to prove that they generate G_a. Theorem 1.2 in the introduction is a result of this sort. The following lemma is really the inductive step in the proof,

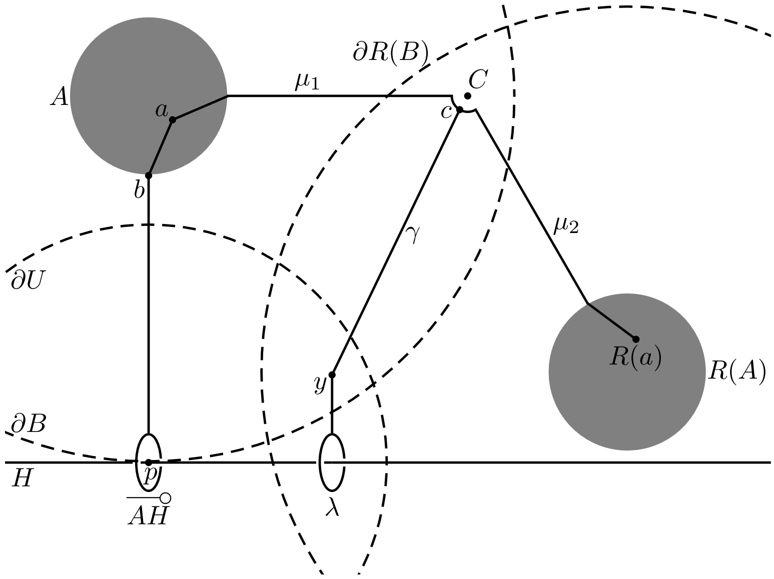

FIGURE 2. Illustration for the proof of lemma 3.1.

so the reader might prefer to read the theorem's proof first. Also, theorem 1.2 uses only case 1 of the lemma; the other cases are for our application to $\mathbb{C}H^{13}$.

Lemma 3.1. *Suppose $\mathcal{C} \subseteq \mathcal{M}$ are the hyperplanes closest to A, and let G be the subgroup of G_a generated by the $(\mu_{a,A,C}, R_C) \in G_a$ with $C \in \mathcal{C}$. Suppose $H \in \mathcal{M}$, write p for the closest point of H to A, r for $d(A,p)$, and B for the open r-neighborhood of A. Suppose G contains $\pi_1(B - \mathcal{H}, A)$ and that there exists a complex reflection $R \in P\Gamma$ with mirror in $\mathcal{C}$, such that one of the following holds:*

1. *R moves A closer to p.*
2. *R moves A closer to H, and no farther from p.*
3. *There exists an open ball U around p such that $U \cap \mathcal{H} = U \cap \mathcal{H}_p$, $B \cap R(B) \cap U \neq \emptyset$, and $R(B) \cap U \cap H \neq \emptyset$.*

Then G contains $\overline{AH}$.

Proof. In every case we have $R(B) \cap H \neq \emptyset$, so $R^{-1}(H)$ is closer to A than H is. Therefore $H \notin \mathcal{C}$, or in other words: the hyperplanes in $\mathcal{C}$ lie at distance $< r$ from A. We will prove the conclusion under hypothesis (3), and then show that the other two cases follow. We hope figure 2 helps the reader. First we introduce the various objects pictured. We write b for the point of A closest to H. Under our identification of $\pi_1(X - \mathcal{H}, A)$ with $\pi_1(X - \mathcal{H}, a)$, the loop $\overline{AH}$ corresponds to $\overline{ab}$ followed by $\overline{bH}$ followed by $\overline{ba}$.

The complex reflection R equals R_C^i for some $C \in \mathcal{C}$. The point marked C in the figure represents the point of C nearest to A. It lies inside B by our remark above that the elements of $\mathcal{C}$ are closer to A than H is. Since it is fixed by R, this point lies inside $R(B)$ too.

Consider the first component of $(\mu_{a,A,C}, R_C)^i$. After a homotopy it may be regarded as a path μ_1 in $B - \mathcal{H}$ from a to a point $c \in \big(B \cap R(B)\big) - \mathcal{H}$, followed by a path μ_2 in $R(B) - \mathcal{H}$ from c to $R(a)$. These paths are marked in the figure. So $(\mu_{a,A,C}, R_C)^i = (\mu_1\mu_2, R)$ in G_a.

We have assumed that $B \cap R(B) \cap U$ is nonempty, so choose a point in it but not in $\mathcal{H}$, say y. Let λ be a loop in $(R(B) \cap U) - \mathcal{H}$, based at y, which has linking number 1 with H. Such a loop exists because H meets $R(B) \cap U$ by hypothesis. Finally, let γ be a path in $(B \cap R(B)) - \mathcal{H}$ from c to y.

Our goal is to prove that G contains $\overset{\multimap}{AH}$. Lemma 2.3 shows that this loop lies in the subgroup of $\pi_1(X - \mathcal{H}, a)$ generated by $\pi_1(B - \mathcal{H}, a)$ and $\mu_1\gamma\lambda\gamma^{-1}\mu_1^{-1}$. This uses our hypothesis $U \cap \mathcal{H} = U \cap \mathcal{H}_p$. Since we assumed G contains the image of $\pi_1(B - \mathcal{H}, a)$, it suffices to show that G contains $\mu_1\gamma\lambda\gamma^{-1}\mu_1^{-1}$, or equivalently the homotopic loop $(\mu_1\mu_2)\big(\mu_2^{-1}\gamma\lambda\gamma^{-1}\mu_2\big)(\mu_2^{-1}\mu_1^{-1})$.

An element of the orbifold fundamental group G_a is really a pair, so we must prove $\big((\mu_1\mu_2)(\mu_2^{-1}\gamma\lambda\gamma^{-1}\mu_2)(\mu_2^{-1}\mu_1^{-1}), 1\big) \in G$. One checks that this equals

$$(\mu_1\mu_2, R) \cdot \Big(R^{-1}\big(\mu_2^{-1}\gamma\lambda\gamma^{-1}\mu_2\big), 1\Big) \cdot \Big(R^{-1}\big(\mu_2^{-1}\mu_1^{-1}\big), R^{-1}\Big)$$

The last term is the inverse of the first, which G contains by definition. So it suffices to show that the middle term lies in G, which is obvious: the loop $\mu_2^{-1}\gamma\lambda\gamma^{-1}\mu_2$ lies in $R(B) - \mathcal{H}$, so its image under R^{-1} lies in $B - \mathcal{H}$. This finishes case (3).

Next we claim that (1) implies (3). Take U to be any ball around p with $U \cap \mathcal{H} = U \cap \mathcal{H}_p$. Then the remaining hypotheses of (3) follow immediately from $p \in R(B)$.

Finally we claim that (2) implies (3). By the previous paragraph it suffices to treat the case that $p \in \partial R(B)$. Take U to be any ball around p with $U \cap \mathcal{H} = U \cap \mathcal{H}_p$. The hypothesis $d(R(A), H) < r$ says that H is not orthogonal to $\overline{R(A), p}$. It follows that $R(B)$ contains elements of H arbitrarily close to p, so $U \cap R(B) \cap H \neq \emptyset$. Similarly, $d(R(A), H) < r$ implies the non-tangency of ∂B and $\partial R(B)$ at p. From this it follows that $B \cap R(B)$ has elements arbitrarily close to p, hence in U. This finishes the proof. □

Proof of theorem 1.2. We will apply lemma 3.1 with $A = \{a\}$, noting that $\mu_{a,A,H} = \mu_{a,H}$ for all $H \in \mathcal{M}$. Write G for the subgroup of G_a generated by the $(\mu_{a,C}, R_C)$'s. By the exact sequence (1) and the assumed surjectivity $G \to P\Gamma$, it suffices to show that G contains $\pi_1(X - \mathcal{H}, a)$. By theorem 1.1 (really, theorem 2.1) it suffices to show that it contains every $\overset{\multimap}{aH}$. We do this by induction on $d(a, H)$.

The base case is $H \in \mathcal{C}$, for which we use the fact that $\overset{\multimap}{aH}$ is a power of $(\mu_{a,H}, R_H)$. So suppose $H \in \mathcal{M} - \mathcal{C}$ and set $r := d(a, H)$. We may assume, by theorem 2.1 and the inductive hypothesis, that G contains $\pi_1(B - \mathcal{H}, a)$, where B is the open r-neighborhood of a. Then case (1) of lemma 3.1 shows that G also contains $\overset{\multimap}{aH}$, completing the inductive step. □

4. A MONSTROUS(?) HYPERPLANE ARRANGEMENT

In this section we give background information on the conjecturally-monstrous hyperplane arrangement in $\mathbb{C}H^{13}$ which is the subject of conjecture 1.3 and theorems 1.4 and 1.5. For more information, see [A3][Ba1][Ba2][A4][H].

We write $\mathbb{C}^{n,1}$ for a complex vector space equipped with a Hermitian form $\langle \cdot \mid \cdot \rangle$ of signature $(n, 1)$. The *norm* v^2 of a vector v means $\langle v \mid v \rangle$. Complex hyperbolic space $\mathbb{C}H^n$ means the set of negative-definite 1-dimensional subspaces. If $V, W \in \mathbb{C}H^{13}$

are represented by vectors v, w then their hyperbolic distance is

$$d(V, W) = \cosh^{-1} \sqrt{\frac{\left|\langle v \,|\, w \rangle\right|^2}{v^2 w^2}}. \tag{2}$$

If s is a vector of positive norm, then $s^\perp \subseteq \mathbb{C}^{n,1}$ defines a hyperplane in $\mathbb{C}H^n$, also written $s^\perp$, and

$$d(V, s^\perp) = \sinh^{-1} \sqrt{-\frac{\left|\langle v \,|\, s \rangle\right|^2}{v^2 s^2}}. \tag{3}$$

These formulas are from [Go], up to an unimportant factor of 2.

A null vector means a nonzero vector of norm 0. If v is one then it represents a point V of the boundary $\partial \mathbb{C}H^n$. For any vector w of non-zero norm we define the *height* of w with respect to v by

$$\mathrm{ht}_v(w) := -\frac{\left|\langle v \,|\, w \rangle\right|^2}{w^2}. \tag{4}$$

This is invariant under scaling w, so it descends to a function on $\mathbb{C}H^n$, which is positive. The height h horosphere (with respect to v) means the set of $W \in \mathbb{C}H^n$ with $\mathrm{ht}_v(W) = h$. We define open and closed horoballs the same way, replacing $=$ by $<$ and $\leq$. (More abstractly, one can define horospheres as the orbits of the unipotent radical of the $\mathrm{PU}(n,1)$-stabilizer of V.)

We think of V as the center of these horospheres and horoballs and h as a sort of generalized radius, even though strictly speaking the distance from any point of $\mathbb{C}H^n$ to V is infinite. In particular, if $W, W' \in \mathbb{C}H^n$ then we say that W is closer to V than W' is, if $\mathrm{ht}_v(W) < \mathrm{ht}_v(W')$. To see that this notion depends on V rather than v, one checks that replacing v by a nonzero scalar multiple of itself does not affect this inequality. (It multiplies both sides by the same positive constant.) Another way to think about this, at least for points outside some fixed closed horoball A, is to regard "closer to V" as alternate language for "closer to A". In any case, in our application there will be a canonical choice for v, up to roots of unity.

Now we describe the particular hyperplane arrangement we will study. We write ω for a primitive cube root of unity and define the Eisenstein integers $\mathcal{E}$ as $\mathbb{Z}[\omega]$. The Eisenstein integer $\omega - \bar{\omega} = \sqrt{-3}$ is so important that it has its own name θ. An $\mathcal{E}$-*lattice* means a free $\mathcal{E}$-module L equipped with a Hermitian form taking values in $\mathcal{E} \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{-3})$, denoted $\langle \cdot \,|\, \cdot \rangle$ and assumed linear in its first argument and antilinear in its second. Sometimes we think of lattice elements as column vectors and $\langle \cdot \,|\, \cdot \rangle$ as specified by a matrix M equal to the transpose of its complex conjugate. Then $\langle v \,|\, w \rangle = v^T M \bar{w}$.

We define L as the $\mathcal{E}$-lattice $\Lambda \oplus \left(\begin{smallmatrix} 0 & \bar{\theta} \\ \theta & 0 \end{smallmatrix}\right)$, where Λ is the complex Leech lattice at the smallest scale at which all inner products lie in $\mathcal{E}$. (At this scale, Λ has minimal norm 6 and all inner products are divisible by θ. At the scale used by Wilson [W] it has minimal norm 9.) Because L has signature $(13, 1)$, we may regard $L \otimes_{\mathcal{E}} \mathbb{C}$ as a copy of $\mathbb{C}^{13,1}$. This is just one concrete description of L; see [A4, §3] for another one. Also, one may describe L abstractly as the unique $\mathcal{E}$-lattice that has this signature and is equal to θ times its dual lattice. See [Ba1] for a proof of this uniqueness.

We define Γ as the isometry group of L, meaning the $\mathcal{E}$-linear automorphisms that respect the inner product. We write $P\Gamma$ for the quotient by scalars. It acts faithfully on $\mathbb{C}H^{13}$. A *root* of L means a lattice vector of norm 3. If s is a root then R_s means the *ω-reflection in s*, which fixes $s^\perp$ pointwise and multiplies s by ω. Its general formula is

$$x \mapsto x - (1-\omega)\frac{\langle x \,|\, s\rangle}{3}s$$

and it preserves L because both $(1-\omega)$ and $\langle x \,|\, s\rangle$ are divisible by θ. One can replace ω by $\bar\omega$ to get the $\bar\omega$-reflection R_s^{-1}. We call these triflections since they are complex reflections of order 3. In [Ba1], it was proved that Γ is generated by the R_s's with s varying over the roots of L, and in [Ba2] that a certain set of 14 of them suffices. The hyperplane $s^\perp \subseteq \mathbb{C}H^{13}$ orthogonal to a root s is called its mirror. We define $\mathcal{M}$ as the set of all these mirrors, and $\mathcal{H}$ as their union.

If $V \in \partial\mathbb{C}H^{13}$ can be represented by a lattice vector v, then we always choose v to be primitive, i.e., generate the 1-dimensional sublattice representing V. This defines v up to multiplication by a sixth root of unity, and the corresponding height function is independent of this factor. Therefore we may regard the function ht_v as intrinsic to V. In particular, if $p \in \mathbb{C}H^{13}$, and both $V, V' \in \partial\mathbb{C}H^{13}$ are represented by lattice vectors, then we sometimes say that p is closer to V than to V'. This means $\mathrm{ht}_v(p) < \mathrm{ht}_{v'}(p)$.

Our goal in section 5 is to find generators for the orbifold fundamental group of $(\mathbb{C}H^{13} - \mathcal{H})/P\Gamma$. We fix the null vector $\rho := (0; 0, 1)$. We would like to use the corresponding point of $\partial\mathbb{C}H^{13}$ as our basepoint, but cannot because it is not in $\mathbb{C}H^{13}$. Instead we will use a "fat basepoint": a closed horoball A centered there which misses $\mathcal{H}$. Such a horoball exists by the following lemma.

Lemma 4.1. *The open horoball $\big\{W \in \mathbb{C}H^{13} \,\big|\, \mathrm{ht}_\rho(W) < 1\big\}$ is disjoint from $\mathcal{H}$, and the mirrors that meet its boundary are the orthogonal complements of the roots s that satisfy $\big|\langle \rho \,|\, s\rangle\big|^2 = 3$.*

Proof. The special property of ρ we need is that it is orthogonal to no roots. This is clear because $\rho^\perp \cong \Lambda \oplus (0)$ has no vectors of norm 3. Now, if s is a root then the point of s's mirror nearest to ρ is represented by the vector projection of ρ to $s^\perp$, namely $p = \rho - \frac{1}{3}\langle \rho \,|\, s\rangle s$. One computes $\mathrm{ht}_\rho(p) = \big|\langle \rho \,|\, s\rangle\big|^2/3$. This is at least 1, with equality just if $\big|\langle \rho \,|\, s\rangle\big|^2 = 3$. □

5. The Leech meridians generate

The purpose of this section is to prove theorem 1.5, giving generators for the orbifold fundamental group of $(\mathbb{C}H^{13} - \mathcal{H})/P\Gamma$, with $\mathcal{M}$, $\mathcal{H}$ and $P\Gamma$ as in the previous section. We call the following roots of L the Leech roots:

$$l = \Big(\lambda; 1, \theta\Big(\frac{\lambda^2 - 3}{6} + \nu_l\Big)\Big) \tag{5}$$

with $\lambda \in \Lambda$ and $\nu_l \in \operatorname{Im}\mathbb{C}$ chosen so that the last coordinate lies in $\mathcal{E}$. The set of possibilities for ν_l is $\frac{1}{\theta}(\frac{1}{2} + \mathbb{Z})$ if 6 divides λ^2 and $\frac{1}{\theta}\mathbb{Z}$ otherwise. What is important about the Leech roots is that their mirrors $l^\perp$, called the Leech mirrors, are the elements of $\mathcal{M}$ coming closest to ρ. By lemma 4.1, these are tangent to the height 1 horoball.

It follows from the same lemma that there exists a closed horoball A centered at $\rho = (0;0,1)$ which misses $\mathcal{H}$. Fix any basepoint $a \in A$. By the Leech meridians we mean the elements of $G_a = \pi_1^{\mathrm{orb}}\big((\mathbb{C}H^{13} - \mathcal{H})/P\Gamma, a\big)$ corresponding to the Leech mirrors, namely the pairs $(\mu_{a,A,l^\perp}, R_l)$ with l varying over the Leech roots. Here we are using the notation of section 3. We write G for the subgroup of G_a they generate. Theorem 1.5 is the assertion that G is all of G_a.

Here is an overview of the proof, which follows that of theorem 1.2. It amounts to showing that the mirror of any non-Leech root s satisfies one of the hypotheses (1)–(3) of lemma 3.1. It turns out (lemma 5.2) that if $\big|\langle \rho \,|\, s\rangle\big|^2 > 21$ then the simplest hypothesis (1) holds. If $\big|\langle \rho \,|\, s\rangle\big|^2 > 3$ then the same method shows that the next simplest hypothesis (2) holds (lemmas 5.2 and 5.3). For the case $\big|\langle \rho \,|\, s\rangle\big|^2 = 3$ we enumerate the orbits of roots $(?;\theta,?)$ under the Γ-stabilizer of ρ (lemma 5.1). There are three orbits, satisfying hypotheses (1), (2) and (3) of lemma 3.1, respectively. The last orbit is especially troublesome (lemma 5.4). The proof of theorem 1.5 is basically a wrapper around these results.

A general vector in $L \otimes \mathbb{C}$ not orthogonal to ρ has the form

$$s = \left(\sigma; m, \frac{\theta}{\bar{m}}\Big(\frac{\sigma^2 - N}{6} + \nu\Big)\right) \tag{6}$$

where $\sigma \in \Lambda \otimes \mathbb{C}$, $m \in \mathbb{C} - \{0\}$, N is the norm s^2, and $\nu \in \operatorname{Im}\mathbb{C}$. Restricting the first coordinate to Λ and the others to $\mathcal{E}$ gives the elements of $L - \rho^\perp$. Further restricting N to 3 gives the roots of L, and finally restricting m to 1 gives the Leech roots. For vectors of any fixed negative (resp. positive) norm, the larger the absolute value of the middle coordinate m, the further from ρ lie the corresponding points (resp. hyperplanes) in $\mathbb{C}H^{13}$.

One should think of s from (6) as being associated to the vector σ/m in the positive-definite Hermitian vector space $\Lambda \otimes_{\mathcal{E}} \mathbb{C}$. By this we mean that the most important part of $\langle s \,|\, s'\rangle$ is governed by the relative positions of σ/m and σ'/m'. Namely, by completing the square and patiently rearranging, one can check

$$\langle s \,|\, s'\rangle = m\bar{m}'\left[\frac{1}{2}\Big(\frac{N'}{|m'|^2} + \frac{N}{|m|^2} - \Big(\frac{\sigma}{m} - \frac{\sigma'}{m'}\Big)^2\Big) + \operatorname{Im}\Big\langle \frac{\sigma}{m} \,\Big|\, \frac{\sigma'}{m'}\Big\rangle + 3\Big(\frac{\nu'}{|m'|^2} - \frac{\nu}{|m|^2}\Big)\right]. \tag{7}$$

In the rest of this section, s will always denote a root. *Caution:* we are using the convention that the imaginary part of a complex number is imaginary; for example $\operatorname{Im}\theta$ is θ rather than $\sqrt{3}$.

Lemma 5.1. *Suppose λ_6, λ_9 are fixed vectors in Λ with norms 6 and 9. Then under the Γ-stabilizer of ρ, every root with $m = \theta$ is equivalent to $(0;\theta,-\omega)$ or $(\lambda_6;\theta,\omega)$ or $(\lambda_9;\theta,-1)$.*

Proof. The Γ-stabilizer of ρ contains the Heisenberg group of "translations"

$$\begin{aligned} &(l;0,0) \mapsto \big(l;0,\bar{\theta}^{-1}\langle l \,|\, \lambda\rangle\big)\\ T_{\lambda,z}:\ &(0;1,0) \mapsto \big(\lambda;1,\theta^{-1}(z - \lambda^2/2)\big)\\ &(0;0,1) \mapsto (0;0,1) \end{aligned}$$

where $\lambda \in \Lambda$ and $z \in \operatorname{Im}\mathbb{C}$ is such that $z - \lambda^2/2 \in \theta\mathcal{E}$. Suppose $s \in L$ has the form (6) with $N = 3$ and $m = \theta$. Applying $T_{\lambda,z}$ to s changes the first coordinate by $\theta\lambda$. By [W, p. 153], every element of Λ is congruent modulo $\theta\Lambda$ to a vector of norm

0, 6 or 9, so we may suppose σ has one of these norms. Since $\operatorname{Aut}\Lambda$ fixes ρ and acts transitively on the vectors of each of these norms [W, p. 155], we may suppose $s = 0$, λ_6 or λ_9. That is, s is one of

$$(0; \theta, \frac{1}{2} - \nu) \qquad (\lambda_6; \theta, -\frac{1}{2} - \nu) \qquad (\lambda_9; \theta, -1 - \nu).$$

In each of the three cases, the possibilities for ν differ by the elements of $\operatorname{Im}\mathcal{E}$. Applying $T_{0,z}$ ($z \in \operatorname{Im}\mathcal{E}$) adds z to the third coordinate of s. Therefore we may take $\nu = \theta/2$, $\bar\theta/2$ and 0 in the three cases, yielding the roots in the statement of the lemma. (These roots are inequivalent under the Γ-stabilizer of ρ, but we don't need this.) □

Lemma 5.2. *Suppose s is the root $(0; \theta, -\omega)$ or a root with $|m| = 2$ or $|m| > \sqrt{7}$, and define p as the point of $s^\perp$ nearest ρ. Then there is a triflection in a Leech root that moves ρ closer to p.*

Proof of lemma 5.2. This proof grew from simpler arguments used for [A1, thm. 4.1] and [Ba1, prop. 4.2].

We have $p = \rho - \frac{1}{3}\langle \rho \,|\, s\rangle s = \rho + \frac{1}{\theta}\bar m s$. We want to choose a Leech root l, and $\zeta = \omega^{\pm 1}$, such that the ζ-reflection in l (call it R) moves ρ closer to p. This is equivalent to $\langle p \,|\, R(\rho)\rangle$ being smaller in absolute value than $\langle p \,|\, \rho\rangle$. We will write down these inner products explicitly and then choose l and ζ appropriately. Direct calculation gives $\langle p \,|\, \rho\rangle = -|m|^2$. Also,

$$R(\rho) = \rho - (1-\zeta)\frac{\langle \rho \,|\, l\rangle}{\langle l \,|\, l\rangle} l = \rho + \frac{1-\zeta}{\theta} l.$$

It turns out that the necessary estimates on $\langle p \,|\, R(\rho)\rangle$ are best expressed in terms of the following parameter:

$$y := \frac{\theta}{|m|^2}\langle p \,|\, l\rangle = \frac{\theta}{|m|^2}\Big\langle \rho + \frac{\bar m}{\theta} s \,\Big|\, l\Big\rangle = -\frac{3}{|m|^2} + \frac{1}{m}\langle s \,|\, l\rangle \tag{8}$$

$$\in -\frac{3}{|m|^2} + \frac{1}{m}\theta\mathcal{E}. \tag{9}$$

First one works out

$$\left|\frac{\langle p \,|\, R(\rho)\rangle}{|\langle p \,|\, \rho\rangle|}\right| = \left|\frac{\langle p \,|\, R(\rho)\rangle}{|m|^2}\right| = \left|\tfrac{1}{3}(1-\bar\zeta)y - 1\right|. \tag{10}$$

Our goal is to choose l and ζ so that this is less than 1. This is equivalent to $|y - (1-\zeta)| < \sqrt{3}$. Because the choices for ζ are $\omega^\pm$, this amount to being able to choose l so that y lies in the union V of the open balls in $\mathbb{C}$ of radius $\sqrt{3}$ around the points $1-\omega$ and $1-\bar\omega$. So our goal is to choose l such that y lies in the shaded region in figure 3.

Now we examine how our choice of l affects y. Choosing l amounts to choosing $\lambda \in \Lambda$, and then choosing $\nu_l \in \operatorname{Im}\mathbb{C}$ subject to the condition that the last coordinate of (5) is in $\mathcal{E}$. Specializing (7) to the case that s has norm $N = 3$ and s' is the Leech root l gives

$$\langle s \,|\, l\rangle = m\Big[\frac{3}{2|m|^2} + \frac{3}{2} - \frac{1}{2}\Big(\frac{\sigma}{m} - \lambda\Big)^2 + \operatorname{Im}\Big\langle \frac{\sigma}{m} \,\Big|\, \lambda\Big\rangle + 3\Big(\nu_l - \frac{\nu}{|m|^2}\Big)\Big].$$

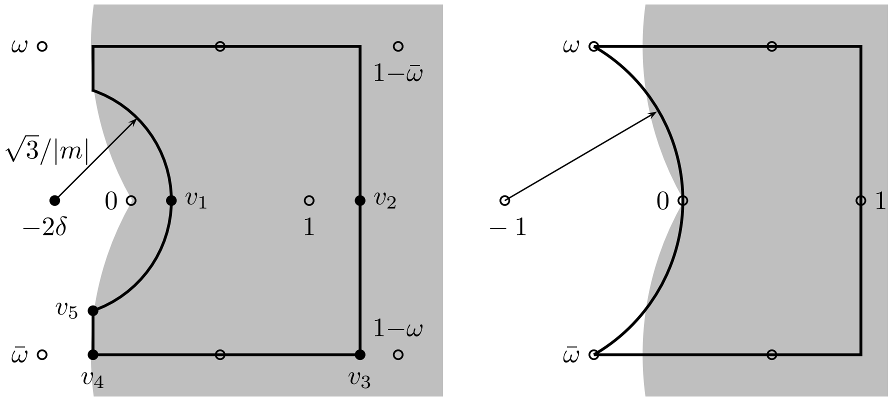

FIGURE 3. See the proof of lemma 5.2. V is the union of the gray (open) disks, which have radius $\sqrt{3}$ and centers $1-\omega^{\pm 1}$. We seek a Leech root l so that y lies in this region. U is the closed region bounded by the solid line, and is where we can arrange for y to be. U varies with $|m|$; we have drawn the case $|m| = \sqrt{7}$, when v_5 is on the boundary of V, and the case $|m| = \sqrt{3}$, when v_4 and v_5 coalesce at $\bar{\omega}$. Hollow circles indicate Eisenstein integers.

Plugging this into formula (8) gives

$$y = -\frac{3}{2|m|^2} + \frac{3}{2} - \frac{1}{2}\Big(\frac{\sigma}{m} - \lambda\Big)^2 + \operatorname{Im}\Big\langle \frac{\sigma}{m} \;\Big|\; \lambda \Big\rangle + 3\Big(\nu_l - \frac{\nu}{|m|^2}\Big). \tag{11}$$

The covering radius of a lattice in Euclidean space is defined as the smallest number such that the closed balls of that radius around lattice points cover Euclidean space. The covering radius of Λ is $\sqrt{3}$, because the underlying real lattice has norms equal to $3/2$ times those of the usual Leech lattice, whose covering radius is $\sqrt{2}$ by [CPS]. Therefore we may take λ so that $0 \leq (\sigma/m - \lambda)^2 \leq 3$. It follows that the real part of (11) lies in $[-\delta, 3/2 - \delta]$ where $\delta := 3/2|m|^2$.

Next we choose ν_l. The only constraint on it is that the last component of $l = (\lambda; 1, ?)$ must lie in $\mathcal{E}$. As mentioned after (5), this amounts to: $\nu_l \in \frac{1}{\theta}(\frac{1}{2} + \mathbb{Z})$ if λ^2 is divisible by 6, and $\nu_l \in \frac{1}{\theta}\mathbb{Z}$ otherwise. In either case, referring to (11) shows that changing our choice of ν_l allows us to change y by any rational integer multiple of θ. So we may take $\operatorname{Im} y \in [-\theta, \theta]$. After these choices we have

$$\operatorname{Re} y \in [-\delta, 3/2 - \delta] \quad \text{and} \quad \operatorname{Im} y \in [-\theta, \theta]. \tag{12}$$

There is an additional constraint on y. We have $y \neq -2\delta$ since -2δ is not in the rectangle (12), and since $y \in -2\delta + \frac{\theta}{m}\mathcal{E}$ by (9), y lies at distance $\geq \sqrt{3}/|m|$ from -2δ. We define U as the closed rectangle (12) in $\mathbb{C}$ minus the open $(\sqrt{3}/|m|)$-disk around -2δ. We have shown that we may choose a Leech root l such that $y \in U$. We have indicated U in outline in figure 3; as $|m|$ increases, the rectangle moves to the right, the center of the removed disk approaches zero, and its radius approaches zero more slowly than the center does.

Now suppose $|m|^2 > 7$. We claim $U \subseteq V$. Since we may choose l such that $y \in U$, and once y is in V we may choose $\zeta = \omega^{\pm 1}$ so that ζ-reflection in l moves ρ closer to p, this will finish the proof in this case. To prove the claim it will suffice

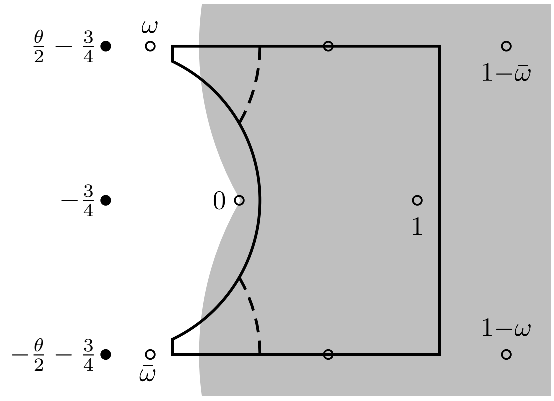

FIGURE 4. The analogue of figure 3 for the special case $|m| = 2$ in the proof of lemma 5.2. The proof shows that y lies in U (bounded by the solid path) but outside two open disks (indicated by the dashed arcs), hence in V (the shaded region).

to show that the lower half of U lies in the open $\sqrt{3}$-ball around $1-\omega$. Obviously it suffices to check this for the points marked $v_1, \ldots, v_5$ in figure 3. These are $v_1 = -2\delta + \sqrt{3}/|m|$, $v_2 = \frac{3}{2} - \delta$, $v_3 = \frac{3}{2} - \delta - \frac{1}{2}i\sqrt{3}$, $v_4 = -\delta - \frac{1}{2}i\sqrt{3}$ and

$$v_5 = -\delta - i\sqrt{\frac{3}{|m|^2} - \frac{9}{4|m|^4}}.$$

Using $|m|^2 > 7$, one can check that each of these lies at distance $< \sqrt{3}$ from $1-\omega$. This finishes the proof of the $|m|^2 > 7$ case. (If $|m|^2 = 7$ then v_5 lies in the boundary of V. If $|m|^2 = 4$ then v_4 and v_5 are outside the boundary; see figure 4. If $|m|^2 = 3$ then $v_1 = 0$ is on the boundary while $v_4 = v_5 = \bar{\omega}$ is outside it; see the second part of figure 3.)

Next we treat the special case $s = (0; \theta, -\omega)$. Choosing $\lambda = 0$ gives $\operatorname{Re} y = 1$ by (11). Then choosing ν_l as above, so that $\operatorname{Im} y$ lies in $i[-\sqrt{3}/2, \sqrt{3}/2]$, yields $y \in V$. So we can move ρ closer to p just as in the $|m|^2 > 7$ case.

Finally, we suppose $|m| = 2$; we may take $m = 2$ by multiplying s by a unit. Recall that once we proved that y lies in the rectangle (12), we could use (9) to show that y lies outside the open disk used in the definition of U. In fact this argument shows more. Since $\delta = \frac{3}{8}$ when $|m| = 2$, (9) shows $y \in -\frac{3}{4} + \frac{\theta}{2}\mathcal{E}$. Since $-\frac{3}{4} \pm \frac{\theta}{2}$ lie in $-\frac{3}{4} + \frac{\theta}{2}\mathcal{E}$ but not in the rectangle (12), y lies at distance $\geq \sqrt{3}/2$ from each of them, just as it lies at distance $\geq \sqrt{3}/2$ from $-\frac{3}{4}$. It is easy to check that U minus the open $\sqrt{3}/2$-balls around $-\frac{3}{4} \pm \frac{\theta}{2}$ lies in V; see figure 4. Therefore $y \in V$, finishing the proof as before. (One can consider the analogues of these extra disks for any m. They are unnecessary if $|m|^2 > 7$, and turn out to be useless if $|m|^2 = 3$ or 7.) □

Lemma 5.3. *Suppose s is the root $(\lambda_6; \theta, \omega)$ or a root with $|m| = \sqrt{7}$, and define p as the point of $s^\perp$ nearest ρ. Then there is a triflection in a Leech root that either moves ρ closer to p, or else moves ρ closer to $s^\perp$ while preserving ρ's distance from p.*

Proof. Suppose first $|m| = \sqrt{7}$. Then the proof of lemma 5.2 goes through unless y is v_5 in figure 3, or its complex conjugate. So suppose $y = v_5$ or $\bar{v}_5$, and take $\zeta = \omega$ or $\bar{\omega}$ respectively. The argument in the proof of lemma 5.2, that R moves ρ closer to p, fails because $\bigl|y - (1-\zeta)\bigr|$ equals $\sqrt{3}$ rather than being strictly smaller. But it does show that $R(\rho)$ is exactly as far from p as ρ is. This is one of our claims, and what remains to show is that R moves ρ closer to $s^\perp$.

To do this we first solve (8) for $\langle s \,|\, l\rangle$ in terms of y, obtaining $\langle s \,|\, l\rangle = (3 + |m|^2 y)/\bar{m}$. Then one works out

$$\left|\frac{\langle s \,|\, R(\rho)\rangle}{\langle s \,|\, \rho\rangle}\right| = \left|\frac{\langle s \,|\, \rho\rangle - \frac{1}{\theta}(1-\bar{\zeta})\langle s \,|\, l\rangle}{\langle s \,|\, \rho\rangle}\right| = \left|1 - \frac{1}{3}(1-\bar{\zeta})\Bigl(\frac{3}{|m|^2} + y\Bigr)\right|.$$

We want this to be less than 1. By copying the argument following (10), this is equivalent to $y + 3/|m|^2$ lying in open $\sqrt{3}$-disk around $1 - \zeta$. This is obvious from the figure because $y + 3/|m|^2$ is $3/7$ to the right of $y = v_5$ or $\bar{v}_5$. This finishes the $|m| = \sqrt{7}$ case.

The case $s = (\lambda_6; \theta, \omega)$ is similar. In this case U appears in figure 3. Taking $\lambda = 0$ leads to $\operatorname{Re} y = 0$, so either $y \in V$ (so the proof of lemma 5.2 applies) or else $y = 0 \in \partial V$. In this case the argument for $|m| = \sqrt{7}$ shows that $R(\rho)$ is just as close to p as ρ is, and that R moves ρ closer to $s^\perp$. □

Lemma 5.4. *Let $s = (\lambda_9; \theta, -1)$, define p as the point of $s^\perp$ nearest ρ, and B as the open horoball centered at ρ, whose bounding horosphere is tangent to $s^\perp$ at p. Then there exists an open ball U around p with $U \cap \mathcal{H} = U \cap \mathcal{H}_p$, and a triflection R in one of the Leech mirrors, such that $B \cap R(B) \cap U \neq \emptyset$ and $R(B) \cap U \cap s^\perp \neq \emptyset$.*

Proof. Since we are verifying hypothesis (3) of lemma 3.1, we will use that lemma's notation H for $s^\perp$. By definition,

$$p = \rho - \tfrac{1}{3}\langle \rho \,|\, s\rangle s = (-\lambda_9; \bar{\theta}, 2).$$

This has norm -3 and lies in L. One computes $\operatorname{ht}_\rho(p) = 3$, so B is the height 3 open horoball around ρ. We take U to have radius $\sinh^{-1}\sqrt{1/3}$. To check that $U \cap \mathcal{H} = U \cap \mathcal{H}_p$, consider a root s' not orthogonal to p. Then $\bigl|\langle p \,|\, s'\rangle\bigr| \geq \sqrt{3}$ since $p \in L$, so

$$d(p, s'^\perp) = \sinh^{-1}\sqrt{-\frac{\bigl|\langle p \,|\, s'\rangle\bigr|^2}{p^2 s'^2}} \geq \sinh^{-1}\sqrt{1/3},$$

as desired.

Next we choose R to be the ω-reflection in the Leech root $l = (0; 1, -\omega)$. (We found l by applying the proof of lemma 5.2 as well as we could. That is, we choose l so that y in that proof equals the lower left corner $\bar{\omega}$ of the second part of figure 3.) This yields $R(\rho) = (0; \bar{\omega}, 0)$. We must verify $R(B) \cap U \cap H \neq \emptyset$ and $B \cap R(B) \cap U \neq \emptyset$.

Our strategy for $R(B) \cap U \cap H \neq \emptyset$ is to define p' as the projection of $R(\rho)$ to H, parameterize $\overline{p'p} \subseteq H$, find the point x where it crosses ∂U, and check that $x \in R(B)$. Here are the details. Computation gives $p' = (\bar{\omega}\lambda_9/\theta; 2\bar{\omega}, -\bar{\omega}/\theta)$. One checks $\langle p' \,|\, p\rangle = 2\bar{\omega}\bar{\theta}$, so $-\omega\theta p'$ and p have negative inner product. Therefore $\overline{p'p} - \{p\}$ is parameterized by $x_t = -\omega\theta p' + tp$ with $t \in [0, \infty)$. One computes $\langle x_t \,|\, p\rangle = -3t - 6$ and $x_t^2 = -3t^2 - 12t - 3$, yielding

$$d(x_t, p) = \cosh^{-1}\sqrt{\frac{\bigl|\langle x_t \,|\, p\rangle\bigr|^2}{x_t^2 p^2}} = \cosh^{-1}\sqrt{\frac{t^2 + 4t + 4}{t^2 + 4t + 1}}$$

Now, x_t lies in ∂U just when this equals $\sinh^{-1}\sqrt{1/3}$, yielding a quadratic equation for t. There is just one nonnegative solution, namely $t = 2\sqrt{3} - 2$. So $x = x_{2\sqrt{3}-2}$. Then one computes $\langle R(\rho)\,|\,x\rangle = \bar{\omega}\bar{\theta}(4\sqrt{3} - 3)$, so

$$\mathrm{ht}_{R(\rho)}(x) = -\frac{\bigl|\langle R(\rho)\,|\,x\rangle\bigr|^2}{x^2} = -\frac{3\bigl(57 - 24\sqrt{3}\bigr)}{-27} < 3.$$

That is, $x \in R(B)$ as desired.

Our strategy for $B \cap R(B) \cap U \neq \emptyset$ is to parameterize the geodesic $\overline{x\rho}$, find the point y where it crosses ∂B, and check that y lies in $R(B)$ and U. Here are the details. Computation shows $\langle x\,|\,\rho\rangle = -6\sqrt{3} < 0$, so $\overline{x\rho} - \{\rho\}$ is parameterized by $y_u = x + u\rho$ with $u \in [0, \infty)$. Further computation shows $\langle y_u\,|\,\rho\rangle = -6\sqrt{3}$ and $y_u^2 = -27 - 12u\sqrt{3}$, so $\mathrm{ht}_\rho(y_u) = 36/(9 + 4u\sqrt{3})$. Setting this equal to 3 yields $u = \sqrt{3}/4$, so $y = y_{\sqrt{3}/4}$. Now one checks that $\mathrm{ht}_{R(\rho)}(y) < 3$, so that $y \in R(B)$. A similar calculation proves $y \in U$. (In fact this calculation can be omitted, because y, p are the projections to ∂B of the two points x, p outside B, but not both in ∂B. By the negative curvature of $\mathbb{C}H^{13}$ we have $d(y,p) < d(x,p) = \sinh^{-1}\sqrt{1/3}$.) $\square$

Proof of theorem 1.5. We will mimic the proof of theorem 1.2 (see the end of section 3), using lemmas 5.2–5.4 in place of the "moves a closer to p" hypothesis of that theorem. Write G for the subgroup of $G_a = \pi_1^{\mathrm{orb}}\bigl((\mathbb{C}H^{13} - \mathcal{H})/P\Gamma, a\bigr)$ generated by the Leech meridians, i.e., the pairs $(\mu_{a,A,l^\perp}, R_l)$ with l a Leech root. We must show that G is all of G_a. It is known [Ba1] (or [A4] for a later proof) that the R_l's generate $P\Gamma$. By the exact sequence (1), it therefore suffices to show that G contains $\pi_1(\mathbb{C}H^{13} - \mathcal{H}, a)$. By theorem 2.1 it suffices to show that G contains every $\overline{AH}$, with H varying over $\mathcal{M}$. We do this by induction on the distance from H to ρ, or properly speaking, on $\bigl|\langle\rho\,|\,s\rangle\bigr|$ where s is a root with $H = s^\perp$. The base case is when s is a Leech root, i.e., $\bigl|\langle\rho\,|\,s\rangle\bigr| = \sqrt{3}$, and we just observe $\overline{AH} = (\mu_{a,A,s^\perp}, R_s)^3$.

Now suppose s is a root but not a Leech root, $H = s^\perp$, p is the point of H closest to ρ, and B is the open horoball centered at ρ and tangent to H at p. We may assume by induction that G contains every $\overline{A, s'^\perp}$ with s' a root satisfying $\bigl|\langle\rho\,|\,s'\rangle\bigr| < \bigl|\langle\rho\,|\,s\rangle\bigr|$. It follows from theorem 2.1 that G contains $\pi_1(B - \mathcal{H}, a)$.

The smallest possible value of $\bigl|\langle\rho\,|\,s\rangle\bigr|$ for a non-Leech root s is 3, occurring when $|m| = \sqrt{3}$ in (6). In the cases $s = (0;\theta,-\omega)$, $(\lambda_6;\theta,\omega)$, resp. $(\lambda_9;\theta,-1)$, hypothesis (1), (2), resp. (3) of lemma 3.1 is satisfied, by lemma 5.2, 5.3, resp. 5.4. If s is any root with $\langle\rho\,|\,s\rangle = 3$ then it is equivalent to one of these examples under the Γ-stabilizer of ρ, by lemma 5.1. Therefore lemma 3.1 applies to $s^\perp$ for every root s with $\bigl|\langle\rho\,|\,s\rangle\bigr| = 3$. It follows that G contains the corresponding loops $\overline{AH}$.

The next possible value of $\bigl|\langle\rho\,|\,s\rangle\bigr|$ is $2\sqrt{3}$, occurring when $|m| = 2$ in (6). In this case lemma 5.2 verifies hypothesis (1) of lemma 3.1, which tells us that G contains $\overline{AH}$. The next possible value of $\bigl|\langle\rho\,|\,s\rangle\bigr|$ is $\sqrt{21}$, occurring when $|m| = \sqrt{7}$. In this case lemma 5.3 verifies hypothesis (2) of lemma 3.1, which tells us that G contains $\overline{AH}$. The general step of the induction is essentially the same. If $\bigl|\langle\rho\,|\,s\rangle\bigr|$ is larger than $\sqrt{21}$, then $|m|$ is larger than $\sqrt{7}$, so lemma 5.2 verifies hypothesis (1) of lemma 3.1. This tells us that G contains $\overline{AH}$, completing the inductive step. $\square$

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