Discrete Mathematics, Chapter 1.1.-1.3: Propositional Logic

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Outline

1. Propositions
2. Logical Equivalences
3. Normal Forms

Propositions

A proposition is a declarative sentence that is either true or false.
Examples of propositions:

- The Moon is made of green cheese.
- Trenton is the capital of New Jersey.
- Toronto is the capital of Canada.
- $1 + 0 = 1$
- $0 + 0 = 2$

Examples that are not propositions.

- Sit down!
- What time is it?
- $x + 1 = 2$
- $x + y = z$

Propositional Logic

Constructing Propositions

- Propositional Variables: $p, q, r, s, \ldots$
- The proposition that is always true is denoted by **T** and the proposition that is always false is denoted by **F**.
- Compound Propositions; constructed from logical connectives and other propositions
- Negation $\neg$
- Conjunction $\wedge$
- Disjunction $\vee$
- Implication $\rightarrow$
- Biconditional $\leftrightarrow$

Disjunction

The disjunction of propositions p and q is denoted by $p \vee q$ and has this truth table:

p	***q***	***p* ∨ *q***
T	T	T
T	F	T
F	T	T
F	F	F

Conjunction

The disjunction of propositions p and q is denoted by $p \wedge q$ and has this truth table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Implication

- If p and q are propositions, then $p \rightarrow q$ is a conditional statement or implication which is read as "if p, then q" and has this truth table:

p	***q***	***p* →*q***
T	T	T
T	F	F
F	T	T
F	F	T

- In $p \rightarrow q$, p is the hypothesis (antecedent or premise) and q is the conclusion (or consequence).
- Implication can be expressed by disjunction and negation:
  $p \rightarrow q \;\equiv\; \neg p \vee q$

Understanding Implication

- In $p \rightarrow q$ there does not need to be any connection between the antecedent or the consequent. The meaning depends only on the truth values of p and q.
- This implication is perfectly fine, but would not be used in ordinary English. "If the moon is made of green cheese, then I have more money than Bill Gates."
- One way to view the logical conditional is to think of an obligation or contract. "If I am elected, then I will lower taxes."

Different Ways of Expressing $p \to q$

if p, then q
if p, q
q unless $\neg p$
q if p
p is sufficient for q
q is necessary for p
a sufficient condition for q is p

p implies q
p only if q
q when p
q whenever p
q follows from p
a necessary condition for p is q

Converse, Contrapositive, and Inverse

- $q \to p$ is the **converse** of $p \to q$
- $\neg q \to \neg p$ is the **contrapositive** of $p \to q$
- $\neg p \to \neg q$ is the **inverse** of $p \to q$

Example: Find the converse, inverse, and contrapositive of
"It is raining is a sufficient condition for my not going to town."
Solution:
converse: If I do not go to town, then it is raining.
inverse: If it is not raining, then I will go to town.
contrapositive: If I go to town, then it is not raining.

How do the converse, contrapositive, and inverse relate to $p \to q$?
Clicker

1. **converse** $\equiv$ **contrapositive** ?
2. **converse** $\equiv$ **inverse** ?
3. **contrapositive** $\equiv$ **inverse** ?

Biconditional

If p and q are propositions, then the biconditional proposition $p \leftrightarrow q$ has this truth table

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

$p \leftrightarrow q$ also reads as

- p if and only if q
- p **iff** q.
- p is necessary and sufficient for q
- if p then q, and conversely
- p implies q, and vice-versa

Precedence of Logical Operators

1. $\neg$
2. $\wedge$
3. $\vee$
4. $\rightarrow$
5. $\leftrightarrow$

Thus $p \vee q \rightarrow \neg r$ is equivalent to $(p \vee q) \rightarrow \neg r$.
If the intended meaning is $p \vee (q \rightarrow \neg r)$ then parentheses must be used.

Satisfiability, Tautology, Contradiction

A proposition is

- satisfiable, if its truth table contains **true** at least once. Example: $p \wedge q$.
- a tautology, if it is always true. Example: $p \vee \neg p$.
- a contradiction, if it always false. Example: $p \wedge \neg p$.
- a contingency, if it is neither a tautology nor a contradiction. Example: p.

Logical Equivalence

Definition

Two compound propositions p and q are logically equivalent if the columns in a truth table giving their truth values agree.
This is written as $p \equiv q$.

It is easy to show:

Fact

$p \equiv q$ if and only if $p \leftrightarrow q$ is a tautology.

De Morgan's Laws

$$\neg(p \land q) \equiv \neg p \lor \neg q$$
$$\neg(p \lor q) \equiv \neg p \land \neg q$$

Truth table proving De Morgan's second law.

p	q	$\neg p$	$\neg q$	$(p \lor q)$	$\neg(p \lor q)$	$\neg p \land \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Important Logical Equivalences

Domination laws:	$p \vee \mathbf{T} \equiv \mathbf{T}$, $p \wedge \mathbf{F} \equiv \mathbf{F}$
Identity laws:	$p \wedge \mathbf{T} \equiv p$, $p \vee \mathbf{F} \equiv p$
Idempotent laws:	$p \wedge p \equiv p$, $p \vee p \equiv p$
Double negation law:	$\neg(\neg p) \equiv p$
Negation laws:	$p \vee \neg p \equiv \mathbf{T}$, $p \wedge \neg p \equiv \mathbf{F}$

The first of the Negation laws is also called "law of excluded middle". Latin: "tertium non datur".

Commutative laws:	$p \wedge q \equiv q \wedge p$, $p \vee q \equiv q \vee p$
Associative laws:	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive laws:	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Absorption laws:	$p \vee (p \wedge q) \equiv p$, $p \wedge (p \vee q) \equiv p$

More Logical Equivalences

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

A Proof in Propositional Logic

To prove: $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

$\neg(p \vee (\neg p \wedge q))$	$\equiv$	$\neg p \wedge \neg(\neg p \wedge q)$	by De Morgan's 2nd law
	$\equiv$	$\neg p \wedge (\neg(\neg p) \vee \neg q)$	by De Morgan's first law
	$\equiv$	$\neg p \wedge (p \vee \neg q)$	by the double negation law
	$\equiv$	$(\neg p \wedge p) \vee (\neg p \wedge \neg q)$	by the 2nd distributive law
	$\equiv$	$\mathbf{F} \vee (\neg p \wedge \neg q)$	because $\neg p \wedge p \equiv \mathbf{F}$
	$\equiv$	$(\neg p \wedge \neg q) \vee \mathbf{F}$	by commutativity of disj.
	$\equiv$	$\neg p \wedge \neg q$	by the identity law for **F**

Conjunctive and Disjunctive Normal Form

- A literal is either a propositional variable, or the negation of one. Examples: p, $\neg p$.
- A clause is a disjunction of literals. Example: $p \vee \neg q \vee r$.
- A formula in conjunctive normal form (CNF) is a conjunction of clauses. Example: $(p \vee \neg q \vee r) \wedge (\neg p \vee \neg r)$

Similarly, one defines formulae in disjunctive normal form (DNF) by swapping the words 'conjunction' and 'disjunction' in the definitions above.
Example: $(\neg p \wedge q \wedge r) \vee (\neg q \wedge \neg r) \vee (p \wedge r)$.

Transformation into Conjunctive Normal Form

Fact

For every propositional formula one can construct an equivalent one in conjunctive normal form.

1. Express all other operators by conjunction, disjunction and negation.
2. Push negations inward by De Morgan's laws and the double negation law until negations appear only in literals.
3. Use the commutative, associative and distributive laws to obtain the correct form.
4. Simplify with domination, identity, idempotent, and negation laws.

(A similar construction can be done to transform formulae into disjunctive normal form.)

Example: Transformation into CNF

Transform the following formula into CNF.

$$\neg(p \to q) \vee (r \to p)$$

1. Express implication by disjunction and negation.

$$\neg(\neg p \vee q) \vee (\neg r \vee p)$$

2. Push negation inwards by De Morgan's laws and double negation.

$$(p \wedge \neg q) \vee (\neg r \vee p)$$

3. Convert to CNF by associative and distributive laws.

$$(p \vee \neg r \vee p) \wedge (\neg q \vee \neg r \vee p)$$

4. Optionally simplify by commutative and idempotent laws.

$$(p \vee \neg r) \wedge (\neg q \vee \neg r \vee p)$$

and by commutative and absorbtion laws

$$(p \vee \neg r)$$