



Abstract mathematics

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MT2116, 2790116

2011

Undergraduate study in
**Economics, Management,
Finance and the Social Sciences**

This is an extract from a subject guide for an undergraduate course offered as part of the University of London International Programmes in Economics, Management, Finance and the Social Sciences. Materials for these programmes are developed by academics at the London School of Economics and Political Science (LSE).

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University of London International Programmes
Publications Office
Stewart House
32 Russell Square
London WC1B 5DN
United Kingdom
Website: www.londoninternational.ac.uk

Published by: University of London

© University of London 2010

Reprinted with minor revisions 2011

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Chapter 1

Introduction

In this very brief introduction, I aim to give you an idea of the nature of this subject and to advise on how best to approach it. I also give general information about the contents and use of this subject guide, and on recommended reading and how to use the textbooks.

1.1 This subject

1.1.1 Relationship to previous mathematics courses

If you are taking this course as part of a BSc Degree you will already have taken a pre-requisite Mathematics subject, either a combination of **05A Mathematics 1** and **05B Mathematics 2** or **174 Calculus**. Any references in the text to these courses for prerequisite material will apply equally to whatever pre-requisite you have taken. Please note: this course may not be taken with **95 Further mathematics for economists**.

In **05A Mathematics 1** and **05B Mathematics 2** you will have learned about techniques of calculus and linear algebra. In **Abstract mathematics** the emphasis is on theory rather than method: we will want to understand why certain techniques work, and how we might be able to prove that they do, for example. The main central topic in this course is *proof*. This course is an introduction to formal mathematical reasoning, in which proof is central. We will meet the fundamental concepts and constructions of mathematics and see how to formulate mathematical statements in precise terms, and we will see how such statements can be proved or disproved.

In this subject, we need to work with *precise definitions* and *statements*, and you will need to know these. Not only will you need to know these, but you will have to understand them, and be able (through the use of them) to demonstrate that you understand them. Simply learning the definitions without understanding what they mean is not going to be adequate. I hope that these words of warning don't discourage you, but I think it's important to make it clear that this is a subject at a higher level than those prerequisite subjects.

In this subject, you will learn how to *prove* mathematical statements precisely. This is a very different sort of mathematics from that which you encountered in **05A Mathematics 1** and **05B Mathematics 2**, where the emphasis is on solving problems through *calculation*. In **Abstract mathematics**, one has to be able to produce convincing mathematical arguments as to why a given mathematical statement is true or false. For example, a prime number is a positive integer greater than 1 that is only divisible by itself and the number 1 (so 7 is a prime number, but 8 is not). The statement 'There are infinitely many prime numbers' is a mathematical statement, and

it is either true (there are infinitely many prime numbers) or false (there are only finitely many prime numbers). In fact, the statement is true. But why? There's no quick 'calculation' we can do to establish the truth of the statement. What is needed is a proof: a watertight, logical argument. This is the type of problem we consider in this subject.

1.1.2 Aims

This course is designed to enable you to:

- develop your ability to think in a critical manner;
- formulate and develop mathematical arguments in a logical manner;
- improve your skill in acquiring new understanding and expertise;
- acquire an understanding of basic pure mathematics, and the role of logical argument in mathematics.

1.1.3 Learning outcomes

At the end of this course and having completed the Essential reading and activities, you should:

- have used basic mathematical concepts in discrete mathematics, algebra and real analysis to solve mathematical problems in this subject
- be able to use formal notation correctly and in connection with precise statements in English
- be able to demonstrate an understanding of the underlying principles of the subject
- be able to solve unseen mathematical problems in discrete mathematics, algebra and real analysis
- be able to prove statements and formulate precise mathematical arguments.

1.1.4 Topics covered

Descriptions of topics to be covered appear in the relevant chapters. However, it is useful to give a brief overview at this stage.

The first half, approximately, of the subject is concerned primarily with proof, logic, and number systems. We shall refer to this part of the subject as the **Numbers and proof** part. The rest of the subject falls into two parts: **Analysis** and **Algebra**. We will be concerned, specifically, with elements of real analysis, and the theory of groups. It is possible to give only a brief overview of these three sections at this stage, since a more detailed description of each inevitably involves technical concepts that have not yet been met.

In the **Numbers and proof** part (Chapters 2 to 8), we will first investigate how precise mathematical statements can be formulated, and here we will use the language and symbols of mathematical logic. We will then study how one can prove or disprove mathematical statements. Next, we look at some important ideas connected with functions, relations, and numbers. For example, we will look at prime numbers and

learn what special properties these important numbers have, and how one may prove such properties.

In the **Analysis** part (Chapters 9 to 11), we will see how the intuitive idea of the ‘limit’ of a sequence of numbers can be made mathematically precise so that certain properties can be proved to hold. We will also look at functions and the key concept of continuity (which is intuitively appealing, but must be precisely mathematically formulated in order to be useful).

The **Algebra** part (Chapters 12 to 14) is about the theory of groups. A group is an abstract mathematical concept, but there are many concrete examples in the earlier part of this subject. In this part of the subject, we study general properties of groups.

Not all chapters of the guide are the same length. It should not be assumed that you should spend the same amount of time on each chapter. We will not try to specify how much relative time should be spent on each: that will vary from person to person and we do not want to be prescriptive.

As a *very rough* guide (bearing in mind that this must vary from individual to individual), we would suggest that the percentages of time spent on each chapter are something along the lines suggested in the table below. (This should not be taken as any indication about the composition of the examination.)

Chapter	Title	% Time
2	Mathematical statements, proof, logic and sets	10
3	Natural numbers and proof by induction	5
4	Functions and counting	5
5	Equivalence relations and the integers	5
6	Divisibility and prime numbers	10
7	Congruence and modular arithmetic	5
8	Rational, real and complex numbers	5
9	Supremum and infimum	5
10	Sequences and limits	15
11	Limits of functions and continuity	10
12	Group	10
13	Subgroups	5
14	Homomorphisms and Lagrange’s theorem	10

1.2 Reading

You will have to read books in order to supplement your reading. This subject guide is just a guide, and is not a textbook.

There are many books that would be useful for this subject, since numbers and proof, analysis and algebra are components of almost all university-level mathematics degree programmes.

For the **Numbers and proof** part of the subject, you should obtain copies of the following two books. It is enough to have one of them, but best to have both. (I will assume you have access to these, as they will be heavily cited in this guide):

✎ Biggs, Norman L. *Discrete Mathematics*. (Oxford Press: Oxford, 2002) second edition [ISBN 9780198507178].

✎ Eccles, P.J. *An Introduction to Mathematical Reasoning: numbers, sets and functions*. (Cambridge University Press: Cambridge and New York, 1997) [ISBN 9780521597180].

For the **Analysis** part, there are many suitable books, with the words ‘analysis’, ‘real analysis’ or ‘mathematical analysis’ in their title. The one I recommend most is the following:

✎ Bryant, Victor *Yet Another Introduction to Analysis*. (Cambridge University Press: Cambridge, 1990) [ISBN 9780521388351].

This book is written informally and entertainingly, and it will be the one I cite in the Analysis chapters. As I indicated, there are many other textbooks with titles such as ‘Real Analysis’ or ‘Mathematical Analysis’ that you will find useful. Here are some:

✎ Binmore, K.G. *Mathematical Analysis: A Straightforward Approach*. (Cambridge University Press: Cambridge, 1982) [ISBN 97805212888278].

✎ Bartle, R.G. and D.R. Sherbert *Introduction to Real Analysis*. (John Wiley and Sons: New York, 1999) fourth edition [ISBN 9780471433316].

For the **Algebra** part of the subject, you should use the Biggs book, cited above.

There is one topic that neither of these covers, which is the topic of complex numbers. However, this is a topic that is well-covered in a number of other textbooks and I have included a fairly full treatment of it in the guide to compensate for the fact that it is not covered in the recommended textbooks.

A text that covers this topic (and will also be very useful for the subject **118 Advanced linear algebra**, a subject you might also be studying) is:

✎ Anton, H. *Elementary Linear Algebra*. (John Wiley: Hoboken, NJ, 2010) tenth edition [ISBN 9789470561577].¹

So the ideal combination of texts consists of three main books: Biggs, Bryant and Eccles, together with access to another book (such as Anton) that covers complex numbers. Your study of this subject will be much enhanced if you have these.

Detailed reading references in this subject guide refer to the editions of the set textbooks listed above. New editions of one or more of these textbooks may have been published by the time you study this course. You can use a more recent edition of any of the books; use the detailed chapter and section headings and the index to identify relevant readings. Also check the VLE regularly for updated guidance on readings.

1.3 Online study resources

In addition to the subject guide and the Essential reading, it is crucial that you take advantage of the study resources that are available online for this course, including the virtual learning environment (VLE) and the Online Library.

¹There are many editions and variants of this book, such as the ‘Applications version’. Any one is equally useful.

You can access the VLE, the Online Library and your University of London email account via the Student Portal at:

<http://my.londoninternational.ac.uk>

You should have received your login details for the Student Portal with your official offer, which was emailed to the address that you gave on your application form. You have probably already logged in to the Student Portal in order to register! As soon as you registered, you will automatically have been granted access to the VLE, Online Library and your fully functional University of London email account.

If you forget your login details at any point, please email uolia.support@london.ac.uk quoting your student number.

1.3.1 The VLE

The VLE, which complements this subject guide, has been designed to enhance your learning experience, providing additional support and a sense of community. It forms an important part of your study experience with the University of London and you should access it regularly.

The VLE provides a range of resources for EMFSS courses:

- Self-testing activities: Doing these allows you to test your own understanding of subject material.
- Electronic study materials: The printed materials that you receive from the University of London are available to download, including updated reading lists and references.
- Past examination papers and *Examiners' commentaries*: These provide advice on how each examination question might best be answered.
- A student discussion forum: This is an open space for you to discuss interests and experiences, seek support from your peers, work collaboratively to solve problems and discuss subject material.
- Videos: There are recorded academic introductions to the subject, interviews and debates and, for some courses, audio-visual tutorials and conclusions.
- Recorded lectures: For some courses, where appropriate, the sessions from previous years' Study Weekends have been recorded and made available.
- Study skills: Expert advice on preparing for examinations and developing your digital literacy skills.
- Feedback forms.

Some of these resources are available for certain courses only, but we are expanding our provision all the time and you should check the VLE regularly for updates.

1.3.2 Making use of the Online Library

The Online Library contains a huge array of journal articles and other resources to help you read widely and extensively.

To access the majority of resources via the Online Library you will either need to use your University of London Student Portal login details, or you will be required to register and use an Athens login:

<http://tinyurl.com/ollathens>

The easiest way to locate relevant content and journal articles in the Online Library is to use the **Summon** search engine.

If you are having trouble finding an article listed in a reading list, try removing any punctuation from the title, such as single quotation marks, question marks and colons.

For further advice, please see the online help pages:

<http://www.external.shl.lon.ac.uk/summon/about.php>

1.4 Using the guide

As already mentioned, it is important that you read textbooks in conjunction with the guide and that you try problems from the textbooks. The Sample examination questions at the end of the chapters of this guide are a very useful resource. You should try them once you think you have mastered a particular chapter. **Really** try them: don't just simply read the solutions provided. Instead, make a serious attempt before consulting the solutions. Note that the solutions are often just sketch solutions, to indicate to you how to answer the questions. However, in the examination, you must show all your reasoning. It is vital that you develop and enhance your problem-solving skills and the only way to do this is to try lots of examples.

Finally, we often use the symbol \square to denote the end of a proof, where we have finished explaining why a particular result is true. This is just to make it clear where the proof ends and the following text begins.

1.5 Examination

Important: the information and advice given here are based on the examination structure used at the time this guide was written. Please note that subject guides may be used for several years. Because of this we strongly advise you to always check both the current *Regulations* for relevant information about the examination, and the virtual learning environment (VLE) where you should be advised of any forthcoming changes. You should also carefully check the rubric/instructions on the paper you actually sit and follow those instructions. Remember, it is important to check the VLE for:

- up-to-date information on examination and assessment arrangements for this course
- where available, past examination papers and *Examiners' commentaries* for the course which give advice on how each question might best be answered.

A Sample examination paper is given as an appendix to this guide. There are no optional topics in this subject: you should study them all. The examination paper will provide some element of choice as to which questions you attempt: see the Sample examination paper at the end of the subject guide for an indication of the structure of the examination paper.

Please do not assume that the questions in a real examination will necessarily be very similar to these sample questions. An examination is designed (by definition) to test

you. You *will* get examination questions unlike questions in this guide. The whole point of examining is to see whether you can apply knowledge in familiar *and* unfamiliar settings. The Examiners (nice people though they are) have an obligation to surprise you! For this reason, it is important that you try as many examples as possible, from the guide and from the textbooks. This is not so that you can cover any possible type of question the Examiners can think of! It's so that you get used to confronting unfamiliar questions, grappling with them, and finally coming up with the solutions.

Do not panic if you cannot completely solve an examination question. There are many marks to be awarded for using the correct approach or method.

1.6 The use of calculators

You will not be permitted to use calculators of any type in the examination. This is not something that you should worry about: the Examiners are interested in assessing that you understand the key concepts, ideas, methods and techniques, and will set questions which do not require the use of a calculator.

Part 1

Numbers and Proof

Chapter 2

Mathematical statements, proof, logic and sets

Essential reading

One or both of the following:

 Biggs, N.L. *Discrete Mathematics*. Chapters 1–3.

 Eccles, P.J. *An Introduction to Mathematical Reasoning*. Chapters 1–4 and 6.

2.1 Introduction

In this important chapter, we set the ground for much of what follows in this course. Abstract mathematics is about making precise mathematical statements and establishing, by proof or disproof, whether these statements are true or false. In this chapter we look at what this means, concentrating on fairly simple types of mathematical statement, in order to emphasise techniques of proof. In later chapters (such as those on numbers, analysis and algebra) we will use these proof techniques extensively. You might think that some of the things we prove in this chapter are very obvious and hardly merit proving, but proving even ‘obvious’ statements can be quite tricky sometimes, and it is good preparation for proving more complicated things later on.

2.2 Mathematical statements and proof

To introduce the topics of mathematical statements and proof, we start by giving some explicit examples. Later in the chapter we give some general theory and principles. Our discussion of the general theory is limited because this is not a course in logic, as such. What we do need is enough logic to understand what mathematical statements mean and how we might prove or disprove them.

2.2.1 Examples of mathematical statements

Consider the following statements (in which, you should recall that the natural numbers are the positive integers):

- (a) 20 is divisible by 4.

- (b) 21 is not divisible by 7.
- (c) 21 is divisible by 4.
- (d) 21 is divisible by 3 or 5.
- (e) 50 is divisible by 2 and 5.
- (f) n^2 is even.
- (g) For every natural number n , the number $n^2 + n$ is even.
- (h) There is a natural number n such that $2n = 2^n$.
- (i) If n is even, then n^2 is even.
- (j) For all odd numbers n , n^2 is odd.
- (k) For natural numbers n , n^2 is even if and only if n is even.
- (l) There are no natural numbers m and n such that $\sqrt{2} = m/n$.

These are all mathematical statements, of different sorts (all of which will be discussed in more detail in the remainder of this chapter).

Statements (a) to (e) are straightforward *propositions* about certain numbers, and these are either true or false. Statements (d) and (e) are examples of *compound statements*. Statement (d) is true precisely when *either one (or both)* of the statements ‘21 is divisible by 3’ and ‘21 is divisible by 5’ is true. Statement (e) is true precisely when *both* of the statements ‘50 is divisible by 2’ and ‘50 is divisible by 5’ are true.

Statement (f) is different, because the number n is not specified and whether the statement is true or false will depend on the value of the so-called ‘free variable’ n . Such a statement is known as a *predicate*.

Statement (g) makes an assertion about *all* natural numbers and is an example of a *universal statement*.

Statement (h) asserts the existence of a particular number and is an example of an *existential* statement.

Statement (i) can be considered as an assertion about all even numbers, and so it is a universal statement, where the ‘universe’ is all even numbers. But it can also be considered as an *implication*, asserting that *if* n happens to be even, *then* n^2 is even.

Statement (j) is a universal statement about all odd numbers. It can also be thought of (or rephrased) as an implication, for it says precisely the same as ‘if n is odd, then n^2 is odd’.

Statement (k) is an ‘if and only if’ statement: what it says is that n^2 is even, for a natural number n , *precisely when* n is even. But this means two things: namely that n^2 is even if n is even, and n is even if n^2 is even. Equivalently, it means that n^2 is even if n is even and that n^2 is odd if n is odd. So statement (k) will be true precisely if (i) and (j) are true.

Statement (l) asserts the non-existence of a certain pair of numbers (m, n) . Another way of thinking about this statement is that it says that for all choices of (m, n) , it is

not the case that $m/n = \sqrt{2}$. (This is an example of the general rule that a non-existence statement can be thought of as a universal statement, something to be discussed later in more detail.)

It's probably worth giving some examples of things that are *not* proper mathematical statements.

For example, '6 is a nice number' is not a mathematical statement. This is because 'nice number' has no mathematical meaning. However, if, beforehand, we had *defined* 'nice number' in some way, then this would not be a problem. For example, suppose we said:

Let us say that a number is *nice* if it is the sum of all the positive numbers that divide it and are less than it.

Then '6 is a nice number' would be a proper mathematical statement, and it would be true, because 6 has positive divisors 1, 2, 3, 6 and $6 = 1 + 2 + 3$. But without defining what 'nice' means, it's not a mathematical statement. Definitions are important.

' $n^2 + n$ ' is not a mathematical statement, because it does not say anything about $n^2 + n$. It is not a mathematical statement in the same way that 'David Cameron' is not a sentence: it makes no assertion about what David Cameron is or does. However, ' $n^2 + n > 0$ ' is an example of a *predicate* with free variable n and, for a particular value of n , this is a mathematical statement. Likewise, 'for all natural numbers n , $n^2 + n > 0$ ' is a mathematical statement.

2.2.2 Introduction to proving statements

We've seen, above, various types of mathematical statement, and such statements are either true or false. But how would we establish the truth or falsity of these?

We can, even at this early stage, prove (by which we mean establish the truth of) or disprove (by which we mean establish the falsity of) most of the statements given above. Here's how we can do this.

- (a) 20 is divisible by 4.

This statement is true. Yes, yes, I know it's 'obvious', but stay with me. To give a proper proof, we need first to understand exactly what the word 'divisible' means. You will probably most likely think that this means that when we divide 20 by 4 we get no remainder. This is correct: in general, for natural numbers n and d , to say that n is divisible by d (or, equivalently, that n is a multiple of d) means precisely that there is some natural number m for which $n = md$. Since $20 = 5 \times 4$, we see that 20 is divisible by 4. And that's a proof! It's utterly convincing, watertight, and not open to debate. Nobody can argue with it, not even a sociologist! Isn't this fun? Well, maybe it's not that impressive in such a simple situation, but we will certainly prove more impressive results later.

- (b) 21 is not divisible by 7.

This is false. It's false because 21 *is* divisible by 7, because $21 = 3 \times 7$.

- (c) 21 is divisible by 4.

This is false, as can be established in a number of ways. First, we note that if the natural number m satisfies $m \leq 5$, then $m \times 4$ will be no more than 20. And if $m \geq 6$ then $m \times 4$ will be at least 24. Well, any natural number m is either at most 5 or at least 6 so, for all possible m , we do not have $m \times 4 = 21$ and hence there is no natural number m for which $m \times 4 = 21$. In other words, 21 is not divisible by 4. Another argument (which is perhaps more straightforward, but which relies on properties of rational numbers rather than just simple properties of natural numbers) is to note that $21/4 = 5.25$, and this is not a natural number, so 21 is not divisible by 4. (This second approach is the same as showing that 21 has remainder 1, not 0, when we divide by 4.)

- (d) 21 is divisible by 3 or 5.

As we noted above, this is a compound statement and it will be true precisely when one (or both) of the following statements is true:

- (i) 21 is divisible by 3
- (ii) 21 is divisible by 5.

Statement (i) is true, because $21 = 7 \times 3$. Statement (ii) is false. Because at least one of these two statements is true, statement (d) is true.

- (e) 50 is divisible by 2 and 5.

This is true. Again, this is a compound statement and it is true precisely if *both* of the following statements are true:

- (i) 50 is divisible by 2
- (ii) 50 is divisible by 5.

Statements (i) and (ii) are indeed true because $50 = 25 \times 2$ and $50 = 10 \times 5$. So statement (e) is true.

- (f) n^2 is even.

As mentioned above, whether this is true or false depends on the value of n . For example, if $n = 2$ then $n^2 = 4$ is even, but if $n = 3$ then $n^2 = 9$ is odd. So, unlike the other statements (which are *propositions*), this is a *predicate* $P(n)$. The predicate will become a proposition when we assign a particular value to n to it, and the truth or falsity of the proposition can then be established. Statements (i), (j), (k) below do this comprehensively.

- (g) For every natural number n , the number $n^2 + n$ is even.

Here's our first non-immediate, non-trivial, proof. How on earth can we prove this, if it is true, or disprove it, if it is false? Suppose it was false. How would you convince someone of that? Well, the statement says that *for every* natural number n , $n^2 + n$ is even. So if you managed (somehow!) to find a particular N for which $N^2 + N$ happened to be odd, you could prove the statement false by simply observing that 'When $n = N$, it is *not* the case that $n^2 + n$ is even.' And that would be the end of it. So, in other words, if a universal statement about natural numbers is false, you can prove it is false by showing that its conclusion is false for *some particular* value of n . But suppose the statement is true. How could you prove it. Well, you could prove it for $n = 1$, then $n = 2$, then $n = 3$, and so on, but at

some point you would expire and there would still be numbers n that you hadn't yet proved it for. And that simply wouldn't do, because if you proved it true for the first 9999 numbers, it might be false when $n = 10000$. So what you need is a more sophisticated, *general* argument that shows the statement is true for any *arbitrary* n .

Now, it turns out that this statement is true. So we need a nice general argument to establish this. Well, here's one approach. We can note that $n^2 + n = n(n + 1)$. The numbers n and $n + 1$ are consecutive natural numbers. So one of them is odd and one of them is even. When you multiply any odd number and any even number together, you get an even number, so $n^2 + n$ is even. Are you convinced? Maybe not? We really should be more explicit. Suppose n is even. What that means is that, for some integer k , $n = 2k$. Then $n + 1 = 2k + 1$ and hence

$$n(n + 1) = 2k(2k + 1) = 2(k(2k + 1)).$$

Because $k(2k + 1)$ is an integer, this shows that $n^2 + n = n(n + 1)$ is divisible by 2; that is, it is even. We supposed here that n was even. But it might be odd, in which case we would have $n = 2k + 1$ for some integer k . Then

$$n(n + 1) = (2k + 1)(2k + 2) = 2((2k + 1)(k + 1)),$$

which is, again, even, because $(2k + 1)(k + 1)$ is an integer.

Right, we're really proving things now. This is a very general statement, asserting something about *all* natural numbers, and we have managed to prove it. I find that quite satisfying, don't you?

- (h) There is a natural number n such that $2n = 2^n$.

This is an *existential statement*, asserting that *there exists* n with $2n = 2^n$. Before diving in, let's pause for a moment and think about how we might deal with such statements. If an existential statement like this is true we would need only to show that its conclusion (which in this case is $2n = 2^n$) holds for some particular n . That is, we need only find an n that works. If the statement is false, we have a lot more work to do in order to prove that it is false. For, to show that it is false, we would need to show that, for *no* value of n does the conclusion hold. Equivalently, for *every* n , the conclusion fails. So we'd need to prove a universal statement and, as we saw in the previous example, that would require us to come up with a suitably general argument.

In fact, this statement is true. This is because when $n = 1$ we have $2n = 2 = 2^1 = 2^n$.

- (i) If n is even, then n^2 is even.

This is true. The most straightforward way to prove this is to assume that n is some (that is, *any*) even number and then show that n^2 is even. So suppose n is even. Then $n = 2k$ for some integer k and hence $n^2 = (2k)^2 = 4k^2$. This is even because it is $2(2k^2)$ and $2k^2$ is an integer.

- (j) For all odd numbers n , n^2 is odd.

This is true. The most straightforward way to prove this is to assume that n is *any* odd number and then show that n^2 is also odd. So suppose n is odd. Then

$n = 2k + 1$ for some integer k and hence $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1$. To establish that this is odd, we need to show that it can be written in the form $2K + 1$ for some integer K . Well, $4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. This is indeed of the form $2K + 1$, where K is the integer $2k^2 + 2k$. Hence n^2 is odd.

Another way to prove this result is to prove that if n^2 is even then n must be even. We won't do that right now, because to do it properly requires a result we meet later concerning the factorisation of numbers into prime numbers. But think about the strategy for a moment. Suppose we were able to prove the following statement, which we'll call Q :

Q : If n^2 is even then n is even.

Why would that establish what we want (namely that if n is odd then n^2 is odd)? Well, one way is to observe that Q is what's called the *contrapositive* of statement (j) that we're trying to prove, and the contrapositive is *logically equivalent* to the initial statement. (This is a bit of formal logic, and we will discuss this in more detail later). But there's another way of thinking about it, which is perhaps easier to understand at this stage. Suppose we have proved statement Q and suppose that n is odd. Then it must be the case that n^2 is odd. For, if n^2 was not odd, it would be even and then Q would tell us that this means n is even. But we have assumed n is odd. It cannot be both even and odd, so we have reached a contradiction. By assuming that the opposite conclusion holds (n^2 even) we have shown that something impossible happens. This type of argument is known as a *proof by contradiction* and it is often very powerful. We will see more about this later.

- (k) For natural numbers n , n^2 is even if and only if n is even.

This is true. What we have shown in proving (i) and (j) is that if n is even then n^2 is even, and if n is odd then n^2 is odd. The first, (statement (i)) establishes that *if* n is even, then n^2 is even. The second of these (statement (j)) establishes that n^2 is even *only if* n is even. This is because it shows that n^2 is odd if n is odd, from which it follows that if n^2 is even, n must not have been odd, and therefore must have been even. 'If and only if' statements of this type are very important. As we see here, the proof of such statements breaks down into the proof of two 'If-then' statements.

- (l) There are no natural numbers m and n such that $\sqrt{2} = m/n$.

This is, in fact, true, though we defer the proof for now, until we know more about factorisation of numbers into prime numbers. We merely comment that the easiest way to prove the statement is to use a proof by contradiction.

These examples hopefully demonstrate that there are a wide range of statements and proof techniques, and in the rest of this chapter we will explore these further.

Right now, one thing I hope comes out very clearly from these examples is that to prove a mathematical statement, you need to know precisely what it means. Well, that sounds obvious, but you can see how detailed we had to be about the meanings (that is, the *definitions*) of the terms 'divisible', 'even' and 'odd'. Definitions are very important.

2.3 Some basic logic

Mathematical statements can be true or false. Let's denote 'true' by T and 'false' by F. Given a statement, or a number of statements, it is possible to form other statements. This was indicated in some of the examples above (such as the compound statements). A technique known as the use of 'truth tables' enables us to define 'logical operations' on statements, and to determine when such statements are true. This is all a bit vague, so let's get down to some concrete examples.

2.3.1 Negation

The simplest way to take a statement and form another statement is to *negate* the statement. The *negation* of a statement P is the statement $\neg P$ (sometimes just denoted 'not P '), which is defined to be true exactly when P is false. This can be described in the very simple truth table, Table 2.1:

P	$\neg P$
T	F
F	T

Table 2.1: The truth table for 'negation' or 'not'

What does the table signify? Quite simply, it tells us that if P is true then $\neg P$ is false and if P is false then $\neg P$ is true.

Example 2.1 If P is '20 is divisible by 3' then $\neg P$ is '20 is not divisible by 3'. Here, P is false and $\neg P$ is true.

It has, I hope, been indicated in the examples earlier in this chapter, that to disprove a universal statement about natural numbers amounts to proving an existential statement. That is, if we want to disprove a statement of the form 'for all natural numbers n , property $p(n)$ holds' (where $p(n)$ is some predicate, such as ' n^2 is even') we need only produce some N for which $p(N)$ fails. Such an N is called a *counterexample*. Equally, to disprove an existential statement of the form 'there is some n such that property $p(n)$ holds', one would have to show that for *every* n , $p(n)$ fails. That is, to disprove an existential statement amounts to proving a universal one. But, now that we have the notion of the negation of a statement we can phrase this a little more formally. Proving that a statement P is false is equivalent to proving that the negation $\neg P$ is true. In the language of logic, therefore, we have the following:

- The negation of a universal statement is an existential statement.
- The negation of an existential statement is a universal statement.

More precisely,

- The negation of the universal statement 'for all n , property $p(n)$ holds' is the existential statement 'there is n such that property $p(n)$ does not hold'.

- The negation of the existential statement ‘there is n such that property $p(n)$ holds’ is the universal statement ‘for all n , property $p(n)$ does not hold’.

We could be a little more formal about this, by defining the negation of a predicate $p(n)$ (which, recall, only has a definitive true or false value once n is specified) to be the predicate $\neg p(n)$ which is true (for any particular n) precisely when $p(n)$ is false. Then we might say that

- The negation of the universal statement ‘for all n , $p(n)$ is true’ is the existential statement ‘there is n such that $\neg p(n)$ is true’.
- The negation of the existential statement ‘there is n such that $p(n)$ is true’ is the universal statement ‘for all n , $\neg p(n)$ is true’.

Now, let’s not get confused here. None of this is really difficult or new. We meet such logic in everyday life. If I say ‘It rains every day in London’ then either this statement is true or it is false. If it is false, it is because on (at least) one day it does not rain. The negation (or disproof) of the statement ‘On every day, it rains in London’ is simply ‘There is a day on which it does not rain in London’. The former is a universal statement (‘On every day, ...’) and the latter is an existential statement (‘there is ...’). Or, consider the statement ‘There is a student who enjoys reading these lecture notes’. This is an existential statement (‘There is ...’). This is false if ‘No student enjoys reading these lecture notes’. Another way of phrasing this last statement is ‘Every student reading these lecture notes does not enjoy it’. This is a more awkward expression, but it emphasises that the negation of the initial, existential statement, is a universal one (‘Every student ...’).

The former is an existential statement (‘there is something I will write that ...’) and the latter is a universal statement (‘everything I write will ...’). This second example is a little more complicated, but it serves to illustrate the point that much of logic is simple common sense.

2.3.2 Conjunction and disjunction

There are two very basic ways of combining propositions: through the use of ‘and’ (known as conjunction) and the use of ‘or’ (known as disjunction).

Suppose that P and Q are two mathematical statements. Then ‘ P and Q ’, also denoted $P \wedge Q$, and called the *conjunction* of P and Q , is the statement that is true precisely when *both* P and Q are true. For example, statement (e) above, which is

‘50 is divisible by 2 and 5’

is the conjunction of the two statements

- 50 is divisible by 2
- 50 is divisible by 5.

Statement (e) is true because *both* of these two statements are true.

Table 2.2 gives the truth table for the conjunction P and Q . What Table 2.2 says is simply that $P \wedge Q$ is true precisely when *both* P and Q are true (and in no other circumstances).

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 2.2: The truth table for ‘and’

Suppose that P and Q are two mathematical statements. Then ‘ P or Q ’, also denoted $P \vee Q$, and called the *disjunction* of P and Q , is the statement that is true precisely when P , or Q , or both, are true. For example, statement (d) above, which is

‘21 is divisible by 3 or 5’

is the disjunction of the two statements

- 21 is divisible by 3
- 21 is divisible by 5.

Statement (d) is true because at least one (namely the first) of these two statements is true.

Note one important thing about the mathematical interpretation of the word ‘or’. It is *always* used in the ‘inclusive-or’ sense. So $P \vee Q$ is true in the case when P is true, or Q is true, or *both*. In some ways, this use of the word ‘or’ contrasts with its use in normal everyday language, where it is often used to specify a choice between mutually exclusive alternatives. (For example ‘You’re either with us or against us’.) But if I say ‘Tomorrow I will wear brown trousers or I will wear a yellow shirt’ then, in the mathematical way in which the word ‘or’ is used, the statement would be true if I wore brown trousers and any shirt, any trousers and a yellow shirt, and also if I wore brown trousers and a yellow shirt. You might have your doubts about my dress sense in this last case, but, logically, it makes my statement true.

Table 2.3 gives the truth table for the disjunction P and Q .

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Table 2.3: The truth table for ‘or’

What Table 2.3 says is simply that $P \vee Q$ is true precisely when *at least one of* P and Q is true.

2.4 If-then statements

It is very important to understand the formal meaning of the word ‘if’ in mathematics. The word is often used rather sloppily in everyday life, but has a very precise

mathematical meaning. Let me give you an example. Suppose I tell you ‘If it rains, then I wear a raincoat’, and suppose that this is a true statement. Well, then, suppose it rains. You can certainly conclude I will wear a raincoat. But what if it does not rain? Well, you can’t conclude anything. My statement only tells you about what happens *if* it rains. If it does not, then I might, or I might not, wear a raincoat: and whether I do or not does not affect the truth of the statement I made. You have to be clear about this: an ‘if-then’ statement only tells you about what follows *if* something particular happens. More formally, suppose P and Q are mathematical statements (each of which can therefore be either true or false). Then we can form the statement denoted $P \Rightarrow Q$ (P implies Q ’ or, equivalently, ‘if P , then Q ’), which has as its truth table Table 2.4. (This type of statement is known as an *if-then* statement or an *implication*.)

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 2.4: The truth table for ‘ $P \Rightarrow Q$ ’

Note that the statement $P \Rightarrow Q$ is false only when P is true but Q is false. (To go back to the previous example, the statement ‘If it rains, I wear a raincoat’ is false precisely if it does rain but I do not wear a raincoat.) This is tricky, so you may have to spend a little time understanding it. As I’ve suggested, perhaps the easiest way is to think about when a statement ‘if P , then Q ’ is false.

The statement $P \Rightarrow Q$ can also be written as $Q \Leftarrow P$. There are different ways of describing $P \Rightarrow Q$, such as:

- if P then Q
- P implies Q
- P is sufficient for Q
- Q if P
- P only if Q
- Q whenever P
- Q is necessary for P .

All these mean the same thing. The first two are the ones I will use most frequently.

If $P \Rightarrow Q$ and $Q \Rightarrow P$ then this means that Q will be true precisely when P is. That is Q is true *if and only if* P is. We use the single piece of notation $P \Longleftrightarrow Q$ instead of the two separate $P \Rightarrow Q$ and $Q \Leftarrow P$. There are several phrases for describing what $P \Longleftrightarrow Q$ means, such as:

- P if and only if Q (sometimes abbreviated to ‘ P iff Q ’)

- P is equivalent to Q
- P is necessary and sufficient for Q
- Q is necessary and sufficient for P .

The truth table is shown in Table 2.5, where we have also indicated the truth or falsity of $P \Rightarrow Q$ and $Q \Rightarrow P$ to emphasise that $P \Longleftrightarrow Q$ is the same as the conjunction $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Longleftrightarrow Q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Table 2.5: The truth table for ' $P \Longleftrightarrow Q$ '

What the table shows is that $P \Longleftrightarrow Q$ is true precisely when P and Q are either both true or both false.

Activity 2.1 Look carefully at the truth table and understand why the values for $P \Longleftrightarrow Q$ are as they are. In particular, try to explain in words why the truth table is the way it is.

2.5 Logical equivalence

Two statements are *logically equivalent* if when either one is true, so is the other, and if either one is false, so is the other. For example, for statements P and Q , the statements $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$ are logically equivalent. We can see this from the truth table, Table 2.6, which shows that, in all cases, the two statements take the same logical value T or F . (This value is highlighted in bold.)

P	Q	$P \vee Q$	$\neg(P \vee Q)$	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Table 2.6: The truth tables for $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$

The fact that $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$ are logically equivalent is quite easy to understand. The statement $P \vee Q$ is true if and only if at least one of P, Q is true. The statement is therefore false precisely when *both* P and Q are false, which means $\neg P$ and $\neg Q$ are both true, which means $\neg P \wedge \neg Q$ is true. Again, we can understand these things fairly easily with some common sense. If I tell you 'I will wear brown trousers or

I will wear a yellow shirt' then this is a false statement only if I *do not* wear brown trousers *and* I *do not* wear a yellow shirt.

Now that we know the meaning of \iff , we can see that to say that $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$ are logically equivalent is to say that $\neg(P \vee Q) \iff \neg P \wedge \neg Q$.

Activity 2.2

Show that the statements $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$ are logically equivalent. [This shows that the negation of $P \wedge Q$ is $\neg P \vee \neg Q$. That is, $\neg(P \wedge Q)$ is equivalent to $\neg P \vee \neg Q$.]

2.6 Converse statements

Given an implication $P \Rightarrow Q$, the 'reverse' implication $Q \Rightarrow P$ is known as its *converse*. Generally, there is no reason why the converse should be true just because the implication is. For example, consider the statement 'If it is Tuesday, then I buy the Guardian newspaper.' The converse is 'If I buy the Guardian newspaper, then it is Tuesday'. Well, I might buy that newspaper on other days too, in which case the implication can be true but the converse false. We've seen, in fact, that if both $P \Rightarrow Q$ and $Q \Rightarrow P$ then we have a special notation, $P \iff Q$, for this situation. Generally, then, the truth or falsity of the converse $Q \Rightarrow P$ has to be determined separately from that of the implication $P \Rightarrow Q$.

Activity 2.3 What is the converse of the statement 'if the natural number n divides 4 then n divides 12'? Is the converse true? Is the original statement true?

2.7 Contrapositive statements

The *contrapositive* of an implication $P \Rightarrow Q$ is the statement $\neg Q \Rightarrow \neg P$. The contrapositive is logically equivalent to the implication, as Table 2.7 shows. (The columns highlighted in bold are identical.)

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg Q$	$\neg Q \Rightarrow \neg P$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

Table 2.7: The truth tables for $P \Rightarrow Q$ and $\neg Q \Rightarrow \neg P$.

If you think about it, the equivalence of the implication and its contrapositive makes sense. For, $\neg Q \Rightarrow \neg P$ says that if Q is false, P is false also. So, it tells us that we cannot have Q false and P true, which is precisely the same information as is given by $P \Rightarrow Q$.

So what's the point of this? Well, sometimes you might want to prove $P \Rightarrow Q$ and it will, in fact, be easier to prove instead the equivalent (contrapositive) statement $\neg Q \Rightarrow \neg P$. See Biggs, section 3.5 for an example.

2.8 Working backwards to obtain a proof

We've already seen, in the examples earlier in this chapter, how some statements may be proved directly. For example, in order to prove a universal statement 'for all n , $P(n)$ ' about natural numbers, we would need to provide a proof that starts by assuming that n is any given (that is, *arbitrary*) natural number and show the desired conclusion holds. To disprove such a statement (which is the same as proving its negation), we would simply need to find a single value of n for which $P(n)$ is false (and such an n is known as a *counterexample*).

However, some statements are difficult to prove directly. It is sometimes easier to 'work backwards'. Suppose you are asked to prove something, such as an inequality or equation. It might be easier to see how to do so if the end-result (the inequality or equation you are required to prove) is simplified, or expanded, or re-written in some way. Here's an example.

Example 2.2 Prove the statement that: 'if a, b are real numbers and $a \neq b$, then $ab < (a^2 + b^2)/2$ '.

It's certainly not immediately obvious how to approach this. But let's start with what we want to prove. This is the inequality $ab < (a^2 + b^2)/2$, which can be rewritten as $a^2 + b^2 - 2ab > 0$. Now, this can be simplified as $(a - b)^2 > 0$ and maybe now you can see why it is true: the given fact that $a \neq b$ means that $a - b \neq 0$ and hence $(a - b)^2$ is a positive number. So we see why the statement is true. To write down a nice proof, we can now reverse this argument, as follows:

Proof

Since $a \neq b$, $a - b \neq 0$ and, hence, $(a - b)^2 > 0$. But $(a - b)^2 = a^2 + b^2 - 2ab$. So we have $a^2 + b^2 > 2ab$ and, therefore, $ab < (a^2 + b^2)/2$, as required. \square

There are a few things to note here. First, mathematics is a language and what you write has to make good sense. Often, it is tempting to make too much use of symbols rather than words. But the words used in this proof, and the punctuation, make it easy to read and give it a structure and an argument. You should find yourself using words like 'so', 'hence', 'therefore', 'since', 'because', and so on. *Do* use words and punctuation and, whatever you do, do not replace them by symbols of your own invention! A second thing to note is the use of the symbol ' \square '. There is nothing particularly special about this symbol: others could be used. What it achieves is that it indicates that the proof is finished. There is no need to use such a symbol, but you will find that textbooks do make much use of symbols to indicate when proofs have ended. It enables the text to be more readable, with proofs not running into the main body of the text. Largely, these are matters of style, and you will develop these as you practise and read the textbooks.

2.9 Sets

2.9.1 Basics

You have probably already met some basic ideas about sets and there is not too much more to add at this stage, but they are such an important idea in abstract mathematics that they are worth discussing here.

Loosely speaking, a set may be thought of as a collection of objects. A set is usually described by listing or describing its *members* inside curly brackets. For example, when we write $A = \{1, 2, 3\}$, we mean that the objects belonging to the set A are the numbers 1, 2, 3 (or, equivalently, the set A consists of the numbers 1, 2 and 3). Equally (and this is what we mean by ‘describing’ its members), this set could have been written as

$$A = \{n \mid n \text{ is a whole number and } 1 \leq n \leq 3\}.$$

Here, the symbol \mid stands for ‘such that’. Often, the symbol ‘:’ is used instead, so that we might write

$$A = \{n : n \text{ is a whole number and } 1 \leq n \leq 3\}.$$

When x is an object in a set A , we write $x \in A$ and say ‘ x belongs to A ’ or ‘ x is a member of A ’. If x is not in A we write $x \notin A$.

As another example, the set

$$B = \{x \in \mathbb{N} \mid x \text{ is even}\}$$

has as its members the set of positive even integers. Here we are specifying the set by *describing* the defining property of its members.

Sometimes it is useful to give a *constructional* description of a set. For example, $C = \{n^2 \mid n \in \mathbb{N}\}$ is the set of natural numbers known as the ‘perfect squares’.

The set which has no members is called the *empty set* and is denoted by \emptyset . The empty set may seem like a strange concept, but it has its uses.

2.9.2 Subsets

We say that the set S is a *subset* of the set T , and we write $S \subseteq T$, if every member of S is a member of T . For example, $\{1, 2, 5\} \subseteq \{1, 2, 4, 5, 6, 40\}$. (Be aware that some texts use \subset where we use \subseteq .) What this means is that the statement

$$x \in S \Rightarrow x \in T$$

is true.

A rather obvious, but sometimes useful, observation is that, given two sets A and B , $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$. So to prove two sets are equal, we can prove that each of these two ‘containments’ holds. That might seem clumsy, but it is, in many cases, the best approach.

For any set A , the empty set, \emptyset , is a subset of A . You might think this is strange, because what it means is that ‘every member of \emptyset is also a member of A ’. But \emptyset has no members! The point, however, is that there is no object in \emptyset that is *not* also in A (because there are no objects at all in \emptyset).

2.9.3 Unions and intersections

Given two sets A and B , the *union* $A \cup B$ is the set whose members belong to A or B (or both A and B): that is,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

Equivalently, to use the notation we've learned,

$$x \in A \cup B \iff (x \in A) \vee (x \in B).$$

Example 2.3 If $A = \{1, 2, 3, 5\}$ and $B = \{2, 4, 5, 7\}$, then $A \cup B = \{1, 2, 3, 4, 5, 7\}$.

Similarly, we define the *intersection* $A \cap B$ to be the set whose members belong to both A and B :

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

So,

$$x \in A \cap B \iff (x \in A) \wedge (x \in B).$$

2.9.4 Universal sets and complements

We've been a little informal about what the possible 'objects' in a set might be. Officially, we always work with respect to some 'universal set' E . For example, if we are thinking about sets of natural numbers, the universal set (the possible candidates for membership of the sets we might want to consider) is the set \mathbb{N} of all natural numbers. This might seem like an unnecessary complication, but it is essential. Suppose I tell you that the set A is the set of all even natural numbers. What are the objects that do not belong to A ? Well, in the context of natural numbers, it is all odd natural numbers. The context is important (and it is this that is encapsulated in the universal set). Without that context (or universal set), then there are many other objects that we could say do not belong to A , such as negative integers, apples, bananas and elephants. (I could go on, but I hope you get the point!)

Given a universal set E and a subset A of E , the *complement* of A (sometimes called the *complement of A in E*) is denoted by $E \setminus A$ and is

$$E \setminus A = \{x \in E \mid x \notin A\}.$$

If the universal set is clear, the complement of A is sometimes denoted by \bar{A} or A^c (with textbooks differing in their notation).

Suppose A is any subset of E . Because each member of E is either a member of A , or is not a member of A , it follows that

$$A \cup (E \setminus A) = E.$$

2.9.5 Sets and logic

There are a great many comparisons and analogies between set theory and logic. Using the shorthand notation for complements, one of the 'De-Morgan' laws of complementation is that

$$\overline{A \cap B} = \bar{A} \cup \bar{B}.$$

This looks a little like the fact (observed in an earlier Activity) that $\neg(P \wedge Q)$ is equivalent to $\neg P \vee \neg Q$. And this is more than a coincidence. The negation operation, the conjunction operation, and the disjunction operation on statements behave entirely in the same way as the complementation, intersection, and union operations (in turn) on sets. In fact, when you start to prove things about sets, you often end up giving arguments that are based in logic.

For example, how would we prove that $\overline{A \cap B} = \bar{A} \cup \bar{B}$? We could argue as follows:

$$\begin{aligned}
 x \in \overline{A \cap B} &\iff x \notin A \cap B \\
 &\iff \neg(x \in A \cap B) \\
 &\iff \neg((x \in A) \wedge (x \in B)) \\
 &\iff \neg(x \in A) \vee \neg(x \in B) \\
 &\iff (x \in \bar{A}) \vee (x \in \bar{B}) \\
 &\iff x \in \bar{A} \cup \bar{B}.
 \end{aligned}$$

What the result says is, in fact, easy to understand: if x is not in *both* A and B , then that's precisely because it fails to be in (at least) one of them.

For two sets A and B (subsets of a universal set E), the *complement of B in A* , denoted by $A \setminus B$, is the set of objects that belong to A but not to B . That is,

$$A \setminus B = \{x \in A \mid x \notin B\}.$$

Activity 2.4 Prove that $A \setminus B = A \cap (E \setminus B)$.

2.9.6 Cartesian products

For sets A and B , the *Cartesian product* $A \times B$ is the set of all *ordered pairs* (a, b) , where $a \in A$ and $b \in B$. For example, if $A = B = \mathbb{R}$ then $A \times B = \mathbb{R} \times \mathbb{R}$ is the set of all ordered pairs of real numbers, usually denoted by \mathbb{R}^2 .

2.9.7 Power sets

For a set A , the set of all subsets of A , denoted $\mathcal{P}(A)$, is called the *power set* of A . Note that the power set is a set of sets. For example, if $A = \{1, 2, 3\}$, then

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Activity 2.5 Write down the power set of the set $A = \{1, 2, 3, 4\}$.

Activity 2.6

Suppose that A has n members, where $n \in \mathbb{N}$. How many members does $\mathcal{P}(A)$ have?

2.10 Quantifiers

We have already met the ideas of universal and existential statements involving natural numbers. More generally, given any set E , a *universal statement* on E is one of the form

‘for all $x \in E$, $P(x)$ ’. This statement is true if $P(x)$ is true for all x in E , and it is false if there is some x in E (known as a *counterexample*) such that $P(x)$ is false. We have a special symbol that is used in universal statements: the symbol ‘ \forall ’ means ‘for all’. So the typical universal statement can be written as

$$\forall x \in E, P(x).$$

(The comma is not necessary, but I think it looks better.) An *existential statement* on E is one of the form ‘there is $x \in E$ such that $P(x)$ ’, which is true if there is some $x \in E$ for which $P(x)$ is true, and is false if for every $x \in E$, $P(x)$ is false. Again, we have a useful symbol, ‘ \exists ’, meaning ‘there exists’. So the typical existential statement can be written as

$$\exists x \in E, P(x).$$

Here, we have omitted the phrase ‘such that’, but this is often included if the statement reads better with it. For instance, we could write

$$\exists n \in \mathbb{N}, n^2 - 2n + 1 = 0,$$

but it would probably be easier to read

$$\exists n \in \mathbb{N} \text{ such that } n^2 - 2n + 1 = 0.$$

Often ‘such that’ is abbreviated to ‘s.t.’. (By the way, this statement is true because $n = 1$ satisfies $n^2 - 2n + 1 = 0$.)

We have seen that the negation of a universal statement is an existential statement and vice versa. In symbols, $\neg(\forall x \in E, P(x))$ is logically equivalent to $\exists x \in E, \neg P(x)$; and $\neg(\exists x \in E, P(x))$ is logically equivalent to $\forall x \in E, \neg P(x)$.

With these observations, we can now form the negations of more complex statements. Consider the statement

$$\forall n \in \mathbb{N}, \exists m \in \mathbb{N}, m > n.$$

Activity 2.7 What does the statement $\forall n \in \mathbb{N}, \exists m \in \mathbb{N}, m > n$ mean? Is it true?

What would the negation of the statement be? Let’s take it gently. First, notice that the statement is

$$\forall n \in \mathbb{N}, (\exists m \in \mathbb{N}, m > n).$$

The parentheses here do not change the meaning. According to the rules for negation of universal statements, the negation of this is

$$\exists n \in \mathbb{N}, \neg(\exists m \in \mathbb{N}, m > n).$$

But what is $\neg(\exists m \in \mathbb{N}, m > n)$? According to the rules for negating existential statements, this is equivalent to $\forall m \in \mathbb{N}, \neg(m > n)$. What is $\neg(m > n)$? Well, it’s just $m \leq n$. So what we see is that the negation of the initial statement is

$$\exists n \in \mathbb{N}, \forall m \in \mathbb{N}, m \leq n.$$

We can put this argument more succinctly, as follows:

$$\begin{aligned} \neg(\forall n \in \mathbb{N}(\exists m \in \mathbb{N}, m > n)) &\iff \exists n \in \mathbb{N}, \neg(\exists m \in \mathbb{N}, m > n) \\ &\iff \exists n \in \mathbb{N}, \forall m \in \mathbb{N}, \neg(m > n) \\ &\iff \exists n \in \mathbb{N}, \forall m \in \mathbb{N}, m \leq n. \end{aligned}$$

2.10.1 Proof by contradiction

We've seen a small example of proof by contradiction earlier in the chapter. Suppose you want to prove $P \Rightarrow Q$. One way to do this is by contradiction. What this means is that you suppose P is true but Q is false (in other words, that the statement $P \Rightarrow Q$ is false) and you show that, somehow, this leads to a conclusion that you know, definitely, to be false.

Here's an example.

Example 2.4 There are no integers m, n such that $6m + 8n = 1099$.

Proof

To prove this by contradiction, we can argue as follows:

Suppose that integers m, n *do* exist such that $6m + 8n = 1099$. Then since 6 is even, $6m$ is also even; and, since 8 is even, $8n$ is even. Hence $6m + 8n$, as a sum of two even numbers, is even. But this means $1099 = 6m + 8n$ is an even number. But, in fact, it is not even, so we have a contradiction. It follows that m, n of the type required do *not* exist. \square

This sort of argument can be a bit perplexing when you first meet it. What's going on in the example just given? Well, what we show is that if such m, n exist, then something impossible happens: namely the number 1099 is both even and odd. Well, this can't be. If supposing something leads to a conclusion you know to be false, then the initial supposition must be false. So the conclusion is that such integers m, n do not exist.

Probably the most famous proof by contradiction is Euler's proof that there are infinitely many prime numbers. A prime number is a natural number greater than 1 which is only divisible by 1 and itself. Such numbers have been historically of huge importance in mathematics, and they are also very useful in a number of important applications, such as information security. The first few prime numbers are 2, 3, 5, 7, 11, ... A natural question is: does this list go on forever, or is there a largest prime number? In fact, the list goes on forever: there are infinitely many prime numbers. We'll mention this result again later. A full, detailed, understanding of the proof requires some results we'll meet later, but you should be able to get the flavour of it at this stage. So here it is, a very famous result:

There are infinitely many prime numbers.

Proof

(Informally written for the sake of exposition) Suppose *not*. That is, suppose there are only a finite number of primes. Then there's a largest one. Let's call it M . Now consider the number

$$X = (2 \times 3 \times 5 \times 7 \times 11 \times \cdots \times M) + 1,$$

which is the product of *all* the prime numbers (2 up to M), with 1 added. Notice that $X > M$, so X is not a prime (because M is the largest prime). If a number X is not prime, that means that it has a divisor p that is a prime number and which satisfies $1 < p < X$. [This is the key observation: we haven't seen this yet, but we will later.] But p must therefore be one of the numbers 2, 3, 5, ..., M . However, X is *not* divisible by any

of these numbers, because it has remainder 1 when divided by any of them. So we have reached a contradiction: on the one hand, X must be divisible by one of these primes, and on the other, it is not. So the initial supposition that there were *not* infinitely many primes simply must be wrong. We conclude there are infinitely many primes. \square

This proof has been written in a fairly informal and leisurely way to help explain what's happening. It could all be written more succinctly.

2.11 Some terminology

At this point, it's probably worth introducing some important terminology. When, in Mathematics, we prove a true statement, we often say we are proving a *Theorem*, or a *Proposition*. (Usually the word 'Proposition' is used if the statement does not seem quite so significant as to merit the description 'Theorem'.) A theorem that is a preliminary result leading up to a Theorem is often called a *Lemma*, and a minor theorem that is a fairly direct consequence of, or special case of, a theorem is called a *Corollary*, if it is not significant enough itself to merit the title Theorem. For your purposes, it is important just to know that these words all mean true mathematical statements. You should realise that these terms are used subjectively: for instance, the person writing the mathematics has to make a decision about whether a particular result merits the title 'Theorem' or is, instead, merely to be called a 'Proposition'.

2.12 General advice

2.12.1 Introduction

Proving things is difficult. Inevitably, when you read a proof, in the textbooks or in these notes, you will ask 'How did the writer know to do that?' and you will often find you asking yourself 'How can I even begin to prove this?'. This is perfectly normal. This is where the key difference between abstract mathematics and more 'methods-based' mathematics lies. If you are asked to differentiate a function, you just go ahead and do it. It might be technically difficult in some cases, but there is no doubt about what approaches you should use. But proving something is more difficult. You might try to prove it, and fail. That's fine: what you should do in that case is try another attack. Keep trying until you crack it. (I suppose this is a little bit like integration. You'll know that there are various methods, but you don't necessarily know which will work on a particular integral, so you should try one, and keep trying until you manage to find the integral.) Abstract mathematics should always be done with a large pile of scrap paper at your disposal. You are unlikely to be able to write down a perfect solution to a problem straightaway: some 'scratching around' to get a feel for what's going on might well be needed, and some false starts might be pursued first. If you expect to be able to envisage a perfect solution in your head and then write it down perfectly, you are placing too much pressure on yourself. Abstract mathematics is simply not done like that.

In this chapter I have tried to indicate that there are methodical approaches to proof (such as proof by contradiction, for example). What you have to always be able to do is to understand *precisely* what it is that you have to prove. That sounds obvious, but it is something the importance of which is often underestimated. Once you understand what you need to show (and, here, working backwards a little from that end-point might be helpful, as we've seen), then you have to try to show it. And you must know when you have done so! So it is inevitable that you will have to take a little time to think about what is required: you cannot simply 'dive in' like you might to a differentiation question.

All this becomes much easier as you practise it. You should attempt problems from the textbooks (and also the problems below). Problems are a valuable resource and you are squandering this resource if you simply turn to the answers (should these be available). It is one thing to 'agree' with an answer, or to understand a proof, but it is quite a different thing to come up with a proof yourself. There is no point in looking at the answer before you have tried hard yourself to answer the problem. By trying (and possibly failing), you will learn more than simply by reading answers. Examination questions will be different from problems you have seen, so there is no point at all in 'learning' answers. You need to understand how to approach problems and how to answer them for yourself.

2.12.2 How to write mathematics

You should write mathematics **in English!!** You shouldn't think that writing mathematics is just using formulae. A good way to see if your writing makes sense is by reading it aloud (where you should only read what you really have written, not adding extra words). If it sounds like nonsense, a sequence of loose statements with no obvious relations, then you probably need to write it again.

Don't use more symbols than necessary.

Since many people seem to think that mathematics involves writing formulae, they often use symbols to replace normal English words. An eternal favourite is the double arrow ' \implies ' to indicate that one thing follows from the other. As in:

$$x^2 = 1 \implies x = 1 \text{ or } x = -1.$$

This is not only pure laziness, since it's just as easy to write:

$$x^2 = 1, \text{ hence } x = 1 \text{ or } x = -1.$$

But it is even probably not what was meant! The implication arrow ' \implies ' has a logical meaning 'if ..., then ...'. So if you write ' $x^2 = 1 \implies x = 1 \text{ or } x = -1$ ', then that really means '**if** $x^2 = 1$, **then** $x = 1 \text{ or } x = -1$ '. And hence this gives no real information about what x is. On the other hand, writing

$$\text{I know } x^2 = 1, \text{ hence } x = 1 \text{ or } x = -1,$$

means that now we know $x = 1 \text{ or } x = -1$ and can use that knowledge in what follows.

Some other unnecessary symbols that are sometimes used are ' \therefore ' and ' \because '. They mean something like 'therefore/hence' and 'since/because'. It is best not to use them, but to write the word instead. It makes things so much easier to read.

Provide all the information required.

A good habit is to start by writing what information is given and what question needs to be answered. For instance, suppose you are asked to prove the following:

For any natural numbers a, b, c with $c \geq 2$, there is a natural number n such that $an^2 + bn + c$ is not a prime.

A good start to an answer would be:

Given: natural numbers a, b, c , with $c \geq 2$.

To prove: there is a natural number n such that $an^2 + bn + c$ is not a prime.

At this point you (and any future reader) has all the information required, and you can start thinking what really needs to be done.

2.12.3 How to do mathematics

In a few words: **by trying** and **by doing it yourself**!!

Try hard

The kind of questions you will be dealing with in this subject often have no obvious answers. There is no standard method to come to an answer. That means that you have to find out what to do yourself. And the only way of doing that is by trial and error.

So once you know what you are asked to do (plus all the information you were given), the next thing is to take a piece of paper and start writing down some possible next steps. Some of them may look promising, so have a better look at those and see if they will help you. Hopefully, after some (or a lot) of trying, you see how to answer the question. Then you can go back to writing down the answer. This rough working is a vital part of the process of answering a question (and, in an examination, you should make sure your working is shown). Once you have completed this part of the process, you will then be in a position to write the final answer in a concise form indicating the flow of the reasoning and the arguments used.

Keep trying

You must get used to the situation that not every question can be answered immediately. Sometimes you immediately see what to do and how to do it. But other times you will realise that after a long time you haven't got any further.

Don't get frustrated when that happens. Put the problem aside, and try to do another question (or do something else). Look back at the question later or another day, and see if it makes more sense then. Often the answer will come to you as some kind of 'ah-ha' flash. But you can't force these flashes. Spending more time improves the chances they happen, though.

Finally, if you need a long time to answer certain questions, you can consider yourself in good company. For the problem known as 'Fermat's Last Theorem', the time between when the problem was first formulated and when the answer was found was about 250 years!

Do it yourself

Here is a possible answer to the previous example: **Given**: natural numbers a, b, c , with $c \geq 2$.

To prove: there is a natural number n such that $an^2 + bn + c$ is not a prime.

By definition (see page 70 of Biggs' book, or the footnote on page 4 of Eccles' book), a number p is **prime** if $p \geq 2$ and the only divisors of p are 1 and p itself.

Hence to prove: there is a natural number n for which $an^2 + bn + c$ is smaller than 2 or it has divisors other than 1 or itself.

Let's take $n = c$. Then we have $an^2 + bn + c = ac^2 + bc + c$.

But we can write $ac^2 + bc + c = c(ac + b + 1)$, which shows that $ac^2 + bc + c$ has c and $ac + b + 1$ as divisors.

Moreover, it's easy to see that neither c nor $ac + b + 1$ can be equal to 1 or to $ac^2 + bc + c$. We've found a value of n for which $an^2 + bn + c$ has divisors other than 1 or itself. \square

The crucial step in the answer above is the one in which I choose to take $n = c$. Why did I choose that? Because it works. How did I get the idea to take $n = c$? Ah, that's far less obvious. Probably some rough paper and lots of trying was involved. In the final answer, no information about how this clever idea was found needs to be given.

You probably have no problems following the reasoning given above, and hence you may think that you understand this problem. But being able to understand the answer, and **being able to find the answer yourself** are two completely different matters. And it is the second skill you are supposed to acquire in this course. (And hence the skill that will be tested in the examination.) Once you have learnt how to approach questions such as the above and come up with the clever trick yourself, you have some hope of being able to answer other questions of a similar type.

But if you only study answers, you will probably never be able to find new arguments for yourself. And hence when you are given a question you've never seen before, how can you trust yourself that you have the ability to see the 'trick' that that particular question requires?

For many, abstract mathematics seems full of clever 'tricks'. But these tricks have always been found by people working very hard to get such a clever idea, not by people just studying other problems and the tricks found by other people.

2.12.4 How to become better in mathematics

One thing you might consider is doing more questions. The books are a good source of exercises. Trying some of these will give you extra practice.

But if you want to go beyond just being able to do what somebody else has written down, you must try to explore the material even further. Try to understand the reason for things that are perhaps not explicitly asked.

As an illustration of thinking that way, look again at the formulation of the example we looked at before:

For any natural numbers a, b, c with $c \geq 2$, there is a natural number n such that $an^2 + bn + c$ is not a prime.

Why is it so important that $c \geq 2$? If you look at the proof in the previous section, you see that that proof goes wrong if $c = 1$. (Since we want to use that c is a divisor different from 1.) Does that mean the statement is wrong if $c = 1$? (No, but a different proof is required.)

And what happens if we allow one or more of a, b, c to be zero or negative?

And what about more complicated expression such as $an^3 + bn^2 + cn + d$ for some numbers a, b, c, d with $d \geq 2$? Could it be possible that there is an expression like this for which all n give prime numbers? If you found the answer to the original question yourself, then you probably immediately see that the answer has to be ‘no’, since similar arguments as before work. But if you didn’t try the original question yourself, and just studied the ready-made answer, you’ll be less well equipped to answer more general or slightly altered versions.

Once you start thinking like this, you are developing the skills required to be good in mathematics. Trying to see beyond what is asked, asking yourself new questions and seeing which you can answer, is the best way to train yourself to become a mathematician.

Learning outcomes

At the end of this chapter and the Essential reading and activities, you should be able to:

- demonstrate an understanding of what mathematical statements are
- prove whether mathematical statements are true or false
- negate statements, including universal statements and existential statements
- construct truth tables for logical statements
- use truth tables to determine whether logical statements are logically equivalent or not
- demonstrate knowledge of what is meant by conjunction and disjunction
- demonstrate understanding of the meaning of ‘if-then’ statements and be able to prove or disprove such statements
- demonstrate understanding of the meaning of ‘if and only if’ statements and be able to prove or disprove such statements
- find the converse and contrapositive of statements
- prove statements by proving their contrapositive
- prove results by various methods, including directly, by the method of proof by contradiction, and by working backwards
- demonstrate understanding of the key ideas and notations concerning sets
- prove results about sets
- use existential and universal quantifiers
- be able to negate statements involving several different quantifiers.

Sample examination questions

2

Question 2.1

Is the following statement about natural numbers n true or false? Justify your answer by giving a proof or a counterexample:

If n is divisible by 6 then n is divisible by 3.

What are the converse and contrapositive of this statement? Is the converse true? Is the contrapositive true?

Question 2.2

Is the following statement about natural numbers n true or false? Justify your answer by giving a proof or a counterexample:

If n is divisible by 2 then n is divisible by 4.

What are the converse and contrapositive of this statement? Is the converse true? Is the contrapositive true?

Question 2.3

Prove that $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$ are logically equivalent.

Question 2.4

Prove that the negation of $P \vee Q$ is $\neg P \wedge \neg Q$.

Question 2.5

Construct the truth tables for $P \Rightarrow (Q \wedge R)$ and $(P \Rightarrow Q) \wedge (P \Rightarrow R)$. Are these two statements logically equivalent?

Question 2.6

Suppose P, Q, R are three statements. Show that $(P \Rightarrow Q) \Rightarrow R$ and $P \Rightarrow (Q \Rightarrow R)$ are *not* logically equivalent.

Question 2.7

Prove that for all real numbers a, b, c , $ab + ac + bc \leq a^2 + b^2 + c^2$.

Question 2.8

Prove by contradiction that there is no largest natural number.

Question 2.9

Prove that there is no smallest positive real number.

Question 2.10

Suppose A and B are subsets of a universal set E . Prove that

$$(E \times E) \setminus (A \times B) = ((E \setminus A) \times E) \cup (E \times (E \setminus B)).$$

Question 2.11

Suppose that $P(x, y)$ is a predicate involving two free variables x, y from a set E . (So, for given x and y , $P(x, y)$ is either true or false.) Find the negation of the statement

$$\exists x \in E, \forall y \in E, P(x, y)$$

Comments on selected activities**Feedback to activity 2.1**

We can do this by constructing a truth table. Consider Table 2.8. This proves that $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$ are equivalent.

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Table 2.8: The truth tables for $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$

Feedback to activity 2.2

The converse is ‘if n divides 12 then n divides 4’. This is false. For instance, $n = 12$ is a counterexample. This is because 12 divides 12, but it does not divide 4. The original statement is true, however. For, if n divides 4, then for some $m \in \mathbb{Z}$, $4 = nm$ and hence $12 = 3 \times 4 = 3nm = n(3m)$, which shows that n divides 12.

Feedback to activity 2.3

We have

$$\begin{aligned} x \in A \setminus B &\iff (x \in A) \wedge (x \notin B) \\ &\iff (x \in A) \wedge (x \in E \setminus B) \\ &\iff x \in A \cap (E \setminus B). \end{aligned}$$

Feedback to activity 2.4

$\mathcal{P}(A)$ is the set consisting of the following sets:

$$\begin{aligned} &\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \\ &\{1, 2, 3\}, \{2, 3, 4\}, \{1, 3, 4\}, \{1, 2, 4\}, \{1, 2, 3, 4\}. \end{aligned}$$

Feedback to activity 2.5

The members of $\mathcal{P}(A)$ are all the subsets of A . A subset S is determined by which of the n members of A it contains. For each member x of A , either $x \in S$ or $x \notin S$. There are therefore two possibilities, for each $x \in A$. It follows that the number of subsets is $2 \times 2 \times \cdots \times 2$ (where there are n factors, one for each element of A). Therefore $\mathcal{P}(A)$ has 2^n members.

Feedback to activity 2.6

The statement means that if we take any natural number n there will be some natural number m greater than n . Well, this is true. For example, $m = n + 1$ will do.

Sketch answers to or comments on sample questions**Answer to question 2.1**

The statement is true. For, suppose n is divisible by 6. Then for some $m \in \mathbb{N}$, $n = 6m$, so $n = 3(2m)$ and since $2m \in \mathbb{N}$, this proves that n is divisible by 3.

The converse is ‘If n is divisible by 3 then n is divisible by 6’. This is false. For example, $n = 3$ is a counterexample: it is divisible by 3, but not by 6.

The contrapositive is ‘If n is not divisible by 3 then n is not divisible by 6’. This is true, because it is logically equivalent to the initial statement, which we have proved to be true.

Answer to question 2.2

The statement is false. For example, $n = 2$ is a counterexample: it is divisible by 2, but not by 4.

The converse is ‘If n is divisible by 4 then n is divisible by 2’. This is true. For, suppose n is divisible by 4. Then for some $m \in \mathbb{N}$, $n = 4m$, so $n = 2(2m)$ and since $2m \in \mathbb{N}$, this proves that n is divisible by 2.

The contrapositive is ‘If n is not divisible by 4 then n is not divisible by 2’. This is false, because it is logically equivalent to the initial statement, which we have proved to be false. Alternatively, you can see that it’s false because 2 is a counterexample: it is not divisible by 4, but it *is* divisible by 2.

Answer to question 2.3

This can be established by using the truth table constructed in Activity 2.2. See the solution above.

Answer to question 2.4

This is established by Table 2.6. That table shows that $\neg(P \vee Q)$ is logically equivalent to $\neg P \wedge \neg Q$. This is the same as saying that the negation of $P \vee Q$ is $\neg P \wedge \neg Q$.

Answer to question 2.5

Table 2.9 is the truth table for $P \Rightarrow (Q \wedge R)$ and Table 2.10 is the table for $(P \Rightarrow Q) \wedge (P \Rightarrow R)$. In all cases, these give the same truth value, so the statements are logically equivalent.

P	Q	R	$Q \wedge R$	$P \Rightarrow (Q \wedge R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	F	F
F	T	T	T	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

Table 2.9: The truth table for $P \Rightarrow (Q \wedge R)$.

P	Q	R	$P \Rightarrow Q$	$P \Rightarrow R$	$(P \Rightarrow Q) \wedge (P \Rightarrow R)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	F
T	F	F	F	F	F
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	T	T

Table 2.10: The truth table for $(P \Rightarrow Q) \wedge (P \Rightarrow R)$.**Answer to question 2.6**

We can show that $(P \Rightarrow Q) \Rightarrow R$ and $P \Rightarrow (Q \Rightarrow R)$ are not logically equivalent by constructing their truth tables and showing that in some cases, they give different truth values. See Table 2.6. This shows, for example, that if P is F , Q is T and R is F , then the two statements take different logical values. For, in this case, $P \Rightarrow Q$ is T and $(P \Rightarrow Q) \Rightarrow R$ is F . On the other hand, $P \Rightarrow (Q \Rightarrow R)$ is T .

P	Q	R	$P \Rightarrow Q$	$Q \Rightarrow R$	$(P \Rightarrow Q) \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	T	T	T	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T
F	F	F	T	T	F	T

Table 2.11: The truth table for Question 2.6

Answer to question 2.7

We work backwards, since it is not immediately obvious how to begin. We note that what we're trying to prove is equivalent to

$$a^2 + b^2 + c^2 - ab - ac - bc \geq 0.$$

This is equivalent to

$$2a^2 + 2b^2 + 2c^2 - 2ab - 2ac - 2bc \geq 0,$$

which is the same as

$$(a^2 - 2ab + b^2) + (b^2 - 2bc + c^2) + (a^2 - 2ac + c^2) \geq 0.$$

You can perhaps now see how this is going to work, for $(a^2 - 2ab + b^2) = (a - b)^2$ and so on. Therefore the given inequality is equivalent to

$$(a - b)^2 + (b - c)^2 + (a - c)^2 \geq 0.$$

We know this to be true because squares are always non-negative. If we wanted to write this proof 'forwards' we might argue as follows. For any a, b, c , $(a - b)^2 \geq 0$, $(b - c)^2 \geq 0$ and $(a - c)^2 \geq 0$, so

$$(a - b)^2 + (b - c)^2 + (a - c)^2 \geq 0$$

and hence

$$2a^2 + 2b^2 + 2c^2 - 2ab - 2ac - 2bc \geq 0,$$

from which we obtain

$$a^2 + b^2 + c^2 \geq ab + ac + bc,$$

as required.

Answer to question 2.8

Let's prove by contradiction that there is no largest natural number. So suppose there is a largest natural number. Let us call it N . (What we want to do now is somehow show that a conclusion, or something we know for sure must be false, follows.) Well, consider the number $N + 1$. This is a natural number. But since N is the largest natural number, we must have $N + 1 \leq N$, which means that $1 \leq 0$, and that's nonsense. So it follows that we must have been wrong in supposing there is a largest natural number. (That's the only place in this argument where we could have gone wrong.) So there is *no* largest natural number. We could have argued the contradiction slightly differently. Instead of using the fact that $N + 1 \leq N$ to obtain the absurd statement that $1 \leq 0$, we could have argued as follows: $N + 1$ is a natural number. But $N + 1 > N$ and this contradicts the fact that N is the largest natural number.

Answer to question 2.9

We use a proof by contradiction. Suppose that there is a smallest positive real number and let's call this r . Then $r/2$ is also a real number and $r/2 > 0$ because $r > 0$. But $r/2 < r$, contradicting the fact that r is the smallest positive real number. (Or, we could argue: because $r/2$ is a positive real number and r is the smallest such number, then we must have $r/2 \geq r$, from which it follows that $1 \geq 2$, a contradiction.)

Answer to question 2.10

We need to prove that

$$(E \times E) \setminus (A \times B) = ((E \setminus A) \times E) \cup (E \times (E \setminus B)).$$

Now,

$$\begin{aligned} (x, y) \in (E \times E) \setminus (A \times B) &\iff \neg((x, y) \in A \times B) \\ &\iff \neg((x \in A) \wedge (y \in B)) \\ &\iff \neg(x \in A) \vee \neg(y \in B) \\ &\iff (x \in E \setminus A) \vee (y \in E \setminus B) \\ &\iff ((x, y) \in (E \setminus A) \times E) \vee ((x, y) \in E \times (E \setminus B)) \\ &\iff (x, y) \in ((E \setminus A) \times E) \cup (E \times (E \setminus B)). \end{aligned}$$

Answer to question 2.11

We deal first with the existential quantifier at the beginning of the statement. So, the negation of the statement is

$$\forall x \in E, \neg(\exists y \in E, P(x, y))$$

which is the same as

$$\forall x \in E, \exists y \in E, \neg P(x, y).$$

