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A MIXED FINITE ELEMENT METHOD ON A STAGGERED MESH FOR NAVIER-STOKES EQUATIONS*

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Abstract

In this paper, we introduce a mixed finite element method on a staggered mesh for the numerical solution of the steady state Navier-Stokes equations in which the two components of the velocity and the pressure are defined on three different meshes. This method is a conforming quadrilateral $Q_1 \times Q_1 - P_0$ element approximation for the Navier-Stokes equations. First-order error estimates are obtained for both the velocity and the pressure. Numerical examples are presented to illustrate the effectiveness of the proposed method.

Mathematics subject classification: 35Q30, 74G15, 74S05.
Key words: Mixed finite element method, Staggered mesh, Navier-Stokes equations, Error estimate.

1. Introduction

It is well known that the simplest conforming low-order elements like the $P_1 - P_0$ (linear velocity vector, constant pressure) triangular element and $Q_1 - P_0$ (bilinear velocity vector, constant pressure) quadrilateral element are not stable when applied to the Navier-Stokes (NS) equations [6]. Therefore, some special treatments are needed in order to keep the schemes stable. During the last two decades, there has been a rapid development in practical stabilization technique for the $P_1 - P_0$ element and the $Q_1 - P_0$ element for solving the NS equations [1, 7, 8, 9, 11]. In [3], an economical finite element scheme is proposed to construct three finite-dimensional subspaces for the two velocity components and the pressure. In [2], a mixed finite element scheme for the Stokes equations is investigated. In this paper, we extend the idea in [3] to construct a mixed finite element scheme for the NS equations, which is more efficient than the scheme given in [3] as the degree of freedom is reduced. The optimal error estimate of this scheme is obtained.

The outline of the paper is as follows. In the next section, we give a formulation of the mixed finite element method for the Navier-Stokes equations. In Section 3, the error estimates will be provided. In Section 4, two numerical examples will be considered. Finally, we end the paper with a short concluding section.

2. A Mixed Finite Element Formulation for the NS Equations

We consider the following boundary value problem of the Navier-Stokes equations:

$$\begin{cases} -\nu\Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, & \text{in } \Omega, \\ \text{div } \mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u} = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

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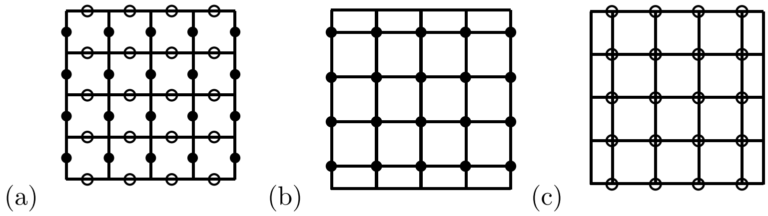

Fig. 2.1. Quadrangulations: (a) $\mathcal{J}_h$, (b) $\mathcal{J}_h^1$, (c) $\mathcal{J}_h^2$.

where $\Omega \subset \mathrm{R}^2$ is a rectangular domain, ν is the viscosity, $\mathbf{u} = (u_1, u_2)^T$ represents the velocity vector, p is the pressure, and $\mathbf{f} = (f_1, f_2)^T$ is the given body force. Let $H^n(\Omega)$ and $H_0^1(\Omega)$ denote the standard Sobolev spaces with the norm $\|\cdot\|_{n,\Omega}$ and $\|\cdot\|_{1,\Omega}$ respectively. Furthermore, let

$$\mathbf{V} \equiv H_0^1(\Omega) \times H_0^1(\Omega), \quad M \equiv \left\{ q : q \in L^2(\Omega) \text{ and } \int_\Omega q dx = 0 \right\}.$$

Then the boundary value problem (2.1) is reduced to the following equivalent variational problem [3]:

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{V} \text{ and } p \in M, \text{ such that} \\ a(\mathbf{u}, \mathbf{v}) + a_1(\mathbf{u}; \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}) & \forall \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u}, q) = 0 & \forall q \in M, \end{cases} \tag{2.2}$$

where

$$a(\mathbf{u}, \mathbf{v}) = \nu \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v} dx,$$

$$a_1(\mathbf{w}; \mathbf{u}, \mathbf{v}) = \frac{1}{2} \sum_{i,j=1}^{2} \int_\Omega w_j \Big(\frac{\partial u_i}{\partial x_j} v_i - \frac{\partial v_i}{\partial x_j} u_i \Big) dx,$$

$$b(\mathbf{v}, q) = -\int_\Omega q \mathrm{div}\ \mathbf{v} dx, \quad (\mathbf{f}, \mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} dx.$$

For simplicity we assume that the domain Ω is a unit square, but the finite element method discussed below can be easily generalized to include the case that the domain Ω is rectangular. Let N be a given integer and $h = 1/N$. We shall construct the finite-dimensional subspaces of $\mathbf{V}$ and M by introducing three different quadrangulations $\mathcal{J}_h$, $\mathcal{J}_h^1$, $\mathcal{J}_h^2$ of Ω. First we divide Ω into equal squares

$$T_{i,j} = \Big\{ (x_1, x_2) : (x_1)_{i-1} \le x_1 \le (x_1)_i,\ (x_2)_{j-1} \le x_2 \le (x_2)_j \Big\}, \quad i, j = 1, \cdots, N,$$

where $(x_1)_i = ih$ and $(x_2)_j = jh$. The corresponding quadrangulation is denoted by $\mathcal{J}_h$. Then for all $T_{i,j} \in \mathcal{J}_h$ we connect all the midpoints of the vertical sides of $T_{i,j}$ by straight line segments if the midpoints have a distance h, and extend the resulting mesh to the boundary Γ. Then Ω is divided into squares and rectangles, and the corresponding quadrangulation is denoted by $\mathcal{J}_h^1$. Similarly, for all $T_{i,j} \in \mathcal{J}_h$ we connect all the midpoints of the horizontal sides of $T_{i,j}$ by straight line segments if the midpoints have a distance h, and extend the resulting mesh to the boundary Γ. Then we obtained the third quadrangulation of Ω, which is denoted by $\mathcal{J}_h^2$ (see Fig. 2.1).

Corresponding to the quadrangulation $\mathcal{J}_h$, let

$$M_h := \left\{ q_h : q_h|_T = constant \quad \forall T \in \mathcal{J}_h \quad \text{and} \int_\Omega q_h dx = 0 \right\},$$

M_h is a subspace of M. Furthermore, using the quadrangulation $\mathcal{J}_h^1$ and $\mathcal{J}_h^2$, we construct two subspaces of $H_0^1(\Omega)$. Set

$$S_h^1 = \Big\{ v_h \in C^{(0)}(\overline{\Omega}) : v_h|_{T^1} \in Q_1(T^1) \quad \forall T^1 \in \mathcal{J}_h^1, \text{ and } v_h|_\Gamma = 0 \Big\},$$
$$S_h^2 = \Big\{ v_h \in C^{(0)}(\overline{\Omega}) : v_h|_{T^2} \in Q_1(T^2) \quad \forall T^2 \in \mathcal{J}_h^2, \text{ and } v_h|_\Gamma = 0 \Big\},$$

where Q_1 denotes the space of all polynomials of degree not exceeding one with respect to each of the two variables x_1 and x_2. Let

$$\mathbf{V}_h = S_h^1 \times S_h^2;$$

obviously, $\mathbf{V}_h$ is a subspace of $\mathbf{V}$. Using the subspaces $\mathbf{V}_h$ and M_h instead of $\mathbf{V}$ and M in the variational problem (2.2), we obtain the discrete problem which is the finite element approximation of the nonlinear variational problem (2.2):

$$\begin{cases} \text{Find } \mathbf{u}_h \in \mathbf{V}_h \text{ and } p_h \in M_h, \text{ such that} \\ a(\mathbf{u}_h, \mathbf{v}_h) + a_1(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h) & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b(\mathbf{u}_h, q_h) = 0 & \forall q_h \in M_h. \end{cases} \tag{2.3}$$

3. Well-Posedness of Problem (2.3) and Optimal Error Estimate

In order to attain the error estimate of the finite element approximation to the nonlinear variational problem (2.2), we first introduce some notations. Let

$$N = \sup_{\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{V}} \frac{|a_1(\mathbf{w}; \mathbf{u}, \mathbf{v})|}{|\mathbf{u}|_{1,\Omega} |\mathbf{v}|_{1,\Omega} |\mathbf{w}|_{1,\Omega}},$$
$$N_h = \sup_{\mathbf{u}_h,\mathbf{v}_h,\mathbf{w}_h \in \mathbf{V}_h} \frac{|a_1(\mathbf{w}_h; \mathbf{u}_h, \mathbf{v}_h)|}{|\mathbf{u_h}|_{1,\Omega} |\mathbf{v}_h|_{1,\Omega} |\mathbf{w}_h|_{1,\Omega}},$$
$$\|\mathbf{f}\|^* = \sup_{\mathbf{v} \in \mathbf{V}} \frac{(\mathbf{f}, \mathbf{v})}{|\mathbf{v}|_{1,\Omega}}, \quad \|\mathbf{f}\|_h^* = \sup_{\mathbf{v}_h \in \mathbf{V_h}} \frac{(\mathbf{f}, \mathbf{v}_h)}{|\mathbf{v}_h|_{1,\Omega}},$$
$$G(\mathbf{u}; \mathbf{v}, \mathbf{w}) = a_1(\mathbf{u}; \mathbf{u}, \mathbf{w}) - a_1(\mathbf{v}; \mathbf{v}, \mathbf{w}),$$

where $|\cdot|_{1,\Omega}$ is the semi-norm of $H^1(\Omega)$. Obviously, the following inequalities hold

$$N_h \le N, \quad \|\mathbf{f}\|_h^* \le \|\mathbf{f}\|^*. \tag{3.1}$$

For the quadrangulation $\mathcal{J}_h$, we divided the edges of all squares into two sets. The first set contains all vertical edges and is denoted by L_V. The second set contains all horizontal edges and is denoted by L_H. We define the operator $\Pi : \mathbf{V} \to \mathbf{V}_h$ by $\Pi\mathbf{u} = (\Pi_h^1 u_1, \Pi_h^2 u_2)^T \in S_h^1 \times S_h^2$ satisfying:

$$\int_l \Pi_h^1 u_1 ds = \int_l u_1 ds \ \ \forall l \in L_V, \quad \int_l \Pi_h^2 u_2 ds = \int_l u_2 ds \ \ \forall l \in L_H.$$

For the problem (2.3), we have the following result. Its proof is omitted as it is basically the same as in continuous problem (see [4] Chapter IV).

Theorem 3.1. *Problem* (2.3) *has at least one solution* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times M_h$. *Moreover, the solution is unique if the following condition holds*

$$N_h\|\mathbf{f}\|_h^*/\nu^2 < 1.$$

Theorem 3.2. *Under the assumption that*

$$N\|\mathbf{f}\|^*/\nu^2 < 1-\delta, \tag{3.2}$$

where $0<\delta<1$ *is a constant, the finite element solution of problem* (2.3) $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times M_h$ *satisfies:*

$$\begin{cases} |\mathbf{u}-\mathbf{u}_h|_{1,\Omega} \le C\left\{|\mathbf{u}-\Pi\mathbf{u}|_{1,\Omega} + \inf\limits_{q_h\in M_h}\|p-q_h\|_{0,\Omega} + \sup\limits_{w_h\in\mathbf{V}_h}\dfrac{|G(\mathbf{u};\Pi\mathbf{u},\mathbf{w}_h)|}{|\mathbf{w}_h|_{1,\Omega}}\right\}, \\ \|p-p_h\|_{0,\Omega} \le C\left\{|\mathbf{u}-\Pi\mathbf{u}|_{1,\Omega} + \inf\limits_{q_h\in M_h}\|p-q_h\|_{0,\Omega}\right. \\ \qquad\qquad \left. + \sup\limits_{w_h\in\mathbf{V}_h}\dfrac{|G(\mathbf{u};\Pi\mathbf{u},\mathbf{w}_h)|}{|\mathbf{w}_h|_{1,\Omega}} + \sup\limits_{\mathbf{w}_h\in\mathbf{V}_h}\dfrac{|G(\mathbf{u};\mathbf{u}_h,\mathbf{w}_h)|}{|\mathbf{w}_h|_{1,\Omega}}\right\}. \end{cases} \tag{3.3}$$

Before the proof we recall some results from [2], which only give the error estimate for the Stokes equations, using the same finite element formulations as in this paper.

Lemma 3.1. (i) *For any* $\mathbf{u}\in\mathbf{V}$, *we have*

$$\int_\Omega q_h \mathrm{div}\ (\mathbf{u}-\Pi\mathbf{u})dx = 0 \qquad \forall q_h\in M_h. \tag{3.4}$$

(ii) *There exist two constants* C_1 *and* C_2 *independent of* h, *such that*

$$\begin{aligned} &\inf_{\mathbf{v}_h\in\mathbf{V}_h}\|\mathbf{u}-\mathbf{v}_h\|_{1,\Omega} \le \|\mathbf{u}-\Pi\mathbf{u}\|_{1,\Omega} \le C_1 h|\mathbf{u}|_{2,\Omega}, \\ &\inf_{q_h\in M_h}\|p-q_h\|_{0,\Omega} \le C_2 h|p|_{1,\Omega}. \end{aligned} \tag{3.5}$$

(iii) *There is a constant* $B>0$ *such that*

$$|b(\mathbf{v},q)| \le B\|\mathbf{v}\|_{1,\Omega}\|q\|_{0,\Omega}. \tag{3.6}$$

(iv) (*Ladyzhenskaya-Babuška-Brezzi condition*) *There exists a constant* $\beta_0>0$ *independent of* h *such that*

$$\sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{b(\mathbf{v}_h,q_h)}{\|\mathbf{v}_h\|_{1,\Omega}} \ge \beta_0\|q_h\|_{0,\Omega} \qquad \forall q_h\in M_h. \tag{3.7}$$

(v) *There exists a constant* C_3 *independent of* h *such that*

$$\|\Pi\mathbf{u}\|_{1,\Omega} \le C_3\|\mathbf{u}\|_{1,\Omega} \qquad \forall\mathbf{u}\in\mathbf{V}.$$

Proof of Theorem 3.2. By inequality (3.2), we know that

$$N_h\|\mathbf{f}\|_h^*/\nu^2 \le N\|\mathbf{f}\|^*/\nu^2 \le 1-\delta. \tag{3.8}$$

It follows from Theorem 3.1 that problem (2.3) has a unique solution $(\mathbf{u}_h, p_h)\in\mathbf{V}_h\times M_h$. Setting $\mathbf{v}_h=\mathbf{u}_h$ in the first equation in (2.3), and observing that $a_1(\mathbf{u}_h;\mathbf{u}_h,\mathbf{u}_h)=0$ and $b(\mathbf{u}_h,p_h)=0$, we obtain

$$(\mathbf{f},\mathbf{u}_h) = a(\mathbf{u}_h,\mathbf{u}_h) + a_1(\mathbf{u}_h;\mathbf{u}_h,\mathbf{u}_h) + b(\mathbf{u}_h,p_h) = a(\mathbf{u}_h,\mathbf{u}_h).$$

Consequently,

$$\nu|\mathbf{u}_h|^2_{1,\Omega} = a(\mathbf{u}_h,\mathbf{u}_h) = (\mathbf{f},\mathbf{u}_h) \le |\mathbf{f}|^*_h|\mathbf{u}_h|_{1,\Omega},$$

which yields

$$|\mathbf{u}_h|_{1,\Omega} \le \frac{\|\mathbf{f}\|^*_h}{\nu}. \tag{3.9}$$

Let $\mathbf{z}_h = \mathbf{u}_h - \Pi\mathbf{u}$ and

$$\Phi = a(\mathbf{u}_h,\mathbf{z}_h) + a_1(\mathbf{u}_h;\mathbf{u}_h,\mathbf{z}_h) - a(\Pi\mathbf{u},\mathbf{z}_h) - a_1(\Pi\mathbf{u};\Pi\mathbf{u},\mathbf{z}_h).$$

Then we have

$$\begin{aligned}\Phi &= a(\mathbf{u}_h,\mathbf{z}_h) + a_1(\mathbf{z}_h;\mathbf{u}_h,\mathbf{z}_h) + a_1(\Pi\mathbf{u};\mathbf{u}_h,\mathbf{z}_h) - a(\Pi\mathbf{u},\mathbf{z}_h) - a_1(\Pi\mathbf{u};\Pi\mathbf{u},\mathbf{z}_h)\\ &= a(\mathbf{z}_h,\mathbf{z}_h) + a_1(\mathbf{z}_h;\mathbf{u}_h,\mathbf{z}_h) + a_1(\Pi\mathbf{u};\mathbf{z}_h,\mathbf{z}_h)\\ &= a(\mathbf{z}_h,\mathbf{z}_h) + a_1(\mathbf{z}_h;\mathbf{u}_h,\mathbf{z}_h) \ge \nu|\mathbf{z}_h|^2_{1,\Omega} - N_h|\mathbf{u}_h|_{1,\Omega}|\mathbf{z}_h|^2_{1,\Omega}\\ &\ge \Big(\nu - \frac{N_h\|\mathbf{f}\|^*_h}{\nu}\Big)|\mathbf{z}_h|^2_{1,\Omega} \ge \Big(\nu - \frac{N\|\mathbf{f}\|^*}{\nu}\Big)|\mathbf{z}_h|^2_{1,\Omega} \ge \nu\delta|\mathbf{z}_h|^2_{1,\Omega},\end{aligned}$$

which gives that

$$|\mathbf{z}_h|^2_{1,\Omega} \le \frac{1}{\nu\delta}|\Phi|. \tag{3.10}$$

On the other hand, we know that

$$\begin{aligned}\Phi &= (\mathbf{f},\mathbf{z}_h) - b(\mathbf{z}_h,p_h) - a(\Pi\mathbf{u},\mathbf{z}_h) - a_1(\Pi\mathbf{u};\Pi\mathbf{u},\mathbf{z}_h)\\ &= a(\mathbf{u},\mathbf{z}_h) + a_1(\mathbf{u};\mathbf{u},\mathbf{z}_h) + b(\mathbf{z}_h,p) - b(\mathbf{z}_h,p_h) - a(\Pi\mathbf{u},\mathbf{z}_h) - a_1(\Pi\mathbf{u};\Pi\mathbf{u},\mathbf{z}_h)\\ &= a(\mathbf{u}-\Pi\mathbf{u},\mathbf{z}_h) + b(\mathbf{z}_h,p-p_h) + G(\mathbf{u};\Pi\mathbf{u},\mathbf{z}_h)\\ &= a(\mathbf{u}-\Pi\mathbf{u},\mathbf{z}_h) + b(\mathbf{z}_h,p-q_h) + b(\mathbf{u}_h-\Pi\mathbf{u},q_h-p_h) + G(\mathbf{u};\Pi\mathbf{u},\mathbf{z}_h)\\ &= a(\mathbf{u}-\Pi\mathbf{u},\mathbf{z}_h) + b(\mathbf{z}_h,p-q_h) + b(\mathbf{u}_h-\mathbf{u},q_h-p_h) + G(\mathbf{u};\Pi\mathbf{u},\mathbf{z}_h)\\ &= a(\mathbf{u}-\Pi\mathbf{u},\mathbf{z}_h) + b(\mathbf{z}_h,p-q_h) + G(\mathbf{u};\Pi\mathbf{u},\mathbf{z}_h) \qquad \forall q_h\in M_h.\end{aligned}$$

Furthermore, we obtain, for any $q_h \in M_h$,

$$|\Phi| \le \left(\nu|\mathbf{u}-\Pi\mathbf{u}|_{1,\Omega} + B\|p-q_h\|_{0,\Omega} + \sup_{w_h\in\mathbf{V}_h}\frac{|G(\mathbf{u};\Pi\mathbf{u},\mathbf{w}_h)|}{|\mathbf{w}_h|_{1,\Omega}}\right)|\mathbf{z}_h|_{1,\Omega}. \tag{3.11}$$

Combining (3.10) and (3.11), we have

$$|\mathbf{z}_h|_{1,\Omega} \le \frac{1}{\nu\delta}\left(\nu|\mathbf{u}-\Pi\mathbf{u}|_{1,\Omega} + B\inf_{q_h\in M_h}\|p-q_h\|_{0,\Omega} + \sup_{w_h\in\mathbf{V}_h}\frac{|G(\mathbf{u};\Pi\mathbf{u},\mathbf{w}_h)|}{|\mathbf{w}_h|_{1,\Omega}}\right).$$

Using the triangle inequality yields

$$\begin{aligned}|\mathbf{u}-\mathbf{u}_h|_{1,\Omega} &\le |\mathbf{u}-\Pi\mathbf{u}|_{1,\Omega} + |\Pi\mathbf{u}-\mathbf{u}_h|_{1,\Omega}\\ &\le C\left\{|\mathbf{u}-\Pi\mathbf{u}|_{1,\Omega} + \inf_{q_h\in M_h}\|p-q_h\|_{0,\Omega} + \sup_{w_h\in\mathbf{V}_h}\frac{|G(\mathbf{u};\Pi\mathbf{u},\mathbf{w}_h)|}{|\mathbf{w}_h|_{1,\Omega}}\right\},\end{aligned} \tag{3.12}$$

where C is a constant dependent only on B, δ and ν. To estimate error $\|p-p_h\|_{0,\Omega}$, consider

$$\begin{aligned} b(\mathbf{v}_h, p_h - q_h) &= b(\mathbf{v}_h, p_h - p) + b(\mathbf{v}_h, p - q_h) \\ &= a(\mathbf{u}-\mathbf{u}_h, \mathbf{v}_h) + G(\mathbf{u};\mathbf{u}_h,\mathbf{v}_h) + b(\mathbf{v}_h, p-q_h), \end{aligned}$$

for any $\mathbf{v}_h \in \mathbf{V}_h$ and $q_h \in M_h$. By (iv) of Lemma 3.1, we obtain, for any $q_h \in M_h$,

$$\begin{aligned} \|p_h - q_h\|_{0,\Omega} &\le \frac{1}{\beta_0} \sup_{\mathbf{v}_h\in\mathbf{V}_h} \frac{|b(\mathbf{v}_h, p_h-q_h)|}{|\mathbf{v}_h|_{1,\Omega}} \\ &\le \frac{1}{\beta_0}\left\{ \nu|\mathbf{u}-\mathbf{u}_h|_{1,\Omega} + B\|p-q_h\|_{0,\Omega} + \sup_{v_h\in\mathbf{V}_h}\frac{|G(\mathbf{u};\mathbf{u}_h,\mathbf{v}_h)|}{|\mathbf{v}_h|_{1,\Omega}} \right\}. \end{aligned} \tag{3.13}$$

Combining inequalities (3.12), (3.13) and using the triangle inequality lead to the conclusion (3.3). □

Now we estimate $G(\mathbf{u};\Pi\mathbf{u},\mathbf{v}_h)$ and $G(\mathbf{u};\mathbf{u}_h,\mathbf{v}_h)$. We have

Lemma 3.2. *Suppose* $(\mathbf{u},p)$ *is the solution of problem* (2.2) *and* $\mathbf{u}\in\mathbf{V}\cap(H^2(\Omega))^2, p\in M\cap H^1(\Omega)$. *Then there exists a constant* C *independent of* h *and* $(\mathbf{u},p)$, *such that*

$$\sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{|G(\mathbf{u};\Pi\mathbf{u},\mathbf{v}_h)|}{|\mathbf{v}_h|_{1,\Omega}} \le Ch\|\mathbf{u}\|_{1,\Omega}|\mathbf{u}|_{2,\Omega}, \tag{3.14}$$

$$\sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{|G(\mathbf{u};\mathbf{u}_h,\mathbf{v}_h)|}{|\mathbf{v}_h|_{1,\Omega}} \le C|\mathbf{u}-\mathbf{u}_h|_{1,\Omega}\big\{\|\mathbf{f}\|^* + |\mathbf{u}|_{1,\Omega}\big\}. \tag{3.15}$$

Proof. Observe that

$$\begin{aligned} |G(\mathbf{u};\Pi\mathbf{u},\mathbf{v}_h)| &= |a_1(\mathbf{u};\mathbf{u},\mathbf{v}_h) - a_1(\Pi\mathbf{u};\Pi\mathbf{u},\mathbf{v}_h)| \\ &= |a_1(\mathbf{u}-\Pi\mathbf{u};\mathbf{u},\mathbf{v}_h) + a_1(\Pi\mathbf{u};\mathbf{u},\mathbf{v}_h) - a_1(\Pi\mathbf{u};\Pi\mathbf{u},\mathbf{v}_h)| \\ &= |a_1(\mathbf{u}-\Pi\mathbf{u};\mathbf{u},\mathbf{v}_h) + a_1(\Pi\mathbf{u};\mathbf{u}-\Pi\mathbf{u},\mathbf{v}_h)| \\ &\le N|\mathbf{u}-\Pi\mathbf{u}|_{1,\Omega}\big\{|\mathbf{u}|_{1,\Omega} + |\Pi\mathbf{u}|_{1,\Omega}\big\}|\mathbf{v}_h|_{1,\Omega} \\ &\le Ch|\mathbf{u}|_{2,\Omega}\|\mathbf{u}\|_{1,\Omega}|\mathbf{v}_h|_{1,\Omega}, \end{aligned}$$

where the last inequality used (ii) and (v) of Lemma 3.1. This ends the proof of (3.14). Similarly, we have

$$\begin{aligned} |G(\mathbf{u};\mathbf{u}_h,\mathbf{v}_h)| &= |a_1(\mathbf{u};\mathbf{u},\mathbf{v}_h) - a_1(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}_h)| \\ &= |a_1(\mathbf{u}-\mathbf{u}_h;\mathbf{u},\mathbf{v}_h) + a_1(\mathbf{u}_h;\mathbf{u},\mathbf{v}_h) - a_1(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}_h)| \\ &= |a_1(\mathbf{u}-\mathbf{u}_h;\mathbf{u},\mathbf{v}_h) + a_1(\mathbf{u}_h;\mathbf{u}-\mathbf{u}_h,\mathbf{v}_h)| \\ &\le N|\mathbf{u}-\mathbf{u}_h|_{1,\Omega}\big\{|\mathbf{u}|_{1,\Omega} + |\mathbf{u}_h|_{1,\Omega}\big\}|\mathbf{v}_h|_{1,\Omega} \\ &\le C|\mathbf{u}-\mathbf{u}_h|_{1,\Omega}\big\{|\mathbf{u}|_{1,\Omega} + \|\mathbf{f}\|^*\big\}|\mathbf{v}_h|_{1,\Omega}, \end{aligned}$$

where we used (3.9) in the last step. The inequality (3.15) is then proved. □

Finally, an application of Theorem 3.2 and Lemma 3.2 yields the following error estimates.

Theorem 3.3. *Suppose that the condition* (3.2) *holds and that the solution of problem* (2.2) *satisfies* $\mathbf{u}\in\mathbf{V}\cap(H^2(\Omega))^2, p\in M\cap H^1(\Omega)$. *Then the following error estimates hold:*

$$|\mathbf{u}-\mathbf{u}_h|_{1,\Omega} \le Ch\big\{|\mathbf{u}|_{2,\Omega} + |p|_{1,\Omega} + \|\mathbf{u}\|_{1,\Omega}|\mathbf{u}|_{2,\Omega}\big\}, \tag{3.16}$$

$$\|p-p_h\|_{0,\Omega} \le Ch\big\{|\mathbf{u}|_{2,\Omega} + |p|_{1,\Omega} + \|\mathbf{u}\|_{1,\Omega}|\mathbf{u}|_{2,\Omega}\big\}\big(1 + |\mathbf{u}|_{1,\Omega} + \|\mathbf{f}\|^*\big). \tag{3.17}$$

Proof. The inequality (3.16) follows from (ii) of Lemma 3.1, Theorem 3.2 and Lemma 3.2. For the inequality (3.17), we have

$$
\begin{aligned}
&\|p-p_h\|_{0,\Omega}\\
\le& C\left\{\|\mathbf{u}-\Pi\mathbf{u}\|_{1,\Omega}+\inf_{q_h\in M_h}\|p-q_h\|_{0,\Omega}+\sup_{w_h\in\mathbf{V}_h}\frac{|G(\mathbf{u};\Pi\mathbf{u},\mathbf{w}_h)|}{\|\mathbf{w}_h\|_{1,\Omega}}\sup_{\mathbf{w}_h\in\mathbf{V}_h}\frac{|G(\mathbf{u};\mathbf{u}_h,\mathbf{w}_h)|}{\|\mathbf{w}_h\|_{1,\Omega}}\right\}\\
\le& C\Big\{h\big(|\mathbf{u}|_{2,\Omega}+|p|_{1,\Omega}+\|\mathbf{u}\|_{1,\Omega}|\mathbf{u}|_{2,\Omega}\big)|\mathbf{u}-\mathbf{u}_h|_{1,\Omega}\big(\|\mathbf{f}\|^*+|\mathbf{u}|_{1,\Omega}\big)\Big\}\\
\le& Ch\Big(|\mathbf{u}|_{2,\Omega}+|p|_{1,\Omega}+\|\mathbf{u}\|_{1,\Omega}|\mathbf{u}|_{2,\Omega}\Big)\big(1+|\mathbf{u}|_{1,\Omega}+\|\mathbf{f}\|^*\big).
\end{aligned}
$$

This completes the proof of the theorem. □

4. Numerical Examples

In this section, we present two numerical examples to show the performance of the FEM described above: one has a known analytical solution that allows a study of the convergence rate and another one is a benchmark problem.

Example 4.1. Let $\Omega=(0,1)\times(0,1)$ and L be the number of partitions for the interval $(0,1)$, i.e., $h=1/L$. We choose the velocity vector $\mathbf{u}(x)=(u_1(x),u_2(x))^T$ and the pressure p as follows:

$$
\begin{cases}
u_1(x_1,x_2)=\sin^2\pi x_1\sin\pi x_2\cos\pi x_2,\\
u_2(x_1,x_2)=-\sin^2\pi x_2\sin\pi x_1\cos\pi x_1,\\
p(x_1,x_2)=-\cos 2\pi x_1\cos 2\pi x_2/16.
\end{cases}
\tag{4.1}
$$

It is straightforward to check that $(\mathbf{u},p)$ is the exact solution of problem (2.1) with a body force $\mathbf{f}=(f_1,f_2)^T$, where

$$
\begin{aligned}
f_1(x_1,x_2)=&\frac{1}{8}\pi\cos 2\pi x_2\sin 2\pi x_1-2\pi^2\cos^2\pi x_1\cos\pi x_2\sin\pi x_2\\
&+6\pi^2\cos\pi x_2\sin^2\pi x_1\sin\pi x_2+\pi\cos\pi x_1\cos^2\pi x_2\sin^3\pi x_1\sin^2\pi x_2\\
&+\pi\cos\pi x_1\sin^3\pi x_1\sin^4\pi x_2,
\end{aligned}
\tag{4.2}
$$

$$
\begin{aligned}
f_2(x_1,x_2)=&\frac{1}{8}\pi\cos 2\pi x_1\sin 2\pi x_2+2\pi^2\cos\pi x_1\cos^2\pi x_2\sin\pi x_1\\
&-6\pi^2\cos\pi x_1\sin\pi x_1\sin^2\pi x_2+\pi\cos^2\pi x_1\cos\pi x_2\sin^2\pi x_1\sin^3\pi x_2\\
&+\pi\cos\pi x_2\sin^4\pi x_1\sin^3\pi x_2.
\end{aligned}
\tag{4.3}
$$

Table 4.1: Numerical error and rate for Example 4.1.

Meshes	H^1 error for $\mathbf{u}$	rate	H^0 error for p	rate
$N=4$	5.39×10^{-1}		1.47×10^{-1}	
$N=8$	3.13×10^{-1}	0.78	5.47×10^{-2}	1.42
$N=16$	1.64×10^{-1}	0.93	1.65×10^{-2}	1.73
$N=32$	8.46×10^{-2}	0.95	4.57×10^{-3}	1.85

Fig. 4.1 shows the pressure and velocity field given by (4.1). Table 4.1 gives the numerical errors and convergence rates obtained on successively refined meshes. These results agree with the optimal theoretical convergence rates except that the convergence rate for p is great than 1.

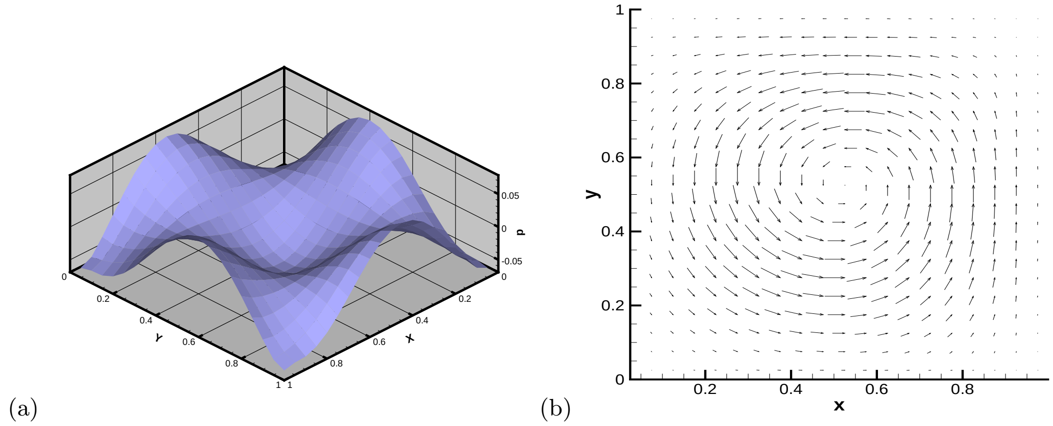

Fig. 4.1. The solution (4.1). (a) the pressure field and (b) the velocity field.

Example 4.2. The second example is the driven cavity problem, in which the incompressible fluid is enclosed in a square box, with an imposed velocity of unity in the horizontal direction on the top boundary, and a no slip condition on the remaining walls.

This problem has been widely used for validating incompressible fluid dynamic algorithms, in spite of the singularities at two of its corners. We will compare our results to those obtained by Ghia et al. [5] and Kaya and Riviere [10].

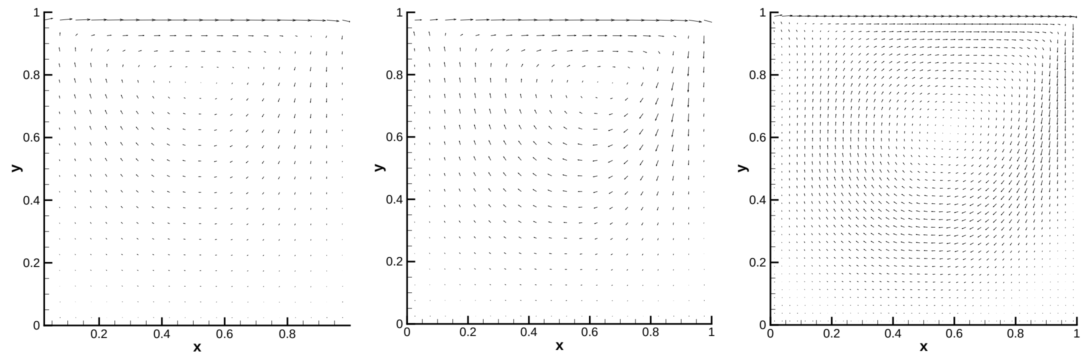

Fig. 4.2. Velocity field for Example 4.2. From left to right: $Re = 1$, 100 and 400.

We consider the flow for different Reynolds numbers on a fixed mesh with $h = 1/20$ or $h = 1/40$. For low Reynolds numbers ($Re = 1$) the flow has only one vortex located above the center. When the Reynolds number increases to $Re = 100$, the flow pattern starts to form reverse circulation cells in two lower corners. It is found that the results for $Re = 1$, 100 and 400 are in good agreement with the solutions presented in [5] and [10].

5. Conclusion

In this paper, we proposed a mixed finite element method on a staggered mesh for the numerical solution of the Navier-Stokes equations. The method is a conforming quadrilateral $Q_1 \times Q_1 - P_0$ element approximation on three different meshes. The optimal error estimates of the numerical solution are given, and two numerical examples are used to show the effectiveness and feasibility of the given method.

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