THE HOCHSCHILD COHOMOLOGY RING OF THE STANDARD PODLEŚ QUANTUM SPHERE

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ABSTRACT. The cup and cap product in twisted Hochschild (co)homology is computed for the standard quantum 2-sphere and used to construct a cyclic 2-cocycle that represents the fundamental Hochschild class.

1. INTRODUCTION

The aim of this article is to compute for the quantised coordinate ring $A = \mathbb{C}_q[S^2]$ (the standard Podleś quantum 2-sphere) the cup and cap product

$$\begin{aligned} \smile &: H^m(A, {}_\sigma A) \otimes H^n(A, {}_\tau A) \to H^{m+n}(A, {}_{\tau\circ\sigma} A), \\ \frown &: H_n(A, {}_\sigma A) \otimes H^m(A, {}_\tau A) \to H_{n-m}(A, {}_{\tau\circ\sigma} A) \end{aligned}$$

in the Hochschild (co)homology of A with coefficients in the bimodules ${}_\sigma A$ that arise by twisting the canonical bimodule structure of A by $\sigma \in \mathrm{Aut}(A)$. In [8] we carried out similar computations for $\mathbb{C}_q[SU(2)]$. For $\mathbb{C}_q[S^2]$ they become much simpler and their conceptual content that was somewhat hidden in [8] between lengthy computations becomes more transparent.

For the coordinate ring of a smooth variety X, the Hochschild-Kostant-Rosenberg theorem identifies $\Lambda(A) := (\bigoplus_{n\geq 0} H^n(A, A), \smile)$ with the algebra of multivector fields on X, and $\bigoplus_{n\geq 0} H_n(A, A)$ as a $\Lambda(A)$-module (via $\frown$) with the differential forms $\Omega(X)$ on X. For noncommutative algebras, $\Lambda(A)$ tends to be fairly degenerate. However, twisting by σ allows one to consider richer cohomology rings that encode more information about A.

For example, the generators of the Drinfeld-Jimbo quantisation of the Lie algebra of $SU(2)$ act via twisted rather than usual derivations on $A = \mathbb{C}_q[S^2]$. These give rise to two cohomology classes $[\partial_{\pm 1}] \in H^1(A, {}_{\sigma_{\mathrm{mod}}^{-1}} A)$, where σ_{mod} is Woronowicz's modular automorphism determined for example by (2) below. We will see that they behave under the cup product similar to the corresponding classical $SU(2)$-invariant vector fields on $S^2 = SU(2)/U(1)$,

$$[\partial_1] \smile [\partial_1] = [\partial_1] \smile [\partial_{-1}] + q^2 [\partial_{-1}] \smile [\partial_1] = [\partial_{-1}] \smile [\partial_{-1}] = 0,$$

and use them to define a functional on $H_2(A, {}_{\sigma_{\mathrm{mod}}} A) \simeq \mathbb{C}$ of the form

$$\varphi([\omega]) := q^{-1} \int [\omega] \frown ([\partial_1] \smile [\partial_{-1}]) \in \mathbb{C}. \tag{1}$$

Here $[\omega] \in H_2(A, {}_{\sigma_{\mathrm{mod}}} A)$ is acted on by $[\partial_1] \smile [\partial_{-1}] \in H^2(A, {}_{\sigma_{\mathrm{mod}}^{-2}} A)$ to produce a class in $H_0(A, {}_{\sigma_{\mathrm{mod}}^{-1}} A)$, and then one applies a certain twisted trace $\int \in (H_0(A, {}_\sigma A))^*$ in order to obtain a numerical invariant of $[\omega]$.

The functional φ provides a dual description of the fundamental class $[\mathsf{d}A] \in H_2(A, {}_{\sigma_{\mathrm{mod}}}A)$ that corresponds under the Poincaré-type duality [12]

$$H_n(A, {}_{\sigma_{\mathrm{mod}}}A) \simeq H^{2-n}(A, A) \tag{2}$$

to $1 \in H^0(A, A)$. From the practical point of view, one can use φ to determine the homology class of a given Hochschild cycle, and for $\mathbb{C}_q[SU(2)]$ this tool allowed us to compute the cyclic homology [13] built upon $H_n(A, {}_{\sigma}A)$ as a special case of Connes-Moscovici's Hopf-cyclic homology [3].

The trace $\int$ in (1) is for $\mathbb{C}_q[S^2]$ actually a character (namely the restriction of the counit ε of $\mathbb{C}_q[SU(2)]$ to $\mathbb{C}_q[S^2]$), so on the level of chains $a_0 \otimes a_1 \otimes a_2$ in the standard Hochschild complex, φ acts as

$$\varphi(a_0, a_1, a_2) = q^{-1}\varepsilon(a_0)F(a_1)E(a_2),$$

where $E, F : A \to k$ are the (untwisted) derivations given by

$$E(a) := \varepsilon(\partial_{-1}(a)), \quad F(a) := \varepsilon(\partial_1(a)).$$

There seems to be a general principle behind this that we observed already in [8]. Therein, the trace $\int$ needed was the integral over the unquantised maximal torus in quantum $SU(2)$. For the quantum 2-sphere the corresponding submanifold of S^2 is simply a point, namely one of the two leaves of the symplectic foliation of the Poisson manifold S^2 quantised by A.

Finally we discuss how to add a counter term η to φ in order to obtain a functional on cyclic homology without changing the functional on $H_2(A, {}_{\sigma_{\mathrm{mod}}}A)$. Schmüdgen and Wagner have defined a cyclic 2-cocycle in [17] that looks like φ, only the trace $\int$ is the Haar functional of $\mathbb{C}_q[SU(2)]$, and this makes their functional trivial on Hochschild homology [6].

The structure of the paper is as follows. In Sections 2-5 we recall the definition of Hochschild (co)homology, of the cup and cap product, give some more background about the above mentioned Poincaré duality and introduce then the algebra $A = \mathbb{C}_q[S^2]$ we want to study. In Section 6 we recall from [6] an explicit formula for the fundamental class $\mathsf{d}A$ of A. In Section 7 we determine the twisted centre of A and its cap product action on the second Hochschild homology: in [6] it was shown that $H_2(A, {}_{\sigma}A) = 0$ except when $\sigma = \sigma^n_{\mathrm{mod}}$ for some $n \geq 1$, and then $H_2(A, {}_{\sigma^n_{\mathrm{mod}}}A) \simeq \mathbb{C}$. Here we identify the sum of all these nontrivial homology groups with the free module of rank 1 over a polynomial ring $k[x_0]$ that constitutes the twisted centre of A. We continue in Section 8 with recalling from [6] (but in a slightly simplified form) the computation of the zeroth Hochschild homology groups of A and of the twisted traces on A, describing in addition the cap product action of the twisted centre. Section 9 is the first really interesting one, here we compute the cup product between twisted derivations of A that arise from the action of the quantised Lie algebra of $SU(2)$ on A. We observe similar as in [8] that these twisted derivations generate a quantised exterior algebra. These computations are then used in Section 10 to define and discuss φ. Section 11 recalls the definition and some key properties of cyclic homology, and finally we show in Section 12 that φ can be altered by a Hochschild coboundary to obtain a cyclic cocycle.

I acknowledge support by the EPSRC fellowship EP/E/043267/1.

2. Hochschild (co)homology with coefficients in ${}_\sigma A$

Let A be a unital associative algebra over a field k and $\sigma \in \mathrm{Aut}(A)$ be an automorphism. We denote by ${}_\sigma A$ the A-bimodule which is A as vector space with left and right A-actions given by $a \triangleright b \triangleleft c := \sigma(a)bc$, $a, b, c \in A$, and by $H_n(A, {}_\sigma A)$ and $H^n(A, {}_\sigma A)$ the Hochschild (co)homology groups of A with coefficients in ${}_\sigma A$. Explicitly, $H_n(A, {}_\sigma A)$ is the homology of the chain complex $C_n^\sigma := A^{\otimes n+1}$ with boundary map $\mathsf{b}_n : C_n^\sigma \to C_{n-1}^\sigma$ given by

$$\begin{aligned}\mathsf{b}_n(a_0 \otimes \ldots \otimes a_n) &= \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n \\ &\quad + (-1)^n \sigma(a_n) a_0 \otimes \ldots \otimes a_{n-1}.\end{aligned}$$

Usually we will write $\mathsf{b}(a_0, \ldots, a_n)$ instead of $\mathsf{b}_n(a_0 \otimes \ldots \otimes a_n)$ and similarly for other multilinear maps. Dually, $H^n(A, {}_\sigma A)$ is the cohomology of the cochain complex $C^n_\sigma := \mathrm{Hom}_k(A^{\otimes n}, A)$ with coboundary map given by

$$\begin{aligned}(\mathsf{b}^n\psi)(a_0, \ldots, a_n) &= \sigma(a_0)\psi(a_1, \ldots, a_n) \\ &\quad + \sum_{i=0}^{n-1} (-1)^{i+1} \psi(a_0, \ldots, a_i a_{i+1}, \ldots, a_n) \\ &\quad + (-1)^{n+1} \psi(a_0, \ldots, a_{n-1}) a_n.\end{aligned}$$

In degree 0, we identify $\psi : A^{\otimes 0} := k \to A$ with $a := \psi(1) \in A$. This is a cocycle precisely when $ab = \sigma(b)a$ for all $b \in A$. Thus $H^0(A, {}_\sigma A)$ consists of the σ-central elements of A. In degree 1, a cocycle is a σ-twisted derivation $\psi : A \to A$, $\psi(ab) = \sigma(a)\psi(b) + \psi(a)b$, and $H^1(A, {}_\sigma A)$ is the space of all such derivations modulo those of the form $\psi(a) = ba - \sigma(a)b$ for some $b \in A$. For more information and details, see e.g. [2, 7, 13, 14, 20].

3. The cup and cap product

The cup product is the map

$$\smile : H^m(A, {}_\sigma A) \otimes H^n(A, {}_\tau A) \to H^{m+n}(A, {}_{\tau\circ\sigma} A), \quad \sigma, \tau \in \mathrm{Aut}(A)$$

given on the level of cochains by

$$(\varphi \smile \psi)(a_1, \ldots, a_{m+n}) = \tau(\varphi(a_1, \ldots, a_m))\psi(a_{m+1}, \ldots, a_{m+n}). \tag{3}$$

For any monoid $G \subset \mathrm{Aut}(A)$, it turns

$$\Lambda_G(A) := \bigoplus_{n\in\mathbb{N}, \sigma\in G} H^n(A, {}_\sigma A)$$

into an $\mathbb{N} \times G$-graded algebra that we would like to view as some analogue of an algebra of multivector fields on a classical space. Twisted derivations play here the role of vector fields, and the following easily checked (see [8]) relations demonstrate their behaviour under $\smile$:

lemma 3.1. *In degree 0, $\smile$ reduces to the opposite product of A,*

$$a \smile b = ba, \quad a \in H^0(A, {}_\sigma A), b \in H^0(A, {}_\tau A), \tag{4}$$

and for $c \in H^0(A, {}_\sigma A)$ and twisted derivations $\varphi \in C^1_\sigma, \psi \in C^1_\tau$ we have

$$\psi \smile c = \tau(c) \smile \psi, \quad [\varphi] \smile [\psi] = -[\sigma^{-1} \circ \psi \circ \sigma] \smile [\varphi] \in H^2(A, {}_{\tau\circ\sigma} A).$$

Dually,

$$\Omega^G(A) := \bigoplus_{n\in\mathbb{N},\sigma\in G} H_n(A, {}_{\sigma^{-1}}A)$$

becomes an $\mathbb{N}\times G$-graded (right) module over $\Lambda_G(A)$ via the cap product

$$\frown : H_n(A, {}_\sigma A)\otimes H^m(A, {}_\tau A) \to H_{n-m}(A, {}_{\tau\circ\sigma}A), \quad m\le n.$$

Explicitly, this is given between $\varphi\in C^m_\tau$ and $a_0\otimes\ldots\otimes a_n\in C^\sigma_n$ by

$$(a_0\otimes\ldots\otimes a_n)\frown\varphi = \tau(a_0)\varphi(a_1,\ldots,a_m)\otimes a_{m+1}\otimes\ldots\otimes a_n\in C^{\tau\circ\sigma}_{n-m}.$$

In particular, the cap product with a twisted central element $c\in H^0(A, {}_\sigma A)$ is simply given by multiplication from the *left*,

$$(a_0\otimes\ldots\otimes a_n)\frown c = \sigma(a_0)c\otimes\ldots\otimes a_n = ca_0\otimes\ldots\otimes a_n. \tag{5}$$

For more information, see e.g. [2, 8, 15].

4. Poincaré duality

The cup and cap product structures are intimately related to Poincaré-type dualities between homology and cohomology. As became clear in recent years, there is for many algebras A a distinguished automorphism σ_{mod} and for all $\tau\in\mathrm{Aut}(A)$ a canonical k-linear isomorphism

$$H^n(A, {}_\tau A)\simeq H_{\dim(A)-n}(A, {}_{\tau\circ\sigma_{\mathrm{mod}}}A), \tag{6}$$

where $\dim(A)$ is the dimension of A in the sense of [2], see e.g. [1, 5, 7, 11, 12] but first of all [19] for this story. Under the above isomorphism, the canonical element $1\in H^0(A,A)$ corresponds to a class $[\mathtt{d}A]\in H_{\dim(A)}(A, {}_{\sigma_{\mathrm{mod}}}A)$, and then the isomorphism is given by the cap product with this fundamental class. In [8] we carried this out explicitly for the standard quantised coordinate ring $\mathbb{C}_q[SU(2)]$ (see e.g. [10] for background on quantum groups), and the aim here is to do the same for the standard quantum 2-sphere of Podleś that we introduce in the next section. For the coordinate ring of a smooth affine variety such a duality will hold if and only if the variety is Calabi-Yau, that is, if the line bundle on X whose sections are the top degree Kähler differentials $\Omega^{\dim(X)}(X)$ is trivial (in general one has to twist not by an automorphism but by this module, see e.g. [11]). This happens if and only if there is a nowhere (i.e. in no localisation at prime ideals) vanishing element in $\Omega^{\dim(X)}(X)$, and under the Hochschild-Kostant-Rosenberg isomorphism $\Omega^{\dim(X)}(X)\simeq H_{\dim(X)}(k[X],k[X])$ such an element will be identified with the fundamental class $[\mathtt{d}k[X]]$.

5. The Podleś sphere

From now on we fix $k=\mathbb{C}$, an element $q\in k\backslash\{0\}$ assumed to be not a root of unity, and A is the standard Podleś quantum 2-sphere [16], that is, the universal k-algebra generated by x_{-1},x_0,x_1 satisfying the relations

$$x_{\pm1}x_0 = q^{\mp2}x_0x_{\pm1}, \quad x_{\pm1}x_{\mp1} = q^{\mp2}x_0^2 + q^{\mp1}x_0.$$

It follows easily from these relations that the elements

$$e_{ij} := \begin{cases} x_0^i x_1^j & j\ge 0,\\ x_0^i x_{-1}^{-j} & j<0,\end{cases} \qquad i\in\mathbb{N}, j\in\mathbb{Z}$$

form a vector space basis of A.

We denote by G the automorphism group of A. The defining relations imply that for any $\lambda \in k\backslash\{0\}$ there is a unique $\sigma_\lambda \in G$ with $\sigma_\lambda(x_n) = \lambda^n x_n$.

LEMMA 5.1. *Any $\sigma \in G$ is of the form σ_λ for some λ, that is, $G \simeq k\backslash\{0\}$.*

Proof. It is straightforward to classify the characters of A and to see that the intersection of their kernels is the ideal generated by x_0. It follows that any automorphism σ maps x_0 to a nonzero scalar multiple of x_0. Hence $x_0\sigma(x_{\pm 1}) = q^{\pm 2}\sigma(x_{\pm 1})x_0$ which implies $\sigma(x_{\pm 1}) = f_\pm x_{\pm 1}$ for some $f_\pm \in k[x_0]$. Inserting into the defining relations gives the claim. □

See also [4, 10] for more information about this algebra.

6. The fundamental class

As shown in [12], A satisfies (6) with $\dim(A) = 2$ (as it probably should be for a quantum 2-sphere), and with σ_{mod} determined uniquely by

$$\sigma_{\mathrm{mod}}(x_n) = q^{2n}x_n.$$

So $H_2(A, {}_{\sigma_{\mathrm{mod}}}A) \simeq H^0(A, A)$, the centre of A. This consists only of the scalars, hence $[\mathsf{d}A]$ is unique up to normalisation. Hadfield has computed explicit vector space bases of all $H_n(A, {}_\sigma A)$ for general automorphisms σ [6] and has given in particular an explicit cycle representing $[\mathsf{d}A] \in H_2(A, {}_{\sigma_{\mathrm{mod}}}A)$:

$$\begin{aligned}\mathsf{d}A \;:=\;& 2x_1 \otimes (x_{-1} \otimes x_0 - q^2 x_0 \otimes x_{-1}) \\ & +2x_{-1} \otimes (q^{-2}x_0 \otimes x_1 - x_1 \otimes x_0) \\ & +1 \otimes (qx_1 \otimes x_{-1} - q^{-1}x_{-1} \otimes x_1 + (q - q^{-1})x_0 \otimes x_0) \\ & +2x_0 \otimes (x_1 \otimes x_{-1} - x_{-1} \otimes x_1 + (q^2 - q^{-2})x_0 \otimes x_0).\end{aligned}$$

7. The twisted centre and $H_2(A, {}_\sigma A)$

As we pointed out above, the fundamental class is classically (meaning for the coordinate ring of a smooth variety) represented by a nowhere vanishing algebraic differential form of top degree, and this form can be multiplied by any regular function to give a new top degree form. Analogously we can act via the cap product by any twisted central element $a \in H^0(A, {}_\tau A)$ on the fundamental class $[\mathsf{d}A] \in H_2(A, {}_{\sigma_{\mathrm{mod}}}A)$ and obtain another homology class in $H_2(A, {}_{\tau\circ\sigma_{\mathrm{mod}}}A)$. This operation clarifies completely the structure of all the other nonvanishing $H_2(A, {}_\sigma A)$ that were computed by Hadfield, since they become altogether identified with a free $k[x_0]$-module of rank 1:

LEMMA 7.1. *The twisted centre $\Lambda^0_G(A)$ of A is the subalgebra generated by $x_0 \in H^0(A, {}_{\sigma_{\mathrm{mod}}}A)$, and for every $[\omega] \in H_2(A, {}_\sigma A)$, σ any automorphism, there exists exactly one polynomial $f \in k[x_0]$ such that $[\omega] = [\mathsf{d}A] \frown f$.*

Proof. Since any automorphism fixes x_0, any twisted central element must commute with x_0 and is hence a polynomial in x_0 (use the vector space basis e_{ij}). Conversely, it is clear that $x_0 \in H^0(A, {}_{\sigma_{\mathrm{mod}}}A)$ and hence $x_0^i = x_0 \smile \ldots \smile x_0 \in H^0(A, {}_{\sigma_{q^{2i}}}A)$ (recall Lemma 3.1). The second part follows by Poincaré duality (6), but of course also from Hadfield's explicit computations of all the nontrivial $H_2(A, {}_\sigma A)$. □

8. Twisted traces and $H_0(A,{}_\sigma A)$

As a vector space, $H_0(A,{}_\sigma A)$ can be described as follows [6]:

lemma 8.1. *For $\sigma=\sigma_\lambda$, the following is a vector space basis of $H_0(A,{}_\sigma A)$:*

$$\{[1]\}\cup\{[x_{\pm1}^j]\,|\,j\neq 0,\lambda=1\}\cup\{[x_0]\,|\,\lambda\neq q^{2i},i>0\}\cup\{[x_0^i]\,|\,\lambda=q^{2i},i>0\}.$$

Proof. By definition, $H_0(A,{}_\sigma A)$ is as a vector space the quotient of A by the subspace spanned by elements of the form $\mathsf{b}(a,b)=ab-\sigma(b)a$. Since

$$a\otimes bc=ab\otimes c+\sigma(c)a\otimes b-\mathsf{b}(a,b,c),$$

one has $\mathsf{b}(a,bc)=\mathsf{b}(ab,c)+\mathsf{b}(\sigma(c)a,b)$, so $\operatorname{im}\mathsf{b}$ is spanned by the elements $b_{ij}^k:=\mathsf{b}(e_{ij},x_k)$, $i\in\mathbb{N}$, $j\in\mathbb{Z}$, $k=-1,0,1$ which are for $\sigma=\sigma_\lambda$ given by

$$\begin{aligned}
b_{ij}^{-1}&=(1-\lambda^{-1}q^{2i})e_{ij-1},\quad j\leq 0\\
b_{ij}^{-1}&=(q^{-4j+2}-\lambda^{-1}q^{2i+2})e_{i+2j-1}+(q^{-2j+1}-\lambda^{-1}q^{2i+1})e_{i+1j-1},\quad j>0,\\
b_{ij}^{0}&=(q^{-2j}-1)e_{i+1j},\\
b_{ij}^{1}&=(1-\lambda q^{-2i})e_{ij+1},\quad j\geq 0\\
b_{ij}^{1}&=(q^{-4j-2}-\lambda q^{-2i-2})e_{i+2j+1}+(q^{-2j-1}-\lambda q^{-2i-1})e_{i+1j+1},\quad j<0.
\end{aligned}$$

Reducing this by sheer inspection gives that $\operatorname{im}\mathsf{b}$ is spanned by the elements

$$\begin{aligned}
&e_{i+1j},\quad(\lambda-1)e_{0j},\quad j\neq 0,\\
&(\lambda-q^{2i+4})q^{-2i-2}e_{i+20}+(\lambda-q^{2i+2})q^{-2i-1}e_{i+10},\quad i\geq 0.
\end{aligned}$$

The claim follows easily. □

Dually, $H_0(A,{}_\sigma A)$ can be described in terms of σ-twisted traces, that is, linear functionals $\int:A\to k$ satisfying

$$\int ab=\int\sigma(b)a,\quad a,b\in A.$$

Such traces obviously descend to well-defined functionals on $H_0(A,{}_\sigma A)$ which we denote for simplicity by the same symbol. For each of the basis elements in Lemma 8.1, we can (and do) define one such trace

$$\begin{aligned}
\int_{[x_{\pm1}^j]}e_{kl}&:=\begin{cases}1 & k=0,\pm j=l,\\ 0 & \text{otherwise,}\end{cases}\qquad j\geq 0,\\
\int_{[x_0]}e_{kl}&:=\begin{cases}1 & k=1,l=0,\\ (-1)^{k+1}q^{1-k}\frac{1-\lambda q^{-2}}{1-\lambda q^{-2k}} & k>1,l=0,\\ 0 & \text{otherwise,}\end{cases}\\
\int_{[x_0^i]}e_{kl}&:=\begin{cases}1 & k=i,l=0,\\ 0 & \text{otherwise,}\end{cases}\qquad i>1.
\end{aligned}$$

Note that $\int_{[x_0]}$ is defined in such a way that the case $\lambda=q^2$ is included. Note also that $\int_{[1]}$ is in fact the character ε determined by $\varepsilon(x_n)=0$. Any automorphism of A leaves $\ker\varepsilon$ invariant, so this is a twisted trace with

respect to all automorphisms of A. Since we have for all elements $[\omega], [\eta]$ of the basis of $H_0(A, {}_\sigma A)$ from Lemma 8.1

$$\int_{[\omega]} [\eta] = \begin{cases} 1 & [\omega] = [\eta], \\ 0 & \text{otherwise}, \end{cases}$$

we can (and will) use the $\int_{[\omega]}$ to determine the homology class of a given 0-cycle. For example, it helps describing the cap product action of the twisted centre on $H_0(A, {}_\sigma A)$:

LEMMA 8.2. *The action of $\Lambda^0_G(A)$ on $H_0(A, {}_{\sigma_\lambda} A)$, is determined by*

$$[x^j_{\pm 1}] \frown x_0 = 0, \quad j > 0,$$
$$[x^i_0] \frown x_0 = [x^{i+1}_0] = \begin{cases} 0 & i = 0, \lambda = q^{2k}, k > 0, \\ -q\frac{1-\lambda}{q^2-\lambda}[x_0] & i = 1, \lambda \neq q^2. \end{cases}$$

Proof. As we remarked in (5), x_0 acts on cycles by multiplication from the left. The claim follows by applying all the above constructed twisted traces to the resulting cycles. □

In particular, the span of the classes $[x^i_0] \in H_0(A, {}_{\sigma^i_{\mathrm{mod}}} A)$ is the orbit of $[1] \in H_0(A, A)$ under the action of the twisted centre. In a sense, the span of the $[x^j_{\pm 1}]$ can be viewed as the orbit of $[1]$ under the cap product with $x_{\pm 1}$, although the latter do not belong to the twisted centre of A: for any subalgebra $B \subset A$ with $\sigma(B) \subset B$, there is a map $H_n(B, {}_\sigma B) \to H_n(A, {}_\sigma A)$ given on the level of cycles by the embedding of B into A in each tensor component. If a class is in the image of this map, then taking the cap product with a twisted central element of B is well-defined, and this applies here to the case B is the subalgebra generated by x_1 or x_{-1}, respectively.

9. Three twisted derivations

The Podleś sphere is a module algebra over the Hopf dual $\mathbb{C}_q[SU(2)]^\circ$ of the quantised coordinate ring of $SU(2)$, hence twisted primitive elements therein act as twisted derivations on A. We will not need Hopf algebra theory later, so we rather state the following lemma that the reader can verify directly by checking compatibility with the defining relations of A:

LEMMA 9.1. *The assignments*

$$\partial_1 : x_{-1}, x_0, x_1 \mapsto 0, qx_{-1}, 1 + (q + q^{-1})x_0,$$
$$\partial_0 : x_{-1}, x_0, x_1 \mapsto -x_{-1}, 0, x_1,$$
$$\partial_{-1} : x_{-1}, x_0, x_1 \mapsto 1 + (q + q^{-1})x_0, q^{-1}x_1, 0$$

can be extended uniquely to 1-cocycles $\partial_i \in C^1_{\sigma^{-|i|}_{\mathrm{mod}}}$,

$$\partial_i(ab) = \sigma^{-|i|}_{\mathrm{mod}}(a)\partial_i(b) + \partial_i(a)b, \quad a, b \in A, \quad i = -1, 0, 1.$$

The following lemma describes the cup product action of $\Lambda^0_G(A)$ on these derivations. Admittedly, the result is slightly weird:

LEMMA 9.2. *The $\Lambda^0_G(A)$-module generated by $[\partial_0]$ is free, but $[\partial_{\pm 1}] \smile x_0 = 0$.*

Proof. One checks directly that one has for all $a \in A$

$$(\partial_{\pm 1} \leftharpoonup x_0)(a) = x_0 \partial_{\pm 1}(a) = \pm \frac{1}{q - q^{-1}}(x_{\mp 1} a - a x_{\mp 1}),$$

so the derivations $\partial_{\pm 1} \leftharpoonup x_0$ are inner. On the other hand, the computation of $e_{jk} x_1 - \sigma^i_{\mathrm{mod}}(x_1) e_{jk} = b^1_{jk}$ in the proof of Lemma 8.1 shows that no inner derivations in $C^1_{\sigma^i_{\mathrm{mod}}}$ can map x_1 to $x_0^i x_1 = (\partial_0 \leftharpoonup x_0^i)(x_1)$. □

For the reason explained at the end of the previous section, it does make sense to take the cup product between $\partial_{\pm 1}$ and $x^j_{\mp 1}$, although these are not twisted central, and this produces new twisted derivations. Acting with them on the fundamental class and comparing the result with the generators of $H_1(A, {}_\sigma A)$ computed in [6] allows one to describe all twisted derivations of A, see [8] where we carried this out for $\mathbb{C}_q[SU(2)]$. However, there is little gain in this for the main purpose of the present paper which is to obtain a functional describing $\mathsf{d}A$ in a dual fashion, so I leave out these calculations.

Let us compute instead the algebra generated by the $[\partial_i]$. Classically, a differential form can be contracted with a vector field to reduce its degree, and in the quantum case this is generalised by the cap product action of (the cohomology class of) a twisted derivation on a homology class. Here is the full orbit of $\mathsf{d}A$ under this action of the ∂_i:

$$\begin{aligned}
\mathsf{d}A \leftharpoonup \partial_0 &= 2q^{-2} x_{-1} x_0 \otimes x_1 + 2q^2 x_1 x_0 \otimes x_{-1} - 2(q^2 + q^{-2}) x_0^2 \otimes x_0 \\
&\quad + q^{-1} x_{-1} \otimes x_1 + q x_1 \otimes x_{-1} - 2(q + q^{-1}) x_0 \otimes x_0, \\
(\mathsf{d}A \leftharpoonup \partial_0) \leftharpoonup \partial_0 &= 2(q^2 - q^{-2}) x_0^3 + 3(q - q^{-1}) x_0^2, \\
(\mathsf{d}A \leftharpoonup \partial_0) \leftharpoonup \partial_{-1} &= 2(-q^5 + q^{-1}) x_1 x_0^2 + (-2q^2 + 1 + q^{-2} 1) x_1 x_0 + q^{-1} x_1, \\
(\mathsf{d}A \leftharpoonup \partial_0) \leftharpoonup \partial_1 &= 2(q - q^{-5}) x_{-1} x_0^2 + (q^2 + 1 - 2q^{-2}) x_{-1} x_0 + q x_{-1},
\end{aligned}$$

$$\begin{aligned}
\mathsf{d}A \leftharpoonup \partial_{-1} &= -2q^{-1} x_1^2 \otimes x_{-1} - 2q^{-1} x_0^2 \otimes x_1 + 2(q^3 + q^{-3}) x_1 x_0 \otimes x_0 \\
&\quad - (1 + q^{-2}) x_0 \otimes x_1 + (1 + q^{-2}) x_1 \otimes x_0 - q^{-1} \otimes x_1, \\
(\mathsf{d}A \leftharpoonup \partial_{-1}) \leftharpoonup \partial_0 &= 2(q^{-3} - q^3) x_1 x_0^2 + (-q^2 - 1 + 2q^{-2}) x_1 x_0 - q^{-1} x_1, \\
(\mathsf{d}A \leftharpoonup \partial_{-1}) \leftharpoonup \partial_{-1} &= 2(q^2 - q^{-6}) x_1^2 x_0 + (q^{-3} - q^{-5}) x_1^2, \\
(\mathsf{d}A \leftharpoonup \partial_{-1}) \leftharpoonup \partial_1 &= 2(q^{-8} - 1) x_0^3 + (-q - 2q^{-1} + q^{-5} + 2q^{-7}) x_0^2 \\
&\quad + (-2 - q^{-2} + q^{-4}) x_0 - q^{-1},
\end{aligned}$$

$$\begin{aligned}
\mathsf{d}A \leftharpoonup \partial_1 &= 2q x_{-1}^2 \otimes x_1 + 2q x_0^2 \otimes x_{-1} - 2(q^3 + q^{-3}) x_{-1} x_0 \otimes x_0 \\
&\quad + (q^2 + 1) x_0 \otimes x_{-1} + (-q^2 - 1) x_{-1} \otimes x_0 + q \otimes x_{-1}, \\
(\mathsf{d}A \leftharpoonup \partial_1) \leftharpoonup \partial_0 &= 2(q^3 - q^{-3}) x_{-1} x_0^2 + (2q^2 - 1 - q^{-2}) x_{-1} x_0 - q x_{-1}, \\
(\mathsf{d}A \leftharpoonup \partial_1) \leftharpoonup \partial_{-1} &= 2(1 - q^8) x_0^3 + (-2q^7 - q^5 + 2q + q^{-1}) x_0^2 \\
&\quad + (-q^4 + q^2 + 2) x_0 + q, \\
(\mathsf{d}A \leftharpoonup \partial_1) \leftharpoonup \partial_1 &= 2(q^6 - q^{-2}) x_{-1}^2 x_0 + (q^5 - q^3) x_{-1}^2.
\end{aligned}$$

In homology, this reduces to:

$$([\mathsf{d}A] \frown [\partial_i]) \frown [\partial_i] = 0,$$
$$([\mathsf{d}A] \frown [\partial_0]) \frown [\partial_{-1}] = -([\mathsf{d}A] \frown [\partial_{-1}]) \frown [\partial_0] = q^{-1}[x_1],$$
$$([\mathsf{d}A] \frown [\partial_0]) \frown [\partial_1] = -([\mathsf{d}A] \frown [\partial_1]) \frown [\partial_0] = q[x_{-1}],$$
$$([\mathsf{d}A] \frown [\partial_1]) \frown [\partial_{-1}] = -q^2([\mathsf{d}A] \frown [\partial_{-1}]) \frown [\partial_1] = (q^2+1)[x_0] + q.$$

From this, we see:

LEMMA 9.3. *The ∂_i satisfy no other relations than those dictated by Lemma 3.1,*

$$[\partial_i] \smile [\partial_j] = -q^{2ij}[\partial_j] \smile [\partial_i], \quad i \leq j.$$

Proof. Poincaré duality tells that the $\Lambda_G(A)$-module $\Omega^G(A)$ is free and generated by $\mathsf{d}A$, so the $[\partial_i]$ satisfy under $\smile$ all the relations they satisfy as linear maps on $\Omega^G(A)$. The result follows from the above computations. □

10. The volume form

Finally, we now put

$$\varphi : \Omega^G(A) \to k, \quad [\omega] \mapsto q^{-1} \int_{[1]} [\omega] \frown ([\partial_1] \smile [\partial_{-1}]),$$

or explicitly on chain level

$$\varphi(a_0, a_1, a_2) = q^{-1} \int_{[1]} \sigma_{\mathrm{mod}}^{-2}(a_0)\sigma_{\mathrm{mod}}^{-1}(\partial_1(a_1))\partial_{-1}(a_2).$$

Our above computation of $([\mathsf{d}A] \frown [\partial_1]) \frown [\partial_{-1}]$ implies

$$\varphi(\mathsf{d}A) = 1,$$

so the functional φ provides a dual description of the fundamental class. It is a very useful tool (and probably the only really applicable one) for checking whether or not a given σ_{mod}-twisted 2-cycle has trivial class in $H_2(A, {}_{\sigma_{\mathrm{mod}}}A)$ or not, which as a result of the above computations (recall that $H_2(A, {}_{\sigma_{\mathrm{mod}}}A) \simeq \mathbb{C}$) is the fact if and only if φ vanishes on the cycle.

Note that we remarked in Section 8 that $\int_{[1]}$ is actually a character that we also denote by ε (when embedding A as usual into the quantised coordinate ring of $SU(2)$, this character becomes the restriction of the counit). Hence φ can also be written as simple as

$$\varphi(a_0, a_1, a_2) = q^{-1}\varepsilon(a_0)F(a_1)E(a_2),$$

where $E, F : A \to k$ are the (untwisted) derivations given by

$$E(a) := \varepsilon(\partial_{-1}(a)), \quad F(a) := \varepsilon(\partial_1(a)).$$

A moment's reflection now gives the explicit formula

$$\varphi(e_{ij}, e_{kl}, e_{mn}) = q^{-1}\delta_{i0}\delta_{j0}\delta_{k0}\delta_{l1}\delta_{m0}\delta_{n-1}. \tag{7}$$

This looks surprisingly simple (not to say banal), usually one expects $\int$ to be some sort of integral. However, we observed already in [8] for the case of quantum $SU(2)$ that the functional appearing when expressing the volume form dual to $\mathsf{d}A$ as above is given by something like the integral

of the restriction of functions to a maximal torus which is a Poisson subgroup of the Poisson group quantised by $\mathbb{C}_q[SU(2)]$. Here we have the same phenomenon, but it appears much sharper since the functional $\int_{[1]}$ is really integration over (meaning evaluation in) a single point. It somehow seems that the homological information about a quantum space can be supported in a classical subspace of smaller dimension.

Note also that one can alternatively define

$$\varphi_\pm : \omega \mapsto \pm q^{\mp 1} \int_{[x_{\mp 1}]} \omega \frown (\partial_0 \smile \partial_{\pm 1}),$$

where $\int_{[\pm 1]}$ is the trace (no twist) dual to $[x_{\pm 1}] \in H_0(A, A)$. Again, the above computations give $\varphi_\pm(\mathsf{d}A) = 1$, hence $\varphi = \varphi_+ = \varphi_-$ as functionals on $H_2(A, {}_{\sigma_{\mathrm{mod}}}A) \simeq \mathbb{C}$ (but not as functionals on $C_2^{\sigma_{\mathrm{mod}}}$), and for some purposes this representation of the functional might be better suited that φ.

11. Cyclic homology

In the classical case $A = k[X]$, X a smooth variety, exterior derivation d turns the algebraic differential forms $\Omega(X)$ into a cochain complex that computes the algebraic de Rham cohomology $H^n(X)$ of X. In the noncommutative case, Connes' cyclic homology provides a subtle analogue of $H^n(X)$. The extension of cyclic homology to the twisted coefficients ${}_\sigma A$ arose in the work of Kustermans, Murphy and Tuset on covariant differential calculi over quantum groups [13], but can also be viewed as a special case of Connes-Moscovici's Hopf-cyclic homology [3], see e.g. our article [8] and the references therein for more background.[1]

The precise relation between $HC_n(k[X])$ ($\sigma = \mathrm{id}$) and $H^n(X)$ is

$$HC_n(k[X]) \simeq \Omega^n(X)/\mathrm{im}\,\mathsf{d} \oplus H^{n-2}(X) \oplus H^{n-4}(X) \oplus \dots, \tag{8}$$

so all differential n-forms (not only the closed ones in $\ker \mathsf{d}$!) have classes in $HC_n(k[X])$, and this gives a map

$$\mathsf{I} : \Omega^n(X) \simeq H_n(k[X], k[X]) \to HC_n(k[X]).$$

Furthermore, d (applied to $\Omega^n(X)/\mathrm{im}\,\mathsf{d}$) gives a well-defined map

$$\mathsf{B} : HC_n(k[X]) \to \Omega^{n+1}(X) \simeq H_{n+1}(k[X], k[X])$$

which kills all the $H^{n-2i}(X)$ summands in (8), and finally there is

$$\mathsf{S} : HC_n(k[X]) \to HC_{n-2}(k[X])$$

that cuts off the first term $\Omega^n(X)/\mathrm{im}\,\mathsf{d}$ and leaves the rest untouched (using the obvious embedding $H^{n-2}(X) \to \Omega^{n-2}(X)/\mathrm{im}\,\mathsf{d}$ in the next summand).

This whole picture carries over to the general noncommutative case and becomes condensed into Connes' SBI-sequence, see [14, 20] for the details. The upshot of this is that $HC_n(A)$ contains a whole lot of ballast, the really interesting part is only the image of the natural map I coming from Hochschild homology, which equals the kernel of the periodicity map $\mathbf{S}$.

One way to define the cyclic theory is in terms of the operator

$$\mathsf{t} : C_n^\sigma \to C_n^\sigma : a_0 \otimes \dots \otimes a_n \mapsto (-1)^n \sigma(a_n) \otimes a_0 \otimes \dots \otimes a_{n-1}.$$

[1] As Jack Shapiro informed me, he will discuss in a forthcoming article the corresponding version of noncommutative de Rham theory [18].

Its coinvariants $C_n^\sigma/\mathrm{im}\,(\mathrm{id}-\mathsf{t})$ form a quotient complex of (C_n^σ, b) whose homology is $HC_n^\sigma(A)$ (k should contain $\mathbb{Q}$ and σ should be diagonalisable, otherwise the result might be not what one wants). As a consequence, a linear functional $\psi : C_n^\sigma \to k$ with $\psi(\mathrm{im}\,\mathsf{b}) = 0$ descends to a functional on $HC_n^\sigma(A)$ if it is invariant under t, $\psi = \psi \circ \mathsf{t}$. Clearly, ψ also induces a functional on $H_n(A, {}_\sigma A)$, but this might vanish even when the one on $HC_n^\sigma(A)$ doesn't (namely when there exists $\chi : C_{n-1}^\sigma \to k$ with $\psi = \chi \circ \mathsf{b}$, but no such χ which is cyclic, $\chi = \chi \circ \mathsf{t}$). Such a functional on $HC_n^\sigma(A)$ that vanishes on $H_n(A, {}_\sigma A)$ corresponds to the above described ballast in cyclic homology, it vanishes in the classical case on the leading term $\Omega^n(X)/\mathrm{im}\,\mathsf{d}$ of $HC_n(k[X])$ and is rather a functional on some $HC_{n-2k}^\sigma(A)$, $k > 0$, that is promoted to a functional on $HC_n^\sigma(A)$ using the periodicity operation S.

12. The case of the Podlsś sphere

Let now A be again the Podleś sphere. Schmüdgen and Wagner have constructed in [17] a nontrivial cyclic 2-cocycle on A which later was shown by Hadfield to be trivial when viewed on Hochschild homology. It is now natural to ask whether our volume form φ constructed above does also give rise to a nontrivial functional on cyclic homology.

It is easily checked that a functional $\psi : C_n^\sigma \to k$ vanishing on $\mathrm{im}\,\mathsf{b}$ is cyclic if and only if $\varphi(1, a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in A$. Using this one sees that our φ itself is not cyclic since by (7) we have

$$\varphi(1, x_1, x_{-1}) = q^{-1} \neq 0.$$

However, we can alter φ by a coboundary to make it cyclic, so the problem is only a matter of representing the functional on $H_2(A, {}_{\sigma_{\mathrm{mod}}} A)$ properly:

LEMMA 12.1. *Define $\eta := \phi \circ \mathsf{b} : C_2^{\sigma_{\mathrm{mod}}} \to k$, where the linear functional $\phi : A \otimes A \to k$ vanishes on all $e_{kl} \otimes e_{mn}$ except on the following ones:*

$$\phi(1, x_0) := \frac{1}{q^{-2}-1}, \quad \phi(1, x_0^2) := \frac{1}{q-q^{-1}}, \quad \phi(x_0, x_0) := \frac{1}{2(q-q^{-1})}.$$

Then η induces the trivial functional on $H_2(A, {}_{\sigma_{\mathrm{mod}}} A)$ and $\varphi + \eta$ is cyclic.

Proof. By its definition, η vanishes on $\mathrm{im}\,\mathsf{b}$ and defines the trivial functional on $H_2(A, {}_{\sigma_{\mathrm{mod}}} A)$. By (7), the only chain of the form $1 \otimes e_{kl} \otimes e_{mn}$ on which φ does not vanish is $1 \otimes x_1 \otimes x_{-1}$, where φ has the value q^{-1}, and one easily checks using

$$\eta(1, a_1, a_2) = \phi(a_1, a_2) - \phi(1, a_1 a_2) + \phi(\sigma_{\mathrm{mod}}(a_2), a_1)$$

that similarly $\eta(1, e_{kl}, e_{mn})$ vanishes except when $e_{kl} \otimes e_{mn} = x_1 \otimes x_{-1}$ and then it equals $-q^{-1}$. The result follows. □

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