Estimation of Isomorphism Degree of Fuzzy Graphs

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Abstract

The questions of isomorphism degree estimation of fuzzy graphs are discussed in this paper. The definition of independent fuzzy set of fuzzy graph is presented. The estimation method of isomorphism degree of fuzzy graphs is suggested and proved. The example of isomorphism degree estimation is considered.

Keywords: Fuzzy Graph, Isomorphism Degree, Independent Degree, Fuzzy Independent Vertex Set, Independent Fuzzy Set.

1 Introduction

A graph isomorphism search is important problem of graph theory. It consists to define of bijective correspondence existence which preserve adjacent relation between vertex sets of two graphs [7]. In a case of fuzzy graphs the notion of isomorphism is fuzzy.

We consider fuzzy graphs [5] $\tilde{G}_X=(X, \tilde{U}_X)$ and $\tilde{G}_Y=(Y, \tilde{U}_Y)$, where X and Y are sets of vertices and $\tilde{U}_X=\{<\mu_X(x_i,x_j)/(x_i,x_j)>|(x_i,x_j)\in X^2\}$ and $\tilde{U}_Y=\{<\mu_Y(y_i,y_j)/(y_i,y_j)>|(y_i,y_j)\in Y^2\}$ are fuzzy sets of edges with membership functions $\mu_X:X^2\to[0,1]$ and $\mu_Y:Y^2\to[0,1]$. Let numbers of vertices coincide ($|X|=|Y|=n$).

Let $\Phi=\{F:X\to Y\}$ be a set of all bijective correspondences which may be define on the set $X\times Y$.

Definition 1 [3]. A value

$$f=\max_{\forall F\in\Phi}\ \underset{i=\overline{1,n}}{\&}\ \underset{j=\overline{1,n}}{\&}(\mu_X(x_i,x_j)\leftrightarrow\mu_Y(y_i,y_j))$$

is called an isomorphism degree of fuzzy graphs $\tilde{G}_X$ and $\tilde{G}_Y$.

A task of search degree isomorphism is NP-complete task as since $|\Phi|=n!$

Here & is conjunction operation, which is defined as $a\&b=\min\{a,b\}$. Operation $\leftrightarrow$ is equivalence operation, which is defined as $a\leftrightarrow b=(a\to b)\&(b\to a)$. Operation $\to$ is implication operation, which has property $0\to b=1$ (falsity implies anything) [4]. In Lukasiewicz logic it is defined as $a\to b=\min\{1,1-a+b\}$.

Crisp graphs may be consider as fuzzy graphs which have membership functions equal *0* or *1*. Consequently, their isomorphism degree *f* may be *0* or *1* also. If $f=1$, then crisp graphs are isomorphous. If $f=0$, then crisp graphs aren't isomorphous.

Crisp graphs are defined some invariants: independent number, dominating vertex number, chromatic number and others [6]. If crisp graphs are isomorphous ($f=1$), then their invariants coincide. If invariants don't coincide, then crisp graphs aren't isomorphous ($f=1$)

When we deal with fuzzy graphs, their invariants are fuzzy values also. A task of fuzzy invariants influence on degree isomorphism of fuzzy graphs arises in this case. A correlation between isomorphism degree and independent fuzzy sets of fuzzy graphs is considered in this paper.

2 Isomorphism degree estimation

Consider a fuzzy subgraph $\widetilde{G}_{X_k}=(X_k,\widetilde{U}_{X_k})$, where $X_i\subseteq X$, $|X_i|=k$ ($k=\overline{1,n}$).

Definition 2 [1]. A value $\alpha_{X_k}=1-\max\limits_{x_i,x_j\in X_k}\mu_X(x_i,x_j)$ is called an independence degree of fuzzy subgraph $\widetilde{G}_{X_k}$.

Definition 3 [1]. A subset of vertices $X_K\subseteq X$ of fuzzy graph $\widetilde{G}_X$ is called a maximal fuzzy independent vertex set with degree $\alpha^0{}_{X_K}$, if the condition $\alpha'_X<\alpha^0{}_{X_K}$ is true for all $X'\supset X_K$.

We define as $\alpha^{\max}_{X_k}=\max\{\alpha^0_{X_k^1},\alpha^0_{X_k^2},...,\alpha^0_{X_k^l}\}$, where $\alpha^0_{X_k^1},\alpha^0_{X_k^2},...,\alpha^0_{X_k^l}$ are the degrees of maximal fuzzy independent vertex sets, which have k elements. A volume $\alpha^{\max}_{X_k}$ means that fuzzy graph $\widetilde{G}_X=(X,\widetilde{U}_X)$ includes a fuzzy subgraph with k vertices and with independent degree $\alpha^{\max}_{X_k}$ and it doesn't include any subgraphs more than k vertices and with independent degree more than $\alpha^{\max}_{X_k}$.

Definition 4. A fuzzy set $\widetilde{A}_X=\{<\alpha^{\max}_{X_1}/1>,<\alpha^{\max}_{X_2}/2>,...,<\alpha^{\max}_{X_n}/n>\}$ is called an independent fuzzy set of fuzzy graph $\widetilde{G}_X$.

Property 1. The following proposition is true:

$$1\ge\alpha^{\max}_{X_1}\ge\alpha^{\max}_{X_2}\ge...\ge\alpha^{\max}_{X_n}\ge 0\cdot$$

Volume $\alpha^{\max}_{X_1}$ equals 1 if set X doesn't contain any vertices with loop, and volume $\alpha^{\max}_{X_n}$ equals 0 if set $\widetilde{U}_X$ contains an edge with membership function equals 1.

We consider a some bijective correspondence $F:X\to Y$ between vertex sets X and Y. Let f be an isomorphism degree of fuzzy graphs $\widetilde{G}_X=(X,\widetilde{U}_X)$ and $\widetilde{G}_Y=(Y,\widetilde{U}_Y)$. Let f_k be an isomorphism degree of fuzzy subgraphs $\widetilde{G}_{X_k}=(X_k,\widetilde{U}_{X_k})$ and $\widetilde{G}_{Y_k}=(Y_k,\widetilde{U}_{Y_k})$.

Property 2. The following proposition is true:

$$(\forall k=\overline{1,n})(f_k\ge f).$$

Proof. We renumber vertices of sets X and Y. We mark vertices of subsets X_K and Y_K as *1,2,…,k* and vertices of subsets $X\backslash X_K$ and $Y\backslash Y_K$ as *k+1, k+2,…,n*. Then the degree of isomorphism may be write as:

$$f=\mathop{\&}_{i=\overline{1,n}}\mathop{\&}_{j=\overline{1,n}}(\mu_X(x_i,x_j)\leftrightarrow\mu_Y(y_i,y_j))=$$

$$=\mathop{\&}_{i=\overline{1,k}}\mathop{\&}_{j=\overline{1,k}}(\mu_X(x_i,x_j)\leftrightarrow\mu_Y(y_i,y_j))\ \&$$

$$\mathop{\&}_{i=\overline{1,n}}\mathop{\&}_{j=\overline{k+1,n}}(\mu_X(x_i,x_j)\leftrightarrow\mu_Y(y_i,y_j))\ \&$$

$$\&\mathop{\&}_{i=\overline{k+1,n}}\mathop{\&}_{j=\overline{1,n}}(\mu_X(x_i,x_j)\leftrightarrow\mu_Y(y_i,y_j))=$$

$$f_k\ \&\mathop{\&}_{i=\overline{1,n}}\mathop{\&}_{j=\overline{k+1,n}}(\mu_X(x_i,x_j)\leftrightarrow\mu_Y(y_i,y_j))\ \&$$

$$\&\mathop{\&}_{i=\overline{k+1,n}}\mathop{\&}_{j=\overline{1,n}}(\mu_X(x_i,x_j)\leftrightarrow\mu_Y(y_i,y_j))\le f_k.$$

Property 2 is proved.

Property 3. Let α_{X_k} and α_{Y_k} be an independence degrees of fuzzy subgraphs $\widetilde{G}_{X_k}$ and $\widetilde{G}_{Y_k}$. Then following proposition is true:

$$(\forall k=\overline{1,n})((\alpha_{X_k}\leftrightarrow\alpha_{Y_k})\ge f_k)\qquad(1)$$

Proof. If α_{X_k} and α_{Y_k} are an independence degrees, then there are some vertices $x_1,x_2\in X_K$, for which $\alpha_{X_K}=1-\mu_X(x_1,x_2)$, and there are some $y_1,y_2\in Y_K$, for which $\alpha_{Y_K}=1-\mu_Y(y_1,y_2)$. This case is presented in Figure1.

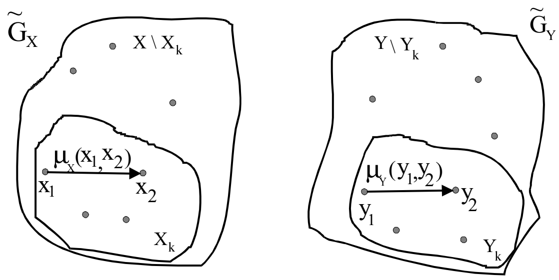

Figure 1: Example of case $\alpha_{X_K}=1-\mu_X(x_1,x_2)$ and $\alpha_{Y_K}=1-\mu_Y(y_1,y_2)$

We consider two cases.

Case 1. Vertex y_1 corresponds vertex x_1, ($F(x_1)=y_1$), vertex y_2 corresponds vertex x_2 ($F(x_2)=y_2$). Then the volume f_k may be estimate as:

$$f_k\le(\mu_{X_k}(x_1,x_2)\leftrightarrow\mu_{Y_k}(y_1,y_2))=$$

$$=(1-\alpha_{X_k})\leftrightarrow(1-\alpha_{Y_k})=\alpha_{X_k}\leftrightarrow\alpha_{Y_k}.$$

Case 2. Vertex y'_1 corresponds vertex x_1 ($F(x_1)=y'_1$), vertex y'_2 corresponds vertex x_2 ($F(x_2)=y'_2$), but $y'_1\ne y_1$ and (or) $y'_2\ne y_2$.

Then the expression $\mu_Y(y_1, y_2) \geq \mu_Y(y_1', y_2')$ is true. Hence we have $t^Y = 1 - \mu_Y(y_1', y_2') \geq \alpha_{Y_k}$. This case is presented in Figure 2.

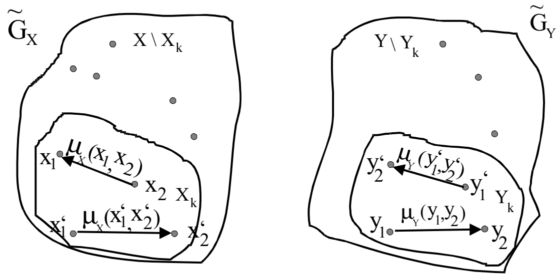

Figure 2: Example of case $y_1' \neq y_1$ and $y_2' \neq y_2$

We consider two cases again.

Case 2.1. The following proposition is true $(\alpha_{X_k} \leftrightarrow \alpha_{Y_k}) \geq (\alpha_{X_k} \leftrightarrow t^Y)$. Then volume f_k may be estimate as:

$$f_k \leq (\mu_X(x_1, x_2) \leftrightarrow \mu_Y(y_1', y_2')) =$$
$$= (1 - \alpha_{X_k}) \leftrightarrow (1 - t^Y) = \alpha_{X_k} \leftrightarrow t^Y =$$
$$= \alpha_{X_k} \leftrightarrow \alpha_{Y_k}$$

Case 2.2. The following proposition is true

$$(\alpha_{X_k} \leftrightarrow \alpha_{Y_k}) < (\alpha_{X_k} \leftrightarrow t^Y). \quad (2)$$

Let vertices $y_1, y_2 \in Y_k$ correspond vertices $x_1', x_2' \in X_k$, ($F^{-1}(y_1) = x_1'$ and $F^{-1}(y_2) = x_2'$). Then $\mu_X(x_1, x_2) \geq \mu_X(x_1', x_2')$ is true and we may write

$$t^X = 1 - \mu_X(x_1', x_2') \geq \alpha_{X_k}. \quad (3)$$

Proposition (2) involves $\alpha_{X_k} \leftrightarrow \alpha_{Y_k} > t^X \leftrightarrow \alpha_{Y_k}$ by condition (3). In this case volume f_k may be estimate as:

$$f_k \leq (\mu_X(x_1', x_2') \leftrightarrow \mu_Y(y_1, y_2)) =$$
$$= (1 - t^X) \leftrightarrow (1 - \alpha_{Y_k}) =$$
$$= t^X \leftrightarrow \alpha_{Y_k} < \alpha_{X_k} \leftrightarrow \alpha_{Y_k}$$

The property 3 is proved for any $k = \overline{1, n}$.

Let $\tilde{A}_X = \{<\alpha_{X_1}^{\max}/1>, <\alpha_{X_2}^{\max}/2>, ..., <\alpha_{X_n}^{\max}/n>\}$ and $\tilde{A}_Y = \{<\alpha_{Y_1}^{\max}/1>, <\alpha_{Y_2}^{\max}/2>, ..., <\alpha_{Y_n}^{\max}/n>\}$ be independent fuzzy sets of fuzzy graphs $\tilde{G}_X$ and $\tilde{G}_Y$ correspondingly, and *f* be an isomorphism degree of these fuzzy graphs.

Property 4. The following proposition is true:

$$f \leq \underset{k=1,n}{\&} (\alpha_{X_k}^{\max} \leftrightarrow \alpha_{Y_k}^{\max}). \quad (4)$$

Proof. Let $\tilde{G}_{X_k} = (X_k, \tilde{U}_{X_k})$ and $\tilde{G}_{Y_k} = (Y_k, \tilde{U}_{Y_k})$ be some fuzzy subgraphs with independence degrees $\alpha_{X_k}^{\max}$ and $\alpha_{Y_k}^{\max}$ correspondingly. We consider two cases.

Case 1. Subset Y_K corresponds to subset X_K ($F(X_k) = Y_k$). This case is presented in Figure3.

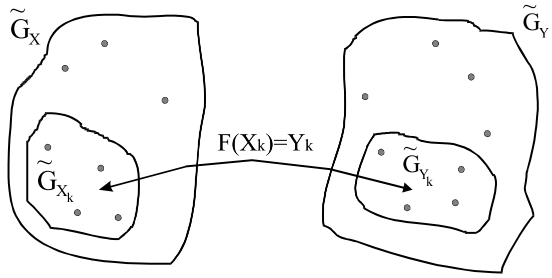

Figure 3: Example of case $F(X_k) = Y_k$

Then we can write $f \leq f_k \leq (\alpha_{X_k}^{\max} \leftrightarrow \alpha_{Y_k}^{\max})$ on the base of properties 2 and 3.

Case 2. Subset Y_K doesn't correspond to subset X_K ($F(X_k) = Y_k' \neq Y_k$). This case is presented in Figure4.

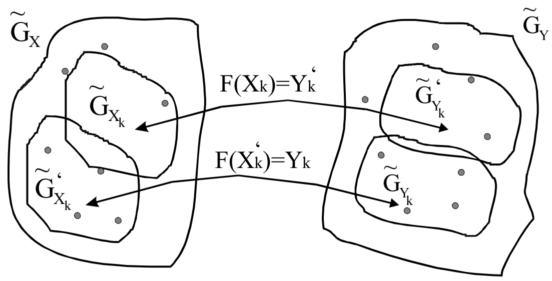

Figure 4: Example of case $F(X_k) = Y_k' \neq Y_k$

Then independence degree α_{Y_k}' of fuzzy subgraph $\tilde{G}_{Y_k}' = (Y_k', \tilde{U}_{Y_k}')$ may be estimate as $\alpha_{Y_k}' \leq \alpha_{Y_k}^{\max}$. We also consider two cases.

Case 2.1. The following proposition is true $(\alpha_{X_k}^{\max} \leftrightarrow \alpha_{Y_2}^{\max}) \geq (\alpha_{X_k}^{\max} \leftrightarrow \alpha_{Y_2}')$. Then isomorphism degree may be estimate as:

$$f \le f_k \le \alpha_{X_k}^{\max} \leftrightarrow \alpha'_{Y_2} \le \alpha_{X_k}^{\max} \leftrightarrow \alpha_{Y_2}^{\max}.$$

Case 2.2. The following proposition is true

$$(\alpha_{X_k}^{\max} \leftrightarrow \alpha_{Y_2}^{\max}) < (\alpha_{X_k}^{\max} \leftrightarrow \alpha'_{Y_2}). \qquad (5)$$

Let subset X'_k corresponds to subset Y_k ($F(X'_k) = Y_k$). Let α'_{X_k} be an independence degree of subset X'_k. Then we write

$$\alpha'_{X_k} \le \alpha_{X_k}^{\max}. \qquad (6)$$

Proposition (5) involves inequality $(\alpha_{X_k}^{\max} \leftrightarrow \alpha_{Y_2}^{\max}) < (\alpha'_{X_k} \leftrightarrow \alpha_{Y_2}^{\max})$ by condition (6). In this case isomorphism degree may be estimate as:

$$f \le f_k \le \alpha'_{X_k} \leftrightarrow \alpha_{Y_2}^{\max} \le \alpha_{X_k}^{\max} \leftrightarrow \alpha_{Y_2}^{\max}.$$

Property 4 is proved because we consider all cases.

Example. Estimate of isomorphism degree of the fuzzy graphs, shown in the Figure 5.

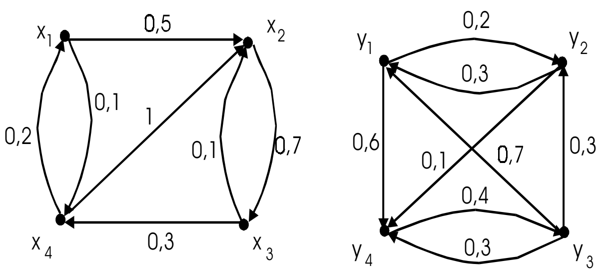

Figure 5: Example of fuzzy graphs $\tilde{G}_X$ and $\tilde{G}_Y$

Independent fuzzy sets of these fuzzy graphs are defined as $\tilde{A}_X = \{<1/1>, <1/2>, <0{,}7/3>, <0/4>\}$ and $\tilde{A}_Y = \{<1/1>, <0{,}9/2>, <0{,}6/3>, <0{,}2/4>\}$.

Then isomorphism degree is estimated as:

$$f \le \underset{k=\overline{1,4}}{\&} (\alpha_{X_k}^{\max} \leftrightarrow \alpha_{Y_k}^{\max}) =$$

$$= \mathbf{min}(1 \leftrightarrow 1; 1 \leftrightarrow 0{,}9; 0{,}7 \leftrightarrow 0{,}6;$$

$$0 \leftrightarrow 0{,}2) = 0{,}8.$$

Hence, we may assert that isomorphism degree of these fuzzy graphs will not be larger then 0,8.

3 Conclusion

Proven proposition (4) allows estimate a possible isomorphism degree of fuzzy graphs by fuzzy independent sets. It is necessary to remark that the estimate of isomorphism degree may be receive by another invariants of fuzzy graphs, such as dominating vertex sets, fuzzy graph kernels [1], fuzzy chromatic set [2].

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