FINITENESS THEOREMS IN STOCHASTIC INTEGER PROGRAMMING

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Dedicated to the memory of C. St. J. A. Nash-Williams, 1932–2001.

Abstract. We study Graver test sets for families of linear multi-stage stochastic integer programs with varying number of scenarios. We show that these test sets can be decomposed into finitely many "building blocks", independent of the number of scenarios, and we give an effective procedure to compute them. The paper includes an introduction to Nash-Williams' theory of better-quasi-orderings, which is used to show termination of our algorithm. We also apply this theory to finiteness results for Hilbert functions.

Contents

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Introduction

A wide range of practical optimization problems can be modeled as (mixed-integer) linear programs. While some methods for solving problems like these assume all data to be known in advance, others deal with uncertainty. Such uncertainty is often inherent to the problem, e.g., demands, prices, or network structure. To be able to treat optimization problems of this type, one requires stochastic information about the uncertain components. With this knowledge, the problem can then be formulated as a *stochastic* linear program: for example, one that minimizes *expected* costs, or one that minimizes the *risk* that the costs exceed a given threshold. (Other popular methods to deal with problems involving uncertainty also exist, for example *robust optimization*; see [2] and the references therein.)

A typical situation is the following: In the first step, one has to make a decision x without knowing the outcome of a random event that lies in the future. For example, one needs to decide where to open new production facilities without exactly knowing the demand in each geographical area. In a second stage, after observing the uncertain data (here: demands), one can make a second (recourse/repair) decision y. For example, one may relocate facilities, decide on production or transportation plans, etc. The goal is now to maximize revenues or profits from these facilities. These quantities can be calculated from the immediate costs $c^\intercal x$ for opening a new location (first stage decision) plus the expected costs and revenues $\mathcal{Q}(x)$ from relocating and running the facilities (second stage decision). This is a typical instance of an *expectation problem,* which can be modeled as

$$\min\left\{c^\intercal x + \mathcal{Q}(x) : Ax = b,\ x \in \mathbb{N}^m\right\},$$

where $A \in \mathbb{Z}^{l\times m}$, $b \in \mathbb{Z}^l$, $c \in \mathbb{R}^m$, and

$$\mathcal{Q}(x) = \mathbb{E}_P \min\left\{d(\omega)^\intercal y : Wy = h(\omega) - Tx,\ y \in \mathbb{N}^n\right\}$$

where $T \in \mathbb{Z}^{l\times m}$, $W \in \mathbb{Z}^{l\times n}$, and d, h are random variables on some probability space $(\Omega, \mathcal{F}, P)$ taking values in $\mathbb{R}^n$ and $\mathbb{Z}^l$, respectively. Note that in the formulation of the problem we assume that the outcome of the random event ω does not depend on the first-stage decision x that has been made.

The computation of $\mathcal{Q}(x)$ involves the symbolic computation of an integral, which is often hard (even for relatively few variables). In practice, when no integrality constraints are imposed on y, the computation can sometimes be simplified by exploiting continuity and convexity of $\mathcal{Q}(x)$. However, if y is required to be integer, $\mathcal{Q}(x)$ is only semi-continuous, and usually not convex. Hence, to make computations possible, the probability distribution for ω is usually approximated by N *scenarios* $\omega_1, \dots, \omega_N$ with respective probabilities $\pi_1, \dots, \pi_N$. In this way, the integral involved in the computation of $\mathcal{Q}(x)$ becomes a sum, and we obtain an integer program having a separate set of y-variables corresponding to each scenario:

$$\min\left\{c^\intercal x + \sum_{i=1}^{N} \pi_i \cdot (d^\intercal y_i) : Ax = a,\ Wy_i = h_i - Tx,\ x \in \mathbb{N}^m,\ y_i \in \mathbb{N}^n\right\}. \tag{0.1}$$

The coefficient matrix of this problem has a nice block structure:

$$\begin{pmatrix} A & & & \\ T & W & & \\ \vdots & & \ddots & \\ T & & & W \end{pmatrix}$$

This two-stage setup (making a decision, observing the outcome of the random event, and then making a recourse decision) can be iterated, leading to the notion of a *multi-stage stochastic integer program.* (See [30] for a recent survey of this topic.) In these programs, information is revealed only at certain points in time, and decisions have to be made without knowing the outcome of future random events. Again we assume that each random event neither depends on the outcome of previous random events nor on the decisions made in the previous stages. The integer programs which we obtain by discretizing the probability distributions of the random variables involved quickly become very big and hard to solve in practice. However, as in the two-stage example above, the non-anticipativity assumption leads to highly structured problem matrices.

In [12], Hemmecke and Schultz exploited this structure to construct a novel algorithm for the solution of two-stage stochastic integer programs, which is based on successive augmentation of a given feasible solution. The main goal of this paper is to extend the ideas presented in [9, 12] from two-stage to multi-stage programs. Instead of decomposing the problem itself, as essentially all other decomposition approaches to two- and multi-stage stochastic integer programming do, we will decompose an object that is closely related to the given problem matrix: its *Graver test set* (or *Graver basis*). The structure of the problem matrix also imposes a lot of structure on the elements in the Graver basis: for a given family of $(k+1)$-stage stochastic integer programs ($k \in \mathbb{N}$) we can define a certain (*a priori* infinite) set $\mathcal{H}_{k,\infty}$ of "building blocks," from which all elements in the Graver basis of the coefficient matrix can be reconstructed, *independent* of the number N of scenarios. (We refer to Section 8 for the precise definition.) We will prove that this set of building blocks, in fact, is always finite. We also show how to compute $\mathcal{H}_{k,\infty}$ (even if only theoretically), and how it can be employed to solve any given particular instance of the given family of $(k+1)$-stage stochastic integer programs.

It is perhaps remarkable that our proof of the finiteness of $\mathcal{H}_{k,\infty}$ rests on some non-trivial properties of the set $\mathbb{N}^n$ of n-tuples of natural numbers. Some of those have appeared before, under various guises, in computational algebra. The most prominent one, known as "Dickson's Lemma," can formulated like this:

> *For every infinite sequence $X^{\nu^{(1)}}, X^{\nu^{(2)}}, \ldots$ of monomials in the polynomial ring $\mathbb{Q}[X] = \mathbb{Q}[X_1, \ldots, X_n]$, where $X^\nu = X_1^{\nu_1} \cdots X_n^{\nu_n}$ for $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}^n$, there exist indices $i < j$ such that $X^{\nu^{(i)}}$ divides $X^{\nu^{(j)}}$.*

This simple combinatorial fact alone is at the heart of many finiteness phenomena in commutative algebra, since it readily implies Hilbert's Basis Theorem: every ideal of the polynomial ring $K[X] = K[X_1, \ldots, X_n]$, where K is a field, is finitely generated. Dickson's Lemma yields that every monomial ideal of $K[X]$ (that is, an ideal generated by monomials) is finitely generated (see, e.g., Proposition 3.1 below), and Gordan's famous proof of the Hilbert Basis Theorem extends this to all ideals of $K[X]$. Now Dickson's Lemma in turn is a consequence of a somewhat more powerful (and less obvious) finiteness statement:

> *There is no infinite sequence $I^{(1)}, I^{(2)}, \dots$ of monomial ideals in the polynomial ring $\mathbb{Q}[X]$ such that $I^{(i)} \not\supseteq I^{(j)}$ for all $i < j$.*

This principle can be easily shown using known techniques in the subject of "well-quasi-orderings". Maclagan [18] rediscovered this fact (by primary decomposition of monomial ideals) and demonstrated how it can be used to give short proofs of several other finiteness statements like the existence of universal Gröbner bases and the finiteness of the number of atomic fibers of a matrix with non-negative integer entries. (Other applications can be found in [8, 11, 19].) The connections to well-quasi-orderings have been made explicit and further explored by the first author and Pong in [1]. See also [31] for an instance where the theory of well-quasi-orderings can be employed to give a quick proof of an interesting finiteness statement in algebraic statistics.

In this paper we exploit an infinite hierarchy of finiteness principles, of which the statements above only represent the two bottom levels. For a precise formulation we refer to Theorem 5.9 below. Here, let us just state an attractive consequence of the next statement in the hierarchy:

Theorem. *Let $\mathcal{S}$ be a collection of monomial ideals in the polynomial ring $\mathbb{Q}[X]$, and let $\mathcal{M}_1, \mathcal{M}_2, \dots$ be an infinite sequence of collections of monomial ideals from $\mathcal{S}$, where each $\mathcal{M}_i$ is closed under inclusion (if $I \in \mathcal{M}_i$ and $J \in \mathcal{S}$ is a monomial ideal such that $J \subseteq I$, then $J \in \mathcal{M}_i$). Then $\mathcal{M}_i \subseteq \mathcal{M}_j$ for some indices $i \neq j$.*

(The statement remains true if "closed under inclusion" is replaced by "closed under reverse inclusion.")

We construct this hierarchy using Nash-Williams' beautiful theory of "better-quasi-ordered sets". This theory, although of a fundamentally combinatorial nature, is probably less well-known in the field of algorithmic algebra than among logicians, who explored its connections to descriptive set theory and computability theory [32], and, more recently, investigated its logical strength [3, 21]. It is for this reason that we include an introduction to this subject in Part 1 of the paper (Sections 1–5), with the hope that it will become useful as a general guide for proving finiteness statements and for establishing termination of algebraic algorithms. We finish Part 1 by applying Nash-Williams' theory to prove a few finiteness properties for Hilbert functions (in Section 6), some of which are known (Corollary 6.4), and some of which might be new (Proposition 6.5).

In Part 2 of the paper we then apply the theorems of Part 1 to establish finiteness and computability in the decomposition approach to solve multi-stage stochastic integer programs mentioned above. We begin by giving a brief introduction to Graver test sets for integer linear programs. For a more thorough treatment see, e.g., [10]. After introducing $\mathcal{H}_{k,\infty}$ in Section 8, we show that this set is always finite, and give an algorithm to compute a set of vectors containing it (in Section 9). The set $\mathcal{H}_{k,\infty}$ holds an enormous amount of information. We finish the paper by showing how knowledge of (an object related to) $\mathcal{H}_{k,\infty}$ allows one to solve any given instance of our family of $(k+1)$-stage stochastic integer programs, for any given number of scenarios.

Part 1. Noetherian Orderings and Monomial Ideals

1. Preliminaries

We first introduce some notations about sets of natural numbers which are constantly used throughout Part 1. We then discuss an infinitary version of Ramsey's Theorem due to Galvin and Prikry, which, together with the notion of a "barrier" (introduced below) is at the base of Nash-Williams' theory.

Sets and sequences of natural numbers. Throughout this paper, m and n range over the set $\mathbb{N} = \{0, 1, 2, \dots\}$ of natural numbers. For a set X, we denote the power set of X (the set of all subsets of X) by $\mathcal{P}(X)$. Given $X \subseteq \mathbb{N}$ and n, we denote by $[X]^n$ the set of all subsets of X consisting of exactly n elements. We let

$$[X]^{<\omega} := \bigcup_{n\in\mathbb{N}} [X]^n,$$

the set of all finite subsets of X. By $[X]^\omega$ we denote the set of all infinite subsets of X. (So $\mathcal{P}(X) = [X]^{<\omega} \cup [X]^\omega$.)

For every subset of $\mathbb{N}$, there is a unique sequence enumerating it in strictly increasing order. We will identify subsets of $\mathbb{N}$ and strictly increasing sequences of natural numbers in this way. (The empty set $\emptyset$ corresponds to the empty sequence.) For example, for $X \in [\mathbb{N}]^\omega$ and $a \in \mathbb{N}$, this identification allows us to define $X^{>a} \in [\mathbb{N}]^\omega$ by

$$X^{>a} := \{x \in X : x > a\}.$$

In the rest of this section, s, t, u range over $[\mathbb{N}]^{<\omega}$, and U, V, W, X over $[\mathbb{N}]^\omega$. We denote by $l(s)$ the cardinality of s. So if s is identified with the corresponding strictly increasing sequence, then $l(s)$ is its **length.** For every $0 \leqslant i < l(s)$ we write s_i for the $(i+1)$-st element of s; therefore, we can write s as $s = (s_0, \dots, s_{l(s)-1})$, with $s_0 < s_1 < \cdots < s_{l(s)-1}$. We write $s \preceq t$ if s is an initial segment of t, that is, $l(s) \leqslant l(t)$ and $s_i = t_i$ for all $0 \leqslant i < l(s)$. We put $s \prec t$ if s is a proper initial segment of t, i.e., $s \preceq t$ and $s \neq t$. These relations extend in a natural way also to the case where t is replaced by an infinite subset of $\mathbb{N}$. Clearly $s \preceq t$ implies $s \subseteq t$.

If $l(s) = n \geqslant 1$, then $s \setminus \{\min s\}$ is the sequence $(s_1, \dots, s_{n-1})$ obtained from $s = (s_0, s_1, \dots, s_{n-1})$ by leaving out its first element s_0. Similarly if $l(s) \geqslant 1$, then $t \setminus \{\max t\}$ is the sequence obtained from t by leaving out its last element. For non-empty s, t we set

$$s \lhd t \qquad :\Longleftrightarrow \qquad s \setminus \{\min s\} \preceq t \setminus \{\max t\}.$$

Note that the relation $\prec$ on $[\mathbb{N}]^{<\omega}$ is irreflexive ($s \not\prec s$ for all $s \in [\mathbb{N}]^{<\omega}$) and transitive (if $s \prec t$ and $t \prec u$, then $s \prec u$), whereas the relation $\lhd$ on the set $[\mathbb{N}]^{<\omega}$ is neither irreflexive nor transitive.

The Ellentuck topology. We write $s < U$ if $\max s < \min U$. Here and below, $\max \emptyset := -\infty < a$ for all $a \in \mathbb{N}$. If $s < U$, we put

$$[s, U] := \big\{X \in [s \cup U]^\omega : s \prec X\big\}.$$

We endow $[\mathbb{N}]^\omega$ with the **Ellentuck topology,** whose basic open sets are the sets of the form $[s, U]$ for $s < U$ as above. (This topology was first introduced in [5].) For $X \in [\mathbb{N}]^\omega$, we consider each $[X]^\omega$ as a subspace of $[\mathbb{N}]^\omega$, equipped with the induced topology.

A theorem of Galvin and Prikry. If $\mathbb{N} = P \cup Q$ is a partition of $\mathbb{N}$, then one of P or Q is infinite, by the familiar (Dirichlet) pigeon-hole principle. Ramsey's Theorem [29] is an extension of this principle: if $[\mathbb{N}]^n = P \cup Q$ is a partition of $[\mathbb{N}]^n$, then there is $H \in [\mathbb{N}]^\omega$ such that $[H]^n \subseteq P$ or $[H]^n \subseteq Q$. Such a set H is called a **homogeneous** set for the partition $[\mathbb{N}]^n = P \cup Q$. Later on, we will need a far-reaching generalization of this theorem, concerning partitions $[\mathbb{N}]^\omega = P \cup Q$ of $[\mathbb{N}]^\omega$. We have to place some restrictions on the nature of the partitioning sets P and Q, since the natural analogue of Ramsey's Theorem for partitions of $[\mathbb{N}]^\omega$ fails for pathological partitions constructed using the Axiom of Choice. (See [15], Section 19.)

Theorem 1.1. (Galvin-Prikry [6].) *Let $X \in [\mathbb{N}]^\omega$, and suppose $[X]^\omega = P \cup Q$ is a partition of $[X]^\omega$, where P is an open set (in the Ellentuck topology). Then there exists $H \in [X]^\omega$ such that $[H]^\omega \subseteq P$ or $[H]^\omega \subseteq Q$.*

We refer to [15], Section 19 or [32] for a proof.

Blocks and barriers. The definition of Nash-Williams ordering uses the language of "blocks" and "barriers." The reader may skip this subsection at first reading and come back to it when it is really needed (in Section 5).

Definition 1.2. A subset B of $[X]^{<\omega}$ is called a **block** (with **base** X) if

(1) $[X]^\omega = \bigcup_{s\in B} [s, X^{>\max s}]$, and
(2) $s \not\sqsubset t$ for all $s, t \in B$.

In other words, a subset B of $[X]^{<\omega}$ is a block with base X if and only if for every strictly increasing infinite sequence $(x_0, x_1, \dots)$ of elements of X, there exists a unique n such that $(x_0, x_1, \dots, x_n) \in B$. It follows that every strictly increasing sequence over X (finite or infinite) has at most one initial segment which lies in B. (Since every element of $[X]^{<\omega}$ occurs as an initial segment of some element of $[X]^\omega$.) Moreover, note that for a block B, its base is given by $X = \bigcup\{b : b \in B\}$.

Examples.

(1) For any $n > 0$, $[X]^n$ is a block with base X.
(2) Suppose B is a block with base X, and let $C \subseteq B$, $Y \in [X]^\omega$. Then C is a block with base Y if and only if $C = B \cap [Y]^{<\omega}$.

A subset B of $[X]^{<\omega}$ is called a **barrier** if it satisfies condition (1) of Definition 1.2 and

(2′) $s \not\subset t$ for all $s, t \in B$.

Every barrier is a block, and the statements in the examples above remain true if "block" is replaced by "barrier". Theorem 1.1 has the following important consequence:

Corollary 1.3. *If B is a block with base X and $B = B_1 \cup B_2$, then there is a block $B' \subseteq B$ such that $B' \subseteq B_1$ or $B' \subseteq B_2$.*

Proof. We may assume that B_1 and B_2 are disjoint, so

$$P = \bigcup_{s\in B_1} [s, X^{>\max s}], \quad Q = \bigcup_{s\in B_2} [s, X^{>\max s}]$$

give a partition of $[X]^\omega$ into open sets P and Q. Hence by Theorem 1.1 there exists $H \in [X]^\omega$ with $[H]^\omega \subseteq P$ or $[H]^\omega \subseteq Q$, and it follows that $B_1 \cap [H]^{<\omega}$ or $B_2 \cap [H]^{<\omega}$, respectively, is a block. □

Remark. Since any block contained in a barrier is itself a barrier, the corollary remains true with "block" replaced by "barrier."

The following construction turns out to be very useful:

Proposition 1.4. *If B is a barrier with base $X \in [\mathbb{N}]^\omega$, then so is*

$$B(2) := \{b(1) \cup b(2) : b(1), b(2) \in B, b(1) \lhd b(2)\}.$$

For every $b \in B(2)$ there exist unique $b(1), b(2) \in B$ such that $b = b(1) \cup b(2)$ and $b(1) \lhd b(2)$.

In the proof, we need:

Lemma 1.5. *For every $Y \in [X]^\omega$ and $b(1) \in B$ with $b(1) \preceq Y$ there exists $b(2) \in B$ such that $b(1) \lhd b(2)$ and $b(1) \cup b(2) \preceq Y$.*

Proof. The infinite sequence $Y \setminus \{\min Y\} \in [X]^\omega$ has an initial segment $b(2)$ in B, by condition (1) of Definition 1.2. By condition (2) in that definition, $b(2) \not\prec b(1)$, and therefore $b(1) \lhd b(2)$. □

Proof (of Proposition 1.4). Let $Y \in [X]^\omega$, and let $b(1)$ be an initial segment of Y in B. By the lemma, there exists $b(2) \in B$ with $b(1) \lhd b(2)$ and $b(1) \cup b(2) \preceq Y$. Thus $B(2)$ satisfies condition (1) in the definition of a block. Now let $b(1), b(2), c(1), c(2) \in B$, with $b(1) \lhd b(2)$, $c(1) \lhd c(2)$, and suppose that $b \subseteq c$, where $b = b(1) \cup b(2)$, $c = c(1) \cup c(2)$. Then $b(2) = b \setminus \{\min b\} \subseteq c \setminus \{\min c\} = c(2)$, hence $b(2) = c(2)$, by condition (2′) in the definition of a barrier. It follows that $b = c$ and $c(1) = b(1)$. This shows that $B(2)$ is a barrier, and also implies the last statement. □

Corollary 1.6. *Let B be a barrier with base X.*

1. *If $b, c \in B(2)$, then $b \lhd c$ if and only if $b(2) = c(1)$.*
2. *If $C \subseteq B(2)$ is a barrier with base $Y \subseteq X$, then*

$$C^* := \{c(1), c(2) : c \in C\} \subseteq B$$

is a barrier with base $Y \subseteq X$. If $c_1, c_2 \in C^*$ satisfy $c_1 \lhd c_2$, then there exists a unique $c \in C$ such that $c_1 = c(1)$ and $c_2 = c(2)$.

Proof. Part (1) follows from condition (2) in Definition 1.2 and the fact that $b(2) = b \setminus \{\min b\}$ and $c(1) \preceq c \setminus \{\max c\}$. For part (2), suppose $C \subseteq B(2)$ is a barrier with base $Y \subseteq X$, so $C = B(2) \cap [Y]^{<\omega}$. Let $b(1) \in B \cap [Y]^{<\omega}$. By Lemma 1.5, there exists $b(2) \in B \cap [Y]^{<\omega}$ with $b(1) \lhd b(2)$. Hence $b = b(1) \cup b(2)$ is an element of $B(2) \cap [Y]^{<\omega} = C$, and thus $b(1), b(2) \in C^*$. This shows that $C^* = B \cap [Y]^{<\omega}$, that is, C^* is a barrier. Finally, if $c_1 \lhd c_2$ are elements of C^*, let $c = c_1 \cup c_2 \in B(2)$; then $c(1) = c_1$, $c(2) = c_2$ as required. □

2. Orderings

We state some definitions and facts concerning (partially) ordered sets and maps between them, and give some examples.

Ordered sets. A **quasi-ordering** on a set S is a binary relation $\leqslant$ on S which is reflexive and transitive; in this case, we call $(S, \leqslant)$ a **quasi-ordered set.** (If no confusion is possible, we will omit $\leqslant$ from the notation, and just call S a quasi-ordered set.) If in addition the relation $\leqslant$ is anti-symmetric, then $\leqslant$ is called an **ordering** on the set S, and $(S, \leqslant)$ (or S) is called an **ordered set.** If moreover $x \leqslant y$ or $y \leqslant x$ for all $x, y \in S$, then $\leqslant$ is called a **total ordering** on S. In the literature, what we call an ordering and an ordered set is often called a partial ordering and a partially ordered set (or poset), respectively. We write as usual $x < y$ if $x \leqslant y$ and $y \nleqslant x$.

Maps between ordered sets. A function $\varphi \colon S \to T$ between quasi-ordered sets $(S, \leqslant_S)$ and $(T, \leqslant_T)$ is called **increasing** if

$$x \leqslant_S y \Rightarrow \varphi(x) \leqslant_T \varphi(y), \qquad \text{for all } x, y \in S,$$

and **strictly increasing** if

$$x <_S y \Rightarrow \varphi(x) <_T \varphi(y), \qquad \text{for all } x, y \in S.$$

Given a quasi-ordering $\leqslant$ on a set S, there exists a unique ordering on the set $S/{\sim} := \{a/{\sim} : a \in S\}$ of equivalence classes of the equivalence relation

$$x \sim y \qquad \Longleftrightarrow \qquad x \leqslant y \text{ and } y \leqslant x$$

on S such that the surjective map $a \mapsto a/{\sim} \colon S \to S/{\sim}$ is increasing. Hence there is usually no loss in generality when working with orderings rather than quasi-orderings. In the following, we shall therefore concentrate on *ordered* sets, and mostly leave it to the reader to adapt the definitions and results to the quasi-ordered case.

Quasi-embeddings, embeddings, and isomorphisms. A **quasi-embedding** between ordered sets $(S, \leqslant_S)$ and $(T, \leqslant_T)$ is a map $\varphi \colon S \to T$ such that

$$\varphi(x) \leqslant_T \varphi(y) \Rightarrow x \leqslant_S y, \qquad \text{for all } x, y \in S,$$

and if in addition φ is increasing, then φ is called an **embedding.** Any quasi-embedding between ordered sets is injective, and any embedding is strictly increasing. A surjective embedding $S \to T$ is an **isomorphism** of the ordered sets $(S, \leqslant_S)$ and $(T, \leqslant_T)$. We say that an ordering $\preceq_S$ on S **extends** the ordering $\leqslant_S$ if the identity on S is an increasing map between $(S, \leqslant_S)$ and $(S, \preceq_S)$, that is, if $\leqslant_S \subseteq \preceq_S$ (as subsets of $S \times S$). We write $(S, \leqslant_S) \subseteq (T, \leqslant_T)$ if $S \subseteq T$ and the natural inclusion $S \to T$ is an embedding (i.e., $\leqslant_S$ is the restriction of $\leqslant_T$ to S).

Examples. Here are some methods for constructing new ordered sets from old ones.

(1) Any subset of an ordered set $(S, \leqslant)$ can be naturally made into an ordered set by restricting the ordering $\leqslant$ to this subset.
(2) The disjoint union $S \amalg T$ of ordered sets $(S, \leqslant_S)$ and $(T, \leqslant_T)$ can be naturally made into an ordered set via the relation $\leqslant_S \cup \leqslant_T$.
(3) The cartesian product $S \times T$ of ordered sets $(S, \leqslant_S)$ and $(T, \leqslant_T)$ can be naturally made into an ordered set by the product ordering

$$(x, y) \leqslant_{S \times T} (x', y') \quad :\Longleftrightarrow \quad x \leqslant_S x' \text{ and } y \leqslant_T y'.$$

(4) Given a set S we denote by $S^\diamond$ the free commutative monoid generated by S. An ordering $\leqslant$ on S extends naturally to an ordering $\leqslant^\diamond$ on $S^\diamond$ as follows:

$$s_1 \cdots s_m \leqslant^\diamond t_1 \cdots t_n \;:\Longleftrightarrow\; \begin{cases} \text{there exists an injective map} \\ \varphi\colon \{1,\dots,m\} \to \{1,\dots,n\} \text{ such} \\ \text{that } s_i \leqslant_S t_{\varphi(i)} \text{ for } i = 1,\dots,m. \end{cases}$$

The simple example of the set $\mathbb{N}^n$ of n-tuples of natural numbers, ordered by the product ordering, will play an important role in further sections. (Another key example will be given at the end of Section 5.) It is sometimes convenient to identify the elements of $\mathbb{N}^n$ with *monomials* in a polynomial ring as follows: Let $X = \{X_1, \dots, X_n\}$ be a set of distinct indeterminates, so $X^\diamond = \{X^\nu : \nu \in \mathbb{N}^n\}$ where $X^\nu := X_1^{\nu_1} \cdots X_n^{\nu_n}$ for $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{N}^n$. Order $X^\diamond$ by divisibility:

$$X^\nu \leqslant X^\mu \quad :\Longleftrightarrow \quad \mu = \nu + \lambda \text{ for some } \lambda \in \mathbb{N}^n.$$

That is, in the context of example (4) above (for $S = X$), the ordering on $X^\diamond$ is $\leqslant^\diamond$ where $\leqslant$ is the trivial ordering on X. The map $\nu \mapsto X^\nu \colon \mathbb{N}^n \to X^\diamond$ is an isomorphism of ordered sets. The elements of $X^\diamond$ can be seen as the monomials in the polynomial ring $R[X] = R[X_1, \dots, X_n]$ over an arbitrary commutative ring R.

Initial and final segments. An **initial segment** of an ordered set $(S, \leqslant)$ is a subset $I \subseteq S$ such that

$$x \leqslant y \text{ and } y \in I \Rightarrow x \in I, \qquad \text{for all } x, y \in S.$$

Dually, $F \subseteq S$ is called a **final segment** if $S \setminus F$ is an initial segment. (In the combinatorial literature, initial and final segments are sometimes called order ideals and dual order ideals, respectively.) Given a subset X of S, we denote by

$$(X) := \big\{ y \in S : \exists x \in X (x \leqslant y) \big\}$$

the final segment of S generated by X, and by

$$[X] := \big\{ y \in S : \exists x \in X (x \geqslant y) \big\}$$

the initial segment generated by X. The set $\mathcal{I}(S)$ of initial segments of S is naturally ordered by inclusion: $I \leqslant J \iff I \subseteq J$, for $I, J \in \mathcal{I}(S)$. Dually, we construe the set $\mathcal{F}(S)$ of final segments of S as an ordered set, with the ordering given by *reverse* inclusion: $F \leqslant G \iff F \supseteq G$, for $F, G \in \mathcal{F}(S)$. The intersection and union of an arbitrary family of initial (final) segments of S is also an initial (resp., final) segment of S.

Example. The isomorphism $\nu \mapsto X^\nu$ identifies $\mathbb{N}^n$ and the monoid of monomials in the polynomial ring $R[X]$. The ordered set $\big(\mathcal{F}(\mathbb{N}^n), \supseteq\big)$ of final segments of $\mathbb{N}^n$ may be identified with the set of *monomial ideals* of $R[X]$ (that is, ideals of $R[X]$ which are generated by monomials), ordered by reverse inclusion.

Given a quasi-ordered set $(S, \leqslant)$, we can define a quasi-ordering $\leqslant_{\mathcal{P}(S)}$ on the power set $\mathcal{P}(S)$ of S as follows: for $X, Y \in \mathcal{P}(S)$,

(2.1) $$X \leqslant_{\mathcal{P}(S)} Y \quad \iff \quad \text{for every } y \in Y \text{ there exists } x \in X \text{ such that } x \leqslant y.$$

Note that $F \supseteq G \iff F \leqslant_{\mathcal{P}(S)} G$ for all $F, G \in \mathcal{F}(S)$, and $X \leqslant_{\mathcal{P}(S)} \langle X \rangle$ for every $X \subseteq S$. This implies that the map

$$F \mapsto F/{\sim} \colon \mathcal{F}(S) \to \mathcal{P}(S)/{\sim}$$

is an isomorphism of ordered sets. Here $\sim$ is the equivalence relation on $\mathcal{P}(S)$ associated to $\leqslant_{\mathcal{P}(S)}$ as in the beginning of this section.

Pullback of final segments. For any function $\varphi\colon S \to T$ between ordered sets $(S, \leqslant_S)$ and $(T, \leqslant_T)$, we get an induced increasing function

$$\varphi^*\colon F \mapsto \big(\varphi^{-1}(F)\big),\ \mathcal{F}(T) \to \mathcal{F}(S)$$

between the ordered sets $\big(\mathcal{F}(T), \supseteq\big)$ and $\big(\mathcal{F}(S), \supseteq\big)$.

Example. Let $(S, \leqslant_S)$, $(T, \leqslant_T)$, and $(U, \leqslant_U)$ be ordered sets with $(S, \leqslant_S), (T, \leqslant_T) \subseteq (U, \leqslant_U)$, so the natural inclusions $i_S\colon S \to U$ and $i_T\colon T \to U$ are increasing. Then

$$i_S^*(F) = F \cap S, \quad i_T^*(F) = F \cap T \qquad \text{for any } F \in \mathcal{F}(U).$$

Hence if in addition $U = S \cup T$, then $i_S^* \times i_T^*$ gives an embedding $\mathcal{F}(U) \to \mathcal{F}(S) \times \mathcal{F}(T)$.

The following rules will be used later:

Lemma 2.1. *Let S and T be ordered sets, and $\varphi\colon S \to T$.*

1. *If φ is a quasi-embedding, then φ^* is surjective.*
2. *If φ is increasing and surjective, then φ^* is a quasi-embedding.*

Proof. Since part (2) is clear, we just prove (1). Suppose φ is a quasi-embedding, and let $G \in \mathcal{F}(S)$. Let F be the final segment of T generated by $\varphi(G)$. Then clearly $G \subseteq \varphi^{-1}(F)$. Conversely, if $x \in S$ satisfies $\varphi(x) \in F$, then $\varphi(x) \geqslant \varphi(g)$ for some $g \in G$. Since φ is a quasi-embedding, we have $x \geqslant g$, so $x \in G$. This shows $\varphi^*(F) = G$. $\square$

Antichains. For elements x, y of an ordered set S, we write $x \| y$ if $x \nleqslant y$ and $y \nleqslant x$. An **antichain** of S is a subset $A \subseteq S$ such that $x \| y$ for all $x \neq y$ in A. An element x of S is called a **minimal** element of S if $y \leqslant x \Rightarrow y = x$ for all $y \in S$. The minimal elements of a subset $X \subseteq S$ form an antichain, denoted by $X_{\min}$.

Well-founded orderings. An ordered set S is **well-founded** if there is no infinite strictly decreasing sequence $x_0 > x_1 > \cdots$ in S. For any element x of a subset X of a well-founded ordered set S, there is at least one minimal element $y \in X_{\min}$ with $y \leqslant x$. It follows that any final segment F of S is generated by its antichain $F_{\min}$ of minimal elements; in fact, $F_{\min}$ is the (unique) smallest generating set for F.

3. Noetherian Orderings

We say that an ordered set S is **Noetherian** if it is well-founded and every antichain of S is finite. More generally, we say that a quasi-ordered set S is Noetherian if the associated ordered set $S/{\sim}$ is Noetherian. Since every antichain of a totally ordered set consists of at most one element, a totally ordered set S is Noetherian if and only if it is well-founded; in this case S is called **well-ordered.**

Remark. Noetherian orderings are usually called "well-quasi-orderings" in the literature (see, e.g., [17]). Following a proposal by Joris van der Hoeven [14] we use the more concise term "Noetherian".

An infinite sequence $x_0, x_1, \ldots$ in S is **good** if $x_i \leqslant x_j$ for some $i < j$, and **bad,** otherwise. Clearly, if $\{x_0, x_1, \ldots\}$ is an antichain, then $x_0, x_1, \ldots$ is a bad sequence. The following characterization of Noetherian orderings is folklore (see, e.g., [22]).

Proposition 3.1. *Let S be an ordered set. The following are equivalent:*

1. *S is Noetherian.*
2. *Every infinite sequence $x_0, x_1, \ldots$ in S contains an increasing subsequence.*
3. *Every infinite sequence $x_0, x_1, \ldots$ in S is good.*
4. *Any subset $X \subseteq S$ has only finitely many minimal elements, and for every $x \in X$ there is a minimal element y of X with $x \geqslant y$.*
5. *Any final segment of S is finitely generated.*
6. *$\big(\mathcal{F}(S), \supseteq\big)$ is well-founded (i.e., the ascending chain condition with respect to inclusion holds for final segments of S).*
7. *$\big(\mathcal{I}(S), \subseteq\big)$ is well-founded (i.e., the descending chain condition with respect to inclusion holds for initial segments of S).*

Proof. The implication (1) $\Rightarrow$ (2) follows from applying Ramsey's Theorem to the partition $[\mathbb{N}]^2 = P \cup Q \cup R$, where

$$P = \big\{\{i, j\} : x_i \| x_j\big\}, \quad Q = \big\{\{i, j\} : i < j, x_i > x_j\big\},$$

and $R = [\mathbb{N}]^2 \setminus (P \cup Q)$. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (1) are trivial, and (1) $\Rightarrow$ (4) $\Rightarrow$ (5) follows from the remarks at the end of the last section. If $F_0 \subseteq F_1 \subseteq \cdots$ is an ascending chain of final segments of S, then $F = \bigcup_n F_n$ is a final segment of S. If F is finitely generated, say by $X \subseteq F$, then $X \subseteq F_n$ for some n; thus $F_n = F_{n+1} = \cdots$. This shows (5) $\Rightarrow$ (6); by passing to complements, we obtain (6) $\iff$ (7). For (6) $\Rightarrow$ (3), let $x_0, x_1, \ldots$ be a sequence in S. By (6), the sequence $(x_0) \subseteq (x_0, x_1) \subseteq \cdots$ of final segments of S becomes stationary: for some n, we have $x_j \in (x_0, \ldots, x_n)$ for all $j > n$. In particular, $x_i \leqslant x_{n+1}$ for some $i \in \{0, \ldots, n\}$. ☐

The proposition now immediately provides the following construction methods for Noetherian orderings:

Examples. Suppose $(S, \leqslant_S)$ and $(T, \leqslant_T)$ are ordered sets. Then:

1. If there exists an increasing surjection $S \to T$, and $(S, \leqslant_S)$ is Noetherian, then $(T, \leqslant_T)$ is Noetherian. In particular, if $(S, \leqslant_S)$ is Noetherian, then any ordering on S which extends $\leqslant_S$ is Noetherian.
2. If there exists a quasi-embedding $S \to T$, and $(T, \leqslant_T)$ is Noetherian, then $(S, \leqslant_S)$ is Noetherian. In particular, if $(T, \leqslant_T)$ is Noetherian, then any subset of T with the induced ordering is Noetherian.
3. If $(S, \leqslant_S)$ and $(T, \leqslant_T)$ are Noetherian and $(U, \leqslant_U)$ is an ordered set such that $(S, \leqslant_S), (T, \leqslant_T) \subseteq (U, \leqslant_U)$, then $S \cup T$ is Noetherian. In particular, it follows that $S \amalg T$ is Noetherian.
4. If $(S, \leqslant_S)$ and $(T, \leqslant_T)$ are Noetherian, then so is $S \times T$. Inductively, it follows that if the ordered set $(S, \leqslant_S)$ is Noetherian, then so is S^n equipped with the product ordering, for every $n > 0$.

Applying Example (4) to $S = \mathbb{N}$, we obtain:

Corollary 3.2. (Dickson's Lemma.) *For each $n > 0$, the ordered set $\mathbb{N}^n$ is Noetherian.* ☐

4. Strongly Noetherian Orderings

Besides the results stated in the examples following Proposition 3.1, several other preservation theorems for Noetherian orderings are known. For example, if $(S, \leqslant)$

is a Noetherian ordered set, then so is $(S^\diamond, \leqslant^\diamond)$ as defined in Section 2. (This was first proved by Higman [13], with a simplified proof given by Nash-Williams [24].) Similar theorems can be proved for other ordered sets built from S, for example the collection of all finite trees whose nodes are labelled by elements of S, ordered by the homeomorphic embedding ordering (Kruskal [16]). The common feature of all these constructions is their *finitary* character. If one builds ordered sets by allowing operations of an infinite nature, the situation changes drastically: for example, if S is a Noetherian ordered set, then $\big(\mathcal{F}(S), \supseteq\big)$ is in general *not* Noetherian. An example for this phenomenon was first given by Rado [28]: Let $R := [\mathbb{N}]^2$, ordered by the rule

$$(i,j) \leqslant_R (k,l) \qquad \Longleftrightarrow \qquad \text{either } i = k \text{ and } j \leqslant l, \text{ or } j < k.$$

It is quickly verified (see, e.g., [18]) that $(R, \leqslant_R)$ is a Noetherian ordered set, but $\big(\mathcal{F}(R), \supseteq\big)$ is not Noetherian: the sequence $F_1, F_2, \dots$ where F_j is the final segment of R generated by all $(i,j) \in \mathbb{N}^2$ with $i < j$, is an infinite antichain in $\mathcal{F}(R)$. This example is archetypical in the following sense:

Theorem 4.1. *For a Noetherian ordered set $(S, \leqslant_S)$, the following are equivalent:*

1. $\big(\mathcal{F}(S), \supseteq\big)$ *is not Noetherian.*
2. $\big(\mathcal{I}(S), \subseteq\big)$ *is not Noetherian.*
3. *There exists a function $f\colon [\mathbb{N}]^2 \to S$ with $f(i,j) \not\leqslant_S f(j,k)$ for all $i < j < k$.*
4. *There exists an embedding $(R, \leqslant_R) \to (S, \leqslant_S)$.*
5. *There exists a quasi-embedding $(R, \leqslant_R) \to (S, \leqslant_S)$.*

Proof. The implication (1) $\Rightarrow$ (2) is trivial. Suppose $I_0, I_1, \dots$ is a bad sequence in $\big(\mathcal{I}(S), \subseteq\big)$. So for each $i < j$ there exists $a \in I_i$ with $a \not\leqslant b$ for all $b \in I_j$. Hence it is possible to choose, for each $i < j$, an element $f(i,j) \in I_i$ such that for all $i < j < k$, we have $f(i,j) \not\leqslant_S f(j,k)$. This shows (2) $\Rightarrow$ (3). For (3) $\Rightarrow$ (4), let $f\colon [\mathbb{N}]^2 \to S$, $f(i,j) = a_{ij}$, be as in (3). Consider the partitions

$$[\mathbb{N}]^3 = P \cup Q, \qquad [\mathbb{N}]^4 = P' \cup Q'$$

given by

$$P = \big\{\{i,j,k\} : i < j < k, a_{ij} \leqslant a_{ik}\big\}, \qquad Q = [\mathbb{N}]^3 \setminus P$$

and

$$P' = \big\{\{i,j,k,l\} : i < j < k < l, a_{ij} \leqslant a_{kl}\big\}, \qquad Q' = [\mathbb{N}]^4 \setminus P'.$$

By applying Ramsey's Theorem twice we obtain an infinite set $H \subseteq \mathbb{N}$ which is homogeneous for both partitions. Since S is Noetherian, we must have $[H]^3 \subseteq P$, $[H]^4 \subseteq P'$. It follows that

$$(i,j) \leqslant_R (k,l) \qquad \Longleftrightarrow \qquad a_{ij} \leqslant_S a_{kl},$$

for all $i < j$, $k < l$ in H. Therefore, $\{a_{ij} : i < j, i,j \in H\}$ (with the ordering induced from S) is isomorphic to $(R, \leqslant_R)$. The implication (4) $\Rightarrow$ (5) is again trivial, and (5) $\Rightarrow$ (1) follows from Lemma 2.1, (1) and Example (1) following Proposition 3.1. $\square$

Let us call an ordered set $(S, \leqslant_S)$ **strongly Noetherian** if $\big(\mathcal{F}(S), \supseteq\big)$ is Noetherian. (So *a forteriori,* $(S, \leqslant_S)$ is Noetherian, by Proposition 3.1.) The following facts follow easily from Lemma 2.1 and the examples after Proposition 3.1:

Examples. Suppose $(S, \leqslant_S)$ and $(T, \leqslant_T)$ are ordered sets. Then:

(1) If there exists an increasing surjection $S \to T$ and $(S, \leqslant_S)$ is strongly Noetherian, then $(T, \leqslant_T)$ is strongly Noetherian. In particular, if $(S, \leqslant_S)$ is strongly Noetherian, then any ordering on S which extends $\leqslant_S$ is strongly Noetherian.
(2) If there exists a quasi-embedding $S \to T$ and $(T, \leqslant_T)$ is strongly Noetherian, then $(S, \leqslant_S)$ is strongly Noetherian. In particular, if $(S, \leqslant_S)$ is strongly Noetherian, then any subset of S with the induced ordering is strongly Noetherian.
(3) If $(S, \leqslant_S)$ and $(T, \leqslant_T)$ are strongly Noetherian and $(U, \leqslant_U)$ is an ordered set with $(S, \leqslant_S), (T, \leqslant_T) \subseteq (U, \leqslant_U)$, then $S \cup T$ is strongly Noetherian. In particular, it follows that if $(S, \leqslant_S)$ and $(T, \leqslant_T)$ are strongly Noetherian, then so is $S \amalg T$.

Strong Noetherianity is also preserved under cartesian products:

Proposition 4.2. *If $(S, \leqslant_S)$ and $(T, \leqslant_T)$ are strongly Noetherian ordered sets, then $S \times T$ is strongly Noetherian.*

(This fact was stated without proof in [18] and attributed there to Farley and Schmidt. For another proof see [1].)

Proof. Let $f\colon [\mathbb{N}]^2 \to S \times T$. We denote by $\pi_S\colon S \times T \to S$ the projection $(s,t) \mapsto s$ onto the first component, and we put $f_S := \pi_S \circ f$. Similarly we define $f_T\colon [\mathbb{N}]^2 \to T$. Since S is strongly Noetherian, there are $i < j < k$ with $f_S(i,j) \leqslant_S f_S(j,k)$, by the previous theorem. Now consider the partition $[\mathbb{N}]^3 = P \cup Q$, where

$$P = \big\{\{i,j,k\} : i < j < k,\ f_S(i,j) \leqslant_S f_S(j,k)\big\}, \qquad Q = [\mathbb{N}]^3 \setminus P.$$

By Ramsey's Theorem, we find an infinite homogeneous set $H \subseteq \mathbb{N}$ for this partition. Since S is strongly Noetherian, we must have $[H]^3 \subseteq P$. Changing from $\mathbb{N}$ to H, we thus may assume that $f_S(i,j) \leqslant_S f_S(j,k)$ *for all* $i < j < k$. Since T is strongly Noetherian, there are $i < j < k$ with $f_T(i,j) \leqslant_T f_T(j,k)$. Hence $f(i,j) \leqslant f(j,k)$. Thus $S \times T$ is strongly Noetherian. □

Corollary 4.3. (Maclagan, [18].) *The ordered set $\mathbb{N}^n$ is strongly Noetherian, for every $n > 0$.* □

5. Nash-Williams Orderings

By the equivalence of (1) and (3) in Theorem 4.1, strong Noetherianity may be expressed using the terminology introduced in Section 1: *An ordered set S is strongly Noetherian if and only if for every function $f\colon B \to S$, where $B \subseteq [\mathbb{N}]^2$ is a barrier, there exist $b_1, b_2 \in B$ with $b_1 \lhd b_2$ and $f(b_1) \leqslant f(b_2)$.* The search for a combinatorial condition on an ordered set S which ensures that $\big(\mathcal{F}(S), \supseteq\big)$ is not only Noetherian, but *strongly* Noetherian, therefore led Nash-Williams [25] to introduce the following concept (under the name of "better well-quasi-ordering"). Below, we shall call a function $f\colon B \to S$, whose domain $B \subseteq [\mathbb{N}]^{<\omega}$ is a barrier, an **S-array.** (So in particular, every sequence $x_0, x_1, \dots$ of elements of S can be considered as an S-array.) We say that an S-array $f\colon B \to S$ is **good** if there are $b_1, b_2 \in B$ such that $b_1 \lhd b_2$ and $f(b_1) \leqslant f(b_2)$, **bad** if it is not good, and **perfect** if $f(b_1) \leqslant f(b_2)$ *for all* $b_1 \lhd b_2$ in B.

Definition 5.1. An ordered set S is **Nash-Williams** if every S-array is good. A quasi-ordered set S is Nash-Williams if $S/{\sim}$ is a Nash-Williams ordered set.

Clearly if S is Nash-Williams, then S is strongly Noetherian. Moreover:

Lemma 5.2. *If S is well-ordered, then S is Nash-Williams.*

Proof. Suppose $f\colon B \to S$ is an S-array. Let $b_1 \in B$ be such that $f(b_1) = \min f(B)$. By Lemma 1.5 there exists $b_2 \in B$ such that $b_1 \lhd b_2$, and we have $f(b_1) \leqslant f(b_2)$. $\square$

As we did for strong Noetherianity, we will now successively show that each of the constructions exhibited in the examples following Proposition 3.1 preserves the Nash-Williams property as well.

Lemma 5.3. *Let $(S, \leqslant_S)$ and $(T, \leqslant_T)$ be ordered sets.*

1. *If there exists an increasing surjection $S \to T$ and S is Nash-Williams, then T is Nash-Williams.*
2. *If there exists a quasi-embedding $S \to T$ and T is Nash-Williams, then S is Nash-Williams.*

Proof. Let $\varphi\colon S \to T$. For part (1), suppose φ is increasing and surjective, and let $f\colon B \to T$ be a T-array. Choose any function $\psi\colon T \to S$ such that $\varphi \circ \psi = \mathrm{id}_T$. Then $\psi \circ f\colon B \to S$ is an S-array. Since S is Nash-Williams, there exist $b_1, b_2 \in B$ with $b_1 \lhd b_2$ and $\psi(f(b_1)) \leqslant_S \psi(f(b_2))$, hence $f(b_1) \leqslant_S f(b_2)$. For part (2), assume that φ is a quasi-embedding, and let $g\colon B \to S$ be an S-array. Then $\varphi \circ g\colon B \to T$ is a T-array. Since T is Nash-Williams, there exist $b_1, b_2 \in B$ with $b_1 \lhd b_2$ and $\varphi(g(b_1)) \leqslant_T \varphi(g(b_2))$, hence $g(b_1) \leqslant_S g(b_2)$, since φ is a quasi-embedding. $\square$

In particular, if $(S, \leqslant)$ is a Nash-Williams ordered set, then so is any subset of S with the induced ordering, and any ordering on S which extends the ordering $\leqslant$ on S.

Lemma 5.4. *Suppose that $(S, \leqslant_S)$, $(T, \leqslant_T)$ and $(U, \leqslant_U)$ are ordered sets such that $(S, \leqslant_S), (T, \leqslant_S) \subseteq (U, \leqslant_U)$. If S and T are Nash-Williams, then $S \cup T$ is also Nash-Williams.*

Proof. Let $f\colon B \to S \cup T$ be a bad $S \cup T$-array. Let $B_S := f^{-1}(S)$ and $B_T := f^{-1}(T)$. Then there exists a barrier $B' \subseteq B$ such that $B' \subseteq B_S$ or $B' \subseteq B_T$, by Corollary 1.3. So either $f|B'$ is a bad S-array, or $f|B'$ is a bad T-array, a contradiction. $\square$

By the previous lemma, every finite ordered set is Nash-Williams, and if $(S, \leqslant_S)$ and $(T, \leqslant_T)$ are Nash-Williams, then $S \amalg T$ is Nash-Williams.

The following fact distinguishes Nash-Williams orderings among (strongly) Noetherian orderings.

Proposition 5.5. *Suppose that the ordered set S is Nash-Williams. Then the ordered sets $\big(\mathcal{F}(S), \supseteq\big)$ and $\big(\mathcal{I}(S), \subseteq\big)$ are Nash-Williams.*

Proof. Let $f\colon B \to \mathcal{I}(S)$ be a bad $\mathcal{I}(S)$-array. The set $B(2)$ defined in Proposition 1.4 is a barrier. We construct a bad S-array $g\colon B(2) \to S$ as follows: For every $b \in B(2)$, we have $f(b(1)) \not\subseteq f(b(2))$; so we can choose $g(b) \in f(b(1)) \setminus f(b(2))$. Now suppose $c \in B(2)$ with $b \lhd c$. Then $c(1) = b(2)$ (by Corollary 1.6, (1)), hence $g(b) \not\leqslant g(c)$. Therefore, g is bad. This shows that $\mathcal{I}(S)$ is Nash-Williams. Suppose $h\colon C \to \mathcal{F}(S)$ is an $\mathcal{F}(S)$-array. We then consider the $\mathcal{I}(S)$-array $h'\colon C \to \mathcal{I}(S)$ defined by $h'(c) = S \setminus h(c)$, for all $c \in C$. Since $\mathcal{I}(S)$ is Nash-Williams, we find $c_1 \lhd c_2$ in C with $h'(c_1) \subseteq h'(c_2)$, and hence $h(c_1) \supseteq h(c_2)$. Thus h is good, and $\mathcal{F}(S)$ is Nash-Williams. $\square$

For the definition of the quasi-ordering $\leqslant_{\mathcal{P}(S)}$ in the statement of the next corollary, see (2.1).

Corollary 5.6. *If S is a Nash-Williams quasi-ordered set, then the quasi-ordered set $\big(\mathcal{P}(S), \leqslant_{\mathcal{P}(S)}\big)$ is Nash-Williams.*

Proof. The canonical increasing surjection $\varphi\colon S \to S/{\sim}$ induces an isomorphism

$$\varphi^*\colon \mathcal{F}(S/{\sim}) \to \mathcal{F}(S)$$

of ordered sets. By Proposition 5.5, $\mathcal{F}(S/{\sim})$, and hence $\mathcal{F}(S)$, is Nash-Williams. Since $\mathcal{P}(S)/{\sim} \cong \mathcal{F}(S)$, it follows that $\mathcal{P}(S)$ is Nash-Williams. ☐

Next, we want to show that the cartesian product of any two Nash-Williams ordered sets is again Nash-Williams. The proof is similar to the one of Proposition 4.2 and uses the following lemma. Here, an S**-subarray** of an S-array $f\colon B \to S$ is an S-array $g\colon C \to S$ where $C \subseteq B$, $g = f|C$.

Lemma 5.7. *Let $(S, \leqslant)$ be an ordered set. Every S-array contains either a bad S-subarray or a perfect S-subarray.*

Proof. Let $f\colon B \to S$ be an S-array. Consider the partition $B(2) = P \cup Q$ of the barrier $B(2)$ from Proposition 1.4, where

$$P = \big\{b \in B(2) : f\big(b(1)\big) \leqslant f\big(b(2)\big)\big\}, \qquad Q = B(2) \setminus P.$$

By Corollary 1.3, there is a barrier $C \subseteq B(2)$ such that $C \subseteq P$ or $C \subseteq Q$. The set C^* defined in Corollary 1.6, (2) is a barrier, and $f|C^*$ is either perfect or bad, depending on whether $C \subseteq P$ or $C \subseteq Q$. ☐

Proposition 5.8. *If $(S, \leqslant_S)$ and $(T, \leqslant_T)$ are Nash-Williams ordered sets, then $S \times T$ is Nash-Williams.* ([25], Corollary 22A.)

Proof. Let $f\colon B \to S \times T$ be an $S \times T$-array; we want to show that f is good. By the previous lemma, and since S is Nash-Williams, the S-array $f_S = \pi_S \circ f$ has a perfect S-subarray $B' \to S$. Restricting f to the barrier B', if necessary, we may assume that $f_S\colon B \to S$ is perfect. Since T is Nash-Williams, there exist $b_1, b_2 \in B$ with $b_1 \lhd b_2$ and $f_T(b_1) \leqslant_T f_T(b_2)$, whence $f(b_1) \leqslant f(b_2)$. So f is good. ☐

It follows that $\mathbb{N}^n$ is Nash-Williams, for any $n > 0$. Defining inductively

$$\mathcal{F}_0(\mathbb{N}^n) := \mathbb{N}^n, \qquad \mathcal{F}_{k+1}(\mathbb{N}^n) := \mathcal{F}\big(\mathcal{F}_k(\mathbb{N}^n)\big) \text{ for all } k \in \mathbb{N},$$

we get, by Proposition 5.5:

Theorem 5.9. *Each of the sets*

$$\mathcal{F}_1(\mathbb{N}^n) = \mathcal{F}(\mathbb{N}^n),\ \mathcal{F}_2(\mathbb{N}^n) = \mathcal{F}\big(\mathcal{F}_1(\mathbb{N}^n)\big),\ \ldots,$$

ordered by reverse inclusion, is Noetherian. ☐

Remark. Theorem 5.9 above implies that every subset of $\mathcal{F}(\mathbb{N}^n)$ is strongly Noetherian, and hence yields the theorem in the introduction.

A key example. Consider the ordering $\sqsubseteq$ on the set $\mathbb{Z}^n$ of n-tuples of integers defined as follows: if $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ are elements of $\mathbb{Z}^n$, we put $a \sqsubseteq b$ if $a_j b_j \geqslant 0$ and $|a_j| \leqslant |b_j|$ for all $j = 1, \dots, n$. That is, $a \sqsubseteq b$ if and only if a belongs to the same orthant of $\mathbb{R}^n$ as b, and each of its components is not greater in absolute value than the corresponding component of b. Note that $\sqsubseteq$ extends the product ordering on $\mathbb{N}^n \subseteq \mathbb{Z}^n$. Moreover, we have $0 \sqsubseteq a$ for all $a \in \mathbb{Z}^n$, and if $a \sqsubseteq b$, then $-a \sqsubseteq -b$, for all $a, b \in \mathbb{Z}^n$. The ordered set $(\mathbb{Z}^n, \sqsubseteq)$ can be identified in a natural way with the n-fold cartesian product of $(\mathbb{Z}, \sqsubseteq)$ with itself. We have $(\mathbb{Z}^{\leqslant 0}, \geqslant) \subseteq (\mathbb{Z}, \sqsubseteq)$ and $(\mathbb{Z}^{\geqslant 0}, \leqslant) \subseteq (\mathbb{Z}, \sqsubseteq)$, where $\mathbb{Z}^{\leqslant 0}$ ($\mathbb{Z}^{\geqslant 0}$) denotes the set of non-positive integers (non-negative integers, respectively). This somewhat roundabout way of describing $(\mathbb{Z}^n, \sqsubseteq)$, together with Lemma 5.4 and Proposition 5.8, shows:

Corollary 5.10. *For each $n > 0$, the ordered set $(\mathbb{Z}^n, \sqsubseteq)$ is Nash-Williams (in particular, Noetherian).* □

As above, this corollary implies that each set

$$\mathcal{F}(\mathbb{Z}^n, \sqsubseteq),\ \mathcal{F}\big(\mathcal{F}(\mathbb{Z}^n, \sqsubseteq)\big),\ \dots,$$

ordered by reverse inclusion, is Noetherian. Here is a slight reformulation of this fact which will be used in Part 2. Let $\sqsubseteq_0$ denote the ordering $\sqsubseteq$ on $\mathcal{P}_0(\mathbb{Z}^n) := \mathbb{Z}^n$, and inductively define $\mathcal{P}_{k+1}(\mathbb{Z}^n) := \mathcal{P}\big(\mathcal{P}_k(\mathbb{Z}^n)\big)$ and a quasi-ordering $\sqsubseteq_{k+1}$ of $\mathcal{P}_{k+1}(\mathbb{Z}^n)$ as follows: for $X, Y \in \mathcal{P}_{k+1}(\mathbb{Z}^n)$ put

$$X \sqsubseteq_{k+1} Y \quad \Longleftrightarrow \quad \text{for each } y \in Y \text{ there is some } x \in X \text{ with } x \sqsubseteq_k y. \tag{5.1}$$

(That is, $\sqsubseteq_{k+1} \, = \, \leqslant_{\mathcal{P}(\mathcal{P}_k(\mathbb{Z}^n), \sqsubseteq_k)}$ for all k, in the notation of (2.1).) By induction on k, Corollary 5.6 implies:

Corollary 5.11. *The quasi-ordered set $\big(\mathcal{P}_k(\mathbb{Z}^n), \sqsubseteq_k\big)$ is Noetherian, for each $k \in \mathbb{N}$ and $n > 0$.* □

6. Applications to Hilbert Functions

As an illustration, let us show how Nash-Williams' theory as outlined above can be employed to deduce some finiteness properties for Hilbert functions. We work in a rather general setting. Let $(S, \leqslant)$ be a non-empty ordered set and $\delta\colon S \to A$ a map, where A is any set. We think of δ as providing a **grading** of S by A, and we call $\delta(s) \in A$ the **degree** of $s \in S$. We call a final segment F of S **admissible** if for any degree $a \in A$, there are only finitely many $x \notin F$ with $\delta(x) = a$. If the map δ has finite fibers, then every final segment of S is admissible. (This is the case, for example, if S is Noetherian and $\delta\colon S \to A$ is a strictly increasing map into an *ordered* set A, see Example (1) following Proposition 3.1.)

We extend the ordering of $\mathbb{N}$ to a total ordering of the set $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$ by declaring $n < \infty$ for all $n \in \mathbb{N}$. For a final segment F of S and a degree $a \in A$, we let $h_F(a)$ be the number of elements of S of degree a which are not in F (an element of $\mathbb{N}_\infty$). We shall call $h_F\colon A \to \mathbb{N}_\infty$ the **Hilbert function** of $F \in \mathcal{F}(S)$. (So F is admissible precisely if $h_F(a) \in \mathbb{N}$ for all a.)

Example. Suppose $S = \mathbb{N}^n$, $n > 0$, and $\delta\colon S \to \mathbb{N}$ is given by $\delta(\nu) = \nu_1 + \cdots + \nu_n$ for all $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{N}^n$. Let K be a field, and identify monomial ideals in the polynomial ring $K[X] = K[X_1, \dots, X_n]$ with final segments of $\mathbb{N}^n$ as usual. Given a monomial ideal I in $K[X]$, make $R = K[X]/I$ into a graded K-algebra, grading by degree; then $h_I(a) = \dim_K R_a$ for each $a \in \mathbb{N}$, where R_a is the degree a part of R, so in this case h_I as defined above is the usual Hilbert function of I.

Growth of Hilbert functions. First we show that the growth of Hilbert functions is finitely determined in the following sense. We fix an ordered set S and a grading δ of S by A as above.

Proposition 6.1. *Suppose S is strongly Noetherian, that is, $\mathcal{F}(S)$ is Noetherian, and let $h\colon A \to \mathbb{N}_\infty$ be any function. There exist finite sets $\mathcal{M}_{\geqslant h}$, $\mathcal{M}_{\not\leqslant h}$ and $\mathcal{M}_{\succeq h}$ of final segments of S such that for every final segment F of S, we have*

$$h_F(a) \geqslant h(a) \text{ for all } a \iff F \subseteq E \text{ for some } E \in \mathcal{M}_{\geqslant h},$$
$$h_F(a) > h(a) \text{ for some } a \iff F \subseteq E \text{ for some } E \in \mathcal{M}_{\not\leqslant h},$$

and

$$h_F(a) \geqslant h(a) \text{ for all but finitely many } a \iff F \subseteq E \text{ for some } E \in \mathcal{M}_{\succeq h}.$$

Proof. For $a \in A$, consider the subsets

$$\mathcal{F}_a := \big\{F \in \mathcal{F}(S) : h_F(a) \geqslant h(a)\big\}, \quad \mathcal{F}'_a := \big\{F \in \mathcal{F}(S) : h_F(a) > h(a)\big\}$$

of $\mathcal{F}(S)$. Since for all $E, F \in \mathcal{F}(S)$,

$$E \supseteq F \quad \Rightarrow \quad h_E(a) \leqslant h_F(a) \text{ for all } a \in A,$$

$\mathcal{F}_a$ and $\mathcal{F}'_a$ are final segments of the Noetherian ordered set $\big(\mathcal{F}(S), \supseteq\big)$. Hence there exist finite sets of generators $\mathcal{M}_{\geqslant h}$, $\mathcal{M}_{\not\leqslant h}$ and $\mathcal{M}_{\succeq h}$ for the final segments

$$\bigcap_a \mathcal{F}_a, \quad \bigcup_a \mathcal{F}'_a, \quad \bigcup_{\substack{D \subseteq A \\ \text{finite}}} \bigcap_{a \notin D} \mathcal{F}_a$$

of $\mathcal{F}(S)$, respectively. These sets have the required properties. $\square$

The previous proposition applies to $S = \mathbb{N}^n$, by Corollary 4.3.

A proposition of Haiman and Sturmfels. Next we show how a combinatorial statement due to Haiman and Sturmfels ([8], Proposition 3.2) can be obtained using the techniques above. This statement is crucial in the construction of the multigraded Hilbert scheme given in [8]. If S is strongly Noetherian, then, given any function $h\colon A \to \mathbb{N}_\infty$, there are only finitely many admissible final segments with Hilbert function h. Moreover:

Lemma 6.2. *Suppose S is strongly Noetherian. Then, for any function $h\colon A \to \mathbb{N}_\infty$, there exists a finite set $D \subseteq A$ such that every $F \in \mathcal{F}(S)$ satisfies: if $h_F(a) \leqslant h(a)$ for all $a \in D$, then $h_F(a) \leqslant h(a)$ for all $a \in A$.*

Proof. Let $h\colon A \to \mathbb{N}_\infty$, and let $\mathcal{M}_{\not\leqslant h} = \{F_1, \dots, F_m\}$ be as in Proposition 6.1. For each i there exists $a_i \in A$ such that $h_{F_i}(a_i) > h(a_i)$. Let $D = \{a_1, \dots, a_m\}$, a finite set of degrees. Now suppose $F \in \mathcal{F}(S)$ satisfies $h_F|D \leqslant h|D$. Then $h_F(a) \leqslant h(a)$ for all a: Otherwise, $F \subseteq F_i$ for some i, so $h_F(a) \geqslant h_{F_i}(a)$ for all $a \in A$; but $h_{F_i}(a_i) > h(a_i) \geqslant h_F(a_i)$: a contradiction. $\square$

Remark. In an analogous way, one shows: there exists a finite set $D \subseteq A$ such that for every $F \in \mathcal{F}(S)$, if $h_F(a) < h(a)$ for all $a \in D$, then $h_F(a) < h(a)$ for all $a \in A$. In particular, there exists a finite set $D_{\mathrm{a}} \subseteq A$ such that for any $F \in \mathcal{F}(S)$, if $h_F(a) < \infty$ for all $a \in D_{\mathrm{a}}$, then F is admissible.

Note that the inequality $h_F(a) \geqslant h(a)$ is clearly not determined by finitely many degrees a. (Consider $S = \mathbb{N}$ equipped with the grading $\delta(m) = m$, and $h(a) = 1$ for all a.) However, we do have the following lemma. Given a subset D of A, we say that a final segment F of S is **generated in degrees** D if F is generated by $F \cap \delta^{-1}(D)$. Recall that for a subset X of S we denote by $[X]$ the initial segment of S generated by X.

Lemma 6.3. *Suppose that S is strongly Noetherian. For every function $h\colon A \to \mathbb{N}$, there exists a finite set $D \subseteq A$ such that*

(1) *every final segment of S with Hilbert function h is generated in degrees D;*
(2) *for every final segment F of S generated in degrees D: if $h_F(a) = h(a)$ for all $a \in D$, then $h_F(a) = h(a)$ for all $a \in A$.*

Proof. Given a subset D of A, denote by $\mathcal{F}_D$ the set of all final segments of S which are generated in degrees D and whose Hilbert function agrees with h on D. Then $\mathcal{F}_D$ is an antichain in $\mathcal{F}(S)$, hence finite. If $D' \subseteq D$, then $[\mathcal{F}_{D'}] \supseteq [\mathcal{F}_D]$: if $F \in \mathcal{F}_D$, then the final segment F' generated by the elements of F with degrees in D' is in $\mathcal{F}_{D'}$, and $F' \subseteq F$. By induction on i, we now construct an increasing sequence $D_0, D_1, \dots$ of finite subsets of A as follows: Let $\mathcal{F} = \mathcal{F}_A = \{F : h_F = h\}$, and let D_0 be a finite set of degrees such that every $F \in \mathcal{F}$ is generated in D_0. Suppose that we have already constructed D_i. If $\mathcal{F}_{D_i} = \mathcal{F}$, then we are done: $D = D_i$ works. Otherwise, for every $F \in \mathcal{F}_{D_i} \setminus \mathcal{F}$ we find $a \in A$ such that $h_F(a) \neq h(a)$. Adjoining finitely many such a to D_i we obtain a finite set $D_{i+1} \supseteq D_i$ of degrees such that every $F \in \mathcal{F}_{D_i}$ with $h_F|D_{i+1} = h|D_{i+1}$ is in $\mathcal{F}$. By Proposition 3.1, this construction has to terminate: since $[\mathcal{F}_{D_{i+1}}] \subseteq [\mathcal{F}_{D_i}]$ for all i, we get $\mathcal{F}_{D_{i+1}} = \mathcal{F}_{D_i}$ for some i, hence $\mathcal{F}_{D_{i+1}} = \mathcal{F}$. □

Restricting to the special case $S = \mathbb{N}^n$, $n > 0$, we obtain [8], Proposition 3.2:

Corollary 6.4. *Given any function $h\colon A \to \mathbb{N}$, there exists a finite set $D \subseteq A$ such that*

(1) *every monomial ideal with Hilbert function h is generated by monomials of degree belonging to D, and*
(2) *if I is a monomial ideal such that $h_I(a) = h(a)$ for all $a \in D$, then $h_I(a) = h(a)$ for all $a \in A$.* □

Noetherianity of the set of Hilbert functions. Let $X = \{X_1, X_2, \dots\}$ be a countably infinite set of indeterminates. For $n \geqslant 1$ we write $X^{\langle n\rangle} = \{X_1, \dots, X_n\}^\diamond$, a subset of $X^\diamond$. We identify the set $X^{\langle n\rangle}$, ordered by divisibility $\leqslant$, with the set $\mathbb{N}^n$, ordered by the product ordering, in the usual way (see Section 2). In our last application, we will be concerned with the grading of $X^{\langle n\rangle}$ by $A = \mathbb{N}$ given by $\delta(X^\nu) = \nu_1 + \dots + \nu_n$ for $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{N}^n$. (This corresponds to the usual grading of monomials in a polynomial ring.) For $n \geqslant 1$ let $\mathcal{H}_n$ be the set of all functions $h\colon \mathbb{N} \to \mathbb{N}$ which arise as a Hilbert function $h = h_F$ for some $F \in \mathcal{F}(X^{\langle n\rangle}, \leqslant)$, and put $\mathcal{H} := \bigcup_{n\geqslant 1} \mathcal{H}_n$. We consider $\mathcal{H}$ as an ordered set via the product ordering: $h \leqslant h'$ if and only if $h(a) \leqslant h'(a)$ for all $a \in \mathbb{N}$. We then have:

Proposition 6.5. *$\mathcal{H}$ is Nash-Williams.*

This proposition generalizes Corollary 3.4 in [1]:

Corollary 6.6. *$\mathcal{H}_n$ is Noetherian, for every $n \geqslant 1$.* □

The applications above (e.g., Proposition 6.1) just needed Maclagan's finiteness principle, i.e., $\mathbb{N}^n$ is strongly Noetherian. Proposition 6.5 is a consequence of a deeper result of Nash-Williams' theory, of which we only state the special case needed here.

Theorem 6.7. (Nash-Williams, [26]) *If $(S, \leqslant)$ is a Nash-Williams ordered set, then the ordered set $(S^\diamond, \leqslant^\diamond)$ is Nash-Williams.*

We totally order X by $\preceq$ so that $X_1 \prec X_2 \prec \cdots$. By the previous theorem, the ordered set $\big(\mathcal{F}(X^\diamond, \preceq^\diamond), \supseteq\big)$ is Noetherian. We denote the restriction of the ordering $\preceq^\diamond$ on $X^\diamond$ to $X^{\langle n\rangle}$ also by $\preceq^\diamond$. We have

$$\mathcal{F}\big(X^{\langle n\rangle}, \preceq^\diamond\big) \subseteq \mathcal{F}\big(X^{\langle n\rangle}, \leqslant\big).$$

The monomial ideals $F \in \mathcal{F}\big(X^{\langle n\rangle}, \preceq^\diamond\big)$ are commonly called **strongly stable.** It is a well-known consequence of a theorem of Galligo (see, e.g., [4], Section 15.9) that given a final segment F of $\big(X^{\langle n\rangle}, \leqslant\big)$ there exists a strongly stable final segment of $\big(X^{\langle n\rangle}, \leqslant\big)$ with the same Hilbert function h_F. If $F \in \mathcal{F}\big(X^{\langle n\rangle}, \leqslant\big)$ and F' denotes the final segment of $(X^{\langle n+1\rangle}, \leqslant)$ generated by F, then $h_F \leqslant h_{F'}$, since $h_{F'}(m) = \sum_{i=0}^m h_F(i)$ for all m. We can now prove Proposition 6.5:

Proof (Proposition 6.5). Let $f\colon B \to \mathcal{H}$ be an $\mathcal{H}$-array, say $f(b) = h_{F_b}$ with $F_b \in \mathcal{F}\big(X^{\langle n_b\rangle}, \leqslant\big)$, $n_b \geqslant 1$ for all b. We need to find $b \lhd c$ such that $h_{F_b} \leqslant h_{F_c}$. After passing to a subarray of f if necessary, we may assume that $n_b \leqslant n_c$ for all $b \lhd c$ in B. (By Lemma 5.7.) By the remarks above, we may further assume that each F_b is strongly stable, that is, $F_b \in \mathcal{F}\big(X^{\langle n_b\rangle}, \preceq^\diamond\big)$ for all b. By Theorem 6.7, there exist $b \lhd c$ such that $(F_b) \supseteq (F_c)$ in $\mathcal{F}\big(X^\diamond, \preceq^\diamond\big)$, where (F_b) and (F_c) denote the final segments of $(X^\diamond, \preceq^\diamond)$ generated by F_b and F_c, respectively. Now $F_b' := (F_b) \cap X^{\langle n_c\rangle}$ is the final segment of $(X^{\langle n_c\rangle}, \preceq^\diamond)$ generated by F_b. We have $F_b' \supseteq F_c$, in $\mathcal{F}\big(X^{\langle n_c\rangle}, \preceq^\diamond\big)$, hence $h_{F_b} \leqslant h_{F_b'} \leqslant h_{F_c}$ as required. □

Part 2. Multi-stage Stochastic Integer Programming

7. Preliminaries: Test Sets

For a given matrix $A \in \mathbb{Z}^{l\times d}$, where $d, l \in \mathbb{N}$, $d, l > 0$, consider the family of optimization problems

$$(\mathrm{IP}_{b,c}) \qquad \min\big\{c^\intercal z : Az = b,\ z \in \mathbb{N}^d\big\}$$

as $b \in \mathbb{R}^l$ and $c \in \mathbb{R}^d$ vary. We call A the **coefficient matrix** of this family (or of a particular instance of it). One way to solve such a problem for given c and b is to start with a feasible solution z (i.e., a vector $z \in \mathbb{N}^d$ such that $Az = b$) and to replace it by another feasible solution $z - v$ with smaller objective value $c^\intercal(z - v) \in \mathbb{R}$, as long as we find such a vector $v \in \mathbb{Z}^d$ that improves the current feasible solution z. Such a vector v is called an **improving vector** for z. If the problem instance $(\mathrm{IP}_{b,c})$ is solvable, this augmentation process has to stop (with an optimal solution). Note that for given b and c and any feasible solution z of $(\mathrm{IP}_{b,c})$, a vector $v \in \mathbb{Z}^d$ is an improving vector for z if and only if the following three conditions are satisfied:

(1) $Av = 0$,
(2) $v \leqslant z$ (in the product ordering on $\mathbb{Z}^d$), and

(3) $c^\intercal v > 0$.

The key step in this scheme is to find such an improving vector v. Test sets provide such vectors: a subset of $\mathbb{Z}^d$ is called a **universal test set** for the above problem family if it contains, for any given choice of $c \in \mathbb{R}^d$ and $b \in \mathbb{R}^l$, an improving vector for any non-optimal feasible solution z to the given specific problem. Clearly, if we have a finite universal test set, we can easily find an improving vector in the augmentation procedure.

The notion of a universal test set was introduced by Graver [7] in 1975. He also gave a simple construction of a finite universal test set. The **Graver basis** $G(A)$ associated to A consists of all $\sqsubseteq$-minimal *non-zero* integer solutions to $Az = 0$. Here $\sqsubseteq$ is the ordering of $\mathbb{Z}^d$ defined in Section 2. (Note that 0 is the only $\sqsubseteq$-minimal solution to $Az = 0$.) As we have seen above, $G(A)$ is always finite by Corollary 5.10. Moreover, $G(A)$ is symmetric: if $v \in G(A)$ then also $-v \in G(A)$. Graver showed that $G(A)$ is indeed a universal test set for the above problem family. This leads to the following algorithm to compute, uniform in b and c, an optimal solution to $(\mathrm{IP}_{b,c})$ from a feasible one:

Algorithm 7.1. (Augmentation algorithm)

Input: a feasible solution z to $(\mathrm{IP}_{b,c})$ and a finite set $G \subseteq \mathbb{Z}^d$ containing $G(A)$.

Output: an optimal solution to $(\mathrm{IP}_{b,c})$.

while there is some $v \in G$ such that $c^\intercal v > 0$ and $v \leqslant z$ **do**

  $z := z - v$

return z

How does one find *some* feasible solution to $(\mathrm{IP}_{b,c})$ to begin with? Universal test sets can also be employed to find an initial feasible solution by a construction similar to Phase I in the simplex method for linear optimization.

Notations. For $a \in \mathbb{Z}$ we put

$$a^+ := \max\{a, 0\} \in \mathbb{N}, \qquad a^- := \max\{-a, 0\} \in \mathbb{N},$$

and for $z = (z_1, \dots, z_d) \in \mathbb{Z}^d$, we put

$$z^+ := (z_1^+, \dots, z_d^+) \in \mathbb{N}^d, \qquad z^- := (z_1^-, \dots, z_d^-) \in \mathbb{N}^d,$$

so $z = z^+ - z^-$. We also let $c(z)$ denote the vector in $\mathbb{Z}^d$ whose i-th component is -1 if $z_i > 0$ and 0 otherwise.

Slightly modifying Algorithm 7.1 yields the following algorithm (for whose termination and correctness see [9, 10]):

Algorithm 7.2. (Finding a feasible solution)

Input: a solution $z \in \mathbb{Z}^d$ to $(\mathrm{IP}_{b,c})$ and a finite set $G \subseteq \mathbb{Z}^d$ containing $G(A)$.

Output: a feasible solution to $(\mathrm{IP}_{b,c})$, or "FAIL" if no such solution exists.

while there is some $v \in G$ such that $c(z)^\intercal v > 0$ and $v \leqslant z^+$ **do**

  $z := z - v$

if $z \geqslant 0$ **then** **return** z **else** **return** "FAIL"

Graver proved finiteness of $G(A)$; however, he did not give an algorithm to compute $G(A)$ from A. The following simple completion procedure, due to Pottier [27], solves this problem. We write $\ker(A) := \{z \in \mathbb{Z}^d : Az = 0\}$ (a $\mathbb{Z}$-submodule of $\mathbb{Z}^d$).

Algorithm 7.3. (Completion procedure)

Input: a finite symmetric generating set for the $\mathbb{Z}$-module $\ker(A)$.

Output: a finite subset G of $\mathbb{Z}^d$ containing $G(A)$.

$G := F \setminus \{0\}$
$C := \{f + g : f, g \in G\}$
while $C \neq \emptyset$ **do**
  $s :=$ an element in C
  $C := C \setminus \{s\}$
  $f := \text{normalForm}(s, G)$
  if $f \neq 0$ **then**
    $C := C \cup \{f + g : g \in G\}$
    $G := G \cup \{f\}$
return G.

Behind the function normalForm(s, G) there is the following algorithm, which upon input of a finite set $G = \{g_1, \ldots, g_n\}$ of non-zero vectors in $\mathbb{Z}^d$, returns a vector f with the property that s has a representation as a finite sum

$$s = f + \sum_i a_i g_i \qquad \text{with } a_i \in \mathbb{N},\ g_i \in G,\ a_i g_i \sqsubseteq s \text{ and } g_i \not\sqsubseteq f \text{ for all } i. \tag{7.1}$$

The vector f is called a **normal form** of s with respect to the set G. The algorithm proceeds by successively reducing s by elements of G:

Algorithm 7.4. (Normal form algorithm)

Input: a vector $s \in \mathbb{Z}^d$, a finite set G of non-zero vectors in $\mathbb{Z}^d$.

Output: a normal form f of s with respect to G.

while there is some $g \in G$ such that $g \sqsubseteq s$ **do**
  $s := s - g$
return $f := s$

As $\|s - g\|_1 < \|s\|_1$ for $0 \neq g \sqsubseteq s$ in $\mathbb{Z}^d$, the algorithm always terminates. Here $\|s\|_1 = |s_1| + \cdots + |s_d|$ for $s = (s_1, \ldots, s_d) \in \mathbb{Z}^d$.

In the subsequent sections we will generalize the main ingredients of this completion procedure. The objects in our algorithm, however, will turn out to be more complicated structures than mere vectors. We finish this section with a useful fact needed in the next section. Given matrices A and B with the same number of rows we denote by $(A|B)$ the matrix obtained by joining A and B horizontally (placing B to the right of A).

Lemma 7.5. *Let $d, e, l > 0$ be integers, $A \in \mathbb{Z}^{l \times d}$, $b \in \mathbb{Z}^l$, $y \in \mathbb{Z}^d$. Then $(y, 1) \in G(A|b)$ if and only if $(y, 1) \in \ker(A|b)$, and the only $z \in \mathbb{Z}^d$ with $(z, 1) \in \ker(A|b)$ and $(z, 1) \sqsubseteq (y, 1)$ is $z = y$.*

Proof. The "only if" direction being clear, suppose $(y, 1) \in \ker(A|b)$ and $(y, 1) \notin G(A|b)$; we need to show that there is $z \in \mathbb{Z}^d \setminus \{y\}$ with $(z, 1) \in \ker(A|b)$ and $(z, 1) \sqsubseteq (y, 1)$. Now there are $u \in \mathbb{Z}^d$ and $a \in \mathbb{Z}$ such that $0 \neq (u, a) \in \ker(A|b)$, $(u, a) \sqsubseteq (y, 1)$, and $(u, a) \neq (y, 1)$. If $a = 1$, then $z := u$ does the job; otherwise $a = 0$, and then $z := y - u$ has the required properties. □

8. Building Blocks

In this section we study multistage stochastic integer programs. We begin by describing the basic setup, and then use results from the previous sections (in particular Corollary 5.11) to show that the Graver basis elements of the coefficient matrices of stochastic integer programs of this type can be constructed from only finitely many "building blocks," as the number $N \geqslant 1$ of scenarios varies.

Basic setup. In the following we fix integers $k \geqslant 0$ and $l \geqslant 1$. Let $T_0, T_1, \ldots, T_k$ be a sequence of integer matrices with each T_s having l rows and n_s columns, where $n_s \in \mathbb{N}$, $n_s \geqslant 1$ for $s = 0, \ldots, k$. We set $A_{0,N} := (T_0)$, and we recursively define

$$A_{s,N} := \begin{pmatrix} T_{s,N} & A_{s-1,N} & 0 & \cdots & 0 \\ T_{s,N} & 0 & A_{s-1,N} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T_{s,N} & 0 & 0 & \cdots & A_{s-1,N} \end{pmatrix} \quad \text{for } s = 1, \ldots, k,$$

where $A_{s,N}$ contains N copies of $T_{s,N}$, and $T_{s,N} = \begin{pmatrix} T_s \\ \vdots \\ T_s \end{pmatrix}$ consists of N^{s-1} many copies of the matrix T_s.

Example. For $k = 2$ and $N = 2$ we have $A_{0,2} = (T_0)$, $A_{1,2} = \begin{pmatrix} T_1 & T_0 & \\ T_1 & & T_0 \end{pmatrix}$ and $A_{2,2} = \begin{pmatrix} T_2 & T_1 & T_0 & & & \\ T_2 & T_1 & & T_0 & & \\ T_2 & & & & T_1 & T_0 & \\ T_2 & & & & T_1 & & T_0 \end{pmatrix}$.

As discussed in the introduction, matrices having the form of $A_{k,N}$ arise as the coefficient matrices of $(k+1)$-stage stochastic integer programs, like (0.1).

Remarks. Note that we assume here that the scenario tree of our stochastic optimization problem splits into exactly N subtrees at every stage. This simplifying condition can easily be achieved by introducing additional scenarios with vanishing conditional probabilities. Also note that the coefficient matrix $A_{1,N}$ for a two-stage stochastic integer program with N scenarios differs somewhat from the general form of a coefficient matrix given in the introduction: in the description of a two-stage stochastic program (0.1), we read the constraints $Ax = a$ as $Ax + 0y_i = a$, and in doing so, we may rewrite the problem matrix as

$$\begin{pmatrix} T' & W' & & \\ \vdots & & \ddots & \\ T' & & & W' \end{pmatrix}$$

with $T' = \binom{A}{T}$ and $W' = \binom{0}{W}$. Thus, we can safely avoid stating the constraints $Ax = a$ explicitly. The same holds true in the multi-stage situation, as we can apply a similar reformulation of the problem.

For $s = 0, \ldots, k$, the matrix $A_{s,N}$ has $N^s \cdot l$ rows and

$$d_{s,N} = n_s + N d_{s-1,N} = n_s + N n_{s-1} + N^2 d_{s-2,N} = \cdots = \sum_{i=0}^{s} N^{s-i} n_i$$

columns. In particular, the full $(k+1)$-stage problem matrix $A_{k,N}$ has

$$d_{k,N} = \sum_{i=0}^{k} N^{k-i} n_i$$

columns, which corresponds to the number of variables of the corresponding stochastic integer problem. In the following h and s range over the set $\{0, \dots, k\}$. We also put

$$n(h) := n_0 + \cdots + n_h \quad (= d_{h,1}).$$

Building blocks. Let $N \geqslant 1$ and $[1,N] := \{1, 2, \dots, N\}$. For every s we fix an enumeration of the set $[1,N]^s$ of sequences of elements from $[1,N]$ of length s, say in lexicographic order. By convention $[1,N]^0 = \{\varepsilon\}$ where $\varepsilon = (\ \ )$ denotes the empty sequence. We partition the components of every vector $z \in \mathbb{Z}^{d_{h,N}}$ as follows:

$$z = (z_0, \dots, z_h), \quad \text{where} \quad z_s = \big(z_\alpha : \alpha \in [1,N]^s\big),\ z_\alpha \in \mathbb{Z}^{n_{h-s}}, \tag{8.1}$$

with $\alpha \in [1,N]^s$ indexed according to the enumeration of $[1,N]^s$. For $\alpha \in [1,N]^s$ we call z_α a **building block** of height s of z.

Example. Suppose $k = 2$ and $N = 2$, and enumerate $\{1,2\}^s$ lexicographically. Then $z \in \mathbb{Z}^{d_{2,2}}$ decomposes as $z = (z_0, z_1, z_2)$ where

$$\begin{aligned} z_0 &= (z_\varepsilon) \in \mathbb{Z}^{n_2}, \\ z_1 &= (z_{(1)}, z_{(2)}) \in \mathbb{Z}^{2n_1}, \\ z_2 &= (z_{(1,1)}, z_{(1,2)}, z_{(2,1)}, z_{(2,2)}) \in \mathbb{Z}^{4n_0}. \end{aligned}$$

We have $z \in \ker(A_{h,N})$ if and only if for all $\alpha = (\alpha_1, \dots, \alpha_h) \in [1,N]^h$

$$T_h z_{\alpha|0} + T_{h-1} z_{\alpha|1} + \cdots + T_0 z_{\alpha|h} = \sum_{i=0}^{h} T_i z_{\alpha|h-i} = 0,$$

where $\alpha|i := (\alpha_1, \dots, \alpha_i)$ for $i = 0, \dots, h$, and we think of z_α as a column vector. (Here again $\mathbb{Z}^0 = \{\varepsilon\}$, so $\alpha|0 = \varepsilon$ for all $\alpha \in \mathbb{Z}^h$.) This relation can also be written as

$$A_{h,1} \begin{pmatrix} z_{\alpha|0} \\ z_{\alpha|1} \\ \vdots \\ z_{\alpha|h} \end{pmatrix} = 0.$$

Vector-trees. It is useful to visualize a vector $z \in \mathbb{Z}^{d_{h,N}}$ as a (directed, labeled, unordered) tree of height h whose nodes are labeled by its building blocks z_α, and there is an edge from the node labeled by z_α to the node labeled by z_β exactly if α is an initial segment of β, that is, $\alpha = \beta|s$ for some $s = 0, \dots, h$. (See Figure 1.) We call this tree the **vector-tree** $\mathcal{T}(z)$ associated to z. Such a vector-tree associated to an element of $\mathbb{Z}^{d_{h,N}}$ is a full N-ary tree (i.e., every internal node has exactly N children). In the following we will consider trees of a similar structure which may split into possibly infinitely many sub-trees at each stage:

Definition 8.1. A **vector-tree** of height h is a (non-empty, countable, directed, labeled) tree which is balanced of height h (i.e., every path from the root to a leaf has the same length h) and whose nodes of height s are labeled by integer vectors in $\mathbb{Z}^{n_{h-s}}$, for $s = 0, \ldots, h$. (Sometimes we also call the labels of a vector-tree its **building blocks**.) We denote the label of the root of a vector-tree $\mathcal{T}$ by $\operatorname{root}(\mathcal{T}) \in \mathbb{Z}^{n_h}$.

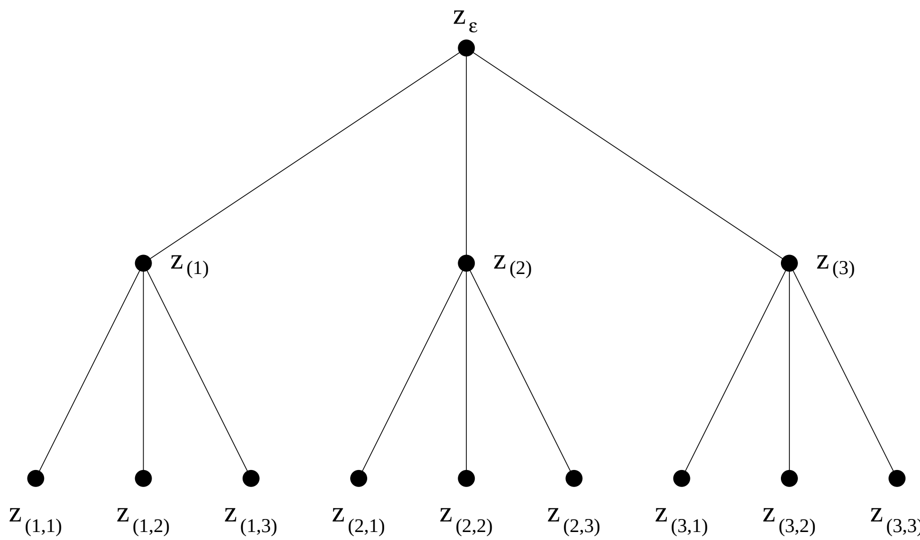

Figure 1: Representing multi-stage solutions as trees

We have an obvious notion of isomorphism of vector-trees (as an isomorphism of directed graphs which preserves the labeling). In the following we identify isomorphic vector-trees; this permits us to speak of "the set of all vector-trees". For every full N-ary vector-tree $\mathcal{T}$ of height h there exists an element z of $\mathbb{Z}^{d_{h,N}}$ whose associated vector-tree is $\mathcal{T}$; we have $\operatorname{root}\big(\mathcal{T}(z)\big) = z_\varepsilon$.

Definition 8.2. If $\mathcal{S}$ and $\mathcal{T}$ are vector-trees (of possibly different heights), we say that $\mathcal{S}$ is a **sub-vector-tree** of $\mathcal{T}$ if

(1) $\mathcal{S}$ is a labeled subtree of $\mathcal{T}$, that is, the underlying graph of $\mathcal{S}$ is a subgraph of the underlying graph of $\mathcal{T}$, and the labeling of the nodes of $\mathcal{S}$ agrees with their corresponding labeling in $\mathcal{T}$;
(2) $\mathcal{S}$ is closed downwards in $\mathcal{T}$, that is, if there is a path in $\mathcal{T}$ from the root of $\mathcal{S}$ to a node a in $\mathcal{T}$, then a is a node of $\mathcal{S}$.

If $\mathcal{S}$ is a sub-vector-tree of $\mathcal{T}$ where $\mathcal{S}$ has height $h-1$ and $\mathcal{T}$ has height h, then $\mathcal{S}$ is called an **immediate** sub-vector-tree of $\mathcal{T}$.

Example. In Figure 1, the labeled subtree consisting of the nodes labeled by $z_{(1)}$, $z_{(1,1)}$, $z_{(1,2)}$, $z_{(1,3)}$ is a sub-vector-tree of the vector-tree associated to z, whereas the labeled subtree consisting of the nodes labeled by z_ε, $z_{(1)}$, $z_{(1,1)}$, $z_{(1,2)}$ is not.

Paths in vector-trees. Every path in a vector-tree $\mathcal{T}$ of height h from its root to one of it leaves is called a **maximal path.** Every maximal path in a vector tree of height h has length h. If $v_0, v_1, \ldots, v_h$ with $v_i \in \mathbb{Z}^{n_{h-i}}$ for $i = 0, \ldots, h$ are the successive labels of the nodes on a path P in $\mathcal{T}$, then we also say that P is **labeled by** $(v_0, \ldots, v_h)$, and we call $(v_0, \ldots, v_h) \in \mathbb{Z}^{n(h)}$ the **label of** P**.** We write $\operatorname{paths}(\mathcal{T})$ for the set of labels of maximal paths in $\mathcal{T}$.

Definition 8.3. We say that a vector-tree $\mathcal{T}$ of height h has **value** $b \in \mathbb{Z}^l$ if for every $(v_0, v_1, \ldots, v_h) \in \operatorname{paths}(\mathcal{T})$ we have

$$T_h v_0 + T_{h-1} v_1 + \cdots + T_0 v_h = b.$$

(Here we think of each v_i as a column vector in $\mathbb{Z}^{n_{h-i}}$.)

Note that if $\mathcal{T}$ is a vector-tree of height $h > 0$ and value b, then each immediate sub-vector-tree of $\mathcal{T}$ has value $b - T_h \operatorname{root}(\mathcal{T})$. An element z of $\mathbb{Z}^{d_{h,N}}$ lies in $\ker(A_{h,N})$ if and only if $\mathcal{T}(z)$ has value 0.

Definition 8.4. Let $\mathcal{T}$ be a vector-tree of height h and N a positive integer. A vector $z \in \mathbb{Z}^{d_{h,N}}$ with the property that $\operatorname{paths}\big(\mathcal{T}(z)\big) \subseteq \operatorname{paths}(\mathcal{T})$ is said to be **constructible from $\mathcal{T}$.** We denote by $\langle \mathcal{T} \rangle_N$ the set of vectors in $\mathbb{Z}^{d_{h,N}}$ which are constructible from $\mathcal{T}$.

Remarks. We have $\langle \mathcal{T} \rangle_1 = \operatorname{paths}(\mathcal{T})$. If $\mathcal{T}$ has value b, then for every $z \in \langle \mathcal{T} \rangle_N$, the vector-tree $\mathcal{T}(z)$ has value b.

We say that a vector-tree $\mathcal{T}$ is **tight** if the labels of the children of internal nodes of $\mathcal{T}$ are pairwise distinct. For example, the vector-tree

$$\begin{tikzcd}[column sep=small] & 0 \arrow[dl] \arrow[dr] & \\ 0 & & 2 \end{tikzcd}$$

is tight, whereas

$$\begin{tikzcd}[column sep=small] & 0 \arrow[dl] \arrow[d] \arrow[dr] & \\ 0 & 0 & 2 \end{tikzcd}$$

is not. The following is easy to show:

Lemma 8.5. *Let S be a non-empty subset of $\mathbb{Z}^{n(h)}$ with $z_\varepsilon = r$ for all $z \in S$. There exists a unique tight vector-tree $\mathcal{T}(S)$ of height h (up to isomorphism) with $S = \operatorname{paths}\big(\mathcal{T}(S)\big)$.*

In particular, for every vector-tree $\mathcal{S}$ of height h there exists a unique tight vector-tree $\mathcal{T}$ of the same height with $\operatorname{paths}(\mathcal{T}) = \operatorname{paths}(\mathcal{S})$, namely $\mathcal{T} = \mathcal{T}\big(\operatorname{paths}(\mathcal{S})\big)$. Note also that in the context of the previous lemma, if $S \subseteq \ker(A_{h,1})$, then $\mathcal{T}(S)$ has value 0.

Reducibility of vector-trees. Next, we define a "reducibility relation" between vector-trees of the same height. This relation bears a formal resemblance to Milner's "simulation quasi-ordering" for transition systems [23].

Definition 8.6. Let $\mathcal{S}$ and $\mathcal{T}$ be vector-trees of height h.

(1) For $h = 0$ we let $\mathcal{S} \sqsubseteq_0 \mathcal{T}$ if $\operatorname{root}(\mathcal{S}) \sqsubseteq \operatorname{root}(\mathcal{T})$ (in $\mathbb{Z}^{n_0}$).
(2) For $h > 0$ we let $\mathcal{S} \sqsubseteq_h \mathcal{T}$ if $\operatorname{root}(\mathcal{S}) \sqsubseteq \operatorname{root}(\mathcal{T})$ (in $\mathbb{Z}^{n_h}$) and for every immediate sub-vector-tree $\mathcal{T}'$ of $\mathcal{T}$ there is an immediate sub-vector-tree $\mathcal{S}'$ of $\mathcal{S}$ such that $\mathcal{S}' \sqsubseteq_{h-1} \mathcal{T}'$.

There is an obvious algorithm to decide, given *finite* vector-trees $\mathcal{S}$ and $\mathcal{T}$ of height h, whether $\mathcal{S} \sqsubseteq_h \mathcal{T}$. Clearly $\sqsubseteq_h$ is a quasi-ordering on the set of vector-trees of height h.

Example. Consider the vector trees

$$\mathcal{S}_1 = \begin{tikzcd}[column sep=small] & (0,0) \arrow[dl] \arrow[dr] & \\ (0,0) \arrow[d] & & (1,0) \arrow[d] \\ (1,0) & & (0,0) \end{tikzcd} \qquad \mathcal{S}_2 = \begin{tikzcd}[column sep=small] & (0,0) \arrow[dl] \arrow[dr] & \\ (0,0) \arrow[d] & & (1,0) \arrow[d] \\ (1,0) & & (1,0) \end{tikzcd}$$

and

$$\mathcal{T} = \begin{tikzcd}[column sep=small] & (0,0) \arrow[dl] \arrow[dr] & \\ (1,1) \arrow[d] & & (0,1) \\ (0,0) & & \end{tikzcd}$$

Then $\mathcal{S}_1 \sqsubseteq_2 \mathcal{T}$ and $\mathcal{S}_2 \not\sqsubseteq_2 \mathcal{T}$.

In the proof of the next lemma we relate $\sqsubseteq_h$ with the quasi-ordering $\sqsubseteq_h$ on $\mathcal{P}_h(\mathbb{Z}^{n(h)})$ defined after Theorem 5.9; we freely use the notations introduced there.

Lemma 8.7. *The quasi-ordering $\sqsubseteq_h$ on the set of vector-trees of height h is Noetherian.*

Proof. First we put

$$\mathcal{P}^*_0(\mathbb{Z}^n) := \mathbb{Z}^n, \quad \mathcal{P}^*_h(\mathbb{Z}^n) := \mathcal{P}\big(\mathcal{P}^*_{h-1}(\mathbb{Z}^n)\big) \setminus \{\emptyset\} \text{ for } h > 0.$$

So $\mathcal{P}^*_h(\mathbb{Z}^n)$ is a subset of $\mathcal{P}_h(\mathbb{Z}^n)$, for each h and $n > 0$. By induction on $h = 0, \dots, k$ we now define for all $m, n \in \mathbb{N}$, $m, n > 0$, an operation

$$(v, X) \mapsto v *_h X \colon \mathbb{Z}^m \times \mathcal{P}^*_h(\mathbb{Z}^n) \to \mathcal{P}^*_h(\mathbb{Z}^{m+n})$$

as follows. For $h = 0$ we have $\mathcal{P}^*_h(\mathbb{Z}^n) = \mathbb{Z}^n$; we let $v *_0 w \in \mathbb{Z}^{m+n}$ be the concatenation of $v \in \mathbb{Z}^m$ and $w \in \mathbb{Z}^n$. For $h > 0$ we let

$$v *_k X := \big\{ v *_{h-1} X' : X' \in X \big\} \quad \text{for all } v \in \mathbb{Z}^m,\ X \in \mathcal{P}^*_h(\mathbb{Z}^n).$$

Note that for $v, w \in \mathbb{Z}^m$ and $X, Y \in \mathcal{P}^*_h(\mathbb{Z}^n)$ we have: if $v *_h X \sqsubseteq_h w *_h Y$ in $\mathcal{P}_h\big(\mathbb{Z}^{m+n}\big)$, then $v \sqsubseteq w$ and $X \sqsubseteq_h Y$ (in $\mathcal{P}_h(\mathbb{Z}^n)$).

Next, we define a map φ_h which associates to every vector-tree $\mathcal{S}$ of height h an element $\varphi_h(\mathcal{S})$ of $\mathcal{P}^*_h(\mathbb{Z}^{n(h)})$ as follows: For $h = 0$ put $\varphi_0(\mathcal{S}) := \operatorname{root}(\mathcal{S}) \in \mathbb{Z}^{n_0}$. For $h > 0$ let

$$\varphi_h(\mathcal{S}) := \big\{ \operatorname{root}(\mathcal{S}) *_h \varphi_{h-1}(\mathcal{S}') : \mathcal{S}' \text{ sub-vector-tree of } \mathcal{S} \text{ of height } h-1 \big\}.$$

By induction on h it is easy to verify that φ_h is a quasi-embedding of the set of vector-trees of height h, quasi-ordered by $\sqsubseteq_h$ as defined above, into $\mathcal{P}^*_h\big(\mathbb{Z}^{n(h)}\big)$, quasi-ordered by the restriction of the quasi-ordering $\sqsubseteq_h$ of $\mathcal{P}_h\big(\mathbb{Z}^{n(h)}\big)$ defined after Theorem 5.9. The lemma now follows from Corollary 5.11. $\square$

Corollary 8.8. *There is no infinite sequence $\mathcal{T}_1, \mathcal{T}_2, \dots$ of vector-trees of height h with $\mathcal{T}_i \not\sqsubseteq_h \mathcal{T}_j$ whenever $i < j$.* $\square$

Given vector-trees $\mathcal{S}$ and $\mathcal{T}$ of height h and an integer $N \geqslant 1$ we write $\mathcal{S} \sqsubseteq_{h,N} \mathcal{T}$ if for every $z \in \langle \mathcal{T} \rangle_N$ there exists $y \in \langle \mathcal{S} \rangle_N$ such that $y \sqsubseteq z$ (in $\mathbb{Z}^{d_{h,N}}$). We put $\mathcal{S} \sqsubseteq_{h,\infty} \mathcal{T}$ if $\mathcal{S} \sqsubseteq_{h,N} \mathcal{T}$ for all $N \geqslant 1$.

Lemma 8.9. *If $\mathcal{S} \sqsubseteq_h \mathcal{T}$, then $\mathcal{S} \sqsubseteq_{h,\infty} \mathcal{T}$.*

Proof. Suppose that $\mathcal{S} \sqsubseteq_h \mathcal{T}$, and let $N \geqslant 1$. By induction on $h = 0, \dots, k$ we show that given $z \in \langle \mathcal{T} \rangle_N$ we can construct a vector $y \in \langle \mathcal{S} \rangle_N$ such that $y \sqsubseteq z$. Suppose first that $h = 0$. Then $z = \operatorname{root}(\mathcal{T})$, and $\operatorname{root}(\mathcal{S}) \sqsubseteq \operatorname{root}(\mathcal{T})$. Hence for y we may take $y = \operatorname{root}(\mathcal{S})$. Now assume that $h > 0$, and let $z_1, \dots, z_N \in \mathbb{Z}^{d_{h-1,N}}$ such that $\mathcal{T}(z_1), \dots, \mathcal{T}(z_N)$ are the immediate sub-vector-trees of $\mathcal{T}(z)$. Then for each $i = 1, \dots, N$ there is an immediate sub-vector-tree $\mathcal{T}_i$ of $\mathcal{T}$ with $z_i \in \langle \mathcal{T}_i \rangle_N$. Since $\mathcal{S} \sqsubseteq_h \mathcal{T}$ we have $\operatorname{root}(\mathcal{S}) \sqsubseteq \operatorname{root}(\mathcal{T})$, and for every i there exists a sub-vector-tree $\mathcal{S}_i$ of $\mathcal{S}$ of height $h-1$ with $\mathcal{S}_i \sqsubseteq_{h-1} \mathcal{T}_i$. By induction hypothesis there exist $y_1, \dots, y_N \in \langle \mathcal{S}_i \rangle_N$ such that $y_i \sqsubseteq z_i$ in $\mathbb{Z}^{d_{h-1,N}}$. Then $y \sqsubseteq z$ for a suitable $y \in \mathbb{Z}^{d_{h,N}}$ whose vector-tree $\mathcal{T}(y)$ has immediate sub-vector-trees $\mathcal{T}(y_1), \dots, \mathcal{T}(y_N)$ and root labeled by $\operatorname{root}(\mathcal{S})$. □

Lemmas 8.7 and 8.9 imply:

Corollary 8.10. *For every h, the quasi-ordering $\sqsubseteq_{h,\infty}$ on the set of vector-trees of height h is Noetherian.* □

Building blocks of Graver bases. Recall that $G(A_{k,N})$ denotes the Graver basis of the matrix $A_{k,N}$ (a finite subset of $\mathbb{Z}^{d_{k,N}}$). We decompose each vector $z \in G(A_{k,N})$ into its building blocks z_α as described in (8.1) and put

$$\mathcal{H}^s_{k,N} := \{ z_\alpha : z \in G(A_{k,N}),\ \alpha \in [1,N]^s \} \subseteq \mathbb{Z}^{n_{k-s}} \qquad \text{for } s = 0, \dots, k.$$

We form the union $\mathcal{H}^s_{k,\infty} := \bigcup_{N=1}^{\infty} \mathcal{H}^s_{k,N}$ and define

$$\mathcal{H}_{k,\infty} := \bigcup_{s=0}^{k} \mathcal{H}^s_{k,\infty} \quad \text{(disjoint union)},$$

the set of building blocks of Graver bases of the matrices $A_{k,N}$ obtained by varying N over the set of positive integers. For $k = 0$ we have $\mathcal{H}_{0,\infty} = G(T_0)$ and thus, $\mathcal{H}_{0,\infty}$ is finite. Finiteness of $\mathcal{H}_{1,\infty}$ was shown in [12]. In what follows, we prove:

Proposition 8.11. *The set $\mathcal{H}_{k,\infty}$ is finite for every k.*

Before we give the proof of this proposition, we first combine the elements of $\mathcal{H}_{k,\infty}$ into (*a priori* possibly infinite) vector-trees: Given $r \in \mathcal{H}^0_{k,\infty}$, we put $\mathcal{T}(r) := \mathcal{T}\big(S(r)\big)$ where

$$S(r) := \bigcup \big\{ \operatorname{paths}\big(\mathcal{T}(z)\big) : z \in G(A_{k,N}) \text{ for some } N \geqslant 1,\ z_\varepsilon = r \big\} \subseteq \mathbb{Z}^{n(k)}.$$

The vector-tree $\mathcal{T}(r)$ of height k has root r and value 0. For any $r \in \mathcal{H}^0_{k,\infty}$ and any $N \geqslant 1$ the set $\langle \mathcal{T}(r) \rangle_N$ contains every vector $z \in G(A_{k,N})$ with $z_\varepsilon = r$. In particular $G(A_{k,N}) \cap \langle \mathcal{T}(r) \rangle_N \neq \emptyset$ for some $N \geqslant 1$. This yields:

Lemma 8.12. *The vector-trees $\mathcal{T}(r)$, where r ranges over all non-zero elements of $\mathcal{H}^0_{k,\infty}$, form a $\sqsubseteq_{k,\infty}$-antichain.*

Proof. For a contradiction suppose that $\mathcal{T}(r') \sqsubseteq_{k,\infty} \mathcal{T}(r)$ for some non-zero $r' \neq r$ in $\mathcal{H}^0_{k,\infty}$. So for every $N \geqslant 1$ and every $z \in \langle \mathcal{T}(r) \rangle_N$ there exists a vector $y \in \langle \mathcal{T}(r') \rangle_N$ such that $y \sqsubseteq z$. Note that $y_\varepsilon = \operatorname{root}\big(\mathcal{T}(r')\big) \neq \operatorname{root}\big(\mathcal{T}(r)\big) = z_\varepsilon$, hence $y, z \neq 0$ and $y \neq z$. Therefore, none of the vectors constructible from $\mathcal{T}(r)$ is an element of a Graver basis $G(A_{k,N})$, for any N. This contradicts the remark preceding the lemma. □

By Corollary 8.10 and Lemma 8.12, the set $\mathcal{H}^0_{k,\infty}$ is finite. We can now prove Proposition 8.11:

Proof (Proposition 8.11). We show, by induction on k, that $\mathcal{H}_{k,\infty}$ is finite, for every choice of matrices $T_0, \dots, T_k$ as in the beginning of this section. We already know that $\mathcal{H}_{0,\infty} = G(T_0)$ is finite. Suppose $k > 0$. We have seen above that $\mathcal{H}^0_{k,\infty}$ is finite. For $r \in \mathcal{H}^0_{k,\infty}$, the labels of the nodes of $\mathcal{T}(r)$ are the building blocks of Graver basis elements $z \in G_{k,N}$, $N \geqslant 1$, with $z_\varepsilon = r$. Hence it suffices to show that for each $r \in \mathcal{H}^0_{k,\infty}$, the vector-tree $\mathcal{T}(r)$ is finite.

Suppose first that $r = 0$. By induction hypothesis, $\mathcal{H}_{k-1,\infty}$ is finite. Hence it is enough to show that all labels of non-root nodes of $\mathcal{T}(0)$ are in $\mathcal{H}_{k-1,\infty}$. Let v be the label of a node of height $s > 0$ of $\mathcal{T}(0)$. We may assume $v \neq 0$. There exists an integer $N \geqslant 1$, a vector $z \in G(A_{k,N})$, and $\alpha = (\alpha_1, \dots, \alpha_s) \in [1, N]^s$ such that $z_\varepsilon = 0$ and $z_\alpha = v$. Let $z' \in \mathbb{Z}^{d_{k-1,N}}$ such that $\mathcal{T}(z')$ is a sub-vector-tree of $\mathcal{T}(z)$ and $z'_{\alpha'} = v$, where $\alpha' = (\alpha_2, \dots, \alpha_s)$ (i.e., $\mathcal{T}(z')$ contains our node labeled by v). Then $z' \in G(A_{k-1,N})$: we have $z' \in \ker(A_{k-1,N})$, since $z \in \ker(A_{k,N})$ and $z_\varepsilon = 0$; $z' \neq 0$, since $v \neq 0$; and z' is $\sqsubseteq$-minimal among the nonzero elements of $\ker(A_{k-1,N})$, since any nonzero $y' \in \ker(A_{k-1,N})$ with $y' \sqsubseteq z'$ and $y' \neq z'$ gives rise to a nonzero $y \in \ker(A_{k,N})$ with $y \sqsubseteq z$ and $y \neq z$ (by replacing the sub-vector-tree of $\mathcal{T}(z)$ corresponding to $\mathcal{T}(z')$ by $\mathcal{T}(y')$), and this contradicts $z \in G(A_{k,N})$. Thus $v \in \mathcal{H}^s_{k-1,\infty}$ as desired.

Now suppose $r \neq 0$. Let T'_{k-1} be the $l \times (1 + n_{k-1})$-matrix $(T_k r | T_{k-1})$ and put $T'_s := T_s$ for $s = 0, \dots, k-2$. Define $A'_{s,N}$ in the same way as $A_{s,N}$ at the beginning of this section, with the matrices T'_s replacing T_s. Let $\mathcal{H}'_{k-1,\infty}$ be the set of building blocks of Graver bases of $A'_{k-1,N}$, for $N \geqslant 1$. By induction hypothesis, $\mathcal{H}'_{k-1,\infty}$ is finite, so it is enough to show that for every node of height $s > 0$ of $\mathcal{T}(r)$ with label v, we have $v \in \mathcal{H}'_{k-1,\infty}$. Let $N \geqslant 1$, $z \in G(A_{k,N})$, and $\alpha = (\alpha_1, \dots, \alpha_s) \in [1, N]^s$ with $z_\varepsilon = r$ and $z_\alpha = v$. Let $z' \in \mathbb{Z}^{d_{k-1,N}}$ such that $\mathcal{T}(z')$ is a sub-vector-tree of $\mathcal{T}(z)$ with $z'_{\alpha'} = v$, where $\alpha' = (\alpha_2, \dots, \alpha_s)$. Then $(1, z') \in G(A'_{k-1,N})$: we have $(1, z') \in \ker(A'_{k-1,N})$ since $z \in \ker(A_{k,N})$ and $z_\varepsilon = r$; and $(1, z')$ is $\sqsubseteq$-minimal among the elements of $\ker(A'_{k-1,N})$ of the form $(1, z'')$ where $z'' \in \mathbb{Z}^{d_{k-1,N}}$ (shown similarly to $\sqsubseteq$-minimality of z in $\ker(A_{k,N}) \setminus \{0\}$ in the case $r = 0$ above); hence $(1, z') \in G(A'_{k-1,N})$ by Lemma 7.5. Therefore the building blocks of z' are in $\mathcal{H}'_{k-1,\infty}$. In particular $v \in \mathcal{H}'_{k-1,\infty}$ as required. $\square$

The proof of the proposition above suggests a procedure for constructing $\mathcal{H}_{k,\infty}$ for $k = 0, 1, 2, \dots$ inductively. In the next section we describe such an algorithm. We finish this section by a few remarks about the choice of the reducibility relation $\sqsubseteq_k$. First note the following immediate consequence of the fact that $\mathrm{paths}(\mathcal{T}) = \langle \mathcal{T} \rangle_1$ for every vector-tree $\mathcal{T}$:

Lemma 8.13. *For vector-trees $\mathcal{S}$, $\mathcal{T}$ of height h we have: if $\mathcal{S} \sqsubseteq_{h,\infty} \mathcal{T}$, then for every $w \in \mathrm{paths}(\mathcal{T})$ there exists $v \in \mathrm{paths}(\mathcal{S})$ such that $v \sqsubseteq w$ (in $\mathbb{Z}^{n(h)}$).*

The converses of the implications in Lemmas 8.9 and 8.13 are false in general, as the following two simple examples show. In particular, this (at least partly) explains why in the algorithm for computing $\mathcal{H}_{k,\infty}$ in the next section, we cannot simply replace $\sqsubseteq_k$ by the quasi-ordering $\leqslant_k$ on vector-trees of height k given by

$$\mathcal{S} \leqslant_k \mathcal{T} \quad :\Longleftrightarrow \quad \begin{cases} \text{for every } w \in \mathrm{paths}(\mathcal{T}) \text{ there is } v \in \mathrm{paths}(\mathcal{S}) \\ \text{such that } v \sqsubseteq w, \end{cases} \tag{8.2}$$

whose Noetherianity is much easier to show than the Noetherianity of the quasi-ordering $\sqsubseteq_k$ (cf. Corollary 4.3 in the case $n = n(k)$). In both examples $k = 2$ and $n_0 = n_1 = n_2 = 2$.

Example. The vector-trees

$$\begin{tikzcd}[column sep=small] & (0,0) \arrow[dl] \arrow[dr] & \\ (0,0) \arrow[d] & & (0,0) \arrow[d] \\ (0,1) & & (1,0) \\ & \mathcal{S} & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & (0,0) \arrow[d] & \\ & (0,0) \arrow[dl] \arrow[dr] & \\ (0,1) & & (1,0) \\ & \mathcal{T} & \end{tikzcd}$$

show that we may have $\mathcal{S} \sqsubseteq_{2,\infty} \mathcal{T}$ and $\mathcal{S} \not\sqsubseteq_2 \mathcal{T}$.

Example. Consider the vector-trees

$$\begin{tikzcd}[column sep=small] & (0,0) \arrow[dl] \arrow[dr] & \\ (0,0) \arrow[d] & & (0,1) \arrow[d] \\ (1,0) & & (0,1) \\ & \mathcal{S} & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & (0,0) \arrow[d] & \\ & (0,1) \arrow[dl] \arrow[dr] & \\ (1,0) & & (0,1) \\ & \mathcal{T} & \end{tikzcd}$$

We have $\mathcal{S} \leqslant_k \mathcal{T}$ (as defined in (8.2)). Suppose that $z \in \mathbb{Z}^{14}$ has associated vector-tree

$$\begin{tikzcd}[column sep=small] & & & (0,0) \arrow[dll] \arrow[drr] & & & \\ & (0,1) \arrow[dl] \arrow[dr] & & & & (0,1) \arrow[dl] \arrow[dr] & \\ (1,0) & & (0,1) & & (1,0) & & (0,1) \end{tikzcd}$$

Then $z \in \langle \mathcal{T} \rangle_2$, but there is no $y \in \langle \mathcal{S} \rangle_2$ with $y \sqsubseteq z$.

9. Computation of Building Blocks

Our algorithm for computing a finite set of vector containing $\mathcal{H}_{k,\infty}$ follows the pattern of a completion procedure, similar to Buchberger's algorithm for computing Gröbner bases of ideals in polynomial rings over fields. Instead of with (finite sets of) polynomials, our procedure operates with finite vector-trees. Before we describe our algorithm, we need to specify some crucial ingredients for this completion procedure, among them the input set and a notion of normal form.

Adding and subtracting vector trees. We begin by defining operations which allow us to construct new vector-trees from old ones. For this we use the following notations, for subsets V and W of $\mathbb{Z}^m$, $m \geqslant 1$:

$$\begin{aligned} -V &:= \{-v : v \in V\}, \\ V + W &:= \{v + w : v \in V,\ w \in W\}, \\ V - W &:= \{v - w : v \in V,\ w \in W,\ w \sqsubseteq v\}. \end{aligned}$$

Note that in general $V - W \neq V + (-W)$. In the following $\mathcal{S}$ and $\mathcal{T}$ range over the set of vector-trees of height k.

Definition 9.1. We put

$$-\mathcal{S} := \mathcal{T}\big(-\operatorname{paths}(\mathcal{S})\big), \quad \mathcal{S} + \mathcal{T} := \mathcal{T}\big(\operatorname{paths}(\mathcal{S}) + \operatorname{paths}(\mathcal{T})\big).$$

If $\mathcal{T} \sqsubseteq_k \mathcal{S}$, then $\operatorname{paths}(\mathcal{S}) - \operatorname{paths}(\mathcal{T}) \neq \emptyset$ by Lemmas 8.9 and 8.13, and in this case we put

$$\mathcal{S} - \mathcal{T} := \mathcal{T}\big(\operatorname{paths}(\mathcal{S}) - \operatorname{paths}(\mathcal{T})\big).$$

Remarks.

(1) We have $\operatorname{paths}(-\mathcal{S}) = -\operatorname{paths}(\mathcal{S})$ and $\operatorname{paths}(\mathcal{S}+\mathcal{T}) = \operatorname{paths}(\mathcal{S})+\operatorname{paths}(\mathcal{T})$. If $\mathcal{T} \sqsubseteq_k \mathcal{S}$ then $\operatorname{paths}(\mathcal{S} - \mathcal{T}) = \operatorname{paths}(\mathcal{S}) - \operatorname{paths}(\mathcal{T})$.
(2) If $\mathcal{S}$ has value $a \in \mathbb{Z}^l$ and $\mathcal{T}$ has value $b \in \mathbb{Z}^l$, then $-\mathcal{S}$ has value $-a$, $\mathcal{S} + \mathcal{T}$ has value $a + b$, and if in addition $\mathcal{T} \sqsubseteq_k \mathcal{S}$, then $\mathcal{S} - \mathcal{T}$ has value $a - b$.
(3) There are obvious algorithms to compute, given finite $\mathcal{S}$ and $\mathcal{T}$, the vector-trees $-\mathcal{S}$, $\mathcal{S} + \mathcal{T}$, and $\mathcal{S} - \mathcal{T}$ (provided $\mathcal{T} \sqsubseteq_k \mathcal{S}$).

Normal forms. We say that $\mathcal{S}^*$ is a **normal form** of $\mathcal{S}$ with respect to a set G of vector-trees of height k if

(1) $\mathcal{T} \not\sqsubseteq_k \mathcal{S}^*$ for all $\mathcal{T} \in G$ with $\operatorname{root}(\mathcal{T}) \neq 0$,
(2) there exists a sequence $\mathcal{S}_0, \dots, \mathcal{S}_n$ of vector-trees of height k such that $\mathcal{S}_0 = \mathcal{S}$, $\mathcal{S}_n = \mathcal{S}^*$, and for every $i = 0, \dots, n-1$ there exists $\mathcal{T}_i \in G$ with $\mathcal{T}_i \sqsubseteq_k \mathcal{S}_i$ and $\mathcal{S}_{i+1} = \mathcal{S}_i - \mathcal{T}_i$.

Note that for $k = 0$, if we identify each vector-tree of height 0 with the label of its root, this notion corresponds to the notion of normal form for elements of $\mathbb{Z}^{n_0}$ introduced in Section 7. The following algorithm computes a normal form:

Algorithm 9.2. (Normal form algorithm)

Input: a finite vector-tree $\mathcal{S}$ of height k and a finite set G of finite vector-trees of height k.

Output: a normal form $\operatorname{normalForm}(\mathcal{S}, G)$ of $\mathcal{S}$ with respect to G.

while there is some $\mathcal{T} \in G$ such that $\mathcal{T} \sqsubseteq_k \mathcal{S}$ and $\operatorname{root}(\mathcal{T}) \neq 0$ **do**
  $\mathcal{S} := \mathcal{S} - \mathcal{T}$
return $\mathcal{S}$

The algorithm above terminates: if $\mathcal{T} \sqsubseteq_k \mathcal{S}$, then $\operatorname{root}(\mathcal{S}-\mathcal{T}) = \operatorname{root}(\mathcal{S})-\operatorname{root}(\mathcal{T}) \sqsubseteq \operatorname{root}(\mathcal{S})$, and if in addition $\operatorname{root}(\mathcal{T}) \neq 0$, then $\|\operatorname{root}(\mathcal{S} - \mathcal{T})\|_1 < \|\operatorname{root}(\mathcal{S})\|_1$. Note that if every vector-tree in G has value 0, and $\mathcal{S}$ has value a, then the output $\operatorname{normalForm}(\mathcal{S}, G)$ also has value a. Algorithm 9.2 will be employed as a subprogram in our algorithm for computing the set $\mathcal{H}_{k,\infty}$. In the proof of the correctness of the latter we will use the following crucial lemma and its corollary below.

Lemma 9.3. *Let $N \geqslant 1$.*

(1) *If $y \in \langle \mathcal{S} \rangle_N$ and $z \in \langle \mathcal{T} \rangle_N$, then $y + z \in \langle \mathcal{S} + \mathcal{T} \rangle_N$.*
(2) *If $\mathcal{T} \sqsubseteq_k \mathcal{S}$ and $y \in \langle \mathcal{S} \rangle_N$, then there exists $z \in \langle \mathcal{T} \rangle_N$ such that $z \sqsubseteq y$ and $y - z \in \langle \mathcal{S} - \mathcal{T} \rangle_N$.*

Proof. For part (1), suppose that $y \in \langle \mathcal{S} \rangle_N$ and $z \in \langle \mathcal{T} \rangle_N$, that is, $\operatorname{paths}\big(\mathcal{T}(y)\big) \subseteq \operatorname{paths}(\mathcal{S})$ and $\operatorname{paths}\big(\mathcal{T}(z)\big) \subseteq \operatorname{paths}(\mathcal{T})$. The label of every maximal path in the vector tree $\mathcal{T}(y+z)$ is the sum of the labels of the corresponding paths in $\mathcal{T}(y)$ and $\mathcal{T}(z)$, respectively. Thus

$$\begin{aligned}\text{paths}\big(\mathcal{T}(y+z)\big) &\subseteq \text{paths}\big(\mathcal{T}(y)\big) + \text{paths}\big(\mathcal{T}(z)\big)\\ &\subseteq \text{paths}(\mathcal{S}) + \text{paths}(\mathcal{T}) = \text{paths}(\mathcal{S}+\mathcal{T}),\end{aligned}$$

hence $y+z \in \langle \mathcal{S}+\mathcal{T}\rangle_N$ as claimed. We show (2) by induction on k. The case $k=0$ being easy, suppose that $k>0$ and the claim holds with $k-1$ in place of k. Since $\mathcal{T} \sqsubseteq_k \mathcal{S}$, we have $\text{root}(\mathcal{T}) \sqsubseteq \text{root}(\mathcal{S}) = y_{0,\varepsilon}$. Consider $y' \in \mathbb{Z}^{d_{k-1,N}}$ whose vector-tree $\mathcal{T}(y')$ is an immediate sub-vector-tree of $\mathcal{T}(y)$. Then there exists an immediate sub-vector-tree $\mathcal{S}'$ of $\mathcal{S}$ with $y' \in \langle \mathcal{S}'\rangle_N$. Since $\mathcal{T} \sqsubseteq_k \mathcal{S}$ we find an immediate sub-vector-tree $\mathcal{T}'$ of $\mathcal{T}$ with $\mathcal{T}' \sqsubseteq_{k-1} \mathcal{S}'$. By the inductive hypothesis there exists $z' \in \langle \mathcal{T}'\rangle_N$ such that $z' \sqsubseteq y'$ and $y'-z' \in \langle \mathcal{S}'-\mathcal{T}'\rangle_N$. These remarks suffice to construct a vector $z \in \mathbb{Z}^{d_{k,N}}$ with the required properties. ☐

For a set G of vector-trees of height k and an integer $N \geqslant 1$ we put

$$\langle G\rangle_N := \bigcup_{\mathcal{T}\in G} \langle \mathcal{T}\rangle_N.$$

Part (2) of the last lemma (which strengthens Lemma 8.9) immediately implies:

Corollary 9.4. *Let G be a set of vector-trees of height k, and let $\mathcal{S}^*$ be a normal form of $\mathcal{S}$ with respect to G. For all $N \geqslant 1$ and $s \in \langle \mathcal{S}\rangle_N$ there exist $f \in \langle \mathcal{S}^*\rangle_N$ and $g_1,\dots,g_n \in \langle G\rangle_N$ such that*

$$s = f + \sum_i g_i, \qquad g_i \sqsubseteq s \text{ for all } i.$$

Choosing an input set. The following lemma justifies the choice of input set for Algorithm 9.8 below.

Lemma 9.5. *Suppose that $k>0$, and let F be a set of generators for the $\mathbb{Z}$-submodule $K := \ker(A_{k,1})$ of $\mathbb{Z}^{n(k)}$ which contains a set of generators for the submodule*

$$K_0 := K \cap \big(\{0\}\times \mathbb{Z}^{n(k-1)}\big)$$

of K. Then for every $N \geqslant 1$, the set

$$F_N := \big\langle \{\mathcal{T}(v) : v \in F\}\big\rangle_N$$

generates $\ker(A_{k,N})$.

Proof. We use the following notations. Let $N \geqslant 1$, $\alpha = (\alpha_1,\dots,\alpha_k) \in [1,N]^k$. For $z \in \mathbb{Z}^{d_{k,N}}$ we put

$$v(\alpha,z) := (z_{\alpha|0},\dots,z_{\alpha|k}) \in \text{paths}\big(\mathcal{T}(z)\big),$$

and for $v = (v_0,\dots,v_k) \in \mathbb{Z}^{n(k)}$, we denote by $z(\alpha,v)$ the vector $z \in \mathbb{Z}^{d_{k,N}}$ which satisfies, for $s = 0,\dots,k$ and $\beta \in [1,N]^s$:

$$z_\beta := \begin{cases} v_s & \text{if } \beta = \alpha|s,\\ 0 & \text{otherwise.}\end{cases}$$

We also let $z(v)$ be the unique element of $\mathbb{Z}^{d_{k,N}}$ whose vector-tree $\mathcal{T} = \mathcal{T}\big(z(v)\big)$ satisfies $\text{paths}(\mathcal{T}) = \{v\}$. Clearly, for $v \in \mathbb{Z}^{n(k)}$ and $z \in \mathbb{Z}^{d_{k,N}}$:

$$z \in \ker(A_{k,N}) \quad\Rightarrow\quad v(\alpha,z) \in \ker(A_{k,1}), \tag{9.1}$$

and

$$v \in \ker(A_{k,1}) \iff z(\alpha, v) \in \ker(A_{k,N}) \iff z(v) \in \ker(A_{k,N}).$$

Let now $z \in \ker(A_{k,N})$, where $k, N \geqslant 1$. Fix an arbitrary $\alpha \in [1,N]^k$ and put $v = v(\alpha, z)$; by (9.1) there are $a_1, \dots, a_m \in \mathbb{Z}$ and $v_1, \dots, v_m \in F$ such that

$$v = a_1 v_1 + \cdots + a_m v_m.$$

Then $z(v_i) \in F_N$ for each i, and

$$z(v) = a_1 z(v_1) + \cdots + a_m z(v_m). \tag{9.2}$$

Now if $\beta \in [1,N]^k$, then $v(z,\beta) - v \in K_0$, hence for some $a_{\beta,1}, \dots, a_{\beta,m_\beta} \in \mathbb{Z}$, $v_{\beta,1}, \dots, v_{\beta,m_\beta} \in F \cap K_0$, $m_\beta \in \mathbb{N}$:

$$v(\beta, z) - v = a_{\beta,1} v_{\beta,1} + \cdots + a_{\beta,m_\beta} v_{\beta,m_\beta}$$

and therefore

$$z\big(\beta, v(\beta,z) - v\big) = a_{\beta,1} z(\beta, v_{\beta,1}) + \cdots + a_{\beta,m_\beta} z(\beta, v_{\beta,m_\beta}) \tag{9.3}$$

with $z(\beta, v_{\beta,j}) \in F_N$ for each j. Combining

$$z = z(v) + \sum_\beta z\big(\beta, v(z,\beta) - v\big)$$

with (9.2) and (9.3) yields an expression of z as a $\mathbb{Z}$-linear combination of vectors in F_N, as required. □

A generating set F satisfying the hypothesis of the lemma can be found algorithmically by standard methods (e.g., Hermite normal form).

Computing $\mathcal{H}_{k,\infty}$. We now specify an algorithm which recursively (in k) computes the set $\mathcal{H}_{k,\infty}$. The following completion procedure is at the heart of the k-th step in the algorithm. We say that a set G of vector-trees of height k is **root-complete** if for all $\mathcal{S}, \mathcal{T} \in G$ the sum $\mathcal{S} + \mathcal{T}$ has a normal form $\mathcal{N}$ with respect to G such that $\mathrm{root}(\mathcal{N}) = 0$.

Algorithm 9.6. (Completion procedure)

Input: a finite set G of finite vector-trees of height k.

Output: a finite set of finite vector-trees of height k which contains G and is root-complete.

$C := \{\mathcal{S} + \mathcal{T} : \mathcal{S}, \mathcal{T} \in G\}$
while $C \neq \emptyset$ **do**
  $\mathcal{S} :=$ an element in C
  $C := C \setminus \{\mathcal{S}\}$
  $\mathcal{T} := \mathrm{normalForm}(\mathcal{S}, G)$
  if $\mathrm{root}(\mathcal{T}) \neq 0$ **then**
    $G := G \cup \{\mathcal{T}, -\mathcal{T}\}$
    $C := C \cup \{\mathcal{S} + \mathcal{T}, \mathcal{S} + (-\mathcal{T}) : \mathcal{S} \in G\}$
return G

We turn to termination and correctness of Algorithm 9.6:

Proposition 9.7. *Algorithm 9.6 terminates and satisfies its specification.*

Proof. Let G_i be the value assigned to G and $\mathcal{S}_i$ be the value assigned to $\mathcal{S}$ at the beginning of the i-th pass of the **while**-loop in Algorithm 9.6, and set $\mathcal{T}_i := \operatorname{normalForm}(\mathcal{S}_i, G_i)$. Suppose that $\operatorname{root}(\mathcal{T}_i) \neq 0$ for all i in an infinite subset I of $\mathbb{N} \setminus \{0\}$. Now $\mathcal{T} \not\sqsubseteq_k \mathcal{T}_i$ for all $\mathcal{T} \in G_i$ with $\operatorname{root}(\mathcal{T}) \neq 0$, and $\mathcal{T}_i \in G_j$ for all $i < j$. In particular $\mathcal{T}_i \not\sqsubseteq_k \mathcal{T}_j$ for all $i < j$ in I. This contradicts Corollary 8.8. Hence there is some n such that $\operatorname{root}(\mathcal{T}_i) = 0$ for all $i \geqslant n$. So if m is the size of the set C before the n-th iteration of the **while**-loop, then Algorithm 9.6 terminates after m more iterations. This shows termination of Algorithm 9.6. Correctness is easily shown. □

We say that a set G of vector-trees of height k is **symmetric** if $\mathcal{S} \in G \Rightarrow -\mathcal{S} \in G$ for all $\mathcal{S}$. If G is symmetric, then for each $N \geqslant 1$, the subset $\langle G \rangle_N$ of $\mathbb{Z}^{d_{k,N}}$ is also symmetric: $z \in \langle G \rangle_N \Rightarrow -z \in \langle G \rangle_N$ for every $z \in \mathbb{Z}^{d_{k,N}}$.

Remarks.

(1) If the input set G in Algorithm 9.6 is symmetric, then so is its output set. If every vector-tree in G has value 0, then so does every vector-tree in the output set.

(2) For $k = 0$ and input set $G =$ a finite symmetric generating set for the $\mathbb{Z}$-module $\ker(T_0)$, Algorithm 9.6 reduces to Algorithm 7.3 from Section 7 and computes a finite set of vectors containing $G(T_0)$.

Here now is:

Algorithm 9.8. (Algorithm to compute $\mathcal{H}_{k,\infty}$)

Input: an integer $k \geqslant 0$.

Output: a finite symmetric set G_k of finite vector-trees of height k such that $G(A_{k,N}) \subseteq \langle G_k \rangle_N$ for all $N \geqslant 1$.

for $i = 0, \dots, k$ **do**
  if $i = 0$ **then**
    $F_0 :=$ a finite symmetric generating set for $\ker(A_{0,1})$
    $G := \big\{\mathcal{T}(v) : v \in F_0\big\}$
  else
    $F_i :=$ a finite symmetric generating set for $\ker(A_{i,1})$ satisfying the hypothesis of Lemma 9.5 (for $k = i$)
    $\mathcal{T}_0 := \mathcal{T}\big(\big\{(0, v) : v = 0 \text{ or } v \in \operatorname{paths}(\mathcal{S}) \text{ for some } \mathcal{S} \in G_{i-1}\big\}\big)$
    $G := \{\mathcal{T}_0\} \cup \big\{\mathcal{T}(v) : v \in F_i\big\}$
  $G_i :=$ output of Algorithm 9.6 applied to G

Since termination of Algorithm 9.8 follows from Proposition 9.7, we only need to establish its correctness:

Theorem 9.9. *Let $G_0, \dots, G_k$ be the sets computed by Algorithm 9.8, for given input k. Then each G_i is a finite symmetric set of finite vector-trees of height i with $G(A_{i,N}) \subseteq \langle G_i \rangle_N$ for all $N \geqslant 1$. (In particular, the set consisting of the building blocks of vector-trees in G_k contains $\mathcal{H}_{k,\infty}$.)*

In the proof we use the following immediate consequence of Lemma 9.3, (1) and Corollary 9.4:

Lemma 9.10. *Let G be a root-complete set of vector-trees of height k. Then for every $N \geqslant 1$ and $y, z \in \langle G\rangle_N$ there exist $f \in \mathbb{Z}^{d_{k,N}}$ and $g_1, \dots, g_n \in \langle G\rangle_N$ such that $f_\varepsilon = 0$ and*

$$y + z = f + \sum_j g_j, \qquad g_j \sqsubseteq y + z \text{ for all } j.$$

For the proof of Theorem 9.9, fix integers $k \geqslant 0$ and $N \geqslant 1$. We show, by induction on $i = 0, \dots, k$, that $G(A_{i,N}) \subseteq \langle G_i\rangle_N$. The case $i = 0$ is covered by Remark (2) following the proof of Proposition 9.7. Suppose that $i > 0$. By the inductive hypothesis, G_{i-1} is a finite symmetric set of finite vector-trees of height $i-1$ with $G(A_{i-1,N}) \subseteq \langle G_{i-1}\rangle_N$. In particular, $\mathcal{T}_0$ has value 0 and $-\mathcal{T}_0 = \mathcal{T}_0$. Moreover, by the next lemma, every $f \in \ker(A_{i,N})$ with $f_\varepsilon = 0$ has normal form 0 with respect to $\langle \mathcal{T}_0\rangle_N$. In the proof we denote the concatenation of the finite sequences $\alpha \in [1, N]^s$ and $\beta \in [1, N]^t$ ($s, t \in \{0, \dots, k\}$) by $\alpha\beta \in [1, N]^{s+t}$.

Lemma 9.11. *For every $f \in \ker(A_{i,N})$ with $f_\varepsilon = 0$ there are $h_1, \dots, h_m \in \langle \mathcal{T}_0\rangle_N$ such that*

$$f = \sum_j h_j, \qquad h_j \sqsubseteq f \text{ for all } j.$$

Proof. Let $f_n \in \mathbb{Z}^{d_{i-1,N}}$, $n = 1, \dots, N$, be given by $f_{n,\alpha} := f_{(n)\alpha}$ for $\alpha \in [1, N]^s$, $s = 0, \dots, i-1$. Since $f \in \ker(A_{i,N})$ and $f_\varepsilon = 0$, we have $f_1, \dots, f_N \in \ker(A_{i-1,N})$. Hence for each $n = 1, \dots, N$ there are $M(n) \in \mathbb{N}$ and $g_{nm} \in G(A_{i-1,N})$, where $m = 1, \dots, M(n)$, such that

$$f_n = \sum_m g_{nm} \qquad g_{nm} \sqsubseteq f_n \text{ for all } m.$$

Now let h_{nm} be the vector in $\mathbb{Z}^{d_{i,N}}$ defined as follows: for $\alpha = (\alpha_1, \dots, \alpha_s) \in [1, N]^s$, $s = 0, \dots, i$, let

$$h_{nm,\alpha} = \begin{cases} g_{nm,\alpha'} & \text{if } s > 0 \text{ and } \alpha_1 = n \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha' = (\alpha_2, \dots, \alpha_s)$. Then h_{nm} can be constructed from $\mathcal{T}_0$, since $G(A_{i-1,N}) \subseteq \langle G_{i-1}\rangle_N$ and $0 \in \text{paths}(\mathcal{T}_0)$, and $f = \sum_{n,m} h_{nm}$ with $h_{nm} \sqsubseteq f$ for all n and m. □

Proof of Theorem 9.9. By Lemma 9.5 and the choice of F_i in Algorithm 9.8, $\langle G_i\rangle_N$ is a symmetric generating set for $\ker(A_{i,N})$. Hence $\langle G_i\rangle_N$ can be used as an input set for the computation of a Graver basis for the matrix $A_{i,N}$ (Algorithm 7.3). By Lemmas 9.10 and 9.11, the sum $y + z$ of two elements y and z of $\langle G_i\rangle_N$ has normal form 0 with respect to $\langle G_i\rangle_N$. Hence Algorithm 7.3 applied to $A_{i,N}$ and $\langle G_i\rangle_N$ just returns the input set $\langle G_i\rangle_N$; in particular $G(A_{i,N}) \subseteq \langle G_i\rangle_N$ as desired. □

Remarks.

(1) Theorem 9.9 gives another proof of Proposition 8.11.
(2) For $k = 1$, the algorithm to compute $\mathcal{H}_{k,\infty}$ described above differs slightly from the one given in [12]. This is because Algorithm 3.15 in [12] is slightly defective: to see this, consider (in the notation introduced there) the pairs $s = g = (0, \{0, 1\})$; then $g \sqsubseteq s$ and $s \ominus g = s$, causing Algorithm 3.15 to diverge on the inputs s and $G = \{g\}$.

10. Finding an Optimal Solution

In this final section we outline how the set G_k produced by Algorithm 9.8 can be employed to solve any particular instance

$$(\mathrm{IP}_{N,b,c}) \qquad \min\left\{c^\intercal z : A_{k,N} z = b,\ z \in \mathbb{N}^{d_{k,N}}\right\}$$

of our family of $(k+1)$-stage stochastic integer programs, given a choice of the number N of scenarios, a right-hand side $b \in \mathbb{Z}^{e_{k,N}}$ (where $e_{k,N} := N^k \cdot l =$ number of rows of the coefficient matrix $A_{k,N}$), and a cost vector $c \in \mathbb{R}^{d_{k,N}}$. Throughout we assume that a finite set G_k of finite vector-trees of height k such that $G(A_{k,N}) \subseteq \langle G_k \rangle_N$ for all $N \geqslant 1$ (as computed by Algorithm 9.8) is at our disposal.

We first concentrate on the problem of augmenting a feasible solution to an optimal one. For this we use the recursive algorithm below. Analogous to our practice in the case of integer vectors above, we write a vector $v \in \mathbb{R}^{d_{k,N}}$ as $v = (v_0, \dots, v_k)$ where

$$v_s = \big(v_\alpha : \alpha \in [1,N]^s\big) \qquad \text{for } s = 0, \dots, k,$$

and for $k > 0$ and $i = 1, \dots, N$ we put

$$v_{(i)} := \big(v_{(i)\,\alpha} : \alpha \in [1,N]^s, s = 0, \dots, k-1\big) \in \mathbb{R}^{d_{k-1,N}}.$$

Conversely, if $k > 0$, given $r \in \mathbb{R}^{n_k}$ and N vectors $v_1, \dots, v_N \in \mathbb{R}^{d_{k-1,N}}$, we denote by $v(r, v_1, \dots, v_N)$ the vector $v \in \mathbb{R}^{d_{k,N}}$ with $v_\varepsilon = r$ and $v_{(i)} = v_i$ for all $i = 1, \dots, N$. With this notation, given $\mathcal{T}$ and $N \geqslant 1$, the set $\langle \mathcal{T} \rangle_N$ consists of all vectors $z \in \mathbb{Z}^{d_{k,N}}$ of the form $z = v\big(\mathrm{root}(\mathcal{T}), v_1, \dots, v_N\big)$, where $v_i \in \langle \mathcal{T}_i \rangle_N$ for an immediate sub-vector-tree $\mathcal{T}_i$ of $\mathcal{T}$, for each $i = 1, \dots, N$.

Algorithm 10.1. (Algorithm to find most expensive constructible vector)

Input: an integer $N \geqslant 1$, a finite set G of finite vector-trees of height k, a cost vector $c \in \mathbb{R}^{d_{k,N}}$, and a vector $z \in \mathbb{Z}^{d_{k,N}}$.

Output: a vector $v = \mathrm{mostExpensive}(N, G, c, z)$ with $v \in \langle G \rangle_N$ and $v \leqslant z$ such that $c^\intercal v$ is maximal with these properties, or "FAIL" if no $v \in \langle G \rangle_N$ with $v \leqslant z$ exists.

$G' := \big\{\mathcal{T} \in G : \mathrm{root}(\mathcal{T}) \leqslant z_\varepsilon\big\}$
if $G' = \emptyset$ **then** **return** "FAIL"
while $G' \neq \emptyset$ **do**
  $\mathcal{T} :=$ the element of G' such that $c_\varepsilon^\intercal\, \mathrm{root}(\mathcal{T})$ is maximal
  $G' := G' \setminus \{\mathcal{T}\}$
  if $k = 0$ **then** **return** $\mathrm{root}(\mathcal{T})$
  $G_{\mathcal{T}} :=$ the set of immediate sub-vector-trees of $\mathcal{T}$
  for $i = 1$ **to** N **do**
    $v_i := \mathrm{mostExpensive}(N, G_{\mathcal{T}}, c_{(i)}, z_{(i)})$
  if $v_1, \dots, v_N \neq$ "FAIL" **then** **return** $v\big(\mathrm{root}(\mathcal{T}), v_1, \dots, v_N\big)$
return "FAIL"

Termination and correctness of the procedure above are easily seen. By the discussion in Section 7 this implies termination and correctness of the following algorithm:

Algorithm 10.2. (Augmentation algorithm)

Input: an integer $N \geqslant 1$, vectors $b \in \mathbb{Z}^{e_{k,N}}$, $c \in \mathbb{R}^{d_{k,N}}$, and a feasible solution $z \in \mathbb{N}^{d_{k,N}}$ to $(\mathrm{IP}_{N,b,c})$.

Output: an optimal solution $\mathrm{optimalSolution}(N, b, c, z)$ to $(\mathrm{IP}_{N,b,c})$.

while $v :=$ mostExpensive$(N, G_k, c, z) \neq$ "FAIL" and $c^\intercal v > 0$ **do**
  $z := z - v$
return z

The next algorithm produces an initial feasible solution from a given solution (in $\mathbb{Z}^{d_{k,N}}$) to the equation $A_{k,N} z = b$. Termination and correctness of this procedure follow from results in [10]; see also Algorithm 7.2 above.

Algorithm 10.3. (Finding a feasible solution)

Input: an integer $N \geqslant 1$, vectors $b \in \mathbb{Z}^{e_{k,N}}$, $c \in \mathbb{R}^{d_{k,N}}$, and a solution $z \in \mathbb{Z}^{d_{k,N}}$ to $(\mathrm{IP}_{N,b,c})$.

Output: a feasible solution feasibleSolution(N, b, c, z) to $(\mathrm{IP}_{N,b,c})$, or "FAIL" if no such solution exists.

while $v :=$ mostExpensive$\big(N, G_k, c(z), z^+\big) \neq$ "FAIL" and $c(z)^\intercal v > 0$ **do**
  $z := z - v$
if $z \geqslant 0$ **then** **return** z **else** **return** "FAIL"

Finally, this leads to our algorithm for solving instances of $(\mathrm{IP}_{N,b,c})$ using G_k:

Algorithm 10.4. (Finding an optimal solution)

Input: an integer $N \geqslant 1$ and vectors $b \in \mathbb{Z}^{e_{k,N}}$, $c \in \mathbb{R}^{d_{k,N}}$.

Output: an optimal solution to $(\mathrm{IP}_{N,b,c})$, or "FAIL" if no feasible solution exists.

if there is no $z \in \mathbb{Z}^{d_{N,k}}$ with $A_{k,N} z = b$ **then**
  return "FAIL"
else
  $z :=$ an element of $\mathbb{Z}^{d_{N,k}}$ satisfying $A_{k,N} z = b$
$f :=$ feasibleSolution(N, b, c, z)
if $f =$ "FAIL" **then**
  return "FAIL"
else
  return optimalSolution(N, b, c, f)

Remark. The complexity of Algorithm 9.8 is unclear. For a very crude complexity result related to Maclagan's principle see [1], Proposition 3.25. Some computational experiments in the case $k = 1$ are reported in [12], Section 4. Note that (once G_k is available) the running time of Algorithm 10.1 above is $O(N^k)$.

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