A Method for Symbolic Computation of Abstract Operations*

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Abstract. This paper helps to bridge the gap between (i) the use of logic for specifying program semantics and performing program analysis, and (ii) abstract interpretation. Many operations needed by an abstract interpreter can be reduced to the problem of *symbolic abstraction*: the symbolic abstraction of a formula φ in logic $\mathcal{L}$, denoted by $\widehat{\alpha}(\varphi)$, is the most-precise value in abstract domain $\mathcal{A}$ that over-approximates the meaning of φ. We present a parametric framework that, given $\mathcal{L}$ and $\mathcal{A}$, implements $\widehat{\alpha}$. The algorithm computes successively better over-approximations of $\widehat{\alpha}(\varphi)$. Because it approaches $\widehat{\alpha}(\varphi)$ from "above", if it is taking too much time, a safe answer can be returned at any stage.

Moreover, the framework is "dual-use": in addition to its applications in abstract interpretation, it provides a new way for an SMT (Satisfiability Modulo Theories) solver to perform unsatisfiability checking: given $\varphi \in \mathcal{L}$, the condition $\widehat{\alpha}(\varphi) = \bot$ implies that φ is unsatisfiable.

1 Introduction

This paper concerns the connection between abstract interpretation and logic. Like several previous papers [29, 37, 21, 12], our work is based on the insight that many of the key operations needed by an abstract interpreter can be reduced to the problem of *symbolic abstraction* [29].

Suppose that $\mathcal{A}$ is an abstract domain with concretization function $\gamma : \mathcal{A} \to \mathcal{C}$. Given a formula φ in logic $\mathcal{L}$, let $[\![\varphi]\!]$ denote the meaning of φ—i.e., the set of concrete states that satisfy φ. The *symbolic abstraction* of φ, denoted by $\widehat{\alpha}(\varphi)$, is the best $\mathcal{A}$ value that over-approximates $[\![\varphi]\!]$: $\widehat{\alpha}(\varphi)$ is the unique value $A \in \mathcal{A}$ such that (i) $[\![\varphi]\!] \subseteq \gamma(A)$, and (ii) for all $A' \in \mathcal{A}$ for which $[\![\varphi]\!] \subseteq \gamma(A')$, $A \sqsubseteq A'$.

This paper presents a new framework for performing symbolic abstraction, discusses its properties, and presents several instantiations for various logics and abstract domains. In addition to providing insight on fundamental limits, the

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new algorithm for $\widehat{\alpha}$ also performs well: our experiments show that it is 11.3 times faster than a competing method [29, 21, 12], while finding dataflow facts (i.e., invariants) that are equally precise at 76.9% of a program's basic blocks, better (tighter) at 19.8% of the blocks, and worse (looser) at only 3.3% of the blocks.

Most-Precise Abstract Interpretation. Suppose that $\mathcal{G} = \mathcal{C} \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \mathcal{A}$ is a Galois connection between concrete domain $\mathcal{C}$ and abstract domain $\mathcal{A}$. Then the "best transformer" [7], or best abstract post operator for transition τ, denoted by $\widehat{\text{Post}}[\tau] : \mathcal{A} \to \mathcal{A}$, is the most-precise abstract operator possible, given $\mathcal{A}$, for the concrete post operator for τ, $\text{Post}[\tau] : \mathcal{C} \to \mathcal{C}$. $\widehat{\text{Post}}[\tau]$ can be expressed in terms of α, γ, and $\text{Post}[\tau]$, as follows [7]: $\widehat{\text{Post}}[\tau] = \alpha \circ \text{Post}[\tau] \circ \gamma$. This equation defines the limit of precision obtainable using abstraction $\mathcal{A}$. However, it is non-constructive; it does not provide an *algorithm*, either for applying $\widehat{\text{Post}}[\tau]$ or for finding a representation of the function $\widehat{\text{Post}}[\tau]$. In particular, in many cases, the application of γ to an abstract value would yield an intermediate result—a set of concrete states—that is either infinite or too large to fit in computer memory.

Symbolic Abstract Operations. The aforementioned problem with applying γ can be side-stepped by working with *symbolic* representations of sets of states (i.e., using formulas in some logic $\mathcal{L}$). The use of $\mathcal{L}$ formulas to represent sets of states is convenient because logic can also be used for specifying a language's concrete semantics; i.e., the concrete semantics of a transformer $\text{Post}[\tau]$ can be stated as a formula $\varphi_\tau \in \mathcal{L}$ that specifies the relation between input states and output states. However, the symbolic approach introduces a new challenge: how to bridge the gap between $\mathcal{L}$ and $\mathcal{A}$ [29]. In particular, we need to develop (i) algorithms to handle interconversion between formulas of $\mathcal{L}$ and abstract values in $\mathcal{A}$, and (ii) symbolic versions of the operations that form the core repertoire at the heart of an abstract interpreter.

1. $\widehat{\gamma}(A)$: Given an abstract value $A \in \mathcal{A}$, the *symbolic concretization* of A, denoted by $\widehat{\gamma}(A)$, maps A to a formula $\widehat{\gamma}(A)$ such that A and $\widehat{\gamma}(A)$ represent the same set of concrete states (i.e., $\gamma(A) = [\![\widehat{\gamma}(A)]\!]$).
2. $\widehat{\alpha}(\varphi)$: Given $\varphi \in \mathcal{L}$, the symbolic abstraction of φ, denoted by $\widehat{\alpha}(\varphi)$, maps φ to the best value in $\mathcal{A}$ that over-approximates $[\![\varphi]\!]$ (i.e., $\widehat{\alpha}(\varphi) = \alpha([\![\varphi]\!])$).
3. $\widehat{\text{Assume}}[\varphi](A)$: Given $\varphi \in \mathcal{L}$ and $A \in \mathcal{A}$, $\widehat{\text{Assume}}[\varphi](A)$ returns the best value in $\mathcal{A}$ that over-approximates the meaning of φ in concrete states described by A. That is, $\widehat{\text{Assume}}[\varphi](A)$ equals $\alpha([\![\varphi]\!] \cap \gamma(A))$.
4. Creation of a representation of $\widehat{\text{Post}}[\tau]$: Some intraprocedural [15] and many interprocedural [32, 22] dataflow-analysis algorithms operate on instances of an abstract datatype $\mathcal{T}$ that (i) represents a family of abstract functions (or relations), and (ii) is closed under composition and join. By "creation of a representation of $\widehat{\text{Post}}[\tau]$", we mean finding the best instance in $\mathcal{T}$ that over-approximates $\text{Post}[\tau]$.

Several other symbolic abstract operations are discussed in §6.

Experience shows that, for most abstract domains, it is easy to write a $\widehat{\gamma}$ function (item 1) [29]. The other three operations are inter-related. $\widehat{\alpha}$ (item 2)

can be reduced to $\widehat{\text{Assume}}$ (item 3) as follows: $\widehat{\alpha}(\varphi) = \widehat{\text{Assume}}[\varphi](\top)$. Item 4 can be reduced to item 2 as follows: The concrete post operator $\text{Post}[\tau]$ corresponds to a formula $\varphi_\tau \in \mathcal{L}$ that expresses the transition relation between input states and output states. An instance of abstract datatype $\mathcal{T}$ in item 4 represents an abstract-domain element that denotes an over-approximation of $[\![\varphi_\tau]\!]$. $\widehat{\alpha}(\varphi_\tau)$ computes the best instance in $\mathcal{T}$ that over-approximates $[\![\varphi_\tau]\!]$.

This paper presents a parametric framework that, for some abstract domains, is capable of performing most-precise abstract operations in the limit. Because the method approaches its result from "above", if the computation takes too much time, it can be stopped to yield a safe result—i.e., an over-approximation to the best abstract operation—at any stage. Thus, the framework provides a tunable algorithm that offers a performance-versus-precision trade-off. We replace "$\widehat{\ }$" with "$\widetilde{\ }$" to denote over-approximating operators—e.g., $\widetilde{\alpha}(\varphi)$, $\widetilde{\text{Assume}}[\varphi](A)$, and $\widetilde{\text{Post}}[\tau](A)$.[3]

Key Insight. In [35], we showed how Stålmarck's method [33], an algorithm for satisfiability checking of propositional formulas, can be explained using abstract-interpretation terminology—in particular, as an instantiation of a more general algorithm, Stålmarck[$\mathcal{A}$], that is parameterized by a (Boolean) abstract domain $\mathcal{A}$ and operations on $\mathcal{A}$. The algorithm that goes by the name "Stålmarck's method" is one instantiation of Stålmarck[$\mathcal{A}$] with a certain abstract domain.

Abstract value A' is a *semantic reduction* [7] *of A with respect to φ* if (i) $\gamma(A') \cap [\![\varphi]\!] = \gamma(A) \cap [\![\varphi]\!]$, and (ii) $A' \sqsubseteq A$. At each step, Stålmarck[$\mathcal{A}$] holds some $A \in \mathcal{A}$; each of the so-called "propagation rules" employed in Stålmarck's method improves A by finding a semantic reduction of A with respect to φ.

The key insight of the present paper is that there is a connection between Stålmarck[$\mathcal{A}$] and $\widetilde{\alpha}_{\mathcal{A}}$. In essence, to check whether a formula φ is unsatisfiable, Stålmarck[$\mathcal{A}$] computes $\widetilde{\alpha}_{\mathcal{A}}(\varphi)$ and performs the test "$\widetilde{\alpha}_{\mathcal{A}}(\varphi) = \bot_{\mathcal{A}}$?" If the test succeeds, it establishes that $[\![\varphi]\!] \subseteq \gamma(\bot_{\mathcal{A}}) = \emptyset$, and hence that φ is unsatisfiable.

In this paper, we present a generalization of Stålmarck's algorithm to *richer logics*, such as quantifier-free linear rational arithmetic (QF_LRA) and quantifier-free bit-vector arithmetic (QF_BV). Instead of only using a Boolean abstract domain, the generalized method of this paper also uses *richer abstract domains*, such as the polyhedral domain [8] and the bit-vector affine-relations domain [12]. By this means, we obtain algorithms for computing $\widetilde{\alpha}$ for these richer abstract domains. The bottom line is that our algorithm is "dual-use": (i) it can be used by an abstract interpreter to compute $\widetilde{\alpha}$ (and perform other symbolic abstract operations), and (ii) it can be used in an SMT (Satisfiability Modulo Theories) solver to determine whether a formula is satisfiable.

Because we are working with more expressive logics, our algorithm uses several ideas that go beyond what is used in either Stålmarck's method [33] or in Stålmarck[$\mathcal{A}$] [35]. The methods described in this paper are also quite dif-

[3] $\widetilde{\text{Post}}[\tau]$ is used by Graf and Saïdi [14] to mean a different state transformer from the one that $\widetilde{\text{Post}}[\tau]$ denotes in this paper. Throughout the paper, we use $\widetilde{\text{Post}}[\tau]$ solely to mean an over-approximation of $\widehat{\text{Post}}[\tau]$; thus, our notation is not ambiguous.

ferent from the huge amount of recent work that uses decision procedures in program analysis. It has become standard to reduce program paths to formulas by encoding a program's actions in logic (e.g., by symbolic execution) and calling a decision procedure to determine whether a given path through the program is feasible. In contrast, the techniques described in this paper adopt—and adapt—the key ideas from Stålmarck's method to create new algorithms for key program-analysis operations. Finally, the methods described in this paper are quite different from previous methods for symbolic abstraction [29, 37, 21, 12], which all make repeated calls to an SMT solver.

Contributions. The contributions of the paper can be summarized as follows:

- We present a connection between symbolic abstraction and Stålmarck's method for checking satisfiability (§2).
- We present a generalization of Stålmarck's method that lifts the algorithm from propositional logic to richer logics (§3).
- We present a new parametric framework that, for some abstract domains, is capable of performing most-precise abstract operations in the limit, including $\widehat{\alpha}(\varphi)$ and $\widehat{\text{Assume}}[\varphi](A)$, as well as creating a representation of $\widehat{\text{Post}}[\tau]$. Because the method approaches most-precise values from "above", if the computation takes too much time it can be stopped to yield a sound result.
- We present instantiations of our framework for two logic/abstract-domain pairs: QF_BV/KS and QF_LRA/Polyhedra, and discuss completeness (§4).
- We present experimental results that illustrate the dual-use nature of our framework. One experiment uses it to compute abstract transformers, which are then used to generate invariants; another experiment uses it for checking satisfiability (§5).

§6 discusses other symbolic operations. §7 discusses related work. Proofs can be found in [36].

2 Overview

We now illustrate the key points of our Stålmarck-inspired technique using two examples. The first shows how our technique applies to computing abstract transformers; the second describes its application to checking unsatisfiability.

The top-level, overall goal of Stålmarck's method can be understood in terms of the operation $\widetilde{\alpha}(\psi)$. However, Stålmarck's method is recursive (counting down on a parameter k), and the operation performed at each recursive level is the slightly more general operation $\widetilde{\text{Assume}}[\psi](A)$. Thus, we will discuss $\widetilde{\text{Assume}}$.

Example 1. Consider the following x86 assembly code

```
L1: cmp eax, 2   L2: jz L4   L3: ...
```

The instruction at `L1` sets the zero flag (`zf`) to true if the value of register `eax` equals 2. At instruction `L2`, if `zf` is true the program jumps to location `L4` (not seen in the code snippet) by updating the value of the program counter (`pc`) to `L4`; otherwise, control falls through to program location `L3`. The transition formula that expresses the state transformation from the beginning of `L1` to the

beginning of L4 is thus $\varphi = (\texttt{zf} \Leftrightarrow (\texttt{eax} = 2)) \wedge (\texttt{pc}' = \mathit{ITE}(\texttt{zf}, \texttt{L4}, \texttt{L3})) \wedge (\texttt{pc}' = \texttt{L4}) \wedge (\texttt{eax}' = \texttt{eax})$. ($\varphi$ is a QF_BV formula.)

Let $\mathcal{A}$ be the abstract domain of affine relations over the x86 registers. Let $A_0 = \top_{\mathcal{A}}$, the empty set of affine constraints over input-state and output-state variables. We now describe how our algorithm creates a representation of the $\mathcal{A}$ transformer for φ by computing $\widetilde{\text{Assume}}[\varphi](A_0)$. The result represents a sound abstract transformer for use in affine-relation analysis (ARA) [27, 21, 12]. First, the ITE term in φ is rewritten as $(\texttt{zf} \Rightarrow (\texttt{pc}' = \texttt{L4})) \wedge (\neg\texttt{zf} \Rightarrow (\texttt{pc}' = \texttt{L3}))$. Thus, the transition formula becomes $\varphi = (\texttt{zf} \Leftrightarrow (\texttt{eax} = 2)) \wedge (\texttt{zf} \Rightarrow (\texttt{pc}' = \texttt{L4})) \wedge (\neg\texttt{zf} \Rightarrow (\texttt{pc}' = \texttt{L3})) \wedge (\texttt{pc}' = \texttt{L4}) \wedge (\texttt{eax}' = \texttt{eax})$.

Next, *propagation rules* are used to compute a semantic reduction with respect to φ, starting from A_0. The main feature of the propagation rules is that they are "local"; that is, they make use of only a small part of formula φ to compute the semantic reduction.

1. Because φ has to be true, we can conclude that each of the conjuncts of φ are also true; that is, $\texttt{zf} \Leftrightarrow (\texttt{eax} = 2)$, $\texttt{zf} \Rightarrow (\texttt{pc}' = \texttt{L4})$, $\neg\texttt{zf} \Rightarrow (\texttt{pc}' = \texttt{L3})$, $\texttt{pc}' = \texttt{L4}$, and $\texttt{eax}' = \texttt{eax}$ are all true.
2. Suppose that we have a function $\mu\widetilde{\alpha}_{\mathcal{A}}$ such that for a literal $l \in \mathcal{L}$, $A' = \mu\widetilde{\alpha}_{\mathcal{A}}(l)$ is a sound overapproximation of $\widehat{\alpha}(l)$. Because the literal $\texttt{pc}' = \texttt{L4}$ is true, we conclude that $A' = \mu\widetilde{\alpha}_{\mathcal{A}}(\texttt{pc}' = \texttt{L4}) = \{\texttt{pc}' - \texttt{L4} = 0\}$ holds, and thus $A_1 = A_0 \sqcap A' = \{\texttt{pc}' - \texttt{L4} = 0\}$, which is a semantic reduction of A_0.
3. Similarly, because the literal $\texttt{eax}' = \texttt{eax}$ is true, we obtain $A_2 = A_1 \sqcap \mu\widetilde{\alpha}_{\mathcal{A}}(\texttt{eax}' = \texttt{eax}) = \{\texttt{pc}' - \texttt{L4} = 0, \texttt{eax}' - \texttt{eax} = 0\}$.
4. We know that $\neg\texttt{zf} \Rightarrow (\texttt{pc}' = \texttt{L3})$. Furthermore, $\mu\widetilde{\alpha}_{\mathcal{A}}(\texttt{pc}' = \texttt{L3}) = \{\texttt{pc}' - \texttt{L3} = 0\}$. Now $\{\texttt{pc}' - \texttt{L3} = 0\} \sqcap A_2$ is $\bot$, which implies that $[\![\texttt{pc}' = \texttt{L3}]\!] \cap \gamma(\{\texttt{pc}' - \texttt{L4} = 0, \texttt{eax}' - \texttt{eax} = 0\}) = \emptyset$. Thus, we can conclude that $\neg\texttt{zf}$ is false, and hence that zf is true. This value of zf, along with the fact that $\texttt{zf} \Leftrightarrow (\texttt{eax} = 2)$ is true, enables us to determine that $A'' = \mu\widetilde{\alpha}_{\mathcal{A}}(\texttt{eax} = 2) = \{\texttt{eax} - 2 = 0\}$ must hold. Thus, our final semantic-reduction step produces $A_3 = A_2 \sqcap A'' = \{\texttt{pc}' - \texttt{L4} = 0, \texttt{eax}' - \texttt{eax} = 0, \texttt{eax} - 2 = 0\}$.

Abstract value A_3 is a set of affine constraints over the registers at L1 (input-state variables) and those at L4 (output-state variables), and can be used for affine-relation analysis using standard techniques (e.g., see [19] or [12, §5]). □

The above example illustrates how our technique propagates truth values to various subformulas of φ. The process of repeatedly applying propagation rules to compute $\widetilde{\text{Assume}}$ is called 0-assume. The next example illustrates the *Dilemma Rule*, a more powerful rule for computing semantic reductions.

Example 2. Let $\mathcal{L}$ be QF_LRA, and let $\mathcal{A}$ be the polyhedral abstract domain [8]. Consider the formula $\psi = (a_0 < b_0) \wedge (a_0 < c_0) \wedge (b_0 < a_1 \vee c_0 < a_1) \wedge (a_1 < b_1) \wedge (a_1 < c_1) \wedge (b_1 < a_2 \vee c_2 < a_2) \wedge (a_2 < a_0) \in \mathcal{L}$ (see Fig. 1(a)). Suppose that we want to compute $\widetilde{\text{Assume}}[\psi](\top_{\mathcal{A}})$.

To make the communication between the truth values of subformulas and the abstract value explicit, we associate a fresh Boolean variable with each subformula of ψ to give a set of *integrity constraints* $\mathcal{I}$. In this case,

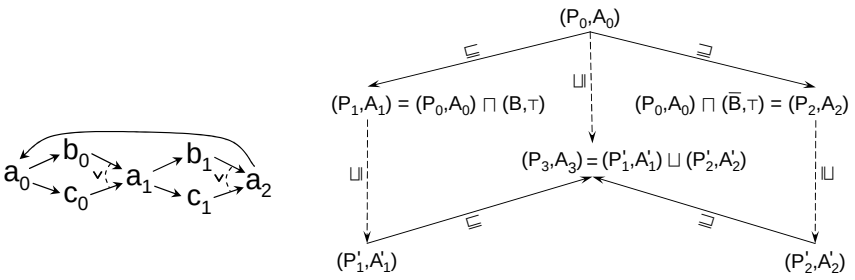


Fig. 1. (a) Inconsistent inequalities in the (unsatisfiable) formula used in Ex. 2. (b) Application of the Dilemma Rule to abstract value (P_0, A_0). The dashed arrows from (P_i, A_i) to (P'_i, A'_i) indicate that (P'_i, A'_i) is a semantic reduction of (P_i, A_i).

$\mathcal{I}_\psi = \{u_1 \Leftrightarrow \bigwedge_{i=2}^{8} u_i, u_2 \Leftrightarrow (a_0 < b_0), u_3 \Leftrightarrow (a_0 < c_0), u_4 \Leftrightarrow (u_9 \vee u_{10}), u_5 \Leftrightarrow (a_1 < b_1), u_6 \Leftrightarrow (a_1 < c_1), u_7 \Leftrightarrow (u_{11} \vee u_{12}), u_8 \Leftrightarrow (a_2 < a_0), u_9 \Leftrightarrow (b_0 < a_1), u_{10} \Leftrightarrow (c_0 < a_1), u_{11} \Leftrightarrow (b_1 < a_2), u_{12} \Leftrightarrow (c_1 < a_2)\}$. The integrity constraints encode the structure of ψ via the set of Boolean variables $\mathcal{U} = \{u_1, u_2, \ldots, u_{12}\}$. When $\mathcal{I}$ is used as a formula, it denotes the conjunction of the individual integrity constraints.

We now introduce an abstraction over $\mathcal{U}$; in particular, we use the Cartesian domain $\mathcal{P} = (\mathcal{U} \to \{0, 1, *\})_\perp$ in which $*$ denotes "unknown", and each element in $\mathcal{P}$ represents a set of assignments in $\mathbb{P}(\mathcal{U} \to \{0, 1\})$. We denote an element of the Cartesian domain as a mapping, e.g., $[u_1 \mapsto 0, u_2 \mapsto 1, u_3 \mapsto *]$, or $[0, 1, *]$ if u_1, u_2, and u_3 are understood. $\top_\mathcal{P}$ is the element $\lambda u.*$. The "single-point" partial assignment in which variable v is set to b is denoted by $\top_\mathcal{P}[v \mapsto b]$.

The variable $u_1 \in \mathcal{U}$ represents the root of ψ; consequently, the single-point partial assignment $\top_\mathcal{P}[u_1 \mapsto 1]$ corresponds to the assertion that ψ is satisfiable. In fact, the models of ψ are closely related to the concrete values in $[\![\mathcal{I}]\!] \cap \gamma(\top_\mathcal{P}[u_1 \mapsto 1])$. For every concrete value in $[\![\mathcal{I}]\!] \cap \gamma(\top_\mathcal{P}[u_1 \mapsto 1])$, its projection onto $\{a_i, b_i, c_i \mid 0 \le i \le 1\} \cup \{a_2\}$ gives us a model of ψ; that is, $[\![\psi]\!] = ([\![\mathcal{I}]\!] \cap \gamma(\top_\mathcal{P}[u_1 \mapsto 1]))|_{(\{a_i, b_i, c_i | 0 \le i \le 1\} \cup \{a_2\})}$. By this means, the problem of computing $\widetilde{\text{Assume}}[\psi](\top_\mathcal{A})$ is reduced to that of computing $\widetilde{\text{Assume}}[\mathcal{I}]((\top_\mathcal{P}[u_1 \mapsto 1], \top_\mathcal{A}))$, where $(\top_\mathcal{P}[u_1 \mapsto 1], \top_\mathcal{A})$ is an element of the reduced product of $\mathcal{P}$ and $\mathcal{A}$.

Because u_1 is true in $\top_\mathcal{P}[u_1 \mapsto 1]$, the integrity constraint $u_1 \Leftrightarrow \bigwedge_{i=2}^{8} u_i$ implies that $u_2 \ldots u_8$ are also true, which refines $\top_\mathcal{P}[u_1 \mapsto 1]$ to $P_0 = [1, 1, 1, 1, 1, 1, 1, 1, *, *, *, *]$. Because u_2 is true and $u_2 \Leftrightarrow (a_0 < b_0) \in \mathcal{I}$, $\top_\mathcal{A}$ can be refined using $\mu\widetilde{\alpha}_\mathcal{A}(a_0 < b_0) = \{a_0 - b_0 < 0\}$. Doing the same for u_3, u_5, u_6, and u_8, refines $\top_\mathcal{A}$ to $A_0 = \{a_0 - b_0 < 0, a_0 - c_0 < 0, a_1 - b_1 < 0, a_1 - c_1 < 0, a_2 - a_0 < 0\}$. These steps refine $(\top_\mathcal{P}[u_1 \mapsto 1], \top_\mathcal{A})$ to (P_0, A_0) via 0-assume.

To increase precision, we need to use the Dilemma Rule, a branch-and-merge rule, in which the current abstract state is split into two (disjoint) abstract states, 0-assume is applied to both abstract values, and the resulting abstract values are merged by performing a join. The steps of the Dilemma Rule are shown schematically in Fig. 1(b) and described below.

In our example, the value of u_9 is unknown in P_0. Let $B \in \mathcal{P}$ be $\top_{\mathcal{P}}[u_9 \mapsto 0]$; then $\overline{B}$, the abstract complement of B, is $\top_{\mathcal{P}}[u_9 \mapsto 1]$. Note that $\gamma(B) \cap \gamma(\overline{B}) = \emptyset$, and $\gamma(B) \cup \gamma(\overline{B}) = \gamma(\top)$. The current abstract value (P_0, A_0) is split into

$$(P_1, A_1) = (P_0, A_0) \sqcap (B, \top) \quad \text{and} \quad (P_2, A_2) = (P_0, A_0) \sqcap (\overline{B}, \top).$$

Now consider 0-assume on (P_1, A_1). Because u_9 is false, and u_4 is true, we can conclude that u_{10} has to be true, using the integrity constraint $u_4 \Leftrightarrow (u_9 \vee u_{10})$. Because u_{10} holds and $u_{10} \Leftrightarrow (c_0 < a_1) \in \mathcal{I}$, A_1 can be refined with the constraint $c_0 - a_1 < 0$. Because $a_0 - c_0 < 0 \in A_1$, $a_0 - a_1 < 0$ can be inferred. Similarly, when performing 0-assume on (P_2, A_2), $a_0 - a_1 < 0$ is inferred. Call the abstract values computed by 0-assume (P_1', A_1') and (P_2', A_2'), respectively. At this point, the join of (P_1', A_1') and (P_2', A_2') is taken. Because $a_0 - a_1 < 0$ is present in both branches, it is retained in the join. The resulting abstract value is $(P_3, A_3) = ([1, 1, 1, 1, 1, 1, 1, 1, *, *, *, *], \{a_0 - b_0 < 0, a_0 - c_0 < 0, a_1 - b_1 < 0, a_1 - c_1 < 0, a_2 - a_0 < 0, a_0 - a_1 < 0\}$. Note that although P_3 equals P_0, A_3 is strictly more precise than A_0 (i.e., $A_3 \sqsubset A_0$), and hence (P_3, A_3) is a semantic reduction of (P_0, A_0) with respect to ψ.

Now suppose (P_3, A_3) is split using u_{11}. Using reasoning similar to that performed above, $a_1 - a_2 < 0$ is inferred on both branches, and hence so is $a_0 - a_2 < 0$. However, $a_0 - a_2 < 0$ contradicts $a_2 - a_0 < 0$; consequently, the abstract value reduces to $(\bot_{\mathcal{P}}, \bot_{\mathcal{A}})$ on both branches. Thus, $\widetilde{\text{Assume}}[\psi](\top_{\mathcal{A}}) = \bot_{\mathcal{A}}$, and hence ψ is unsatisfiable. In this way, $\widetilde{\text{Assume}}$ instantiated with the polyhedral domain can be used to decide the satisfiability of a QF_LRA formula. □

The process of repeatedly applying the Dilemma Rule is called 1-assume. That is, repeatedly some variable $u \in \mathcal{U}$ is selected whose truth value is unknown, the current abstract value is split using $B = \top_{\mathcal{P}}[u \mapsto 0]$ and $\overline{B} = \top_{\mathcal{P}}[u \mapsto 1]$, 0-assume is applied to each of these values, and the resulting abstract values are merged via join (Fig. 1(b)). Different policies for selecting the next variable on which to split can affect how quickly an answer is found; however, any fair selection policy will return the same answer. The efficacy of the Dilemma Rule is partially due to case-splitting; however, the real power of the Dilemma Rule is due to the fact that it *preserves information learned in both branches when a case-split is "abandoned" at a join point.*

The generalization of the 1-assume algorithm is called k-assume: repeatedly some variable $u \in \mathcal{U}$ is selected whose truth value is unknown, the current abstract value is split using $B = \top_{\mathcal{P}}[u \mapsto 0]$ and $\overline{B} = \top_{\mathcal{P}}[u \mapsto 1]$; (k–1)-assume is applied to each of these values; and the resulting values are merged via join. However, there is a trade-off: higher values of k give greater precision, but are also computationally more expensive.

For certain abstract domains and logics, $\widetilde{\text{Assume}}[\psi](\top_{\mathcal{A}})$ is complete—i.e., with a high-enough value of k for k-assume, $\widetilde{\text{Assume}}[\psi](\top_{\mathcal{A}})$ always computes the most-precise $\mathcal{A}$ value possible for ψ. However, our experiments show that $\widetilde{\text{Assume}}[\psi](\top_{\mathcal{A}})$ has very good precision with $k = 1$ (see §5)—which jibes with the observation that, in practice, with Stålmarck's method for propositional validity (tautology) checking "a formula is either [provable with $k = 1$] or not a tautology at all!" [18, p. 227].

Algorithm 1: $\widetilde{\text{Assume}}[\varphi](A)$

```
1 ⟨I, u_φ⟩ ← integrity(φ)
2 P ← ⊤_P[u_φ ↦ 1]
3 (P̃, Ã) ← k-assume[I]((P, A))
4 return Ã
```

Algorithm 2: 0-assume$[\mathcal{I}]((P, A))$

```
1  repeat
2    (P′, A′) ← (P, A)
3    foreach J ∈ I do
4      if J has the form u ⇔ ℓ then
5        (P, A) ← LeafRule(J, (P, A))
6      else
7        (P, A) ← InternalRule(J, (P, A))
8  until ((P, A) = (P′, A′)) ∥ timeout
9  return (P, A)
```

Algorithm 3: k-assume$[\mathcal{I}]((P, A))$

```
1  repeat
2    (P′, A′) ← (P, A)
3    foreach u ∈ U such that P(u) = ∗ do
4      (P_0, A_0) ← (P, A)
5      (B, B̄) ← (⊤_P[u ↦ 0], ⊤_P[u ↦ 1])
6      (P_1, A_1) ← (P_0, A_0) ⊓ (B, ⊤)
7      (P_2, A_2) ← (P_0, A_0) ⊓ (B̄, ⊤)
8      (P′_1, A′_1) ← (k–1)-assume[I]((P_1, A_1))
9      (P′_2, A′_2) ← (k–1)-assume[I]((P_2, A_2))
10     (P, A) ← (P′_1, A′_1) ⊔ (P′_2, A′_2)
11 until ((P, A) = (P′, A′)) ∥ timeout
12 return (P, A)
```

$$\frac{\varphi := \ell \qquad \ell \in \text{literal}(\mathcal{L})}{u_\varphi \Leftrightarrow \ell \ \in \mathcal{I}}\ \textsc{Leaf} \qquad\qquad \frac{\varphi := \varphi_1 \text{op}\ \varphi_2}{u_\varphi \Leftrightarrow (u_{\varphi_1}\ \text{op}\ u_{\varphi_2})\ \in \mathcal{I}}\ \textsc{Internal}$$

Fig. 2. Rules used to convert a formula $\varphi \in \mathcal{L}$ into a set of integrity constraints $\mathcal{I}$. op represents any binary connective in $\mathcal{L}$, and literal$(\mathcal{L})$ is the set of atomic formulas and their negations.

3 Algorithm for $\widetilde{\text{Assume}}[\varphi](A)$

This section presents our algorithm for computing $\widetilde{\text{Assume}}[\varphi](A) \in \mathcal{A}$, for $\varphi \in \mathcal{L}$. The assumptions of our framework are as follows:

1. There is a Galois connection $\mathcal{C} \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \mathcal{A}$ between $\mathcal{A}$ and concrete domain $\mathcal{C}$.
2. There is an algorithm to perform the join of arbitrary elements of $\mathcal{A}$.
3. Given a literal $l \in \mathcal{L}$, there is an algorithm $\mu\widetilde{\alpha}$ to compute a safe (overapproximating) "micro-$\widetilde{\alpha}$"—i.e., $A' = \mu\widetilde{\alpha}(l)$ such that $\gamma(A') \supseteq [\![l]\!]$.
4. There is an algorithm to perform the meet of an arbitrary element of $\mathcal{A}$ with an arbitrary element of $\{\mu\widetilde{\alpha}(l) \mid \ell \in \text{literal}(\mathcal{L})\}$.

Note that $\mathcal{A}$ is allowed to have infinite descending chains; because $\widetilde{\text{Assume}}$ works from above, it is allowed to stop at any time, and the value in hand is an overapproximation of the most precise answer.

Alg. 1 presents the algorithm that computes $\widetilde{\text{Assume}}[\varphi](A)$ for $\varphi \in \mathcal{L}$ and $A \in \mathcal{A}$. Line (1) calls the function integrity, which converts φ into integrity constraints $\mathcal{I}$ by assigning a fresh Boolean variable to each subformula of φ, using the rules described in Fig. 2. The variable u_φ corresponds to formula φ. We use $\mathcal{U}$ to denote the set of Boolean variables created when converting φ to $\mathcal{I}$. Alg. 1 also uses a second abstract domain $\mathcal{P}$, each of whose elements represents a set of Boolean assignments in $\mathbb{P}(\mathcal{U} \to \{0, 1\})$. For simplicity, in this paper $\mathcal{P}$ is the Cartesian domain $(\mathcal{U} \to \{0, 1, *\})_\perp$, but other more-expressive Boolean domains could be used [35].

On line (2) of Alg. 1, an element of $\mathcal{P}$ is created in which u_φ is assigned the value 1, which asserts that φ is true. Alg. 1 is parameterized by the value of k (where $k \geq 0$). Let $\gamma_{\mathcal{I}}((P, A))$ denote $\gamma((P, A)) \cap [\![\mathcal{I}]\!]$. The call to k-assume on line (3) returns $(\widetilde{P}, \widetilde{A})$, which is a semantic reduction of (P, A) with respect to $\mathcal{I}$; that is, $\gamma_{\mathcal{I}}((\widetilde{P}, \widetilde{A})) = \gamma_{\mathcal{I}}((P, A))$ and $(\widetilde{P}, \widetilde{A}) \sqsubseteq (P, A)$. In general, the greater the value of k, the more precise is the result computed by Alg. 1. The next theorem states that Alg. 1 computes an over-approximation of Assume$[\varphi](A)$.

Theorem 1 ([36]). *For all $\varphi \in \mathcal{L}, A \in \mathcal{A}$, if $\widetilde{A} = A\widetilde{ssume}[\varphi](A)$, then $\gamma(\widetilde{A}) \supseteq [\![\varphi]\!] \cap \gamma(A)$, and $\widetilde{A} \sqsubseteq A$.* □

Alg. 3 presents the algorithm to compute k-assume, for $k \geq 1$. Given the integrity constraints $\mathcal{I}$, and the current abstract value (P, A), k-assume$[\mathcal{I}]((P, A))$ returns an abstract value that is a semantic reduction of (P, A) with respect to $\mathcal{I}$. The crux of the computation is the inner loop body, lines (4)–(10), which implements an analog of the Dilemma Rule from Stålmarck's method [33].

The steps of the Dilemma Rule are shown schematically in Fig. 1(b). At line (3) of Alg. 3, a Boolean variable u whose value is unknown is chosen. $B = \top_{\mathcal{P}}[u \mapsto 0]$ and its complement $\overline{B} = \top_{\mathcal{P}}[u \mapsto 1]$ are used to split the current abstract value (P_0, A_0) into two abstract values $(P_1, A_1) = (P, A) \sqcap (B, \top)$ and $(P_2, A_2) = (P, A) \sqcap (\overline{B}, \top)$, as shown in lines (6) and (7).

The calls to (k–1)-assume at lines (8) and (9) compute semantic reductions of (P_1, A_1) and (P_2, A_2) with respect to $\mathcal{I}$, which creates (P_1', A_1') and (P_2', A_2'), respectively. Finally, at line (10) (P_1', A_1') and (P_2', A_2') are merged by performing a join. (The result is labeled (P_3, A_3) in Fig. 1(b).)

The steps of the Dilemma Rule (Fig. 1(b)) are repeated until a fixpoint is reached, or some resource bound is exceeded. The next theorem states that k-assume$[\mathcal{I}]((P, A))$ computes a semantic reduction of (P, A) with respect to $\mathcal{I}$.

Theorem 2 ([36]). *For all $P \in \mathcal{P}$ and $A \in \mathcal{A}$, if $(P', A') =$ k-assume$[\mathcal{I}]((P, A))$, then $\gamma_{\mathcal{I}}((P', A')) = \gamma_{\mathcal{I}}((P, A))$ and $(P', A') \sqsubseteq (P, A)$.* □

Alg. 2 describes the algorithm to compute 0-assume: given the integrity constraints $\mathcal{I}$, and an abstract value (P, A), 0-assume$[\mathcal{I}]((P, A))$ returns an abstract value (P', A') that is a semantic reduction of (P, A) with respect to $\mathcal{I}$. It is in this algorithm that information is passed between the component abstract values $P \in \mathcal{P}$ and $A \in \mathcal{A}$ via *propagation rules*, like the ones shown in Figs. 3 and 4. In lines (4)–(7) of Alg. 2, these rules are applied by using a single integrity constraint in $\mathcal{I}$ and the current abstract value (P, A).

Given $J \in \mathcal{I}$ and (P, A), the net effect of applying any of the propagation rules is to compute a semantic reduction of (P, A) with respect to $J \in \mathcal{I}$. The propagation rules used in Alg. 2 can be classified into two categories:

1. Rules that apply on line (7) when J is of the form $p \Leftrightarrow (q \text{ op } r)$, shown in Fig. 3. Such an integrity constraint is generated from each internal subformula of formula φ. These rules compute a non-trivial semantic reduction of P with respect to J by only using information from P. For instance, rule

$$\frac{J = (u_1 \Leftrightarrow (u_2 \vee u_3)) \in \mathcal{I} \qquad P(u_1) = 0}{(P \sqcap \top[u_2 \mapsto 0, u_3 \mapsto 0], A)} \textsc{Or1}$$

$$\frac{J = (u_1 \Leftrightarrow (u_2 \wedge u_3)) \in \mathcal{I} \qquad P(u_1) = 1}{(P \sqcap \top[u_2 \mapsto 1, u_3 \mapsto 1], A)} \textsc{And1}$$

Fig. 3. Boolean rules used by Alg. 2 in the call InternalRule$(J, (P, A))$.

$$\frac{J = (u \Leftrightarrow l) \in \mathcal{I} \qquad P(u) = 1}{(P, A \sqcap \mu\widetilde{\alpha}_{\mathcal{A}}(l))} \textsc{PtoA-1} \qquad \frac{J = (u \Leftrightarrow l) \in \mathcal{I} \qquad P(u) = 0}{(P, A \sqcap \mu\widetilde{\alpha}_{\mathcal{A}}(\neg l))} \textsc{PtoA-0}$$

$$\frac{J = (u \Leftrightarrow \ell) \in \mathcal{I} \qquad A \sqcap \mu\widetilde{\alpha}_{\mathcal{A}}(l) = \bot_{\mathcal{A}}}{(P \sqcap \top[u \mapsto 0], A)} \textsc{AtoP-0}$$

Fig. 4. Rules used by Alg. 2 in the call LeafRule$(J, (P, A))$.

And1 says that if J is of the form $p \Leftrightarrow (q \wedge r)$, and p is 1 in P, then we can infer that both q and r must be 1. Thus, $P \sqcap \top[q \mapsto 1, r \mapsto 1]$ is a semantic reduction of P with respect to J. (See Ex. 1, step 1.)

2. Rules that apply on line (5) when J is of the form $u \Leftrightarrow \ell$, shown in Fig. 4. Such an integrity constraint is generated from each leaf of the original formula φ. This category of rules can be further subdivided into
   (a) Rules that propagate information from abstract value P to abstract value A; viz., rules PtoA-0 and PtoA-1. For instance, rule PtoA-1 states that given $J = u \Leftrightarrow l$, and $P(u) = 1$, then $A \sqcap \mu\widetilde{\alpha}(l)$ is a semantic reduction of A with respect to J. (See Ex. 1, steps 2 and 3.)
   (b) Rule AtoP-0, which propagates information from abstract value A to abstract value P. Rule AtoP-0 states that if $J = (u \Leftrightarrow \ell)$ and $A \sqcap \mu\widetilde{\alpha}(l) = \bot_{\mathcal{A}}$, then we can infer that u is false. Thus, the value of $P \sqcap \top[u \mapsto 0]$ is a semantic reduction of P with respect to J. (See Ex. 1, step 4.)

Alg. 2 repeatedly applies the propagation rules until a fixpoint is reached, or some resource bound is reached. The next theorem states that 0-assume computes a semantic reduction of (P, A) with respect to $\mathcal{I}$.

Theorem 3 ([36]). *For all $P \in \mathcal{P}, A \in \mathcal{A}$, if $(P', A') = 0\text{-assume}[\mathcal{I}]((P, A))$, then $\gamma_{\mathcal{I}}((P', A')) = \gamma_{\mathcal{I}}((P, A))$ and $(P', A') \sqsubseteq (P, A)$.* □

4 Instantiations

In this section, we describe instantiations of our framework for two logical-language/abstract-domain pairs: QF_BV/KS and QF_LRA/Polyhedra. We say that an $\widetilde{\text{Assume}}$ algorithm is *complete* for a logic $\mathcal{L}$ and abstract domain $\mathcal{A}$ if it is guaranteed to compute the best value $\widehat{\text{Assume}}[\varphi](A)$ for $\varphi \in \mathcal{L}$ and $A \in \mathcal{A}$. We give conditions under which the two instantiations are complete.

Bitvector Affine-Relation Domain (QF_BV/KS). King and Søndergaard [21] gave an algorithm for $\widehat{\alpha}$ for an abstract domain of Boolean affine relations. Elder et al. [12] extended the algorithm to arithmetic modulo 2^w (i.e., bitvectors of width w). Both algorithms work from below, making repeated calls on a SAT solver (King and Søndergaard) or an SMT solver (Elder et al.), performing joins to create increasingly better approximations of the desired answer. We call this family of domains KS, and call the (generalized) algorithm $\widehat{\alpha}^{\uparrow}_{\mathrm{KS}}$.

Given a literal $l \in$ QF_BV, we compute $\mu\widetilde{\alpha}_{\mathrm{KS}}(l)$ by invoking $\widehat{\alpha}^{\uparrow}_{\mathrm{KS}}(l)$. That is, we harness $\widehat{\alpha}^{\uparrow}_{\mathrm{KS}}$ in service of $\widetilde{\mathrm{Assume}}_{\mathrm{KS}}$, but only for $\mu\widetilde{\alpha}_{\mathrm{KS}}$, which means that $\widehat{\alpha}^{\uparrow}_{\mathrm{KS}}$ is only applied to literals. If an invocation of $\widehat{\alpha}^{\uparrow}_{\mathrm{KS}}$ does not return an answer within a specified time limit, we use $\top_{\mathrm{KS}}$.

Alg. 1 is not complete for QF_BV/KS. Let x be a bitvector of width 2, and let $\varphi = (x \neq 0 \wedge x \neq 1 \wedge x \neq 2)$. Thus, $\widehat{\mathrm{Assume}}[\varphi](\top_{\mathrm{KS}}) = \{x - 3 = 0\}$. The KS domain is not expressive enough to represent disequalities. For instance, $\mu\widetilde{\alpha}(x \neq 0)$ equals $\top_{\mathrm{KS}}$. Because Alg. 1 considers only a single integrity constraint at a time, we get $\widetilde{\mathrm{Assume}}[\varphi](\top_{\mathrm{KS}}) = \mu\widetilde{\alpha}(x \neq 0) \sqcap \mu\widetilde{\alpha}(x \neq 1) \sqcap \mu\widetilde{\alpha}(x \neq 2) = \top_{KS}$.

The current approach can be made complete for QF_BV/KS by making 0-assume consider multiple integrity constraints during propagation (in the limit, having to call $\mu\widetilde{\alpha}(\varphi)$). For the affine subset of QF_BV, an alternative approach would be to perform a 2^w-way split on the KS value each time a disequality is encountered, where w is the bit-width—in effect, rewriting $x \neq 0$ to $(x = 1 \vee x = 2 \vee x = 3)$. Furthermore, if there is a $\mu\widetilde{\mathrm{Assume}}$ operation, then the second approach can be extended to handle all of QF_BV: $\mu\widetilde{\mathrm{Assume}}[\ell](A)$ would be used to take the current KS abstract value A and a literal ℓ, and return an over-approximation of $\widetilde{\mathrm{Assume}}[\ell](A)$. All these approaches would be prohibitively expensive. Our current approach, though theoretically not complete, works very well in practice (see §5).

Polyhedral Domain (QF_LRA/Polyhedra). The second instantiation that we implemented is for the logic QF_LRA and the polyhedral domain [8]. Because a QF_LRA disequality $t \neq 0$ can be normalized to $(t < 0 \vee t > 0)$, every literal l in a normalized QF_LRA formula is merely a half-space in the polyhedral domain. Consequently, $\mu\widetilde{\alpha}_{\mathrm{Polyhedra}}(l)$ is exact, and easy to compute. Furthermore, because of this precision, the $\widetilde{\mathrm{Assume}}$ algorithm is complete for QF_LRA/Polyhedra. In particular, if $k = |\varphi|$, then k-assume is sufficient to guarantee that $\widetilde{\mathrm{Assume}}[\varphi](A)$ returns $\widehat{\mathrm{Assume}}[\varphi](A)$. For polyhedra, our implementation uses PPL [28].

The observation in the last paragraph applies in general: if $\mu\widetilde{\alpha}_{\mathcal{A}}(l)$ is exact for all literals $l \in \mathcal{L}$, then Alg. 1 is complete for logic $\mathcal{L}$ and abstract domain $\mathcal{A}$.

5 Experiments

Bitvector Affine-Relation Analysis (ARA). We compare two methods for computing the abstract transformers for the KS domain for ARA [21]:

- the $\widehat{\alpha}^{\uparrow}$-based procedure described in Elder et al. [12].
- the $\widetilde{\alpha}$-based procedure described in this paper ("$\widetilde{\alpha}^{\downarrow}$"), instantiated for KS.

Prog.	Measures of size				$\widehat{\alpha}^{\uparrow}$ Performance			
name	instrs	CFGs	BBs	brs	WPDS	t/o	post*	query
finger	532	18	298	48	110.9	4	0.266	0.015
subst	1093	16	609	74	204.4	4	0.344	0.016
label	1167	16	573	103	148.9	2	0.344	0.032
chkdsk	1468	18	787	119	384.4	16	0.219	0.031
convert	1927	38	1013	161	289.9	9	1.047	0.062
route	1982	40	931	243	562.9	14	1.281	0.046
logoff	2470	46	1145	306	621.1	16	1.938	0.063
setup	4751	67	1862	589	1524.7	64	0.968	0.047

Fig. 5. WPDS experiments ($\widehat{\alpha}^{\uparrow}$). The columns show the number of instructions (instrs); the number of procedures (CFGs); the number of basic blocks (BBs); the number of branch instructions (brs); the times, in seconds, for WPDS construction with $\widehat{\alpha}^{\uparrow}_{\mathrm{KS}}$ weights, running post*, and finding one-vocabulary affine relations at blocks that end with branch instructions (query). The number of basic blocks for which $\widehat{\alpha}^{\uparrow}_{\mathrm{KS}}$-weight generation timed out is listed under "t/o".

Our experiments were designed to answer the following questions:

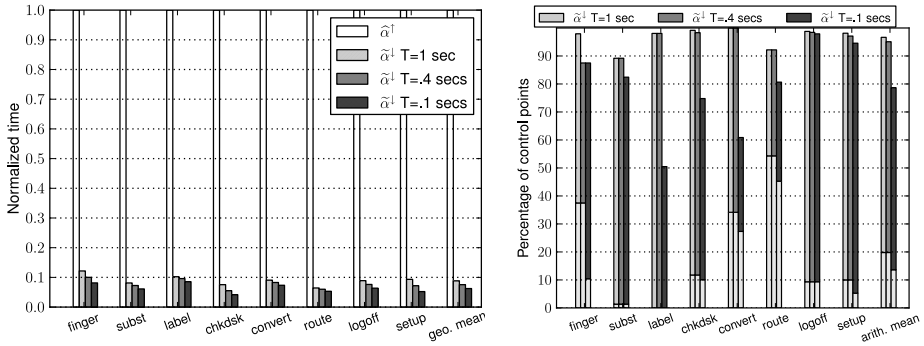
1. How does the speed of $\widetilde{\alpha}^{\downarrow}$ compare with that of $\widehat{\alpha}^{\uparrow}$?
2. How does the precision of $\widetilde{\alpha}^{\downarrow}$ compare with that of $\widehat{\alpha}^{\uparrow}$?

To address these questions, we performed ARA on x86 machine code, computing affine relations over the x86 registers. Our experiments were run on a single core of a quad-core 3.0 GHz Xeon computer running 64-bit Windows XP (SP2), configured so that a user process has 4GB of memory. We analyzed a corpus of Windows utilities using the WALi [20] system for weighted pushdown systems (WPDSs). For the baseline $\widehat{\alpha}^{\uparrow}$-based analysis we used a weight domain of $\widehat{\alpha}^{\uparrow}$-generated KS transformers. The weight on each WPDS rule encodes the KS transformer for a basic block B of the program, including a jump or branch to a successor block. A formula φ_B is created that captures the concrete semantics of B, and then the KS weight for B is obtained by performing $\widehat{\alpha}^{\uparrow}(\varphi_B)$ (cf. Ex. 1). We used EWPDS merge functions [24] to preserve caller-save and callee-save registers across call sites. The post* query used the FWPDS algorithm [23].

Fig. 5 lists several size parameters of the examples (number of instructions, procedures, basic blocks, and branches) along with the times for constructing abstract transformers and running post*.[4] Col. 6 of Fig. 5 shows that the calls to $\widehat{\alpha}^{\uparrow}$ during WPDS construction dominate the total time for ARA.

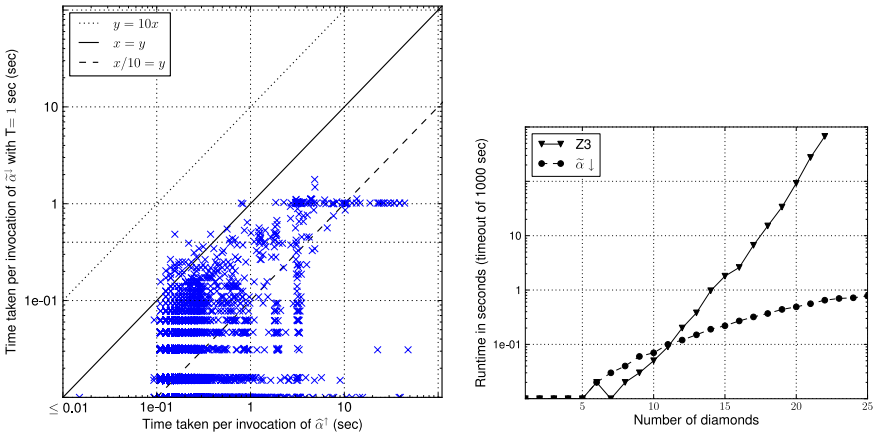
Each call to $\widehat{\alpha}^{\uparrow}$ involves repeated invocations of an SMT solver. Although the overall time taken by $\widehat{\alpha}^{\uparrow}$ is not limited by a timeout, we use a 3-second timeout for each invocation of the SMT solver (as in Elder et al. [12]). Fig. 5 lists the number of such SMT solver timeouts for each benchmark. In case the invocation of the SMT solver times out, $\widehat{\alpha}^{\uparrow}$ is forced to return $\top_{\mathrm{KS}}$ in order to be sound. (Consequently, it is possible for $\widetilde{\alpha}^{\downarrow}$ to return a more precise answer than $\widehat{\alpha}^{\uparrow}$.)

[4] Due to the high cost of the $\widehat{\alpha}^{\uparrow}$-based WPDS construction, all analyses excluded the code for libraries. Because register `eax` holds the return value from a call, library functions were modeled approximately (albeit unsoundly, in general) by "`havoc(eax)`".



(a) (b)

Fig. 6. (a) Performance: $\widetilde{\alpha}^{\downarrow}$ vs. $\widehat{\alpha}^{\uparrow}$. (b) Precision: % of control points at which $\widetilde{\alpha}^{\downarrow}$ has as good or better precision as $\widehat{\alpha}^{\uparrow}$; the lighter-color lower portion of each bar indicates the % of control points at which the precision is strictly greater for $\widetilde{\alpha}^{\downarrow}$.



(a) (b)

Fig. 7. (a) Log-log scatter plot of transformer-construction time. (b) Semilog plot of Z3 vs. $\widetilde{\alpha}^{\downarrow}$ on χ_d formulas.

The setup for the $\widetilde{\alpha}^{\downarrow}$-based analysis is the same as the baseline $\widehat{\alpha}^{\uparrow}$-based analysis, except that we call $\widetilde{\alpha}^{\downarrow}$ when calculating the KS weight for a basic block. We use 1-assume in this experiment. Each basic-block formula φ_B is rewritten to a set of integrity constraints, with ITE-terms rewritten as illustrated in Ex. 1. The priority of a Boolean variable is its postorder-traversal number, and is used to select which variable is used in the Dilemma Rule. We bound the total time taken by each call to $\widetilde{\alpha}^{\downarrow}$ to a fixed timeout $\mathtt{T}$. Note that even when the call to $\widetilde{\alpha}^{\downarrow}$ times out, it can still return a sound non-$\top_{\mathrm{KS}}$ value. We ran $\widetilde{\alpha}^{\downarrow}$ using $\mathtt{T} = 1$ sec, $\mathtt{T} = 0.4$ secs, and $\mathtt{T} = 0.1$ secs.

Fig. 6(a) shows the normalized time taken for WPDS construction when using $\widetilde{\alpha}^{\downarrow}$ with $\mathtt{T} = 1$ sec, $\mathtt{T} = 0.4$ secs, and $\mathtt{T} = 0.1$ secs. The running time is normalized to the corresponding time taken by $\widehat{\alpha}^{\uparrow}$; lower numbers are better.

WPDS construction using $\widetilde{\alpha}^{\downarrow}$ with $\mathtt{T} = 1$ sec. is about 11.3 times faster than $\widehat{\alpha}^{\uparrow}$ (computed as the geometric mean), which answers question 1.

Decreasing the timeout $\mathtt{T}$ makes the $\widetilde{\alpha}^{\downarrow}$ WPDS construction only slightly faster: on average, going from $\mathtt{T} = 1$ sec. to $\mathtt{T} = .4$ secs. reduces WPDS construction time by only 17% (computed as the geometric mean). To understand this behavior better, we show in Fig. 7(a) a log-log scatter-plot of the times taken by $\widehat{\alpha}^{\uparrow}$ versus the times taken by $\widetilde{\alpha}^{\downarrow}$ (with $\mathtt{T} = 1$ sec.), to generate the transformers for each basic block in the benchmark suite. As shown in Fig. 7(a), the times taken by $\widetilde{\alpha}^{\downarrow}$ are bounded by 1 second. (There are a few calls that take more than 1 second; they are an artifact of the granularity of operations at which we check whether the procedure has timed out.) Most of the basic blocks take less than 0.4 seconds, which explains why the overall time for WPDS construction does not decrease much as we decrease $\mathtt{T}$ in Fig. 6(a). We also see that the $\widehat{\alpha}^{\uparrow}$ times are not bounded, and can be as high as 50 seconds.

To answer question 2 we compared the precision of the WPDS analysis when using $\widetilde{\alpha}^{\downarrow}$ with $\mathtt{T}$ equal to 1, 0.4, and 0.1 seconds with the precision obtained using $\widehat{\alpha}^{\uparrow}$. In particular, we compare the affine relations (i.e., invariants) computed by the $\widetilde{\alpha}^{\downarrow}$-based and $\widehat{\alpha}^{\uparrow}$-based analyses for each *control point*—i.e., the beginning of a basic block that ends with a branch. Fig. 6(b) shows the percentage of control points for which the $\widetilde{\alpha}^{\downarrow}$-based analysis computes a better (tighter) or equally precise affine relation. On average, when using $\mathtt{T}$= 1 sec, $\widetilde{\alpha}^{\downarrow}$-based analysis computes an *equally precise* invariant at 76.9% of the control points (computed as the arithmetic mean). Interestingly, the $\widetilde{\alpha}^{\downarrow}$-based analysis computes an answer that is *more precise* compared to that computed by the $\widehat{\alpha}^{\uparrow}$-based analysis. That is not a bug in our implementation; it happens because $\widehat{\alpha}^{\uparrow}$ has to return $\top_{\mathrm{KS}}$ when the call to the SMT solver times out. In Fig. 6(b), the lighter-color lower portion of each bar shows the percentage of control points for which $\widetilde{\alpha}^{\downarrow}$-based analysis provides strictly more precise invariants when compared to $\widehat{\alpha}^{\uparrow}$-based analysis; on average, $\widetilde{\alpha}^{\downarrow}$-based analysis is more precise for 19.8% of the control points (arithmetic mean, for $\mathtt{T} = 1$ second). $\widetilde{\alpha}^{\downarrow}$-based analysis is less precise at only 3.3% of the control points. Furthermore, as expected, when the timeout for $\widetilde{\alpha}^{\downarrow}$ is reduced, the precision decreases.

Satisfiability Checking. The formula used in Ex. 2 is just one instance of a family of unsatisfiable QF_LRA formulas [25]. Let $\chi_d = (a_d < a_0) \wedge \bigwedge_{i=0}^{d-1}((a_i < b_i) \wedge (a_i < c_i) \wedge ((b_i < a_{i+1}) \vee (c_i < a_{i+1})))$. The formula ψ in Ex. 2 is χ_2; that is, the number of "diamonds" is 2 (see Fig. 1(a)). We used the QF_LRA/Polyhedra instantiation of our framework to check whether $\widetilde{\alpha}(\chi_d) = \bot$ for $d = 1 \ldots 25$ using 1-assume. We ran this experiment on a single processor of a 16-core 2.4 GHz Intel Zeon computer running 64-bit RHEL Server release 5.7. The semilog plot in Fig. 7(b) compares the running time of $\widetilde{\alpha}^{\downarrow}$ with that of Z3, version 3.2 [11]. The time taken by Z3 increases exponentially with d, exceeding the timeout threshold of 1000 seconds for $d = 23$. This corroborates the results of a similar experiment conducted by McMillan et al. [25], where the reader can also find an in-depth explanation of this behavior.

On the other hand, the running time of $\widetilde{\alpha}^{\downarrow}$ increases linearly with d taking 0.78 seconds for $d = 25$. The cross-over point is $d = 12$. In Ex. 2, we saw how two successive applications of the Dilemma Rule suffice to prove that ψ is unsatisfiable. That explanation generalizes to χ_d: d applications of the Dilemma Rule are sufficient to prove unsatisfiability of χ_d. The order in which Boolean variables with unknown truth values are selected for use in the Dilemma Rule has no bearing on this linear behavior, as long as no variable is starved from being chosen (i.e., a fair-choice schedule is used). Each application of the Dilemma Rule is able to infer that $a_i < a_{i+1}$ for some i.

We do not claim that $\widetilde{\alpha}^{\downarrow}$ is better than mature SMT solvers such as Z3. We do believe that it represents another interesting point in the design space of SMT solvers, similar in nature to the GDPLL algorithm [25] and the k-lookahead technique used in the DPLL($\sqcup$) algorithm [4].

6 Applications to Other Symbolic Operations

The symbolic operations of $\widehat{\gamma}$ and $\widehat{\alpha}$ can be used to implement a number of other useful operations, as discussed below. In each case, over-approximations result if $\widehat{\alpha}$ is replaced by $\widetilde{\alpha}$.

- The operation of containment checking, $A_1 \sqsubseteq A_2$, which is needed by analysis algorithms to determine when a post-fixpoint is attained, can be implemented by checking whether $\widehat{\alpha}(\widehat{\gamma}(A_1) \wedge \neg\widehat{\gamma}(A_2))$ equals $\bot$.
- Suppose that there are two Galois connections $\mathcal{G}_1 = \mathcal{C} \underset{\alpha_1}{\overset{\gamma_1}{\leftrightarrows}} \mathcal{A}_1$ and $\mathcal{G}_2 = \mathcal{C} \underset{\alpha_2}{\overset{\gamma_2}{\leftrightarrows}} \mathcal{A}_2$, and one wants to work with the reduced product of $\mathcal{A}_1$ and $\mathcal{A}_2$ [7, §10.1]. The semantic reduction of a pair (A_1, A_2) can be performed by letting ψ be the formula $\widehat{\gamma}_1(A_1) \wedge \widehat{\gamma}_2(A_2)$, and creating the pair $(\widehat{\alpha}_1(\psi), \widehat{\alpha}_2(\psi))$.
- Given $A_1 \in \mathcal{A}_1$, one can find the most-precise value $A_2 \in \mathcal{A}_2$ that over-approximates A_1 in $\mathcal{A}_2$ as follows: $A_2 = \widehat{\alpha}_2(\widehat{\gamma}_1(A_1))$.
- Given a loop-free code fragment F, consisting of one or more blocks of program statements and conditions, one can obtain a representation of its best transformer by symbolically executing F to obtain a transition formula ψ_F, and then performing $\widehat{\alpha}(\psi_F)$.

7 Related Work

Extensions of Stålmarck's Method. Björk [3] describes extensions of Stålmarck's method to first-order logic. Like Björk, our work goes beyond the classical setting of Stålmarck's method [33] (i.e., propositional logic) and extends the method to more expressive logics, such as QF_LRA or QF_BV. However, Björk is concerned solely with validity checking, and—compared with the propositional case—the role of abstraction is less clear in his method. Our algorithm not only uses an abstract domain as an explicit datatype, the goal of the algorithm is to compute an abstract value $A' = \widetilde{\text{Assume}}[\varphi](A)$.

Our approach was influenced by Granger's method of using (in)equation solving as a way to implement semantic reduction and Assume as part of his

technique of *local decreasing iterations* [16]. Granger describes techniques for performing reductions with respect to (in)equations of the form $x_1 \bowtie F(x_1, \ldots, x_n)$ and $(x_1 * F(x_1, \ldots, x_n)) \bowtie G(x_1, \ldots, x_n)$, where $\bowtie$ stands for a single relational symbol of $\mathcal{L}$, such as $=$, $\neq$, $<$, $\leq$, $>$, $\geq$, or $\equiv$ (arithmetical congruence). Our framework is not limited to literals of these forms; all that we require is that for a literal $l \in \mathcal{L}$, there is an algorithm to compute an overapproximating value $\mu\widetilde{\alpha}(l)$. Moreover, Granger has no analog of the Dilemma Rule, nor does he present any completeness results (cf. §4).

SMT Solvers. Most methods for SMT solving can be classified according to whether they employ *lazy* or *eager* translations to SAT. (The SAT procedure then employed is generally based on the DPLL procedure [10, 9].) In contrast, the algorithm for SMT described in this paper is not based on a translation to SAT; instead, it generalizes Stålmarck's method for propositional logic to richer logics.

Lazy approaches abstract each atom of the input formula to a distinct propositional variable, use a SAT solver to find a propositional model, and then check that model against the theory [1, 13, 11]. The disadvantage of the lazy approach is that it cannot use theory information to prune the search. In contrast, our algorithm is able to use theory-specific information to make deductions—in particular, in the LeafRule function (Fig. 4) used in Alg. 2. The use of theory-specific information is the reason why our approach outperformed Z3, which uses the lazy approach, on the diamond example (§5).

Eager approaches [5, 34] encode more of the theory into the propositional formula that is given to the SAT solver, and hence are able to constrain the solution space with theory-specific information. The challenge in designing such solvers is to ensure that the propositional formula does not blow up in size. In our approach, such an explosion in the set of literals in the formula is avoided because our learned facts are restricted by the abstract domain in use.

A variant of the Dilemma Rule is used in DPLL($\sqcup$), and allows the theory solver in a lazy DPLL-based SMT solver to produce joins of facts deduced along different search paths. However, as pointed out by Bjørner et al. [4, §5], their system is weaker than Stålmarck's method, because Stålmarck's method can learn equivalences between literals.

Another difference between our work and existing approaches to SMT is the connection presented in this paper between Stålmarck's method and the computation of best abstract operations for abstract interpretation.

Best Abstract Operations. Several papers about best abstract operations have appeared in the literature [14, 29, 37, 21, 12]. Graf and Saïdi [14] showed that decision procedures can be used to generate best abstract transformers for predicate-abstraction domains. Other work has investigated more efficient methods to generate approximate transformers that are not best transformers, but approach the precision of best transformers [2, 6].

Several techniques work from *below* [29, 21, 12]—performing joins to incorporate more and more of the concrete state space—which has the drawback that if they are stopped before the final answer is reached, the most-recent approxima-

tion is an *under-approximation* of the desired value. In contrast, our technique works from *above*. It can stop at any time and return a safe answer.

Yorsh et al. [37] developed a method that works from above to perform $\widetilde{\text{Assume}}[\varphi](A)$ for the kind of abstract domains used in shape analysis (i.e., "canonical abstraction" of logical structures [30]). Their method has a splitting step, but no analog of the join step performed at the end of an invocation of the Dilemma Rule. In addition, their propagation rules are much more heavyweight.

Template Constraint Matrices (TCMs) are a parametrized family of linear-inequality domains for expressing invariants in linear real arithmetic. Sankaranarayanan et al. [31] gave a parametrized meet, join, and set of abstract transformers for all TCM domains. Monniaux [26] gave an algorithm that finds the best transformer in a TCM domain across a straight-line block (assuming that concrete operations consist of piecewise linear functions), and good transformers across more complicated control flow. However, the algorithm uses quantifier elimination, and no polynomial-time elimination algorithm is known for piecewise-linear systems.

Cover algorithms. Gulwani and Musuvathi [17] defined the "cover problem", which addresses *approximate existential quantifier elimination*: Given a formula φ in logic $\mathcal{L}$, and a set of variables V, find the strongest quantifier-free formula $\overline{\varphi}$ in $\mathcal{L}$ such that $[\![\exists V : \varphi]\!] \subseteq [\![\overline{\varphi}]\!]$. They presented cover algorithms for the theories of uninterpreted functions and linear arithmetic, and showed that covers exist in some theories that do not support quantifier elimination.

The notion of a cover has similarities to the notion of symbolic abstraction, but the two notions are distinct. Our technical report [36] discusses the differences in detail, describing symbolic abstraction as over-approximating a formula φ using an impoverished logic fragment, while a cover algorithm only removes variables V from the vocabulary of φ. The two approaches yield different over-approximations of φ, and the over-approximation obtained by a cover algorithm does not, in general, yield suitable abstract values and abstract transformers.

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