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Exact Solutions for The Space-Time Fractional SRLW and STO Equationsby The $\frac{D^{\alpha}G}{G}$-Expansion Method

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Abstract A new application of the remarkable $\frac{D^{\alpha}G}{G}$-expansion method based on a fractional order ordinary differential equation is used to find exact solutions of the space-time fractionalsymmetric regularized long wave (SRLW) equation and the space-time fractional Sharma-Tasso-Olver (STO) equation. This method involves Jumarie's modified Riemann-Liouville derivative and uses some of its basic properties. Exact solutions for both equations are obtained.

Keywords: *fractional differential equations, improved $D^{\alpha}G/G$ expansion method, Jumarie's modified Riemann-liouville derivative, SRLW equation, STO equation, analytical solutions*

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1. Introduction

Nonlinear fractional partial differential equations (FPDEs) are generalization of the classical nonlinear partial differential equations (PDEs) of integer order. In recent years, nonlinear FPDEs become one of the hottest topics for mathematician and other scientists because they are widely used to describe large number of new complex phenomena in many fields such as engineering, physics, biology, signal processing, systems identification, control theory, finance and others [1-9]. In the past, scientists defined and established a lot of powerful methods to find numerical and exact solutions of nonlinear FPDEs, such as the finite difference method [10,11], the finite element method [12,13,14], the Adomian decomposition method [15,16], the variational iteration method [17,18,19,20], the homotopy perturbation method [21,22], the fractional sub-equation method [23,24,25], the $\frac{G'}{G}$-expansion method [26] and many others.

In this paper, we will apply the $\frac{D^{\alpha}G}{G}$-expansion method [26], which is an improvement of the fractional $\frac{G'}{G}$-expansion method, to solve two nonlinear FPDEs, namely SRLW and STO equations. The fractional derivatives in these equations are described in the sense of Jumarie's modified Riemann-Liouville derivative which is defined as follows:

$$D_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-z)^{-\alpha-1} \left[f(z) - f(0) \right] dz, \\ \qquad \alpha < 0, \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-z)^{-\alpha} \left[f(z) - f(0) \right] dz, \\ \qquad 0 < \alpha < 1, \\ \left(D_t^{(\alpha-n)} f(t) \right)^{(n)} = D_t^{(\alpha-n)} \left(f^{(n)}(t) \right), \\ \qquad 1 \le n \le \alpha < n+1, n \in \mathbb{N}, \end{cases}$$

where the Gamma function is defined for $\Re(z) > 0$ by $\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} \, dt.$

Using simple calculations, we can obtain

$$\Gamma(z+1) = z\Gamma(z), \Gamma(z+1) = z!$$

Here we summarize some basic properties of the Jumarie's modified Riemann-Liouville derivative:

$$D_x^{\alpha} x^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}, \beta > 0 \quad \text{and } D_x^{\alpha} c = 0, c \text{ is acostant}. \tag{1}$$

$$D_x^{\alpha} \left(c f(x) + g(x) \right) = c\, D_x^{\alpha} f(x) + D_x^{\alpha} g(x), \quad c \text{ is a constant}. \tag{2}$$

$$D_x^{\alpha} \left(f(x) g(x) \right) = g(x) D_x^{\alpha} f(x) + f(x) D_x^{\alpha} g(x). \tag{3}$$

$$D_x^{\alpha}(f\circ g)(x)=D_x^{\alpha}f(g(x))$$
$$=f_g^{'}(g(x))D_x^{\alpha}g(x)=D_g^{\alpha}f(g(x))(g'(x))^{\alpha}. \quad (4)$$

2. Description of The $\frac{D^{\alpha}G}{G}$ Expansion Method

Step 1. Assume that we have the following nonlinear FPDE in the form:

$$P\left(u,u_x,u_t,D_x^{\alpha}u,D_t^{\alpha}u,\ldots\right)=0,\ 0<\alpha\le 1, \quad (5)$$

where $D_x^{\alpha}u$ and $D_t^{\alpha}u$ are Jumarie's modified Riemann-Liouville derivatives of $u, u=u(x,t)$ is an unknown function, P is a polynomial in u and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved.

Step 2. Using the wave transformation:

$$u(x,t)=U(z),\ z=k\,x+c\,t, \quad (6)$$

where k and c are constants to be determined later, the nonlinear FPDE in Eq. (5) is reduced to the following nonlinear fractional ordinary differential equation (FODE) for $U=U(z)$:

$$P\left(U,kU',cU',k^{\alpha}D_z^{\alpha}U,c^{\alpha}D_z^{\alpha}U,\ldots\right)=0,0<\alpha\le 1. \quad (7)$$

Step 3. Suppose that Eq. (7) has the solution in the following form:

$$U(z)=\sum_{i=0}^{n}a_i\left(\frac{D_z^{\alpha}G(z)}{G(z)}\right)^i \quad (8)$$

where $a_i(i=0,1,2,\ldots,n)$ are coefficient constants to be determined later, n is a positive integer determined by balancing the highest order derivatives and nonlinear terms in Eq. (5) or Eq. (7), while $G(z)$ satisfies the following fractional ordinary equation (FODE):

$$D_z^{2\alpha}G(z)+\lambda D_z^{\alpha}G(z)+\mu G(z)=0,0<\alpha\le 1, \quad (9)$$

where λ and μ are constants.

The following solutions of fractional Eq. (9) in the form of $W(z)=\frac{D^{\alpha}G(z)}{G(z)}$ are as follows:

$$W(z)=\frac{D^{\alpha}G(z)}{G(z)}=\begin{cases}\frac{\sqrt{\lambda^2-4\mu}}{2}\left[\frac{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}z^{\alpha}\right)+C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}z^{\alpha}\right)}{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}z^{\alpha}\right)+C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}z^{\alpha}\right)}\right]-\frac{\lambda}{2}, if\ \lambda^2-4\mu>0,\\ \frac{\sqrt{4\mu-\lambda^2}}{2}\left[\frac{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}z^{\alpha}\right)-C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}z^{\alpha}\right)}{C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}z^{\alpha}\right)+C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}z^{\alpha}\right)}\right]-\frac{\lambda}{2}, if\ \lambda^2-4\mu<0,\\ \frac{C_2\Gamma(1+\alpha)}{C_2z^{\alpha}+C_1\Gamma(1+\alpha)}-\frac{\lambda}{2}, if\ \lambda^2-4\mu=0.\end{cases} \quad (10)$$

Step 4. Substituting Eq. (8) along with Eq. (9) into Eq. (7) and using the properties of Jumarie's modified Riemann-Liouville derivative (2), (3) and (4), we can get a polynomial in $W(z)=\frac{D^{\alpha}G(z)}{G(z)}$. Setting all these coefficients of $W^i(i=0,1,2,\ldots,n)$ to zero, yields a set of over determined nonlinear algebraic system of equations for $a_i(i=0,1,2,\ldots,n),\lambda,\mu,k$ and c.

Step 5. Finally, assuming that the constants $a_i(i=0,1,2,\ldots,n),\lambda,\mu,k$ and c can be obtained by solving the algebraic system of equations in Step 4, substituting these constants and the solutions of Eq. (9) into Eq. (8), then by Eq. (6) we can obtain the explicit solutions of Eq. (5) immediately.

3. Applications

3.1. The Space-Time-Fractional SRLW Equation

The space-time-fractional SRLW equation is given by

$$D_t^{2\alpha}u(x,t)+D_x^{2\alpha}u(x,t)+u(x,t)D_t^{\alpha}\left(D_x^{\alpha}u(x,t)\right)$$
$$+D_t^{\alpha}u(x,t)D_x^{\alpha}u(x,t)+D_t^{2\alpha}\left(D_x^{2\alpha}u(x,t)\right)=0, \quad (11)$$

where $0<\alpha\le 1, t>0$.

This equation arises in many nonlinear problems of mathematical physics and applied mathematics including ion sound waves in plasma. It is symmetrical with respect to x and t. see [27].

Using the wave transformationin Eq. (6), we get the following:

$$D_x^{\alpha}u(z)=k^{\alpha}D_z^{\alpha}U(z),D_t^{\alpha}u(z)=c^{\alpha}D_z^{\alpha}U(z). \quad (12)$$

Substituting Eq. (6) and Eq. (12) in Eq. (11) we get:

$$c^{2\alpha}D_z^{2\alpha}U(z)+k^{2\alpha}D_z^{2\alpha}U(z)+c^{\alpha}k^{\alpha}U(z)D_z^{2\alpha}U(z)$$
$$+c^{\alpha}k^{\alpha}(D_z^{\alpha}U(z))^2+c^{2\alpha}k^{2\alpha}D_z^{4\alpha}U(z)=0. \quad (13)$$

Balancing the order of the highest derivative term $D_z^{4\alpha}U(z)$ and the highest nonlinear term $U(z)D_z^{2\alpha}U(z)$ in Eq. (13), we obtain $n = 2$. Thus, Eq. (8) reduces to:

$$U(z) = a_0 + a_1\frac{D_z^{\alpha}G(z)}{G(z)} + a_2\left(\frac{D_z^{\alpha}G(z)}{G(z)}\right)^2.$$

If we let $W(z) = \frac{D^{\alpha}G(z)}{G(z)}$, then

$$U(z) = a_0 + a_1W(z) + a_2W(z)^2 \quad \text{or simply, } U = a_0 + a_1W + a_2W^2. \tag{14}$$

Therefore, we can compute the fractional derivatives of $U(z) = U, D_z^{\alpha}U, D_z^{2\alpha}U, D_z^{3\alpha}U$ *and* $D_z^{4\alpha}U$ and substituting them in Eq. (13), we get the coefficients of powers of W are as follows:

$$W^0 = \mu\begin{pmatrix} 14c^{2\alpha}k^{2\alpha}\mu\lambda^2a_2 + c^{2\alpha}k^{2\alpha}\lambda^3a_1 \\ +2c^{\alpha}k^{\alpha}\mu a_0a_2 + c^{\alpha}k^{\alpha}a_1^2\mu + c^{\alpha}k^{\alpha}\mu a_0a_1 \\ +16c^{2\alpha}k^{2\alpha}\mu^2a_2 + 8c^{2\alpha}k^{2\alpha}\mu\lambda a_1 \\ +2c^{2\alpha}\mu a_2 + c^{2\alpha}\lambda a_1 + 2k^{2\alpha}\mu a_2 + k^{2\alpha}\lambda a_1 \end{pmatrix} \tag{15}$$

$$\begin{aligned} W^1 &= 30c^{2\alpha}k^{2\alpha}\mu\lambda^3a_2 + c^{2\alpha}k^{2\alpha}\lambda^4a_1 \\ &+120c^{2\alpha}k^{2\alpha}\mu^2\lambda a_2 + 22c^{2\alpha}k^{2\alpha}\mu\lambda^2a_1 \\ &+6c^{\alpha}k^{\alpha}\mu^2a_1a_2 + 6c^{\alpha}k^{\alpha}\mu\lambda a_0a_2 \\ &+3c^{\alpha}k^{\alpha}\mu\lambda a_1^2 + c^{\alpha}k^{\alpha}\lambda^2a_0a_1 + 16c^{2\alpha}k^{2\alpha}\mu^2a_1 \\ &+2c^{\alpha}k^{\alpha}\mu a_0a_1 + 6c^{2\alpha}\mu\lambda a_2 + c^{2\alpha}\lambda^2a_1 \\ &+6k^{2\alpha}\mu\lambda a_2 + k^{2\alpha}\lambda^2a_1 + 2c^{2\alpha}\mu a_1 + 2k^{2\alpha}\mu a_1 \end{aligned} \tag{16}$$

$$\begin{aligned} W^2 &= 16c^{2\alpha}k^{2\alpha}\lambda^4a_2 + 6c^{\alpha}k^{\alpha}\mu^2a_2^2 \\ &+15c^{\alpha}k^{\alpha}\mu\lambda a_1a_2 + 4c^{\alpha}k^{\alpha}\lambda^2a_0a_2 \\ &+2c^{\alpha}k^{\alpha}\lambda^2a_1^2 + 232c^{2\alpha}k^{2\alpha}\mu\lambda^2a_2 \\ &+15c^{2\alpha}k^{2\alpha}\lambda^3a_1 + 8c^{\alpha}k^{\alpha}\mu a_0a_2 + 4c^{\alpha}k^{\alpha}a_1^2\mu \\ &+3c^{\alpha}k^{\alpha}\lambda a_0a_1 + 136c^{2\alpha}k^{2\alpha}\mu^2a_2 + 60c^{2\alpha}k^{2\alpha}\mu\lambda a_1 \\ &+4c^{2\alpha}\lambda^2a_2 + 4k^{2\alpha}\lambda^2a_2 + 8c^{2\alpha}\mu a_2 \end{aligned} \tag{17}$$

$$\begin{aligned} W^3 &= 130c^{2\alpha}k^{2\alpha}\lambda^3a_2 + 14c^{\alpha}k^{\alpha}\mu\lambda a_2^2 \\ &+9c^{\alpha}k^{\alpha}\lambda^2a_1a_2 + 440c^{2\alpha}k^{2\alpha}\mu\lambda a_2 \\ &+50c^{2\alpha}k^{2\alpha}\lambda^2a_1 + 18c^{\alpha}k^{\alpha}\mu a_1a_2 + 10c^{\alpha}k^{\alpha}\lambda a_0a_2 \\ &+5c^{\alpha}k^{\alpha}\lambda a_1^2 + 40c^{2\alpha}k^{2\alpha}\mu a_1 + 2c^{\alpha}k^{\alpha}a_0a_1 \\ &+10c^{2\alpha}\lambda a_2 + 10k^{2\alpha}\lambda a_2 + 2c^{2\alpha}a_1 + 2k^{2\alpha}a_1 \end{aligned} \tag{18}$$

$$\begin{aligned} W^4 &= 8c^{\alpha}k^{\alpha}\lambda^2a_2^2 + 330c^{2\alpha}k^{2\alpha}\lambda^2a_2 + 16c^{\alpha}k^{\alpha}\mu a_2^2 \\ &+21c^{\alpha}k^{\alpha}\lambda a_1a_2 + 240c^{2\alpha}k^{2\alpha}\mu a_2 + 60c^{2\alpha}k^{2\alpha}\lambda a_1 \\ &+6c^{\alpha}k^{\alpha}a_0a_2 + 3c^{\alpha}k^{\alpha}a_1^2 + 6c^{2\alpha}a_2 + 6k^{2\alpha}a_2 \end{aligned} \tag{19}$$

$$\begin{aligned} W^5 &= 18c^{\alpha}k^{\alpha}\lambda a_2^2 + 336c^{2\alpha}k^{2\alpha}\lambda a_2 \\ &+12c^{\alpha}k^{\alpha}a_1a_2 + 24c^{2\alpha}k^{2\alpha}a_1 \end{aligned} \tag{20}$$

$$W^5 = 10c^{\alpha}k^{\alpha}a_2^2 + 120c^{2\alpha}k^{2\alpha}a_2 \tag{21}$$

Equating the coefficients (15) to (21) to zero, then solving the resulting system of these equations for a_0, a_1 *and* a_2 by Maple, we get the following solutions:

$$a_0 = -(c^{\alpha}k^{\alpha}\lambda^2 + 8c^{\alpha}k^{\alpha}\mu + c^{\alpha}k^{-\alpha} + c^{-\alpha}k^{\alpha}) \tag{22}$$

$$a_1 = -12c^{\alpha}k^{\alpha}\lambda \tag{23}$$

$$a_2 = 12c^{\alpha}k^{\alpha}. \tag{24}$$

Therefore, by substituting Eq. (10) and Eq. (22) to Eq. (24) in Eq. (14) we can write the following solutions for Eq. (13):

$$\begin{aligned} U_1(z) = &-\left(c^{\alpha}k^{\alpha}\lambda^2 + 8c^{\alpha}k^{\alpha}\mu + c^{\alpha}k^{-\alpha} + c^{-\alpha}k^{\alpha}\right) \\ &-12c^{\alpha}k^{\alpha}\lambda\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\left[\frac{\left[C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}z^{\alpha}\right) + C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}z^{\alpha}\right)\right]}{\left[C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}z^{\alpha}\right) + C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}z^{\alpha}\right)\right]}\right] - \frac{\lambda}{2}\right] \\ &-12c^{\alpha}k^{\alpha}\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\left[\frac{\left[C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}z^{\alpha}\right) + C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}z^{\alpha}\right)\right]}{\left[C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}z^{\alpha}\right) + C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}z^{\alpha}\right)\right]}\right] - \frac{\lambda}{2}\right]^2, \end{aligned} \tag{25}$$

if $\lambda^2 - 4\mu > 0$

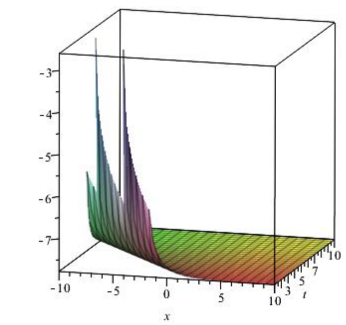
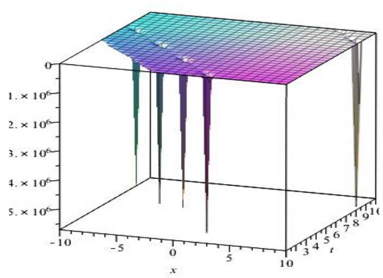
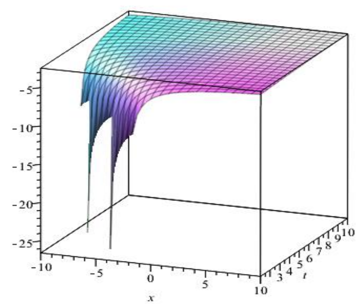
$$\begin{aligned} U_2(z) = &-\left(c^{\alpha}k^{\alpha}\lambda^2 + 8c^{\alpha}k^{\alpha}\mu + c^{\alpha}k^{-\alpha} + c^{-\alpha}k^{\alpha}\right) \\ &-12c^{\alpha}k^{\alpha}\lambda\left[\frac{\sqrt{4\mu-\lambda^2}}{2}\left[\frac{\left[C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}z^{\alpha}\right) - C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}z^{\alpha}\right)\right]}{\left[C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}z^{\alpha}\right) + C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}z^{\alpha}\right)\right]}\right] - \frac{\lambda}{2}\right] \end{aligned}$$

$$-12c^{\alpha}k^{\alpha}\left[\frac{\sqrt{4\mu-\lambda^2}}{2}\left[\frac{\left[\begin{array}{l}C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}z^{\alpha}\right)\\ -C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}z^{\alpha}\right)\end{array}\right]}{\left[\begin{array}{l}C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}z^{\alpha}\right)\\ +C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}z^{\alpha}\right)\end{array}\right]}\right]-\frac{\lambda}{2}\right]^2, \quad (26)$$

$if\ \lambda^2-4\mu<0$

$$\begin{aligned}U_3(z)=&-\left(c^{\alpha}k^{\alpha}\lambda^2+8c^{\alpha}k^{\alpha}\mu+c^{\alpha}k^{-\alpha}+c^{-\alpha}k^{\alpha}\right)\\&-12c^{\alpha}k^{\alpha}\lambda\left[\frac{C_2\Gamma(1+\alpha)}{C_2z^{\alpha}+C_1\Gamma(1+\alpha)}-\frac{\lambda}{2}\right]\\&-12c^{\alpha}k^{\alpha}\left[\frac{C_2\Gamma(1+\alpha)}{C_2z^{\alpha}+C_1\Gamma(1+\alpha)}-\frac{\lambda}{2}\right]^2, if\ \lambda^2-4\mu=0.\end{aligned} \quad (27)$$

As an illustration, the graphs of the solutions $u(x,t)$ of Eq. 11are shown, with the following assumptions:

<table>
<tr><td colspan="2">$Let\ \alpha=\frac{1}{2},\lambda=4,\mu=3,k=1,c=2,C_1=5,C_2=6,\ then\ a_0=-\frac{83\sqrt{2}}{2},a_1=-48\sqrt{2},a_2=-12\sqrt{2}.$</td></tr>
<tr><td>$u_1(x,t)=-12\left(\frac{5\cosh\left(\frac{2\sqrt{2t+x}}{\sqrt{\pi}}\right)+6\sinh\left(\frac{2\sqrt{2t+x}}{\sqrt{\pi}}\right)}{5\sinh\left(\frac{2\sqrt{2t+x}}{\sqrt{\pi}}\right)+6\cosh\left(\frac{2\sqrt{2t+x}}{\sqrt{\pi}}\right)}-2\right)^2\sqrt{2}-48\left(\frac{5\cosh\left(\frac{2\sqrt{2t+x}}{\sqrt{\pi}}\right)+6\sinh\left(\frac{2\sqrt{2t+x}}{\sqrt{\pi}}\right)}{5\sinh\left(\frac{2\sqrt{2t+x}}{\sqrt{\pi}}\right)+6\cosh\left(\frac{2\sqrt{2t+x}}{\sqrt{\pi}}\right)}-2\right)\sqrt{2}-\frac{83}{2}\sqrt{2}$</td><td></td></tr>
<tr><td colspan="2">$Let\ \alpha=\frac{1}{2},\lambda=4,\mu=5,k=1,c=2,C_1=3,C_2=6,then\ a_0=-\frac{115\sqrt{2}}{2},a_1=-48\sqrt{2},a_2=-12\sqrt{2}.$</td></tr>
<tr><td>$u_2(x,t)=12\left(\frac{-3\sin\left(\frac{2\sqrt{2t+x}}{\sqrt{\pi}}\right)+6\cos\left(\frac{2\sqrt{2t+x}}{\sqrt{\pi}}\right)}{3\cos\left(\frac{2\sqrt{2t+x}}{\sqrt{\pi}}\right)+6\sin\left(\frac{2\sqrt{2t+x}}{\sqrt{\pi}}\right)}-2\right)^2\sqrt{2}-48\left(\frac{-3\sin\left(\frac{2\sqrt{2t+x}}{\sqrt{\pi}}\right)+6\cos\left(\frac{2\sqrt{2t+x}}{\sqrt{\pi}}\right)}{3\cos\left(\frac{2\sqrt{2t+x}}{\sqrt{\pi}}\right)+6\sin\left(\frac{2\sqrt{2t+x}}{\sqrt{\pi}}\right)}-2\right)\sqrt{2}-\frac{115}{2}\sqrt{2}$</td><td></td></tr>
<tr><td colspan="2">$Let\ \alpha=\frac{1}{2},\lambda=4,\mu=4,k=1,c=2,C_1=5,C_2=6,\ then\ a_0=-\frac{99\sqrt{2}}{2},a_1=-48\sqrt{2},a_2=-12\sqrt{2}.$</td></tr>
<tr><td>$u_3(x,t):=-12\left(\frac{3\sqrt{\pi}}{6\sqrt{2t+x}+\frac{5}{2}\sqrt{\pi}}-2\right)^2\sqrt{2}-48\left(\frac{3\sqrt{\pi}}{6\sqrt{2t+x}+\frac{5}{2}\sqrt{\pi}}-2\right)\sqrt{2}-\frac{99}{2}\sqrt{2}$</td><td></td></tr>
</table>

3.2. The Space-Time-Fractional STO Equation

The space-time-fractional STO equation is given by

$$\begin{aligned}&D_t^{\alpha}u(x,t)+3A(D_x^{\beta}u(x,t))^2+3Au(x,t)^2D_x^{\beta}u(x,t)\\&+3Au(x,t)D_x^{2\beta}u(x,t)+AD_x^{3\beta}u(x,t)=0,\end{aligned} \quad (28)$$

Where $\mathbf{0}<\alpha,\beta\leq 1,t>0$, see [28]

When $\beta=\alpha$, then Eq. (28) becomes

$$\begin{aligned}&D_t^{\alpha}u(x,t)+3A(D_x^{\alpha}u(x,t))^2+3Au(x,t)^2D_x^{\alpha}u(x,t)\\&+3Au(x,t)D_x^{2\alpha}u(x,t)+AD_x^{3\alpha}u(x,t)=0.\end{aligned} \quad (29)$$

Using the wave transformation (6) and Eq. (12) in Eq. (29), we get the following:

$$c^{\alpha} D_z^{\alpha} U(z) + 3Ak^{2\alpha} (D_z^{\alpha} U(z))^2 + 3Ak^{\alpha} U(z)^2 D_z^{\alpha} U(z) + 3Ak^{2\alpha} U(z) D_z^{2\alpha} U(z) + Ak^{3\alpha} D_z^{3\alpha} U(z) = 0. \quad (30)$$

Now, by balancing the order of the highest derivative term $D_z^{3\alpha}U(z)$ and the highest nonlinear term $U(z)D_z^{2\alpha}U(z)$, we get $n = 1$. Thus, Eq. (8) reduces to:

$$U(z) = a_0 + a_1 W(z) \text{ or simply, } U = a_0 + a_1 W. \quad (31)$$

Similar to section 3.1, we can compute the fractional derivatives of $U(z) = U, D_z^{\alpha}U, D_z^{2\alpha}U \; and \; D_z^{3\alpha}U$ and substituting them in Eq. (30), we get the coefficients of powers of W as follow:

$$W^0 : -a_1 c^{\alpha} \mu - 3Aa_0^2 a_1 k^{\alpha} \mu + 3Aa_0 a_1 k^{2\alpha} \lambda\mu + 3Aa_1^2 k^{2\alpha} \mu^2 - Aa_1 k^{3\alpha} \mu(\lambda^2 + 2\mu) \quad (32)$$

$$W^1 : -a_1 c^{\alpha} \lambda - 3Aa_0^2 a_1 k^{\alpha} \lambda - 6Aa_0 a_1^2 k^{\alpha} \mu + 9Aa_1^2 k^{2\alpha} \lambda\mu + 3Aa_0 a_1 k^{2\alpha} (\lambda^2 + 2\mu) - Aa_1 k^{3\alpha} \lambda(\lambda^2 + 8\mu) \quad (33)$$

$$W^2 : -a_1 c^{\alpha} - 3Aa_0^2 a_1 k^{\alpha} - 6Aa_0 a_1^2 k^{\alpha} \lambda + 9Aa_0 a_1 k^{2\alpha} \lambda + 3Aa_1^2 k^{2\alpha} \lambda^2 - 3Aa_1^3 k^{\alpha} \mu + 6Aa_1^2 k^{2\alpha} \mu + 3Aa_1^2 k^{2\alpha} (\lambda^2 + 2\mu) - Ak^{3\alpha} a_1 (7\lambda^2 + 8\mu) \quad (34)$$

$$W^3 : -6Aa_0 a_1^2 k^{\alpha} a_1^2 + 6Aa_0 a_1 k^{2\alpha} - 3Aa_1^3 k^{\alpha} \lambda + 15Aa_1^2 k^{2\alpha} \lambda - 12Aa_1 k^{3\alpha} \lambda \quad (35)$$

$$W^4 : -3Aa_1^3 k^{\alpha} + 9Aa_1^2 k^{2\alpha} - 6Aa_1 k^{3\alpha} \quad (36)$$

Equating the coefficients of powers of W from (32) to (36) to zero, then solving the resulting system for a_0, a_1 and λ by Mathematica, we get the following of solutions:

Table 1

case	a_0	a_1	λ	$\lambda^2 - 4\mu$
1	$\frac{3Ak^{2\alpha}\lambda + \sqrt{12A^2k^{4\alpha}\mu - 3A^2k^{4\alpha}\lambda^2 - 12Ac^{\alpha}k^{\alpha}}}{6Ak^{\alpha}}$	k^{α}	λ	Any real number (3 solutions)
2	$\frac{3Ak^{2\alpha}\lambda - \sqrt{12A^2k^{4\alpha}\mu - 3A^2k^{4\alpha}\lambda^2 - 12Ac^{\alpha}k^{\alpha}}}{6Ak^{\alpha}}$	k^{α}	λ	Any real number (3 solutions)
3	$\frac{3k^{\alpha}\sqrt{A\mu - c^{\alpha}k^{-3\alpha}}}{\sqrt{13A}}$	k^{α}	$\frac{2\sqrt{A\mu - c^{\alpha}k^{-3\alpha}}}{\sqrt{13A}}$	$-\frac{4(c^{\alpha}k^{-3\alpha} + 12A\mu)}{13A}$ (3 solutions)
4	$-\frac{3k^{\alpha}\sqrt{A\mu - c^{\alpha}k^{-3\alpha}}}{\sqrt{13A}}$	k^{α}	$-\frac{2\sqrt{A\mu - c^{\alpha}k^{-3\alpha}}}{\sqrt{13A}}$	$-\frac{4(c^{\alpha}k^{-3\alpha} + 12A\mu)}{13A}$ (3 solutions)
5	$\frac{k^{\alpha}\sqrt{4A\mu - c^{\alpha}k^{-3\alpha}}}{\sqrt{A}}$	$2k^{\alpha}$	$\frac{\sqrt{4A\mu - c^{\alpha}k^{-3\alpha}}}{\sqrt{A}}$	$-\frac{c^{\alpha}k^{-3\alpha}}{A}$ (2 solutions)
6	$-\frac{k^{\alpha}\sqrt{4A\mu - c^{\alpha}k^{-3\alpha}}}{\sqrt{A}}$	$2k^{\alpha}$	$-\frac{\sqrt{4A\mu - c^{\alpha}k^{-3\alpha}}}{\sqrt{A}}$	$-\frac{c^{\alpha}k^{-3\alpha}}{A}$ (2 solutions)
	The total number of solutions is			16

So the solutions of Eq. (30) in case 1and 2 are as follows: $U(z) = a_0 + a_1 W(z)$ becomes

$$U_1(z) = \frac{3Ak^{2\alpha}\lambda + \sqrt{\left(12A^2k^{4\alpha}\mu - 3A^2k^{4\alpha}\lambda^2 - 12Ac^{\alpha}k^{\alpha}\right)}}{6Ak^{\alpha}} + k^{\alpha}\left[\frac{\sqrt{\lambda^2 - 4\mu}}{2}\left[\frac{\left[C_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2\Gamma(1+\alpha)} z^{\alpha}\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2\Gamma(1+\alpha)} z^{\alpha}\right)\right]}{\left[C_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2\Gamma(1+\alpha)} z^{\alpha}\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2\Gamma(1+\alpha)} z^{\alpha}\right)\right]}\right] - \frac{\lambda}{2}\right], if\ \lambda^2 - 4\mu > 0 \quad (37)$$

$$U_2(z) = \frac{3Ak^{2\alpha}\lambda + \sqrt{12A^2k^{4\alpha}\mu - 3A^2k^{4\alpha}\lambda^2 - 12Ac^{\alpha}k^{\alpha}}}{6Ak^{\alpha}} + k^{\alpha}\left[\frac{\sqrt{4\mu - \lambda^2}}{2}\left[\frac{C_1 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2\Gamma(1+\alpha)} z^{\alpha}\right) - C_2 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2\Gamma(1+\alpha)} z^{\alpha}\right)}{C_1 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2\Gamma(1+\alpha)} z^{\alpha}\right) + C_2 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2\Gamma(1+\alpha)} z^{\alpha}\right)}\right] - \frac{\lambda}{2}\right], if\ \lambda^2 - 4\mu < 0 \quad (38)$$

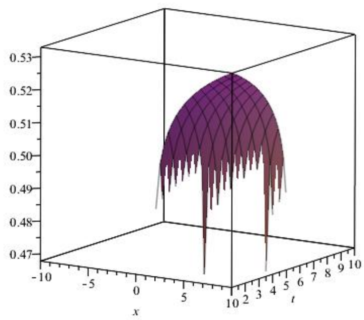
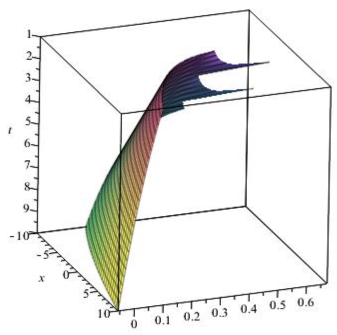
$$U_3(z)=\frac{3Ak^{2\alpha}\lambda+\sqrt{12A^2k^{4\alpha}\mu-3A^2k^{4\alpha}\lambda^2-12Ac^{\alpha}k^{\alpha}}}{6Ak^{\alpha}}+k^{\alpha}\left[\frac{C_2\Gamma(1+\alpha)}{C_2z^{\alpha}+C_1\Gamma(1+\alpha)}-\frac{\lambda}{2}\right], if\ \lambda^2-4\mu=0. \qquad (39)$$

$$U_4(z)=\frac{3Ak^{2\alpha}\lambda-\sqrt{12A^2k^{4\alpha}\mu-3A^2k^{4\alpha}\lambda^2-12Ac^{\alpha}k^{\alpha}}}{6Ak^{\alpha}}+k^{\alpha}\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\left[\frac{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}z^{\alpha}\right)+C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}z^{\alpha}\right)}{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}z^{\alpha}\right)+C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2\Gamma(1+\alpha)}z^{\alpha}\right)}\right]-\frac{\lambda}{2}\right], if\ \lambda^2-4\mu>0 \qquad (40)$$

$$U_5(z)=\frac{3Ak^{2\alpha}\lambda-\sqrt{12A^2k^{4\alpha}\mu-3A^2k^{4\alpha}\lambda^2-12Ac^{\alpha}k^{\alpha}}}{6Ak^{\alpha}}+k^{\alpha}\left[\frac{\sqrt{4\mu-\lambda^2}}{2}\left[\frac{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}z^{\alpha}\right)-C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}z^{\alpha}\right)}{C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}z^{\alpha}\right)+C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2\Gamma(1+\alpha)}z^{\alpha}\right)}\right]-\frac{\lambda}{2}\right], if\ \lambda^2-4\mu<0 \qquad (41)$$

$$U_6(z)=\frac{3Ak^{2\alpha}\lambda-\sqrt{12A^2k^{4\alpha}\mu-3A^2k^{4\alpha}\lambda^2-12Ac^{\alpha}k^{\alpha}}}{6Ak^{\alpha}}+k^{\alpha}\left[\frac{C_2\Gamma(1+\alpha)}{C_2z^{\alpha}+C_1\Gamma(1+\alpha)}-\frac{\lambda}{2}\right], if\ \lambda^2-4\mu=0. \qquad (42)$$

and by a similar way, the remaining solutions can be found.
As an illustration, the graphs of two solutions $u(x,t)$ of Eq. 29are shown, with the following assumptions:

<table>
<tr><td colspan="2">$Case5(i), Let\ \alpha=\frac{1}{3}, A=4, \mu=3, k=1, c=-2, C_1=5, C_2=6, \text{then}, a_0=\frac{\sqrt{48+\sqrt[3]{2}}}{2}, a_1=2, \lambda=\frac{\sqrt{48+\sqrt[3]{2}}}{2}$</td></tr>
<tr><td>$u_1(x,t)=\frac{1}{2}\left(\sqrt[6]{2}\left(5\cosh\left(\frac{3}{8}\frac{\sqrt[6]{2}\sqrt{3}\,\Gamma\left(\frac{2}{3}\right)(-2t+x)^{1/3}}{\pi}\right)+6\sinh\left(\frac{3}{8}\frac{\sqrt[6]{2}\sqrt{3}\,\Gamma\left(\frac{2}{3}\right)(-2t+x)^{1/3}}{\pi}\right)\right)\Big/\left(5\sinh\left(\frac{3}{8}\frac{\sqrt[6]{2}\sqrt{3}\,\Gamma\left(\frac{2}{3}\right)(-2t+x)^{1/3}}{\pi}\right)+6\cosh\left(\frac{3}{8}\frac{\sqrt[6]{2}\sqrt{3}\,\Gamma\left(\frac{2}{3}\right)(-2t+x)^{1/3}}{\pi}\right)\right)\right)$</td><td>
</td></tr>
<tr><td colspan="2">$Case\ 5(ii), Let\ \alpha=\frac{1}{3}, A=4, \mu=3, k=1, c=2, C_1=5, C_2=6, \text{then}, a_0=\frac{\sqrt{48+\sqrt[3]{2}}}{2}, a_1=2, \lambda=\frac{\sqrt{48+\sqrt[3]{2}}}{2}$</td></tr>
<tr><td>$u_2(x,t)=\frac{1}{2}\left(\sqrt[6]{2}\left(-5\sin\left(\frac{3}{8}\frac{\sqrt[6]{2}\,(2t+x)^{1/3}\sqrt{3}\,\Gamma\left(\frac{2}{3}\right)}{\pi}\right)+6\cos\left(\frac{3}{8}\frac{\sqrt[6]{2}\,(2t+x)^{1/3}\sqrt{3}\,\Gamma\left(\frac{2}{3}\right)}{\pi}\right)\right)\Big/\left(5\cos\left(\frac{3}{8}\frac{\sqrt[6]{2}\,(2t+x)^{1/3}\sqrt{3}\,\Gamma\left(\frac{2}{3}\right)}{\pi}\right)+6\sin\left(\frac{3}{8}\frac{\sqrt[6]{2}\,(2t+x)^{1/3}\sqrt{3}\,\Gamma\left(\frac{2}{3}\right)}{\pi}\right)\right)\right)$</td><td>
</td></tr>
</table>

4. Conclusion

In this paper, the $\frac{D^{\alpha}G}{G}$ expansion method which is one of the powerful fractional sub-equation method has been successfully used to find exact solutions for the well-known SRLW and STO equations in an efficient way. Even though this method is not easy to implement, however, it produces many convenient solutions to nonlinear FPDEs.

Finally, we believe that this method provides a powerful and remarkable mathematical tool to obtain exact analytical solutions for a large number of nonlinear FPDEs in physics, biology and engineering.

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