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DRBEM Applied to the 3D Helmholtz Equation and Its Particular Solutions with Various Radial Basis Functions

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Abstract This paper presents to solve the 3D Helmholtz equation using dual reciprocity boundary element method (DRBEM) and its particular solutions with various radial basis functions (RBFs). The important function in this method is to employ the RBF. Here, we find the particular solution of the Helmholtz equation $(\nabla^2 \pm k^2)h = f(r)$, where $f(r)$ is the RBF. Various RBFs are chosen and the particular solutions are obtained. The dual reciprocity method (DRM) is a method that converts the domain integral into the boundary integral. Mathematical formulations and discretization forms are described and discussed. Numerical results with three RBF with and without polynomial terms are presented and discussed. Algorithm of the method is also presented.

Keywords: *3D Helmholtz equation, radial basis function, particular solution, dual reciprocity method*

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1. Introduction

The boundary element method is a numerical method for solving partial differential equations encountered in mathematical physics and engineering [1].The boundary element method can be viewed as some sort of half-way house between analytical and numerical methods [2]. According to work of Nardini and Brebbia (1982) [3], there has been an increasing interest in using the Dual Reciprocity Method (DRM) to solve partial differential equations (PDEs) by boundary element methods (Partidge et al. 1992 [4] and Golberg and Chen 1997 [5]). The attractiveness of the DRM is its capability to transfer domain integration to the boundary integration. In the past, 1+r has been chosen as the ad-hoc basis function in the DRM [6,7]. In 1994, Golberg and Chen [7,8] provided theoretical evidence for the choice of the basic functions by using the RBFs in the DRM. On the issue of applicability, the DRM has been only applied to the case when the major differential operator is kept as the Laplace or in harmonic operator and the rest of the terms in the original differential are treated as a forcing term. This is primarily due to the difficulty in obtaining particular solutions in a closed form. As a result, the DRM is less effective when the forcing terms become too complicated [10,11]. In general, we would like to keep forcing term as simple as possible to make better approximation by RBF. Meanwhile, the simpler the forcing term, the more complicated the differential operator becomes, and also the fundamental solution become more involved. So far, the choice of the main differential operator seems to be limited to Laplace and in harmonic operators due to difficulty of producing particular solutions in a closed form. A particular solution to the governing differential equation is then determined for each basis function [12]. In this regard, Zhu [13] attempted to using Helmholtz operators as the main differential operator in the DRM. The key ingredient in doing this is the ability to analytically calculate particular solutions for various linear PDEs, $L\phi = b$. This is usually done by approximating f by a series $\sum_{j=1}^{N} a_j f_j$, and then solving $Lh_j = f_j, 1 < j < N$, where $\{f_j\}$ is an appropriate set of linearly independent basis function. Hence, the choice of $\{f_j\}$ is important, and the analysis given in Golberg et al. (1998b). They defined that $\{f_j\}$ needs to provide an accurate approximation to b, and also it should be as a form that $Lh_j = f_j$ be solved analytically.

The following sections are organized as follows. Section 2 is described the physical problem and obtain the governing equation. Section 3 shows that how the DRM method converts the domain integral into the boundary integral. Section 4 finds analytical particular solutions for various RBF like simple function $f(r) = 1 + r$, thin plate Splines (TPS), $f(r) = r^2 logr$ and higher-order polyharmonic Splines. Then, applying the particular solutions is described in Section 5. Section 6 is discussed the numerical results and finally conclusions are given in Section 7.

2. Helmholtz Equation

The complete dynamic basic equations of the fluid are the mass continuity, Navier-stokes and energy equations [14]:

$$\nabla^2 \pm k^2\phi = b(\phi),\ in\ \Omega \tag{1}$$

$$B.C. = known,\ on$$

Eq. (1) is the governing equation for the sound field in the three-dimensional flow.

The drawbacks of using the Laplace operator instead of the Helmholtz operator are:

(i) The information in the original differential equation is partially lost.

(ii) The forcing term becomes more complicated and difficult to interpolate by radial basis functions.

(iii) The solution may not even converge when k becomes large [15].

3. Dual Reciprocity Method (DRM)

In the boundary element method, for a body of the boundary (Γ) and domain (Ω), the integral formulation of the Eq. (1) may be expressed as:

$$e(p)\phi(p) = \int_\Gamma (G\frac{\partial\phi}{\partial n} - \phi\frac{\partial G}{\partial n})ds - \int_\Omega Gb(\phi)d\Omega \tag{2}$$

where:

$$e(p) = \begin{cases} 1 & for\ P\ outside\ \ Surface \\ 0.5 & for\ P\ on\ \ Surface \\ 0 & for\ P\ outside\ \ Surface \end{cases} \tag{3}$$

And G is the Green's function from the Helmholtz equation. For the 3D problems:

$$\begin{aligned} \nabla^2\phi + k^2\phi = b(\phi), \quad & G = \frac{\exp(-jkr)}{4\pi r} \\ \nabla^2\phi - k^2\phi = b(\phi), \quad & G = \frac{\exp(kr)}{4\pi r} \end{aligned} \tag{4}$$

Eq. (2) contains the volume integral, which is a difficult problem. Therefore, in order to overcome this difficulty, one of the easiest way is converting volume integral into the boundary integral via DRM [3-16]. This method focuses to the term $b(\phi)$ which may be approximated by the following expression,

$$b(X) = \sum_{i=1}^{N+L} f_i\,\alpha_i \tag{5}$$

where α_i, f_i are interpolation coefficients and radial basis function (RBF), respectively. N is the number of collocation nodes along the boundary, L is the number of collocation points inside the domain, and r_i is defined as the distance between the node under consideration and the node i.

For each simple source function f_i, a particular solution h_i needs to be found and satisfied as:

$$\nabla^2 h_i + k^2 h_i = f_i \tag{6}$$

Hereafter, the particular solution of the Eq. (6) was obtained by some various RBFs.

Substituting equations (6) and (5) into Eq. (2) yields:

$$\begin{aligned} e(p)\phi(p) &= \int_\Gamma (G\frac{\partial\phi}{\partial n} - \phi\frac{\partial G}{\partial n})ds \\ &- \sum_{i=1}^{N+L}\alpha_i\left\{e(p)h_i(p) - \int_\Gamma (Gh_i' - h_i\frac{\partial G}{\partial n})ds\right\} \end{aligned} \tag{7}$$

Discretization form of the Eq. (7) can be represented as follows:

$$\begin{aligned} e_l\phi_l &= \sum_{j=1}^{N} L_{lj}\phi_{n_j} - \sum_{j=1}^{N}\hat{H}_{lj}\phi_j \\ &- \sum_{i=1}^{N+L}\alpha_i\left\{e_l h_{li} - \sum_{j=1}^{N} L_{lj}h_{ji}' - \sum_{j=1}^{N}\hat{H}_{lj}h_{ji}\right\} \end{aligned} \tag{8}$$

where L_{ij} and $\hat{H}_{ij}$ are influence coefficients, and defined as follows:

$$\begin{cases} L_{lj} = \int_\Gamma G ds \\ \hat{H}_{lj} = \int_\Gamma \frac{\partial G}{\partial n} ds \end{cases} \tag{9}$$

These integrals can be evaluated by numerical and analytical methods.

By verifying it in detail:

$$H_{lj} = 0.5\delta_{lj} + \hat{H}_{lj} \tag{10}$$

where δ_{ij} is the Kronecker delta, which is defined as $\delta_{ij} = 0$ for $i \neq j$, and $\delta_{ij} = 1$ for $i = j$.

Eq. (8) may further be written as:

$$\sum_{j=1}^{N} H_{lj}\phi_j = \sum_{j=1}^{N} L_{lj}\phi_{n_j} - \sum_{i=1}^{N+L}\alpha_i\left\{\sum_{j=1}^{N} H_{lj}h_{ji} - \sum_{j=1}^{N} L_{lj}h_{ji}'\right\} \tag{11}$$

where h_i is obtained in Section 4.

4. Particular Solutions

First, we introduce some RBFs as given:

Linear classic:

$$f_i = 1 + r_i \tag{12}$$

TPS:

$$f_j = r^2 \ln r \tag{13}$$

Higher-order Spline:

$$f_j^{[n]} = r_j^{2n-1} \tag{14}$$

4.1. Classical RBF

By substituting Eq. (12) into Eq. (6), the particular solution of the Eq. (6) can be found as follows:

$$(\nabla^2 + k^2)h_i = 1 + r_i \tag{15}$$

Since this solution is axisymmetric with respect to the source, it is independent of the polar angle θ, and thus Eq. (15) becomes [17]:

$$\left(\frac{d^2}{dr_i^2}+\frac{2}{r_i}\frac{d}{dr_i}+k^2\right)h_i=1+r_i \tag{16}$$

A regular solution of Eq. (16) is obtained [13,14]:

$$h_i=\frac{1+r_i}{k^2}-\frac{2}{k^4}\left(\frac{1-\cos(kr_i)}{r_i}\right) \tag{17}$$

Where:

$$\vec{r}=\vec{q}-\vec{p} \tag{18}$$

4.2. Thin Plate Spline (TPS)

We consider an approach, usually called the "annihilator method" as described in Ref. [18]. Here, it is assumed that there is a linear partial differential operator M which satisfies

$$Mf_i=0 \tag{19}$$

And commutes with L; i.e. $ML=LM$, and then:

$$MLh_i=Mf_i=0=LMh_i \tag{20}$$

If the solution sets, $V=\{v:Lv=0\}$ and $W=\{w:Mw=0\}$, are finite and disjoint, then

$$h_i=\sum_{k=1}^{s}b_k\beta_k+\sum_{k=1}^{t}c_k\gamma_k+z \tag{21}$$

where $\{\beta_k\}$ is a basis for V, $\{\gamma_k\}$ is a basis for W, and $Lz=0$. The coefficients $\{b_k\}$ and $\{c_k\}$ are determined by requiring $Lh_i=f_i$ and additional regularity conditions (Golberg and Chen 1997 [5] and Golberget al. 1998a [21]).

By substituting Eq. (13) into Eq. (6), the particular solution of the Eq. (6) in 2D can be found as follows:

$$(\nabla^2+k^2)h_i=r^2\ln r \tag{22}$$

And for $L\_$ we have:

$$L\_h(p)=r^2\log r, r=\| P \|, \tag{23}$$

Now, with respect to Eqs. (19, 20), and Eq. (21):

$$\nabla^4 r^2\log r=0,\ r>0, \tag{24}$$

h can be obtained by solving $\nabla^4 L\_h(p)=0$. For radially symmetric solutions, it is equivalent to solve:

$$\nabla_r^4\left(\nabla_r^2-k^2\right)h=0 \tag{25}$$

h can be obtained by solving:

$$\nabla_r^4 w=0,\left(\nabla_r^2-k^2\right)v=0 \tag{26}$$

Since $\left(\nabla_r^2-k^2\right)$ is a Bessel operator (Derrick and Grossman 1976 [19]):

$$v(r)=AI_0(kr)+BK_0(kr) \tag{27}$$

where I_0 and K_0 are Bessel functions of order zero.

Since ∇_r^4 is a multiple of an Euler operator, $\nabla_r^2 r^p=p^2r^{p-2}, \nabla_r^4 r^p=p^2(p-2)^2 r^{p-4}$, p must satisfy the characteristic equation $p^2(p-2)^2=0$, (Derrick and Grossman 1976 [19])

$$w(r)=a+b\log r+cr^2+dr^2\log r \tag{28}$$

Thus,

$$\begin{aligned} h(r)=&AI_0(kr)+BK_0(kr)+a\\ &+b\log r+cr^2+dr^2\log r \end{aligned} \tag{29}$$

The coefficients $\{A,B,a,b,c,d\}$ are found by requiring $\left(\nabla_r^2-k\right)h=r^2\log r$ and the condition that h should be continuous at $r=0$. One solution is given in Chen and Rashed (1998a) by:

$$h(r)=\begin{cases}-\dfrac{4}{k^2}-\dfrac{4\log r}{k^4}-\dfrac{r^2\log r}{k^2}-\dfrac{4K_0(kr)}{k^4}, & r\neq 0\\ -\dfrac{4}{k^4}+-\dfrac{4\gamma}{k^4}+-\dfrac{4}{k^4}\log\left(\dfrac{k}{2}\right), & r=0\end{cases} \tag{30}$$

where $\gamma\cong 0.5772156649015328$ is Euler's constant.

4.3. Higher-order Spline

Here, we want to obtain the particular solutions for the higher-order Spline, i.e defined as follows:

$$L_{\pm}h(r)=r^{2n}\log r\ in\ \mathbb{R}^2 \tag{31}$$

$$L_{\pm}h(r)=r^{2n-1}\ in\ \mathbb{R}^3 \tag{32}$$

To calculate the particular solution for $L_{\pm}\ in\ R^3$, we have to solve $\left(\nabla_r^2\pm k^2\right)v=0$ and $\nabla_r^{n+3}w=0$, therefore $\nabla_r^{n+3}r^{2n-1}=0$. It is easily shown that the solution to $\left(\nabla_r^2-k^2\right)v=0$ is (Golberg et al. [7]):

$$v(r)=\frac{A\cosh(kr)}{r}+\frac{B\sinh(kr)}{r} \tag{33}$$

And for $\left(\nabla_r^2+k^2\right)v=0$:

$$v(r)=\frac{A\cos(kr)}{r}+\frac{B\sin(kr)}{r} \tag{34}$$

To obtain solutions, which are regular at $r=0$, we use the Taylor series expansions of $\cosh(kr)$ and $\sinh(kr)$ at $r=0$, and comparing coefficients gives:

$$h(r)=\frac{(2n)!\cosh(kr)}{rk^{2n+3}}-\sum_{i=0}^{n}\frac{(2n)!}{(2i)!}\frac{r^{2i-1}}{k^{2n-2i+2}} \tag{35}$$

A similar argument for L_+ gives:

$$h(r)=\frac{(-1)^{n+1}(2n)!}{rk^{2n+2}}\cos(kr)+\sum_{i=0}^{n}\frac{(2n)!}{(2i)!}\frac{(-1)^{n+i}\,r^{2i-1}}{k^{2n-2i+2}} \tag{36}$$

The results for different orders are given in Table 1 and Table 2.

Table 1. Results for different order for L_

$f(r)$	h_i for L_-
r^1	$\frac{2\cosh(kr)}{rk^5}-\frac{2}{rk^4}-\frac{r}{k^2}$
r^3	$\frac{24\cosh(kr)}{rk^7}-\frac{24}{rk^6}-\frac{12r}{k^4}-\frac{r^3}{k^2}$
r^5	$\frac{720\cosh(kr)}{rk^9}-\frac{720}{rk^8}-\frac{360r}{k^6}-\frac{30r^3}{k^4}-\frac{r^5}{k^2}$
r^7	$\frac{40320\cosh(kr)}{rk^{11}}-\frac{40320}{rk^{10}}-\frac{20160r}{k^8}-\frac{1680r^3}{k^6}-\frac{56r^5}{k^4}-\frac{r^7}{k^2}$

Table 2. Results for different order for L₊

$f(r)$	h_i for L_+
r^1	$\frac{2\cos(kr)}{rk^4}-\frac{2}{rk^4}+\frac{r}{k^2}$
r^3	$-\frac{24\cos(kr)}{rk^6}+\frac{24}{rk^6}-\frac{12r}{k^4}+\frac{r^3}{k^2}$
r^5	$\frac{720\cos(kr)}{rk^8}-\frac{720}{rk^8}+\frac{360r}{k^6}-\frac{30r^3}{k^4}+\frac{r^5}{k^2}$
r^7	$-\frac{40320\cos(kr)}{rk^{10}}+\frac{40320}{rk^{10}}-\frac{20160r}{k^8}+\frac{1680r^3}{k^6}-\frac{56r^5}{k^4}+\frac{r^7}{k^2}$

Finally, the flowchart algorithm of the DRBEM is shown in Figure 1. First, geometry of the body is modeled. DRBEM method defines some internal and boundary nodes. Using RBF, the domain integral can be inverted to the boundary integral. This is the main advantage of DRM to deal with boundary of the body.

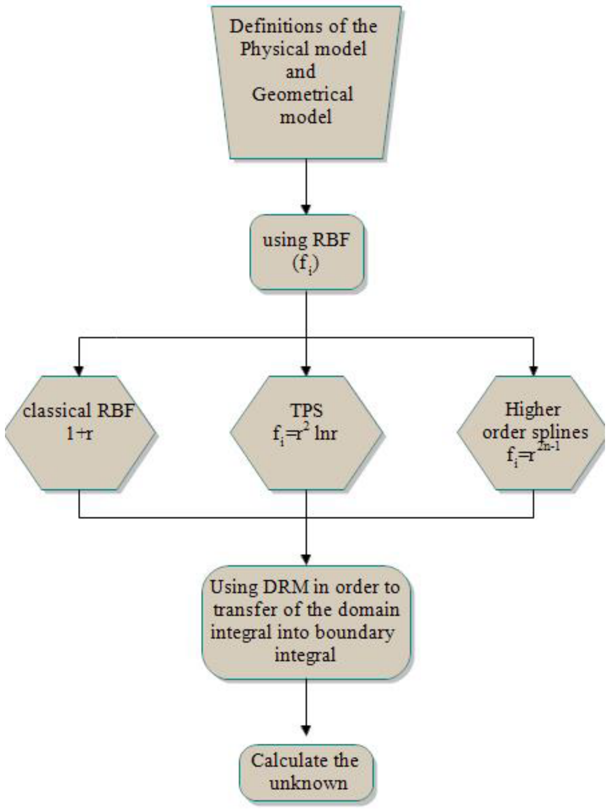

Figure 1. Flowchart algorithm of the DRBEM

5. Numerical Results

The geometry of problems is a Square with side of unit length. These examples are chosen since their analytical solutions can be obtained. More complex problems can be handled in the same DRM fashion without any extra difficulty.

The inhomogeneous 2D Helmholtz problem is governed by equation:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+u=x \tag{37}$$

$$u(x,y)=D(x,y) \tag{38}$$

The analytical solution is:

$$u(x,y)=\sin x+\sin y+x \tag{39}$$

The Dirichlet boundary conditions $D(x,y)$ can be determined from the exact solution (39).

5.1. Without Polynomial Terms

For all different RBF in Table 3, the solution is obtained using 200 boundary elements (50 on each side) and 25 internal nodes.

Table 3. Comparisons various RBF Without polynomial terms

x	y	RBF	Exact	DRM	% error
0.5	0.5	$1+r$	1.4589	1.4632	0.294743
0.5	0.5	$r^2\log r$	1.4589	1.477	1.240661
0.5	0.5	$r^4\log r$	1.4589	1.5382	5.435602

As it is seen, the result of *1+r* without polynomial term is more accurate, while the results of TPS and poly-harmonic Splines of order 2 without polynomial terms are so bad.

5.2. With Polynomial Terms

To obtain a more accurate answer, polynomial terms have been added to radial basis functions:

$$f=f_j+\zeta_{(p)} \tag{40}$$

where $\zeta_{(p)}$ is obtained from the following equation:

$$L_{\pm}\zeta_{(p)}=p_n \tag{41}$$

where p_n can be any polynomial function. Some of the functions p and $\zeta_{(p)}$ are given in Table 4.

Table 4. Particular solutions for polynomial terms in L₊

p_n	$\zeta_{(p)}$	p_n	$\zeta_{(p)}$
1	$\frac{1}{k^2}$	xy^2	$\frac{xy^2}{k^2}-\frac{2x}{k^4}$
x	$\frac{x}{k^2}$	y^3	$\frac{y^3}{k^2}-\frac{6y}{k^4}$
y	$\frac{y}{k^2}$	x^4	$\frac{x^4}{k^2}-\frac{12x^2}{k^4}+\frac{24}{k^6}$
x^2	$\frac{x^2}{k^2}-\frac{2}{k^4}$	x^3y	$\frac{x^3y}{k^2}-\frac{6xy}{k^4}$
xy	$\frac{xy}{k^2}$	x^2y^2	$\frac{x^2y^2}{k^2}-\frac{2x^2}{k^4}-\frac{2y^2}{k^4}+\frac{8}{k^6}$
y^2	$\frac{y^2}{k^2}-\frac{2}{k^4}$	xy^3	$\frac{xy^3}{k^2}-\frac{6xy}{k^4}$
x^3	$\frac{x^3}{k^2}-\frac{6x}{k^4}$	y^4	$\frac{y^4}{k^2}-\frac{12y^2}{k^4}+\frac{24}{k^6}$
x^2y	$\frac{x^2y}{k^2}-\frac{2y}{k^4}$		

Results for some points, are compared in Table 5 by applying polynomial term, where p_n is a polynomial of total degree n.

Table 5. Comparisons of the results for various RBFs with polynomial terms

(x, y)	exact	acbrf	TPS+p (n=1)	TPS+p (n=2)
(0.15, 0.2)	0.4981	0.4974	0.4913	0.5013
(0.15, 0.5)	0.7789	0.7782	0.7647	0.7841
(0.15, 0.75)	0.9811	0.9806	0.971	0.9869
(0.5, 0.5)	1.4589	1.4587	1.4568	1.4614
(0.7, 0.25)	1.5916	1.5915	1.5947	1.5914
(0.7, 0.65)	1.9494	1.9496	1.9513	1.9515
(0.75, 0.8)	2.149	2.1493	2.1475	2.152

Advanced classical RBF shown in Table 5, is obtained from contrariwise method. In this method at first we choice any particular solutions (f) , for example $f=\frac{r^2}{4}+\frac{r^3}{9}$, then with solving this equation $Lf=rbf$, as a result, the advanced classical radial basis function (*acrbf*) is obtained.

$$acrbf=1+r+k^2\left(\frac{r^2}{4}+\frac{r^3}{9}\right) \tag{42}$$

According to Table 5, it's seen that the result from this method is very accurate, but it's so difficult to guessing a particular solution that gives an exact answer.

Finally, we compared the relative error (RE) with various RBFs, as shown in Figure 2. Although for all cases the RE is less that 0.03%, but for acrbf is very small error.

6. Conclusions

In this paper, a DRBEM formulation for axisymmetric Helmholtz-type equation is briefly presented. We have generalized pervious work using TPS for finding particular solutions to Helmholtz-type operators by using higher-order Splines, substantially increased accuracy can be obtained using higher Splines. Based on this research, following conclusions can be drawn:

- Higher accuracy can be obtained by using higher-order Splines with polynomial term. Acrbf is much better than others.
- The polynomial terms of RBF are important as shown in Table 3 and Table 5.
- With annihilator approach, we can obtain particular solutions of any order of TPS in 2D and 3D. Table 1 and Table 2 shows the particular solutions of 3D Helmholtz-type equation and Table 4 shows the particular solutions of 2D polynomials terms for L_+.

Numerical examples for Helmholtz-type equation using higher-order Splines in 3D will be examined and compare with lower order Splines in the next research, and that will be our future plan.

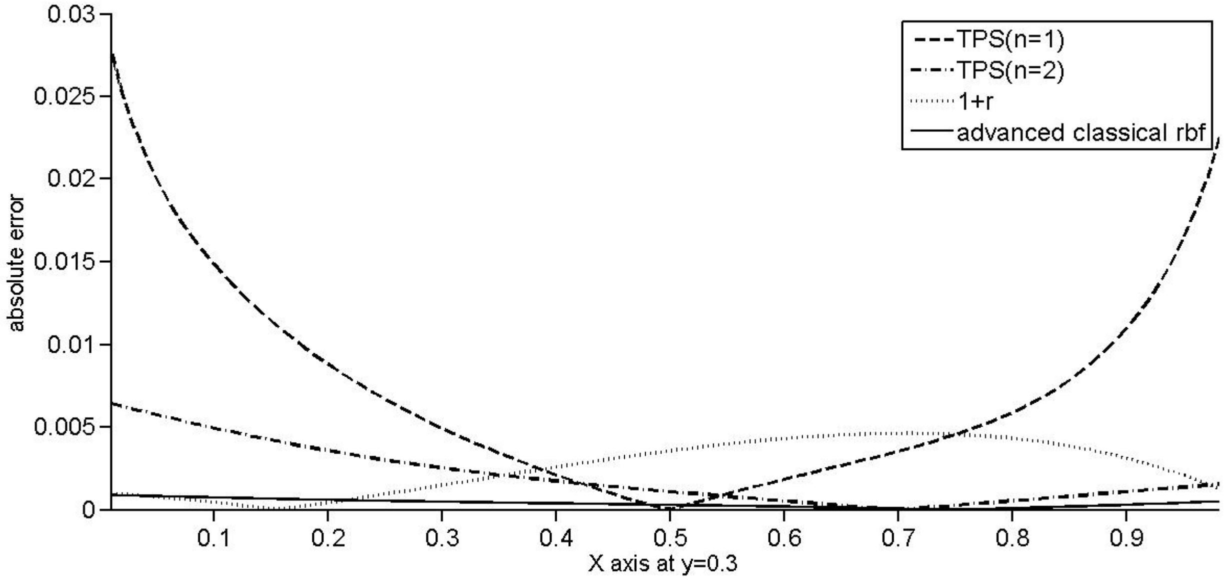

Figure 2. Relative error using various RBFs

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