RECENT DEVELOPMENTS IN FINITE ELEMENT ANALYSIS
A book dedicated to Robert L. Taylor
T.J.R. Hughes, E. Oñate and O.C. Zienkiewicz (Eds.)
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PARALLEL ADAPTIVE hp-FINITE ELEMENT METHODS FOR PROBLEMS IN FLUID AND SOLID MECHANICS

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Dedicated to Professor Robert Taylor on the Occasion of his 60th Birthday. *To Bob Taylor, with warmest regards and best wishes and in hope that his excellent work and innovative contributions to the theory and application of finite element methods will continue for many years to come.*

This note summarizes recent work by me and my graduate students and is co-authored by

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ABSTRACT

This note is a brief review of on-going work on parallel, adaptive, hp-methods and on domain decomposition and preconditioning schemes for such methods.

1 INTRODUCTION

For the better part of a decade, we have been working on hp-version finite element methods with the hope that they could be used to create a significant advance in computational mechanics which featured a number of lofty goals:

- exponential convergence (thereby giving good answers with orders-of-magnitude fewer degrees of freedom than conventional FEMs)
- error estimation and control (thereby giving users control of the computational process and a measure of solutions quality)

- optimal meshes and very fast solutions

One does not have to study such methods very long to appreciate the significant pitfalls that stand in the way of accomplishing each of these goals, and it may be that these difficulties have been responsible for the fact that very few investigators have attempted to tackle them except for the simplest applications. The difficulties include

1. production of an efficient data structure that allows h (mesh size) and p (polynomial order) to be varied as independent elementwise parameters, for only then can the full power of hp-meshing and the super-algebraic convergence rates be realized;

2. derivation of rigorous and practical a posteriori error estimators that give good estimates of elementwise errors in appropriate norms;

3. development of an adaptive strategy that can efficiently produce near optimal hp-meshes to reduce error as rapidly as possible;

4. control the stability (conditioning) and computational overhead in hp-methods, including memory requirements, which can be orders-of-magnitude more serious than conventional h-version schemes, and

5. development of parallel hp-schemes, which by their nature require very unconventional approaches to load balancing and domain decomposition.

We have worked on all five of these issues and have made some progress in each, although much remains to be done and significant improvements are certainly possible. For problem areas 1 and 2, two- and three-dimensional data structures supporting general hp-approximations on quadrilateral and hexahedral meshes are discussed in [1,2] and effective techniques for error estimation have been developed in [3, 4, 5]. The first hp-adaptive strategy was reported in [6] and variants of it are used in commercial codes [7]. The 3-step scheme, introduced in [8], is perhaps our preferred approach to date, and extensions of the ideas are reported in a forthcoming work [9]. Consideration of difficulties 4 and 5 has been the subject of more recent work; both theoretical and experimental results on two-dimensional elliptic problems pertain to domain decomposition, parallel adaptive schemes, and pre-conditioning of hp-matrices and are encouraging.

In the remainder of this note, a summary of some of the results in areas 3, 4, and 5 are given.

2 THE TEXAS 3-STEP: A PARALLEL hp-ADAPTIVE STRATEGY

A more complete account is to appear in [9], which generalizes and extends [8]. The idea is to use both *a posteriori* and *a priori* error estimates to choose a distribution of h and p to arrive at a target error. One begins by equating the error (say $\|e\|_{1,\Omega} = \|u - u^{hp}\|_{1,\Omega}$) to the estimated error θ (obtained by an error estimator effective for hp-methods) and then equating this to the error bound in an a priori estimate:

$$\theta \approx \|e\|_{1,\Omega} \approx \sum_{k=1}^{N_0} \Lambda_k \frac{h_k^{\mu_k}}{p_k^{\nu_k}} \tag{1}$$

Here N_0 is the number of elements Ω_k in an initial mesh, $\Lambda_k = C\|u\|_{r,\Omega_k}, h_k =$ dia $(\Omega_k), p_k =$ polynomial degree of shape functions over element $\Omega_k, \mu_k = \min(p_k, r-1), \nu_k = r - 1$.

Equipped with a reliable error estimator, we inquire if it is possible to determine the quantities Λ_k, h_k, p_k (and μ_k, ν_k) on a series of meshes in order to attain a target error index

$$\eta_T = \frac{\|e\|_{1,\Omega}}{\|u\|_{1,\Omega}} \tag{2}$$

With a few approximations, this goal can be attained in three steps:

Step 1. An initial coarse hp-mesh, $\Omega^0_{h_0p_0}$ is generated and used to obtain a cheap initial solution u_0^{hp} and estimates of $\|u\|_{1,\Omega}$ and ν_k, μ_k are calculated.

Step 2. An intermediate mesh, with target error index $\eta_I = \gamma\eta_T(\gamma \approx .10)$ is generated by purely h-refinements keeping the initial p-distributions fixed, but choosing the intermediate mesh sizes h_I to roughly equidistribute the error, and then,

Step 3. keeping the intermediate h_I-distribution fixed, compute a non-uniform distribution of orders p_k needed to produce the target error η_T.

This process can be remarkably accurate for two-dimensional linear elliptic problems and computed target errors often agree closely with those specified. If there is significant deviation, the process is repeated. Parallel versions have been developed and implemented successfully on a 32-processor Intel iPSC/860. For hyperbolic problems, a 4-step variation of the scheme has also been implemented [10].

3 DOMAIN DECOMPOSITION AND PRECONDITIONING OF hp-MESHES

We wish to develop a program of domain decomposition and a preconditioning scheme in preparation for iterative solvers, that controls the condition number for hp-schemes while balancing the computational load in a multiprocessor environment.

In each step of the 3-step adaptive strategy, and particularly the final step, a non-uniform hp-mesh Ω_{hp} is obtained over which a discrete solution u^{hp} is sought. For each such mesh, we seek to partition Ω_{hp} into P-subdomains Ω_i , P being the number of processors, and then over each Ω_i, we wish to construct an effective preconditioner to be used, for example, in a preconditioned conjugate-gradient solver.

hp-Domain Decomposition

We have developed a new domain decomposition scheme based on what we call the RLBBO (the Recursive Load Based Balance of Ordering) for hp-meshes. The idea is based on two operations:

1) given Ω_{hp}, a Hilbert-Peano curve is generated which connects the centroids of each element into an unbroken path over the mesh in which each element is traversed only once

2) a recursive bisection scheme is then implemented on the Hilbert-Peano curve which partitions the mesh so as to equidistribute load measured by either degrees-of-freedom, estimated element error, or a comparable measure.

This process produces a well-balanced decomposition of the mesh with minimal or small interfaces, meaning low communication costs between processors.

Pre-Conditioners for hp-Meshes

Suppose Ω_i, $i = 1, 2, \cdots, N_D$, is a subdomain obtained using the preconditioning scheme described above. Let

$$H_i = \mathrm{dia}(\Omega_i) \tag{3}$$

and suppose

$$\Omega_i = \cup_{k=1}^{N_i} \omega_k^i \tag{4}$$

where ω_k^i are the elements making up Ω_i. Denote

$$h_i = \max_{1 \leq k \leq N_i} h_k^i \, , \; p_i = \min_{1 \leq k \leq N_i} p_k^i \tag{5}$$

where

$$h_k^i = \mathrm{dia}\,(\omega_k^i)\,, \; p_k^i = \text{ minimum degree of complete polynomial defined on } \omega_k^i \tag{6}$$

Then it can be proved [11] that a systematic partial orthogonalization scheme can be implemented which permits the condition number to grow no faster that

$$C(1 + log p_i)(1 + \log H_i p_i / h_i) \tag{7}$$

This is a quite reasonable control of condition numbers and leads to effective iterative methods for each subdomain. Moreover, the preconditioning process is parallelizable.

For an hp-mesh on a subdomain Ω_i such as is shown in Fig. 1, the degrees of freedom are partitioned into 5 classes: N (*nodes* on the boundary $\partial\Omega_i$ that support piecewise linear functions), S (nodes on *sides* on $\partial\Omega_i$ that support polynomials of higher degree, V (interior *vertices* that support linear or bilinear shape functions), E (interior *edges* with p-shape functions) and B (interior *bubble* shape functions

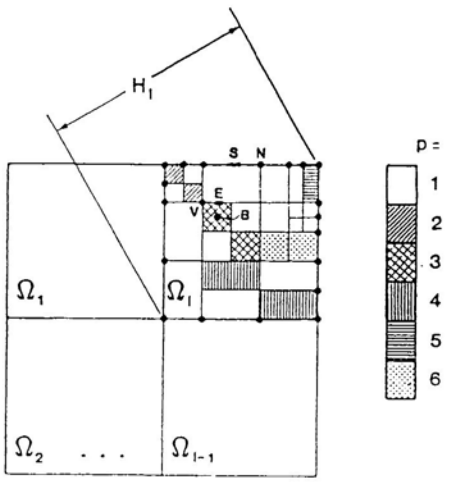

Figure 1: A load-balanced hp-mesh with non-uniform hp-distributions

that vanish on element boundaries). Symbolically, the structure of the matrix for the subdomain is of the following form:

$$K = \begin{bmatrix} NN & NS & NI \\ SN & SS & SI \\ IN & IS & II \end{bmatrix} \text{ where II } = \begin{bmatrix} VV & VE & VB \\ EV & EE & EB \\ BV & BE & BB \end{bmatrix} \qquad (8)$$

Two orthogonalization operations are done:

$$1.\ \boldsymbol{M}_1^T \boldsymbol{K} \boldsymbol{M}_1 = \widetilde{\boldsymbol{K}} = \begin{bmatrix} NN & NS & NI \\ SN & SS & SI \\ IN & IS & \widetilde{II} \end{bmatrix}, \widetilde{II} = \begin{bmatrix} \widetilde{VV} & \widetilde{VE} & 0 \\ \widetilde{EV} & \widetilde{EE} & 0 \\ 0 & 0 & BB \end{bmatrix}$$

$$2.\ \boldsymbol{M}_2^T \widetilde{\boldsymbol{K}} \boldsymbol{M}_2 = \widetilde{\widetilde{\boldsymbol{K}}} = \begin{bmatrix} \widetilde{NN} & \widetilde{NS} & 0 \\ \widetilde{SN} & \widetilde{SS} & 0 \\ 0 & 0 & \widetilde{II} \end{bmatrix}$$

Among preconditioners are the submatrix,

$$\begin{bmatrix} \widetilde{NN} & 0 & 0 \\ 0 & \text{diag}\,(\widetilde{SS}) & 0 \\ 0 & 0 & \text{diag}\,(\widetilde{II}) \end{bmatrix} \qquad (9)$$

Other choices are possible. Numerical experiments indicate that such procedures are not excessively expensive, since they are parallelizable, and that they result in well-conditioned systems.

4 BRIEF EXAMPLES

Figure 2 contains results of applying the RLBBO scheme to a complex hp-mesh generated on a non-convex domain. The method leads to a clean decomposition of the domain into four subdomains with a minimal interface, no domain splitting, and a roughly balanced load.

Figure 3 shows an adapted hp mesh for a Poisson problem on a square subdomain Ω_i with an irregular solution. The model had 568 degrees of freedom with a maximum p of $p = 4$. The preconditioned matrix had a condition number of only 3.67 (compared with 27.65 for a Jacobi preconditioner and $O(10^4)$ with no preconditioning). The PCG scheme converged in 5 iterations. Further details can be found in [11].

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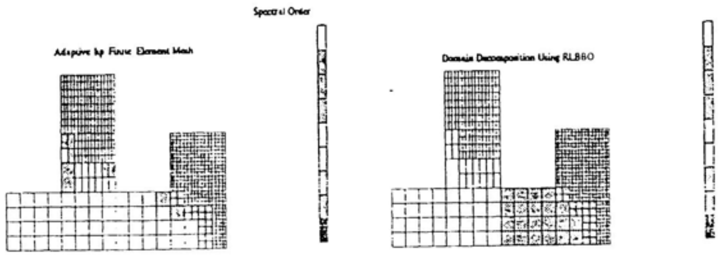

Figure 2: (a) Adaptive *hp* mesh, (b) Partitioning generated by RLBBO

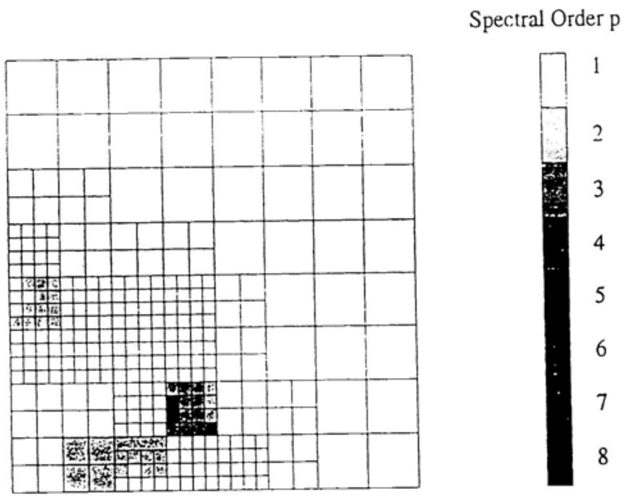

Figure 3: *hp* Adaptive Mesh (dof=568)