PRODUCTS OF DEGENERATE QUADRATIC FORMS

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Abstract. We challenge the classical belief that products of degenerate quadratic forms must remain degenerate and we show that this fails in general, *e.g.* over tensor triangulated categories with duality. This opens new ways of constructing non-degenerate quadratic forms and hence classes in Witt groups. In addition, we encapsulate in a Leibniz-type formula the behaviour of the product with respect to the symmetric cone construction. We illustrate these ideas by computing the total Witt group of regular projective spaces.

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Introduction

Perpetuation of degeneracy is the following well-known phenomenon: Given a degenerate symmetric form α_1 on a finite dimensional vector space V_1 and any symmetric form α_2 on a space V_2, the tensor product symmetric form $\alpha_1 \otimes \alpha_2$ on $V_1 \otimes V_2$ is again degenerate, except, of course, in the trivial case where the space V_2 is zero, *i.e.* when $V_1 \otimes V_2 = 0$. Perpetuation of degeneracy is not specific to vector spaces and holds similarly in all classical frameworks, like for finitely generated projective modules over rings with involution, or for vector bundles over schemes.

The present work builds on the surprising observation that perpetuation of degeneracy does *not* hold in more flexible frameworks, like in triangulated categories with duality [2, 3]. To formalize this observation, we introduce a topological invariant of the forms α_1 and α_2, called the *consanguinity* of α_1 and α_2, which captures their inclination for a degenerate product. In particular, we prove:

Theorem. *If α_1 and α_2 have no consanguinity then $\alpha_1 \otimes \alpha_2$ is non-degenerate.*

This is Corollary 4.6. The definition of consanguinity is given in Section 4.

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We want to interpret the above in terms of Witt groups, so let us briefly sketch the definition of triangular Witt groups. To do this, we need a notion introduced in [2], namely *the symmetric cone* of a possibly degenerate symmetric form α. This symmetric cone is a non-degenerate symmetric form associated to α, that we denote by $\mathrm{d}(\alpha)$ in the present paper. Symmetric spaces of the form $\mathrm{d}(\alpha)$ are precisely the *metabolic* ones and a symmetric form α is non-degenerate if and only if $\mathrm{d}(\alpha)=0$. In particular $\mathrm{d}(\mathrm{d}(\alpha))=0$, which means that d behaves like a differential: $\mathrm{d}\circ\mathrm{d}=0$. So, *triangular Witt groups*, which classify non-degenerate symmetric forms modulo metabolic ones, can be remembered as the homology of the complex defined by this symmetric cone construction $\alpha\mapsto\mathrm{d}\,\alpha$. This is explained in Section 2. We have established in [3] that all classical Witt groups can be recovered as some triangular Witt groups, for suitable derived categories, at least if 2 is invertible in the original setting. The reader can find in the survey [5] basic notions and motivations for the theory of classical and triangular Witt groups, in particular in algebraic geometry.

We now translate into Witt group language the appearance of non-degenerate symmetric forms as products of forms with no consanguinity. Indeed, this method allows us to *construct Witt classes* $[\alpha_1\otimes\alpha_2]$ out of two symmetric forms α_1 and α_2 which might be degenerate and hence might not define Witt classes themselves. In the form of a slogan, this reads:

$$\nexists\,[\alpha_1]\quad\text{or}\quad\nexists\,[\alpha_2]\qquad\text{but still}\qquad\exists\,[\alpha_1\otimes\alpha_2]\,. \tag{1}$$

Before moving towards geometric applications, let us make a second general observation. Namely, assume that one of the forms, say α_2, is *metabolic*, then it may happen that not only the product $\alpha_1\otimes\alpha_2$ is non-degenerate, as explained above, but is quite surprisingly *non-metabolic.* In some sense, the degeneracy of the form α_1 can compensate the metabolicity of the form α_2. Sloganized, this becomes:

$$\nexists\,[\alpha_1]\quad\text{and}\quad[\alpha_2]=0\qquad\text{but}\qquad[\alpha_1\otimes\alpha_2]\neq0\,. \tag{2}$$

These two observations (1) and (2) will be illustrated by geometric examples. They both imply that the Witt class of the product should not be understood as the product of the classes $[\alpha_1\otimes\alpha_2]\neq[\alpha_1]\cdot[\alpha_2]$, at least in this generality.

In the presence of a differential $\alpha\mapsto\mathrm{d}(\alpha)$ and of a product $(\alpha_1,\alpha_2)\mapsto\alpha_1\otimes\alpha_2$, it is legitimate to wonder if these structures satisfy some type of Leibniz formula: $\mathrm{d}(\alpha_1\otimes\alpha_2)\overset{?}{=}\mathrm{d}(\alpha_1)\otimes\alpha_2\,\pm\,\alpha_1\otimes\mathrm{d}(\alpha_2)$. For symmetric forms, this is not true in general for the simple reason that the left-hand side $\mathrm{d}(\alpha_1\otimes\alpha_2)$ is always non-degenerate, as is $\mathrm{d}(\alpha)$ for all α, whereas the right-hand side is only conditionally non-degenerate. Here again, the consanguinity obstruction can be used for $\mathrm{d}(\alpha_1)$ and α_2, or, for α_1 and $\mathrm{d}(\alpha_2)$. Indeed, the consanguinity of $\mathrm{d}(\alpha_1)$ and α_2 is exactly equal to the consanguinity α_1 and $\mathrm{d}(\alpha_2)$ and coincides with the locus of common degeneracy of α_1 and α_2 (Prop. 5.1). When this obstruction is empty, we have:

Theorem (**Leibniz-type formula**). *Let α and β be symmetric forms whose degeneracy loci do not intersect. Then, we have an isometry:*

$$\mathrm{d}\,(\alpha\otimes\beta)\quad=\quad\mathrm{d}\,(\alpha)\otimes\beta\quad+\quad\alpha\otimes\mathrm{d}\,(\beta)$$

up to signs which are made precise in Section 5.

This is Theorem 5.2, where + of course means the orthogonal sum. Unfortunately, the signs are not as easy as in the usual Leibniz formula.

We now want to see these abstract considerations at work in algebraic geometry.

In the last few years, triangular Witt theory [2, 3] led to a certain number of applications (see a survey in [5]), among which the recent computation of the total Witt group of projective bundles by Charles Walter [16], who considerably generalized Arason's famous theorem $\mathrm{W}(\mathbb{P}^n_k) = \mathrm{W}(k)$, see [1], where k was a field. The total Witt group of a scheme X

$$\mathrm{W}^{\mathrm{Tot}}(X) = \bigoplus_{i\in\mathbb{Z}/4}\ \bigoplus_{\mathcal{L}\in\mathrm{Pic}(X)/2} \mathrm{W}^i(X,\mathcal{L})$$

is the graded ring of all (derived) Witt groups for all possible shifts i and all possible twists $\mathcal{L}$ of the duality. Walter's computation constitutes a real *tour de force*, involving a precise description of derived categories of projective bundles. Over regular schemes though, this level of technicality is not always necessary and Witt groups can sometimes be computed by means of more geometric results like Mayer-Vietoris, homotopy invariance and the like, as developed in [4]. Such a geometric strategy would typically consist in guessing the answer, in constructing a homomorphism globally, between this conjectured answer and the Witt groups under study, and in proving it an isomorphism locally. The crucial step, namely the construction of the custom-tailored global homomorphism, is where consanguinity might be used. We illustrate these ideas in Section 7 where we give a very simple geometric proof of Walter's theorem [16] in the special but emblematic case of $\mathbb{P}^n_X$ with X regular:

Theorem. *Let X be a regular $\mathbb{Z}[\frac{1}{2}]$-scheme. Then the total graded Witt ring $\mathrm{W}^{\mathrm{Tot}}(\mathbb{P}^n_X)$ is canonically a free $\mathrm{W}^{\mathrm{Tot}}(X)$-module of rank 2 generated by the unit $\langle 1\rangle \in \mathrm{W}(\mathbb{P}^n_X)$ and one other class $[\beta^{(n)}_X] \in \mathrm{W}^n(\mathbb{P}^n_X, \mathcal{O}(n+1))$, whose square is zero.*

This is Theorem 7.4 and we now explain how to construct the symmetric space $\beta^{(n)}_X$ defining the crucial generator $[\beta^{(n)}_X]$ by means of products of symmetric forms with no consanguinity. Indeed, our method is not specific to projective bundles and does not use regularity. (The regularity assumption comes from the geometric theorems of [4] mentioned above.) The general method goes as follows.

Let X be a scheme. Consider $\mathcal{L} \in \mathrm{Pic}(X)$ a line bundle over X and $s \in \Gamma(X,\mathcal{L})$ a global section. This can be seen as a one-dimensional "diagonal" symmetric form $\alpha(s\,;\,\mathcal{L})$ on the vector bundle $\mathcal{O}_X$ with respect to the $\mathcal{L}$-twisted (unshifted) duality:

$$\alpha(s\,;\,\mathcal{L}) := \big(\,\mathcal{O}_X\,,\,\mathcal{O}_X \xrightarrow{s} \mathcal{L}\,\big)\,.$$

Such a symmetric form $\alpha(s\,;\,\mathcal{L})$ is usually degenerate, unless s yields a trivialization of the line bundle $\mathcal{L}$. The product of a finite number of such symmetric forms $\alpha(s_1;\mathcal{L}_1)\otimes\ldots\otimes\alpha(s_m;\mathcal{L}_m) = \alpha(s_1\otimes\ldots\otimes s_m\,;\,\mathcal{L}_1\otimes\ldots\otimes\mathcal{L}_m)$ still has the same nature and degeneracy clearly tends to increase in this process. So, this cannot lead us to a non-degenerate symmetric space unless we are simply considering a good old one-dimensional form $\langle u\rangle$ for a global unit $u \in \Gamma(X,\mathcal{O}_X)^\times$ – and this would not be worth the trouble. However, we can also consider mixed products involving one diagonal form $\alpha(s_0\,;\,\mathcal{L}_0)$ as well as symmetric cones $\mathrm{d}\,\alpha(s_i\,;\,\mathcal{L}_i)$ for $i = 1,\ldots,n$. Orthogonal sums of such mixed products are the *pseudo-diagonal forms* of Section 6. Pseudo-diagonal forms may be non-degenerate without necessarily being mere diagonal forms $\langle u_1,\ldots,u_n\rangle$ for global units $u_1,\ldots,u_n$. Indeed, using consanguinity methods, we establish in Corollary 6.13 the following result:

Theorem. *Let $n \geq 0$, let $\mathcal{L}_0, \ldots, \mathcal{L}_n$ be $n+1$ line bundles over a scheme X and let $s_i \in \Gamma(X, \mathcal{L}_i)$, for $i = 0, \ldots, n$, be global sections which do not vanish simultaneously : $\bigcap_{i=0}^{n} Z(s_i) = \varnothing$. Then*

$$\alpha(s_0\,;\,\mathcal{L}_0) \;\otimes\; \mathbf{d}\,\alpha(s_1\,;\,\mathcal{L}_1) \otimes \ldots \otimes \mathbf{d}\,\alpha(s_n\,;\,\mathcal{L}_n)$$

is a non-degenerate symmetric space and defines a class in the Witt group $\mathrm{W}^n(X, \mathcal{L})$ where $\mathcal{L} = \mathcal{L}_0 \otimes \ldots \otimes \mathcal{L}_n$.

It is an interesting open question to know for which schemes the total Witt group is generated by such pseudo-diagonal spaces. This would be a global version of the well-known diagonalization theorems over fields and local rings.

In any case, if we apply this to the scheme $\mathbb{P}^n_X$, to $\mathcal{L}_0 = \ldots = \mathcal{L}_n = \mathcal{O}(1)$ and to $s_i = T_i$ (the homogeneous coordinates), then it is clear that $\cap_{i=0}^{n}\{T_i = 0\} = \varnothing$ and that the above result provides us with a non-degenerate space. This is nothing but the announced generator of the total Witt group of $\mathbb{P}^n_X$:

$$\beta^{(n)}_X = \alpha(T_0\,;\,\mathcal{O}(1)) \;\otimes\; \mathrm{d}\,\alpha(T_1\,;\,\mathcal{O}(1)) \otimes \ldots \otimes \mathrm{d}\,\alpha(T_n\,;\,\mathcal{O}(1))\,.$$

This quite non-trivial application illustrates the strength of the abstract machinery of consanguinity. It also provides examples of the above "surprises" (1) and (2). Let us also stress that consanguinity needs not to be applied only to diagonal forms but is a very general concept.

Although slightly beside the point of this article, let us briefly comment of the various projective bundle theorems for Witt groups. This is easy to summarize : only Walter [16] reaches maximal generality. Note that for non-trivial projective bundles $\mathbb{P}(\mathcal{E})$, Walter's description is not always as simple as above and can involve a non-split exact sequence of Witt groups. As already mentioned, Walter does not use regularity of the ground scheme. For the history between Arason and Walter, we refer to [16]. For the very recent post-Walter times, let us mention Nenashev current series of articles, see [14] and more references there, which also provides a geometric approach to Witt groups of projective bundles over a regular basis, using non-oriented cohomology theories and deformation to the normal cone techniques. Note that Nenashev also considers $\mathbb{P}(\mathcal{E})$ for some vector bundles $\mathcal{E}$.

However, the goal of the present article is certainly *not* the projective bundle theorem itself. This only appears as a nice by-product of our main theme : the study of non-degenerate products of possibly degenerate symmetric forms.

Let us briefly review the part of the material not mentioned so far.

Our natural language is the one of algebraic geometry, that is, the reader could have in mind his favorite scheme X and various derived categories over X. Although everything could be expressed at this level of generality, we introduce a more abstract language, namely the one of a *triangulated category defined over a topological space* X, see Section 1. This has the following advantages. First, even in the above algebro-geometric examples, it will avoid making a different story for each type of derived category we can associate to X (of vector bundles, of coherent modules, of perfect complexes, ...) and it also makes clear which geometric properties are really needed. Moreover, of course, our general formalism can possibly serve outside this algebro-geometric context.

The short Sections 2 and 3 contain basic notions about symmetric forms, like degeneracy, support, symmetric cones, Witt groups and the like. Although not revolutionary, the presentation of Witt groups as the homology groups of the graded semiring of possibly degenerate symmetric forms, see 2.5, ideally prepares the reader's mind to the Leibniz formula of Section 5.

Section 4 deals with products and consanguinity. We essentially use the product in triangular Witt theory of Gille and Nenashev [10], except for the simplifying trick they introduced in their main definition, a trick which only works if the forms are non-degenerate, that is precisely what we cannot afford to assume here. We use instead a more natural construction, but in the same framework, namely the one of "triangulated categories with product and duality" or TPD-categories, recalled in Appendix A for the reader's convenience. In order to prove the above Leibniz formula in Section 5, we also need to control the behaviour of the tensor product with respect to the triangulation in a more precise way than just requiring the product be exact in each variable. For this, we need May's recent axiomatization [12], revamped *à la* Keller-Neeman [11] and also transcripted in Appendix A.

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Preliminaries and conventions

Remark 0.1. In mathematics, some things must be made explicit, some other things *must absolutely not*. This circumspection applies in particular to natural isomorphisms. Here, our rule is to label those natural isomorphisms which are relevant to the current argument and to consider as identities those which are not.

Convention 0.2. We assume without mention that 2 is invertible, *i.e.* our schemes are $\mathbb{Z}[\frac{1}{2}]$-schemes and our categories with duality are $\mathbb{Z}[\frac{1}{2}]$-categories.

Convention 0.3. A scheme is called *regular* if it is noetherian, separated and locally regular.

Notation 0.4. Grothendieck-Verdier's notion of *triangulated category* is defined in [15]. We usually assume that our triangulated categories are essentially small. We denote by $T : \mathcal{K} \longrightarrow \mathcal{K}$ the *translation* functor in the triangulated category $\mathcal{K}$ (a.k.a. the "suspension" in topology or the "shift" in homological algebra).

Definition 0.5. Let $u : A \to B$ be a morphism in a triangulated category. We call *cone of* u any object C, or more precisely any triple (C, u_1, u_2), such that the triangle $A \xrightarrow{u} B \xrightarrow{u_1} C \xrightarrow{u_2} TA$ is distinguished. For a fixed morphism u, its cone is unique up to non-unique isomorphism and we denote it by $\mathrm{cone}(u)$.

1. Triangulated categories defined over a topological space

Definition 1.1. Let X be a topological space. A *triangulated category defined over* X is a pair $(\mathcal{K}, \mathrm{supp})$ where $\mathcal{K}$ is a triangulated category and supp assigns to each object $A \in \mathcal{K}$ a closed subset of X

$$\mathrm{supp}(A) \subset X$$

called *the support of* A and subject to the following four elementary rules :

(S1) Only the support of zero is empty : $\mathrm{supp}(A) = \varnothing \iff A \cong 0.$

(S2) The support respects direct sums : $\mathrm{supp}(A \oplus B) = \mathrm{supp}(A) \cup \mathrm{supp}(B).$

(S3) The support respects translation: $\operatorname{supp}(A) = \operatorname{supp}(TA)$.

(S4) The support respects distinguished triangles: if there exists a distinguished triangle $A \longrightarrow B \longrightarrow C \longrightarrow TA$ then $\operatorname{supp}(C) \subset \operatorname{supp}(A) \cup \operatorname{supp}(B)$.

Remark 1.2. By the rotation axiom and by (S3), we can equivalently say in (S4) that $\operatorname{supp}(A) \subset \operatorname{supp}(B) \cup \operatorname{supp}(C)$ or that $\operatorname{supp}(B) \subset \operatorname{supp}(A) \cup \operatorname{supp}(C)$. It follows from this and from (S1) that the support respects isomorphisms: $A \simeq B \Rightarrow \operatorname{supp}(A) = \operatorname{supp}(B)$.

Example 1.3. Let X be a scheme. The following are triangulated categories over the underlying topological space of X. In all cases, the support is the *homological support*, that is, the usual support of the total homology $\mathcal{O}_X$-module.

(a) Let $\mathcal{K} = \mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$ the derived category of bounded complexes of vector bundles over X. (We use $\mathbb{D}$ not to confuse below with the D of dualities.)

(b) Assume X noetherian and let $\mathcal{K} = \mathbb{D}^{\mathrm{b}}(\mathrm{Coh}_X)$ the derived category of bounded complexes of coherent $\mathcal{O}_X$-modules.

(c) Let $\mathcal{K} = \mathbb{D}^{\mathrm{perf}}(X)$ the derived category of perfect complexes over X.

In all derived categories, for shifts and mapping cones, we follow the (homological) sign conventions of Weibel [17].

Definition 1.4. Let $(\mathcal{K}, \operatorname{supp})$ be a triangulated category defined over X as in Def. 1.1. Assume that $\mathcal{K}$ carries a structure $(\mathcal{K}, D, \delta, \varpi)$ of triangulated category with duality as recalled in Def. A.2. We say that $\mathcal{K}$ is a *triangulated category with duality defined over* X if

(S5) the support respects the duality: $\operatorname{supp}(DA) = \operatorname{supp}(A)$.

Definition 1.5. Let $(\mathcal{K}, \operatorname{supp}_{\mathcal{K}}), (\mathcal{L}, \operatorname{supp}_{\mathcal{L}})$ and $(\mathcal{M}, \operatorname{supp}_{\mathcal{M}})$ be triangulated categories defined over X as in Def. 1.1. Assume that $\boxtimes : \mathcal{K} \times \mathcal{L} \to \mathcal{M}$ is a pairing of triangulated categories as recalled in Def. A.1. We say that *the pairing* $\boxtimes$ *is defined over* X if

(S6) supports respect products: $\operatorname{supp}_{\mathcal{M}}(A \boxtimes B) = \operatorname{supp}_{\mathcal{K}}(A) \cap \operatorname{supp}_{\mathcal{L}}(B)$.

Definition 1.6. *A pairing of triangulated categories with duality defined over* X is the data of three triangulated categories $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{M}$, all equipped with a duality, all defined over X, and of a pairing $\boxtimes : \mathcal{K} \times \mathcal{L} \to \mathcal{M}$ of triangulated categories with duality in the sense of [10] (see Def. A.4), such that both axioms (S5) and (S6) are satisfied.

In the special case where $\mathcal{K} = \mathcal{L} = \mathcal{M}$, we say that $\mathcal{K}$ is a *TPD-category defined over* X or longer: *a triangulated category with product and duality, defined over* X.

Example 1.7. In Ex. 1.3 (a) and (c), the categories are equipped with product and duality without further assumption, by simply deriving the usual ones on vector bundles. They define TPD-categories over X in the sense of Def. 1.6. We shall also consider dualities twisted by line bundles, as usual. In Ex. 1.3 (b), it is recommended to assume X be Gorenstein of finite Krull dimension to get a duality (see [7]). The author does not know of a good condition for the existence of a reasonable tensor product on $\mathbb{D}^{\mathrm{b}}(\mathrm{Coh}_X)$ itself. More common is the pairing $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X) \times \mathbb{D}^{\mathrm{b}}(\mathrm{Coh}_X) \to \mathbb{D}^{\mathrm{b}}(\mathrm{Coh}_X)$ which is defined over X in the sense of Def. 1.6. This external pairing turns $\mathbb{D}^{\mathrm{b}}(\mathrm{Coh}_X)$ into a module over $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$ and illustrates why external pairings are really necessary.

Remark 1.8. It is clear from Prop. A.7 and from axiom (S3) that shifting the dualities on $\mathcal{K}$, $\mathcal{L}$ and coherently on $\mathcal{M}$ still produces a pairing of triangulated categories with duality defined over X.

Remark 1.9. A naive misconception would be to think that the cone of a morphism of the form $u \boxtimes u'$ simply is the product $\boxtimes$ of the cones of u and u'. The situation is more complicated, as explained in App. A. The following statement gives some control on supports of such cones.

Lemma 1.10. *Let $\boxtimes : \mathcal{K} \times \mathcal{L} \longrightarrow \mathcal{M}$ be a pairing of triangulated categories over X. Consider two morphisms $u : A \to B$ in $\mathcal{K}$ and $u' : A' \to B'$ in $\mathcal{L}$. Let us denote their cones by $C := \mathrm{cone}(u) \in \mathcal{K}$ and $C' := \mathrm{cone}(u') \in \mathcal{L}$. Then the support of the cone of $u \boxtimes u'$ is contained in the following 4-term union:*

$$\begin{array}{rcl} \mathrm{supp}_{\mathcal{M}}(\mathrm{cone}(u \boxtimes u')) & \subset & \mathrm{supp}_{\mathcal{K}}(A) \cap \mathrm{supp}_{\mathcal{K}}(B) \cap \mathrm{supp}_{\mathcal{L}}(C') \\ & \cup & \mathrm{supp}_{\mathcal{L}}(A') \cap \mathrm{supp}_{\mathcal{L}}(B') \cap \mathrm{supp}_{\mathcal{K}}(C) \\ & \cup & \mathrm{supp}_{\mathcal{K}}(A) \cap \mathrm{supp}_{\mathcal{L}}(A') \cap \mathrm{supp}_{\mathcal{K}}(C) \cap \mathrm{supp}_{\mathcal{L}}(C') \\ & \cup & \mathrm{supp}_{\mathcal{K}}(B) \cap \mathrm{supp}_{\mathcal{L}}(B') \cap \mathrm{supp}_{\mathcal{K}}(C) \cap \mathrm{supp}_{\mathcal{K}}(C') \, . \end{array}$$

If moreover $\boxtimes$ is a pairing of triangulated categories with duality *over X and if $B = D_{\mathcal{K}} A$ and $B' = D_{\mathcal{L}} A'$, then the above reduces to:*

$$\mathrm{supp}_{\mathcal{M}}(\mathrm{cone}(u \boxtimes u')) \ \subset \ \mathrm{supp}_{\mathcal{K}}(A) \cap \mathrm{supp}_{\mathcal{L}}(C') \ \cup \ \mathrm{supp}_{\mathcal{L}}(A') \cap \mathrm{supp}_{\mathcal{K}}(C) \, .$$

Proof. Let us abbreviate $C'' := \mathrm{cone}(u \boxtimes u') \in \mathcal{M}$. We have by definition two distinguished triangles in $\mathcal{K}$ and $\mathcal{L}$ as follows:

$$A \xrightarrow{u} B \longrightarrow C \longrightarrow TA \quad \text{and} \quad A' \xrightarrow{u'} B' \longrightarrow C' \longrightarrow T(A') \, .$$

From the relation

$$u \boxtimes u' = (u \boxtimes \mathrm{id}_{B'}) \circ (\mathrm{id}_A \boxtimes u') \, ,$$

the octahedron axiom guarantees the existence of a distinguished triangle relating the cones of these three morphisms. The cone of $u \boxtimes u'$ is our object C'' by definition and the cones of $u \boxtimes \mathrm{id}_{B'}$ and $\mathrm{id}_A \boxtimes u'$ are simply obtained from the above distinguished triangles by applying the exact functors $- \boxtimes B'$ and $A \boxtimes -$ respectively. So, we have from the octahedron a distinguished triangle as follows:

$$A \boxtimes C' \longrightarrow C'' \longrightarrow C \boxtimes B' \longrightarrow T(A \boxtimes C') \, .$$

By axiom (S4), we deduce that $\mathrm{supp}_{\mathcal{M}}(C'') \subset \mathrm{supp}_{\mathcal{M}}(A \boxtimes C') \cup \mathrm{supp}_{\mathcal{M}}(C \boxtimes B') \overset{\text{(S6)}}{=} \mathrm{supp}_{\mathcal{K}}(A) \cap \mathrm{supp}_{\mathcal{L}}(C') \cup \mathrm{supp}_{\mathcal{K}}(C) \cap \mathrm{supp}_{\mathcal{L}}(B')$. Using the other decomposition $u \boxtimes u' = (\mathrm{id}_B \boxtimes u') \circ (u \boxtimes \mathrm{id}_{A'})$, we establish similarly the inclusion $\mathrm{supp}_{\mathcal{M}}(C'') \subset \mathrm{supp}_{\mathcal{K}}(B) \cap \mathrm{supp}_{\mathcal{L}}(C') \cup \mathrm{supp}_{\mathcal{K}}(C) \cap \mathrm{supp}_{\mathcal{L}}(A')$. Hence $\mathrm{supp}_{\mathcal{M}}(C'')$ is contained in the intersection of the two sets we just found, which gives the set of the statement.

For the last part, it suffices to use (S5) to replace $\mathrm{supp}_{\mathcal{K}}(B)$ by $\mathrm{supp}_{\mathcal{K}}(A)$ and similarly with B' and A'. In this case, the above 4-term union boils down to the announced one, as is easily checked. $\square$

2. Symmetric forms, cones and Witt groups

For this section, $(\mathcal{K}, D, \delta, \varpi)$ is a triangulated category with duality as recalled in Def. A.2. Although quite standard, we fix the following terminology since the distinction between degenerate and non-degenerate forms is essential here.

Definition 2.1. We define a *symmetric pair* to be a couple (A, a) formed by an object $A \in \mathcal{K}$ and a symmetric morphism $a : A \to DA$. As usual, symmetry means $D(a) \circ \varpi_A = a$. Note that we do not require a to be an isomorphism. The morphism a is referred to as *the form* of the symmetric pair (A, a).

When a is moreover an isomorphism, we say that the form $a : A \to DA$ is *non-degenerate* and that the symmetric pair (A, a) is a *symmetric space*.

Let $i \in \mathbb{Z}$ be an integer. A *symmetric i-pair*, a *symmetric i-space*, an *i-form*, with respect to the duality D, respectively mean a symmetric pair, a symmetric space, a form for the i^{th} shifted duality $(D^{(i)}, \delta^{(i)}, \varpi^{(i)})$ over $\mathcal{K}$, as recalled in Def. A.3.

Notations 2.2. We define isometries of symmetric pairs as usual and denote by

$$\mathrm{Symm}(\mathcal{K}) \qquad \text{or} \qquad \mathrm{Symm}(\mathcal{K}, D, \delta, \varpi)$$

the monoid of isometry classes of symmetric forms over the considered category with duality. Our assumption about essential smallness of $\mathcal{K}$ implies that $\mathrm{Symm}(\mathcal{K})$ is a set. It is a monoid with the usual orthogonal sum. We shall not adopt a new notation for the class of a symmetric pair (A, a) in $\mathrm{Symm}(\mathcal{K})$ and simply write it (A, a). For an integer $i \in \mathbb{Z}$, we denote by

$$\mathrm{Symm}^{(i)}(\mathcal{K}) := \mathrm{Symm}(\mathcal{K}, D^{(i)}, \delta^{(i)}, \varpi^{(i)})$$

the corresponding monoid for the i^{th} shifted duality.

Definition 2.3. We now recall from [2, § 2] the notion of *symmetric cone* of a symmetric form. The cone of a morphism $a : A \to B$ is recalled in Def. 0.5. If $B = DA$ and if the morphism a is symmetric, then its cone $C = \mathrm{cone}(a)$ also becomes symmetric, in the sense that it carries a non-degenerate symmetric form, but for the 1-shifted duality $D^{(1)}$. Namely there exists a symmetric 1-space (C, Φ), unique up to isometry [2, Thm. 2.6], such that the following triangle is distinguished:

$$A \xrightarrow{a} DA \xrightarrow{a_1} C \xrightarrow{a_2} TA \tag{3}$$

and is symmetric in the sense that the following equation holds:

$$\Phi \circ a_1 = -T(D(a_2))\,. \tag{4}$$

It is equivalent to say that (C, Φ) is a 1-space such that the following diagram with distinguished rows commutes:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "\delta\cdot\varpi_A"', "\cong"] & DA \arrow[r, "a_1"] \arrow[d, equal] & C \arrow[r, "a_2"] \arrow[d, "\Phi"', "\simeq"] & T(A) \arrow[d, "\cong"', "\delta\cdot T(\varpi_A)"] \\
D(DA) \arrow[r, "\delta\cdot D(a)"'] & DA \arrow[r, "-T(D(a_2))"'] & T(D(C)) \arrow[r, "T(D(a_1))"'] & T(D(DA))\,,
\end{tikzcd}$$

in which the second row is the dual of the first.

The symmetry of $\Phi : C \longrightarrow D^{(1)}(C)$ for the 1-shifted duality $D^{(1)} = T \circ D$ reads by definition: $D^{(1)}(\Phi) \circ \varpi_C^{(1)} = \Phi$. Unfolding Def. A.3 gives: $-\delta \cdot T(D(\Phi)) \circ \varpi_C = \Phi$. This new symmetric space is denoted by:

$$\mathrm{Cone}(A, a) := (C, \Phi)$$

and is the *symmetric cone* of the symmetric pair (A, a). See an example in 3.5.

Remark 2.4. In the next statement, we use freely the language of TWG [2, § 2]. The reader unfamiliar with TWG can consider the following as (quite conceptual) definitions for the monoids $\mathrm{MW}^i(\mathcal{K})$, $\mathrm{NW}^i(\mathcal{K})$ and the groups $\mathrm{W}^i(\mathcal{K})$.

Proposition and Definition 2.5. *Let $(\mathcal{K}, D, \delta, \varpi)$ be a triangulated category with duality. The symmetric cone construction (see 2.3) induces for all $i \in \mathbb{Z}$ a well-defined homomorphism of monoids:*

$$\begin{array}{rccc} \mathrm{d}: & \mathrm{Symm}^{(i)}(\mathcal{K}) & \longrightarrow & \mathrm{Symm}^{(i+1)}(\mathcal{K}) \\ & (A, a) & \longmapsto & \mathrm{Cone}(A, a) \end{array}$$

which enjoys the following properties:

(a) *The homomorphism* d *is a differential:* $\mathrm{d} \circ \mathrm{d} = 0$.

$$\cdots \xrightarrow{\mathrm{d}} \mathrm{Symm}^{(i-1)}(\mathcal{K}) \xrightarrow{\mathrm{d}} \mathrm{Symm}^{(i)}(\mathcal{K}) \xrightarrow{\mathrm{d}} \mathrm{Symm}^{(i+1)}(\mathcal{K}) \xrightarrow{\mathrm{d}} \cdots$$

(b) *Its kernel coincides exactly with the submonoid of* $\mathrm{Symm}^{(i)}(\mathcal{K})$ *made of symmetric i*-spaces*:*

$$\mathrm{MW}^i(\mathcal{K}) = \ker(\mathrm{d}) := \mathrm{d}^{-1}(0)\,.$$

(c) *Its image coincides exactly with the submonoid of* $\mathrm{Symm}^{(i+1)}(\mathcal{K})$ *made of* metabolic *or* neutral *$(i+1)$-spaces:*

$$\mathrm{NW}^{i+1}(\mathcal{K}) = \mathrm{im}(\mathrm{d}) := \mathrm{d}\left(\mathrm{Symm}^{(i)}(\mathcal{K})\right).$$

(d) *Its homology is the i^{th}* triangular Witt group *of $\mathcal{K}$:*

$$\mathrm{W}^i(\mathcal{K}) = \mathrm{MW}^i(\mathcal{K}) / \mathrm{NW}^i(\mathcal{K}) = \ker(\mathrm{d}) / \mathrm{im}(\mathrm{d})\,.$$

Proof. The fact that the isometry class of $\mathrm{Cone}(A, a)$ only depends on the isometry class of (A, a) is immediate from the definition, see 2.3, and the fact that the symmetric space (C, Φ) constructed there is unique up to isometry [2, Thm. 2.6].

Part (a) is clear since $\mathrm{Cone}(A, a)$ is a space, so it has a trivial symmetric cone. (In the above notation, we have $\mathrm{cone}(\Phi) = 0$.) Conversely, $\mathrm{Cone}(A, a) = 0$ implies that the form a is an isomorphism, which proves (b). Parts (c) and (d) are transcriptions of the definitions, see [2, § 2]. □

Remark 2.6. Observe that the "homology" of a complex of monoids is probably as slippery a notion as the one of "exact sequence" of monoids. We do not know if there is a reasonable version of the above complex $\mathrm{Symm}^{(\bullet)}(\mathcal{K})$ made of abelian groups (its group completion for instance) whose homology coincides with the above Witt groups. Although elements in $\mathrm{Symm}^{(i)}(\mathcal{K})$ do not admit an opposite, we can define $-\alpha := (A, -a)$ for any symmetric pair $\alpha = (A, a)$ in $\mathrm{Symm}^{(i)}(\mathcal{K})$ and we have $\mathrm{d}(-\alpha) = -\,\mathrm{d}(\alpha)$. This does provide the opposite in the Witt *group*.

Remark 2.7. We have 4-periodicity of Witt groups, which is already visible on the level of $\mathrm{Symm}^{(i)}(\mathcal{K})$ and is simply induced by the translation:

$$\begin{aligned} \mathrm{Symm}^{(i)}(\mathcal{K}) &\xrightarrow{\sim} \mathrm{Symm}^{(i+4)}(\mathcal{K}) \\ (A,a) &\longmapsto \big(T^2(A), T^2(a)\big)\,, \end{aligned}$$

using that $TD = DT^{-1}$ which gives in particular $T^2 \circ D = D^{(4)} \circ T^2$. Note also that the signs involved in $\delta^{(i)}$ and $\varpi^{(i)}$ are 4-periodic, as recalled in Def. A.3.

Example 2.8. Returning to our geometric examples 1.3 and 1.7, we can now define Witt groups of the respective derived categories with duality associated to the scheme X. In case (a), that is for $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$, and for any line bundle $\mathcal{L} \in \mathrm{Pic}(X)$, we obtain the so-called *(derived) Witt groups* $\mathrm{W}^i(X,\mathcal{L}) := \mathrm{W}^i(\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X), D_{\mathcal{L}})$, *with i-shifted and $\mathcal{L}$-twisted duality*, where $D_{\mathcal{L}}$ is the derived duality twisted by $\mathcal{L}$:

$$D_{\mathcal{L}}(-) = \mathcal{H}om_{\mathcal{O}_X}(-,\mathcal{O}_X) \otimes \mathcal{L}\,.$$

The same duality applies in case (c), that is on $\mathbb{D}^{\mathrm{perf}}(X)$, yielding what could be called *perfect (derived) Witt groups* of X. In case (b), that is for $\mathbb{D}^{\mathrm{b}}(\mathrm{Coh}_X)$, and under the assumptions insuring the existence of the duality, we obtain the *coherent (derived) Witt groups* of X, see [7, § 2.5]. We shall not use here the latter two examples but only derived Witt groups (of vector bundles).

3. Support and degeneracy locus

Let $\mathcal{K}$ be a triangulated category with duality defined over a topological space X (Def. 1.4). We have the following concepts.

Definition 3.1. We define the *support* of a symmetric pair $\alpha = (A,a)$ as the support of the object A:

$$\mathrm{Supp}(\alpha) := \mathrm{supp}(A) \subset X\,.$$

Definition 3.2. We define the *degeneracy locus* of a symmetric pair $\alpha = (A,a)$ to be the support of the cone of the morphism a:

$$\mathrm{DegLoc}(\alpha) := \mathrm{supp}(\mathrm{cone}(a))\,.$$

Combined with Definitions 2.5 and 3.1, this gives:

$$\mathrm{DegLoc}\,(\alpha) = \mathrm{Supp}\,(\mathrm{d}(\alpha))\,. \tag{5}$$

Proposition 3.3. *The degeneracy locus of a symmetric form is contained in its support:* $\mathrm{DegLoc}(\alpha) \subset \mathrm{Supp}(\alpha)$.

Proof. Write $\alpha = (A,a)$ and use the distinguished triangle (3) of Def. 2.3. Then

$$\mathrm{DegLoc}(\alpha) \overset{\text{def.}}{=} \mathrm{supp}(C) \overset{\text{(S4)}}{\subset} \mathrm{supp}(A) \cup \mathrm{supp}(DA) \overset{\text{(S5)}}{=} \mathrm{supp}(A) \overset{\text{def.}}{=} \mathrm{Supp}(\alpha)\,. \qquad \square$$

Proposition 3.4. *A symmetric pair is a symmetric space if and only if its degeneracy locus is empty.*

Proof. In a triangulated category, a morphism is an isomorphism if and only if its cone is zero. The statement follows from (S1) of Def. 1.1. $\square$

Example 3.5. Let X be a scheme and let $s \in \Gamma(X, \mathcal{O}_X)$ be a global section of the structure sheaf $\mathcal{O}_X$. Let $\alpha(s) = (\mathcal{O}_X, s)$ be the obvious symmetric pair for the unshifted untwisted duality on $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$, *i.e.* consider $\mathcal{O}_X$ as a complex concentrated in degree 0 and s as a morphism from $\mathcal{O}_X$ to its dual, which is $\mathcal{O}_X$ again. Then the support of α is the support of $\mathcal{O}_X$, that is, the whole of X. The degeneracy locus of α is the zero set $Z(s)$ of s. The symmetric cone 2.5 of $\alpha(s)$, is the following symmetric 1-space (C, Φ) :

$$\mathrm{d}\,\alpha(s) = \left(\begin{tikzcd} C := \arrow[d, "\Phi"', ":="] & \cdots 0 \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r, "-s"] \arrow[d, "-1"] & \mathcal{O}_X \arrow[r] \arrow[d, "+1"] & 0 \arrow[r] \arrow[d] & 0 \cdots \arrow[d] \\ TD(C) = & \cdots 0 \arrow[r] & \mathcal{O}_X \arrow[r, "s"'] & \mathcal{O}_X \arrow[r] & 0 \arrow[r] & 0 \cdots \end{tikzcd}\right)$$

with the objects $\mathcal{O}_X$ in homological degrees 1 and 0. This metabolic space has support $Z(s)$, compare (5), and its degeneracy locus is empty, as for any space. We shall generalize this example in Section 6, see Def. 6.4 and Prop. 6.5.

4. Product and consanguinity

In this section, $\boxtimes : \mathcal{K} \times \mathcal{L} \to \mathcal{M}$ is a pairing of triangulated categories with duality defined over X as in Def. 1.6. We write supp for the three support-assignments, independently of the category $\mathcal{K}$, $\mathcal{L}$ or $\mathcal{M}$.

Let $\alpha = (A, a)$ and $\beta = (B, b)$ be two symmetric pairs in $\mathcal{K}$ and $\mathcal{L}$ respectively.

Definition 4.1. We define *the product* of the symmetric pairs α and β to be the symmetric pair $\alpha \star \beta := \big(A \boxtimes B\,,\ \mu_{A,B} \circ (a \boxtimes b)\big)$. The same notation $\star$ applies also to the form itself. So, we have

$$(A, a) \star (B, b) = (A \boxtimes B\,,\ a \star b)$$

where we use μ to identify the product of the duals with the dual of the product :

$$\begin{tikzcd} A \boxtimes B \arrow[r, "a \boxtimes b"] \arrow[rr, bend right, "a \star b"'] & D_{\mathcal{K}} A \ \boxtimes\ D_{\mathcal{L}} B \arrow[r, "\mu_{A,B}", "\cong"'] & D_{\mathcal{M}}(A \boxtimes B)\,. \end{tikzcd}$$

Remark 4.2. See Def. A.6 for how to define $\mu^{(i,j)}$ so that $(\boxtimes, \mu^{(i,j)})$ is again a pairing of triangulated categories with duality, when using the i-shifted duality $(\mathcal{K}, D^{(i)}, \delta^{(i)}, \varpi^{(i)})$ on $\mathcal{K}$, the j-shifted duality $(\mathcal{L}, D^{(j)}, \delta^{(j)}, \varpi^{(j)})$ on $\mathcal{L}$ and $(i + j)$-shifted duality $(\mathcal{M}, D^{(i+j)}, \delta^{(i+j)}, \varpi^{(i+j)})$ on $\mathcal{M}$. With this in mind, Definition 4.1 also applies to the shifted dualities. So, for all $i, j \in \mathbb{Z}$, we have a bi-additive pairing of monoids :

$$\star : \mathrm{Symm}^{(i)}(\mathcal{K}) \times \mathrm{Symm}^{(j)}(\mathcal{L}) \to \mathrm{Symm}^{(i+j)}(\mathcal{M})\,.$$

(Distributivity with respect to orthogonal sum, is obvious.) It is clear that this pairing respects 4-periodicity, see Rem. 2.7. We shall see in Theorem 5.2 how this pairing behaves with respect to the differential $\mathrm{d} : \mathrm{Symm}^{(i)} \to \mathrm{Symm}^{(i-1)}$.

Definition 4.3. We define the *consanguinity* of the symmetric pairs α and β to be the following closed subset of X :

$$\mathrm{Cons}(\alpha,\beta) := \big(\mathrm{Supp}(\alpha)\cap\mathrm{DegLoc}(\beta)\big)\ \cup\ \big(\mathrm{DegLoc}(\alpha)\cap\mathrm{Supp}(\beta)\big)\,.$$

With $\alpha=(A,a)$, $\beta=(B,b)$ and Defs. 3.1 and 3.2, the above subset of X is

$$\mathrm{Cons}(\alpha,\beta) = \big(\,\mathrm{supp}(A)\ \cap\ \mathrm{supp}\,\big(\mathrm{cone}(b)\big)\big)\ \cup\ \big(\mathrm{supp}\,\big(\mathrm{cone}(a)\big)\ \cap\ \mathrm{supp}(B)\big)\,.$$

We say that the symmetric pairs α and β *have no consanguinity* if $\mathrm{Cons}(\alpha,\beta)=\varnothing$.

Proposition 4.4. *We have* $\mathrm{DegLoc}(\alpha\star\beta)\subset\mathrm{Cons}(\alpha,\beta)$*.*

Proof. This is the last statement of Lemma 1.10. □

Remark 4.5. In algebraic geometry, we have in fact equality $\mathrm{DegLoc}(\alpha\star\beta) = \mathrm{Cons}(\alpha,\beta)$. We do not know whether this can be proved for all triangulated categories defined over a topological space without further assumptions. Anyway, we only need the inclusion of Prop. 4.4 to apply Prop. 3.4 and obtain the following:

Corollary 4.6. *If the symmetric pairs α and β have no consanguinity then $\alpha\star\beta$ is a symmetric space.* □

* * *

Remark 4.7. We now want to extend the above considerations to products of several symmetric pairs $\alpha_1\star\ldots\star\alpha_n$. We find it too cumbersome to consider a multiple-entry product $\boxtimes:\mathcal{K}_1\times\ldots\times\mathcal{K}_n\longrightarrow\mathcal{M}$ and to unfold all the relevant natural isomorphisms. Therefore, we now restrict attention to TPD-categories $(\mathcal{K},\otimes)$ defined over X in the sense of Def. 1.6, that is, to the case where all the categories involved coincide. We do not assume the tensor product to be associative, although it will be so in the geometric examples, for the reason that the sign conventions hidden in associativity isomorphisms would unnecessarily overburden the presentation.

Definition 4.8. We extend the definition of the product $\alpha\star\beta$ given in Def. 4.1 to several symmetric pairs by induction over $n\geq 2$:

$$\alpha_1\star\ldots\star\alpha_n := (\alpha_1\star\ldots\star\alpha_{n-1})\star\alpha_n\,.$$

Definition 4.9. Let $\alpha_1,\ldots,\alpha_n$ be symmetric forms in our TPD-category $\mathcal{K}$ defined over the topological space X. We define the *consanguinity* of $\alpha_1,\ldots,\alpha_n$ to be the following closed subset of X :

$$\mathrm{Cons}(\alpha_1,\ldots,\alpha_n) := \bigcap_{i=1}^{n}\mathrm{Supp}(\alpha_i)\ \cap\ \Big(\bigcup_{j=1}^{n}\mathrm{DegLoc}(\alpha_j)\Big)\,.$$

Observe that this definition is symmetric in $\alpha_1,\ldots,\alpha_n$. We say that the forms $\alpha_1,\ldots,\alpha_n$ *have no consanguinity* if this set is empty: $\mathrm{Cons}(\alpha_1,\ldots,\alpha_n)=\varnothing$.

Lemma 4.10. *Let $\alpha_1,\ldots,\alpha_n$ be symmetric pairs.*

(a) *For $n=1$, we have* $\mathrm{Cons}(\alpha_1)=\mathrm{DegLoc}(\alpha_1)$*.*

(b) *For $n\geq 2$, we have the following inductive formula:*

$$\mathrm{Cons}(\alpha_1,\ldots,\alpha_n) =$$
$$\big(\mathrm{Supp}(\alpha_1\star\ldots\star\alpha_{n-1})\ \cap\ \mathrm{Cons}(\alpha_n)\big)\ \cup\ \big(\mathrm{Cons}(\alpha_1,\ldots,\alpha_{n-1})\ \cap\ \mathrm{Supp}(\alpha_n)\big)$$

which, by (a), coincides with Definition 4.3 when $n=2$.

Proof. Statement (a) is immediate from the definition and from Prop. 3.3. Let us compute directly the right-hand side of (b):

$$\begin{aligned}
&\mathrm{Supp}(\alpha_1\star\ldots\star\alpha_{n-1})\cap\mathrm{Cons}(\alpha_n)\cup\mathrm{Cons}(\alpha_1,\ldots,\alpha_{n-1})\cap\mathrm{Supp}(\alpha_n)\\
&\overset{\text{(S6)}}{=}\bigcap_{j=1}^{n-1}\mathrm{Supp}(\alpha_j)\cap\mathrm{Cons}(\alpha_n)\cup\mathrm{Cons}(\alpha_1,\ldots,\alpha_{n-1})\cap\mathrm{Supp}(\alpha_n)\\
&\overset{(a)\,\&\,4.9}{=}\bigcap_{j=1}^{n-1}\mathrm{Supp}(\alpha_j)\cap\mathrm{DegLoc}(\alpha_n)\cup\bigcap_{j=1}^{n}\mathrm{Supp}(\alpha_j)\cap\Big(\bigcup_{i=1}^{n-1}\mathrm{DegLoc}(\alpha_i)\Big)\\
&\overset{3.3}{=}\bigcap_{j=1}^{n}\mathrm{Supp}(\alpha_j)\cap\mathrm{DegLoc}(\alpha_n)\cup\bigcap_{j=1}^{n}\mathrm{Supp}(\alpha_j)\cap\Big(\bigcup_{i=1}^{n-1}\mathrm{DegLoc}(\alpha_i)\Big)\\
&\overset{4.9}{=}\mathrm{Cons}(\alpha_1,\ldots,\alpha_n)\,.
\end{aligned}$$

□

Proposition 4.11. *Let $\alpha_1,\ldots,\alpha_n$ be symmetric forms. Then, we have:*

$$\mathrm{DegLoc}(\alpha_1\star\ldots\star\alpha_n)\subset\mathrm{Cons}(\alpha_1,\ldots,\alpha_n)\,.$$

Proof. By induction over n. For $n=1$ both sides are equal to $\mathrm{DegLoc}(\alpha_1)$ by Lem. 4.10 (a). Assume that $n\geq 2$ and that the result holds for $n-1$. We have

$$\begin{aligned}
&\mathrm{DegLoc}(\alpha_1\star\ldots\star\alpha_n)\overset{4.4}{\subset}\mathrm{Cons}((\alpha_1\star\ldots\star\alpha_{n-1})\,,\,\alpha_n)\\
&\overset{4.3}{=}\mathrm{Supp}(\alpha_1\star\ldots\star\alpha_{n-1})\cap\mathrm{DegLoc}(\alpha_n)\cup\mathrm{DegLoc}(\alpha_1\star\ldots\star\alpha_{n-1})\cap\mathrm{Supp}(\alpha_n)\\
&\overset{\text{I.H.}}{\subset}\mathrm{Supp}(\alpha_1\star\ldots\star\alpha_{n-1})\cap\mathrm{DegLoc}(\alpha_n)\cup\mathrm{Cons}(\alpha_1,\ldots,\alpha_{n-1})\cap\mathrm{Supp}(\alpha_n)\\
&\overset{4.10}{=}\mathrm{Cons}(\alpha_1,\ldots,\alpha_n)\,,
\end{aligned}$$

where the inclusion labelled "I.H." holds by induction hypothesis. □

Remark 4.12. The above proof shows that if equality holds in Prop. 4.4 then it holds in Prop. 4.11 as well. This is in particular the case in algebraic geometry as mentioned in Rem. 4.5. Still, we only need the above inclusion for the following:

Corollary 4.13. *Let $\alpha_1,\ldots,\alpha_n$ be symmetric forms with no consanguinity. Then $\alpha_1\star\ldots\star\alpha_n$ is non-degenerate.* □

5. Leibniz formula

We return to the general situation of a pairing $\boxtimes:\mathcal{K}\times\mathcal{L}\longrightarrow\mathcal{M}$ of triangulated categories with duality, defined over a topological space X, as in Def. 1.6.

Proposition and Definition 5.1. *Let α and β be symmetric forms. Then the following are equivalent:*

(a) *The degeneracy loci of the forms α and β do not intersect.*
(b) *The forms $\mathrm{d}(\alpha)$ and β have no consanguinity.*
(c) *The forms α and $\mathrm{d}(\beta)$ have no consanguinity.*

In this case, we say that α and β have no common degeneracy, *which implies in particular that $\mathrm{d}(\alpha)\star\beta$ and $\alpha\star\mathrm{d}(\beta)$ are non-degenerate.*

Proof. We have $\mathrm{DegLoc}(\mathrm{d}(\alpha)) = \varnothing$ and $\mathrm{Supp}(\mathrm{d}(\alpha)) = \mathrm{DegLoc}(\alpha)$, see Eq. (5). By Definition 4.3, it follows that $\mathrm{Cons}(\mathrm{d}(\alpha), \beta) = \mathrm{DegLoc}(\alpha) \cap \mathrm{DegLoc}(\beta) = \mathrm{Cons}(\alpha, \mathrm{d}(\beta))$. This proves the equivalence of the three conditions. The conclusion about the non-degeneracy of the products comes from Cor. 4.6. $\square$

For the next result, we need to assume that the pairing $\boxtimes : \mathcal{K} \times \mathcal{L} \longrightarrow \mathcal{M}$ is compatible with the octahedron axiom, as recalled in Def. A.11. This axiomatization follows May [12] and holds of course for any pairing observable in nature.

Theorem 5.2 (Leibniz-type formula for symmetric spaces)**.** *Let α and β be symmetric forms with no common degeneracy (see 5.1). Then, we have an isometry:*

$$\delta_{\mathcal{M}} \cdot \quad \mathrm{d}(\alpha \star \beta) \quad \simeq \quad \delta_{\mathcal{K}} \cdot \quad \mathrm{d}(\alpha) \star \beta \quad + \quad \delta_{\mathcal{L}} \cdot \quad \alpha \star \mathrm{d}(\beta)\,,$$

where we recall that the signs $\delta_{\mathcal{K}}, \delta_{\mathcal{L}}, \delta_{\mathcal{M}} = \pm 1$ express the exactness of the three dualities involved $D_{\mathcal{K}}, D_{\mathcal{L}}$ and $D_{\mathcal{M}}$.

Proof. Write the symmetric forms $\alpha = (A, a)$ and $\beta = (B, b)$ and consider distinguished triangles in $\mathcal{K}$ and $\mathcal{L}$ respectively:

$$A \xrightarrow{\ a\ } DA \xrightarrow{\ a_1\ } C \xrightarrow{\ a_2\ } TA \tag{6}$$

$$B \xrightarrow{\ b\ } DB \xrightarrow{\ b_1\ } C' \xrightarrow{\ b_2\ } TB \tag{7}$$

as well as the cone symmetric forms $\Phi : C \overset{\sim}{\to} D^{(1)}_{\mathcal{K}}(C)$ and $\Phi' : C' \overset{\sim}{\to} D^{(1)}_{\mathcal{L}}(C')$ which satisfy the following equations, see (4) in Def. 2.3 if necessary:

$$\Phi \circ a_1 = -D^{(1)}_{\mathcal{K}}(a_2) \qquad \text{and} \qquad \Phi' \circ b_1 = -D^{(1)}_{\mathcal{L}}(b_2)\,. \tag{8}$$

From now on, we shall write D for $D_{\mathcal{K}}$, $D_{\mathcal{L}}$ and $D_{\mathcal{M}}$ since it is always clear which duality is meant from the object or the morphism it is applied to.

The proof will consist in finding a distinguished triangle over the morphism $a \star b$ and in showing that the symmetric form Φ'' on its cone satisfying an equation of type (8) can be chosen to be $\Phi'' = \delta_{\mathcal{K}} \delta_{\mathcal{M}} \cdot (\Phi \star b) \perp \delta_{\mathcal{L}} \delta_{\mathcal{M}} \cdot (a \star \Phi')$ as announced in the statement. This will be the symmetric cone $\mathrm{d}(\alpha \star \beta)$ by uniqueness of the construction, see 2.3. Indeed, it is not hard to see that the cone of $a \star b$, which is the cone of $a \boxtimes b$, is isomorphic to the direct sum of $C \boxtimes B$ and $A \boxtimes C'$ as predicted by the Theorem. It is harder to get the right morphisms in this distinguished triangle in order to check the equation of type (4) for Φ''. We proceed as follows.

The assumption $\varnothing = \mathrm{DegLoc}(\alpha) \cap \mathrm{DegLoc}(\beta) = \mathrm{supp}(C) \cap \mathrm{supp}(C') \overset{(S6)}{=} \mathrm{supp}(C \boxtimes C')$ implies by (S1) the vanishing of the product $C \boxtimes C' = 0$. We use this in the next diagram. Applying the bi-exact functor $- \boxtimes -$ to the above distinguished triangles (6) and (7), we obtain a diagram with distinguished rows and columns, which commutes except for the lower-right square which anti-commutes:

(9)
$$\begin{tikzcd}
A\boxtimes B \arrow[r, "a\boxtimes\mathrm{id}"] \arrow[d, "\mathrm{id}\boxtimes b"] & DA\boxtimes B \arrow[r, "a_1\boxtimes\mathrm{id}"] \arrow[d, "\mathrm{id}\boxtimes b"] & C\boxtimes B \arrow[r, "\lambda\cdot(a_2\boxtimes\mathrm{id})"] \arrow[d, "\mathrm{id}\boxtimes b"] & T(A\boxtimes B) \arrow[d, "T(\mathrm{id}\boxtimes b)"] \\
A\boxtimes DB \arrow[r, "a\boxtimes\mathrm{id}"] \arrow[d, "\mathrm{id}\boxtimes b_1"] & DA\boxtimes DB \arrow[r, "a_1\boxtimes\mathrm{id}"] \arrow[d, "\mathrm{id}\boxtimes b_1"] & C\boxtimes DB \arrow[r, "\lambda\cdot(a_2\boxtimes\mathrm{id})"] \arrow[d, "\mathrm{id}\boxtimes b_1"] & T(A\boxtimes DB) \arrow[d, "T(\mathrm{id}\boxtimes b_1)"] \\
A\boxtimes C' \arrow[r, "a\boxtimes\mathrm{id}"] \arrow[d, "\rho\cdot(\mathrm{id}\boxtimes b_2)"] & DA\boxtimes C' \arrow[r, "a_1\boxtimes\mathrm{id}"] \arrow[d, "\rho\cdot(\mathrm{id}\boxtimes b_2)"] & C\boxtimes C' \arrow[r, "\lambda\cdot(a_2\boxtimes\mathrm{id})"] \arrow[d, "\rho\cdot(\mathrm{id}\boxtimes b_2)"] \arrow[dr, phantom, "(-1)"] & T(A\boxtimes C') \arrow[d, "-T(\rho\cdot(\mathrm{id}\boxtimes b_2))"] \\
T(A\boxtimes B) \arrow[r, "T(a\boxtimes\mathrm{id})"'] & T(DA\boxtimes B) \arrow[r, "T(a_1\boxtimes\mathrm{id})"'] & T(C\boxtimes B) \arrow[r, "-T(\lambda\cdot(a_2\boxtimes\mathrm{id}))"'] & T^2(A\boxtimes B)
\end{tikzcd}$$

Plugging $C\boxtimes C'=0$ in this diagram, we immediately deduce the following:

(10) $$\bar{a}:=a\boxtimes\mathrm{id}_{C'}\ :\ A\boxtimes C' \xrightarrow{\sim} DA\boxtimes C' \qquad \text{is an isomorphism}$$

(11) $$\bar{b}:=\mathrm{id}_C\boxtimes b\ :\ C\boxtimes B \xrightarrow{\sim} C\boxtimes DB \qquad \text{is an isomorphism}$$

from exactness of the third rows and columns, whereas commutativity of the squares "(2,3)" and "(3,2)" gives:

(12) $$a_2\boxtimes b_1=0 \qquad \text{and} \qquad a_1\boxtimes b_2=0\,.$$

From compatibility of the pairing $\boxtimes$ with the octahedron axiom (Def. A.11) there exist in $\mathcal{M}$ three distinguished octahedra:

$$\begin{tikzcd}
 & & & DA\boxtimes DB \arrow[dl, "\mathrm{id}_{DA}\boxtimes b_1"'] \arrow[ddrr, "c"] & & \\
DA\boxtimes B \arrow[urrr, "\mathrm{id}_{DA}\boxtimes b"] \arrow[ddrr, "a_1\boxtimes\mathrm{id}_B"'] & & DA\boxtimes C' \arrow[ll, "\rho\circ(\mathrm{id}_{DA}\boxtimes b_2)"'] \arrow[dd, "0"] & & & \\
 & & & A\boxtimes B \arrow[uu, "a\boxtimes b"] \arrow[ulll, "a\boxtimes\mathrm{id}_B"'] & & (C\boxtimes B)\oplus(DA\boxtimes C') \arrow[ulll, "\begin{pmatrix}0 & 1\end{pmatrix}"'] \arrow[ll, "d"'] \\
 & & C\boxtimes B \arrow[ur, "\lambda\circ(a_2\boxtimes\mathrm{id}_B)"] \arrow[urrr, "\begin{pmatrix}1\\0\end{pmatrix}"'] & & &
\end{tikzcd}$$

$$
\begin{tikzcd}
 & & & DA \boxtimes DB \arrow[dl, "a_1 \boxtimes \mathrm{id}_{DB}"'] \arrow[ddrr, "c"] & & \\
A \boxtimes DB \arrow[urrr, "a \boxtimes \mathrm{id}_{DB}"] \arrow[dddrr, "\mathrm{id}_A \boxtimes b_1"'] & & C \boxtimes DB \arrow[ll, dashed, "\lambda\circ(a_2 \boxtimes \mathrm{id}_{DB})"'] \arrow[dd, "0"'] & & & \\
 & & & A \boxtimes B \arrow[uu, "a \boxtimes b"] \arrow[ulll, "\mathrm{id}_A \boxtimes b"'] & & (C \boxtimes B) \oplus (DA \boxtimes C') \arrow[ll, dashed, "d"'] \arrow[ulll, "h"'] \\
 & & A \boxtimes C' \arrow[ur, dashed, "\rho\circ(\mathrm{id}_A \boxtimes b_2)"] \arrow[urrr, "g"'] & & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & & A \boxtimes C' \arrow[dl, "\bar{a}"'] \arrow[ddrr, "g"] & & \\
0 \arrow[urrr] \arrow[dddrr] & & DA \boxtimes C' \arrow[ll, dashed] \arrow[dd, "0"'] & & & \\
 & & & T^{-1}(C \boxtimes DB) \arrow[uu, "0"] \arrow[ulll] & & (C \boxtimes B) \oplus (DA \boxtimes C') \arrow[ll, dashed, "h"'] \arrow[ulll, "\begin{pmatrix} 0 & 1 \end{pmatrix}"'] \\
 & & C \boxtimes B \arrow[ur, dashed, "\bar{b}"] \arrow[urrr, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"'] & & &
\end{tikzcd}
$$

These three octahedra are simply (37), (38) and (39) of Definition A.11 applied to the distinguished triangles (6) and (7), in which we used the above information $C \boxtimes C' = 0$, as well as $a_2 \boxtimes b_1 = 0$ and $a_1 \boxtimes b_2 = 0$. As explained in Rem. A.13, we also allowed ourselves to choose *one* of the distinguished triangles (34), (35), (36), namely (35), the one over $a_1 \boxtimes b_2 = 0$, which we chose to be

$$
C \boxtimes B \xrightarrow{e := \begin{pmatrix} 1 \\ 0 \end{pmatrix}} (C \boxtimes B) \oplus (DA \boxtimes C') \xrightarrow{f := \begin{pmatrix} 0 & 1 \end{pmatrix}} DA \boxtimes C' \xrightarrow{0} T(C \boxtimes B)\,.
$$

This is also how we know that the object E of A.11 is here $(C \boxtimes B) \oplus (DA \boxtimes C')$.

From the commutativity in the third octahedron, we immediately compute one entry, in matrix notation, of each of the morphisms g and h. Since $h \circ g = 0$ and since $\bar{a}$ and $\bar{b}$ are isomorphisms, we deduce that there exists a morphism

$$
(13) \qquad k : A \boxtimes C' \to C \boxtimes B \quad \text{such that} \quad g = \begin{pmatrix} k \\ \bar{a} \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} \bar{b} & -\bar{b}\, k\, \bar{a}^{-1} \end{pmatrix}\,.
$$

We are now going to use the trick of A.13 again, namely the one explained in the second part of A.13, carefully keeping our two morphisms $e = \begin{pmatrix}1\\0\end{pmatrix}$ and $f = \begin{pmatrix}0 & 1\end{pmatrix}$. For this, consider the automorphism

$$\ell := \begin{pmatrix}1 & -k\bar{a}^{-1}\\ 0 & 1\end{pmatrix} \;:\; (C \boxtimes B) \oplus (DA \boxtimes C') \xrightarrow{\sim} (C \boxtimes B) \oplus (DA \boxtimes C')\,.$$

Composing the above octahedra with this isomorphism ℓ gives three new octahedra, which are of course as distinguished as the above ones. Let us see what happens to the morphisms involved in this composition, namely the six morphisms having source or target equal to the modified object. They become:

$$\ell \circ \begin{pmatrix}1\\0\end{pmatrix},\ \begin{pmatrix}0 & 1\end{pmatrix} \circ \ell^{-1},\ \ell \circ c,\ d \circ \ell^{-1},\ \ell \circ g \text{ and } h \circ \ell^{-1}\,.$$

Now, by choice of the automorphism ℓ and by (13) these six morphisms simply are:

$$\begin{pmatrix}1\\0\end{pmatrix},\ \begin{pmatrix}0 & 1\end{pmatrix},\ \ell \circ c,\ d \circ \ell^{-1},\ \begin{pmatrix}0\\ \bar{a}\end{pmatrix} \text{ and } \begin{pmatrix}\bar{b} & 0\end{pmatrix}\,.$$

Let us rebaptise the last four morphisms c, d, g and h, respectively. So, we now have three octahedra exactly as above, with in addition

$$g = \begin{pmatrix}0\\ \bar{a}\end{pmatrix} \text{ and } h = \begin{pmatrix}\bar{b} & 0\end{pmatrix}\,.$$

Using this in the second octahedron it follows that

$$c = \begin{pmatrix}\bar{b}^{-1} \circ (a_1 \boxtimes \mathrm{id}_{DB})\\ ?\end{pmatrix} \quad \text{and} \quad d = \begin{pmatrix}\,?? & \rho \circ (\mathrm{id}_A \boxtimes b_2) \circ \bar{a}^{-1}\end{pmatrix} \tag{14}$$

whereas the first octahedron gives us:

$$c = \begin{pmatrix}???\\ \mathrm{id}_{DA} \boxtimes b_1\end{pmatrix} \quad \text{and} \quad d = \begin{pmatrix}\lambda \circ (a_2 \boxtimes \mathrm{id}_B) & ????\,\end{pmatrix}\,. \tag{15}$$

Since it is the same c and the same d in both octahedra (this is the whole point of this proof!), we can put (14) and (15) together and obtain:

$$c = \begin{pmatrix}\bar{b}^{-1} \circ (a_1 \boxtimes \mathrm{id}_{DB})\\ \mathrm{id}_{DA} \boxtimes b_1\end{pmatrix} \quad \text{and} \quad d = \begin{pmatrix}\lambda \circ (a_2 \boxtimes \mathrm{id}_B) & \rho \circ (\mathrm{id}_A \boxtimes b_2) \circ \bar{a}^{-1}\end{pmatrix}\,, \tag{16}$$

that is, we have the complete description of a distinguished triangle over $a \boxtimes b$:

$$A \boxtimes B \xrightarrow{a \boxtimes b} DA \boxtimes DB \xrightarrow{c} (C \boxtimes B) \oplus (DA \boxtimes C') \xrightarrow{d} T(A \boxtimes B)\,.$$

From this we deduce the distinguished triangle over $a \star b = \mu_{A,B} \circ a \boxtimes b$ in the obvious way, since $\mu_{A,B} : DA \boxtimes DB \xrightarrow{\sim} D(A \boxtimes B)$ is an isomorphism:

$$A \boxtimes B \xrightarrow{a \star b} D(A \boxtimes B) \xrightarrow{m_1} (C \boxtimes B) \oplus (A \boxtimes C') \xrightarrow{m_2} T(A \boxtimes B)\,, \tag{17}$$

in which we also replaced $DA \boxtimes C'$ by the isomorphic $A \boxtimes C'$, using the isomorphism $\bar{a}$. The morphisms m_1 and m_2 are explicitly given by:

$$m_1 = \begin{pmatrix}\mathrm{id} & 0\\ 0 & \bar{a}^{-1}\end{pmatrix} \circ c \circ \mu^{-1} \overset{(16)}{=} \begin{pmatrix}\bar{b}^{-1} \circ (a_1 \boxtimes \mathrm{id}_{DB})\\ \bar{a}^{-1} \circ (\mathrm{id}_{DA} \boxtimes b_1)\end{pmatrix} \circ \mu^{-1}\,, \tag{18}$$

and

$$m_2 = d \circ \begin{pmatrix} \mathrm{id} & 0 \\ 0 & \bar{a} \end{pmatrix} \overset{(16)}{=} \big(\lambda \circ (a_2 \boxtimes \mathrm{id}_B) \quad \rho \circ (\mathrm{id}_A \boxtimes b_2)\big) \ . \tag{19}$$

Consider now the two symmetric 1-forms:

$$\Phi \star b : C \boxtimes B \xrightarrow{\sim} D^{(1)}(C \boxtimes B) \qquad \text{and} \qquad a \star \Phi' : A \boxtimes C' \xrightarrow{\sim} D^{(1)}(A \boxtimes C')$$

which are non-degenerate by Proposition 5.1. The claim of the Theorem is that their orthogonal sum Φ'' (up to the signs $\delta_{\mathcal{K}}, \delta_{\mathcal{L}}, \delta_{\mathcal{M}}$ announced in the statement)

$$\Phi'' := \delta_{\mathcal{K}}\, \delta_{\mathcal{M}} \cdot (\Phi \star b) \perp \delta_{\mathcal{L}}\, \delta_{\mathcal{M}} \cdot (a \star \Phi') \tag{20}$$

$$\Phi'' \ : \ (C \boxtimes B) \oplus (A \boxtimes C') \xrightarrow{\sim} D^{(1)}\big((C \boxtimes B) \oplus (A \boxtimes C')\big)$$

is isomorphic to $\mathrm{Cone}(a \star b)$. To check this, using the definition of the symmetric cone 2.3, we have to find a distinguished triangle over $a \star b$, which we indeed already have in (17), and we then have to establish the analogue of equation (4), namely:

$$\Phi'' \circ m_1 = -D^{(1)}(m_2)$$

or equivalently, since $\mu_{A,B} : DA \boxtimes DB \xrightarrow{\sim} D(A \boxtimes B)$ is an isomorphism:

$$\Phi'' \circ m_1 \circ \mu = -D^{(1)}(m_2) \circ \mu \, . \tag{21}$$

To show this, first observe that

$$\Phi \star b \overset{\text{def.}}{=} \mu \circ (\Phi \boxtimes\, b) = \mu \circ (\Phi \boxtimes \mathrm{id}_{DB}) \circ (\mathrm{id}_C \boxtimes b) \overset{(11)}{=} \mu \circ (\Phi \boxtimes \mathrm{id}_{DB}) \circ \bar{b} \, . \tag{22}$$

Similarly, using (10), we get that

$$a \star \Phi' = \mu \circ (\mathrm{id}_{DA} \boxtimes \Phi') \circ \bar{a} \, . \tag{23}$$

Hence, the left-hand side of (21) becomes in matrix notation:

$$\begin{aligned}
\Phi'' \circ m_1 \circ \mu &\overset{(20)}{=} (\delta_{\mathcal{K}}\, \delta_{\mathcal{M}} \cdot \Phi \star b \perp \delta_{\mathcal{L}}\, \delta_{\mathcal{M}} \cdot a \star \Phi') \circ m_1 \circ \mu \\
&\overset{(18)}{=} \begin{pmatrix} \delta_{\mathcal{K}}\, \delta_{\mathcal{M}} \cdot \Phi \star b & 0 \\ 0 & \delta_{\mathcal{L}}\, \delta_{\mathcal{M}} \cdot a \star \Phi' \end{pmatrix} \cdot \begin{pmatrix} \bar{b}^{-1} \circ (a_1 \boxtimes \mathrm{id}_{DB}) \\ \bar{a}^{-1} \circ (\mathrm{id}_{DA} \boxtimes b_1) \end{pmatrix} \\
&\overset{(22),(23)}{=} \begin{pmatrix} \delta_{\mathcal{K}}\, \delta_{\mathcal{M}} \cdot \mu \circ (\Phi \boxtimes \mathrm{id}_{DB}) \circ (a_1 \boxtimes \mathrm{id}_{DB}) \\ \delta_{\mathcal{L}}\, \delta_{\mathcal{M}} \cdot \mu \circ (\mathrm{id}_{DA} \boxtimes \Phi') \circ (\mathrm{id}_{DA} \boxtimes b_1) \end{pmatrix} \\
&\overset{(8)}{=} - \begin{pmatrix} \delta_{\mathcal{K}}\, \delta_{\mathcal{M}} \cdot \mu \circ (D^{(1)}(a_2) \boxtimes \mathrm{id}_{DB}) \\ \delta_{\mathcal{L}}\, \delta_{\mathcal{M}} \cdot \mu \circ (\mathrm{id}_{DA} \boxtimes D^{(1)}(b_2)) \end{pmatrix} \, .
\end{aligned}$$

We are almost done except that we need to move the natural isomorphism μ around. At this stage, it is necessary to add decorations $\mu^{(i,j)}$ to specify the considered natural isomorphism between $D^{(i)}(-) \boxtimes D^{(j)}(-)$ and $D^{(i+j)}(- \boxtimes -)$, induced by $\mu^{(0,0)} = \mu$ as defined in A.6. Otherwise, it is impossible to understand the appearance of the signs. We have indeed established:

$$\Phi'' \circ m_1 \circ \mu = - \begin{pmatrix} \delta_{\mathcal{K}}\, \delta_{\mathcal{M}} \cdot \mu^{(1,0)} \circ (D^{(1)}(a_2) \boxtimes \mathrm{id}_{DB}) \\ \delta_{\mathcal{L}}\, \delta_{\mathcal{M}} \cdot \mu^{(0,1)} \circ (\mathrm{id}_{DA} \boxtimes D^{(1)}(b_2)) \end{pmatrix} \, . \tag{24}$$

Using naturality of the transformation $\mu^{(i,j)}$ our left-hand side of (21) becomes:

$$
\begin{aligned}
\Phi'' \circ m_1 \circ \mu &= -\begin{pmatrix} \delta_{\mathcal{K}}\,\delta_{\mathcal{M}}\cdot D^{(1)}(a_2\boxtimes \mathrm{id}_B)\circ \mu^{(1,0)}_{TA,B} \\ \delta_{\mathcal{L}}\,\delta_{\mathcal{M}}\cdot D^{(1)}(\mathrm{id}_A\boxtimes b_2)\circ \mu^{(0,1)}_{A,TB}\end{pmatrix} \\
&\overset{\mathrm{A.6}}{=} -\begin{pmatrix} D^{(1)}(a_2\boxtimes \mathrm{id}_B)\circ \delta_{\mathcal{K}}\,\delta_{\mathcal{M}}\cdot T(\mu_{TA,B})\circ \lambda_{DTA,DB} \\ D^{(1)}(\mathrm{id}_A\boxtimes b_2)\circ \delta_{\mathcal{L}}\,\delta_{\mathcal{M}}\cdot T(\mu_{A,TB})\circ \rho_{DA,DTB}\end{pmatrix} \\
&\overset{\mathrm{(PD2)\ in\ A.4}}{=} -\begin{pmatrix} D^{(1)}(a_2\boxtimes \mathrm{id}_B)\circ D^{(1)}(\lambda_{A,B})\circ \mu_{A,B} \\ D^{(1)}(\mathrm{id}_A\boxtimes b_2)\circ D^{(1)}(\rho_{A,B})\circ \mu_{A,B}\end{pmatrix} \\
&= -\begin{pmatrix} D^{(1)}\big(\lambda_{A,B}\circ (a_2\boxtimes \mathrm{id}_B)\big) \\ D^{(1)}\big(\rho_{A,B}\circ (\mathrm{id}_A\boxtimes b_2)\big)\end{pmatrix}\circ \mu_{A,B} \overset{(19)}{=} -D^{(1)}(m_2)\circ \mu_{A,B}\,.
\end{aligned}
$$

This establishes the wanted equation (21) and finishes the proof. □

Corollary 5.3. *Suppose that α and β have no common degeneracy (5.1). Then the two symmetric spaces $\mathrm{d}(\alpha)\star\beta$ and $\alpha\star\mathrm{d}(\beta)$ define, up to a sign, the same Witt class:*

$$[\,\mathrm{d}(\alpha)\star\beta\,] = -\delta_{\mathcal{K}}\,\delta_{\mathcal{L}}\cdot[\,\alpha\star\mathrm{d}(\beta)\,]$$

in the suitable Witt group of $\mathcal{M}$. □

Remark 5.4. Of course, if we choose β to be non-degenerate in Theorem 5.2, the formula simply says

$$\mathrm{d}(\alpha\star\beta) = \pm\,\mathrm{d}(\alpha)\star\beta\,.$$

This also proves that $\mathrm{d}(\alpha)\star\beta$ is metabolic for any non-degenerate symmetric form β, *i.e.* that $\star$ induces a well-defined product $\star$ on Witt groups, as already established in [10]. The linearity of the connecting homomorphism in the localization long exact sequences follows from this same equation. The verification of the details is left to the reader, simplifying slightly [10, § 2.3].

6. Pseudo-diagonal forms

Notation 6.1. In this section, we move towards geometric applications. So, we fix a $\mathbb{Z}[\frac{1}{2}]$-scheme X and, as explained in Examples 1.3 (a), 1.7 and 2.8, we consider the bounded derived category $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$ of vector bundles over X as a TPD-category (A.5) with the usual product $\otimes_{\mathcal{O}_X}$. Here, support, degeneracy locus and consanguinity are closed subsets of the underlying topological space of X. We shall consider dualities twisted by various line bundles $\mathcal{L}\in\mathrm{Pic}(X)$. Recall that the whole theory is 2-periodic in the twists, exactly as for classical dualities, and 4-periodic in the shifts, see Rem. 2.7. So, if we abbreviate the monoid of symmetric pairs (Def. 2.2) for the triangulated category with $\mathcal{L}$-twisted duality $(\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X), D_{\mathcal{L}})$ by

$$\mathrm{Symm}^{(i)}(X,\mathcal{L}) := \mathrm{Symm}^{(i)}\left(\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)\,,\,D_{\mathcal{L}}\right),$$

we obtain a bi-graded monoid:

$$\mathrm{Symm}^{\mathrm{Tot}}(X) := \bigoplus_{i\in\mathbb{Z}/4}\ \bigoplus_{\mathcal{L}\in\mathrm{Pic}(X)/2}\ \mathrm{Symm}^{(i)}(X,\mathcal{L})$$

and the product of symmetric pairs of Def. 4.1 defines a product:

$$
\begin{array}{rccc}
\star\,: & \mathrm{Symm}^{(i)}(X,\mathcal{L}_1)\times\mathrm{Symm}^{(j)}(X,\mathcal{L}_2) & \longrightarrow & \mathrm{Symm}^{(i+j)}(\,X\,,\,\mathcal{L}_1\otimes\mathcal{L}_2) \\
& (\ \alpha\ ,\ \beta\ ) & \longmapsto & \alpha\star\beta\,.
\end{array}
$$

Recall from Def. 4.8 that the product of several symmetric pairs is $\alpha_1 \star \ldots \star \alpha_n = (\alpha_1 \star \ldots \star \alpha_{n-1}) \star \alpha_n$, with shifts and twists adding up as above.

Remark 6.2. It might be useful to think of $\mathrm{Symm}^{\mathrm{Tot}}(X)$, equipped with orthogonal sum and product, as a *graded semiring*, *i.e.* almost a graded ring but no opposite for the addition. Note that, although it comes equipped with a differential (the symmetric cone d of 2.5), $\mathrm{Symm}^{\mathrm{Tot}}(X)$ is not a *differential* graded semiring since the Leibniz rule holds only conditionally by Thm. 5.2.

We do not formalize commutativity and associativity of $\otimes$ in the abstract triangular framework, but they hold in this geometric situation for $\otimes_{\mathcal{O}_X}$. Note however that signs might be involved in the associativity of the product, depending on conventions. We renounce these rather arid considerations here, since we can moreover circumvent them in applications, see Rem.7.8. So, we leave the proof of the following rather obvious statement to the careful reader. For $\langle 1 \rangle$, recall that it is the symmetric pair $(\mathcal{O}_X, 1)$ which is clearly a unit for $\star$.

Lemma 6.3. *Up to signs (see 2.6), the product on* $\mathrm{Symm}^{\mathrm{Tot}}(X)$ *is commutative and associative. Moreover, it admits a unit* $\langle 1 \rangle \in \mathrm{Symm}^{(0)}(X, \mathcal{O}_X)$. □

We generalize Example 3.5 as follows:

Definition 6.4. Let $\mathcal{L}$ be a line bundle and let $s : \mathcal{O}_X \to \mathcal{L}$ be a global section.

(a) We denote by $Z(s) \subset X$ the *zero locus* of s, that is, the smallest closed subset of X outside of which s is an isomorphism:
$$s\big|_{X \smallsetminus Z(s)} \ : \ \mathcal{O}_{X \smallsetminus Z(s)} \xrightarrow{\sim} \mathcal{L}\big|_{X \smallsetminus Z(s)} .$$

(b) We denote by $\alpha(s\,;\mathcal{L}) := (\mathcal{O}_X\ ,\ s)\ \in \mathrm{Symm}^{(0)}(X, \mathcal{L})$ the symmetric pair
$$\alpha(s;\mathcal{L}) = \left(\begin{array}{ccccccccc} \cdots 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 & \longrightarrow & 0 \cdots \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle s} & & \downarrow & & \downarrow \\ \cdots 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{L} & \longrightarrow & 0 & \longrightarrow & 0 \cdots \end{array} \right)$$
formed by the object $\mathcal{O}_X$, considered in $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$ as a complex concentrated in degree 0, and by the form $s\ :\ \mathcal{O}_X \to D_{\mathcal{L}}(\mathcal{O}_X) = \mathcal{L}$. For simplicity, we might write $\alpha(s)$ instead of $\alpha(s;\mathcal{L})$.

(c) We call *diagonal symmetric pair* any (orthogonal) sum of symmetric pairs as above $\alpha(s_1;\mathcal{L}_1) + \ldots + \alpha(s_n;\mathcal{L}_n)\ \in \mathrm{Symm}^{\mathrm{Tot}}(X)$.

Proposition 6.5. *With the above notation, the symmetric cone of* $\alpha(s\,;\mathcal{L})$ *is*
$$\mathrm{d}\big(\alpha(s\,;\mathcal{L})\big) = \left[\begin{array}{ccccccccc} \cdots 0 & \longrightarrow & \mathcal{O}_X & \xrightarrow{-s} & \mathcal{L} & \longrightarrow & 0 & \longrightarrow & 0 \cdots \\ \downarrow & & \downarrow{\scriptstyle -1} & & \downarrow{\scriptstyle +1} & & \downarrow & & \downarrow \\ \cdots 0 & \longrightarrow & \mathcal{O}_X & \xrightarrow[s]{} & \mathcal{L} & \longrightarrow & 0 & \longrightarrow & 0 \cdots \end{array} \right] \in \mathrm{Symm}^{(1)}(X, \mathcal{L})$$
with $\mathcal{L}$ *in degree* 0. *(The object is the complex in the first row, its* $D_{\mathcal{L}}$*-dual the object in the second row and the symmetric 1-form the vertical morphism of complexes.) Therefore*
$$\begin{array}{ll} \mathrm{Supp}\,\big(\alpha(s)\big) = X & \qquad \mathrm{DegLoc}\,\big(\alpha(s)\big) = Z(s) \\ \mathrm{Supp}\,\big(\,\mathrm{d}\,\alpha(s)\big) = Z(s) & \qquad \mathrm{DegLoc}\,\big(\,\mathrm{d}\,\alpha(s)\big) = \varnothing\,. \end{array}$$

Proof. By definition, the cone C of the morphism s is the complex depicted in the first row, see [17, § 1.5]. One has to check that the vertical morphism defines the form $\Phi : C \to TD_{\mathcal{L}}(C)$ of formula (4) in 2.3. This is easy since the morphism a_1 is $\mathrm{id}_{\mathcal{L}}$ in degree zero and 0 elsewhere, whereas a_2 is $-\mathrm{id}_{\mathcal{O}_X}$ in degree 1 and zero elsewhere, with the same convention [17, 1.5.2].

Therefore: $\mathrm{Supp}\big(\alpha(s;\mathcal{L})\big) \overset{3.1}{=} \mathrm{supp}(\mathcal{O}_X) = X$. The next two equalities come from $\mathrm{DegLoc}\big(\alpha(s;\mathcal{L})\big) \overset{(5)}{=} \mathrm{Supp}\big(\mathrm{d}\,\alpha(s;\mathcal{L})\big) \overset{3.1}{=} \mathrm{supp}(\mathrm{cone}(s)) \overset{6.4(a)}{=} Z(s)$. Finally, $\mathrm{d}(\alpha)$ is non-degenerate for any symmetric pair α. $\square$

Definition 6.6. We denote by

$$\mathrm{PDiag}^{\mathrm{Tot}}(X) = \bigoplus_{i\in\mathbb{Z}/4}\ \bigoplus_{\mathcal{L}\in\mathrm{Pic}(X)/2} \mathrm{PDiag}^{(i)}(X,\mathcal{L})$$

the graded sub-semiring of $\mathrm{Symm}^{\mathrm{Tot}}(X)$ generated by all symmetric pairs $\alpha(s;\mathcal{L})$ for all global sections $s : \mathcal{O}_X \to \mathcal{L}$ of line bundles and by all their cones $\mathrm{d}(\alpha(s;\mathcal{L}))$. We call it the total sub-semiring of *pseudo-diagonal symmetric pairs*.

Here are examples:

Notation 6.7. Let $n \geq 0$. Consider $n+1$ line bundles $\mathcal{L}_0,\ldots,\mathcal{L}_n \in \mathrm{Pic}(X)$ and consider global sections $s_j : \mathcal{O}_X \to \mathcal{L}_j$ for $j = 0,\ldots,n$. Consider the following pseudo-diagonal pair involving one symmetric pair and n symmetric cones:

$$\beta(s_0,\ldots,s_n\,;\,\mathcal{L}_0,\ldots,\mathcal{L}_n) := \alpha(s_0;\mathcal{L}_0) \star \mathrm{d}\,\alpha(s_1;\mathcal{L}_1) \star \ldots \star \mathrm{d}\,\alpha(s_n;\mathcal{L}_n)\,.$$

This symmetric pair, sometimes only written $\beta(s_0,\ldots,s_n)$, defines an element of $\mathrm{PDiag}^{(n)}(X,\mathcal{L}) \subset \mathrm{Symm}^{(n)}(X,\mathcal{L})$ where $\mathcal{L} := \mathcal{L}_0 \otimes \ldots \otimes \mathcal{L}_n \in \mathrm{Pic}(X)/2$ since all factors contribute to a twist by $\mathcal{L}_i$ but only the last n factors contribute to a shift by 1. For $n = 0$, we simply have $\beta(s_0) = \alpha(s_0)$. For instance, $\beta(1;\mathcal{O}_X) = \langle 1\rangle$.

Remark 6.8. The support $\mathrm{Supp}\big(\beta(s_0,\ldots,s_n\,;\,\mathcal{L}_0,\ldots,\mathcal{L}_n)\big)$ of such a form is equal to $\mathrm{Supp}(\alpha(s_0;\mathcal{L}_0)) \cap \bigcap_{i=1}^n \mathrm{Supp}(\mathrm{d}\,\alpha(s_i;\mathcal{L}_i)) = \cap_{i=1}^n Z(s_i)$ by Prop. 6.5. We shall consider its degeneracy locus in Prop. 6.11.

Proposition 6.9. *Let $i \in \mathbb{Z}/4$ and $\mathcal{L} \in \mathrm{Pic}(X)/2$. Then, any element of the monoid of pseudo-diagonal $\mathcal{L}$-twisted symmetric i-pairs $\mathrm{PDiag}^{(i)}(X,\mathcal{L})$, as defined in 6.6, is an orthogonal sum of symmetric pairs $\beta(s_0,\ldots,s_n\,;\,\mathcal{L}_0,\ldots,\mathcal{L}_n)$ as defined in 6.7, for various integers $n \equiv i$ modulo 4 and for various families of n line bundles $\mathcal{L}_0,\ldots,\mathcal{L}_n \in \mathrm{Pic}(X)$ such that $\mathcal{L}_0 \otimes \ldots \otimes \mathcal{L}_n \equiv \mathcal{L}$ modulo $2\,\mathrm{Pic}(X)$.*

Proof. Observe the following two properties:

(a) $-\alpha(s_0;\mathcal{L}_0) = \alpha(-s_0;\mathcal{L}_0)$ by definition of $-\alpha$, see 2.6;

(b) $\alpha(s_1;\mathcal{L}_1) \star \alpha(s_2;\mathcal{L}_2) = \alpha(s_1 \otimes s_2;\mathcal{L}_1 \otimes \mathcal{L}_2)$ by definition of the product.

Now, by construction, $\mathrm{PDiag}^{(i)}(X,\mathcal{L})$ is the $(i,\mathcal{L})$-graded part of $\mathrm{PDiag}^{\mathrm{Tot}}(X)$ and the latter is the sub-semiring of $\mathrm{Symm}^{\mathrm{Tot}}(X)$ generated by diagonal symmetric pairs $\alpha(s_j;\mathcal{L}_j)$, see 6.4, and by their cones $\mathrm{d}\,\alpha(s_j;\mathcal{L}_j)$. So, *a priori*, an element of $\mathrm{PDiag}^{(i)}(X,\mathcal{L})$ is a sum of products of such pairs. Using commutativity and associativity up to signs (Lem. 6.3), we can regroup such a product as

$$\pm\,\alpha(s_1;\mathcal{L}_1) \star \ldots \star \alpha(s_m;\mathcal{L}_m) \star \mathrm{d}\,\alpha(s_{m+1};\mathcal{L}_{m+1}) \star \ldots \star \mathrm{d}\,\alpha(s_{m+n};\mathcal{L}_{m+n})\,.$$

Using (a) and (b), we can regroup the m factors with no “d” into only one and we can even incorporate the possible sign into it. Such a product is a symmetric pair β as in the statement. $\square$

Remark 6.10. This says that the symmetric forms $\beta(s_0,\ldots,s_n\,;\,\mathcal{L}_0,\ldots,\mathcal{L}_n)$ of 6.7 essentially describe all pseudo-diagonal forms, in the sense of Def. 6.6. These pseudo-diagonal forms constitute the natural generalization to arbitrary schemes of the usual diagonal forms over fields (where no "d" intervenes). It would be interesting to know the answer to the following two open questions:

Q 1 : When (*i.e.* over which schemes) is any symmetric space pseudo-diagonal ?

Q 2 : When is any symmetric space Witt-equivalent to a pseudo-diagonal one ?

We only know that the weaker Question 2 is stable by passing from X to $\mathbb{A}^1_X$ or to $\mathbb{P}^n_X$ when X is regular. We now decide when such a pseudo-diagonal form $\beta(s_0,\ldots,s_n)$ is non-degenerate:

Proposition 6.11. *We have* $\mathrm{DegLoc}\big(\beta(s_0,\ldots,s_n)\big)\subset\bigcap_{i=0}^{n}Z(s_i)$.

Proof. By Prop. 6.5, $\mathrm{Supp}(\alpha(s_0))\cap\mathrm{Supp}(\mathrm{d}\,\alpha(s_1))\cap\ldots\cap\mathrm{Supp}(\mathrm{d}\,\alpha(s_n))=Z(s_1)\cap\ldots\cap Z(s_n)$ whereas $\mathrm{DegLoc}(\alpha(s_0))\cup\mathrm{DegLoc}(\mathrm{d}\,\alpha(s_1))\cup\ldots\cup\mathrm{DegLoc}(\mathrm{d}\,\alpha(s_n))=Z(s_0)$. So, by Def. 4.9, we obtain $\mathrm{Cons}(\alpha(s_0),\mathrm{d}\,\alpha(s_1),\ldots,\mathrm{d}\,\alpha(s_n))=\cap_{i=0}^{n}Z(s_i)$. Since $\beta(s_0,\ldots,s_n)$ is the product $\alpha(s_0)\star\mathrm{d}\,\alpha(s_1)\star\ldots\star\mathrm{d}\,\alpha(s_n)$ by definition. The result now follows from Prop. 4.11. □

Remark 6.12. Continuing Rems. 4.5 and 4.12, observe that equality holds in this statement. However, the above inclusion suffices to apply Prop. 3.4 and to obtain:

Corollary 6.13. *For each $n\geq 0$, for each collection of $n{+}1$ line bundles $\mathcal{L}_0,\ldots,\mathcal{L}_n$ and for each family of global sections $s_i\in\Gamma(X,\mathcal{L}_i)$ such that $\bigcap_{i=0}^{n}Z(s_i)=\varnothing$, the pseudo-diagonal symmetric pair*

$$\beta(s_0,\ldots,s_n)=\alpha(s_0)\,\star\,\mathrm{d}\,\alpha(s_1)\star\ldots\star\mathrm{d}\,\alpha(s_n)$$

is non-degenerate and hence defines a class in the Witt group $\mathrm{W}^n(X,\mathcal{L})$ where $\mathcal{L}=\mathcal{L}_0\otimes\ldots\otimes\mathcal{L}_n$. □

Remark 6.14. Observe that Corollary 6.13 allows us to determine all non-degenerate pseudo-diagonal forms $\mathrm{MW}^i(X,\mathcal{L})\cap\mathrm{PDiag}^{(i)}(X,\mathcal{L})$ by means of Prop. 6.9. They will be sums of pseudo-diagonal spaces as in the corollary.

Remark 6.15. In Corollary 6.13, we do not say that $\alpha(s_0)\star\ldots\star\alpha(s_n)$ is non-degenerate since this is completely wrong. Indeed, the consanguinity of the forms $\alpha(s_0),\ldots,\alpha(s_n)$ is the *union* of the $Z(s_i)$ for $i=0,\ldots,n$ and so this product is non-degenerate only if every s_i is an isomorphism. We do not either consider $\mathrm{d}\,\alpha(s_0)\star\ldots\star\mathrm{d}\,\alpha(s_n)$, which is always non-degenerate, without assumption on the $Z(s_i)$, but is also always metabolic and hence of little interest for Witt groups, although they may define useful non-zero classes in Witt groups with support.

Let us draw the attention of the hurried reader to the asymmetry of the definition of $\beta(s_0,\ldots,s_n)$ in $s_0,\ldots,s_n$. The choice of having no d only in front of $\alpha(s_0)$ is not so important up to Witt equivalence though. Indeed, suppose that $\cap_{i=0}^{n}Z(s_i)=\varnothing$, then it follows from commutativity and associativity of $\star$ up to signs (Lem. 6.3) and from the Leibniz formula (Cor. 5.3) that, up to signs again, the Witt class of the symmetric space $\beta(s_0,\ldots,s_n)$ in $\mathrm{W}^n(X,\mathcal{L})$ does not depend on the order of the s_i, that is, for every permutation σ of $\{0,\ldots,n\}$, we have

$$[\beta(s_0,\ldots,s_n\,;\,\mathcal{L}_0,\ldots,\mathcal{L}_n)]=\pm\,[\beta(s_{\sigma(0)},\ldots,s_{\sigma(n)}\,;\,\mathcal{L}_{\sigma(0)},\ldots,\mathcal{L}_{\sigma(n)})]\,.$$

Remark 6.16. It is clear that our graded semirings $\mathrm{Symm}^{\mathrm{Tot}}(X)$, $\mathrm{PDiag}^{\mathrm{Tot}}(X)$, and so on, are functorial in X, in a contravariant way. Both definitions of $\alpha(s;\mathcal{L})$ and of $\beta(s_0,\ldots,s_n\,;\,\mathcal{L}_0,\ldots,\mathcal{L}_n)$ are natural in the obvious sense.

Proposition 6.17. *For $n\geq 1$, under the condition that $\cap_{i=0}^{n}Z(s_i)=\varnothing$, the symmetric space $\beta(s_0,\ldots,s_n\,;\,\mathcal{L}_0,\ldots,\mathcal{L}_n)$ is locally trivial in the Witt group, namely each point of X has a neighborhood on which this space is metabolic.*

Proof. We have by assumption a covering of X by the complements of the $Z(s_i)$. On the first open $U_0:=X\smallsetminus Z(s_0)$ the space becomes a product of a *space* $\alpha(s_0;\mathcal{L}_0)\big|_{U_0}$ with the metabolic space $\mathrm{d}(\alpha(s_1;\mathcal{L}_1))\star\ldots\star\mathrm{d}(\alpha(s_n;\mathcal{L}_n))\big|_{U_0}$ and the product is therefore metabolic since $n\geq 1$. On the other open subsets $X\smallsetminus Z(s_i)$ for $i=1,\ldots,n$ the object supporting the space is indeed zero since one of the factors is zero $\mathrm{d}\,\alpha(s_i;\mathcal{L}_i)\big|_{X\smallsetminus Z(s_i)}=0$. Hence the result. ☐

7. Explicit examples over projective spaces

In this section, X is a scheme. Recall Conventions 0.2 and 0.3.

Notation 7.1. Recall that $\mathbb{P}^n_X=\mathbb{P}^n_{\mathbb{Z}}\times_{\mathrm{Spec}(\mathbb{Z})}X$ where $\mathbb{P}^n_{\mathbb{Z}}=\mathrm{Proj}(\mathbb{Z}[T_0,\ldots,T_n])$. For each $i=0,\ldots,n$, we also denote by T_i the corresponding global section of $\mathcal{O}(1)$ over $\mathbb{P}^n_{\mathbb{Z}}$ and over $\mathbb{P}^n_X$ as well. We denote by

$$Z_i:=Z(T_i)\ \subset\ \mathbb{P}^n_X\qquad\text{and}\qquad U_i:=\mathbb{P}^n_X\smallsetminus Z_i\ \subset\ \mathbb{P}^n_X$$

the closed subscheme $Z_i\simeq\mathbb{P}^{n-1}_X$ corresponding to "$T_i=0$" and its open complement $U_i\simeq\mathbb{A}^n_X$. We shall also consider the closed subset

$$Y:=Z_1\cap\ldots\cap Z_n=\{T_1=0,\ldots,T_n=0\}\ \subset\ U_0\ \subset\ \mathbb{P}^n_X$$

corresponding to the point $[1:0:\ldots:0]$ of $\mathbb{P}^n$ and its open complement

$$V:=\mathbb{P}^n_X\smallsetminus Y\ \subset\ \mathbb{P}^n_X\,.$$

For simplicity we denote by the same letter π all projection morphisms to X :

$$\begin{tikzcd}
\mathbb{A}^n_X \arrow[drr, "\pi"'] & U_i \arrow[l, "\simeq"'] \arrow[r, hook] \arrow[dr, "\pi"] & \mathbb{P}^n_X \arrow[d, "\pi"] & V \arrow[l, hook'] \arrow[dl, "\pi"] \\
& & X &
\end{tikzcd}$$

and even $\pi:\mathbb{P}^{n-1}_X\to X$. It is always clear from the context which projection is meant. For $n\geq 2$, we have a morphism over X

$$\begin{array}{rccc}\eta: & V & \longrightarrow & \mathbb{P}^{n-1}_X\\ & [t_0:\ldots:t_n] & \longmapsto & [t_1:\ldots:t_n]\end{array}\tag{25}$$

which is obtained by base change to X from the integral morphism described in the second line. That is, η can be defined for $X=\mathrm{Spec}(\mathbb{Z})$ by the above formula in homogenous coordinates and then pull-backed to any scheme X. For $n=1$ we make the convention that $\eta:V\to\mathbb{P}^0_X=X$ is the structure morphism π.

Remark 7.2. We adopt the following notation to drop unnecessary mentions of π^*.

(a) For $\mathcal{M}\in\mathrm{Pic}(X)$, we simply write $\mathrm{W}^i(\mathbb{P}^n_X,\mathcal{M})$ to mean $\mathrm{W}^i\left(\mathbb{P}^n_X\,,\,\pi^*\mathcal{M}\right)$.

(b) For any class $w \in \mathrm{W}^{\mathrm{Tot}}(\mathbb{P}^n_X)$, the homomorphism $\mathrm{W}^{\mathrm{Tot}}(X) \longrightarrow \mathrm{W}^{\mathrm{Tot}}(\mathbb{P}^n_X)$ consisting in π^* followed by multiplication by w will simply be denoted by $\cdot w$ and will be called multiplication by w (say, on the right).

Definition 7.3. We apply the constructions of the previous sections:

(a) For any $i = 0, \ldots, n$, following Def. 6.4, we define the symmetric pair

$$\alpha_i := \alpha(\,T_i\,;\,\mathcal{O}(1)\,) \quad \in \ \mathrm{Symm}^{(0)}(\mathbb{P}^n_X, \mathcal{O}(1))\,.$$

(b) In the notation of 6.7, we define the symmetric pair

$$\begin{aligned}\beta^{(n)}_X : &= \beta(T_0, \ldots, T_n\,;\,\mathcal{O}(1), \ldots, \mathcal{O}(1))\\ &= \alpha_0 \star \mathrm{d}\,\alpha_1 \star \ldots \star \mathrm{d}\,\alpha_n \qquad \in \quad \mathrm{Symm}^{(n)}(\mathbb{P}^n_X, \mathcal{O}(n+1))\,.\end{aligned}$$

Observing that $\cap_{i=0}^n Z_i = \varnothing$ we know from Cor. 6.13 that the above $\beta^{(n)}_X$ is non-degenerate and therefore defines a Witt class

$$[\beta^{(n)}_X] \in \mathrm{W}^n(\mathbb{P}^n_X\,,\,\mathcal{O}(n+1))\,.$$

(c) Using the short notation of 7.2 (b), we define a homomorphism:

$$\begin{array}{rccl}\big(1 \quad [\beta^{(n)}_X]\big): & \mathrm{W}^{\mathrm{Tot}}(X) \oplus \mathrm{W}^{\mathrm{Tot}}(X) & \longrightarrow & \mathrm{W}^{\mathrm{Tot}}(\mathbb{P}^n_X)\\ & (\ \phi\ ,\ \psi\ ) & \longmapsto & \pi^*(\phi)\ +\ \pi^*(\psi) \star [\beta^{(n)}_X]\,.\end{array}$$

Theorem 7.4. *Let X be a regular scheme and $n \geq 1$. The above homomorphism*

$$\big(1 \quad [\beta^{(n)}_X]\big): \quad \mathrm{W}^{\mathrm{Tot}}(X) \oplus \mathrm{W}^{\mathrm{Tot}}(X) \ \longrightarrow\ \mathrm{W}^{\mathrm{Tot}}(\mathbb{P}^n_X)$$

is an isomorphism. The ring structure is determined by the property that

$$[\beta^{(n)}_X] \star [\beta^{(n)}_X] = 0$$

in $\mathrm{W}^{2n}(\mathbb{P}^n_X)$ and by the fact that $\pi^* : \mathrm{W}^{\mathrm{Tot}}(X) \to \mathrm{W}^{\mathrm{Tot}}(\mathbb{P}^n_X)$ is a homomorphism.

Proof. We proceed by induction on $n \geq 1$. For any line bundle $\mathcal{L} \in \mathrm{Pic}(\mathbb{P}^n_X)$, we have a localization long exact sequence, see [2, Thm. 6.2 & 6.8] or [4, Thm. 1.6]:

$$(26) \quad \cdots \longrightarrow \mathrm{W}^i_Y(\mathbb{P}^n_X, \mathcal{L}) \longrightarrow \mathrm{W}^i(\mathbb{P}^n_X, \mathcal{L}) \longrightarrow \mathrm{W}^i(V, \mathcal{L}\big|_V) \xrightarrow{\partial_{\mathcal{L}}} \mathrm{W}^{i+1}_Y(\mathbb{P}^n_X, \mathcal{L}) \longrightarrow \cdots$$

where the connecting homomorphism $\partial_{\mathcal{L}} : \mathrm{W}^i(V, \mathcal{L}\big|_V) \longrightarrow \mathrm{W}^{i+1}_Y(\mathbb{P}^n_X, \mathcal{L})$ is induced by the cone construction, see [2, § 5]. Before proceeding to a term-by-term analysis of (26), we recall, for those readers who might fear the loss of some twists in the sequel, that we have an isomorphism $\mathrm{Pic}(X) \oplus \mathbb{Z} \xrightarrow{\sim} \mathrm{Pic}(\mathbb{P}^n_X)$ given by $(\mathcal{M}, m) \mapsto \pi^*(\mathcal{M})(m) = \pi^*(\mathcal{M}) \otimes \mathcal{O}(m)$. Also observe that the global section $T_i : \mathcal{O}_{\mathbb{P}^n_X} \to \mathcal{O}_{\mathbb{P}^n_X}(1)$ is an isomorphism outside $Z_i = Z(T_i)$, that is, on $U_i \simeq \mathbb{A}^n_X$. So, we have the following situation for Picard groups (written as abelian groups):

$$(27) \quad \begin{array}{ccccc}\mathrm{Pic}(X)/2 & \xleftarrow{\ (\mathrm{id}\ \ 0)\ } & \mathrm{Pic}(X)/2\ \oplus\ \mathbb{Z}/2 & \longrightarrow & \mathrm{Pic}(\mathbb{P}^{n-1}_X)/2\\ {\scriptstyle \pi^*}\big\downarrow{\scriptstyle \simeq} & & {\scriptstyle (\pi^*\ \ \mathcal{O}(1))}\big\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\big\downarrow{\scriptstyle \eta^*}\\ \mathrm{Pic}(U_i)/2 & \xleftarrow[\mathrm{res}_{U_i}]{} & \mathrm{Pic}(\mathbb{P}^n_X)/2 & \xrightarrow[\mathrm{res}_V]{} & \mathrm{Pic}(V)/2\,.\end{array}$$

Note that, for $n = 1$, the right-hand groups are isomorphic to $\mathrm{Pic}(X)/2$ whereas for $n \geq 2$, we have $\mathrm{codim}_{\mathbb{P}^n_X}(\mathbb{P}^n_X \smallsetminus V) = \mathrm{codim}_{\mathbb{P}^n_X}(Y) = n \geq 2$ so the restriction $\mathrm{res}_V : \mathrm{Pic}(\mathbb{P}^n_X) \to \mathrm{Pic}(V)$ is an isomorphism. Therefore, when $n \geq 2$, all morphisms in the right-hand square of (27) are isomorphisms.

Lemma 7.5. *Consider the metabolic symmetric space* $\gamma := \mathrm{d}\,\alpha_1 \star \ldots \star \mathrm{d}\,\alpha_n$ *in* $\mathrm{Symm}^{(n)}(\mathbb{P}^n_X, \mathcal{O}(n))$. *Let* $\mathcal{M} \in \mathrm{Pic}(X)/2$. *Then, we have two isomorphisms:*

$$\mathrm{W}^{i-n}(X, \mathcal{M}) \xrightarrow[\simeq]{\cdot[\gamma]} \mathrm{W}^i_Y\left(\mathbb{P}^n_X, \mathcal{M}(n)\right)$$

and

$$\mathrm{W}^{i-n}(X, \mathcal{M}) \xrightarrow[\simeq]{\cdot[\beta^{(n)}_X]} \mathrm{W}^i_Y\left(\mathbb{P}^n_X, \mathcal{M}(n+1)\right)$$

given by multiplication by the classes of γ *and of* $\beta^{(n)}_X$, *respectively.*

Proof. First observe that $\mathrm{Supp}(\gamma) = \cap_{i=1}^n Z_i = Y$ and that therefore the first homomorphism is well-defined. Similarly we have $\mathrm{Supp}(\beta^{(n)}_X) = Y$, see Rem. 6.8. Indeed, in the notation of 6.7, we have $\gamma = \beta(1, T_1, \ldots, T_n\,;\, \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(1))$.

On $\mathbb{A}^n_X := \mathrm{Spec}(\mathbb{Z}[T'_1, \ldots, T'_n]) \times X$, consider the Koszul symmetric space $\kappa := \mathrm{d}\,\alpha'_1 \star \ldots \star \mathrm{d}\,\alpha'_n$, where $\alpha'_i := \alpha(T_i, \mathcal{O})$ following the notation of 6.4. Gille has proved in [9, Thm. 9.3] that

$$\mathrm{W}^{i-n}(X, \mathcal{M}) \xrightarrow{\cdot[\kappa]} \mathrm{W}_{Y'}(\mathbb{A}^n_X, \mathcal{M})$$

is an isomorphism where $Y' = \{T'_1 = 0, \ldots, T'_n = 0\}$, at least in the case of X affine and regular and of $\mathcal{M} = \mathcal{O}_X$. The global case is an immediate corollary of Gille's result by applying Mayer-Vietoris on the base X. Consider the morphisms:

$$\mathrm{W}^{i-n}(X, \mathcal{M}) \xrightarrow[\simeq]{\cdot[\kappa]} \mathrm{W}^i_{Y'}(\mathbb{A}^n_X, \mathcal{M}) \xrightarrow[\simeq]{} \mathrm{W}^i_Y(U_0, \mathcal{M}) \xleftarrow[\simeq]{\mathrm{res}_{U_0}} \mathrm{W}^i_Y(\mathbb{P}^n_X, \mathcal{M}(m))\,.$$

The first one is an isomorphism by the above result of Gille. The second one is an isomorphism since $U_0 \simeq \mathbb{A}^n_X$. The last isomorphism follows by Zariski excision (which is only a question of underlying categories, not of dualities), see [4, Cor. 2.3]. Note that $\mathcal{O}(1)\big|_{U_0} \simeq \mathcal{O}_{U_0}$ and so the $m \in \mathbb{Z}/2$ disappears on U_0.

We are left to show that the two following diagrams commute, one for γ (with $m = n$) and one for $\beta^{(n)}_X$ (with $m = n+1$):

$$\begin{tikzcd}
\mathrm{W}^{i-n}(X, \mathcal{M}) \arrow[r, shift left, "\cdot[\gamma]"] \arrow[r, shift right, "\cdot[\beta^{(n)}_X]"'] \arrow[d, "\cdot[\kappa]"', "\simeq"] & \mathrm{W}^n_Y\left(\mathbb{P}^n_X, \mathcal{M}(m)\right) \arrow[d, "\simeq"', "\mathrm{res}_{U_0}"] \\
\mathrm{W}^i_{Y'}(\mathbb{A}^n_X, \mathcal{M}) \arrow[r, "\simeq"'] & \mathrm{W}^i_Y(U_0, \mathcal{M})\,.
\end{tikzcd}$$

To see this, recall that the classical isomorphism $U_0 \xrightarrow{\sim} \mathbb{A}^n_X$ corresponds to $T'_i \mapsto T_i/T_0$ and that we use multiplication with T_0 to identify $\mathcal{O}_{U_0} \simeq \mathcal{O}(1)\big|_{U_0}$. Therefore, this isomorphism $U_0 \xrightarrow{\sim} \mathbb{A}^n_X$, which of course sends Y to Y', also sends the symmetric pair α'_i to α_i for $i = 1, \ldots, n$. Via this isomorphism, we have *a fortiori* $[\kappa] \longmapsto [\gamma\big|_{U_0}] = [\beta^{(n)}_X\big|_{U_0}]$. This last equality follows from $(\alpha_0)\big|_{U_0} = \langle 1 \rangle$. This is the claimed commutativity and the lemma follows. $\square$

Lemma 7.6. *Let* $\mathcal{M} \in \mathrm{Pic}(X)/2$, $m \in \mathbb{Z}/2$. *Suppose that* $n \geq 2$. *Then we have an isomorphism:*

$$\eta^* \;:\; \mathrm{W}^i\left(\mathbb{P}^{n-1}_X, \mathcal{M}(m)\right) \xrightarrow{\sim} \mathrm{W}^i\left(V, \mathcal{M}(m)\right).$$

For $n = 1$, *we simply have an isomorphism* $\eta^* : \mathrm{W}^i(X, \mathcal{M}) \xrightarrow{\sim} \mathrm{W}^i\left(V, \mathcal{M}\right)$.

Proof. The morphism $\eta : V \to \mathbb{P}^{n-1}_X$ has $\mathbb{A}^1$-fibers and the result follows by generalized homotopy invariance, see Gille [8, Cor. 4.2]. $\square$

Lemma 7.7. *Suppose that $n \geq 2$. Recall $\gamma = \mathrm{d}\,\alpha_1 \star \ldots \star \mathrm{d}\,\alpha_n$ from Lemma 7.5. The image of the class $[\beta^{(n-1)}_X] \in \mathrm{W}^{n-1}(\mathbb{P}^{n-1}_X, \mathcal{O}(n))$ via the composition*

$$\mathrm{W}^{n-1}(\mathbb{P}^{n-1}_X, \mathcal{O}(n)) \xrightarrow{\eta^*} \mathrm{W}^{n-1}(V, \mathcal{O}(n)) \xrightarrow{\partial} \mathrm{W}^n_Y(\mathbb{P}^n_X, \mathcal{O}(n))$$

is given by $\partial(\eta^*[\beta^{(n-1)}_X]) = \pm\,[\gamma]$. Here of course $\partial = \partial_{\mathcal{O}(n)}$.

Proof. The definition of the connecting homomorphism ∂ is as follows. To compute $\partial\big([\eta^*(\beta^{(n-1)}_X)]\big)$ we need to find a symmetric pair on $\mathbb{P}^n_X$ whose restriction to V is the symmetric space $\eta^*(\beta^{(n-1)}_X)$ and then apply the symmetric cone construction d to this "lift". See details in [2, 5.16]. In formula, it means that we have

$$\partial([\eta^*(\beta^{(n-1)}_X)]) = [\mathrm{d}(\alpha_1 \star \mathrm{d}\,\alpha_2 \star \ldots \star \mathrm{d}\,\alpha_n)]$$

as soon as observe that $\alpha_1 \star \mathrm{d}\,\alpha_2 \star \ldots \star \mathrm{d}\,\alpha_n$ is a symmetric pair on $\mathbb{P}^n_X$ whose restriction to V is $\eta^*(\beta^{(n-1)}_X)$. The latter is obvious by definition of η, see (25), and by definition of $\beta^{(n-1)}_X$, see Def. 7.3. Therefore we have

$$[\mathrm{d}(\alpha_1 \star \mathrm{d}\,\alpha_2 \star \ldots \star \mathrm{d}\,\alpha_n)] \overset{5.4}{=} \pm\,[\mathrm{d}\,\alpha_1 \star \mathrm{d}\,\alpha_2 \star \ldots \star \mathrm{d}\,\alpha_n] = \pm\,[\gamma]$$

which gives the Lemma. $\square$

End of proof of Theorem 7.4: For $n \geq 2$, consider the diagram:

(28)
$$\begin{tikzcd}
\mathrm{W}^{\mathrm{Tot}}_Y(\mathbb{P}^n_X) \arrow[r] & \mathrm{W}^{\mathrm{Tot}}(\mathbb{P}^n_X) \arrow[r] & \mathrm{W}^{\mathrm{Tot}}(V) \arrow[r, "\partial"] & \mathrm{W}^{\mathrm{Tot}}_Y(\mathbb{P}^n_X) \\
 & & \mathrm{W}^{\mathrm{Tot}}(\mathbb{P}^{n-1}_X) \arrow[u, "\eta^*"', "\simeq"] & \\
\mathrm{W}^{\mathrm{Tot}}(X)^2 \arrow[r, "\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}"'] \arrow[uu, "\big({[\gamma]} \;\; {[\beta^{(n)}_X]}\big)", "\simeq"'] & \mathrm{W}^{\mathrm{Tot}}(X)^2 \arrow[r, "\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}"'] \arrow[uu, "\big(1 \;\; {[\beta^{(n)}_X]}\big)"] & \mathrm{W}^{\mathrm{Tot}}(X)^2 \arrow[r, "\begin{pmatrix} 0 & \pm 1 \\ 0 & 0 \end{pmatrix}"'] \arrow[u, "\big(1 \;\; {[\beta^{(n-1)}_X]}\big)", "\simeq"'] & \mathrm{W}^{\mathrm{Tot}}(X)^2 \arrow[uu, "\big({[\gamma]} \;\; {[\beta^{(n)}_X]}\big)", "\simeq"']
\end{tikzcd}$$

where $\mathrm{W}^{\mathrm{Tot}}(X)^2 = \mathrm{W}^{\mathrm{Tot}}(X) \oplus \mathrm{W}^{\mathrm{Tot}}(X)$ and where we use the notation $\mathrm{W}^{\mathrm{Tot}}_Y(\mathbb{P}^n_X)$ for $\oplus_{i\in\mathbb{Z}/4} \oplus_{\mathcal{L}\in\mathrm{Pic}(\mathbb{P}^n_X)/2} \mathrm{W}^i_Y(\mathbb{P}^n_X, \mathcal{L})$.

Exactness of the first row is a compact form of the localization exact sequence (26). The second row is trivially exact.

The first (hence the last) vertical morphism is an isomorphism by Lemma 7.5. The vertical morphisms in the third column are isomorphism by induction hypothesis and by Lemma 7.6 for η^*.

To see commutativity of the first square it suffices to prove that $[\gamma] = 0$ in $\mathrm{W}^{\mathrm{Tot}}(\mathbb{P}^n_X)$, which is obvious since $\gamma = \mathrm{d}\,\alpha_1 \star \ldots \star \mathrm{d}\,\alpha_n$ is a product of metabolic forms. To see commutativity of the second square, it suffices to see that $\mathrm{res}_V([\beta^{(n)}_X]) = 0$ in $\mathrm{W}^{\mathrm{Tot}}(V)$ which is obvious since $[\beta^{(n)}_X]$ comes from $\mathrm{W}^{\mathrm{Tot}}_Y(\mathbb{P}^n_X)$, that is, $\beta^{(n)}_X$ is supported on $\cap_{i=1}^n Z_i = Y$ as we already checked in Remark 6.8. To see commutativity of the third square it suffices to prove $\partial(\eta^*([\beta^{(n-1)}_X]) = \pm\,[\gamma]$, which is Lemma 7.7.

We get the wanted isomorphism by the 5-Lemma, since diagram (28) is periodic.

For $n = 1$, we have $V = U_1$ and hence α_1 is non-degenerate on V and defines a class in $\mathrm{W}^0(V, \mathcal{O}(1)\big|_V)$. Consider the diagram:

$$
\begin{tikzcd}[column sep=large, row sep=huge]
\mathrm{W}^{\mathrm{Tot}}_Y(\mathbb{P}^1_X) \arrow[r] & \mathrm{W}^{\mathrm{Tot}}(\mathbb{P}^1_X) \arrow[r] & \mathrm{W}^{\mathrm{Tot}'}(V) \arrow[r, "\partial"] & \mathrm{W}^{\mathrm{Tot}}_Y(\mathbb{P}^1_X) \\
\mathrm{W}^{\mathrm{Tot}}(X)^2 \arrow[r, "\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}"'] \arrow[u, "\begin{pmatrix} [\gamma] & [\beta^{(1)}_X] \end{pmatrix}", "\simeq"'] & \mathrm{W}^{\mathrm{Tot}}(X)^2 \arrow[r, "\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}"'] \arrow[u, "\begin{pmatrix} 1 & [\beta^{(1)}_X] \end{pmatrix}"] & \mathrm{W}^{\mathrm{Tot}}(X)^2 \arrow[r, "\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}"'] \arrow[u, "\begin{pmatrix} \eta^* & [\alpha_1] \end{pmatrix}", "\simeq"'] & \mathrm{W}^{\mathrm{Tot}}(X)^2 \arrow[u, "\begin{pmatrix} [\gamma] & [\beta^{(1)}_X] \end{pmatrix}", "\simeq"']
\end{tikzcd}
\tag{29}
$$

where we use $\mathrm{W}^{\mathrm{Tot}'}(V)$ to mean $\oplus_{i\in\mathbb{Z}/4} \oplus_{\mathcal{L}\in \mathrm{Pic}(\mathbb{P}^1_X)/2} \mathrm{W}^i(V, \mathcal{L}\big|_V)$, which is not $\mathrm{W}^{\mathrm{Tot}}(V)$ but rather two copies of it, since $\mathcal{O}(1)\big|_V \cong \mathcal{O}_V$, see also (27). We need $\mathrm{W}^{\mathrm{Tot}'}(V)$ because the first line of (29) is the sum over all shifts and all possible twists over $\mathbb{P}^n_X$ (not over V !) of the localization exact sequence (26). Note also that the connecting homomorphism $\partial_{\mathcal{L}}$ of the localization exact sequence (26) depends on the "ambient" category with duality, here $\big(\mathbb{D}^{\mathrm{b}}(\mathbb{P}^n_X), D_{\mathcal{L}}\big)^{(i)}$. So, although $[\alpha_1] = \langle 1 \rangle$ if we identify $\mathrm{W}^0(V, \mathcal{O}(1)\big|_V)$ with $\mathrm{W}^0(V)$, the connecting homomorphism which applies to α_1 is the connecting homomorphism $\partial_{\mathcal{O}(1)}$ with respect to the twisted duality $D_{\mathcal{O}(1)}$. So, we get $\partial(\alpha_1) = [\mathrm{d}\,\alpha_1] = [\gamma]$ as wanted. The rest of the proof is as above: the diagram commutes and has two isomorphisms out of three by the previous Lemmas.

To prove that $[\beta^{(n)}_X]^2 = 0$ observe that $\alpha_0 \star \mathrm{d}\,\alpha_1 \star \ldots \star \mathrm{d}\,\alpha_n \star \alpha_0$ is non-degenerate since $\alpha_0, \mathrm{d}\,\alpha_1, \ldots, \mathrm{d}\,\alpha_n$ have no consanguinity and since consanguinity does not change if we repeat some of the symmetric pairs (here α_0) as can be verified directly on the definition, see 4.9. Therefore, we are allowed to consider the Witt class of this space $[\alpha_0 \star \mathrm{d}\,\alpha_1 \star \ldots \star \mathrm{d}\,\alpha_n \star \alpha_0] \in \mathrm{W}^n(\mathbb{P}^n_X, \mathcal{O}(n))$ and to make the following computation in $\mathrm{W}^{\mathrm{Tot}}(\mathbb{P}^n_X)$:

$$
\begin{aligned}
[\beta^{(n)}_X]^2 &= \big[\alpha_0 \star \mathrm{d}\,\alpha_1 \star \ldots \star \mathrm{d}\,\alpha_n \star \alpha_0 \star \mathrm{d}\,\alpha_1 \star \ldots \star \mathrm{d}\,\alpha_n\big] \\
&= \big[\alpha_0 \star \mathrm{d}\,\alpha_1 \star \ldots \star \mathrm{d}\,\alpha_n \star \alpha_0\big] \star \underbrace{\big[\,\mathrm{d}\,\alpha_1 \star \ldots \star \mathrm{d}\,\alpha_n\big]}_{=0} = 0\,.
\end{aligned}
$$

The latter class vanishes since the spaces $\mathrm{d}\,\alpha_i$ are metabolic (and since $n \geq 1$). $\square$

Remark 7.8. It is a triviality that to prove $\begin{pmatrix} 1 & [\beta^{(n)}_X] \end{pmatrix}$ an isomorphism we can as well replace $[\beta^{(n)}_X]$ by its opposite $-[\beta^{(n)}_X]$. Therefore, any variation in the definitions leading to a sign change of $\beta^{(n)}_X$ does not really affect the presentation of the total Witt group of $\mathbb{P}^n_X$.

Remark 7.9. It is also immediate from the general considerations of Proposition 6.17 that the generator $[\beta^{(n)}_X]$ of $\mathrm{W}^{\mathrm{Tot}}(\mathbb{P}^n_X)$ is locally trivial on $\mathbb{P}^n_X$.

Appendix A. Recalling products, dualities and octahedra

Definition A.1. Consider three triangulated categories $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{M}$. A *pairing of triangulated categories* is triple $(\boxtimes, \lambda, \rho)$ formed by a bifunctor

$$\boxtimes : \mathcal{K} \times \mathcal{L} \longrightarrow \mathcal{M}\,.$$

which is exact in each variables with the natural isomorphisms

$$\rho_{A,B} : A \boxtimes (T_{\mathcal{L}} B) \xrightarrow{\sim} T_{\mathcal{M}}(A \boxtimes B) \qquad \text{and} \qquad \lambda_{A,B} : (T_{\mathcal{K}} A) \boxtimes B \xrightarrow{\sim} T_{\mathcal{M}}(A \boxtimes B)$$

expressing compatibility with translation for the exact functors $A \boxtimes -$ and $- \boxtimes B$. The following square is moreover assumed to be skew-commutative:

$$\begin{tikzcd}
(T_{\mathcal{K}} A) \boxtimes (T_{\mathcal{L}} B) \arrow[r, "\lambda_{A,TB}"] \arrow[d, "\rho_{TA,B}"'] \arrow[dr, phantom, "(-1)"] & T_{\mathcal{M}}\big(A \boxtimes (T_{\mathcal{L}} B)\big) \arrow[d, "T(\rho_{A,B})"] \\
T_{\mathcal{M}}\big((T_{\mathcal{K}} A) \boxtimes B\big) \arrow[r, "T(\lambda_{A,B})"'] & T^2_{\mathcal{M}}(A \boxtimes B)\,.
\end{tikzcd} \tag{30}$$

Definition A.2. A *duality* on a triangulated category $\mathcal{K}$ is a triple (D, δ, ϖ) where

- $D : \mathcal{K}^{\mathrm{op}} \to \mathcal{K}$ is a ± 1-exact contravariant functor (the *duality*); exactness means in particular that $D \circ T = T^{-1} \circ D$.
- $\delta = \pm 1$ gives the exactness of D. So, 1-exact means exact and -1-exact means that distinguished triangles are sent to skew-distinguished ones (those which are distinguished after changing the sign of the three morphisms).
- $\varpi : \mathrm{Id}_{\mathcal{K}} \to D \circ D$ is an isomorphism of functors, such that $D(\varpi_A) \circ \varpi_{D(A)} = \mathrm{id}_{D(A)}$ and $T(\varpi_A) = \varpi_{T(A)}$ for all $A \in \mathcal{K}$.

A triangulated category with duality is a quadruple $(\mathcal{K}, D, \delta, \varpi)$. See details in [2].

Definition A.3. Given a triangulated category with duality $(\mathcal{K}, D, \delta, \varpi)$ and an integer $i \in \mathbb{Z}$, the *i-th shifted duality*

$$(D, \delta, \varpi)^{(i)} = (D^{(i)}, \delta^{(i)}, \varpi^{(i)})$$

on the same category $\mathcal{K}$ is defined by

$$D^{(i)} := T^i \circ D\,, \qquad \delta^{(i)} := (-1)^i \cdot \delta \qquad \text{and} \qquad \varpi^{(i)} := (-1)^{\frac{i(i+1)}{2}} \cdot \delta^i \cdot \varpi\,.$$

It is easy to see that $(D, \delta, \varpi)^{(i+j)} = \big((D, \delta, \varpi)^{(i)}\big)^{(j)}$ for all $i, j \in \mathbb{Z}$.

Definition A.4 (Gille-Nenashev)**.** Consider $(\mathcal{K}, D_{\mathcal{K}}, \delta_{\mathcal{K}}, \varpi_{\mathcal{K}})$, $(\mathcal{L}, D_{\mathcal{L}}, \delta_{\mathcal{L}}, \varpi_{\mathcal{L}})$ and $(\mathcal{M}, D_{\mathcal{M}}, \delta_{\mathcal{M}}, \varpi_{\mathcal{M}})$ three triangulated categories with duality. Following [10, Def. 1.11], a *pairing of triangulated categories with dualities* between the three considered categories is a pair $(\boxtimes, \mu)$ where:

- $\boxtimes : \mathcal{K} \times \mathcal{L} \longrightarrow \mathcal{M}$ is a pairing of triangulated categories (Def. A.1),
- μ is a natural isomorphism

$$\mu_{A,B} : \quad D_{\mathcal{K}} A \boxtimes D_{\mathcal{L}} B \xrightarrow{\sim} D_{\mathcal{M}}(A \boxtimes B)$$

such that the following two properties are satisfied:

(PD 1) The following diagram commutes:

$$\begin{tikzcd}
A \boxtimes B \arrow[r, "\varpi_A \boxtimes \varpi_B"] \arrow[d, "\varpi_{A\boxtimes B}"'] & D^2_{\mathcal{K}}(A) \boxtimes D^2_{\mathcal{L}}(B) \arrow[d, "\mu_{DA,DB}"] \\
D^2_{\mathcal{M}}(A \boxtimes B) \arrow[r, "D_{\mathcal{M}}(\mu_{A,B})"'] & D_{\mathcal{M}}\big((D_{\mathcal{K}}A) \boxtimes (D_{\mathcal{L}}B)\big) .
\end{tikzcd}$$

(PD 2) The following diagram commutes up to signs (given in the center):

$$\begin{tikzcd}
T_{\mathcal{M}}(D_{\mathcal{K}}T_{\mathcal{K}}A \boxtimes D_{\mathcal{L}}B) \arrow[d, "T_{\mathcal{M}}(\mu_{TA,B})"'] \arrow[dr, phantom, "(\delta_{\mathcal{K}}\cdot\delta_{\mathcal{M}})"] & D_{\mathcal{K}}A \boxtimes D_{\mathcal{L}}B \arrow[l, "\lambda_{DTA,DB}"'] \arrow[r, "\rho_{DA,DTB}"] \arrow[d, "\mu_{A,B}"] \arrow[dr, phantom, "(\delta_{\mathcal{L}}\cdot\delta_{\mathcal{M}})"] & T_{\mathcal{M}}(D_{\mathcal{K}}A \boxtimes D_{\mathcal{L}}T_{\mathcal{L}}B) \arrow[d, "T_{\mathcal{M}}(\mu_{A,TB})"] \\
T_{\mathcal{M}}D_{\mathcal{M}}(T_{\mathcal{K}}A \boxtimes B) & D_{\mathcal{M}}(A \boxtimes B) \arrow[l, "T_{\mathcal{M}}D_{\mathcal{M}}(\lambda_{A,B})"] \arrow[r, "T_{\mathcal{M}}D_{\mathcal{M}}(\rho_{A,B})"'] & T_{\mathcal{M}}D_{\mathcal{M}}(A \boxtimes T_{\mathcal{L}}B) .
\end{tikzcd}$$

In the special case where $\mathcal{K} = \mathcal{L} = \mathcal{M}$, this gives:

Definition A.5. A *triangulated category with product and duality*, or in short a *TPD-category* is a triple $\big(\mathcal{K}, (D, \delta, \varpi), (\otimes, \mu)\big)$ where $(\mathcal{K}, D, \delta, \varpi)$ is a triangulated category with duality (Def. A.2) and $(\otimes, \mu)$ with

$$\otimes : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$$

is a pairing compatible with the duality as in Def. A.4.

Definition A.6. Consider a pairing $(\boxtimes, \mu)$ of triangulated categories with duality $\boxtimes : \mathcal{K} \times \mathcal{L} \longrightarrow \mathcal{M}$ as in Def. A.4. Let $i, j \in \mathbb{Z}$ be two integers. Define a new pairing $(\boxtimes, \mu)^{(i,j)} := (\boxtimes, \mu^{(i,j)})$ by the formula:

$$\mu^{(i,j)}_{A,B} \;:=\; T^{i+j}_{\mathcal{M}}(\mu_{A,B}) \;\circ\; T^i_{\mathcal{M}}(\rho^{(j)}_{DA,DB}) \;\circ\; \lambda^{(i)}_{DA,D^{(j)}B}$$

where $\lambda^{(i)}_{A,B} : (T^i_{\mathcal{K}}A) \boxtimes B \overset{\sim}{\to} T^i_{\mathcal{M}}(A \boxtimes B)$ and $\rho^{(j)}_{A,B} : A \boxtimes (T^j_{\mathcal{K}}B) \overset{\sim}{\to} T^j_{\mathcal{M}}(A \boxtimes B)$ are the obvious iterations of λ and ρ. More explicitly:

$$\begin{tikzcd}
(D^{(i)}_{\mathcal{K}}A) \boxtimes (D^{(j)}_{\mathcal{L}}B) \arrow[ddd, "\mu^{(i,j)}_{A,B} :="'] & = & (T^i_{\mathcal{K}}D_{\mathcal{K}}A) \boxtimes (T^j_{\mathcal{L}}D_{\mathcal{L}}B) \arrow[d, "\lambda^{(i)}_{DA,T^jDB}"] \\
 & & T^i_{\mathcal{M}}\big((D_{\mathcal{K}}A) \boxtimes (T^j_{\mathcal{L}}D_{\mathcal{L}}B)\big) \arrow[d, "T^i_{\mathcal{M}}(\rho^{(j)}_{DA,DB})"] \\
 & & T^{i+j}_{\mathcal{M}}(D_{\mathcal{K}}A \boxtimes D_{\mathcal{L}}B) \arrow[d, "T^{i+j}_{\mathcal{M}}(\mu_{A,B})"] \\
D^{(i+j)}_{\mathcal{M}}(A \boxtimes B) & = & T^{i+j}_{\mathcal{M}}D_{\mathcal{M}}(A \boxtimes B) .
\end{tikzcd}$$

Proposition A.7. *With notation of Def. A.6, the pairing $\boxtimes : \mathcal{K} \times \mathcal{L} \to \mathcal{M}$ is also compatible with the shifted dualities of $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{M}$ in the sense that the above $(\boxtimes, \mu^{(i,j)})$ is again a pairing of triangulated categories with duality from the pair $(\mathcal{K}, D^{(i)}, \delta^{(i)}, \varpi^{(i)})$, $(\mathcal{L}, D^{(j)}, \delta^{(j)}, \varpi^{(j)})$ and with values in $(\mathcal{M}, D^{(i+j)}, \delta^{(i+j)}, \varpi^{(i+j)})$.*

Proof. We do not give all details but a pattern the cautious reader can follow.

First, we prove that $\mu^{(1,0)}$ turns $\boxtimes$ into a pairing of triangulated categories with duality between $(\mathcal{K}, D^{(1)}, \delta^{(1)}, \varpi^{(1)}) \times (\mathcal{L}, D^{(0)}, \delta^{(0)}, \varpi^{(0)})$ and $(\mathcal{M}, D^{(1)}, \delta^{(1)}, \varpi^{(1)})$.

To check the new (PD 1), proceed as follows: first write it down; then replace $\mu^{(1,0)}$ by its definition (write this inside the diagram); then the central diagram is the juxtaposition of the old (PD 1) and of the old (PD 2) for (DA, DB) instead of (A, B); use also that $\varpi^{(1)} = (-\delta) \cdot \varpi$.

To check the new (PD 2), proceed as follows: first write it down; then replace $\mu^{(1,0)}$ by its definition (write this inside the diagram); then the left-hand diagram is the translation $T_{\mathcal{M}}$ of the left square of the old (PD 2) plus the fact that $TDT = D$, whereas the right-hand diagram is obtained by the juxtaposition of the translation $T_{\mathcal{M}}$ of the right square of the old (PD 2) with diagram (30).

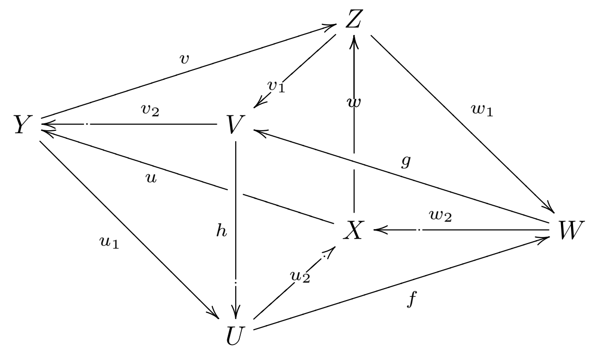
The similar statement for $\mu^{(0,1)}$ can be established by following the mirror argument. Then, by induction, the statement holds for $\mu^{(i,0)}$ and $\mu^{(0,j)}$. Finally, we use that $\mu^{(i,j)} = (\mu^{(0,j)})^{(i,0)}$. $\square$

Remark A.8. Two words of caution about the definition of $\mu^{(i,j)}$: There is no sign choice hidden in the definitions of $\lambda^{(i)}$ and $\rho^{(j)}$. They can be given explicitly as in [10, Rem. 1.1]. There are sign choices in the definition of $\mu^{(i,j)}$ coming from the *order* in which we apply the natural isomorphism λ, ρ and μ. This roots back to the possible skew-commutativity of diagrams (30) and (PD 2), which roughly say that these natural isomorphisms only commute up to signs. With this in mind, there is not really a distinction between a left and a right product as in [10] but rather lots of choices for the order of stage appearance of λ, ρ and μ in the definition of the natural isomorphism $\mu^{(i,j)}$, all choices giving the same result up to sign.

We now turn to the compatibility of product and triangulation. First, recall:

Definition A.9. In a triangulated category, an *octahedron* is a diagram as follows:

(31)



in which the morphisms pictured with a broken arrow $\dashrightarrow$ are of degree one, namely $u_2 : U \to TX$, $v_2 : V \to TY$, $w_2 : W \to TX$ and $h : V \to TU$. This octahedron is called *distinguished* if the following conditions hold:

(Oct1) the four triangles which can commute ( ) do commute;

(Oct2) the four triangles which can be distinguished ( ) are distinguished;

(Oct3) both ways from Y to W coincide: $w_1\, v = f\, u_1$;

(Oct4) both ways from W to $T(Y)$ coincide: $T(u)\, w_2 = v_2\, g$.

If moreover, to close the ring, the following two triangles containing the morphisms of (Oct3) and (Oct4) are distinguished:

$$\text{(Oct5)}\quad Y \xrightarrow{\ s\ } W \xrightarrow{\begin{pmatrix} g \\ w_2 \end{pmatrix}} V \oplus TX \xrightarrow{\begin{pmatrix} v_2 & -Tu \end{pmatrix}} TY \quad \text{where } s := w_1\, v \overset{\text{Oct3}}{=} f\, u_1\,,$$

$$\text{(Oct6)}\quad Y \xrightarrow{\begin{pmatrix} u_1 \\ v \end{pmatrix}} U \oplus Z \xrightarrow{\begin{pmatrix} -f & w_1 \end{pmatrix}} W \xrightarrow{\ t\ } TY \quad \text{where } t := T(u)\, w_2 \overset{\text{Oct4}}{=} v_2\, g\,,$$

then we say that (31) is a *very distinguished* octahedron. In this case, all morphisms extractable from octahedron (31) have an explicit distinguished triangle to live in.

In a triangulated category, the octahedron axiom (TR4), or composition axiom, asserts that any commutative triangle as in the left margin can be completed into a distinguished octahedron (31). The enriched octahedron axiom (TR4$^+$) asserts the same with a very distinguished octahedron. Our triangulated categories are assumed to satisfy (TR4$^+$), as all known triangulated categories do. This enrichment is due to Beilinson–Bernstein–Deligne [6].

$$\begin{tikzcd} & Z \\ Y \arrow[ur, "v"] & \\ & X \arrow[ul, "u"] \arrow[uu, "w"'] \end{tikzcd}$$

* * *

Remark A.10. The following axiomatization is adapted from May [12] and Keller–Neeman [11]. Consider a pairing of triangulated categories $\boxtimes : \mathcal{K} \times \mathcal{L} \longrightarrow \mathcal{M}$ as in Def. A.1 and two distinguished triangles in $\mathcal{K}$ and $\mathcal{L}$ respectively:

$$A \xrightarrow{\ a\ } A' \xrightarrow{\ a_1\ } A'' \xrightarrow{\ a_2\ } TA \tag{32}$$

$$B \xrightarrow{\ b\ } B' \xrightarrow{\ b_1\ } B'' \xrightarrow{\ b_2\ } TB\,. \tag{33}$$

Choose one morphism in each triangle, say a and b, and write their product as

$$a \boxtimes b = (\mathrm{id}_{A'} \boxtimes b) \circ (a \boxtimes \mathrm{id}_B)\,.$$

This produces an octahedron. Note however that the above product can also be decomposed as $a \boxtimes b = (a \boxtimes \mathrm{id}_{B'}) \circ (\mathrm{id}_A \boxtimes b)$, yielding another octahedron. In good logic, since there are 9 such pairs of morphisms, we can *a priori* produce 18 octahedra. As the reader would expect, there is some redundancy in this way of axiomatizing the relation between $\boxtimes$ and octahedra. Minimizing the redundancy is precisely the point of the following definition.

Definition A.11. We say that a pairing of triangulated categories $\boxtimes : \mathcal{K} \times \mathcal{L} \longrightarrow \mathcal{M}$ as in Def. A.1 is *compatible with the octahedron axiom* if for any two distinguished triangles (32) and (33) in $\mathcal{K}$ and $\mathcal{L}$ respectively, there exists an object E in $\mathcal{M}$ and distinguished triangles in $\mathcal{M}$:

$$A \boxtimes B \xrightarrow{\ a \boxtimes b\ } A' \boxtimes B' \xrightarrow{\ c\ } E \xrightarrow{\ d\ } T(A \boxtimes B)\,, \tag{34}$$

$$A'' \boxtimes B \xrightarrow{\ e\ } E \xrightarrow{\ f\ } A' \boxtimes B'' \xrightarrow{\ \rho \circ (a_1 \boxtimes b_2)\ } T(A'' \boxtimes B) \tag{35}$$

$$A \boxtimes B'' \xrightarrow{\ g\ } E \xrightarrow{\ h\ } A'' \boxtimes B' \xrightarrow{\ \lambda \circ (a_2 \boxtimes b_1)\ } T(A \boxtimes B'')\,, \tag{36}$$

involving the same object E and six morphisms c, d, e, f, g and h, such that the following three octahedra are (very) distinguished:

(37)
$$\begin{tikzcd}
 & & & A'\boxtimes B' \arrow[dl, "\mathrm{id}_{A'}\boxtimes b_1"'] \arrow[ddr, "c"] & \\
A'\boxtimes B \arrow[urrr, "\mathrm{id}_{A'}\boxtimes b"] \arrow[ddrr, "a_1\boxtimes \mathrm{id}_B"'] & & A'\boxtimes B'' \arrow[ll, "\rho\circ(\mathrm{id}_{A'}\boxtimes b_2)"'] \arrow[dd, "\rho\circ(a_1\boxtimes b_2)"'] & & \\
 & & & A\boxtimes B \arrow[uu, "a\boxtimes b"] \arrow[ulll, "a\boxtimes \mathrm{id}_B"'] & E \arrow[l, "d"'] \arrow[ull, "f"'] \\
 & & A''\boxtimes B \arrow[ur, "\lambda\circ(a_2\boxtimes \mathrm{id}_B)"] \arrow[urr, "e"'] & &
\end{tikzcd}$$

(38)
$$\begin{tikzcd}
 & & & A'\boxtimes B' \arrow[dl, "a_1\boxtimes \mathrm{id}_{B'}"'] \arrow[ddr, "c"] & \\
A\boxtimes B' \arrow[urrr, "a\boxtimes \mathrm{id}_{B'}"] \arrow[ddrr, "\mathrm{id}_A\boxtimes b_1"'] & & A''\boxtimes B' \arrow[ll, "\lambda\circ(a_2\boxtimes \mathrm{id}_{B'})"'] \arrow[dd, "\lambda\circ(a_2\boxtimes b_1)"'] & & \\
 & & & A\boxtimes B \arrow[uu, "a\boxtimes b"] \arrow[ulll, "\mathrm{id}_A\boxtimes b"'] & E \arrow[l, "d"'] \arrow[ull, "h"'] \\
 & & A\boxtimes B'' \arrow[ur, "\rho\circ(\mathrm{id}_A\boxtimes b_2)"] \arrow[urr, "g"'] & &
\end{tikzcd}$$

(39)
$$\begin{tikzcd}
 & & & A\boxtimes B'' \arrow[dl, "a\boxtimes \mathrm{id}_{B''}"'] \arrow[ddr, "g"] & \\
T^{-1}(A''\boxtimes B'') \arrow[urrr, "-T^{-1}(\lambda\circ(a_2\boxtimes \mathrm{id}_{B''}))"] \arrow[ddrr, "T^{-1}(\rho\circ(\mathrm{id}_{A''}\boxtimes b_2))"'] & & A'\boxtimes B'' \arrow[ll, "a_1\boxtimes \mathrm{id}_{B''}"'] \arrow[dd, "\rho\circ(a_1\boxtimes b_2)"'] & & \\
 & & & T^{-1}(A''\boxtimes B') \arrow[uu, "-T^{-1}(\lambda\circ(a_2\boxtimes b_1))"] \arrow[ulll, "T^{-1}(\mathrm{id}_{A''}\boxtimes b_1)"'] & E \arrow[l, "h"'] \arrow[ull, "f"'] \\
 & & A''\boxtimes B \arrow[ur, "\mathrm{id}_{A''}\boxtimes b"] \arrow[urr, "e"'] & &
\end{tikzcd}$$

Remark A.12. The information encapsulated in the (very) distinction of those three octahedra is exactly equivalent to May's axiom (TC3) of [12]. (Easier to check with May's (TC3') of [12, Lem. 4.7].) For more on tensor triangulated categories, see Keller-Neeman [11]. Note that distinguished octahedra with objects exactly as above always exist without any extra compatibility axiom. The real point of Def. A.11 is that these three octahedra can be built with the *same* six morphisms c, d, e, f, g and h, each of them staging in two different octahedra.

Remark A.13. We can always replace the object E obtained in Definition A.11 via an isomorphism $\ell : E \xrightarrow{\sim} E'$, changing accordingly the morphisms c, d, e, f, g and h having source or target equal to E, into the six morphisms

$$\ell \circ c\,, \quad d \circ \ell^{-1}\,, \quad \ell \circ e\,, \quad f \circ \ell^{-1}\,, \quad \ell \circ g \quad \text{and} \quad h \circ \ell^{-1}\,.$$

This allows us to *choose one* distinguished triangle among (34), (35) and (36). Moreover, once one of these triangles is chosen, say the second one (35), that is, if we want to keep e and f as they are, then we can still apply the above procedure a second time to modify the four other morphisms c, d, g and h, but only with an automorphism $\ell : E \to E$ such that $\ell \circ e = e$ and $f \circ \ell^{-1} = f$. This is what we do in the proof of Thm. 5.2. The new octahedra are again (very) distinguished, since this property is preserved by isomorphism of octahedra.

References

[1] J. Arason, Der Wittring projektiver Räume, *Math. Ann.* **253**, no. 3 (1980), pp. 205-212.

[2] P. Balmer, Triangular Witt groups, Part I : The 12-term localization exact sequence, *K-theory* **19**, no. 4 (2000), pp. 311-363.

[3] P. Balmer, Triangular Witt groups, Part II : From usual to derived, *Math. Z.* **236**, no. 2 (2001), pp. 351-382.

[4] P. Balmer, Witt cohomology, Mayer-Vietoris, homotopy invariance and the Gersten Conjecture, *K-theory* **23**, no. 1 (2001), pp. 15-30.

[5] P. Balmer, Witt groups, to appear in the K-theory handbook, see the author's webpage.

[6] A. Beilinson, J. Bernstein, P. Deligne Faisceaux pervers, Astérisque **100** (1982).

[7] S. Gille, On Witt groups with support, *Math. Ann.* **322**, no. 1 (2002), pp. 103-137.

[8] S. Gille, Homotopy invariance of coherent Witt groups, *Math. Z.* **244**, no. 2 (2003), pp. 211-233.

[9] S. Gille, A transfer morphism for Witt groups, *J. reine angew. Math.* **564** (2003), pp. 215-233.

[10] S. Gille and A. Nenashev, Pairings in triangular Witt theory, *J. Algebra* **261**, no. 2 (2003), pp. 292-309.

[11] B. Keller and A. Neeman, The connection between May's axioms for a triangulated tensor product and Happel's description of the derived category of the Quiver D_4, *Doc. Math.* **7** (2002), pp. 535-560.

[12] P. May, The additivity of traces in triangulated categories, *Adv. in Math.* **163** (2001), pp. 34-73.

[13] F. Morel, On the motivic stable π_0 of the sphere spectrum, In: Axiomatic, Enriched and Motivic Homotopy Theory, pp. 219-260, J.P.C. Greenlees (ed.), 2004 Kluwer.

[14] A. Nenashev, On the Witt groups of projective bundles and split quadrics geometric reasoning, preprint (2004), available online at `http://www.math.uiuc.edu/K-theory/0696/` .

[15] J.-L. Verdier, Catégories dérivées des catégories abéliennes, Astérisque **239**, Soc. math. de France (1996).

[16] C. Walter, Grothendieck-Witt groups of projective bundles, preprint (2003), available online at `http://www.math.uiuc.edu/K-theory/0644/` .

[17] C. Weibel, An introduction to homological algebra, Cambridge University Press.

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