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Cordiality of a Star of the Complete Graph and a Cycle Graph $C(N \cdot K_N)$

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Abstract

In this paper we prove that a star of K_n and a cycle of n copies of K_n are cordial. We also get condition for maximum value of $e_f(1) - e_f(0)$ and highest negative value of $e_f(1) - e_f(0)$ in K_n, where f is the binary vertex labeling function on the vertex set of K_n.

Keywords: complete graph, binary vertex labeling, star of a graph and cycle of a graph

AMS subject classification number: 05C78

1. Introduction

Let $G = (V, E)$ be a simple, undirected finite graph with $|V| = p$ vertices and $|E| = q$ edges. For all basic terminology and standard notations we follow Harary (1972). Here are some of the definitions which are useful in this paper.

Definition 1.1 If the vertices of the graph are assigned values subject to certain conditions then it is known as *graph labeling*.

Definition 1.2 A function $f \colon V \to \{0, 1\}$ is called *binary vertex labeling* of a graph G and $f(v)$ is called *label of the vertex* v of G under f.

For an edge $e = (u, v)$, the induced function $f^* \colon E \to \{0, 1\}$ defined as $f^*(e) = |f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$ be number of vertices of G having labels 0 and 1 respectively under f and let $e_f(0)$, $e_f(1)$ be number of edges of G having labels 0 and 1 respectively under f^*.

A binary vertex labeling f of a graph G is called *cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph which admits cordial labeling is called *cordial graph*.

Definition 1.3 A graph obtained by replacing each vertex of star $K_{1,n}$ by a connected graph G of n vertices is called *star of* G and we shall denote it by $G^\star$. The graph G which replaced at the center of $K_{1,n}$ we call it as central copy of $G^\star$.

Above definition was introduced by Vaidya et al. (2008, p. 54-64).

Definition 1.4 For a cycle C_n, each vertex of C_n is replace by connected graphs $G_1, G_2, \ldots, G_n$ is known as *cycle of graphs* and we shall denote it by $C(G_1, G_2, \ldots, G_n)$. If we replace each vertices by a graph G i.e. $G_1 = G$, $G_2 = G$, $\ldots G_n = G$, such cycle of a graph G, we shall denote it by $C(n \cdot G)$.

Gallian (2013) survey provide vast amount of literature on different type of graph labeling. Labeled graph has many diversified applications. The cordial labeling introduced by Cahit (1987, p. 201-207) is a weaker version of graceful labeling,also he proved that K_n is cordial if and only if $n \leq 3$. After this, many researchers have studied cordial graphs.

Kaneria and Vaidya (2010, p. 38-46) discussed cordiality of graphs in different context. They introduced the index of cordiality for a graph G and proved that the index of cordiality for K_{t^2} ($t \in N$) is precisely 2. They also raised the following conjecture.

Conjecture 1.5 For any $n \in N$, $K_n^\star$ is cordial.

Kaneria et al. (2014, p. 173-178, IJMR) introduced cycle of graphs and proved that cycle of cycles and cycle of complete bipartite graphs are cordial.

In this paper we prove that $K_n^\star$ (the star of the complete graph), $C(n \cdot K_n)$ (cycle of n copies of the complete graph) are cordial.

1.6 Discussion on cordiality of K_n: Let f be a binary vertex labeling on K_n. We know that if $f^{-1}(0) = v_f(0) = l$ then $f^{-1}(1) = v_f(1) = n - l$ and in this case $e_f(0) = {}^lC_2 +^{n-l} C_2$, $e_f(1) = l(n - l)$ holds for K_n. If we take $v_f(1) = l$ then $v_f(0) = n - l$, while $e_f(0) = {}^lC_2 +^{n-l} C_2$, $e_f(1) = l(n - l)$ would remains same for K_n. Moreover we observe that K_n, $e_f(0)$ and $e_f(1)$ depends on the value of $v_f(0)$ and $v_f(1)$. Particularly these values are $e_f(1) = v_f(0) \cdot v_f(1)$ and $e_f(0) = \frac{n}{2}(n - 1) - e_f(1)$. Using this fact Kaneria and Vaidya (2010, p. 38-46) proved that $K_n \cup K_n$ is a cordial graph, when $n = t^2$, for some $t \in N - \{1\}$. They also proved star of K_n is cordial, when $n = t^2 + 2$ or t^2 or $t^2 - 2$, for some $t \in N - \{1\}$.

2. Main Results

Theorem 2.1 *Let n be an even positive integer and f be a binary vertex labeling on the vertex set of K_n. If $v_f(0) = v_f(1)$ in K_n then $e_f(1) - e_f(0)$ has maximum value $\frac{n}{2}$.*

Proof. Let $n = 2m$, for some $m \in N$.

Take $v_f(0) = v_f(1) = m$. In this case $e_f(1) = v_f(0)v_f(1) = m^2$ and

$$e_f(0) = |E(K_n)| - e_f(1) = \frac{n}{2}(n-1) - m^2 = m(2m-1) - m^2 = m^2 - m.$$

$$\Rightarrow e_f(1) - e_f(0) = m^2 - (m^2 - m) = m.$$

If we take $v_f(0) = m + k$, $v_f(1) = m - k$ or $v_f(0) = m - k$, $v_f(1) = m + k$, for some k ($1 \le k \le m$) then $e_f(1) = m^2 - k^2 < m^2$ and

$$e_f(0) = 2m^2 - m - (m^2 - k^2) = m^2 - m + k^2$$

$$\Rightarrow e_f(1) - e_f(0) = m - 2k^2 < m, \text{ as } k > 0.$$

Thus $e_f(1) - e_f(0)$ has maximum value $m = \frac{n}{2}$ in K_n, when $v_f(1) = v_f(0)$. □

Theorem 2.2 *Let n be an odd positive integer and f be a binary vertex labeling on the vertex set of K_n. If $|v_f(0) - v_f(1)| = 1$ in K_n, then $e_f(1) - e_f(0)$ has maximum value $\frac{n-1}{2}$.*

Proof. Let $n = 2m - 1$, for some $m \in N$.

Take $v_f(0) = m - 1$, $v_f(1) = m$ or $v_f(0) = m$, $v_f(1) = m - 1$. In this case

$$e_f(1) = m^2 - m \text{ and } e_f(0) = m^2 - 2m + 1$$

$$\Rightarrow e_f(1) - e_f(0) = m - 1 = \frac{n-1}{2}.$$

If we take $\{v_f(1), v_f(0)\} = \{(m - 1 - k), (m + k)\}$, for some k ($1 \le k \le m - 1$), then

$$e_f(1) = m^2 - m - (k^2 + k) < m^2 - m \text{ as } k > 0$$

$$e_f(0) = m^2 - 2m + (k^2 + k + 1)$$

$$\Rightarrow e_f(1) - e_f(0) = m - 1 - 2(k^2 + k) < m - 1 \text{ as } k > 0.$$

Thus $e_f(1) - e_f(0)$ has maximum value $m - 1 = \frac{n-1}{2}$, when $|v_f(1) - v_f(0)| = 1$. □

Remark 2.3 Let f be a binary vertex labeling on the vertex set of K_n. Then in Theorem 2.1 and 2.2, we proved that $e_f(1) - e_f(0)$ has maximum value

$\frac{n}{2}$, when n is even and $v_f(1) = v_f(0)$;

$\frac{n-1}{2}$, when n is odd and $|v_f(1) - v_f(0)| = 1$.

We shall denote this maximum value for $e_f(1) - e_f(0)$ by d_1. i.e.

$$d_1 = \frac{n}{2}, \quad \text{when } n \text{ is even and } v_f(1) = v_f(0);$$
$$= \frac{n-1}{2}, \quad \text{when } n \text{ is odd and } |v_f(1) - v_f(0)| = 1.$$

We also see that in Theorem 2.1, if we take $\{v_f(1), v_f(0)\} = \{\frac{n}{2} - k, \frac{n}{2} + k\}$ in K_n, for some k $(1 \le k \le \frac{n}{2})$, we shall have $e_f(1) - e_f(0) = \frac{n}{2} - 2k^2$, when n is even as well as in Theorem 2.2, if we take $\{v_f(1), v_f(0)\} = \{\frac{n-1}{2} - k, \frac{n+1}{2} + k\}$ in K_n, for some k, we shall have $e_f(1) - e_f(0) = \frac{n-1}{2} - 2(k^2 + k)$, when n is odd.

This is a decreasing sequence and it stops at $-|E(K_n)|$ by taking $\{v_f(1), v_f(0)\} = \{0, n\}$. What will be the first negative(highest negative) value? when the above sequence $e_f(1) - e_f(0)$ comes. The difference we call as d_{-1}. □

Theorem 2.4 *If $n = 4t^2 + 2r$, for some $t, r \in N$ and $1 \le r \le 4t + 1$, then in K_n, $d_{-1} = -(4t + 2) + r$, where d_{-1} is the first negative value of $e_f(1) - e_f(0)$.*

Proof. When n is an even positive integer, then $\exists$ $t, r \in N$ such that $n = 4t^2 + 2r$ and $1 \le r \le 4t + 1$ (See more detail in proof of Theorem 2.6).

By taking $\{v_f(1), v_f(0)\} = \{(2t^2 + r - t), (2t^2 + r + t)\}$, we shall have

$$e_f(1) = (2t^2 + r)^2 - t^2 \text{ and } e_f(0) = 4t^4 + 4t^2 r + r^2 - r - t^2.$$

$$\Rightarrow e_f(1) - e_f(0) = r > 0.$$

Next we take $\{v_f(1), v_f(0)\} = \{(2t^2 + r - t - 1), (2t^2 + r + t + 1)\}$, we shall have

$$e_f(1) = (2t^2 + r)^2 - (t + 1)^2 \text{ and } e_f(0) = (2t^2 + r)^2 + (t + 1)^2 - (2t^2 + r).$$

$$\Rightarrow e_f(1) - e_f(0) = -(4t + 2) + r < 0 \text{ as } r \le 4t + 1.$$

Therefore $d_{-1} = -(4t + 2) + r$ in K_n, when $n = 4t^2 + 2r$ and $1 \le r \le 4t + 1$. □

Theorem 2.5 *If $n = (2t - 1)^2 + 2r$, for some $t, r \in N$ and $1 \le r \le 4t - 1$, then in K_n, $d_{-1} = -4t + r$, where d_{-1} is the first negative value of $e_f(1) - e_f(0)$.*

Proof. When n is an odd positive integer, then $\exists$ $t, r \in N$ such that $n = (2t - 1)^2 + 2r$ and $1 \le r \le 4t$. (See more detail in proof of Theorem 2.6).

By taking $\{v_f(1), v_f(0)\} = \{(2t^2 + r - 3t + 1), (2t^2 + r - t)\}$, we shall have

$$\begin{aligned} e_f(1) &= [(2t^2 + r) - 3t + 1][(2t^2 + r) - t] \\ &= 4t^4 + 4t^2 r - 8t^3 + r^2 - 4tr + 5t^2 + r - t \end{aligned}$$

and

$$e_f(0) = 4t^4 + 4t^2 r - 8t^3 + r^2 - 4tr + 5t^2 - t.$$

$$\Rightarrow e_f(1) - e_f(0) = r > 0.$$

Next we take $\{v_f(1), v_f(0)\} = \{(2t^2 + r) - 3t, (2t^2 + r) - t + 1\}$, we shall have

$$e_f(1) = 4t^4 + 4t^2 r - 8t^3 + r^2 - 4tr + 5t^2 + r - 3t$$

and

$$e_f(0) = 4t^4 + 4t^2 r - 8t^3 + r^2 - 4tr + 5t^2 + t.$$

$$\Rightarrow e_f(1) - e_f(0) = -4t + r < 0 \text{ as } r \le 4t - 1.$$

Therefore $d_{-1} = -4t + r$ in K_n, when $n = (2t - 1)^2 + 2r$ and $1 \le r \le 4t - 1$. □

Theorem 2.6 *$K_n^\star$ is cordial, $\forall$ $n \in N$.*

Proof. We know that $K_1^\star = K_2$, $K_2^\star$ = Path on six vertices, which both are cordial graphs. $K_3^\star$ and its cordial labeling shown in Figure 1. □

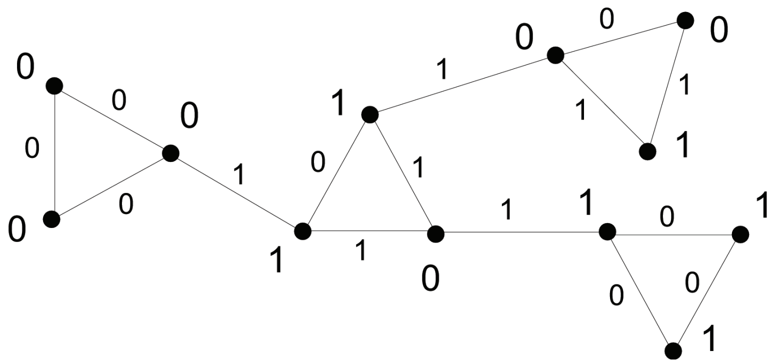

Figure 1. $K_3^\star$ and its cordial labeling ($v_f(1) = 6 = v_f(0)$, $e_f(1) = 7, e_f(0) = 8$)

Also Kaneria and Vaidya (2010, p. 38-46) proved that $K_{t^2}^\star$ ($t \in N$) is a cordial graph.

So we assume that $n \geq 5$ and $n \neq t^2$, for some $t \in N$. At this stage we shall consider following two cases for n.

Case I n is even

$\Rightarrow \exists\, t \in N$ such that $(2t)^2 < n < (2t + 2)^2$.

$\Rightarrow 4t^2 < n < 4t^2 + 8t + 4$.

$\Rightarrow 0 < n - 4t^2 < 8t + 4$.

$\Rightarrow n = 4t^2 + 2r$, for some $r \in N$ $(1 \leq r \leq 4t + 1)$.

$\Rightarrow d_1 = \frac{n}{2} = 2t^2 + r$ and $d_{-1} = -(4t + 2) + r$.

$\Rightarrow d_1 + |d_{-1}| = 2t^2 + r - r + (4t + 2) = 2(t + 1)^2 \leq n = 4t^2 + 2r$.

We know that $K_n^\star$ contains $n + 1$ copies of K_n. If we take r_1 copies of K_n, which produces d_1 and r_2 copies of K_n, which produces d_{-1}, then union of $n + 1$ copies of K_n contains $e_f(1) - e_f(0) = r_1 d_1 + r_2 d_{-1}$. Also to preserv $|v_f(1) - v_f(0)| \leq 1$ in union of $n + 1$ copies of K_n, we would take r_2 even. If $r_1 = -d_{-1}, r_2 = d_1$ then union of $n + 1$ copies of K_n satisfies $v_f(0) = v_f(1)$, $e_f(0) = e_f(1)$. Since n is even we shall take central copy of $K_n^\star$ with $v_f(0) = v_f(1) = \frac{n}{2}$ and we shall join each vertices of central copy with other copies of $K_n^\star$ whose vertex label is 1 by an edge such edge get 1 edge label if vertex of the central copy has vertex label 0, otherwise the edge get 0 edge label. This produce $e_f(0) = e_f(1)$ for $K_n^\star$ and it becomes a cordial graph.

When $r_1 \neq d_{-1}$ or $r_2 \neq d_1$ in which case we choose d_1 copies of K_n which produce d_{-1} and $|d_{-1}|$ copies of K_n which produce d_1. Then remaining copies of K_n is

$$\begin{aligned} Rcopy &= n + 1 - (d_1 + |d_{-1}|) \\ &= n + 1 - 2(t + 1)^2 \\ &= 4t^2 + 2r + 1 - 2t^2 - 4t - 2 \\ &= 2t^2 - 4t - 1 + 2r \end{aligned}$$

i.e. $Rcopy = 2(t - 1)^2 + 2r - 3 = x$ (say).

Now this $x = Rcopy$ we have to make two parts say y and $x - y$, so that

$$\frac{y}{x - y} \approx \frac{d_1}{|d_{-1}|} \Rightarrow y \approx \frac{(2t^2 + r)(2(t - 1)^2 + 2r - 3)}{2(t + 1)^2} = \frac{d_1 x}{d_1 - d_{-1}}.$$

Now $r_2 = d_1 + y$, which we take even to maintain $v_f(0) = v_f(1)$ in $K_n^\star$. So we shall take $y_1 = \frac{d_1 x}{d_1 - d_{-1}}$ and

$$\begin{aligned} y &= \lfloor y_1 \rfloor + 1, \quad \text{when } \lfloor y_1 \rfloor + d_1 \text{ is odd;} \\ &= \lfloor y_1 \rfloor, \quad \text{when } \lfloor y_1 \rfloor + d_1 \text{ is even.} \end{aligned}$$

Take $r_2 = d_1 + y$ and $r_1 = (n + 1) - r_2$.

By choosing r_1, r_2 copies of K_n in $K_n^\star$, we shall have $e_f(1) - e_f(0) = r_1 d_1 + r_2 d_{-1}$, if its absolute value is less than or equal to $n + 1$, we can maintain $|e_f(1) - e_f(0)| \leq 1$ in $K_n^\star$, when n is even as shown in Table 1.

Table 1. Shows for even n to produce d_1 and d_{-1} in K_n and to compute r_1, r_2 in $K_n^\star$

t	r	n	d_1	d_{-1}	y_1	y	r_2	r_1	$r_1d_1+r_2d_2$
1	1	6	3	-5	-0.4	-1	2	5	5
1	2	8	4	-4	0.5	0	4	5	4
1	3	10	5	-3	1.9	1	6	5	7
1	4	12	6	-2	3.8	4	10	3	-2
1	5	14	7	-1	6.1	7	14	1	-7
2	1	18	9	-9	0.5	1	10	9	-9
2	2	20	10	-8	1.7	2	12	9	-6
2	3	22	11	-7	3.1	3	14	9	1
2	4	24	12	-6	4.7	4	16	9	12
2	5	26	13	-5	6.5	7	20	7	-9
2	6	28	14	-4	8.6	8	22	7	10
2	7	30	15	-3	10.8	11	26	5	-3
2	8	32	16	-2	13.3	14	30	3	-12
2	9	34	17	-1	16.1	17	34	1	-17
3	1	38	19	-13	4.2	5	24	15	-27
3	2	40	20	-12	5.6	6	26	15	-12
3	3	42	21	-11	7.2	7	28	15	7
3	4	44	22	-10	8.9	8	30	15	30
3	5	46	23	-9	10.8	11	34	13	-7
3	6	48	24	-8	12.8	12	36	13	24
3	7	50	25	-7	14.8	15	40	11	-5
3	8	52	26	-6	17.1	18	44	9	-30
3	9	54	27	-5	19.4	19	46	9	13
3	10	56	28	-4	21.9	22	50	7	-4
3	11	58	29	-3	24.5	25	54	5	-17
3	12	60	30	-2	27.2	28	58	3	-26
3	13	62	31	-1	30	31	62	1	-31
4	1	66	33	-17	11.2	11	44	23	11
4	2	68	34	-16	12.9	12	46	23	46
4	3	70	35	-15	14.7	15	50	21	-15
4	4	72	36	-14	16.6	16	52	21	28
4	5	74	37	-13	18.5	19	56	19	-25
4	6	76	38	-12	20.5	20	58	19	26
4	7	78	39	-11	22.6	23	62	17	19
4	8	80	40	-10	24.8	24	64	17	40
7	27	250	125	-3	120.1	121	246	5	-113
7	28	252	126	-2	123	124	250	3	-122
7	29	254	127	-1	126	127	254	1	-127
8	1	258	129	-33	77.2	77	206	53	39
8	2	260	130	-32	79.4	80	210	51	-90
8	25	306	153	-9	136.9	137	290	17	-9
8	33	322	161	-1	160	161	322	1	-161

Where $n = 4t^2 + 2r$, $d_1 = 2t^2 + r$, $d_{-1} = -(4t + 2) + r$, y taken as computation of the case, $r_2 = d_1 + y$ and $r_1 = (n + 1) - r_2$.

Case II n is odd

$\Rightarrow \exists\, t \in N$ such that $(2t - 1)^2 < n < (2t + 1)^2$.

$\Rightarrow 0 < n - 4t^2 + 4t - 1 < 8t$.

$\Rightarrow 2 \leq n - (2t - 1)^2 \leq 8t - 2$.

$\Rightarrow n = (2t - 1)^2 + 2r$, for some $r \in N$ $(1 \leq r \leq 4t - 1)$.

$\Rightarrow d_1 = \frac{n-1}{2} = 2t(t-1) + r$ and $d_{-1} = -4t + r$.

$\Rightarrow d_1 + |d_{-1}| = 2t(t+1)$.

If we take $Rcopy = n + 1 - (d_1 + |d_{-1}|)$ like Case I, we must have $Rcopy = 2t^2 - 6t + 2r + 2 = x$ (say).

Now here we have to make two parts say y and $x - y$, so that

$$\frac{y}{x-y} \approx \frac{d_1}{|d_{-1}|} \Rightarrow y \approx \frac{d_1(2t^2 - 6t + 2r + 2)}{2t^2 + 2t} = \frac{d_1 x}{d_1 - d_{-1}}.$$

Here $r_2 = d_1 + y$, we shall take even to maintain $v_f(0) = v_f(1)$ in $K_n^\star$. For this we shall take $y_2 = \frac{d_1 x}{d_1 - d_{-1}}$ and

$$\begin{aligned} y &= \lfloor y_2 \rfloor + 1, \quad \text{when } \lfloor y_2 \rfloor + d_1 \text{ is odd;} \\ &= \lfloor y_2 \rfloor, \quad \text{when } \lfloor y_2 \rfloor + d_1 \text{ is even.} \end{aligned}$$

By choosing $r_2 = d_1 + y, r_1 = (n+1) - r_2$ copies of K_n in $K_n^\star$, we shall have $e_f(1) - e_f(0) = r_1 d_1 + r_2 d_{-1}$. If its absolute value is less than or equal to $n + 1$, we can maintain $|e_f(1) - e_f(0)| \leq 1$ in $K_n^\star$, when n is odd as shown in Table 2.

Table 2. Shows for odd n to produce d_1 and d_{-1} in K_n and to compute r_1, r_2 in $K_n^\star$

t	r	n	d_1	d_{-1}	y_2	y	r_2	r_1	$r_1d_1+r_2d_2$
1	1	3	1	-3	0	1	2	2	-4
1	2	5	2	-2	1	2	4	2	-4
1	3	7	3	-1	3	3	6	2	0
2	1	11	5	-7	0	1	6	6	-12
2	2	13	6	-6	1	2	8	6	-12
2	3	15	7	-5	2.33	3	10	6	-8
2	4	17	8	-4	4	4	12	6	0
2	5	19	9	-3	6	7	16	4	-12
2	6	21	10	-2	8.33	8	18	4	4
2	7	23	11	-1	11	11	22	2	0
3	1	27	13	-11	2.17	3	16	12	-20
3	2	29	14	-10	3.5	4	18	12	-12
3	3	31	15	-9	5	5	20	12	0
3	4	33	16	-8	6.67	6	22	12	16
3	5	35	17	-7	8.5	9	26	10	-12
3	6	37	18	-6	10.5	10	28	10	12
3	7	39	19	-5	12.7	13	32	8	-8
3	8	41	20	-4	17	16	36	6	-24
3	9	43	21	-3	17.5	17	38	6	12
3	10	45	22	-2	20.17	20	42	4	4
3	11	47	23	-1	23	23	46	2	0
4	1	51	25	-15	7.5	7	32	20	20
4	2	53	26	-14	9.1	10	36	18	-36
4	3	55	27	-13	10.8	11	38	18	-8
4	4	57	28	-12	12.6	12	40	18	24
4	5	59	29	-11	14.5	15	44	16	-20
4	6	61	30	-10	23	16	46	16	20
5	17	115	57	-3	53.2	53	110	6	12
5	18	117	58	-2	56.1	56	114	4	4
5	19	119	59	-1	59	59	118	2	0
6	1	123	61	-23	29.1	29	90	34	4
6	2	125	62	-22	31	32	94	32	-84
6	14	149	74	-10	58.1	58	132	18	12
6	23	167	83	-1	83	83	166	2	0

Where $n = (2t-1)^2 + 2r$, $d_1 = \frac{n-1}{2}$, $d_{-1} = -4t + r$, y taken as computation of the case, $r_2 = d_1 + y$ and $r_1 = (n+1) - r_2$.

Tables 1 and 2 show that $r_1d_1 + r_2d_{-1}$ is too small when n becomes large. Also $|r_1d_1 + r_2d_{-1}| \leq n, \forall\, n \in N$. Thus $K_n^\star$ can be made a cordial graph according to Tables 1 and 2.

Illustrative example 2.7 $K_5^\star$ and cordial labeling is shown in Figure 2. According to Table 2, we have following data.

$$n = 5, d_1 = 2, d_{-1} = -2, y_2 = 1, y = 2, r_2 = 4, r_1 = 2 \text{ and } r_1d_1 + r_2d_{-1} = -4.$$

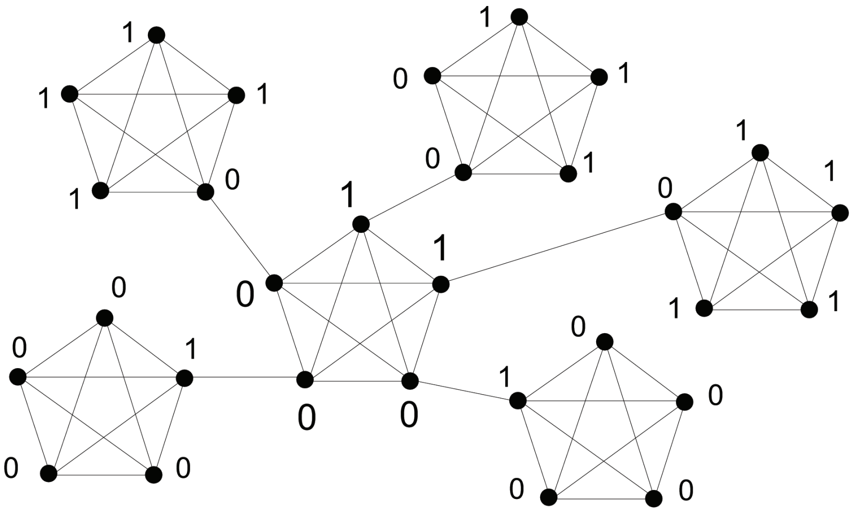

Figure 2. $K_5^\star$ and its cordial labeling ($v_f(1) = 15 = v_f(0), e_f(1) = 32, e_f(0) = 33$)

Let $u_{0,i}$ $(1 \leq i \leq 5)$ be vertices of the central copy K_5 of $K_5^\star$ and $u_{l,i}$ $(1 \leq i, l \leq 5)$ be vertices of other copies of $K_5^\star$. According to above data we shall define $f\colon V(K_5^\star) \longrightarrow \{0, 1\}$ as follows:

$$
\begin{aligned}
f(u_{0,i}) &= 1, \quad \text{when } i = 1, 2 \\
&= 0, \quad \text{when } i = 3, 4, 5; \\
f(u_{1,i}) &= 0, \quad \text{when } i = 1, 2 \\
&= 1, \quad \text{when } i = 3, 4, 5; \\
f(u_{l,i}) &= 0, \quad \text{when } i = 1 \text{ and } l = 2 \text{ or } l = 5 \\
&= 1, \quad \text{when } i = 2, 3, 4, 5 \text{ and } l = 2 \text{ or } l = 5; \\
f(u_{l,i}) &= 1, \quad \text{when } i = 1 \text{ and } l = 3 \text{ or } l = 4 \\
&= 0, \quad \text{when } i = 2, 3, 4, 5 \text{ and } l = 3 \text{ or } l = 4.
\end{aligned}
$$

To join each copies K_5 with the central copy in $K_5^\star$, we have to produce four more 1 edge labels. So we can join $u_{0,i}$ with $u_{i,1}$, $\forall\, i = 1, 2, 3, 4, 5$.

Above labeling function give rises to $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$, as $e_f(0) = 33, e_f(1) = 32, v_f(0) = 15, v_f(1) = 15$ in $K_5^\star$. Thus $K_5^\star$ is a cordial graph.

Illustrative example 2.8 For $K_{22}^\star$ and its cordial labeling, according to Table 1, we have following data.

$$n = 22, d_1 = 11, d_{-1} = -7, y_1 = 3.1, y = 3, r_2 = 14, r_1 = 9 \text{ and } r_1d_1 + r_2d_{-1} = 1.$$

Table 3. Shows binary vertex labeling for $K_{22}^\star$

Order of copy	vf(0)	vf(1)	ef(1)	ef(0)	ef(1)-ef(0)
Central copy	11x1=11	11x1=11	121	110	11
1 to 8	11x8=88	11x8=88	121x8	110x8	11x8=88
9 to 15	7x8=56	14x7=98	112x7	119x7	-7x7= - 49
16 to 22	14x7=98	7x8=56	112x7	119x7	-7x7= - 49
Other outer edges	0	0	11	11	0
Total	253	253	2668	2667	1

Let v_i $(1 \le i \le 22)$ be vertices of the central copy of $K^{\star}_{22}$ and $u_{i,j}$ $(1 \le i, j \le 22)$ be vertices of other copies of $K^{\star}_{22}$. We shall join v_i of the central copy with $u_{i,i}$ the vertex of i^{th} copy of $K^{\star}_{22}$, $\forall\ i = 1, 2, \ldots, 22$.

To define required labeling function $f\colon V(K^{\star}_{22}) \longrightarrow \{0, 1\}$, we use Table 3 and vertex labels which are given below:

$$
\begin{aligned}
f(v_i) &= 0, \quad \forall i = 1, 2, \ldots, 11; \\
f(v_j) &= 1, \quad \forall j = 12, 13, \ldots, 22; \\
f(u_{i,j}) &= 1, \quad \forall i = 1, 2, \ldots, 8, \qquad \forall j = 1, 2, \ldots, 11; \\
f(u_{i,j}) &= 0, \quad \forall i = 1, 2, \ldots, 8, \qquad \forall j = 12, 13, \ldots, 22; \\
f(u_{i,j}) &= 1, \quad \forall i = 9, 10, \ldots, 15, \qquad \forall j = 1, 2, \ldots, 14; \\
f(u_{i,j}) &= 0, \quad \forall i = 9, 10, \ldots, 15, \qquad \forall j = 15, 16, \ldots, 22; \\
f(u_{i,j}) &= 1, \quad \forall i = 16, 17, \ldots, 22, \quad \forall j = 1, 2, \ldots, 8; \\
f(u_{i,j}) &= 0, \quad \forall i = 16, 17, \ldots, 22, \quad \forall j = 9, 10, \ldots, 22.
\end{aligned}
$$

So above labeling pattern give rises to $|v_f(1) - v_f(0)| \le 1$ and $|e_f(1) - e_f(0)| \le 1$, as $e_f(0) = 2667, e_f(1) = 2668, v_f(0) = 253, v_f(1) = 253$ in $K^{\star}_{22}$. Thus $K^{\star}_{22}$ is a cordial graph.

Theorem 2.9 *$C(n \cdot K_n)$ is cordial, $\forall\ n \in N - \{1\}$.*

Proof. We know that $C(2 \cdot K_2) = C_4$, which is a cordial graph.

Case I n is even

$\Rightarrow \exists\ t \in N$ such that $(2t)^2 < n \le (2t + 2)^2$

$\Rightarrow n = 4t^2 + 2r$, for some r $(1 \le r \le 4t + 2)$ and $d_1 = 2t^2 + r$, $d_{-1} = -(4t + 2) + r$ with $d_1 + |d_{-1}| = 2(t + 1)^2$.

Since $C(n \cdot K_n)$ contain n copies of K_n, take r_1 copies of K_n, which produces d_1 and r_2 copies of K_n, which produces d_{-1}. In this case $C(n \cdot K_n)$ contains $e_f(1) - e_f(0) = r_1 d_1 + r_2 d_{-1}$ and we shall take r_2 even to preserve $|v_f(1) - v_f(0)| \le 1$.

First we shall choose d_1 copies of K_n, which produces d_{-1} and $|d_{-1}|$ copies of K_n, which produce d_1. Then (the remaining copy of K_n)

$$Rcopy = n - (d_1 - d_{-1}) = 2(t - 1)^2 + 2r - 4 = x \text{ (say)}$$

Here we have to make $x = Rcopy$ as two parts say y and $x - y$, so that

$$\frac{y}{x - y} \approx \frac{d_1}{|d_{-1}|} \Rightarrow y \approx \frac{(2t^2 + r)(2(t - 1)^2 + 2r - 4)}{2(t + 1)^2} = \frac{d_1 x}{d_1 - d_{-1}}.$$

Now $r_2 = d_1 + y$, which we take even to maintain $v_f(0) = v_f(1)$ in $C(n \cdot K_n)$. So we shall take $y_3 = \frac{d_1 x}{d_1 - d_{-1}}$ and

$$
\begin{aligned}
y &= \lfloor y_3 \rfloor, && \text{when } \lfloor y_3 \rfloor + d_1 \text{ is even} \\
&= \lfloor y_3 \rfloor + 1, && \text{when } \lfloor y_3 \rfloor + d_1 \text{ is odd.}
\end{aligned}
$$

By choosing $r_2 = d_1 + y$, $r_1 = n - r_2$ copies of K_n in $C(n \cdot K_n)$, we shall have $e_f(1) - e_f(0) = r_1 d_1 + r_2 d_{-1}$, if $|r_1 d_1 + r_2 d_{-1}| \le n$, we can maintain $e_f(1) = e_f(0)$ in $C(n \cdot K_n)$, when n is even, as shown in Table 4.

Case II n is odd

$\Rightarrow \exists\ t \in N$ such that $(2t - 1)^2 < n \le (2t + 1)^2$

$\Rightarrow n = (2t - 1)^2 + 2r$, for some r $(1 \le r \le 4t - 1)$ and $d_1 = 2t(t - 1) + r$, $d_{-1} = -4t + r$ with $d_1 + |d_{-1}| = 2t(t + 1)$.

If we take $Rcopy = n - (d_1 - d_{-1})$ like *Case*–I, we must have $Rcopy = 2t^2 - 6t + 2r + 1 = x$ (say).

Now this $x = Rcopy$ we have to make two parts say y and $x - y$, so that

$$\frac{y}{x - y} \approx \frac{d_1}{|d_{-1}|}. \Rightarrow y \approx \frac{d_1(2t^2 - 6t + 2r + 1)}{d_1 - d_{-1}}$$

We shall take $r_2 = d_1 + y$ even to preserve $|v_f(1) - v_f(0)| \le 1$ in $C(n \cdot K_n)$. For this we shall take $y_4 = \frac{d_1 x}{d_1 - d_{-1}}$ and

$$
\begin{aligned}
y &= \lfloor y_4 \rfloor + 1, && \text{when } \lfloor y_4 \rfloor + d_1 \text{ is odd} \\
&= \lfloor y_4 \rfloor, && \text{when } \lfloor y_4 \rfloor + d_1 \text{ is even.}
\end{aligned}
$$

We shall see that $n = t^2$, for some $t \in N$, $d_1 = 0$ and in this case we shall choose $r_2 = n$, $r_1 = 0$ an exceptional case due to n is odd and we can preserve $|v_f(1) - v_f(0)| = 1$.

By choosing $r_2 = d_1 + y$, $r_1 = n - r_2$ copies of K_n in $C(n \cdot K_n)$, we shall have $e_f(1) - e_f(0) = r_1 d_1 + r_2 d_{-1}$. If its absolute value is less than or equal to n, we can maintain $e_f(1) - e_f(0) \leq 1$ in $C(n \cdot K_n)$, when n is odd, as shown in Table 5.

Table 4. Shows for even n to produce d_1 and d_{-1} in K_n and to compute r_1, r_2 in $C(n \cdot K_n)$

t	r	n	d_1	d_{-1}	y_3	y	r_2
0	2	4	2	0	2	2	4
1	1	6	3	-5	-0.75	-1	2
1	2	8	4	-4	0	0	4
1	3	10	5	-3	1.25	1	6
1	4	12	6	-2	3	4	10
1	5	14	7	-1	5.25	5	12
1	6	16	8	0	8	8	16
2	1	18	9	-9	0	1	10
2	8	32	16	-2	12.44	12	28
2	9	34	17	-1	15.11	15	32
2	10	36	18	0	18	18	36
3	1	38	19	-13	3.56	3	22
3	2	40	20	-12	5	6	26
3	13	62	31	-1	29.06	29	60
3	14	64	32	0	32	32	64

Where $n = 4t^2 + 2r$, $d_1 = \frac{n}{2}$, $d_{-1} = -(4t + 2) + r$, y taken as computation of the case, $r_2 = d_1 + y$ and $r_1 = n - r_2$.

Table 5. Shows for odd n to produce d_1 and d_{-1} in K_n and to compute r_1, r_2 in $C(n \cdot K_n)$

t	r	n	d_1	d_{-1}	y_4	y	r_2	r_1
1	1	3	1	-3	-0.25	-1	0	3
1	2	5	2	-2	0.5	0	2	3
1	3	7	3	-1	2.25	3	6	1
1	4	9	4	0	5	5	9	0
2	1	11	5	-7	-0.42	-1	4	7
2	2	13	6	-6	0.5	0	6	7
2	3	15	7	-5	1.75	1	8	7
2	4	17	8	-4	3.33	4	12	5
2	5	19	9	-3	5.25	5	14	5
2	6	21	10	-2	7.5	8	18	3
2	7	23	11	-1	10.08	11	22	1
2	8	25	12	0	13	13	25	0
3	1	27	13	-11	1.63	1	14	13
3	2	29	14	-10	2.92	2	16	13
3	3	31	15	-9	4.38	5	20	11
3	4	33	16	-8	6	6	22	11
3	5	35	17	-7	7.79	7	24	11

Where $n = (2t - 1)^2 + 2r$, $d_1 = \frac{n-1}{2}$, $d_{-1} = -4t + r$, y taken as computation of the case, $r_2 = d_1 + y$ and $r_1 = n - r_2$.

Above Tables 4 and 5 shows that $r_1 d_1 + r_2 d_{-1}$ is too small, when n is becoming large. Thus $C(n \cdot K_n)$ can be made a cordial graph, according to Tables 4 and 5.

Illustrative example 2.10 For $C(12 \cdot K_{12})$ and its cordial labeling, according to Table 4, we have following data.

$$n = 12, d_1 = 6, d_{-1} = -2, y_3 = 3, y = 4, r_2 = 10, r_1 = 2 \text{ and } r_1d_1 + r_2d_{-1} = -8.$$

Let $u_{i,j}$ $(1 \leq i, j \leq 12)$ be vertices of $C(12 \cdot K_{12})$. We shall define require labeling $f\colon V(C(12 \cdot K_{12})) \longrightarrow \{0, 1\}$ by taking help of Table 6 as follows.

$$\begin{aligned}
f(u_{i,j}) &= 0, \quad \forall j = 1, 2, \ldots, 6, \qquad \forall i = 1, 2;\\
f(u_{i,j}) &= 1, \quad \forall j = 7, 8, \ldots, 12, \qquad \forall i = 1, 2;\\
f(u_{i,j}) &= 0, \quad \forall j = 1, 2, 3, 4,\\
&\qquad \forall i = 3, 5, 7, 9, 11;\\
f(u_{i,j}) &= 1, \quad \forall j = 5, 6, \ldots, 12,\\
&\qquad \forall i = 3, 5, 7, 9, 11;\\
f(u_{i,j}) &= 1, \quad \forall j = 1, 2, 3, 4,\\
&\qquad \forall i = 4, 6, 8, 10, 12;\\
f(u_{i,j}) &= 0, \quad \forall j = 5, 6, \ldots, 12,\\
&\qquad \forall i = 4, 6, 8, 10, 12.
\end{aligned}$$

Also we shall join $u_{i,2}$ with $u_{i+1,1}$, $\forall\, i = 1, 2, \ldots, 11$ and $u_{12,2}$ with $u_{1,1}$ by an edge to form the cycle graph $C(12 \cdot K_{12})$. Above labeling pattern give rises to $|v_f(1) - v_f(0)| = 0$, $|e_f(1) - e_f(0)| = 0$ for $C(12 \cdot K_{12})$, as shown in Table 6 and Figure 3 and so $C(12 \cdot K_{12})$ is a cordial graph.

Table 6. Shows binary vertex labeling for $C(12 \cdot K_{12})$

Order of copy	vf(0)	vf(1)	ef(1)	ef(0)	ef(1)-ef(0)
1 and 2	6 x 2	6 x 2	36 x 2	30 x 2	6 x 8 = 12
3,5,7,9,11	4 x 5	8 x 5	32 x 5	34 x 5	-2 x 5 = -10
4,6,8,10,12	8 x 5	4 x 5	32 x 5	34 x 5	-2 x 5 = -10
Other outer edges	0	0	10	2	8
Total	72	72	402	402	0

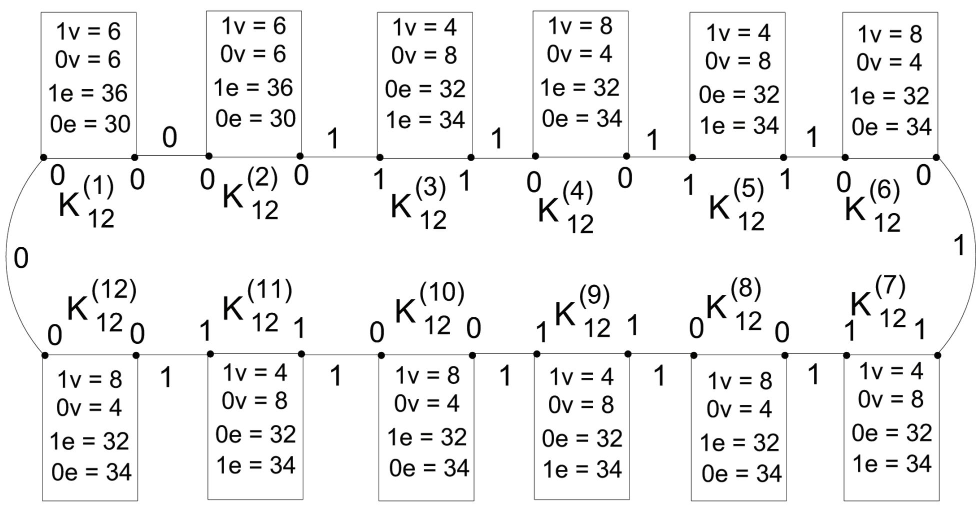

Figure 3. $C(12 \cdot K_{12})$ and its cordial labeling ($v_f(1) = 72 = v_f(0)$, $e_f(1) = 402 = e_f(0)$)

3. Concluding Remarks

In the present work cordial labeling for $K_n^\star$ and $C(n \cdot K_n)$ are discussed. This work rule out the impression of cordial labeling being a weak labeling. The labeling pattern is demonstrated by means of illustrations, which provide better understanding of derived results. The combination of Number Theory and Graph Labeling is a real beauty of this investigations.

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