A combinatorial problem related to Mahler's measure

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ABSTRACT. We give a generalization of a result of Myerson on the asymptotic behavior of norms of certain Gaussian periods. The proof exploits properties of the Mahler measure of a trinomial.

1. Introduction

This paper was motivated by the following remarkable asymptotic result of Myerson [**8**] about the norm of a Gaussian period. Let $p \equiv 1 \bmod 3$ be a prime and ω a primitive cube root of unity in the finite field $\mathbb{F}_p = \{0, 1, \ldots, p-1\}$. Also let $K = \mathbb{Q}(\zeta)$ be the p-th cyclotomic field, where $\zeta = e^{2\pi i/p}$. Then, as $p \to \infty$,

$$\tag{1} \tfrac{1}{p} \log |N_{\mathbb{Q}}^{K}(\zeta + \zeta^{\omega} + \zeta^{\omega^2})| \to L'(-1, \chi) = .3231\ldots,$$

where $L(s, \chi)$ is the Dirichlet L-function with χ the nontrivial character $\bmod 3$. As a consequence of a more general result we will give the following refinement of (1).

THEOREM 1. *For* $p \equiv 1 \bmod 3$

$$\tfrac{1}{p} \log |N_{\mathbb{Q}}^{K}(\zeta + \zeta^{\omega} + \zeta^{\omega^2})| = L'(-1, \chi) + \mathrm{O}(p^{-1/2} \log p),$$

with an absolute implied constant.

The method of proof behind Theorem 1 differs from that of Myerson and develops further an interesting relationship between a certain combinatorial problem and Mahler's measure. In the next section we introduce this combinatorial problem, briefly describe Myerson's approach to (1), state the general result, Theorem 2 and show

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that it implies Theorem 1. The five sections that follow contain results of independent interest that lead up to the proof of Theorem 2.

2. The combinatorial problem

Let S be an arbitrary subset of cardinality $|S|$ of $\mathbb{F}_p$, for p an odd prime. For a given $t \in \mathbb{F}_p$ denote by N_t the number of solutions $(n_1, \ldots, n_{p-1}) \in S^{p-1}$ of the equation

$$\sum_{\ell \in \mathbb{F}_p^*} \ell \, n_\ell = t,$$

where $\mathbb{F}_p^* = \mathbb{F}_p \setminus \{0\}$. Clearly $N_t = N_{at}$ for $a \in \mathbb{F}_p^*$ so that N_t takes on only the two values N_0 and N_1. Furthermore, N_0 and N_1 are invariant under affine maps $S \mapsto aS + b$ for $a \in \mathbb{F}_p^*$ and $b \in \mathbb{F}_p$. The combinatorial problem of interest here is to determine N_0 and N_1 as precisely as possible in terms of p and (the affine equivalence class of) S.

Since it is obvious that $N_0 + (p-1)N_1 = |S|^{p-1}$, this problem reduces to the determination of $\Delta = N_0 - N_1$, in terms of which

$$N_1 = \tfrac{1}{p} \, |\, S|^{p-1} - \tfrac{1}{p} \, \Delta \;\text{ and }\; N_0 = \tfrac{1}{p} |\, S|^{p-1} + (1 - \tfrac{1}{p})\Delta.$$

We have that $\Delta = \sum_{t \in \mathbb{F}_p} N_t \, \zeta^t$, which yields the basic formula

$$\Delta = \prod_{\ell \in \mathbb{F}_p^*} \sum_{n \in S} \zeta^{n\ell}. \tag{2}$$

The counting problem is thus equivalent to computing the norm of the cyclotomic integer $\sum_{n \in S} \zeta^n$.

As a consequence of the arithmetic-geometric inequality applied in (2) it follows that

$$|\Delta| \le |S|^{\frac{p}{2}-1}.$$

This shows that the values $\sum_\ell \ell \, n_\ell$ are very well distributed among the values of $\mathbb{F}_p$. It also follows easily from (2) that for $|S| \le 2$ we have $\Delta = 1$ so we may restrict attention to S where $|S| \ge 3$.

In case S is a subgroup of $\mathbb{F}_p^*$ such problems were discussed by Myerson **[7, 8]**. Here the values of ℓS run over the cosets of S exactly $|S|$ times. He actually considered the problem of counting the number of representations of t as a sum of distinct elements of these cosets. It is easily seen from (2) that this counting problem reduces to the determination of a natural $|S|$-th root of Δ. This is a well known problem of cyclotomy in that it entails the explicit evaluation of the norm of a Gaussian period. Except for some subgroups of small index (see **[7]**) or of size $|S| \leq 2$, such evaluations are apparently unknown.

In lieu of an explicit evaluation of Δ when S is a subgroup of $\mathbb{F}_p^*$ of fixed size, Myerson introduced the idea of determining the asymptotic behavior of $|\Delta|$ as a function of $p \equiv 1 \bmod |S|$ as $p \to \infty$. When $|S| = 3$ so that $S = \{1, \omega, \omega^2\}$, he proved in **[8]** that

(3) $$\tfrac{1}{p} \log |\Delta| \to \int_0^1 \int_0^1 \log |\, e(u) + e(v) + e(-u-v)|\, du\, dv,$$

as $p \to \infty$ with $p \equiv 1 \bmod 3$. The statement (1) then follows from (2) and the evaluation of the integral in (3), which follows from **[11]**. The idea of the proof of (3) is to interpret the formula from (2),

$$\tfrac{1}{p} \log |\Delta| = \tfrac{1}{p} \sum_{\ell} \log |\sum_{n \in S} e(n\ell/p)|,$$

as giving an approximation to the integral in (3) by using the fact that $(\ell/p, \omega\ell/p)$ becomes uniformly distributed $(\operatorname{mod} 1)$ as ℓ runs over $\mathbb{F}_p^*$. The difficulty lies in the fact that the integrand has singularities, but it is tractable since they are isolated from the points $(\ell/p, \omega\ell/p)$. Using special arguments he was able to quantify this statement well enough to prove (3). Myerson also conjectured in **[8]** a result like (3) for subgroups of any fixed size. However, for $|S| > 4$ (the case $|S| = 4$ being simpler), the singular set is infinite and this seems to present a serious obstacle in the way of a proof. See **[10]** and chapter 10 of the book of Konyagin and Shparlinski **[4]** for some further developments of these ideas.

This paper will show that $\frac{1}{p} \log |\Delta|$ is actually well approximated by $L'(-1, \chi)$ for any set S with $|S| = 3$, provided that the "width" $\mathrm{w}(S)$ is large. Here, for any S, $\mathrm{w}(S)$ is defined to be the length of the smallest interval in $\{0, 1, 2, \ldots, p-1\}$ that contains an image $aS + b$ of S under an affine transformation.

THEOREM 2. *Suppose that* $|S| = 3$. *Then*

$$\tfrac{1}{p} \log |\Delta| = L'(-1, \chi) + O\Big(\tfrac{\log \mathrm{w}(S)}{\mathrm{w}(S)}\Big),$$

with an absolute implied constant.

The proof of Theorem 2 uses a uniform asymptotic estimate for $\frac{1}{p} \log |\Delta|$ given in terms of the Mahler measure of a certain trinomial. This is then related to a two dimensional Mahler measure like the integral in (3). Although this method differs significantly from that described above, it also encounters difficulties with larger sets S.

To see that Theorem 1 follows from Theorem 2, it is enough to observe that $\mathrm{w}(\{1,\omega,\omega^2\}) \gg p^{1/2}$. Suppose that for some $a \in \mathbb{F}_p^*$ and some $c > 0$ the three elements $a, a\omega, a\omega^2$ are all contained in some interval of length $cp^{1/2}$, where this is interpreted in the obvious way. Since $a + a\omega + a\omega^2 = 0$ this interval must contain either $0, (p-1)/3$ or $(2p-2)/3$. Hence $3a, 3a\omega, 3a\omega^2$ must be contained in an interval around 0 of length $3cp^{1/2}$. Let n and m be the integers of smallest absolute value representing $3a$ and $3a\omega$, respectively. Then

$$0 < n^2 + nm + m^2 \equiv 0 \bmod p.$$

This implies that $\frac{3}{2}(n^2 + m^2) \geq p$, which is impossible for $c > 0$ sufficiently small. Thus $\mathrm{w}(\{1,\omega,\omega^2\}) \gg p^{1/2}$. We remark that any set S with $|S| = 3$ satisfies $\mathrm{w}(S) \ll p^{1/2}$, as will be seen in the proof of Theorem 5 below.

3. The asymptotic problem for fixed S

Before turning to the proof of Theorem 2, consider first the asymptotic problem when S is a *fixed* subset of $\mathbb{Z}$, interpreted for each (sufficiently large) p as a subset of $\mathbb{F}_p$. Recall that the Mahler measure of a monic $f \in \mathbb{C}[x]$ is given by

$$\mathrm{M}(f) = \prod_{\alpha} \max(1, |\,\alpha|\,),$$

where α runs over the zeros of f, counted with multiplicity. As is standard we write

$$\mathrm{m}(f) = \log \mathrm{M}(f). \tag{4}$$

Associated to $S = \{n_1, n_2, \ldots, n_{|S|}\}$ with $n_1 < n_2 < \cdots < n_{|S|}$, is the polynomial

$$f_S(x) = x^{n_1} + x^{n_2} + \cdots + x^{n_{|S|}}.$$

The following result shows that for fixed S the basic asymptotic problem has a simple solution in terms of $\mathrm{m}(f_S)$.

THEOREM 3. *Suppose that S is fixed. Then for p sufficiently large*

$$\tfrac{1}{p} \log|\Delta| = \mathrm{m}(f_S) + O(p^{-1}\log p),$$

where the implied constant depends only on S.

PROOF. Writing $f_S(x) = \prod_\alpha (x - \alpha)$, we see from (2) that

$$\Delta = \prod_{\alpha} \prod_{\ell \in \mathbb{F}_p^*} (\alpha - \zeta^\ell) = \prod_{\alpha} \frac{1 - \alpha^p}{1 - \alpha},$$

using that $p-1$ is even. Since $\prod_\alpha(1-\alpha) = f_S(1) = |S|$ we obtain

$$|S|\,\Delta = \prod_\alpha (1-\alpha^p). \tag{5}$$

This quantity was studied by D.H. Lehmer in his influential paper of 1933 [**6**]. Thus we have for p sufficiently large

$$\tfrac{1}{p}\log|\Delta| = \tfrac{1}{p}\sum_\alpha \log|1-\alpha^p| - \tfrac{1}{p}\log|S|. \tag{6}$$

If $|\alpha|>1$ then $\frac{1}{p}\log|1-\alpha^p| = \log|\,\alpha| + O(|\,\alpha|^{-p})$ while if $|\alpha|<1$ then $\frac{1}{p}\log|1-\alpha^p| = O(|\alpha|^p)$. Only in case $|\alpha|=1$ is there any difficulty, and this may be handled as in [**3**, Lemma 1.10] by an application of Baker's Theorem, giving that for p sufficiently large

$$\log|1-\alpha^p| \ll_\alpha \log p.$$

Note that the restriction in [**3**, Lemma 1.11] that α not be a root of unity is unnecessary here since p is a prime. Thus

$$\tfrac{1}{p}\sum_\alpha \log|1-\alpha^p| = \mathrm{m}(f_S) + O(p^{-1}\log p).$$

By (6) we finish the proof. □

This proof serves to underscore that the main challenge in understanding the behavior of $|\,\Delta|$ when S is not fixed, at least by this method, is to control small values of $|1-\alpha^p|$ for α a zero of f_S in terms of p. Baker's theorem does not seem to yield enough information without making some very restrictive assumptions on $\mathrm{w}(S)$.

4. Zeros of a trinomial

The assumption that $|S|=3$ allows us to control how close a root of f_S, which is a trinomial, can be to a p-th root of unity. This in turn can be used to estimate how small $|1-\alpha^p|$ can be. The following theorem formulates this idea precisely.

THEOREM 4. *Suppose that $p>3$ and $0<m<n$. Then there is an absolute constant $c>0$ so that for α a root of $f(x) = x^n + x^m + 1 = 0$ with $|\alpha|\le 1$ we have*

$$|1-\alpha^p| > \frac{c}{n}.$$

PROOF. As before let $\zeta = e^{2\pi i/p}$. We first show that there is an absolute constant $c_1>0$ so that for all $\ell\in\mathbb{Z}$

$$|\,\zeta^\ell - \alpha| > c_1(pn)^{-1}. \tag{7}$$

Fix ℓ and α and let L denote the line segment from α to ζ^ℓ. By the complex mean value theorem given in **[2]** there are $z_1, z_2 \in L$ so that

$$\mathrm{Re}\, \frac{f(\zeta^\ell) - f(\alpha)}{\zeta^\ell - \alpha} = \mathrm{Re}\, f'(z_1) \quad \text{and} \quad \mathrm{Im}\, \frac{f(\zeta^\ell) - f(\alpha)}{\zeta^\ell - \alpha} = \mathrm{Im}\, f'(z_2).$$

Thus

$$\left| \frac{f(\zeta^\ell) - f(\alpha)}{\zeta^\ell - \alpha} \right| \leq 2 \max_{z \in L} |f'(z)| \leq 4n,$$

which yields

$$4n|\zeta^\ell - \alpha| \geq |f(\zeta^\ell)| = |\zeta^{n\ell} + \zeta^{m\ell} + 1|.$$

Since $p > 3$ this sum three of p-th roots of unity cannot vanish and in fact must satisfy $|\zeta^{n\ell} + \zeta^{m\ell} + 1| \geq c_2 p^{-1}$ for some $c_2 > 0$ (see **[9]**), giving (7).

Write $\alpha = re^{i\theta}$. By (7) there are constants $c_3, c_4 > 0$ so that at least one of the following holds:

$$r < 1 - \frac{c_3}{pn} \quad \text{or} \quad \left| \theta - \frac{2\pi\ell}{p} \right| > \frac{c_4}{pn} \quad \text{for all} \quad \ell \in \mathbb{Z}.$$

In the first case

$$|1 - \alpha^p| > 1 - \Big(1 - \frac{c_3}{pn}\Big)^p = \frac{c_3}{n} + O(n^{-2}),$$

while in the second $|p\theta - 2\pi\ell| > c_4 n^{-1}$ and hence

$$|1 - \alpha^p| \geq \sin(c_4 n^{-1}) = \frac{c_4}{n} + O(n^{-3}),$$

since $\sin(c_4 n^{-1})$ is the minimal distance from 1 to the ray at angle $c_4 n^{-1}$. This finishes the proof. □

5. The Mahler measure of a trinomial

Suppose that for some $p > 3$ we have a set $S \subset \mathbb{F}_p$ with $|S| = 3$. This S can be transformed by an affine transformation to one of the form $\{0, m, n\}$, where $0 < m < n$ and n is minimal: that is $n = \mathrm{w}(S)$. Write $f(x) = x^n + x^m + 1$ for some such choice of m.

THEOREM 5. *Given S with $|S| = 3$ and f as above, we have*

$$\tfrac{1}{p} \log |\Delta| = \mathrm{m}(f) + O(p^{-1/2} \log n)$$

with an absolute implied constant.

PROOF. By (5) we are reduced to considering

$$\Delta = \tfrac{1}{3}\prod_{\alpha}(1-\alpha^p), \tag{8}$$

where α runs over all the zeros of f. Now

$$\sum_{\alpha}\log|1-\alpha^p| - p\,\mathrm{m}(f) = \sum_{|\alpha|\le 1}\log|1-\alpha^p| + \sum_{|\alpha|>1}\log|1-\alpha^{-p}| \ll n\log n$$

by Theorem 4 applied to $f(x)$ and its reciprocal

$$x^n f(x^{-1}) = x^n + x^{n-m} + 1.$$

Thus by (8) we have

$$\tfrac{1}{p}\log|\Delta| = \mathrm{m}(f) + O\big(\frac{n\log n}{p}\big).$$

To finish the proof of Theorem 5 we will show that $n \ll p^{1/2}$. By performing a suitable affine transformation we may suppose that $S_p = \{0,1,\ell\}$ where $1<\ell<p$. Let $s = \lceil\sqrt{p}\,\rceil = [\sqrt{p}]+1$ and observe that at least one difference $k\ell - j\ell$ for $0\le j<k\le s$ must lie in $\{1,2,\dots,s\}$ or $\{p-s,p-s+1,\dots,p-1\}$. Taking $a=k-j$ and some b we see that $aS_p+b\subset\{0,1,\dots,2s\}$. It follows that $n\ll p^{1/2}$. □

6. Limits of Mahler measures

Recall that the Mahler measure of a non-zero polynomial $f\in\mathbb{C}[x,y]$ is defined by

$$\mathrm{m}(f) = \int_0^1\int_0^1 \log|\,f(e(u),e(v))|\,du\,dv. \tag{9}$$

This reduces to (4) for monic $f\in\mathbb{C}[x]$ by Jensen's formula [**3**, p.7], which states that for any $z\in\mathbb{C}$,

$$\int_0^1 \log|\,e(u)-z|\,du = \log^+|z|, \tag{10}$$

where $\log^+ z = \log\max(1,|z|\,)$.

The following result expresses $\mathrm{m}(x+y+1)$ as a uniform limit of Mahler measures of trinomials. The proof is modeled after that of Boyd, who gave the case $m=1$ in [**1**, p.463].

THEOREM 6. *For* $0<m<n$ *with* $(m,n)=1$

$$\mathrm{m}(x^n+x^m+1) = \mathrm{m}(x+y+1) + \alpha(m+n)n^{-2} + O(mn^{-3}),$$

where

$$\alpha(n) = \begin{cases} -\frac{\sqrt{3}\pi}{6}, & \text{if } n \equiv 0 \bmod 3; \\ \frac{\sqrt{3}\pi}{18}, & \text{otherwise.} \end{cases} \tag{11}$$

PROOF. For $x = e(u) = e^{2\pi i u}$ with $\cos(2\pi m u) > -1/2$ we have

$$\log(1 + x^m + x^n) = \log(1 + x^m) + \sum_{\ell \geq 1} \frac{(-1)^{\ell-1}}{\ell} \Big(\frac{x^n}{1 + x^m} \Big)^{\ell}, \tag{12}$$

while when $\cos(2\pi m u) < -1/2$ we have that

$$\log(1 + x^m + x^n) = \log(x^n) + \sum_{\ell \geq 1} \frac{(-1)^{\ell-1}}{\ell} \Big(\frac{1 + x^m}{x^n} \Big)^{\ell}. \tag{13}$$

By (10) applied in (9)

$$\begin{aligned} \mathrm{m}(x + y + 1) &= \int_0^1 \log^+ |1 + e(u)| \, du \\ &= \int_0^1 \log^+ |1 + e(mu)| \, du = \int \log |\, 1 + e(mu)| du, \end{aligned} \tag{14}$$

where the range of the last integral consists of those subintervals of $[0, 1]$ that satisfy $\cos(2\pi m u) > -1/2$. Thus set

$$I_1(\ell) = \int e(n\ell\, u)(1 + e(mu))^{-\ell} du, \tag{15}$$

where the range of integration is over those subintervals of $[0, 1]$ that satisfy $\cos(2\pi m u) > -1/2$, and

$$I_2(\ell) = \int e(-n\ell\, u)(1 + e(mu))^{\ell} du, \tag{16}$$

where the range of integration is over those subintervals of $[0, 1]$ that satisfy $\cos(2\pi m u) < -1/2$. By (12-14) we have the identity

$$\mathrm{m}(x^n + x^m + 1) = \mathrm{m}(x + y + 1) + \mathrm{Re} \sum_{\ell \geq 1} \frac{(-1)^{\ell-1}}{\ell} (I_1(\ell) + I_2(\ell)). \tag{17}$$

After changing variables $u \mapsto mu$ we have from (15)

$$I_1(\ell) = \tfrac{1}{m}\sum_{k=0}^{m-1}\Big(\int_k^{k+1/3} + \int_{k+2/3}^{k+1}\Big)e(\tfrac{n\ell\, u}{m})(1+e(u))^{-\ell}du$$
$$= \tfrac{2}{m}\,\mathrm{Re}\sum_{k=0}^{m-1} e(\tfrac{kn\ell}{m})\int_0^{1/3} e(\tfrac{n\ell\, u}{m})(1+e(u))^{-\ell}du$$
$$= 2\,\mathrm{Re}\int_0^{1/3} e(nq\,u)(1+e(u))^{-qm}du \tag{18}$$

when $\ell = qm$ and $I_1(\ell) = 0$ otherwise. Here we use the assumption that $(n,m) = 1$. Similarly, from (16)

$$I_2(\ell) = 2\,\mathrm{Re}\int_{1/3}^{1/2} e(-nq\,u)(1+e(u))^{qm}du \tag{19}$$

when $\ell = qm$ and $I_2(\ell) = 0$ otherwise. Integrating by parts three times in (18) and (19) we get after some calculation that

$$I_1(qm) + I_2(qm) = \frac{\sqrt{3}}{\pi}\frac{m}{n^2 q}(-1)^{qm}\cos\big(\tfrac{2\pi}{3}(m+n)q\big) + O\Big(\frac{m^2}{n^3 q}\Big). \tag{20}$$

Here the real part of the sum of the boundary terms obtained after the first integration by parts vanishes and we use that $(nq \pm 1)^{-1} = (nq)^{-1} + O((nq)^{-2})$ in the boundary terms after the second integration by parts. Also, we estimate the final integrals that arise by

$$\int_0^{1/3} |1+e(u)|^{-qm-3}du = \int_0^{1/3}(2+2\cos u)^{(-qm-3)/2}du$$
$$\le \int_0^{1/3}(4-9u)^{(-qm-3)/2}du \ll (qm)^{-1},$$

and similarly

$$\int_{1/3}^{1/2} |1+e(u)|^{qm-3}du \ll (qm)^{-1}.$$

Theorem 6 now follows easily from (17) and (20). □

7. A uniform result

Theorem 2 is a consequence of the the following more precise result together with the fact that $n = \mathrm{w}(S) \ll p^{1/2}$, which was proved in Theorem 5.

THEOREM 7. *Suppose that $|S| = 3$ and that $n = \mathrm{w}(S)$. Then*

$$\tfrac{1}{p}\log|\Delta| = L'(-1,\chi) + O(p^{-1/2}\log n + n^{-2}),$$

with an absolute implied constant.

PROOF. In the notation of §5, since $\{0, m, n\}$ is assumed minimal we must have that $(m, n) = 1$. Thus Theorem 7 follows from Theorems 5 and 6 together with Smyth's evaluation **[11]** (see also **[1**, p.462]):

$$\mathrm{m}(x + y + 1) = \tfrac{3\sqrt{3}}{4\pi}L(2,\chi) = L'(-1,\chi).$$

□

Some of the ingredients in the proof of Theorem 7 generalize to sets S with $|S| > 3$. For instance, an analogue of Theorem 4 holds for quadrinomials, since one has a lower bound for the non-zero sum of four p-th roots of unity. Also, nontrivial upper bounds for the width of a set can be given more generally and an analogue of Theorem 5 can be proved for sets S with $|S| = 4$. Also, Theorem 6 can be generalized in certain ways (see **[1]** and **[5]**). However, the next interesting case of Myerson's conjectured asymptotic when S is a subgroup occurs for $|S| = 5$, and here no sufficiently strong lower bound for non-zero sums of five p-th roots of unity is known (see **[9]**). In fact, this interesting problem seems to be the central difficulty in extending the methods given in this paper to prove this conjecture.

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References

[1] D.W. Boyd, *Speculations concerning the range of Mahler's measure.* Canad. Math. Bull. 24 (1981), 453–469.

[2] J.-Cl. Evard and F. Jafari, *A complex Rolle's theorem.* Amer. Math. Monthly 99 (1992), 858–861.

[3] G. Everest and T. Ward, *Heights of polynomials and entropy in algebraic dynamics.* Universitext. Springer-Verlag London, Ltd., London, 1999.

[4] S.V. Konyagin and I. E. Shparlinski, *Character sums with exponential functions and their applications.* Cambridge University Press, Cambridge, 1999.

[5] W. M. Lawton, *A problem of Boyd concerning geometric means of polynomials.* J. Number Theory 16 (1983), 356–362.

[6] D.H. Lehmer, *Factorization of certain cyclotomic functions.* Ann. of Math. (2) 34 (1933), 461–479.

[7] G. Myerson, *A combinatorial problem in finite fields. I.* Pacific J. Math. 82 (1979), 179–187.

[8] G. Myerson, *A combinatorial problem in finite fields. II.* Quart. J. Math. Oxford Ser. (2) 31 (1980), 219–231.

[9] G. Myerson, *Unsolved Problems: How Small Can a Sum of Roots of Unity Be?* Amer. Math. Monthly 93 (1986), 457–459.

[10] G. Myerson, *A sampler of recent developments in the distribution of sequences.* Number theory with an emphasis on the Markoff spectrum (Provo, UT, 1991), 163–190, Lecture Notes in Pure and Appl. Math., 147, Dekker, New York, 1993.

[11] C.J. Smyth, *On measures of polynomials in several variables.* Bull. Austral. Math. Soc. 23 (1981), 49–63.

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