A Discontinuous Sturm-Liouville Operator With Indefinite Weight

Qiuxia Yang (Corresponding author)
Mathematics Science college, Inner Mongolia University
235 University of West Street, Huhhot 010021, China
&
Department of Computer, Dezhou University
2 University Street, Dezhou 253000, China
E-mail: dlzxyanglixia@163.com

Wanyi Wang
Mathematics Science college, Inner Mongolia University
235 University of West Street, Huhhot 010021, China
Tel: 86-471-499-4310 E-mail: wwy@imu.edu.cn

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Abstract

In this paper, we consider an indefinite Sturm-Liouville operator with eigenparameter-dependent boundary conditions and transmission conditions. In an appropriate space K, we define a new self-adjoint operator A such that the eigenvalues of A coincide with those of such a problem and obtain asymptotic approximation for its eigenvalues and eigenfunctions.

Keywords: Indefinite Sturm-Liouville operator, Eigenvalue, Eigenfunction, Transmission condition

1. Introduction

In recent years, more and more researchers are interested in the discontinuous Sturm-Liouville problem for its application in physics (Demirci M., 2004, p.101-113 and Buschmann D., 1995, p.169-186). The various physics applications of this kind of problem are found in many literature, including some boundary value with transmission conditions that arise in the theory of heat and mass transfer (Aiping W., 2006, p.66-74 and Akdoğan Z., 2007, p.1719-1738).

Here we consider a discontinuous Sturm-Liouville problem with the indefinite weight function $r(x)$. By using the technics of (Kadakal M., 2005, p.229-245 and Kadakal M., 2006, p.1519-1528) and some new approaches, we define a new linear operator A associated with the problem on an appropriate Krein space K. We discuss its eigenvalues and eigenfunctions, and derive asymptotic approximation formulas for eigenvalues and eigenfunctions.

In this study, we consider a discontinuous eigenvalue problem consisting of indefinite Sturm-Liouville equation

$$lu := -(a(x)u'(x))' + q(x)u(x) = \lambda r(x)u(x), \ x \in I \tag{1}$$

where $I = [-1,0) \cup (0,1]$, $a(x) = a_1^2$ for $x \in [-1,0)$ and $a(x) = a_2^2$ for $x \in (0,1]$, a_1, a_2 are positive real constants, $xr(x) > 0$ *a.e.*, $r(x), q(x) \in L^1[I,\mathbb{R}]$, and $\lambda \in \mathbb{C}$ is a complex eigenparameter; with the eigenparameter-dependent boundary conditions at the endpoints

$$l_1u := \lambda(\alpha_1' u(-1) - \alpha_2' u'(-1)) - (\alpha_1 u(-1) - \alpha_2 u'(-1)) = 0 \tag{2}$$

$$l_2u := \lambda(\beta_1' u(1) - \beta_2' u'(1)) + (\beta_1 u(1) - \beta_2 u'(1)) = 0 \tag{3}$$

and the transmission conditions at the point of discontinuity

$$l_3u := u(0+) - \alpha_3 u(0-) - \beta_3 u'(0-) = 0 \tag{4}$$

$$l_4u := u'(0+) - \alpha_4 u(0-) - \beta_4 u'(0-) = 0 \tag{5}$$

where the coefficients $\alpha_i, \beta_i, \alpha_j'$ and β_j' ($i = \overline{1,4}$ and $j = 1,2$) are real numbers. Throughout this paper, we assume that

$$\theta = \begin{vmatrix} \alpha_3 & \beta_3 \\ \alpha_4 & \beta_4 \end{vmatrix} > 0, \ \rho_1 = \begin{vmatrix} \alpha_1' & \alpha_1 \\ \alpha_2' & \alpha_2 \end{vmatrix} \neq 0, \ \rho_2 = \begin{vmatrix} \beta_1' & \beta_1 \\ \beta_2' & \beta_2 \end{vmatrix} \neq 0$$

We define the inner product in $L_r^2(I)$ as

$$[f,g]_1 = \frac{\theta}{a_1^2}\int_{-1}^{0} f_1\overline{g}_1 r dx + \frac{1}{a_2^2}\int_0^1 f_2\overline{g}_2 r dx,\ \forall f,g \in L_{|r|}^2(I)$$

where $f_1(x) = f(x)|_{[-1,0)}$ and $f_2(x) = f(x)|_{(0,1]}$. Obviously $(L_r^2(I), [\cdot,\cdot]_1)$ is a Krein space.

2. An operator formulation in the adequate Krein space

In this section, we introduce the special inner product in Krein space $K := L_r^2(I) \oplus \mathbb{C}_{\rho_1} \oplus \mathbb{C}_{\rho_2}$ and a symmetric operator A defined on K such that (1)-(5). Namely, we define an inner product on K by

$$[F,G] := \frac{\theta}{a_1^2}\int_{-1}^{0} f_1\overline{g}_1 r dx + \frac{1}{a_2^2}\int_0^1 f_2\overline{g}_2 r dx + \frac{\theta}{\rho_1}\langle h,k\rangle + \frac{1}{\rho_2}\langle q,s\rangle \tag{6}$$

for $F := (f,h,q), G := (g,k,s) \in K$. Then $(L_r^2(I)\oplus\mathbb{C}_{\rho_1}\oplus\mathbb{C}_{\rho_2}, [\cdot,\cdot])$ is a Krein space, which we denote by $L_r^2(I)\oplus\mathbb{C}_{\rho_1}\oplus\mathbb{C}_{\rho_2}$. A fundamental symmetry on the Krein space is given by

$$J := \begin{bmatrix} J_0 & 0 & 0 \\ 0 & \mathrm{sgn}\rho_1 & 0 \\ 0 & 0 & \mathrm{sgn}\rho_2 \end{bmatrix}$$

where $\mathrm{sgn}\rho_i \in \{-1,1\}, (i = 1,2)$ and $J_0 : L_r^2(I) \to L_r^2(I)$ is defined by

$$(J_0 f)(x) = f(x)\mathrm{sgn}(r(x)), x \in [-1,1]$$

Let $\langle\cdot,\cdot\rangle = [J\cdot,\cdot]$, then $\langle\cdot,\cdot\rangle$ is a positive definite inner product which turns K into a Hilbert space $H = (L_{|r|}^2(I) \oplus \mathbb{C}_{|\rho_1|} \oplus \mathbb{C}_{|\rho_2|}, [J\cdot,\cdot])$.
We define the operator A in K as follows:

$$D(A) = \{(f(x),h,q) \in K | f_1, f_1' \in AC_{loc}((-1,0)), f_2, f_2' \in AC_{loc}((0,1)), lf \in L_r^2(I)$$

$$l_3 f = l_4 f = 0, h = \alpha_1' f(-1) - \alpha_2' f'(-1),\ q = \beta_1' f(1) - \beta_2' f'(1)\}$$

$$AF = (lf, \alpha_1 f(-1) - \alpha_2 f'(-1), -(\beta_1 f(1) - \beta_2 f'(1)))\ \text{ for } F = (f, \alpha_1' f(-1) - \alpha_2' f'(-1), \beta_1' f(1) - \beta_2' f'(1)) \in D(A)$$

For convenience, for $(f,h,q) \in D(A)$, set

$$N_1(f) = \alpha_1 f(-1) - \alpha_2 f'(-1),\ N_1'(f) = \alpha_1' f(-1) - \alpha_2' f'(-1)$$

$$N_2(f) = \beta_1 f(1) - \beta_2 f'(1),\ N_2'(f) = \beta_1' f(1) - \beta_2' f'(1)$$

Now we can rewrite the considered problem (1)-(5) in the operator form $AF = \lambda r F$.

Lemma 2.1 The eigenvalues and eigenfunctions of the problem (1)-(5) are defined as the eigenvalues and the first components of the corresponding eigenelements of the operator A, respectively.

Lemma 2.2 The domain $D(A)$ is dense in H.

Proof: Let $F = (f(x),h,q) \in H$, $F \perp D(A)$ and $\widetilde{C}_0^\infty$ be a functional set such that

$$\varphi(x) = \begin{cases} \varphi_1(x), x \in [-1,0) \\ \varphi_2(x), x \in (0,1] \end{cases}$$

where $\varphi_1(x) \in \widetilde{C}_0^\infty[-1,0)$ and $\varphi_2(x) \in \widetilde{C}_0^\infty(0,1]$. Since $\widetilde{C}_0^\infty \oplus 0 \oplus 0 \subset D(A)$ $(0 \in \mathbb{C})$, any $U = (u(x),0,0) \in \widetilde{C}_0^\infty \oplus 0 \oplus 0$ is orthogonal to F, namely

$$\langle F,U\rangle = \frac{\theta}{a_1^2}\int_{-1}^{0} f_1\overline{u}|r|dx + \frac{1}{a_2^2}\int_0^1 f_2\overline{u}|r|dx = \langle f,u\rangle_1$$

This implies that $f(x)$ is orthogonal to $\widetilde{C}_0^\infty$ in $L_{|r|}^2(I)$ and hence $f(x) = 0$. Next suppose that $G_1 = (g(x),k,0) \in D(A)$, then $\langle F,G_1\rangle = \frac{\theta}{|\rho_1|}h\overline{k}$, so $h = 0$. Similarly we have $q = 0$. So $F = (0,0,0)$. Hence, $D(A)$ is dense in H.

Theorem 2.3 The linear operator A is self-adjoint in K.

Proof: The operator A is self-adjoint in Krein space K if and only if the operator JA is self-adjoint in Hilbert space H.

For all $F,G \in D(A)$. By two partial integrations we obtain

$$[AF,G] = [F,AG] + \theta W(f,\overline{g};0-) - \theta W(f,\overline{g};-1) + W(f,\overline{g};1) - W(f,\overline{g};0+)$$

$$+\frac{\theta}{\rho_1}(N_1(f)\overline{N_1'(g)} - N_1'(f)\overline{N_1(g)}) + \frac{1}{\rho_2}(N_2'(f)\overline{N_2(g)} - N_2(f)\overline{N_2'(g)})$$

where, as usual, by $W(f, g; x)$ we denote the Wronskians $f(x)g'(x) - f'(x)g(x)$.

Since f and g satisfy the boundary conditions (2)-(3) and transmission conditions (4)-(5), we get

$$\theta W(f, \overline{g}; -1) = \frac{\theta}{\rho_1}(N_1(f)\overline{N_1'(g)} - N_1'(f)\overline{N_1(g)})$$

$$W(f, \overline{g}; 1) = \frac{1}{\rho_2}(N_2(f)\overline{N_2'(g)} - N_2'(f)\overline{N_2(g)})$$

$$W(f, \overline{g}; 0+) = \theta W(f, \overline{g}; 0-)$$

Then we have $[AF, G] = [F, AG]$ $(F, G \in D(A))$, so A is symmetric, JA is also symmetric.

Let $JA = B$, $J_0 l = L$. If $\rho_1 < 0$, then

$$L_1 u := \lambda(\alpha_1' u(-1) - \alpha_2' u'(-1)) + (\alpha_1 u(-1) - \alpha_2 u'(-1)) = 0$$

If $\rho_1 > 0$, then $L_1 u = l_1 u$. Similarly, if $\rho_2 < 0$, then

$$L_2 u := \lambda(\beta_1' u(1) - \beta_2' u'(1)) - (\beta_1 u(1) - \beta_2 u'(1)) = 0$$

If $\rho_2 > 0$, then $L_2 u := l_2 u$. And $L_3 u := l_3 u$, $L_4 u := l_4 u$. For simplicity, we set $\rho_1 > 0$ and $\rho_2 > 0$.

In the following, we show that for all $F = (f, N_1'(f), N_2'(f)) \in D(A)$, s.t. $\langle BF, W\rangle = \langle F, U\rangle$. Then $W \in D(A)$ and $BW = U$, here $W = (w(x), h, q)$, $U = (u(x), k, s)$, i.e. (i) $w_1, w_1' \in AC_{loc}((-1, 0))$, $w_2, w_2' \in AC_{loc}((0, 1))$, $lw \in L^2_{|r|}(I)$; (ii) $h = \alpha_1' w(-1) - \alpha_2' w'(-1)$, $q = \beta_1' w(1) - \beta_2' w'(1)$; (iii) $l_3 w = l_4 w = 0$; (iv)$u(x) = lw$; (v) $k = \alpha_1 w(-1) - \alpha_2 w'(-1)$, $s = \beta_1 w(1) - \beta_2 w'(1)$.

For all $F \in \widetilde{C}_0^\infty \oplus 0 \oplus 0 \subset D(A)$, we obtain

$$\frac{\theta}{a_1^2}\int_{-1}^{0}(lf)\overline{w}|r|dx + \frac{1}{a_2^2}\int_{0}^{1}(lf)\overline{w}|r|dx = \frac{\theta}{a_1^2}\int_{-1}^{0} f\overline{u}|r|dx + \frac{1}{a_2^2}\int_{0}^{1} f\overline{u}|r|dx$$

namely, $\langle lf, w\rangle = \langle f, u\rangle$. Hence, by standard Sturm-Liouville theory, (i) and (iv) hold. By (iv), the equation $\langle AF, W\rangle = \langle F, U\rangle$ becomes

$$\frac{\theta}{a_1^2}\int_{-1}^{0}(lf)\overline{w}|r|dx + \frac{1}{a_2^2}\int_{0}^{1}(lf)\overline{w}|r|dx + \frac{\theta N_1(f)\overline{h}}{\rho_1} - \frac{N_2(f)\overline{r}}{\rho_2} = \frac{\theta}{a_1^2}\int_{-1}^{0} f\overline{u}|r|dx + \frac{1}{a_2^2}\int_{0}^{1} f\overline{u}|r|dx + \frac{\theta N_1'(f)\overline{k}}{\rho_1} - \frac{N_2'(f)\overline{s}}{\rho_2}$$

Then

$$\langle lf, w\rangle_1 = \langle f, lw\rangle_1 + \frac{\theta N_1'(f)\overline{k}}{\rho_1} - \frac{\theta N_1(f)\overline{h}}{\rho_1} + \frac{N_2(f)\overline{r}}{\rho_2} - \frac{N_2'(f)\overline{s}}{\rho_2}$$

However

$$\langle lf, w\rangle_1 = \langle f, lw\rangle_1 + \theta W(f, \overline{w}; 0-) - \theta W(f, \overline{w}; -1) + W(f, \overline{w}; 1) - W(f, \overline{w}; 0+)$$

So

$$\frac{\theta N_1'(f)\overline{k}}{\rho_1} - \frac{\theta N_1(f)\overline{h}}{\rho_1} + \frac{N_2(f)\overline{r}}{\rho_2} - \frac{N_2'(f)\overline{s}}{\rho_2} = \theta(f(0-)\overline{w}'(0-) - f'(0-)\overline{w}(0-))$$
$$-\theta(f(-1)\overline{w}'(-1) - f'(-1)\overline{w}(-1)) + (f(1)\overline{w}'(1) - f'(1)\overline{w}(1)) - (f(0+)\overline{w}'(0+) - f'(0+)\overline{w}(0+)) \quad (7)$$

By Naimark's Patching Lemma (Naimark M.A. (1968)) that there exists an $F \in D(A)$ such that

$$f(0-) = f'(0-) = f(0+) = f'(0+) = 0,\ f(-1) = \alpha_2',\ f'(-1) = \alpha_1',\ f(1) = \beta_2',\ f'(1) = \beta_1'$$

Thus $N_1'(f) = 0$, $N_2'(f) = 0$. Then from (7), (ii) be true. Similarly (v) is proved.

Next choose function $F \in D(A)$ and satisfies

$$f(1) = f'(1) = f(-1) = f'(-1) = f(0+) = 0,\ f(0-) = -\beta_3,\ f'(0-) = -\alpha_3,\ f'(0+) = \theta$$

Thus $N_1'(f) = N_2'(f) = N_1(f) = N_2(f) = 0$. Then from (7), we can have

$$w(0+) = \alpha_3 w(0-) + \beta_3 w'(0-)$$

Similarly we can have

$$w'(0+) = \alpha_4 w(0-) + \beta_4 w'(0-)$$

Corollary 2.4 All eigenvalues of the operator JA are real, and if λ_1 and λ_2 be the two different eigenvalues of the problem (1)-(5), then the corresponding eigenfunctions $f(x)$ and $g(x)$ are orthogonal in the sense of

$$\frac{\theta}{a_1^2}\int_{-1}^{0} f\overline{g}|r| + \frac{1}{a_2^2}\int_{0}^{1} f\overline{g}|r| + \frac{\theta}{|\rho_1|}(\alpha_1' f(-1) + \alpha_2' f'(-1))(\alpha_1'\overline{g}(-1) - \alpha_2'\overline{g'}(-1)) + \frac{1}{|\rho_2|}(\beta_1' f(1) - \beta_2' f'(1))(\beta_1'\overline{g}(1) - \beta_2'\overline{g}'(1)) = 0$$

3. Asymptotic approximations of fundamental solutions

Let $\varphi_{1\lambda}(x)$ be the solution of equation (1) on the interval $[-1, 0)$, satisfying the initial conditions

$$\varphi_{1\lambda}(-1) = -\alpha_2 + \lambda\alpha_2', \ \varphi_{1\lambda}'(-1) = -\alpha_1 + \lambda\alpha_1' \tag{8}$$

After defining this solution we can define the solution $\varphi_{2\lambda}(x)$ of equation (1) on the interval $(0, 1]$ by the initial conditions

$$\varphi_{2\lambda}(-1) = \alpha_3\varphi_{1\lambda}(0) + \beta_3\varphi_{1\lambda}'(0), \ \varphi_{2\lambda}'(-1) = \alpha_4\varphi_{1\lambda}(0) + \beta_4\varphi_{1\lambda}'(0) \tag{9}$$

Analogously we shall define the solutions $\chi_{2\lambda}(x), \chi_{1\lambda}(x)$ by initial conditions

$$\chi_{2\lambda}(1) = \beta_2 + \lambda\beta_2', \ \chi_{2\lambda}'(1) = \beta_1 + \lambda\beta_1' \tag{10}$$

$$\chi_{1\lambda}(0) = \frac{\beta_4\chi_{2\lambda}(0) - \beta_3\chi_{2\lambda}'(0)}{\theta}, \ \chi_{1\lambda}'(0) = \frac{\alpha_4\chi_{2\lambda}(0) - \alpha_3\chi_{2\lambda}'(0)}{\theta} \tag{11}$$

Let us consider the Wronskians $\omega_i(\lambda) := W_\lambda(\varphi_i, \chi_i; x)$ $(i = 1, 2)$ which are independent of $x \in \Omega_i$ and are entire functions, where $\Omega_1 = [-1, 0)$ and $\Omega_2 = (0, 1]$. This sort of calculation gives $\omega_2(\lambda) = \theta\omega_1(\lambda)$. Now we introduce the characteristic function $\omega(\lambda)$ as $\omega(\lambda) := \omega_2(\lambda)$.

Theorem 3.1 The eigenvalues of the problem (1)-(5) consist of the zeros of function $\omega(\lambda)$.

Proof: Let $u_0(x)$ be any eigenfunction corresponding to eigenvalue λ_0. Then the function $u_0(x)$ may be represented in the form

$$u_0(x) = \begin{cases} C_1\varphi_{1\lambda_0}(x) + C_2\chi_{1\lambda_0}(x), x \in [-1, 0) \\ C_3\varphi_{2\lambda_0}(x) + C_4\chi_{2\lambda_0}(x), x \in (0, 1] \end{cases}$$

where at least one of the constants c_i $(i = \overline{1, 4})$ is not zero.

Consider the true function $l_v(u_0(x)) = 0, v = \overline{1, 4}$ as the homogenous system of linear equations in the variables c_i $(i = \overline{1, 4})$ and talking into account (8)-(11), it follows that the determinant of this system is

$$\begin{vmatrix} 0 & \omega_1(\lambda_0) & 0 & 0 \\ 0 & 0 & \omega_2(\lambda_0) & 0 \\ -\varphi_{2\lambda_0}'(0) & -\chi_{2\lambda_0}'(0) & \varphi_{2\lambda_0}(0) & \chi_{2\lambda_0}(0) \\ -\varphi_{2\lambda_0}'(0) & -\chi_{2\lambda_0}'(0) & \varphi_{2\lambda_0}'(0) & \chi_{2\lambda_0}'(0) \end{vmatrix} = \frac{\omega(\lambda_0)^3}{\theta} = 0$$

Lemma 3.2 Let $\lambda \mathrm{sgn} x = s^2$, $s = \sigma + it$. Then the following integral equations hold for $k = 0, 1$

$$\frac{d^k}{dx^k}\varphi_{1\lambda}(x) = (-\alpha_2 - s^2\alpha_2')\frac{d^k}{dx^k}\cos\frac{s(x+1)}{a_1} + \frac{a_1}{s}(-\alpha_1 - s^2\alpha_1')\frac{d^k}{dx^k}\sin\frac{s(x+1)}{a_1} + \frac{1}{a_1 s}\int_{-1}^{x}\frac{d^k}{dx^k}\sin\frac{s(x-y)}{a_1}q(y)\varphi_{1\lambda}(y)dy \tag{12}$$

$$\frac{d^k}{dx^k}\varphi_{2\lambda}(x) = (\alpha_3\varphi_{1\lambda}(0) + \beta_3\varphi_{1\lambda}'(0))\frac{d^k}{dx^k}\cos\frac{sx}{a_2} + \frac{a_2}{s}(\alpha_4\varphi_{1\lambda}(0) + \beta_4\varphi_{1\lambda}'(0))\frac{d^k}{dx^k}\sin\frac{sx}{a_2} + \frac{1}{a_2 s}\int_{0}^{x}\frac{d^k}{dx^k}\sin\frac{s(x-y)}{a_2}q(y)\varphi_{2\lambda}(y)dy \tag{13}$$

Proof: Regard $\varphi_{1\lambda}(x)$ as the solution of the following non-homogeneous Cauchy problem

$$\begin{cases} a_1^2 u''(x) + s^2 u(x) = q(x)\varphi_{1\lambda}(x) \\ \varphi_{1\lambda}(-1) = -\alpha_2 - s^2\alpha_2', \ \varphi_{1\lambda}'(-1) = -\alpha_1 - s^2\alpha_1' \end{cases}$$

Using the method of constant changing, $\varphi_{1\lambda}(x)$ satisfies

$$\varphi_{1\lambda}(x) = (-\alpha_2 - s^2\alpha_2')\cos\frac{s(x+1)}{a_1} + \frac{a_1}{s}(-\alpha_1 - s^2\alpha_1')\sin\frac{s(x+1)}{a_1} + \frac{1}{a_1 s}\int_{-1}^{x}\sin\frac{s(x-y)}{a_1}(y)\varphi_{1\lambda}(x)dy$$

Then differentiating it with respect to x, we have (12). The proof for (13) is similar.

Lemma 3.3 Let $\lambda \mathrm{sgn} x = s^2,\ \mathrm{Im} s = t$. Then for $\alpha_2' \neq 0$

$$\frac{d^k}{dx^k}\varphi_{1\lambda}(x) = -\alpha_2' s^2 \frac{d^k}{dx^k}\cos\frac{s(x+1)}{a_1} + O(|s|^{k+1}e^{|t|\frac{x+1}{a_1}}) \tag{14}$$

$$\frac{d^k}{dx^k}\varphi_{2\lambda}(x) = \frac{\beta_3\alpha_2' s^3}{a_1}\sin\frac{s}{a_1}\frac{d^k}{dx^k}\cos\frac{sx}{a_2} + O(|s|^{k+2}e^{|t|(\frac{1}{a_1}+\frac{x}{a_2})}) \tag{15}$$

while if $\alpha_2' = 0$

$$\frac{d^k}{dx^k}\varphi_{1\lambda}(x) = -a_1\alpha_1' s\frac{d^k}{dx^k}\sin\frac{s(x+1)}{a_1} + O(|s|^k e^{|t|\frac{x+1}{a_1}}) \tag{16}$$

$$\frac{d^k}{dx^k}\varphi_{2\lambda}(x) = -\beta_3\alpha_1' s^2\cos\frac{s}{a_1}\frac{d^k}{dx^k}\cos\frac{sx}{a_2} + O(|s|^{k+1}e^{|t|(\frac{1}{a_1}+\frac{x}{a_2})}) \tag{17}$$

$k = 0, 1$. Each of this asymptotic equalities hold uniformly for x as $|\lambda| \to \infty$.

Theorem 3.4 Let $\lambda \mathrm{sgn} x = s^2,\ \mathrm{Im} s = t$. Then the characteristic function $\omega(\lambda)$ has the following asymptotic representations:

Case 1 $\alpha_2' \neq 0,\ \beta_2' \neq 0$

$$\omega(\lambda) = -\frac{\beta_3\beta_2'\alpha_2' s^6}{a_1 a_2\theta}\sin\frac{s}{a_1}\sin\frac{s}{a_2} + O(|s|^5 e^{|t|(\frac{1}{a_2}-\frac{1}{a_1})})$$

Case 2 $\alpha_2' \neq 0,\ \beta_2' = 0$

$$\omega(\lambda) = \frac{\beta_3\beta_2'\alpha_1' s^5}{a_2\theta}\cos\frac{s}{a_1}\sin\frac{s}{a_2} + O(|s|^4 e^{|t|(\frac{1}{a_2}-\frac{1}{a_1})})$$

Case 3 $\alpha_2' = 0,\ \beta_2' \neq 0$

$$\omega(\lambda) = \frac{\beta_3\beta_1'\alpha_2' s^5}{a_1\theta}\sin\frac{s}{a_1}\cos\frac{s}{a_2} + O(|s|^4 e^{|t|(\frac{1}{a_2}-\frac{1}{a_1})})$$

Case 4 $\alpha_2' = 0,\ \beta_2' = 0$

$$\omega(\lambda) = -\frac{\beta_3\beta_1'\alpha_1' s^4}{\theta}\cos\frac{s}{a_1}\cos\frac{s}{a_2} + O(|s|^3 e^{|t|(\frac{1}{a_2}-\frac{1}{a_1})})$$

Proof: The proof is obtained by substituting (15) into the representation

$$\omega(\lambda) = (\beta_1 + \lambda\beta_1')\varphi_{2\lambda}(1) - (\beta_2 + \lambda\beta_2')\varphi_{2\lambda}'(1)$$

Corollary 3.5 The eigenvalues of the problem (1)-(5) are semibounded below.

Theorem 3.6 The following asymptotic formulae hold for the real eigenvalues of the problem (1)-(5) with $r(x) = \mathrm{sgn} x$:

Case 1 $\alpha_2' \neq 0,\ \beta_2' \neq 0$

$$\sqrt{-\lambda_n'} = a_1(n-1)\pi + O(\frac{1}{n}),\ \sqrt{\lambda_n''} = a_2(n-1)\pi + O(\frac{1}{n})$$

Case 2 $\alpha_2' = 0,\ \beta_2' \neq 0$

$$\sqrt{-\lambda_n'} = a_1(n-\frac{1}{2})\pi + O(\frac{1}{n}),\ \sqrt{\lambda_n''} = a_2(n-1)\pi + O(\frac{1}{n})$$

Case 3 $\alpha_2' \neq 0,\ \beta_2' = 0$

$$\sqrt{-\lambda_n'} = a_1(n-1)\pi + O(\frac{1}{n}),\ \sqrt{\lambda_n''} = a_2(n-\frac{1}{2})\pi + O(\frac{1}{n})$$

Case 4 $\alpha_2' = 0,\ \beta_2' = 0$

$$\sqrt{-\lambda_n'} = a_1(n-\frac{1}{2})\pi + O(\frac{1}{n}),\ \sqrt{\lambda_n''} = a_2(n-\frac{1}{2})\pi + O(\frac{1}{n})$$

Proof: By applying the known Rouche theorem, we can obtain these conclusions.

4. More accuracy asymptotic formulae for eigenvalues and eigenfunctions

In this section, for the sake of simplicity we assume that $a_1 = a_2 = 1,\ \alpha_4 = \beta_3 = 0,\ \alpha_3 = \beta_4$ and the weight function $r(x) = \mathrm{sgn} x$. We will use the following method to obtain more accurate conclusions.

Similarity with the third section, we can get the following three conclusions:

Lemma 4.1 Let $\lambda \mathrm{sgn} x = s^2,\ \mathrm{Im} s = t$. Then $\alpha_2' \neq 0$

$$\frac{d^k}{dx^k}\varphi_{1\lambda}(x) = -\alpha_2' s^2 \frac{d^k}{dx^k}\cos s(x+1) - \alpha_1' s \frac{d^k}{dx^k}\sin s(x+1) + \frac{1}{s}\int_{-1}^{x}\frac{d^k}{dx^k}\sin s(x-y)q(y)\varphi_{1\lambda}(y)dy + O(|s|^k e^{|t|(x+1)})$$

$$\frac{d^k}{dx^k}\varphi_{2\lambda}(x) = -\alpha_2'\alpha_3 s^2 \frac{d^k}{dx^k}\cos s(x+1) - \alpha_1'\alpha_3 s \frac{d^k}{dx^k}\sin s(x+1) + \frac{\alpha_3}{s}\int_{-1}^{0}\frac{d^k}{dx^k}\sin s(x-y)q(y)\varphi_{1\lambda}(y)dy$$
$$+\frac{1}{s}\int_{0}^{x}\frac{d^k}{dx^k}\sin s(x-y)q(y)\varphi_{2\lambda}(y)dy + O(|s|^k e^{|t|(x+1)})$$

while if $\alpha_2' = 0$

$$\frac{d^k}{dx^k}\varphi_{1\lambda}(x) = -\alpha_2 \frac{d^k}{dx^k}\cos s(x+1) - \alpha_1' s \frac{d^k}{dx^k}\sin s(x+1) + \frac{1}{s}\int_{-1}^{x}\frac{d^k}{dx^k}\sin s(x-y)q(y)\varphi_{1\lambda}(y)dy + O(|s|^{k-2} e^{|t|(x+1)})$$

$$\frac{d^k}{dx^k}\varphi_{2\lambda}^{(k)}(x) = -\alpha_2\alpha_3 \frac{d^k}{dx^k}\cos s(x+1) - \alpha_1'\alpha_3 s \frac{d^k}{dx^k}\sin s(x+1) + \frac{\alpha_3}{s}\int_{-1}^{0}\frac{d^k}{dx^k}\sin s(x-y)q(y)\varphi_{1\lambda}(y)dy$$
$$+\frac{1}{s}\int_{0}^{x}\frac{d^k}{dx^k}\sin s(x-y)q(y)\varphi_{2\lambda}(y)dy + O(|s|^{k-2} e^{|t|(x+1)})$$

$k = 0, 1$. Each of this asymptotic equalities hold uniformly for x as $|\lambda| \to \infty$.

Theorem 4.2 The characteristic function $\omega(\lambda)$ has the following asymptotic representations:

Case 1 $\alpha_2' \neq 0,\ \beta_2' \neq 0$

$$\omega(\lambda) = -\alpha_2'\alpha_3\beta_2' s^5 \sin 2s + (\alpha_1'\alpha_3\beta_2' s^4 - \alpha_2'\alpha_3\beta_1' s^4)\cos 2s + \alpha_2'\alpha_3\beta_2' s^4 \int_0^1 \cos s(1-y)\cos s(1+y)q(y)dy$$
$$+\alpha_2'\alpha_3\beta_2' s^4 \int_{-1}^0 \cos s(1-y)\cos s(1+y)q(y)dy + O(|s|^3 e^{2|t|})$$

Case 2 $\alpha_2' = 0,\ \beta_2' \neq 0$

$$\omega(\lambda) = \alpha_1'\alpha_3\beta_2' s^4 \cos 2s - (\alpha_2\alpha_3\beta_2' s^3 + \alpha_1'\alpha_3\beta_1' s^3)\sin 2s + \alpha_1'\alpha_3\beta_2' s^3 \int_{-1}^0 \cos s(1-y)\sin s(1+y)q(y)dy$$
$$+\alpha_1'\alpha_3\beta_2' s^3 \int_0^1 \cos s(1-y)\sin s(1+y)q(y)dy + O(|s|^2 e^{2|t|})$$

Case 3 $\alpha_2' \neq 0,\ \beta_2' = 0$

$$\omega(\lambda) = -\alpha_2'\alpha_3\beta_1' s^4 \cos 2s + (\alpha_2'\alpha_3\beta_2 s^3 - \alpha_1'\alpha_3\beta_1' s^3)\sin 2s - \alpha_2'\alpha_3\beta_1' s^3 \int_{-1}^0 \sin s(1-y)\cos s(1+y)q(y)dy$$
$$-\alpha_2'\alpha_3\beta_1' s^3 \int_0^1 \sin s(1-y)\cos s(1+y)q(y)dy + O(|s|^2 e^{2|t|})$$

Case 4 $\alpha_2' = 0,\ \beta_2' = 0$

$$\omega(\lambda) = -\alpha_1'\alpha_3\beta_1' s^3 \sin 2s - (\alpha_1'\alpha_3\beta_2 s^2 + \alpha_2\alpha_3\beta_1' s^2)\cos 2s - \alpha_1'\alpha_3'\beta_1' s^2 \int_{-1}^0 \sin s(1-y)\sin s(1+y)q(y)dy$$
$$-\alpha_1'\alpha_3\beta_1' s^2 \int_0^1 \sin s(1-y)\sin s(1+y)q(y)dy + O(|s| e^{2|t|})$$

Theorem 4.3 The following asymptotic formulas hold for the real eigenvalues and eigenfunctions of the boundary value transmission problem (1)-(5):

Case 1 $\alpha_2' \neq 0,\ \beta_2' \neq 0$

$$s_n = \frac{(n-1)\pi}{2} + O\Big(\frac{1}{n}\Big),\ \varphi(x,\lambda_n) = \begin{cases} -\frac{(n-1)^2}{4}\pi^2\alpha_2' \cos\frac{(n-1)\pi(x+1)}{2} + O(n), x \in [-1,0) \\ -\frac{(n-1)^2}{4}\pi^2\alpha_2'\alpha_3 \cos\frac{(n-1)\pi(x+1)}{2} + O(n), x \in (0,1] \end{cases}$$

Case 2 $\alpha_2' = 0,\ \beta_2' \neq 0$

$$s_n = \frac{(n-\frac{1}{2})\pi}{2} + O\Big(\frac{1}{n}\Big),\ \varphi(x,\lambda_n) = \begin{cases} -\frac{(n-\frac{1}{2})^2}{4}\pi^2\alpha_2\cos\frac{(n-\frac{1}{2})\pi(x+1)}{2} + O(n), x\in[-1,0) \\ -\frac{(n-\frac{1}{2})^2}{4}\pi^2\alpha_2\alpha_3\cos\frac{(n-\frac{1}{2})\pi(x+1)}{2} + O(n), x\in(0,1] \end{cases}$$

Case 3 $\alpha_2' \neq 0,\ \beta_2' = 0$

$$s_n = \frac{(n-\frac{1}{2})\pi}{2} + O\Big(\frac{1}{n}\Big),\ \varphi(x,\lambda_n) = \begin{cases} -\frac{(n-\frac{1}{2})^2}{4}\pi^2\alpha_2'\cos\frac{(n-\frac{1}{2})\pi(x+1)}{2} + O(n), x\in[-1,0) \\ -\frac{(n-\frac{1}{2})^2}{4}\pi^2\alpha_2'\alpha_3\cos\frac{(n-\frac{1}{2})\pi(x+1)}{2} + O(n), x\in(0,1] \end{cases}$$

Case 4 $\alpha_2' = 0,\ \beta_2' = 0$

$$s_n = \frac{n\pi}{2} + O\Big(\frac{1}{n}\Big),\ \varphi(x,\lambda_n) = \begin{cases} -\pi^2\alpha_2\cos\frac{n\pi(x+1)}{2} + O(\frac{1}{n}), x\in[-1,0) \\ -\frac{n^2}{4}\pi^2\alpha_2\alpha_3\cos\frac{n\pi(x+1)}{2} + O(\frac{1}{n}), x\in(0,1] \end{cases}$$

Theorem 4.4 The following more accuracy asymptotic formulas hold for the real eigenvalues of the boundary value transmission problem (1)-(5):

Case 1 $\alpha_2' \neq 0,\ \beta_2' \neq 0$

$$s_n = \frac{(n-1)\pi}{2} + \frac{1}{(n-1)\pi}\Big(\frac{\alpha_1'}{\alpha_2'} - \frac{\beta_1'}{\beta_2'} + \frac{Q}{2}\Big) + O\Big(\frac{1}{n^2}\Big) \tag{18}$$

Case 2 $\alpha_2' = 0,\ \beta_2' \neq 0$

$$s_n = \frac{(n-\frac{1}{2})\pi}{2} - \frac{1}{(n-\frac{1}{2})\pi}\Big(\frac{\alpha_2}{\alpha_1'} + \frac{\beta_1'}{\beta_2'} - \frac{Q}{2}\Big) + O\Big(\frac{1}{n^2}\Big) \tag{19}$$

Case 3 $\alpha_2' \neq 0,\ \beta_2' = 0$

$$s_n = \frac{(n-\frac{1}{2})\pi}{2} + \frac{1}{(n-\frac{1}{2})\pi}\Big(\frac{\alpha_1'}{\alpha_2'} - \frac{\beta_2}{\beta_1'} + \frac{Q}{2}\Big) + O(\frac{1}{n^2}) \tag{20}$$

Case 4 $\alpha_2' = 0,\ \beta_2' = 0$

$$s_n = \frac{n\pi}{2} - \frac{1}{n\pi}\Big(\frac{\alpha_2}{\alpha_1'} + \frac{\beta_2}{\beta_1'} - \frac{Q}{2}\Big) + O\Big(\frac{1}{n^2}\Big) \tag{21}$$

where $Q := \int_{-1}^{0} q(y)dy + \int_0^1 q(y)dy$.

Proof: Let us consider Case 1 only.

Putting $s_n = \frac{(n-1)\pi}{2} + \frac{\delta_n}{2}, \delta_n = O\Big(\frac{1}{n}\Big)$ in the equality $\omega(\lambda)$. We find that

$$\sin\delta_n = \frac{\cos\delta_n}{s_n}\Big(\frac{\alpha_1'}{\alpha_2'} - \frac{\beta_1'}{\beta_2'} + \frac{Q}{2} + O(\frac{1}{n})\Big) + O\Big(|s_n|^{-2}\Big) \tag{22}$$

where $Q := \int_{-1}^{0} q(y)dy + \int_0^1 q(y)dy$. Consequently, from (22) it follows that

$$\delta_n = \frac{2}{(n-1)\pi}\Big(\frac{\alpha_1'}{\alpha_2'} - \frac{\beta_1'}{\beta_2'} + \frac{Q}{2}\Big) + O\Big(\frac{1}{n^2}\Big) \tag{23}$$

substituting (23) in $s_n = \frac{(n-1)\pi}{2} + \frac{\delta_n}{2}$, we have (18).

The proof of other cases are similar.

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