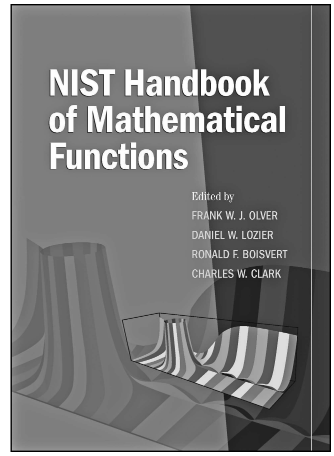
Handbooks of Mathematical Functions, Versions 1.0 and 2.0

Reviewed by Richard Beals



NIST Handbook of Mathematical Functions
Edited by Frank W. J. Olver, Daniel W. Lozier, Ronald F. Boisvert, and Charles W. Clark
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Who in the mathematical community uses handbooks of special functions, and why? And will the newest version make a difference?

By various objective measures, the handbook most used currently is the *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* [AS], universally known as "Abramowitz and Stegun" (A&S). Estimates of the number of copies sold range up to a million. Since 1997, when MathSciNet began compiling citation lists from papers reviewed in *Mathematical Reviews*, A&S has been acknowledged some 2,800 times, outdistancing other compendia by a factor of two (Gradshteyn-Ryzhik), three (Bateman Manuscript Project), five (Prudnikov-Brychkov-Marichev), and more.

So, more specifically, who has been using A&S? A check of a somewhat random list of forty analysts turned up a total of five such citations. Given the research areas and total number of publications, this turns out to be close to what one might predict. Per MSC major classification, the largest number of citations is in numerical analysis (313 as of mid-May), followed by partial differential equations (286). Fifteen classifications show one or fewer citations; manifolds and cell complexes is one of four classifications that show two citations. A more telling statistic is the ratio of papers to citations in each classification. It is no surprise that special functions leads, with about one citation per sixty books and papers. Areas in the one-in-200 to one-in-900 range include integral transforms, number theory, most of the applied mathematics areas, almost all the science classifications, dynamical systems, harmonic analysis, and differential and integral equations. (For manifolds and cell complexes the rate is about one in 12,500.)

This gives us a picture of who cites A&S. Why do they do so, apart from those who specialize in the subject of special functions? Although A&S contains material about combinatorics, which accounts for many of the number theory citations, the larger part is devoted to classical special functions. Most of these functions arose by proposing a model physical equation and separating variables in one coordinate system or another, thereby reducing the problem to a second-order ordinary differential equation such as Bessel's equation. We still often find that the simplest model of a given physical or mathematical problem leads to a classical equation and, therefore, a solution involving special functions. In ODE this includes turning points and some types of singularity. In PDE it includes problems of mixed type in aerodynamics and transport theory, hypoelliptic and degenerate elliptic problems, and weakly hyperbolic equations. An explicit solution of a model problem can point the way to an understanding of more general problems, allow for calculation of asymptotics, and so on—provided one has at hand enough information about the ingredients of the explicit solution. The list of such models continues to grow. Calculation of asymptotics also accounts for some of the usage in number theory.

Why and how might a successor to A&S be useful? Over half of the thousand pages of A&S are devoted to numerical tables (which is the reason that this writer eyed the Dover edition

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many times over many years before finally buying it). The ubiquity of computers and the Internet has made the tables largely obsolete. Most of the rest consists simply of lists of identities and formulas, some graphs, and some rather general references at the ends of chapters. There are competing works, but not as inexpensive, convenient, and easily available: Erdélyi et al. [EMOT] runs to three volumes, and Magnus et al. [MOS], the much more comprehensive successor to [MO], has long been out of print. But still, A&S is basically a cookbook. Some users must wonder whether and how the recipes hang together, where they come from, and how the formulas can be derived. (Such questions might even drive a late arrival to the subject to become involved in textbook writing.) There are recent textbooks and treatises, and though A&S is cited four or more times as often as any one of them, they may well be "used" more often. So, again, is a successor to the point?

A&S was the culmination of a decade-long project of the National Bureau of Standards, which has since been renamed the National Institute of Standards and Technology (NIST). A second decade–long project was undertaken under NIST auspices to update A&S. The result is a multimedia successor: the print and CD-ROM *NIST Handbook of Mathematical Functions* (NHMF) [OLBC1] and the online *Digital Library of Mathematical Functions* (DLMF) [OLBC2], edited by Frank W. J. Olver, Daniel W. Lozier, Ronald F. Boisvert, and Charles W. Clark.

It would take a genuine expert in special functions—or rather a committee of such experts—to give an adequate appraisal of this new effort. However, many of those who are most knowledgeable have been directly involved in the new production: four editors, ten associate editors, twenty-five validators, and twenty-nine authors (albeit with considerable overlap). Moreover, forty-seven members of NIST and fifty-one nonmembers are acknowledged by name as having "contributed to the project in a variety of ways." Therefore it has fallen to an amateur to discuss this production for the *Notices*.

Except for the chapters on constants and scales of notation and some of the material from the chapter on probability functions, all the nontabular content of A&S is incorporated in NHMF, generally in an expanded form. This includes the elementary functions and all the usual special functions, as well as Bernoulli and Euler polynomials and combinatorial analysis. The choice of additional material has been inspired by internal mathematical developments, by a broadening from classical analysis to related areas in algebra and number theory, and by such advances in mathematical physics as integrable models in continuum mechanics and statistical physics. There are new chapters on generalized hypergeometric functions, q–hypergeometric functions, multidimensional theta functions, Lamé functions, Heun functions, $3j$, $6j$, $9j$ symbols, Painlevé transcendents, and integrals with coalescing saddles.

The two methods chapters in A&S have been expanded to three: algebraic and analytic methods, asymptotic approximations, and numerical methods. In addition to definitions and brief summaries of standard facts from real and complex analysis, the first chapter introduces distributions and tempered distributions and gives a number of series and integral representations of the delta distribution. The second chapter is a comprehensive survey of old and new methods for asymptotics, with applications to differential and difference equations. The third chapter, on numerical methods, also contains new developments and is twice the size of the corresponding A&S chapter. The graphics are a major advance from what was feasible at the time of A&S, mostly three-dimensional views in color. (See, for example, the Bessel function graphs on pages 219–221.) Finally, the number of references is about 2,300, nearly an order of magnitude more than A&S, with much new work on asymptotics, approximations, location of zeros, and q–analysis.

All this suggests that NHMF will be a very useful replacement for A&S, with a wider audience. But, all in all, isn't it basically just another cookbook? The answer is yes, and no. Some cookbooks just have recipes, some instruct in the art of cooking, and some make your mouth water.

Consider the organization of a typical chapter. After a brief section on notation, there are sections on various functions, with subsections on definitions, graphics, representations and identities, zeros, integrals, sums, and asymptotics—very much in the spirit of A&S. Following this material is a section on applications—mathematical and physical. Sometimes brief, sometimes quite extensive, but generally very informative, these summaries indicate areas of use, with references. A section on computation contains brief discussions of methods, tables, and software, also with references. The final section contains general references for the chapter, followed by specific references, listed subsection by subsection. For each equation, either a specific reference or sketch of a derivation is given, sometimes noting corrections that need to be made in the source material.

NHMF positively invites browsing, in the methods chapters and throughout. The descriptions of applications and remarks on sources are a mine of information. Do you wonder what q-analysis is all about and where on earth it might be used? Do you want a quick view of combinatorics, Catalan numbers, Stirling numbers, and the like? Is there something you use a lot, and wonder if there is anything new to be learned about it? Are you intrigued by chapters with titles like "Integrals with

Coalescing Saddles" or "Functions with Matrix Argument"? If you have heard of—perhaps even used—Bessel functions, do you have any curiosity at all about Struwe functions and Heun functions? Do you know how the Chinese remainder theorem is used to facilitate numerical computations, or exactly how public key cryptography works? Are you surprised that a century-old topic like Painlevé transcendents has surfaced anew in a twenty-first-century handbook? Or a nearly two-century-old topic such as multidimensional theta functions? Would you like to see three-dimensional graphs of some familiar or less-familiar functions? It's all there, and more.

With a volume this large and ambitious—951 pages, double columned—it would be surprising to find nothing to second-guess, and perhaps even more surprising to find a review that does not indulge in some second-guessing … .

One of the goals of A&S was to further the standardization of notation, such as the use of M and U rather than Φ and Ψ for the Kummer functions. This remains a goal of NHMF. One innovation is particularly noted in the Mathematical Introduction: getting rid of the singularities of the hypergeometric function $F(a,b;c;x)$ by multiplying F by $1/\Gamma(c)$ to get $\mathbf{F}$:

$$\mathbf{F}(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{\Gamma(c+n)\,n!}\,x^n.$$

However, this choice is not used consistently in the relevant chapter. The sections on linear transformations and integral representations use $\mathbf{F}$, whereas those on quadratic transformations, differentiation formulas, and contiguous relations use F—even though the latter two sets of formulas simplify somewhat with $\mathbf{F}$. Similarly, in the confluent case, $\mathbf{M}(a,b,x) = M(a,b,x)/\Gamma(b)$ is introduced but not used throughout, and the companion possibility $\mathbf{U}(a,b,x) = U(a,b,x)/\Gamma(b-a)$ is overlooked entirely.

A less successful venture in A&S was to replace the conventional elliptic function indices k, k' by $m = k^2$ and $m_1 = (k')^2$, putting two chapters out of step with the rest of the world. NHMF returns to k, k' but continues to obscure the historic connection with integration of algebraic functions. The chapter on elliptic integrals systematically changes z to $\sin\phi$, while the chapter on Jacobi elliptic function begins by defining sn, cn, and dn in terms of θ functions.

The chapter on orthogonal polynomials in NHMF is much more extensive than the chapter in A&S. It now includes discrete orthogonal polynomials—not just the "classical" ones of Chebyshev, Charlier, Krawtchouk, and Meixner but all those in the Askey scheme, the additional continuous polynomials in that scheme, and all the q–versions as well. Unlike the orthogonal polynomial chapters in Erdélyi et al. [EMOT] and Magnus et al. [MO], [MOS], where the principal subdivision is by type of polynomial, the various sections here are each arranged by topic: orthogonality relations, series representations, recurrence relations, and so on. Looking for information on, say, Charlier polynomials, one starts with an index entry "*see* Hahn class …" and then has to insert C_n and various constants into schematic formulas spread over five pages. This Olympian view probably represents the way most experts see the subject, but those less expert may find it annoying. In a similar vein, some new notation is introduced to accompany some new normalizations, which are chosen so that in this version each of the classical discrete polynomials is exactly a generalized hypergeometric ${}_pF_q$. Nevertheless I found no mention of the older notations and normalizations. (A user-unfriendly approach is traditional in this area: see [Sz].)

While on the subject of notation and minor annoyances, I note that although the Mathematical Introduction to NHMF defines inf, sup, mod, and res, it does not define ph, which crops up later. It turns out that here, rather than referring to the concentration of hydrogen ions, ph replaces arg.

Quibbles aside, NHMF and the online version DLMF are a treasure for the mathematical and scientific communities, one that will be used and valued for decades. The organization, presentation, and general appearance are excellent. This beautiful book reflects credit on everyone and every organization involved: NIST; the National Science Foundation for funding; those who organized the project and obtained the funding; the advisors, editors, authors and validators; and Cambridge University Press. Above all, NHMF and DLMF are a monument to the efforts of the editor-in-chief, author of one chapter of A&S and author or coauthor of five chapters of this successor volume, Frank Olver.

A few final notes—for the history of the A&S project, see [Gr] and [BL]; for an early view of the DLMF/NHMF project, see Lozier [Lo] and for a later view [BCLO]; for two very personal viewpoints on handbooks and special functions, see Askey [As] and Wolfram [Wo]; for mathematical tables in general, see [CCFR]. And do look closely at the photograph and caption facing the title page of NHMF.

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