# An Introduction to Magnetohydrodynamics

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**Abstract**

Magnetohydrodynamics (MHD) is a field of study which combines elements of electromagnetism and fluid mechanics to describe the flow of electrically conducting fluids. It is generally regarded as a difficult academic discipline, both conceptually as well as mechanically. The purpose of this paper is to give an upbeat and friendly introduction to MHD, by first exploring the basics of electromagnetism and fluid dynamics. The importance of MHD applications as well as current research problems in the field will be also be discussed.

## 1 Introduction

Magnetohydrodynamics (MHD) is not a field of study for those with only high school equivalent mathematics and science knowledge. It takes much mathematical rigor to simply understand its governing equations. In a nutshell, MHD is the study of electrically conducting fluids, combining both principles of fluid dynamics and electromagnetism. This is where it gets tricky (the very beginning..).

The subject of MHD is traditionally studied as a continuum theory, that is to say, attempts at studying discrete particles in the flows are not at a level such that computation in these regards is realistic. To run "realistic simulations" would require computations of flows with many more particles than current computers are able to handle. Thus, the only way to study MHD seems to be in its continuum form- leading us to it's description using the Navier-Stokes fluids equations.

All the obstacles and open problems in classical fluid dynamics are adopted into MHD by definition, e.g.- classical turbulence. Do vortices and other such phenomena exist in MHD? Well, we'll get to those ideas later. Right now I think it's best to become familiar with some basic applications of MHD to answer the question, "Why on earth would anyone want to study this?!" before diving headfirst into the mathematics.

To prelude some applications of MHD, the first historically documented MHD experiment was performed by none other than Michael Faraday in 1832. A quick aside about Faraday- he had no formal mathematical background but was able to perform experiments whose results can regarded as the most appealing and beautiful results in classical electrodynamics (See Section(2.2.3)). In terms of his MHD work, knowing that salt water was a conductor and that flowing salt water is constantly interacting with the Earth's magnetic field, he tried to measure the voltage difference between banks in the Thames river. Unfortunately his equipment was not able to pickup the potential difference, due to the inefficiencies in equipment in the 1830s, but his prediction about an induced current was later confirmed by Dr. William Wollaston, who measured the voltage difference across the English Channel in 1851. [3]

You might ask if MHD has any actually relevance to everyday life. All you have to do it look up and look at the sun (but don't stare at it). Sun spots and solar flares are believed to have magnetic origins. Solar flare research is an active area of research these days, being that they affect all electronics here on Earth. If a large enough one does shoot off the sun, it's possible that it could disrupt (or in a very cliche' Hollywood sense destroy) all telecommunications, computers, and electronic devices on Earth, which could send us back to the stone age (or the late 1800s as it were).

If you manage to look past the sun, this time through the eyes of a telescope you might be looking in the direction of another active area in MHD research. This application being the accretion disks surrounding black holes and neutron stars, which are plasmas, or even interstellar space where magnetic fields have influence over star and galaxy formation.

On your way home from that observatory you were just at, you might get lost. If you're feeling adventurous you might try to get home using a good ole fashioned compass. How does a compass work? It utilizes the Earth's magnetic field to point north. Being overly ambitious and curious you might want to discover why the Earth has this "intrinsic" magnetic field at all. The answer- it's due to the fluid motion in the core, called *dynamo action*. This of course is another modeling job for MHD. [1],[3]

However, perhaps the most significant active area of research using MHD theory is in the area of nuclear fusion. It is not a secret that oil prices have increased and that using it as an energy source seems to be harmful to the environment. Furthermore there is much emphasis on finding alternative energy sources. Nuclear fusion is optimistically one potential candidate for cheap renewable energy in the future. The problem lies in the confinement of nuclear fusion. When you have plasmas of temperatures around $10^8$K floating around, confinement becomes an obvious problem. One model of confinement uses magnetic forces to keep the plasma away from reactor walls. Basically the outer confinement layer is a solenoid that passes a current through it, in which induces a current on the surface of the plasma. The key idea is that this induced current exhibits a Lorentz magnetic force pinching inward on the plasma, keeping it contained. The main problem with this approach of confinement is that hydrodynamic instabilities arise. People are working on ways to remedy such instabilities.

This paper is broken down into 4 sections. The second section will introduce the governing equations of electromagnetism, the Maxwell equations, and the third section is an overview of fluid dynamics and the governing fluids equations, the Navier-Stokes equations. Section 4 will discuss the coupling of the Maxwell equations and the Navier-Stokes equations to give the basic governing equations of MHD.

# 2 Maxwell's Equations

The entire story of electricity and magnetism begins with a single entity of charge, q. Call this charge a source charge. By placing another charge in the universe, call it the test charge Q, the fundamental question becomes,"what force does q exert on Q?" In general, both charges could be in motion.

It would seem that for such a basic two body system, the governing equations should be simple. However, the force on Q can depend on the velocity, acceleration, and of course the separation distance between the charges. We will see later that electromagnetic information travels at the speed of light, that is to say, information is not *not* instantaneous. What concerns Q will then be the velocity, position, and acceleration of q at some earlier time, when the information was sent.

To remedy jumping too fast into the deep end of electromagnetism and drowning, our story begins with electrostatics, where all source charges are stationary, while the test charges may be in motion.

## 2.1 The Electric Field

Recall we assumed that the source charge q is pinned down. Now consider a test charge, which is at rest, a distance r away. Coulomb's law gives the force on q due Q,

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\,\frac{qQ}{r^2}\,\hat{r}. \tag{1}$$

We see that the force is inversely proportional to the square of the distance between the charges, and proportional to the product of the charges. Note that $\epsilon_0$ is called the *permitivity of free space* and is a natural constant, having the value of (in SI units),

$$\epsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{N \cdot m^2}. \tag{2}$$

We see from (1) that the force points directly from q to Q. If the product of the charges is positive, it should be understood that the force between the charges is positive and hence the force is repulsive. On the other hand, if the product is negative, then we see the force is attractive. This is a difference between the Newtonian Gravitational Force and the Coulomb Force.

What if we were to dump a bunch of other source charges into this system, with only the single test charge? From experimental observation, it's been seen that Coulomb's Law obeys the principle of superposition. We suppose $\{q_i\}_{i=1}^n$ are the source charges with separation distances from the test charge located at point P of $\{r_i\}_{i=1}^n$, respectively. The principle of superposition gives the total force $\mathbf{F}$ on the test charge as

$$\begin{aligned} \mathbf{F} &= \mathbf{F_1} + \mathbf{F_2} + \ldots + \mathbf{F_n} \\ &= \frac{Q}{4\pi\epsilon_0} \left[ \frac{q_1 \hat{r_1}}{r_1^2} + \frac{q_2 \hat{r_2}}{r_2^2} + \ldots + \frac{q_n \hat{r_n}}{r_n^2} \right]. \end{aligned}$$

We now define the electric field, $\mathbf{E}$, of the source charges to be

$$\mathbf{F} = Q\mathbf{E}, \tag{3}$$

where we call

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^{n} \frac{q_j \hat{r_j}}{r_j^2}. \tag{4}$$

The easiest way to think of the electric field is that the it is the force per unit charge that would be exerted on a test charge if it was at that particular point in space. We will see that the electric field will be a much more useful mathematical concept than always trying to compute the Coulomb forces.

However, this story about the electric field and Coulomb force is cute and all, but in the real world it is not very likely a scientist will ever be able to count a discrete number of point charges. This is where we begin to consider continuous charge distributions, or by saying the charge fills a volume and we get a charge density, $\rho$. Very naturally, our definition of electric field in (4) can be extended into this type of scenario by an integral,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r'})}{r^2} d\tau'. \tag{5}$$

### 2.1.1 The Divergence of E and Gauss's Law

Consider once again, a single source charge, q, at the origin. We wish to find the flux of its corresponding electric field through a "Gaussian" surface, which we take to be a sphere of radius r. We see that

$$\oint \mathbf{E} \cdot d\mathbf{a} = \int \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r^2}\right) \cdot r^2 \sin\theta \; d\theta \; d\phi \; \hat{r} = \frac{q}{\epsilon_0}. \tag{6}$$

What's important to see is that the integral didn't have to be over a sphere- any old surface would have given the same result! Physically, this is understood that any closed surface would have trapped the same amount of electric field lines. Through this simple calculation we have stumbled upon a profound result- the flux through any surface that encloses charge is $\frac{q}{\epsilon_0}$. We can now extend this result to a system that has n source charges scattered about using the principle of superposition. We see the flux that encloses them all will be:

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \sum_{i=1}^{n} \left[\oint_S \mathbf{E_i} \cdot d\mathbf{a}\right] = \sum_{i=1}^{n} \left(\frac{q_i}{\epsilon_0}\right) = \frac{Q_{enc}}{\epsilon_0}. \tag{7}$$

The above equation is usually referred to as the Integral version of *Gauss's Law*, where $Q_{enc}$ denotes the total charge enclosed within the surface. Using the Divergence theorem, we can easily transform this integral equation into a differential one. First we use the divergence theorem to get:

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \int_V \nabla \cdot \mathbf{E} d\tau. \tag{8}$$

Now using our good friend, the charge density $\rho$, we see that

$$Q_{enc} = \int_V \rho \; d\tau. \tag{9}$$

Hence it is seen that by equating (8) and (9) we get the Differential Version of Gauss's Law,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}. \tag{10}$$

However, the differential version of Gauss's Law is not very useful in its own right. It will give us a partial differential equation to solve. Even in the most simple problems, ones where we can solve using symmetry in the integral version, may not be possible to perform by hand using the differential form of Gauss's Law.

### 2.1.2 The Curl of E

Returning to our example of a single source charge at the origin, we can see physically that the curl of its electric field must be zero. Why? Well simply, the electric field of this configuration only admits a radial component. It has no angular components. But as scientists, we cannot assume this is true from just this one toy example. Once again, consider a bunch of source charges scattered about. We see that from the principle of superposition we have

$$\nabla \times \mathbf{E} = \nabla \times \left( \sum_{i=1}^{n} \mathbf{E_i} \right) = \sum_{i=1}^{n} \nabla \times \mathbf{E_i}. \tag{11}$$

Now we see that the result of our toy scenario from above is actually quite profound. It is from this and the principle of superposition that we get the result

$$\nabla \times \mathbf{E} = 0, \tag{12}$$

since we can look at the single source charge system and do that n-times. This result says that for a static charge distribution, the electric field if *irrotational*.

## 2.2 The Magnetic Field

We begin our journey into magnetism with the study of *magnetostatics*. This term may seem vauge and hypocritical, since magnetic forces inherently depend on the motion of charges. That is to say, when charges are not pinned down but are moving. In magnetostatics we assume all currents are steady. This in turn will make magnetic fields be constant in time, just like how in electrostatics the electric fields produced were time independent. Before we jump into the idea of currents, we consider the idea of moving charges. From experimental observation is has been found that the foundational force relation in magnetism is

$$\mathbf{F}_{mag} = Q(\mathbf{v} \times \mathbf{B}). \tag{13}$$

This law describes the force on a test charge Q, when it is moving with velocity $\mathbf{v}$ in magnetic field $\mathbf{B}$. One interesting tidbit regarding (13) is that we see *magnetic forces do no work.* We can prove this by assuming that a charge Q moves an amount $d\mathbf{l} = \mathbf{v}dt$, then we see

$$dW_{mag} = \mathbf{F}_{mag} \cdot d\mathbf{l} = Q(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}dt = 0,$$

since $\mathbf{v} \times \mathbf{B}$ is perpendicular to $\mathbf{v}$. This says that although magnetic forces can change the direction in which charges move, but they cannot alter its velocity whatsoever. Furthermore if that test charge is in the presence of both a magnetic field $\mathbf{B}$ and electric field $\mathbf{E}$, we have the following force law

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{14}$$

This is known as the *Lorentz Force Law.* Again, it is a fundamental axiom of electricity and magnetism so the only justification for such a law is through experimental verification. However, the story is nowhere near finished. To keep the ball rolling, we now introduce the idea of a current. A current is the flow of charge per unit time in a wire passing by at a specific point. We can rewrite the magnetic force law, using currents, as

$$\mathbf{F}_{mag} = \int (\mathbf{v} \times \mathbf{B}) dq = \int (\mathbf{I} \times \mathbf{B}) dl, \tag{15}$$

but for steady currents in which the magnitude of $\mathbf{I}$ doesn't change from point to point we see the above becomes

$$\mathbf{F}_{mag} = I \int (d\mathbf{l} \times \mathbf{B}). \tag{16}$$

Furthermore a lot of times we will be concerned with the total charge flowing through a volume. We define the volume current density to be $\mathbf{J}$. We then see that $\mathbf{J} = \frac{d\mathbf{I}}{da_\perp}$. Hence we find that $\mathbf{J} = \rho\mathbf{v}$, where $\rho$ is the mobile charge density and $\mathbf{v}$ is the velocity. We then see the magnetic force can be written as

$$\mathbf{F}_{mag} = \int (d\mathbf{v} \times \mathbf{B}) \ \rho \ d\tau = \int (d\mathbf{J} \times \mathbf{B}) d\tau. \tag{17}$$

We will see later that (17) is the force law we will use when coupling Maxwell's E&M equations with the Navier-Stokes Fluid equations in forming the equations of MHD.

### 2.2.1 The Continuity Equation for Charge

We are concerned with deriving a conservation equation for charge. For this short aside we are no longer assuming magnetostatics, rather the situation where charges are able to bunch up, or in the very least are time dependent. The basic premise is the "what goes out must come in" idea. We see that the total charge leaving a volume V can be written as a surface integral as the current density **J** flowing out of the surface,

$$\oint_S \mathbf{J} \cdot d\mathbf{a} = \int_V (\nabla \cdot \mathbf{J}) d\tau = -\frac{d}{dt}\int_V \rho d\tau = -\int_V \left(\frac{\partial \rho}{\partial t}\right) d\tau.$$

The negative sign is due to the fact that whatever flows out of the surface is coming from what is inside. Now we have the charge continuity equation,

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}. \tag{18}$$

Note that for magnetostatics, we have $\frac{\partial \rho}{\partial t} = 0$ since the currents are continuous and time independent. Therefore we get that $\nabla \cdot \mathbf{J} = 0$ in magnetostatics.

### 2.2.2 The Curl of B and Ampere's Law

Now that we've described how to calculate the force from a magnetic field, as scientists it is always imperative to find out how to compute such quantities. Unfortunately, there is no straight-forward "$8^{th}$ grade" method to calculate **B**. Thankfully French scientists Jean-Baptiste Biot and Felix Savart derived a fundamental law of physics, now known as the *Biot-Savart law*, out of their experimental observations of the magnetic field arising from a steady current in a wire.

This law can be written in its intimidating integral form as

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\mathbf{l}' \times \mathbf{r}'}{r^2}, \tag{19}$$

where $\mu_0$ is a constant called the *permeability of free space* with value in SI units of $4\pi \times 10^{-7} N/A^2$. This constant describes the material's ability to more or less contain a magnetic field. More or less you can think of it as the measure of how the magnetization the material will hold when an external magnetic field is applied to it. We also note that although this force law looks hairier than other force laws in physics, it shares the common feature that it holds an inverse square relation, just like Coulomb's law or Newton's gravity law.

Perhaps the most famous magnetic field is the one calculated from a steady current running through an infinite straight wire. The result can be found to be

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}. \tag{20}$$

This says that the magnetic field arising from such a situation will be concentric circles of radius wire, where r is the radius away from the wire. Hence as r increases, we see that B will decrease in magnitude. (Likewise, B is much stronger when we are calculated the field 'close' to the wire.) This simple, yet profound, result says something about the magnetic field- *it has a non-zero curl!*. This is completely different that the electric field in electrostatics, which always has a vanishing curl.

We will now give a brief derivation of Ampere's law. Using (20) we see we can use the result to obtain,

$$\oint \mathbf{B}\cdot d\mathbf{l} = \frac{\mu_0 I}{2\pi r}\oint dl = \mu_0 I.$$

We find that the answer is independent of r! This enlightens us to think that as the circumference of the path increases, B will decreases at the same rate. So we did not have to integrate over a circular path around the wire, but any closed loop would have given the same result. Taking this idea one step further, we suppose the situation where we have a bundle of straight wires. Next we just draw a loop around our favorite wires, to note that each wire passing through the loop will contribute $\mu_0 I$, while those outside cannot contribute anything. We then get

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{enc} = \int_S \mathbf{J}\cdot d\mathbf{A}, \tag{21}$$

where $I_{enc}$ is the total current enclosed within our loop. Using Stoke's theorem, we find

$$\oint \mathbf{B}\cdot d\mathbf{l} = \int_S (\nabla\times\mathbf{B})\cdot d\mathbf{A}.$$

Now combining the last two equations will give us the integral equation,

$$\int (\nabla\times\mathbf{B})\cdot d\mathbf{A} = \mu_0\int_S \mathbf{J}\cdot d\mathbf{A}, \tag{22}$$

which we can trivially declare that

$$(\nabla\times\mathbf{B}) = \mu_0\mathbf{J}. \tag{23}$$

This is known as Ampere's law. Note that although we have restricted this derivation to straight-line currents, it turns out this result holds for any steady-current configuration. Just like Gauss's law in electrostatics, we find the integral version of Ampere's Law is easier to work with under certain symmetries. Who wants to solve all the coupled PDEs arising from (23) anyway, for what could be done in a few steps using symmetrical arguments!?

#### 2.2.3 Faraday's law and the Electromotive Force

We've talked about wires having currents and magnetic fields arising from them, but we have not talked about what happens when an external field is placed onto the wires. Imagine we have a loop of wire that encloses upon itself that is hooked up to a battery. We then have two 'forces' responsible for driving the current in the ciruit- the force per charge from the electric field, $\mathbf{E}$, and the force per charge from the source, $\mathbf{f}_s$. Here we can think of it as a battery, thermocouple, or etc. We now define a quantity called the electromotive force, or EMF, of the circuit to be

$$\mathcal{E} \equiv \oint (\mathbf{E} + \mathbf{f}_s) \cdot d\mathbf{l} = \oint \mathbf{f}_s \cdot d\mathbf{l}. \tag{24}$$

You can think of the EMF as the work done per unit charge. Now what happens if this same loop of wire is moving through a steady magnetic field? In 1831 the scientist Michael Faraday set up and performed various experiments to understand this situation. [5]

#### 2.2.4 Faraday's First Experiment

In Faraday's first experiment, he pulled a loop of wire to the right, through a constant magnetic field that was pointing down into the paper. What he noticed was that a current seemed to flow in the loop! Was this the EMF?

Suppose we have a loop of wire, that is of fixed height h, where only x of its total length is immersed in the magnetic field pointing straight into the table. Suppose we pull the loop of wire to the right with a speed v. We can calculated the Lorentz force as

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} = qvB\hat{k}.$$

Hence we find the EMF is

$$\mathcal{E} = \oint \mathbf{f}_{mag} \cdot d\mathbf{l} = \int_0^h \frac{qvB}{q} \hat{k} \cdot dh \ \hat{k} = vhB. \tag{25}$$

We now construct an alternative method to calculating the EMF passed through this loop from pulling it to the right. We define a quantity called the flux of $\mathbf{B}$ through the loop, $\Phi$,

$$\Phi \equiv \int \mathbf{B} \cdot d\mathbf{A}. \tag{26}$$

Now going back to our example from above we find that, $\Phi = Bhx$, since initially only a rectangle that is $h$ by $x$ is immersed in $\mathbf{B}$. Moving that wire loop to the right implies that

$$\frac{d\Phi}{dt} = Bh\frac{dx}{dt} = -Bhv,$$

since we assume that $\mathbf{B}$ is a steady current.

Now putting (25) and the above equation together we find that

$$\mathcal{E} = -\frac{d\Phi}{dt}. \tag{27}$$

Once again, this definition of $\mathcal{E}$ works for any loop moving in any direction. It is not restricted to only rectangular loops.

So what Faraday found in his first experiment was in fact a confirmation of this flux rule for motional EMFs.

#### 2.2.5 Faraday's Second Experiment

Essentially he must have been curious after the results of his first experiment and asked himself, "Well what happens in the loop is fixed in place, but I move the magnetic field?" His experiment involved moving the magnet to the left while leaving the loop stationary. He found that still a current flowed through the wire loop!

Recall from his earlier experiment he found that the loop moving is what causes the magnetic force that induces an EMF. But since the loop is stationary, there cannot be a magnetic field producing the EMF. He knew that electric fields exert force on other charges at rest. Hence Faraday ingeniously postulated that *changing magnetic fields induce an electric field.*

He then assumed that the EMF must be equal to the flux, and concluded that

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A}. \tag{28}$$

This is now known as Faraday's Law in its integral form. Using Stoke's Theorem we can recover the law into its differential form by noting that

$$\oint \mathbf{E} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{E}.$$

Hence Faraday's Law in its differential form is

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{29}$$

Note that in the static case, that is to say when $\mathbf{B}$ is constant, it reduces to $\nabla \times \mathbf{E} = 0$. Going further, we now see that there are two kinds of electric fields- those coming from static charges and those fields attributed to changing magnetic fields.

### 2.2.6 Faraday's Third Experiment

The only experiment left with this configuration of a loop of wire and an applied magnetic field from a magnet was to keep the loop and magnet both fixed in place and change increase the strength of the magnetic field. He was able to do this by using an electromagnet and increasing the current through it.

Guess what? He found that once again a current flowed through the loop of wire. However, this is not unexpected since he simply changed the flux of the magnetic field through the loop. Once again it induces an EMF of

$$\mathcal{E} = -\frac{d\Phi}{dt},$$

in the loop.

At this point there is not much left to say about Faraday and the induced fields; however, there is a law of nature that says *nature abhors a change in flux.* This is known as Lenz's Law. Basically it is intended to help sort out any sign ambiguity that might arise from Faraday's law. It pretty much says that a loop wants to maintain a constant flux through it and when you try to change that flux, the loop responds by introducing a current in such a direction to resist the change. Lenz's law unfortunately can tell you nothing more than the direction of the flow.

#### 2.2.7 The divergence of B

We now consider the divergence of **B**. There are two ways to go about this. We can calculate it directly using the Biot-Savart formulation, or we can use Faraday's Law. We will derive the result using Faraday's law.

Recall that the divergence of any curl is equal to zero, ie- for a vector field **F**,

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

This is a well known vector calculus identity. So applying the divergence operator to (29), we see

$$\nabla \cdot (\nabla \times \mathbf{E}) = 0,$$

and hence

$$\nabla \cdot \frac{\partial \mathbf{B}}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}) = 0.$$

We have just unveiled the fundamental result that the magnetic field **B** is solenoidal. That is to say,

$$\nabla \cdot \mathbf{B} = 0. \tag{30}$$

This is seemingly innocent fact about the magnetic field that deserves more careful attention. Unlike the electric field, which has a curl of zero in the electrostatics case, we find that the magnetic field always has a divergence of zero. This tells us that all the magnetic field lines close upon another, which we can then infer to mean that there do not exist any magnetic monopoles! In electrostatics we can talk about single positive charges or single negative charges, but in magnetostatics, we cannot talk about any magnetic monopoles. For example, if you break a bar magnet in half, you do not get half that is the 'north' side and half that is the 'south' side. Instead you will see that you will get two separate magnets each with a north and south side. So far magnetic monopoles are not believed to exist...although certain particle physicists are praying that one day repeated experimental observations can confirm their existence, but that's another story.

## 2.3 Maxwell's Equations

Electrodynamics before Maxwell can be summarized into four fundamental laws,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \text{Gauss's Law,} \tag{31}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{32}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{Faraday's Law,} \tag{33}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \qquad \text{Ampere's Law.} \tag{34}$$

However when Maxwell was putting all the equations together and studying them, he found one fatal inconsistency- that was dealing with Ampere's Law. We note that when we took the divergence of Faraday's Law, we then in turn uncovered that the divergence of a magnetic field was zero. This makes sense experimentally since no one has observed magnetic monopoles. Now let's see what happens when we take the divergence of Ampere's Law,

$$\nabla \cdot (\nabla \times \mathbf{B}) = \nabla \cdot (\mu_0 \mathbf{J}) = \mu_0 (\nabla \cdot \mathbf{J}).$$

We note that from basic vector calculus identities that $\nabla \cdot (\nabla \times \mathbf{B}) = 0$; however, there is no reason that we would have $(\nabla \cdot \mathbf{J}) = 0$ in general. Sure, for steady currents we find that Ampere's Law is okay in its current form, but what about non-stead currents? Or dynamics beyond magnetostatics?

Maxwell's insight came next when we looked about the divergence of the current density and looked at it through Gauss's Law,

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\epsilon_0 \nabla \cdot \mathbf{E}).$$

We can then write this as

$$\frac{\partial}{\partial t}(\epsilon_0 \nabla \cdot \mathbf{E}) = \nabla \cdot \left( \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right).$$

Now let's see what happens if we combine the last term on the right in the above equations with Ampere's Law as is stands, and then take the divergence the whole quantity,

$$\nabla \cdot (\nabla \times \mathbf{B}) = \nabla \cdot (\mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}) = \mu_0 \epsilon_0 (\nabla \cdot \mathbf{J} + \nabla \cdot \left( \frac{\partial \mathbf{E}}{\partial t} ) \right) = 0.$$

Hence we found the term that will kill off the extra divergence in Ampere's Law.

Maxwell's contribution was not just simple mathematical trick to wipe away an ambiguity in the electrodynamics equations. Furthermore it introduced a new fundamental physics concept- *a changing electric field induces a magnetic field.* Just like how a changing magnetic field induces an electric field, the converse is also true. If this isn't the epitome of beautiful symmetry in the universe, this author does not know what else it could be!

Using Maxwell's contribution to Ampere's Law, we can now summarize the notorious equations of electromagnetism, known as the Maxwell Equations, as follows,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad \text{Gauss's Law,} \tag{35}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{36}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{Faraday's Law,} \tag{37}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad \text{Ampere's Law.} \tag{38}$$

## 2.4 A Quick Aside- Electromagnetic Waves

We now consider the Maxwell's equations written in free-space, that is to say there are no points charges or currents, and so we are left with the equations,

$$\nabla \cdot \mathbf{E} = 0 \quad \text{Gauss's Law,} \tag{39}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{40}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{Faraday's Law,} \tag{41}$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad \text{Ampere's Law.} \tag{42}$$

Next we will derive the wave equations of electricity and magnetism in their usual differential form. First we begin by taking the curl of Faraday's Law,

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{E}) &= \nabla \times \left( \frac{\partial \mathbf{B}}{\partial t} \right) \\ \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} &= -\frac{\partial}{\partial t} (\nabla \times \mathbf{B}), \end{aligned}$$

since we note that for any vectors $\mathbf{A}$,$\mathbf{B}$, and $\mathbf{C}$ the triple cross product is $\mathbf{A} \times \mathbf{B} \times \mathbf{C} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$. Here we would consider $\mathbf{A} = \mathbf{B} = \nabla$ and $\mathbf{C} = \mathbf{E}$. We can further simplify the above equation by invoking

Ampere's Law as well as Gauss's Law in free space to obtain,

$$-\nabla^2 \mathbf{E} = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

Hence we find the result that

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}, \tag{43}$$

which is the usual form of a wave equation.

We can do a similar calculation for **B**, by starting with Ampere's Law and taking the curl of it to get:

$$\begin{aligned}
\nabla \times (\nabla \times \mathbf{B}) &= \nabla \times \left( \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \\
\nabla (\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} &= -\mu_0 \epsilon_0 \frac{\partial}{\partial t} (\nabla \times \mathbf{B}), \\
-\nabla^2 \mathbf{B} &= -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}.
\end{aligned}$$

Hence we find that the wave equation for **B** exactly parallels that of the equation for **E**,

$$\nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}. \tag{44}$$

Now we note that the wave speed is equal to

$$\frac{1}{\sqrt{\mu_0 \epsilon_0}} = \frac{1}{\sqrt{(4\pi \times 10^{-7})(8.85 \times 10^{-12}) s/m}} \approx 2.99863 \times 10^8 m/s.$$

This tells us the waves produced by electromagnetic waves travel at the speed of light. This leads us to discover their identity as light waves, which we all have some familiarity from day to day life.

# 3 Navier-Stokes Equations

The story of the motion of fluids has no clear place beginning. Let's first set out with the initial goal of defining what we mean by a fluid. As compared to solids, a fluid cannot maintain its shape under any amount of shear stress for any amount of time. That is to say, forces parallel to the surface of the fluid will affect its overall shape and behavior. The way to think about a fluid is as a continuum of particles, whether they are gas particles, liquid particles, or even plasma elements. One then tries to understand the overall flow of the continuum rather than the single flows of any one particle. Having just studied electromagnetism, you could think of a fluid like a current, in that you do not study a current with discrete moving charges but rather as a whole.

Therefore in asking oneself, what it means to have a single quantity of fluid is a vague and esoterically silly question. Instead let our adventure into the world of fluid dynamics begins with the next closest idea- a single cubic blob of fluid. We first wish to describe the forces that can act upon this fluid blob. To simplify this even more, let's assume the fluid is incompressible. This allows us to believe that the fluid has a constant density. The incompressible condition states that

$$\nabla \cdot \mathbf{v} = 0, \tag{45}$$

where $\mathbf{v}$ is the velocity of the fluid. The mass of the fluid blob is the density of the fluid, $\rho$, multiplied by the volume of the blob, $\delta V$. Hence the mass is $\rho(\delta V)$.

Now what forces are acting on the blob? Obviously, one thinks that there is a force term due to the pressure surrounding the blob. This turns out to be correct. What about some of the other forces? Well there could be externals forces that act at a distance like gravity or electromagnetism. There could also be terms due to the shear-stress at the surface, or viscosity forces. Let's first take a peek at the force due to the pressure.

Recall that pressure $P = \frac{force}{area}$. So to get the force, we need to multiply the pressure by the area. Now consider the pressure on the left wall of the blob at x. We find it to be $pdydz$. The pressure on the face at an infinitesimal distance $\Delta x$ further we get $-\left(p + \frac{\partial p}{\partial x}dx\right)dydz$. Hence the net pressure in the x direction is:

$$p_{net_x} = pdydz - \left(p + \frac{\partial p}{\partial x}dx\right)dydz = -\frac{\partial p}{\partial x}dydz.$$

Hence looking at the pressure on the remaining faces of the cube lead us to get that the force due to pressure per unit volume is $-\nabla p$. Now suppose there are some other forces acting on the cubic blob, which are distance dependent forces. Whatever these forces may be, let's define the potential per unit mass that will generalize for all of them, call is $\phi$. (Here we assuming under the rug that these forces are conservative). From mechanics we note that from a potential, we can define a force as $F = -\nabla\phi$. Also, let's call any force per unit mass coming from viscosity, or shear-force, or even external forces as $\mathbf{f}_{ext}$.

Therefore the from Newton's $2^{nd}$ Law, we find that force on the blob is

$$(\rho\delta V)\ \frac{D\mathbf{v}}{Dt} = -\nabla p\ (\delta V) - \nabla\phi\ (\rho\delta V) + \mathbf{f}_{ext}\ (\rho\delta V). \tag{46}$$

Dividing by the volume, $\delta V$, and constant density, $\rho$, we get that the acceleration on the cubic blob will be

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla p - \nabla\phi\ + \mathbf{f}_{ext}. \tag{47}$$

At this point, you might be scratching your head and wondering why we are writing the acceleration operator as $\frac{D}{Dt}$ rather than the traditional $\frac{d}{dt}$. Typo? Nope! This is one of the very subtle, but fundamental points in fluid mechanics. We found the acceleration for this fluid blob at one particular point in time, not the entire fluid itself. We have no way of knowing what the acceleration of the fluid is anywhere else besides at this particular point in time. Luckily we can easily remedy this.

Note that if the velocity of the blob is $v(x, y, z, t)$, the velocity of the same blob at a time $\Delta t$ later will be $v(x + \Delta x, y + \Delta y, z + \Delta z, t + \Delta t)$, where

$$\Delta x = v_x \Delta t\ ,\quad \Delta y = v_y \Delta t\ ,\quad \Delta z = v_z \Delta t.$$

Hence to first order we get the following approximation,

$$v(x + \Delta x, y + \Delta y, z + \Delta z, t + \Delta t) = v(x, y, z, t) + \frac{\partial \mathbf{v}}{\partial x} v_x \Delta t + \frac{\partial \mathbf{v}}{\partial y} v_y \Delta t + \frac{\partial \mathbf{v}}{\partial z} v_z \Delta t + \frac{\partial \mathbf{v}}{\partial t} \Delta t.$$

Therefore from the above we find the acceleration to be

$$\frac{D\mathbf{v}}{Dt} = \frac{\Delta \mathbf{v}}{\Delta t} = \frac{v(x + \Delta x, y + \Delta y, z + \Delta z, t + \Delta t) - v(x, y, z, t)}{\Delta t} = \frac{\partial \mathbf{v}}{\partial x} v_x + \frac{\partial \mathbf{v}}{\partial y} v_y + \frac{\partial \mathbf{v}}{\partial z} v_z + \frac{\partial \mathbf{v}}{\partial t}.$$

Because this is exactly the definition of partial derivative from elementary Calculus, pushing this into a more sophisticated mathematical language we find that

$$\frac{D\mathbf{v}}{Dt} = (\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}. \tag{48}$$

Traditionally the operator $\frac{D}{Dt}$ is called the *material derivative*. The material derivative in our case describes the evolution of a particular fluid blob as it moves along a certain trajectory as it flows alongside the rest of the fluid. To describe the overall velocity field of the fluid, we need (48) to connect the evolution of our particular fluid blob to the overall evolution of the entire fluid.

Therefore we can finally put together a skeletal version of the Navier-Stokes equations,

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p - \nabla \phi \ + \mathbf{f}_{ext}. \tag{49}$$

As a quick aside, it is interesting to note that the fluid may experience an acceleration even under constant velocity. We consider the case when $\frac{\partial \mathbf{v}}{\partial t} = 0$. Hence the only term left from the Material Derivative is $(\mathbf{v} \cdot \nabla)\mathbf{v}$. It is not obvious how this term describes the acceleration of fluid, but imagine what would be one of the fascinating flows to witness- water flowing in a circle with constant velocity. Even though the velocity at a given point in the flow is constant, the velocity of a particular blob in the flow points in a different direction any short time later. This of course can be summarized in introductory physics as centripetal acceleration.

Next we will dive into some conservation laws in incompressible fluid mechanics and witness how the universe prefers recycling certain quantities.

## 3.1 Conservation of Mass

First, let's throw aside all of the fluid dynamics we have been looking at. Now if you're a human, you've probably dreamt about being a fireman at some point in your life, if not, you know how much fun it is to spray stuff with a hose. Think about if you just take a hose, turn it on and decide to spray your driveway with it. You begin to think about the conservation of mass because, hey you're a scientist after-all. You know generally this type of conservation of mass can be summed up as, "what goes out, must come in." Hence the fluid coming out of the end of the hose can be written as

$$\int_S (\rho \mathbf{v}) \cdot d\mathbf{A} = \int_V (\nabla \cdot (\rho \mathbf{v})) \ d\tau,$$

using the Divergence Theorem. We then note that this flow of the fluid can cause a change in density, so we have,

$$\frac{\partial}{\partial t} \int_V \rho d\tau = \int_V \frac{\partial \rho}{\partial t} \ d\tau.$$

Hence putting these together, we find the continuity equation for mass in fluid dynamics can be written as,

$$\int_V (\nabla \cdot (\rho \mathbf{v}))\ d\tau = -\int_V \frac{\partial \rho}{\partial t}\ d\tau,$$

or more explicitly as

$$\nabla \cdot \rho \mathbf{v} = -\frac{\partial \rho}{\partial t}. \tag{50}$$

We note that this continuity has a similar form to the continuity equation for charge in electromagnetism. Also, recall since our discussion of fluid dynamics has focused on incompressible fluids, we have $\rho$ is a constant, therefore we find for incompressible fluids,

$$\nabla \cdot \mathbf{v} = 0,$$

which was the incompressibility equation we mentioned earlier.

## 3.2 Vorticity and the Conservation of Angular Momentum

Let's first recall our friend, the skeletal Navier-Stokes equation: $\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p - \nabla \phi\ + \mathbf{f}_{ext}$.

For our considerations, suppose that there are no external forces acting on the fluid, so $\mathbf{f}_{ext} = 0$. Now on the back of everyone's mind are vector calculus identities, so naturally the next thing to do is to use the identity that for any vector field,

$$(\mathbf{F} \cdot \nabla)\mathbf{F} = (\nabla \times \mathbf{F}) \times \mathbf{F} + \frac{1}{2}\nabla(\mathbf{F} \cdot \mathbf{F}).$$

Using this result into the Navier-Stokes equation gives us,

$$\frac{\partial \mathbf{v}}{\partial t} + (\nabla \times \mathbf{v}) \times \mathbf{v} + \frac{1}{2}\nabla(\mathbf{v} \cdot \mathbf{v}) = -\frac{1}{\rho}\nabla p - \nabla \phi.$$

We define a new quantity called the vorticity,

$$\omega = \nabla \times \mathbf{v}. \tag{51}$$

It might be tempting to say that $\omega$ describes the global rotation of the fluid, but this infact is misleading. Although many flows are characterized by local regions of intense rotation like smoke rings, whirlpools, tornadoes, or even the red spot on jupiter, some flows that have no global rotation, but do have vorticity. Usually in these cases vorticity is due to shear-stresses along the boundary. We will get there when we begin talking about shear-stresses and external forces on the fluids.

Substituting our definition of vorticity into the Navier-Stokes equation, we get

$$\frac{\partial \mathbf{v}}{\partial t} + \omega \times \mathbf{v} + \frac{1}{2}\nabla(v^2) = -\frac{1}{\rho}\nabla p - \nabla\phi.$$

Now taking the curl of the above equation we get an equation for the evolution of the vorticity,

$$\frac{\partial \omega}{\partial t} + \nabla \times (\omega \times \mathbf{v}) = 0. \tag{52}$$

We note that the pressure terms drop out because the force from pressure acts perpendicular to the surface of the fluid blobs and not parallel to it. Also note that in this case, where shear-forces and all external forces are absent, that is $\omega = 0$ everywhere at any particular point in time, that $\omega = 0$ for any time in the future. Hence we would call the fluid *irrotational* in this case, having $\omega = \nabla \times \mathbf{v} = 0$.

#### 3.2.1 The Analogy between $\omega$ and B

We note that in both the case for the magnetic field and the case of vorticity that both fields are solenoidal. That is to say, all there respective field lines will tend for closed loops. We see this from the solenoidal equation in Maxwell's equation, $\nabla \cdot \mathbf{B} = 0$. as well as by noting the divergence of vorticity, which is the divergence of the curl of a vector, is equal to zero, ie-

$$\nabla \cdot \omega = \nabla \cdot (\nabla \times \mathbf{v}) = 0.$$

We also note that from $\omega$ we can find the velocity $\mathbf{v}$ by borrowing a law from electromagnetism. That law is the Biot-Savart Law. Recall in electromagnetism we have $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$, where we can find $\mathbf{B}$ given a current density as

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}dV \times \mathbf{r}}{r^3}.$$

Likewise in fluid mechanics, we can find $\mathbf{v}$ using

$$\mathbf{v}(\mathbf{r}) = \frac{1}{4\pi}\int_V \frac{\omega dV \times \mathbf{r}}{r^3}.$$

Just as magnetic fields have magnetic field lines, we define an analogous quantity for voriticity called *vortex lines*. Vortex lines are vorticity field lines that have the same direction as $\omega$ and whose density in any region is proportional to the magnitude of vorticity. It is these vortex lines which close upon themselves just as the magnetic field lines form closed loops.

### 3.2.2 Helmholtz and Conservation of Angular Momentum

The German physicist, Hermann von Helmholtz, discovered the following statement- that vortex lines move with the fluid. This may not seem like a big deal, but it is a rather remarkable result. For example, if we somehow managed to mark a blob of fluid and trace out its flows, we would find it will always mark the new positions of vortex lines. That is to say, if we know $\mathbf{v}$ initially, we can find $\omega$ by definition, then from the original $\mathbf{v}$ you can tell where the vortex lines will be a 'short time' later. Then just using the definition of voriticity equation and the solenoidal property of voriticity, we can find a new $\mathbf{v}$. Hence if we have the flow pattern at any instant, we can characterize the flow for all times in the future. Disclaimer- this only works for inviscid flow, or flows with no viscosity. [4]

Let's talk turkey, or the conservation of angular momentum. Recall that we have briefly touched upon how there are no forces present which are tangential to the surface of our fluid blob. The pressure forces or forces like gravity all act perpendicular to the surface of a blob of fluid. Now consider our blob of fluid is in the shape of a cylinder. Because there is no viscosity or any other forces tangential present, the angular momentum of the fluid blob cannot change! We note that angular momentum for our cylinder will be given by: $(mass \times radius^2)\omega$. Hence we have

$$(M_0 r_0^2)\omega_0 = (M_t r_t^2)\omega_t, \tag{53}$$

where the subscript 0 represents the initial values and subscript $t$ denotes the values at some time $t$ later. Note that because of all our assumptions the masses must be equal, so $M_0 = M_t$.

Now let's see what Helmholtz has to say about all of this. Suppose we have a cylinder at time 0, where the vortex lines are parallel to the axis of the cylinder. At some time $t$ later, the blob of fluid will be somewhere else with the same volume, but possibly not the same dimensioned cylinder. If the diameter of the cylinder has gotten small, to keep a constant volume, we see that the length of the cylinder has increased, and vice versa. Also, if the diameter has gone down, this would mean that the flux of the vortex lines would increase since vortex lines are stuck with the material. Hence we get that magnitude of $\omega$ times the area of the top of the cylinder should remain constant,

$$\omega_0 A_0 = \omega_t A_t.$$

Now compare the above equation to the conservation of angular momentum, and note that $A \sim r^2$. Because the masses are equal, we find that Helmholtz's statement about vortex lines moving with the fluid is equivalent to the conservation of angular momentum in the absence of tangential forces.

## 3.3 Bernoulli and the Conservation of Energy

We continue our journey into incompressible fluid mechanics with the topic of the conservation of energy. Suppose we have a tube of fluid, with fluid flowing into one end of the tube, of area $A_0$, with velocity $v_0$. The density of the fluid if $\rho_0$ and we associate a possible potential energy with it $\phi_0$. The fluid then flows out the other side of the tube, of area $A_t$, with velocity $v_t$, density $\rho_t$, and potential energy $\phi_t$ some time $\Delta t$ later. From the conservation of mass, we realize that the mass that flows into the tube must equal the mass of the fluid flowing out, or

$$\delta M = \rho_0 A_0 v_0 \Delta t = \rho_t A_t v_t \Delta t,$$

since the fluid at $A_0$ travels a distance of $v_0 \Delta t$, likewise the fluid flowing out of the tube at $A_t$ flows out a distance of $v_t \Delta t$ during the interval $\Delta t$.

Therefore we find that

$$\rho_0 A_0 v_0 = \rho_t A_t v_t,$$

which when the density is constant states that the velocity varies inversely with the cross-sectional area of the tube. Now we will calculate the work done on the fluid flowing into the tube, as well as out, due to the pressure. The work done on the fluid flowing into the tube is $p_0 A_0 v_0 \Delta t$, likewise the work done on the fluid flowing out is $p_t A_t v_t \Delta t$. Recall that the work done by a force is equal to a change in energy.

The energy per unit mass of the fluid can be calculated to be

$$E = \frac{1}{2} v^2 + \phi + U,$$

where $\frac{1}{2} v^2$ is the kinetic energy per unit mass, $\phi$ is the potential energy per unit mass, and $U$ is a term describing the internal energy of the fluid (things like the thermal energy or chemical energy of the fluid). For incompressible fluids, we note that $U = 0$. Now relating the net work to a change in energy we find

$$p_0 A_0 v_0 \Delta t - p_t A_t v_t \Delta t = \delta M \Delta E,$$

so we obtain

$$\frac{p_0 A_0 v_0 \Delta t}{M_0} - \frac{p_t A_t v_t \Delta t}{M_t} = \Delta E.$$

The above relation is true only because of the conservation of mass, giving us that $\delta M = M_0 = M_t$. Substituting $\Delta E$ into the above equation, we find that

$$\frac{p_0}{\rho_0} + \frac{1}{2}v_0^2 + \phi_0 = \frac{p_t}{\rho_t} + \frac{1}{2}v_t^2 + \phi_t, \tag{54}$$

which is exactly the conservation of energy for incompressible, inviscid fluid flow. Now the $\frac{p}{\rho}$ term may look a little foreign, but it is in fact an energy per unit mass to keep consistent with other terms, ie-

$$\frac{p}{\rho} = \frac{pressure}{density} = \frac{force}{area} \times \frac{volume}{mass} = \frac{force \times distance}{mass} = \frac{work}{mass}.$$

Now where does the Dutch-Swiss mathematician Daniel Bernoulli's work come into play? First we must discuss the topic of steady-flow. You can think about steady-flow almost like constant currents in magnetostatics. The underlining idea of steady flow is that the velocity is constant, hence $\frac{\partial \mathbf{v}}{\partial t} = 0$. So similarly to how we drew constant magnetic field lines in magnetostatics, we draw constant velocity paths tangent to the velocity called streamlines. These streamlines trace out the true path of particles in the fluid. We should note that for flows where the velocity changes, there is no reason that streamlines actually represent the paths of the fluid. Our Navier-Stokes equation then looks like

$$\omega \times \mathbf{v} + \frac{1}{2}\nabla(v^2) + \frac{1}{\rho}\nabla p + \nabla\phi = 0$$

. If we take the dot product of velocity into the above equation we obtain

$$\mathbf{v} \cdot \nabla \left( \frac{p}{\rho} + \phi + \frac{1}{2}v^2 \right) = 0. \tag{55}$$

Physically we can interpret (55) as saying that for a small displacement in the direction of the streamline we get a conserved quantity. Bernoulli's theorem describes this integral of motion as,

$$\frac{p}{\rho} + \phi + \frac{1}{2}v^2 = K_{streamline}, \tag{56}$$

for some $K \in \mathbb{R}$. For each streamline, this $K_{streamline}$ will have a different value, since (56) only describes the motion along one particular streamline. It is easily seen that Bernoulli's theorem is nothing more than a statement about the conversation of energy in the flow! Although subtle and quite elementary, it is a fundamentally important fact!

Furthermore, what if the vorticity was identically zero to begin with? Well, we would not have to perform the dot product with velocity as in (55), rather we would just obtain from the Navier-Stokes equation that

$$\nabla\left(\frac{p}{\rho} + \phi + \frac{1}{2}v^2\right) = 0 \quad \Rightarrow \quad \frac{p}{\rho} + \phi + \frac{1}{2}v^2 = K_{everywhere}, \tag{57}$$

where this value of $K_{everywhere}$ is the same everywhere and not just on one particular streamline!

Unfortunately, we must now eject from the study of inviscid fluids into the realm where viscosity reigns supreme. (Well not really, but sometimes...but what viscosity does is throw some surprises and "twists" into the picture which were absent in inviscid fluids.)

## 3.4 Viscous Fluids

Upon venturing down the rabbit hole further to learn about viscous fluids, it is imperative to that get a clear idea of what viscosity is, or moreover, what shear-forces are. As a child, have you ever played a game with someone tries to push you over, but the catch is that you are only allowed to stand still and pray your feet and the ground bonded to form some kind of stable alliance? If your friend (or foe) has any common sense, they will try to push you over somewhere around the head, since that will create the more torque and instability then if they tried to push you over by the knee. Unfortunately, any push by them is going to slightly perturb your stand-still behavior.

In a very crude nutshell, this is the idea of a shear-stress. The analogy makes more sense when we consider what has been discovered experimentally of fluids against a boundary. At the boundary of a flow, like a wall, it's been seen that the fluid velocity on the boundary is zero. So if you think of a single fluid blob "standing" against the boundary, the very bottom of the fluid blob has velocity zero. However, this does not mean that the top of the fluid blob has zero velocity. The top of the fluid blob will have some velocity due to the shear stresses acting on the blob. So the blob will look as though it is being stretched. This is illustrated in Figure(1).

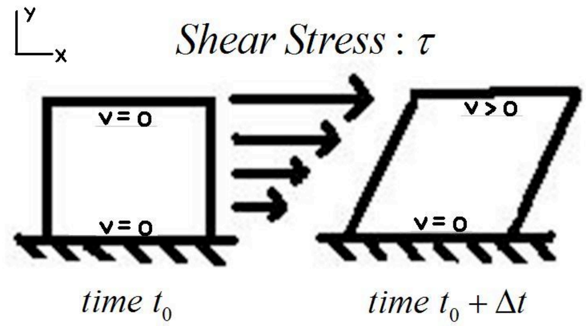


Figure 1: Shear-stresses acting on fluid blob at a boundary.

If we try to quantify the shear-stress on the fluid blob in the above picture, we note that shear-stress is defines as a force per unit area. Say the fluid blob in the above diagram has a density of $\rho$, the shear-stress, $\tau$, is

$$\tau = \rho\nu \frac{\partial v_x}{\partial y}, \tag{58}$$

where $\frac{\partial v_x}{\partial y}$ describes the change in the x component of velocity with respect to the coordinate y and $\nu$ is a parameter called the *kinematic viscosity.* We now note that if the blob was not against the boundary but in the middle of the flow, the blob would still experience shear-stress, and if the flow is not unidirectional, there will be shear-stresses in the y direction as well. In 2 dimensions, the shear-stress on a fluid blob is

$$\tau_{xy} = \rho\nu \left( \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right). \tag{59}$$

Note in 3 dimensions, the quantitative definition of shear-stress follows analogously.

This shear-stress term gives rise to a new force term in the Navier-Stokes equation. First consider the force on a single blob once again, (46), but with an added term for the shear-stress,

$$(\rho\delta V)\,\frac{D\mathbf{v}}{Dt} = -\nabla p\,(\delta V) - \nabla\phi\,(\rho\delta V) + \left(\frac{\partial \tau_{ij}}{\partial x_j}\right)(\rho\delta V), \tag{60}$$

or equivalently,

$$(\rho\delta V)\,\frac{D\mathbf{v}}{Dt} = -\nabla p\,(\delta V) - \nabla\phi\,(\rho\delta V) + \frac{\partial}{\partial x_j}\left[\rho\nu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right](\rho\delta V), \tag{61}$$

for $i, j = \{x, y, z\}$.

We can mathematically massage the force term due to the shear-stress to get a more compact form because we assume the fluid is incompressible. Note that the incompressibility condition gives

$$\frac{\partial u_x}{\partial x} = \frac{\partial u_y}{\partial y} = \frac{\partial u_z}{\partial z} = 0.$$

Using this fact we see that

$$\frac{\partial}{\partial x_j}\left[\rho\nu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right] = \left[\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right]\ \hat{\imath} + \left[\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial z^2}\right]\ \hat{\jmath} + \left[\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2}\right]\ \hat{k}.$$

Hence we get that

$$\left(\frac{\partial \tau_{ij}}{\partial x_j}\right) = \nabla^2 \mathbf{v}.$$

Now we can divide out by the volume of the blob, $\delta V$, and use the definition of the material derivative, (48),

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p - \nabla\phi\ + \nu\ \nabla^2\mathbf{v} \tag{62}$$

Performing the same rigmarole but using the definition of vorticity gives us and taking the curl as in Section(3.2),

$$\frac{\partial \omega}{\partial t} + \nabla\times(\omega\times\mathbf{v}) = \nu\nabla^2\omega. \tag{63}$$

Now if $\omega\times\mathbf{v} = 0$, it is evident that the Navier-Stokes equation describing vorticity yields the form of a traditional diffusion equation,

$$\frac{\partial \omega}{\partial t} = v\nu\nabla^2\omega.$$

Physically this says that if there is a large gradient in the vorticity, vorticity is going to diffuse through the fluid in an analogous manner to how heat transfers throughout a medium. As an example, if a smoker is so inclined to blow a smoke ring, they would witness it grow larger as time progresse until it grows too thin.

The last topic that needs to be addressed before we finally claim to have enough background in MHD to look at a few topics in it, is the key (notorious) dimensionless quantity in fluid dynamics.

### 3.5 The Reynolds Number and Different Regimes of Fluid Mechanics

If you have ever had the pleasure of sitting through a fluid dynamics research seminar, without any prior knowledge of fluids, you might try to focus your attention onto the types of equations being studied, which is probably all you could do. But one thing is for sure, by the end of the presentation, you will probably be scratching your head wishing you understood what a quantity called the *Reynold's Number* is. Essentially depending on what the Reynold's Number, $Re$, is of a particular flow will determine what kind of methods we use to study the flow. That is, $Re$ categorizes different flow regimes.

Quantitatively the Reynold's Number is traditionally defined by

$$Re = \frac{inertial\ forces\ per\ unit\ volume}{viscous\ forces\ per\ unit\ volume} = \frac{vl}{\nu}, \tag{64}$$

where $v$ is the fluid velocity, $\nu$ is the kinematic viscosity, and $l$ is a characteristic length scale. In a nutshell, when $Re$ is small, the fluid flow is considered *laminar*, that is viscous forces play a large role. These types of flow are *smooth* and have relatively low velocities. Essentially laminar flows are stable small perturbations in the flow. Laminar flow is categorized by having $Re \lesssim 2300$.

On the other hand for very large Reynold's Numbers, a very interesting phenomenon occurs- the flow becomes chaotic and *turbulent*. Unlike laminar flow, turbulent flow is unstable to the slightest perturbation and occurs for very high velocities. You can think of turbulent flow as the velocity of a fluid blob at a certain point is constantly changing, both in the direction its flowing and its speed. It is very difficult to understand this flow regime, as it is the main focus of many scientist's professional lives! However, turbulent flow is not some science fiction concept or purely theoretical area of study; examples of turbulence can be seen in everyday life. Things like the flow over a golf ball as it whizzes through the air, the airflow over the wing tips of an airplane, and even blood flow through our arteries are all examples! It has been said that when the great German physicist Werner Heisenbrg was asked what he would ever ask a deity, he responded, "When I meet God, I am going to ask him two questions: Why relativity? And why turbulence? I really believe he will have an answer for the first." [7]

## 4 MHD Equations

Continuing with the fairy tale of MHD, we are finally in a position to introduce it's main characters- the governing equations of incompressible MHD. First we have the two continuity equations, one for charge conservation and one for the fluid density,

$$\nabla \cdot \mathbf{J} = \frac{\partial \rho_c}{\partial t} = 0, \tag{65}$$

$$\nabla \cdot (\rho \mathbf{u}) = \frac{\partial \rho}{\partial t} = 0, \tag{66}$$

where $\rho_c = 0$ because we are assuming the absence of an external charge distribution and $\rho$ is the density of the fluid and $\mathbf{u}$ is the velocity field describing the flow. For our considerations in MHD, the other Maxwell's equations we are interested in are

$$\nabla \times \mathbf{B} = \mu \mathbf{J}, \tag{67}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{68}$$

$$\nabla \times \mathbf{B} = 0. \tag{69}$$

We omit Gauss's Law since we are assuming there is no charge distribution in the system. We're only considering the curl of $\mathbf{E}$ as it is an induced electric field coming from changes in the magnetic field. Furthermore, we note that we can write the Lorentz force as

$$\mathbf{F}_{mag} = \mathbf{J} \times \mathbf{B}. \tag{70}$$

On the other side of the family, we have the Navier-Stokes equations that we now write as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla \phi + \nu \nabla^2 \mathbf{u} + \frac{1}{\rho}(\mathbf{J} \times \mathbf{B}) \tag{71}$$

$$\frac{\partial \omega}{\partial t} + \nabla \times (\omega \times \mathbf{v}) = \nu \nabla^2 \omega + \nabla \times \left( \frac{\mathbf{J} \times \mathbf{B}}{\rho} \right), \tag{72}$$

where we have added one force term due to the Lorentz force. Now that you have been introduced to both sides of the family, it is my hope that neither side has scared you away (yet)! We present one last analogy between vorticity and the magnetic field before complicating the plot further...

## 4.1 Advection and Diffusion of a Magnetic Field

Recall that in Section (3.4) we conjured the equation describing the advection and diffusion of the vorticity field, ie- (63),

$$\frac{\partial \omega}{\partial t} = \nabla \times (\mathbf{u} \times \omega) + \nu \nabla^2 \omega.$$

We will now derive a similar equation for the magnetic field. Starting with induced electric field equation, and then substituting in Ohm's Law, $\mathbf{J} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B})$ we obtain

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \quad (73)$$

$$= -\nabla \left[ \frac{\mathbf{J}}{\sigma} - \mathbf{u} \times \mathbf{B} \right] \quad (74)$$

$$= \nabla \times \mathbf{u} \times \mathbf{B} - \nabla \times \nabla \times \left( \frac{\mathbf{B}}{\sigma \mu} \right), \quad (75)$$

where the last equality comes from Ampere's Law. Recall that the vector identity for a curl of a curl is

$$\nabla \times \nabla \times \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}.$$

Using this identity and the fact that B is solenoidal, $\nabla \cdot \mathbf{B} = 0$, we obtain the advection and diffusion equation for the magnetic field,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{u} \times \mathbf{B} + \frac{1}{\sigma \mu} \nabla^2 \mathbf{B}. \quad (76)$$

Comparing (63) and (76), it is clear they share an almost identical form. However, it is important not to forget that the vorticity is defined explicitly through the curl of the velocity, while the magnetic field is *not* defined in terms of a curl of a velocity. On the other hand, if $\mathbf{u} = 0$ then we have

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{1}{\sigma \mu} \nabla^2 \mathbf{B},$$

which shares the canonical form of a diffusion equation. Like vorticity, or heat for that matter, it appears that the magnetic field diffuses itself through a conductor. It's possible to think of this as the initial state on the boundary of the conductor oozing inwards throughout the conducting material.

We will now discuss one extraordinary (and somewhat counterintuitive) component of MHD - the existence of transverse inertial waves. These waves are called *Alfvén Waves*.

## 4.2 Alfvén Waves

In 1942 Alfvén predicted the existence of a magnetohydrodynamic waves, but due to experimental restrictions of the time, they would not be confirmed observationally until 1949 when Lundquist studied such waves in liquid metal mercury. Since then they have been seen in various experiments with plasmas. The importance of these waves are unparalleled in MHD, as these waves bear induced magnetic fields as well as velocity fields as they propagate. [1], [2]

The formation of Alfvén Waves is much like that of a plucked guitar string. Imagine that we have a steady magnetic field and that a velocity field transversely sweeps the magnetic field sideways. Eventually the inertia of the fluid is overcome by a restoring force arising from the bending magnetic field lines. At that moment the lateral movement of the magnetic field lines will cease, and the Lorentz force will reverse the flow to bring to magnetic field lines back to their equilibrium position. However, as the Lorentz force does this, the inertia of the fluid carries the field lines past their equilibrium position and the whole process starts again, but in the opposite direction. Hence the Alfvén Wave is born. This idea can be seen in the following figure. [3]

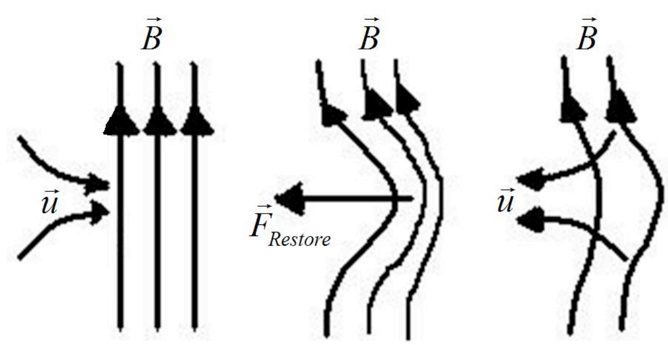


Figure 2: Formation of an Alfvén Wave

Now it's the time to turn physical intuition into a more concrete mathematical form. Assuming we have a steady, uniform magnetic field, $\mathbf{B}$, say a small velocity field, $\mathbf{u}$, perturbs it. This perturbation will cause slight perturbations in the current density and the magnetic field. Call these perturbed fields, $\mathbf{j}$ and $\mathbf{b}$ respectively. Looking at the corresponding Navier-Stokes equation we see that,

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \rho\nu\nabla^2\mathbf{u} + \mathbf{j}\times\mathbf{B}.$$

However, we are mostly interested in the leading order terms in the above equation, since they will be the terms that effectively have the most impact. From this we can then disregard the $\rho(\mathbf{u}\cdot\nabla)\mathbf{u}$ term since it is $2^{nd}$ order in $\mathbf{u}$. Thus the Navier-Stokes equation we are concerned with is

$$\rho\frac{\partial\mathbf{u}}{\partial t} = -\nabla p + \rho\nu\nabla^2\mathbf{u} + \mathbf{j}\times\mathbf{B}.$$

At this point we need to find the corresponding vorticity equation from the above equation. This vorticity equation is essentially going to give us the wave equation we desire once we take another time derivative. Taking the curl of the Navier-Stokes equation we have, gives us

$$\begin{aligned}\rho\frac{\partial\omega}{\partial t} &= -\nabla\times(\nabla p) + \rho\nu\nabla\times(\nabla^2\mathbf{u}) + \nabla\times(\mathbf{j}\times\mathbf{B}) \\ &= \rho\nu\nabla^2(\nabla\times\mathbf{u}) + [\mathbf{j}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{j}) + (\mathbf{B}\cdot\nabla)\mathbf{j} - (\mathbf{j}\cdot\nabla)\mathbf{B}] \\ &= \rho\nu\nabla^2\omega + (\mathbf{B}\cdot\nabla)\mathbf{j}, \qquad (77)\end{aligned}$$

as $(\nabla\cdot\mathbf{B}) = 0$ because $\mathbf{B}$ is solenoidal, $(\nabla\cdot\mathbf{j}) = 0$ because charge is conserved, and $(\mathbf{j}\cdot\nabla)\mathbf{B} = 0$ because $\mathbf{B}$ is constant. It is now time for a basic calculus exercise in taking a time derivative of (77),

$$\begin{aligned}\rho\frac{\partial^2\omega}{\partial t^2} &= \frac{\partial}{\partial t}(\mathbf{B}\cdot\nabla)\mathbf{j} + \rho\nu\frac{\partial}{\partial t}\left(\nabla^2\omega\right) \\ &= (\mathbf{B}\cdot\nabla)\frac{\partial\mathbf{j}}{\partial t} + \rho\nu\nabla^2\left(\frac{\partial\omega}{\partial t}\right). \qquad (78)\end{aligned}$$

Staring at (78), one might see what looks like a kind of wave equation in $\omega$ but with some other pieces in terms of $\mathbf{j}$. We now set out to get (78) solely in terms of $\omega$. To do this, we must return to our old friend, the magnetic induction equation.

In terms of the perturbed fields, we have

$$\frac{\partial\mathbf{b}}{\partial t} = \nabla\times(\mathbf{u}\times\mathbf{B}) + \frac{1}{\mu\sigma}\nabla^2\mathbf{b}.$$

Now where do we go from here? We have to remember our old friend, Ampere's Law, that relates the magnetic field to a current density. In our case we would have $\nabla\times\mathbf{b} = \mu\mathbf{j}$. Now we'll do what it seems we always do- take the curl of the above equation. We obtain

$$\begin{aligned}
\frac{\partial}{\partial t}(\nabla \times \mathbf{b}) &= \nabla \times \Big(\nabla \times (\mathbf{u} \times \mathbf{B})\Big) + \frac{1}{\mu\sigma}\ \nabla \times (\nabla^2 \mathbf{b}) \\
\mu \frac{\partial \mathbf{j}}{\partial t} &= \nabla \times \Big[\mathbf{u}(\nabla \cdot \mathbf{b}) - \mathbf{B}(\nabla \cdot \mathbf{u}) + (\mathbf{B} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{B}\Big] + \frac{1}{\mu\sigma}\ \nabla^2(\nabla \times \mathbf{b}) \\
&= \nabla \times \Big[(\mathbf{B} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{B}\Big] + \frac{\mu}{\mu\sigma}\ \nabla^2 \mathbf{j} \\
&= (\mathbf{B} \cdot \nabla)(\nabla \times \mathbf{u}) - (\mathbf{u} \cdot \nabla)(\nabla \times \mathbf{B}) + \frac{\mu}{\mu\sigma}\ \nabla^2 \mathbf{j} \\
\mu \frac{\partial \mathbf{j}}{\partial t} &= (\mathbf{B} \cdot \nabla)(\nabla \times \mathbf{u}) + \frac{\mu}{\mu\sigma}\ \nabla^2 \mathbf{j}, \qquad (79)
\end{aligned}$$

as $(\nabla \cdot \mathbf{B}) = 0$ because $\mathbf{B}$ is a solenoidal field, $(\nabla \cdot \mathbf{u}) = 0$ because we are only considering incompressible fluids, and $(\nabla \times \mathbf{B}) = 0$ because $\mathbf{B}$ is a steady, uniform field. Note that the perturbed magnetic field, $\mathbf{b}$, is not steady, nor uniform.

Returning back to our ugly wave equation, we substitute (79) into (78). Upon doing so, we obtain,

$$\begin{aligned}
\frac{\partial^2 \omega}{\partial t^2} &= (\mathbf{B} \cdot \nabla)\left[\frac{1}{\rho\mu}(\mathbf{B} \cdot \nabla)\ \omega + \frac{1}{\mu\sigma}\ \nabla^2 \mathbf{j}\right] + \nu \nabla^2 \left(\frac{\partial \omega}{\partial t}\right) \\
&= \frac{1}{\rho\mu}\ (\mathbf{B} \cdot \nabla)^2 \omega + \nu \nabla^2 \left(\frac{\partial \omega}{\partial t}\right) + \frac{1}{\rho\mu\sigma}\ (\mathbf{B} \cdot \nabla)(\nabla^2 \mathbf{j}). \qquad (80)
\end{aligned}$$

We are almost home free in getting our equation in terms of $\omega$ alone; however, the last term on the right hand side still has a $\nabla^2 \mathbf{j}$ term in it. We can use (79) to remedy that, ie-

$$\begin{aligned}
\frac{1}{\rho\mu\sigma}(\mathbf{B} \cdot \nabla)(\nabla^2 \mathbf{j}) &= \frac{1}{\mu\sigma}\ \nabla^2 \left[\frac{1}{\rho}(\mathbf{B} \cdot \nabla)\mathbf{j}\right] \\
&= \frac{1}{\mu\sigma}\ \nabla^2 \left[\frac{\partial \omega}{\partial t} - \nu \nabla^2 \omega\right] \\
&= \frac{1}{\mu\sigma}\ \nabla^2 \left(\frac{\partial \omega}{\partial t}\right) - \frac{\nu}{\mu\sigma}\ \nabla^4 \omega. \qquad (81)
\end{aligned}$$

Just doing a little change up, by substituting (81) into (80), we obtain the following wave-like equation,

$$\frac{\partial^2 \omega}{\partial t^2} = \frac{1}{\rho\mu}\ (\mathbf{B} \cdot \nabla)^2 \omega + \left(\nu + \frac{1}{\mu\sigma}\right) \nabla^2 \left(\frac{\partial \omega}{\partial t}\right) + \frac{\nu}{\mu\sigma}\ \nabla^4 \omega, \qquad (82)$$

which is solely in terms of the vorticity. [3]

It is now our goal to show the existence of some kind of wave-solution arising from (82). In general we can assume a wave solution with the following form,

$$\omega = \omega_0 e^{i(\mathbf{k}\cdot\mathbf{x}-2\pi\ ft)}. \tag{83}$$

where $\mathbf{k}$ is the wave-number and $2\pi f$ gives the angular-frequency of oscillation. We can try substituting (83) into (82) to see if we truly have wave solutions; however, this would be overkill for our considerations. After-all we just wish to show these Alfvén wave solutions exist. Let's look at a much easier problem on the eyes (and brain) and consider the 1-spatial-dimensional version of (82), given as:

$$\frac{\partial^2\omega}{\partial t^2} = \frac{B_{0_x}^2}{\rho\mu}\ \frac{\partial^2\omega}{\partial x^2} + \left(\nu + \frac{1}{\mu\sigma}\right)\left(\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial\omega}{\partial t}\right) + \frac{\nu}{\mu\sigma}\ \left(\frac{\partial^4}{\partial x^4}\right)\omega, \tag{84}$$

We now attempt to find wave solutions of the form,

$$\omega = \omega_0 e^{i(kx-2\pi ft)}.$$

Taking the corresponding derivatives using the above trial wave solution, we obtain

$$\begin{aligned}
\frac{\partial^2\omega}{\partial t^2} &= -(2\pi f)^2\ \omega_0 e^{i(kx-2\pi ft)} \\
\frac{B_{0_x}}{\rho\mu}\ \frac{\partial^2\omega}{\partial x^2} &= -\frac{k^2 B_{0_x}^2}{\rho\mu}\ \omega_0 e^{i(kx-2\pi ft)} \\
\left(\nu + \frac{1}{\mu\sigma}\right)\left(\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial\omega}{\partial t}\right) &= \left(\nu + \frac{1}{\mu\sigma}\right)(2\pi f)\ k^2\ \omega_0 e^{i(kx-2\pi ft)} \\
\frac{\nu}{\mu\sigma}\ \left(\frac{\partial^4}{\partial x^4}\right)\omega &= \frac{\nu}{\mu\sigma}\ k^4 \omega_0 e^{i(kx-2\pi ft)}.
\end{aligned}$$

Putting it all together, we obtain the following equation,

$$(2\pi f)^2\ \omega_0 e^{i(kx-2\pi ft)} = \left[\frac{k^2 B_{0_x}^2}{\rho\mu} - \frac{\nu}{\mu\sigma}\ k^4 - \left(\nu + \frac{1}{\mu\sigma}\right)k^2(2\pi f)\right]\ \omega_0 e^{i(kx-2\pi ft)}.$$

What a bloody mess! Well, we can cancel the exponential terms from both sides. In doing so, those so enlightened in the beauty of mathematics, will find that we are left with nothing more than a disguised freshman year of high school problem- that of course being an equation in the infamous quadratic equation form! Explicitly, we see

$$(2\pi f)^2 + \left(\nu + \frac{1}{\mu\sigma}\right) k^2 \ (2\pi f) + \left(\frac{\nu}{\mu\sigma} \ k^4 - \frac{k^2 B_{0_x}^2}{\rho\mu}\right) = 0. \tag{85}$$

Solving for the natural frequency, $2\pi f$, we obtain

$$2\pi f = -\frac{\left(\nu + \frac{1}{\mu\sigma}\right) k^2}{2} \pm \frac{\sqrt{4\frac{k^2 B_{0_x}^2}{\rho\mu} - \left(\nu - \frac{1}{\mu\sigma}\right)^2 k^4}}{2}. \tag{86}$$

And boom! Just like that, we have shown the existence of transverse inertial waves in MHD. These waves are of great importance is some MHD studies, as they can be responsible for propagating energy and momentum away in a system. For example, when an interstellar cloud is collapsing in the formation of a star, it is thought that these Alfvén waves transport angular momentum away from the core. [3].

# 5 Conclusions

Magnetohydrodynamics is a very rich subject, full of many inherent difficulties, but has very with interesting and equally important applications. The topics and applications touched upon in the paper (most of which are unjustifiably short introductions and anecdotes), are by no means the only important phenomena or applications of MHD. This paper was intended to entice the reader to develop a new and optimistic relationship in the material, and moreover capture the reader's interest. It is my hope that this paper has gotten one excited about MHD, applied mathematics and physics, or in the very least an unrelenting childish curiosity in wanting to understand the natural world.

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