NOTES ON ADJUNCTIONS, MONADS AND LAWVERE THEORIES

FILIP BÁR

1. ADJUNCTIONS

1.1. **Universal constructions and adjunctions.**

Definition 1.1 (Adjunction)**.** A pair of functors $U : C \to D$ and $F : D \to C$ form an *adjoint pair* $F \dashv U$ or *adjunction* if for every $c \in \operatorname{ob} C$ and $d \in \operatorname{ob} D$ there is a bijection
$$C(Fd, c) \cong D(d, Uc)$$
natural in both arguments. The functor F is called the *left adjoint* of U, the functor U is called the *right adjoint* of F.

Remark 1.2 (Adjunctions in terms of unit and counit)*.* Recall that an adjunction $F \dashv U$ is given equivalently by the following data: A pair of functors U, F like above and natural transformations $\eta : I_D \Rightarrow UF$, $\varepsilon : FU \Rightarrow I_C$ s.t. the following so called *triangle identities* hold

$$
\begin{array}{ccc}
F \xRightarrow{\;F\eta\;} FUF & \qquad & U \xRightarrow{\;\eta_U\;} UFU \\
\qquad\searrow \quad \Downarrow{\varepsilon_F} & & \qquad\searrow \quad \Downarrow{U\varepsilon} \\
\qquad\quad F & & \qquad\quad U
\end{array}
$$

The natural transformation η is called the *unit*, the natural transformatin ε is called the *counit* of the adjunction.

Adjunctions arise in the context of universal constructions, i.e. when there is a functor $U : C \to D$ and for any object $d \in \operatorname{ob} D$ a universal arrow $d \to Uc$ from d to U in the sense of Mac Lane. (That is an initial object in the comma category $(d \downarrow U)$.) Choosing a universal arrow for every $d \in \operatorname{ob} D$ yields a functor $F : D \to C$, which is left adjoint to U. An analysis of the notion of adjoint functors reveals that universal constructions always arise in pairs. (See [Mac98, theorem IV.1.2])

Remark 1.3*.* The name '*adjoint functors*' comes from the definition of adjunctions in terms of hom-sets, where the hom-bifunctor

$$\operatorname{hom} : C^{op} \times C \to \operatorname{Set}$$

is compared with a scalar product

$$\langle -, - \rangle : V^c \times V \to \mathbb{C}$$

on a vector space V over the complex numbers $\mathbb{C}$. (Here V^c denotes the vector space V where the $\mathbb{C}$-action is precomposed with complex conjugation.) Note that this comparison is conceptually meaningful. A scalar product is a structure for measuring correlations of two vectors. A hom-set (respectively its cardinality) measures too how two objects are correlated in a category.

In linear algebra adjunctions are closely related to linear optimization problems (see least-square problem and the relation between pseudo-inverses and adjoints). The intuition is that what orthogonality and minimizers are in linear algebra, universal arrows (i.e. universal properties) are in category theory. However, nobody was able to make this analogy precise so far, afaik.

Remark 1.4. This kind of intuition about orthogonality is also used in (othogonal) factorization systems, where we have a notion of when two (sets of) arrows are "othogonal". (Cf. e.g. [AHS04])

1.2. **Adjunctions and limits.** Recall that a category C admits limits of shape J iff the diagonal functor $\Delta_J : C \to [J, C]$ admits a right adjoint. (And colimits of shape J iff Δ_J has a left adjoint.) So limits 'are' adjoints. Conversely, adjoints can be expressed in terms of limits (cf. [Mac98, theorem X.1.2]):

A functor $U : C \to D$ has a left adjoint iff

(i) U preserves *all* limits
(ii) for every $d \in \mathrm{ob}\, D$ the limit of the canonical projection functor $(d \downarrow U) \to C$ exits.

In this case the left adjoint is given on objects by

$$Fd = \varinjlim (d \downarrow U) \to C$$

This formal characterization, however, doesn't give a useful practical criterion for the existence of left adjoints. The purpose of the various adjoint functor theorems (there are more than two!) is to give more practical criteria by restricting

(i) to small (or other classes of) limits, and
(ii) conditions on C and D such that one can infer the existence of these limits.

The most famous ones are *Freyd's Adjoint Functor Theorem*, which considers the class of small limits in (i) and introduces solution set conditions and demands the existence of small limits in C for (ii), and *SAFT*, which also considers the class of small limits and replaces the assumption of a solution set condition with the existence of a small coseparating set and well-powerdness.

1.3. **Adjunctions, equivalences and dualities.** Every equivalence of categories can be made into an adjoint equivalence. In this case the functors are simultaneously left and right adjoint to each other. Conversely, every adjunction restricts to an equivalence of categories:

Theorem 1.5. *Let $F : C \to D$ be a functor, G its right adjoint and η and ϵ the unit and counit of this adjunction. Consider the full subcategory C_0 of C generated by all the $c \in \mathrm{ob}\, C$ s.t. η_c is an isomorphism, and the full subcategory D_0 generated by all the $d \in \mathrm{ob}\, D$ s.t. ϵ_d is an isomorphism. Then F restricts to an (adjoint) equivalence of C_0 and D_0.*

Proof. The triangle identity yields

$$\epsilon_{Fc} \circ F\eta_c = 1_{Fc}.$$

if η_c is an iso, then $F\eta_c$ is an iso, hence η_{Fc} is an iso. This shows that F restricts to a functor $C_0 \to D_0$. Given a $d \in \mathrm{ob}\, D_0$ then the other triangle identity reads

$$G\epsilon_d \circ \eta_{Gd} = 1_{Gd},$$

hence G restricts to a functor $D_0 \to C_0$. The adjunction $F \dashv G$ restricts to an adjunction between C_0 and D_0. By construction the unit is η restricted to C_0 and hence an iso. The same is true for the counit, so F and G restrict to an adjoint equivalence as asserted. $\qquad\square$

If the induced monad is idempotent, then C_0 is a reflective subcategory (cf. example sheet 3 ex 6). If the induced comonad is idempotent, then D_0 is a coreflective subcategory. (See also [MM92, lemma II.6.4])

In the contravariant case this shows that every adjunction restricts to a duality. In fact many dualities in mathematics are restrictions of adjunctions. Two famous examples are the dual vector space construction and the Galois correspondence in field theory.

1.4. **2-Categorical perspective.** If we want to study adjunctions algebraically, we need the algebraic theory of 2-categories to be able to say what an adjoint pair is in terms of equations (the triangle identities). Recall that a 2-category abstracts from the structure we find in the category of (small) categories: we have objects the small categories, arrows are functors, and then we find natural transformations, which obey two composition laws. There is the vertical composition '·' and the horizontal composition '∘'. Both are interelated by the interchange law, and the units of the vertical composition law are units for the horizontal. A way to remember both is that the operation of composing functors is a bifunctor:

$$[D, E] \times [C, D] \to [C, E], \qquad (F, G) \mapsto F \circ G, \quad (\alpha, \beta) \mapsto \alpha \circ \beta$$

The interchange law

$$(\alpha \cdot \alpha') \circ (\beta \cdot \beta') = (\alpha \circ \beta) \cdot (\alpha' \circ \beta')$$

is equivalent to saying that composition of functors preserves composition of arrows in the product category of the two functor categories. The assertion about the units is exactly that this bifunctor maps identity arows to identity arrows.

A *2-category* has objects, 1-cells and 2-cells. The objects and 1-cells together form a category. The 2-cells are defined algebraically following the observations made about natural transformations: we have 2 composition laws, an interchange law relating the two and the vertical identity 2-cells are also identity 1-cells for the horizontal composition.

As with the category of small categories in a 2-category the hom-set of 1-cells $\mathrm{hom}(c, d)$ actually carries a natural structure of a category. The arrows are 2-cells

between the 1-cells $c \to d$ and the composition law is the vertical composition of 2-cells. Composition of 1-cells turns out to be a bifunctor again. In fact, a 2-category can be defined equivalently as a category enriched in the cartesian closed category of small categories.

In a 2-category we can define when two 1-cells form an adjoint pair by just restating the definition of an adjoint pair in terms of unit, counit and the triangle identities.

An example of a 2-category besides Cat is the category HTop, which has as objects topological spaces, 1-cells continuous maps and 2-cells homotopy classes of homotopies. An adjoint pair in HTop are two continuous maps $f : X \to Y$, $g : Y \to X$ and homotopy classes of homotopies $\eta : 1_X \Rightarrow gf$, $\epsilon : fg \Rightarrow 1_Y$. Since all 2-cells are invertible in HTop, this just says that f and g yield a homotopy equivalence of X and Y.

These constructions can be iterated and lead to the notion of strict n-categories. However, it turns out that strict n-categories are not the higher categorical structures which arise in mathematics naturally. Instead one can observe that the composition laws of n-cells, say, are only associative up to an invertible $(n + 1)$-cell. A good example is HTop, We need to consider homotopy classes of homotopies as 2-cells for HTop to have a strictly associative composition law. If we would just consider homotopies, the associativity would only hold up to a homotopy equivalence. In fact, Top together with all the higher homotopies as the higher dimensional cells is considered as the fundamental example of what is called an ∞-groupoid. Another related notion is that of $(\infty, 1)$-category. Both are algebraic structures in the focus of current research in abstract algebraic topology and homotopy theory, in particular.

2. Monads

There are three major perspectives on monads. Comming from the poset case they can be regarded as *generalized closure operators*. This perspective can be made precise with the notion of idempotent monad (cf. example sheet 3, ex 1 and ex 6, or proposition 4.2.3 and corollary 4.2.4 in [Bor94]).

Another perspective is that monads are the *internal monoids* in the strict monoidal category $(\mathrm{End}(C), \circ, I_C)$. This helps to remember the definition and is also of importance in algebraic topology; namely, a monad yields a cosimplicial object (as does any internal monoid in a monoidal category) which one can use to construct cochain complexes and hence resolutions in the sense of (co)homological algebra. (See [Mac98, section VII.6], and for the the treatment of cohomology induced by monads Beck's thesis [Bec67] TAC reprint no.2)

The most important and so far most far reaching perspective on a monad T on C is that of an *algebraic theory* on the category C. Usually models of algebraic theories are considered only in the category Set of small sets. Here the intuition can be made precise in form of the following theorem:

Theorem 2.1. *There is an equivalence between the (meta)category of monads on Set and the (meta)category of infinitary algebraic theories. Furthermore, the category of Eilenberg-Moore algebras of a monad is equivalent to the category of models of its corresponding infinitary algebraic theory.*

Here algebraic theories are considered in the sense of universal algebra, i.e., we have one sort and operations of various arities on that sort satisfying certain equations (see e.g. the notion of "type of algebraic system" in [Mac98]p. 124). Operations of arity zero are constants. An infinitary algebraic theory allows for (small) cardinal numbers as arities. Examples of such theories are (bounded) complete sup-lattices. If L is such a lattice and X a subset of L, then $\sup X$ is considered an operation of arity the cardinality of X. (In fact, we do consider infinitary algebraic theories rather as infinitary Lawvere-theories than variety types of universal algebra in this theorem.)

(*Rough sketch of proof*: For one direction one uses that the forgetful functor from the category of Set-models of an infinitary algebraic theory has a left adjoint, and hence induces a monad. Futhermore, with *Beck's monadicity theorem* one can show the forgetful functor to be monadic. For the other direction one uses the opposite of the Kleisli category Set_T^{op} of the monad T as a syntactic category of an (infinitary) algebraic theory. (In fact one needs a bit more, namely the structure of arities.) One can show that T-algebras in Set correspond one-to-one to (small) product preserving functors from Set_T^{op}. The induced monad of the infinitary algebraic theory Set_T^{op} is isomorphic to T, and the the theory of the monad induced by an infinitary algebraic theory is equivalent to the latter. For all this to make any sense, we need to describe an algebraic theory by means of a syntactic category, and understand which categories arise as syntactic categories of algebraic theories. I will explain this in more detail for the finitary case in the section about *Lawvere theories*, whose infinitary generalizations have been used here secretly.)

2.1. T-algebras.

The most important notion related to a monad T is that of a T-algebra and T-algebra homomorphism. To be able to understand why (and how) monads are considered as algebraic theories, it is important to understand how ordinary algebraic structures like monoids or groups (in Set), for example, can be expressed as T-algebras first.

Recall that $T = UF$ is the monad build from the forgetful functor U and the free monoid (respectively free group) functor F. UFX is the set of finite tuples of elements of X. The unit η of this monad is the "insertion of generators $x \in X$" as 1-tuples, the multiplication μ maps finite tuples of tuples to tuples by removing the inner brackets. In the case of groups UFX is the set of tuples of elements of X and its formal inverses, i.e., tuples of elements of a disjoint union $X + X$. The multiplication μ_X does the same as for monoids with the additional requirement that successive occurences of formal inverses of a k-tuple and the k-tuple itself in a tuple are to be removed both.

How can we encode a monoid (M, m, e) as a T-algebra (M, h)? The idea is to consider the set TM as *the set of formal operations on the elements of M*. In the case of a monoid a finite tuple of elements of M stands for the operation of taking the product of these elements. In the case of a group (G, m, e) this is essentially the same. We have to deal with the formal inverses in addition, which stand for inverting the respective element of G, before multiplying it. The T-algebra multiplication h is then an actual realization of these formal operations on elements of M as actual operations. For a monoid this means that we assign to a k-tuple of elements of M its actual product; the same is true for groups.

Associativity of these operations means in full generality in this context that if we form tuples of tuples of elements of M, then removing the inner brackets and then taking the product or taking the product of the inner tuples first and then the product of the resulting tuple yields the same result. Convince yourselves that this is exactly the commutativity of the associativity diagram of a T-algebra.

What about the unit e in M? The unit law of the T-algebra (M, h) says that the 1-tuples (x) for $x \in M$ have to be mapped to x by h. The formal unit in TM is the empty tuple $()$. h has to map this formal unit to e. The unit and associativity law of the T-algebra together imply that e is the unit. (Conversely, for any T-algebra (M, h) the element $e := h(())$ is the unit of the monoid M.)

What about the inverses in the case of a group G? h has to map the formal inverses of G to their actual inverses in G. μ_G together with the unit and associativity law of the T-algebra (G, h) yield that multiplying $g \in G$ with its inverse gives the unit e.

To sum it up: If we think of the monad T as being the algebraic theory of groups, say, then T constructs all the formal group operations, μ encodes how these operations compose with each other, and η keeps track of the generating set. A T-algebra structure (M, h) on M realizes these formal operations on elements of M as particular operations in M, and makes M into a group.

But what singles out the particular operations of group multiplication, unit and inversion in the monad T, and T-algebra (M, h)? The answer is: nothing. These operations are just one of many possible particular choices of generating operations for all the group operations. (Recall that it is possible to define a group structure by just using one binary operation of 'division'. The binary multiplication, unit and inversion together with their repsective properties can be derived from this operation of 'division'.) A monad doesn't single out any generating operations; it describes all operations on equal footing. This comes of no surprise when bearing in mind that a monad T corresponds to an algebraic theory $\mathcal{A}$ and T-algebras represent any kind of algebra for this theory $\mathcal{A}$.

We can learn the following from these examples:

(1) The algebraic structure of a monoid or group is completely described by the fact how the operations compose. In other words: everything resides in the monad (T, μ, η). We can see this by noting that we actually don't need to make references to particular sets X and its elements. We just need T, μ and η. This comes of no surprise of course, because the axioms for monoids and groups just say how the operations compose. A reference to elements is convenient for our thinking to subtitute variables with something "actual", but it is formally not necessary. (Recall that 0-ary operations are constants. This gives the unit as an operation, and so doesn't depend on elements either.)

(2) We can develop the general perspective on T-algebras (A, h) for a monad (T, μ, η) on a category C as TA being the object of formal operations on (generalized) elements of A, and h being the concrete realization of these operations in A. The unit axiom for (A, h) says that η_A is split monic, hence it is meaningful to speak about the insertion of generators in general.

(3) T constructs the object of formal operations, μ encodes how these formal operations compose (i.e. it corresponds to the axioms of the algebraic theory), η keeps track of the generators.

(For a discussion of monoids as T-algebras see also the introduction to [Bor94, chapter 4])

Recall that generalizing the example of monoids and groups for any (finitary) algebraic theory $\mathcal{A}$ the category Set^T of T-algebras is equivalent (actually isomorphic) to the category algebras of $\mathcal{A}$ in a natural way (namely via the comparison functor) by Beck's theorem. We shall discuss Beck's thereom later, but note that this implies that with monads (on Set) we have a formal tool to deal with universal algebra (of varieties) at hand. However, monads transcend the usual framework of universal algebra (of varieties). A nice example illustrating this is the fact that the category of compact Hausdorff spaces is equivalent to Set^T for the monad $T = UF$, where U is the forgetful functor and FX the Stone-Čech compactification of a set X considered as a discrete topological space.

2.2. **Properties of C^T.** Realizing the importance of monads T and the category C^T of T-algebras we would like to understand what properties C^T inherits from C and properties of T.

(1) The forgetful functor $U^T : C^T \to C$ has a left adjoint $F^T : C \to C^T$ and hence preserves all limits. In fact we have the stronger result that U^T *creates limits*. In particular, C^T has all the J-limits that C has. This is the abstract reason why we always construct limits (like e.g. products and pullbacks) from the underlying limit in Set in algebra. U^T is *conservative*, i.e., it reflects isomorphisms. This is also a generalization of the familiar fact in algebra that any homomorphism, whose underlying map is bijective, is an isomorphism. (From this we can see, for example, that even though the category of compact Hausdorff spaces is monadic over Set, the category Top of topological spaces is not: it has continuous bijective maps, which are not homeomorphisms, i.e., isomorphisms in Top.)

(2) As regards colimits, roughly, U^T creates J-colimits that are preserved by T. More precisely we have the following result: Let $F : J \to C^T$ be a diagram such that UF has a colimit in C, which is preserved by T and T^2; then F has a colimit in C^T which is preserved by U.

   (*sketch of proof*: One proves that given a colimiting cocone $\lambda : U^T F \Rightarrow \Delta X$, s.t. $T\lambda$ is a colimiting cocone in C and $(T^2\lambda_j)_{j \in \mathrm{ob}\, J}$ is a jointly epimorphic family of maps, then there is a unique T-algebra structure $h : TX \to X$ on X s.t. λ has a lift to a colimiting cocone $\lambda' : F \Rightarrow \Delta(X, h)$ in C^T)

One important class of colimits for T-algebras are coequalizers (of reflexive pairs). Recall that every group G has a presentation, i.e., it is a quotient of a free group F by the normal subgroup R generated by certain relations. Diagrammatically we can write this as a coequalizer in Set

$$ R \overset{d_0}{\underset{d_1}{\rightrightarrows}} F \longrightarrow G $$

Here F is the free group generated by G and R is the equivalence relation and subgroup of $F \times F$ generated by the equations which hold between products of elements of G. We can replace R with the free group F_R generated by R and get a presentation of G in terms of a coequalizer of free groups

$$F_R \underset{d_1'}{\overset{d_0'}{\rightrightarrows}} F \longrightarrow G$$

The presentation of groups in terms of free groups generalizes to arbitrary T-algebras. In fact, using the approach of formal operations and their realizations, it becomes a natural construction: Every T-algebra (X, h) has a canonical presentation in terms of a coequalizer diagram of free T-algebras in C

$$T^2 X \underset{Th}{\overset{\mu_X}{\rightrightarrows}} TX \overset{h}{\longrightarrow} X \tag{2.1}$$

In fact it is a split coequalizer. The splitting is given by the unit η_X and η_{TX}. Recall that for a monad T the multiplication μ encodes the axioms of the corresponding algebraic theory, i.e., how the formal operations compose. In a T-algebra (X, h) the arrow h realizes these formal operations and hence encodes the equational relations that hold in the particular X. The coequalizer (2.1) makes this intuition precise.

The canonical presentation of a group G is actually a coequalizer in the category Grp of groups, not just in Set, if we consider the normal subgroup obtained as the image of R under $(g, h) \mapsto gh^{-1}$. Again, every T-algebra (X, h) has such a presentation in C^T as a coequalizer

$$(T^2 X, \mu_{TX}) \underset{Th}{\overset{\mu_X}{\rightrightarrows}} (TX, \mu_X) \overset{h}{\longrightarrow} (X, h) \ , \tag{2.2}$$

which can be obtained from the previous coequalizer and what we've said about colimits in C^T earlier. The pair (Th, μ_X) is a reflexive pair, but the coequalizer isn't split in general. Its U^T-image is a split coequalizer however; so it is a *reflexive U^T-split pair.*

The importance of coequalizers for the colimits in C^T is underpinned by the following result due to Linton:

Theorem 2.2 (Linton)**.** *Let C be (finitely) cocomplete. TFAE*

(i) *C^T has coequalizers for all reflexive pairs (i.e. we have a coequalizer for every pair of parallel T-algebra maps which have a common section in C^T)*
(ii) *C^T is (finitely) cocomplete*

Remark 2.3. The fact that we only need reflexive pairs is because we only need "quotients by equivalence relations" of free algebras. Indeed, the free algebra functor $F^T : C \to C^T$ preserves all colimits of C, and we know that every T-algebra is a canonical quotient of free T-algebras. First we construct coproducts in C^T. For that we write down the canonical presentations of the T-algebras (X_i, h_i); then we use $F^T(X_1 + X_2) = F^T X_1 + F^T X_2$ (recall $F^T X = (TX, \mu_X)$) to obtain a reflexive pair

$$T^2(X_1 + X_2) \underset{\longrightarrow}{\overset{\longrightarrow}{\longrightarrow}} T(X_1 + X_2)$$

for which we can form the coequalizer in C^T. Now a diagram chase shows that this coequalizer is indeed the desired coproduct in C^T. Conceptually speaking we have constructed the coproduct by joining together the defininging canonical relations of (X_1, h_1) and (X_2, h_2) and dividing them out from the free algebra on the sum of generators. (Recall that this is exactly how the coproduct of groups is usually constructed in algebra. There it is called the "free product".) One can show that any category that has (small) coproducts and equalizers of reflexive pairs, has all equalizers and hence all (small) colimits. (Exercise :-))

But when does C^T have coequalizers for all reflexive pairs and a cocomplete C? We approach this by trying to mimick the construction of coequalizers like in Set, namely as quotients of equivalence relations. First let's assume that C is finitely complete and that the reflexive pair is an 'equivalence relation', i.e., it is in addition invariant under the symmetry $X \times Y \to Y \times X$ and it is transitive (one expresses this with help of pullback diagrams). C has an equalizer of the U-image of the reflexive pair. If this coequalizer is moreover split, then we get from (2) that there is a unique T-algebra structure on the coequalizer making it a coequalizer in C^T.

To be able to apply this to any reflexive pair in C^T, we need to form the symmetric and transitive closure of the relation described by the U-image of the reflexive pair in C. For this we need C to be a *complete regular category*. In such a category one can then show (like in Set) that the coequalizer of the symmetric transitive closure is also the coequalizer of the reflexive pair. To be able to construct a split coequalizer we need that every such coequalizer has a section. This is the case if every regular epi in C has a section. Summarizing:

Theorem 2.4 (cf. [Bor94, theorem 4.3.5]). *Let C be a complete and cocomplete regular category, then C^T is also a complete, cocomplete and regular category.*

In layman's terms: If C allows for a nice interpretation of the theory of (equivalence) relations and is cocomplete, then C^T too allows for a nice interpretation of (equivalence) relations and is cocomplete. (Or another way of seeing it: if C admits pullback stable image-factorizations, so does C^T.) One example of such a category C is Set (assuming the axiom of choice, which in categorical terms is equivalent to the statement that every epi in Set splits). We get the following very strong general result:

Corollary 2.5. *For any monad T on Set, Set^T is complete and cocomplete.*

Think for a minute what implications that has: not only is every category of models of an any algebraic theory (in the sense of universal algebra of varieties) complete and cocomplete, but also categories like the category of compact Hausdorff spaces, etc. (Try to think how to prove that the category of compact Hausdorff spaces is closed under small products and coproducts directly. The existence of small products, for example, is known as (a version of) Tychonoff's Theorem. It is easy to proof using the theory of ultrafilters, but I don't know of any easy proof besides that.)

Note that dealing with (equivalence) relations and their quotients in algebra comprises the majority of an introductory course in algebra. The only difference is that in algebra we study representations of equivalence relations by substructures like normal subgroups or ideals etc. In case of T-algebras we deal with the equivalence relations and their quotients in terms of parallel pairs and coequalizers.

2.3. **Universal splittings of monads.** Every adjunction yields a monad. Conversely, the *Eilenberg-Moore construction* shows that every Monad T on C comes from an adjunction $F \dashv U$ with

$$U^T : C^T \to C \qquad (X, h) \mapsto X, \quad f \mapsto f$$
$$F^T : C \to C^T \qquad X \mapsto (TX, \mu_X), \quad f \mapsto Tf$$

The Eilenberg-Moore construction enjoys a universal property in the category of splittings of T into adjoint pairs:

Theorem 2.6. *Let $F : C \to A$, and $U : A \to C$ be functors s.t. $F \dashv U$ and $T = UF$; then there is a unique functor $K : A \to C^T$ with $U^T K = U$, $KF = F^T$. Moreover we get $K\varepsilon = \varepsilon^T K$.*

The 'smallest' subcategory of C^T we have to consider to be able to recover the monad T from an adjoint pair is the full subcategory generated by the free T-algebras (Tx, μ_x), $x \in \mathrm{ob}\,C$; that is, the essential image of F^T. Free algebras and morphisms between them, like e.g. the free monoid FX or the free group FX, and morphisms $FX \to FY$ are uniquely determined by how they map the generators $x \in X$. This is a direct consequence of the universal property of these constructions. But this true for free T-algebras and morphisms between them in general, and a consequence of $F^T \dashv U^T$. (Recall the universal property of the unit of an adjunction) Because of this we can attempt (and actually want) to describe the category of free T-algebras syntactically just in terms of C. This gives one way of looking at the *Kleisli-construction*:

- $\mathrm{ob}\,C_T = \mathrm{ob}\,C$ (We think of each $x \in \mathrm{ob}\,C$ as the 'set' of generators of the free T-algebra (Tx, μ_x))
- Morphisms $f_T : x_T \to y_T$ in C_T are the morphisms $f : x \to Ty$ (since any such morphism uniquely determines a T-algebra morphism $(Tx, \mu_x) \to (Ty, \mu_y)$ and vice versa.)

The composition law can be obtained from the universal property. If we use the counit of $F^T \dashv U^T$, we can express the underlying morphism of the composite $f_T \circ g_T$ of two morphisms f_T, g_T in C_T as $\mu_{\mathrm{cod}\,f} T(f)g$.

Keeping in mind that we're actually describing free T-algebras gives us an adjunction $F_T \dashv U_T$ with $T = U_T F_T$. U_T is the 'forgetful functor of free T-algebras'

$$U_T : \quad x_T \mapsto Tx, \quad f_T \mapsto \mu_{\mathrm{cod}\,f} Tf$$
$$F_T : \quad x \mapsto x_T, \quad f \mapsto (\eta_{\mathrm{dom}\,f} \circ f)_T$$

Moreover one can show (cf ES 3, ex 10) that the syntactically constructed Kleisli-category is indeed equivalent (but, in general, not isomorphic) to the full subcategory of free T-algebras.

The Kleisli construction enjoys a universal property in the category of splittings of T into adjoint pairs dual to that of Eilenberg-Moore:

Theorem 2.7. *Let $F' : C \to A$, and $U' : A \to C$ be functors s.t. $F' \dashv U'$ and $T = U'F'$; then there is a unique functor $L : C_T \to A$ with $U_T = U'L$ and $F' = LF_T$.*

2.4. **Beck's monadicity theorem.** Suppose we have an adjuncion $F \dashv U$ with $U : A \to C$. We can ask the following question: *When is the comparison functor $K : A \to C^T$ an equivalence; i.e. when is U monadic?* This question is answered by the various versions of *Beck's monadicity theorem*. But what is monadicity good for? If U is monadic, we get a representation of A as C^T, and hence a representation in terms of a formally constructed category using just data from C (and endofunctors on C). So A can be seen as a category of structures defined over C. (In particular we like to think about A as a category of models of an algebraic theory T over C.) The main application of monadicity is to establish certain properties of A and of U using what we know about C^T and U^T; e.g., that U creates limits, or that A has certain limits or colimits, etc.

Example 2.8. A prominent example is to use dualities and monadicity to prove the existence of colimits from the existence of limits. In Set, for instance, we have the (contravariant) powerset functor

$$\mathcal{P} : \mathrm{Set}^{op} \to \mathrm{Set}, \qquad X \mapsto \mathcal{P}(X), \quad f \mapsto f^{-1}$$

which has 'itself' as a left adjoint

$$\mathcal{P} : \mathrm{Set} \to \mathrm{Set}^{op}, \qquad X \mapsto \mathcal{P}(X), \quad f \mapsto f^{-1}$$

One can proof $\mathcal{P}$ monadic. Since it creates limits, Set^{op} is complete, and hence Set cocomplete. This kind of argument generalizes to the contravariant powerobject functor on an (elementary) topos E, and proves that a topos E has finite colimits from the existence of finite limits.

What is the key property of C^T characterizing it as a category of T-algebras up to equivalence? It is the canonical presentation of a T-algebra as the coequalizer of free T-algebras in C^T. To see this more clearly we shall sketch the proof of the following version of Beck's theorem.

Theorem 2.9 (Beck)**.** *Let $U : A \to C$ be a functor with a left adjoint F and monad $T = UF$. Let $K : A \to C^T$ be the unique comparison functor*

  (i) *If A has coequalizers of reflexive U-split pairs, then K has a left adjoint L,*
 (ii) *If in addition to (i), U preserves these coequalizers, then the unit of $L \dashv K$ is an isomorphism*
(iii) *If in addition to (i) and (ii) U reflects isomorphisms, then the counit of $L \dashv K$ is an isomorphism.*

Proof. (Sketch)

(i) We apply the technique of wishful thinking. Suppose K has a left adjoint L, then L preserves coequalizers. Hence the canonical presentation of a T-algebra is mapped to a coequalizer in A. If we write (2.2) in terms of the adjunction $F_T \dashv U_T$ it becomes

$$F^T U^T F^T U^T (X, h) \xrightarrow[\varepsilon^T_{F^T U^T (X,h)}]{\overset{F^T U^T \varepsilon^T_{(X,h)}}{\longrightarrow}} F^T U^T (X, h) \xrightarrow{\varepsilon^T_{(X,h)}} (X, h)$$

Applying L to it and using $T = U^T F^T = UF$, $LF^T \cong F$ (by composition and uniqueness of adjoints) together with $L\varepsilon^T = \varepsilon L$ (see [Mac98, proposition IV.7.1], or prove it from $L \dashv K$ and $\varepsilon^T K = K\varepsilon$ directly) we get

$$FUFX \xrightarrow[\varepsilon_{FX}]{\overset{Fh}{\longrightarrow}} FX \longrightarrow L(X, h) \qquad (2.3)$$

to be a coequalizer. Now the pair (Fh, ε_{FX}) is U-split, since we have $UL \cong U^T$ and it is the L-image of a U^T-split pair. It is reflexive, as witnessed by the common section $F\eta_X$. A has coequalizers of reflexive U-split pairs by assumption, so we can define $L(X, h)$ as the coequalizer of (Fh, ε_{FX}). The unit can be constructed as follows: Applying $U^T K$ to the coequalizer diagram gives

$$UFUFX \xrightarrow[U\varepsilon_{FX}]{\overset{UFh}{\longrightarrow}} UFX \longrightarrow UL(X, h)$$

which can be rewritten to (recall $U\varepsilon_{FX} = \mu_X$)

$$T^2 X \xrightarrow[\mu_X]{\overset{Th}{\longrightarrow}} TX \longrightarrow UL(X, h) \qquad (2.4)$$

But since (X, h) is a T-algebra by (2.1) h is a coequalizer of the parallel pair (Th, μ_X) in C. This yields a unique arrow $X \to UL(X, h)$. Using that (2.1) is a split coequalizer in C one can check that this arrow is a T-algebra homomorphism $(X, h) \to (UL(X, h), U\varepsilon_{L(X,h)})$. We take this arrow to be the (X, h)-component of the desired unit. The counit is obtained easily from the defining coequalizer of LKA:

$$FUFUX \xrightarrow[\varepsilon_{FUA}]{\overset{FU\varepsilon_A}{\longrightarrow}} FUA \longrightarrow L(UA, U\varepsilon_A) \qquad (2.5)$$

By naturality of ε we have $\varepsilon_A FU\varepsilon_A = \varepsilon_A \varepsilon_{FUA}$, so ε_A coequalizes $(FU\varepsilon_A, \varepsilon_{FUA})$ and there is a unique arrow $LKA \to A$ in A. We take this arrow to be the A-component of the counit. One can check that both constructions give natural transformations $I_{C^T} \Rightarrow KL$, $LK \Rightarrow I_A$ and that they satisfy the triangle identities.

(ii) If U preserves coequalizers of reflexive U-split pairs then U preserves the defining coequalizer (2.3) of $L(X, h)$. But the U-image of this coequalizer is (2.4). So both $TX \to UL(X, h)$ and $h : TX \to X$ are coequalizers of the

pair (Th, μ_X) in C and the unique arrow $X \to UL(X, h)$ is an isomorphism; but this arrow is the underlying arrow of the (X, h)-component of the unit of $L \dashv K$. This shows the unit to be an isomorphism (U^T reflects isomorphisms).

(iii) By assumption U preserves coequalizers of reflexive U-split pairs, so applying U to the coequalizer (2.5) yields

$$T^2 U A \underset{\mu_{UA}}{\overset{TU\varepsilon_A}{\rightrightarrows}} TUA \longrightarrow UL(UA, U\varepsilon_A)$$

a coequalizer in C. But (2.1) with $(X, h) = (UA, U\varepsilon_A)$ is a split coequalizer, and hence preserved by any functor. Since both are coequalizers of $(TU\varepsilon_A, \mu_{UA})$ the U-image of the counit has to be an isomorphism. Now U is assumed to reflect isomorphisms, so the counit is an isomorphism, too.

Remark 2.10. By considering coequalizers of reflexive U-split pairs, we have considered the smallest class of colimits necessary to construct the left adjoint L. It is obvious from the proof that it would be sufficient to demand only the existence and preservation of coequalizers of U-split pairs only, or the existence and preservation of reflexive pairs only.

The U-split pairs are needed to characterize monadic functors U; namely, a functor U is monadic iff it has a left adjoint and creates coequalizers of U-split pairs by the 'precise' version of Beck's monadicity theorem.

Checking the conditions (i) and (ii) for U-split pairs is inconvenient in practice. It is usually much more convenient to check (i) and (ii) for reflexive pairs. The existence and preservation of reflexive pairs is, however, only sufficient but not necessary for U to be monadic.

$\square$

2.5. **2-Categorical perspective: formal theory of monads.** If we want to understand and study monads algebraically then we need the structure of a 2-category C. As it was the case for adjunctions we can just restate the definition of a monad in Cat in an arbitrary 2-category C: A monad (t, μ, η) is a 1-cell $t : c \to c$ and two 2-cells $\mu : t^2 \Rightarrow t$, $\eta : I_c \Rightarrow t$, such that the respective associativity and unit diagrams commute[1].

How to define Eilenberg-Moore objects or Kleisliobjects of the monad t in C? Firstly, what is a t-algebra? In Cat it is an element x of an object c of the 2-category Cat equipped with an arrow $tx \to x$ of c satisfying the associativity and left unit axiom. In a general 2-category we cannot speak about elements and arrows of 0-cells, but we can use the idea of *generalized elements*. A t-algebra in Cat is, for instance, a 1-cell $x : 1 \to c$ equipped with a 2-cell $\theta : tx \Rightarrow x$ satisfying

$$
\begin{array}{ccc}
t^2 x \xrightarrow{t\theta} tx & \qquad x = I_c x \xrightarrow{\eta x} tx \\
\mu x \Big\downarrow \quad\quad \Big\downarrow \theta & \qquad \qquad \underset{1_x}{\searrow} \Big\downarrow \theta \\
tx \xrightarrow[\theta]{} x & \qquad \qquad\qquad x
\end{array}
\qquad (2.6)
$$

----
[1] We shall denote by I_c the identity 1-cell on c, and by 1_c the identity 2-cell on I_c.

Here 1 denotes the terminal one-object category and by abuse of notation in (horizontal) compositions of 1-cells with 2-cells the 1-cells are identified with their unit 2-cells; e.g. μx stands for $\mu 1_x$. So ordinary t-algebras in Cat are t-algebra structures on a generalized element of c of type 1. But the definition of a t-algebra in terms of these diagrams makes sense for any generalized element $x : a \to c$ of c of type a. The terminal category 1 is a "2-separating" 0-cell in Cat, that is, for any two distinct 1-cells $F : c \to d$, $G : c \to d$ in Cat we can find 1-cells $x, y : 1 \to c$ and a 2-cell $\alpha : x \Rightarrow y$ such that $F\alpha \neq G\alpha$[2]. One doesn't need to deal with t-algebra structures on generalized elements of all types, since studying the one of type 1 already determines everything. In a general 2-category, where we might not have a (2-)separating set of 0-cells[3], we need to consider all the types.

We can restate everything in terms of hom-categories: For every 0-cell a $[a, c]$ is a category and $(t_* = [a, t], \mu_*, \eta_*)$ is a monad on $[a, c]$. The t-algebras of type a are precisely the t_*-algebras. (For the case $C = $ Cat see ES 3, ex 7*.) Since $[a, c]$ is a category, we can construct the category of t_*-algebras $[a, c]^{t_*}$. This construction can be extended to a 2-functor $[-, c]^{t_*} : C^{op} \to $ Cat, where C^{op} stands for the 2-category with the same 0-cells and 2-cells as C, but reversed 1-cells. We say that t has an *Eilenberg-Moore object* c^t if the 2-functor $[-, c]^{t_*}$ is representable, i.e., $[-, c]^{t_*} \cong [-, c^t]$ in Cat. In the case of Cat the Eilenberg-Moore objects of t is isomorphic to the category of t-algebras.

Kleisli objects for t can be defined by following the same line of idea by considering C^{op}. Since C^{op} has the same 0-and 2-cells as C, (t, μ, η) is also a monad on C^{op}. We can ask again about the representability of the 2-functor $[c, -]^{t_*} : C \to $ Cat. If this is the case, we call the representing object c_t the Kleisli object of t. In Cat this coincides (up to isomorphy) with the Kleisli category of t.

Remark 2.11. Recall that we can define limits and colimits in a category C from limits and colimits in Set by utilizing the Yoneda lemma. For example, $c, d \in \text{ob}\, C$ have a product in C iff the functor $C(-, c) \times C(-, d) : C^{op} \to $ Set is representable. This approach generalizes to *enriched categories* utilizing the enriched Yoneda lemma, and to 2-categories in particular, since 2-categories can be considered as categories enriched in Cat. The approach to Eilenberg-Moore objects presented here is an example of this: We have defined what an Eilenberg-Moore object is using representability and that we know what the category of Eilenberg-Moore algebras is in Cat.

How is this construction of the Eilenberg-Moore object c^t of t related to a (universal) splitting of the monad t into an adjunction? We obtain it from the isomorphism

$$\varphi : [-, c^t] \cong [-, c]^{t_*}.$$

The image of $\varphi_{c^t}(I_{c^t})$ is a t-algebra of type c^t; that is, a 1-cell $u^t : c^t \to c$, which we consider the "forgetful" 1-cell and a 2-cell $\alpha : tu^t \Rightarrow u^t$. For the "free t-algebra" 1-cell $f^t : c \to c^t$ we consider the φ_c-preimage of the "free" t-algebra (of type c), i.e., $(t, \mu : t^2 \Rightarrow t)$ in $[c, c]^{t_*}$. The naturality of φ yields

---

[2]This is equivalent to saying that the hom-functor $[1, -] : $ Cat $\to$ Cat is faithful. Note, however, that 1 is not a separator in the category Cat of small categories and functors!

[3]Recall the discussion of generalized elements and separating sets in categories in the supervisions.

$$(t, \mu) = \varphi_c[f^t, c^t](I_{c^t}) = [f^t, c]^{t*}\varphi_{c^t}(I_{c^t}) = (u^t f^t, \alpha f^t).$$

This gives us $t = u^t f^t$ and $\alpha f^t = \mu$. For the unit of the desired adjunction $f^t \dashv u^t$ we take the unit $\eta : I_c \Rightarrow u^t f^t$ of the monad t. To get the counit we use the naturality of φ once more:

$$\varphi_{c^t}(f^t u^t) = \varphi_{c^t}[u^t, c^t](f^t) = [u^t, c]^{t*}\varphi_c(f_t) = (t u^t, \mu u^t).$$

By the associativity law for α the 2-cell $\alpha : t u^t \Rightarrow u^t$ lifts to a t-algebra homomorphism $\alpha : (t u^t, \mu u^t) \to (u^t, \alpha)$. We define the counit $\varepsilon : f^t u^t \Rightarrow I_{c^t}$ to be $\varphi_{c^t}^{-1}(\alpha)$. The triangle equalities are left to the reader as an exercise.

So whenever an Eilenberg-Moore object exists for a monad t in C, then the monad comes from an adjunction in C. Note, however, that Eilenberg-Moore objects need not exist for every monad in an arbitrary 2-category, in general.

Remark 2.12. (a) The definition of Eilenberg-Moore objects in terms of representable 2-functors prompts one to ask about the relationship between the Eilenberg-Moore construction, a 2-adjunction, and (weighted) limits. Indeed if every monad in C has an Eilenberg-Moore object then the Eilenberg-Moore construction is the object part of a 2-functor from the 2-category $\mathrm{Mnd}(C)$ of monads in C to C. One can show that it is a 2-right adjoint to the inclusion 2-functor $C \to \mathrm{Mnd}(C)$, which maps every object c to the trivial monad $(I_c, 1_c : I_c^2 = I_c \Rightarrow I_c)$ [Str72]. One can also obtain Eilenberg-Moore objects as weighted limits [Str76].

(b) Another approach to the Eilenberg-Moore construction of a monad t would have been to translate its univeral property from Cat in the 2-category C. One can show that an Eilenberg-Moore object defined via representatbility has this universal property [Str72]. However, I haven't checked if an Eilenberg-Moore object defined via the universal property is a representing object for $[-, c]^{t*}$.

For more about the formal theory of monads see [Str72], [LS02] and the referenes therein.

3. Lawvere theories

What is an *algebraic theory*? The answer to this question given by *universal algebra* is the *signature* (consisting of sorts, operations, an arity function but no relations) together with axioms in form of equations between terms build up from operations and (typed) variables, which may either be considered to be free or universally quantified over their respective sort. This answer is unsatisfactory, because, as we have seen for groups for example, the signature and the defining equations are just one particular *presentation* of an algebraic theory, and different equivalent presentations of the same theory exist. Passing to the notion of algebraic theory in terms of first-order logic is also of little help, because the notion of first-order theory is a too general type of structure to deal with algebraic theories efficiently.

Why is this a problem? For two reasons. Firstly, in universal algebra we lack a good notion of morphism between algebraic theories, and so cannot study what algebraic

theories have in common or what makes them fundamentally different. Second, it is conceptually unsatisfactory to not have a good notion of algebraic theory.

The signature problem can be circumvented by considering all operations of the theory and all equations between them. With T-algebras of a monad we have also learned of a structure to organise and reason about this data in a convenient way. So is a monad the same as an algebraic theory? If one would be just interested in Set-models of algebraic theories one could be tempted to answer affirmatively. However, as I have explained in the supervisions on the example of groups, we can give models of the group signature in any category with finite products; this is what I called *internal groups*. For each category C with finite products where the forgetful functor from internal groups $\mathrm{Grp}(C) \to C$ is monadic we have a group monad. This monad corresponds to the algebraic theory of groups in C; but what about the algebraic theory of groups as such?

Consider, for example, the monad of groups internal to Set and the monad of topological groups. Even though we can find and study morphisms between these monads, we cannot say what the algebraic theory of groups is as such with the help of monads (if we don't want to consider the group monad on Set as the theory of groups as such and hence Set as the undisputed base category).

This is a general feature of monads: A monad always merges the syntax and semantics of the theory; we cannot say what the theory is without referring to a fixed domain of interpretation C. Furthermore, if C does not admit countable coproducts and coequalizers, then the free group functor is not likely to exist and the forgetful functor is not monadic (example?); but we can still have a non-trivial category of internal groups. Identifying monads with algebraic theories is therefore not fully satisfactory.
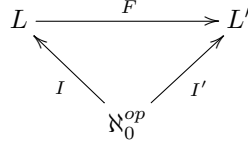
What is needed for a good definition of an algebraic theory? There are two key obsevations we can make from the preceding discussion:

(1) We need to consider all the operations of the theory and how they compose.
(2) We need to take the distinction between *syntax and semantics* seriously. The algebraic theory itself should be defined in syntactical terms only, without any explicit reference to its semantics (i.e. models).

The idea behind a *Lawvere theory* is to encode the data of all the operations of an algebraic theory $\mathcal{A}$ (in the sense of universal algebra) and how they compose in a category $L[\mathcal{A}]$. We restrict to the most important case of one-sorted, finitary algebraic theories, only.

Definition 3.1 (Lawvere theory)**.** Let $\aleph_0$ be a skeleton[4] of the category of finite sets that is closed under finite coproducts and s.t. the coproducts are strictly associative. A *Lawvere theory* L is a small category L with finite products together with a strictly product preserving functor $I : \aleph_0^{op} \to L$ that is a bijection on objects. A *map of Lawvere theories* $F : L \to L'$ is a finite-product preserving functor which renders commutative the diagram

---

[4]A *skeleton* of a category C is a full isomorphism-dense subcategory S which contains no non-trivial isomorphisms; i.e., any two objects isomorphic in S are identical.

$$L \xrightarrow{\quad F \quad} L'$$

(diagram: $L \xrightarrow{F} L'$, with $I : L \to \aleph_0^{op}$ and $I' : \aleph_0^{op} \to L'$ meeting at $\aleph_0^{op}$ below)

One can think of a Lawvere theory L as a category L with objects the natural numbers, s.t. the object n is the n-fold product of 1 with itself. 1 represents the sort. The arrows $n = 1 \times \ldots \times 1 \to 1$ are the projections, which correspond to variables, and the n-ary operations. By the universal property of products the arrows $n \to m$ are uniquely determined by the n-ary operations. We call such arrows *derived operations*. The reason for the more 'complicated' definition is that it gives more flexibility to write down Lawvere theories explicitly and stresses the importance of the aritities.

If we're given an algebraic theory $\mathcal{A}$ (of varieties) in the sense of universal algebra we can construct a Lawvere theory $L[\mathcal{A}]$ purely syntactically as described above: objects are natural numbers and arrows $n \to 1$ are the projections together with all the n-ary operations we obtain from all the compositions of the defining operations given in the signature of $\mathcal{A}$ subject to the equations between those composites as demanded by the axioms (and all the derivable equations). Conversely, Given a Lawvere theory L we obtain an algebraic theory $\mathcal{A}[L]$ in the sense of universal algebra with one sort, no relations and the n-ary operations being the arrows $n \to 1$ not the projections. The axioms are given by the equality of composites of the derived operations. Because of the syntactic nature of the construction one refers to $L[\mathcal{A}]$ as the *syntactic category* of $\mathcal{A}$.

Using the flexibility build in in our definition we can give a more hands-on construction of $L[\mathcal{A}]$, which is also important for applications and for dealing with concrete Lawvere theories, in particular. Let $F_{\mathcal{A}} : \mathrm{Set} \to \mathrm{Mod}(\mathcal{A})$ denote the free $\mathcal{A}$-model/algebra functor. Define the category $L[\mathcal{A}]$ as the dual of the full subcategory in $\mathrm{Mod}(\mathcal{A})$ generated by the $F_{\mathcal{A}}$-image of $\aleph_0$. The functor $I : \aleph_0^{op} \to L[\mathcal{A}]$ is given by the restriction of $F_{\mathcal{A}}^{op}$ to $\aleph_0^{op}$. As $F_{\mathcal{A}}$ preserves coproducts, $F_{\mathcal{A}}^{op}$ preserves products, and the products in $L[\mathcal{A}]$ can be chosen in such a way that the preservation is strict; it is a bijection on objects by construction.

Example 3.2 (Theory of equality)**.** The simplest Lawvere theory we can think of is $L_i := \aleph_0^{op}$ with I being the identity functor. The arrows are generated by the product projections $n \to 1$. $\aleph_0^{op}$ is an algebraic theory with *no* (non-trivial) operations, and hence the algebraic theory of equality. It is an initial Lawvere theory in the category of Lawvere theories.

Example 3.3 (Theory of groups)**.** To write down the Lawvere theory L_G of groups, we use the free functor construction outlined above. The objects of L_G are the free groups of n generators $F_G(n)$. Arrows $F_G(n) \to F_G(1) = \mathbb{Z}$ in L_G are the group homomorphisms $\mathbb{Z} \to F_G(n)$. The product in L_G is the coproduct in Grp restricted to the full subcategory L_G^{op}; i.e. the 'free product' of free groups.

Giving a group homomorphism $\mathbb{Z} \to F_G(n)$ is the same as to give an element of $F_G(n)$ and vice versa. Recall from the section on T-algebras of a monad T that elements of $F_G(n)$ can be regarded as formal group operations on n elements. If

we consider the n generators as formal variables $X_1, \ldots, X_n$, then an element of $F_G(X_1, \ldots, X_n)$ is a formal group operation in n-variables. But these are exactly all the n-ary group operations. So arrows $F_G(n) \to F_G(1) = \mathbb{Z}$ are indeed the n-ary group operations.

Example 3.4 (Theory of commutative rings)**.** We use the free functor construction once more. The free commutative ring on n generators is the polynomial ring $\mathbb{Z}[X_1, \ldots, X_n] =: \mathbb{Z}[n]$. These form the set of objects of the Lawvere theory of commutative rings L_R. The arrows $\mathbb{Z}[n] \to \mathbb{Z}[1] = \mathbb{Z}[X]$ in L_R are the ring homomorphisms $\mathbb{Z}[X] \to \mathbb{Z}[n]$. Recall (or check) that the coproduct in the category of commutative rings CRng is the tensor product $\otimes_\mathbb{Z}$. As it was the case for groups the coproduct restricts to a coproduct in the full subcategory L_R^{op}, and hence becomes a product in L_R.

Giving a ring homomorphism $\mathbb{Z}[X] \to \mathbb{Z}[n]$ is the same as to give an element of $\mathbb{Z}[n]$ and vice versa. The elements $\mathbb{Z}[n]$ are the polynomials in n-variables, but these are exactly the n-ary operations for commutative rings.

A set R admits a commutative ring structure iff it admits interpretations of all the n-ary operations compatible with their composition laws; that is, R admits a commutative ring structure iff for any polynomial p in n-variables we can find maps $f_p : R^n \to R$ s.t. whenever we substitute a family of n polynomials $q_1, \ldots, q_n$ for the formal variables $X_1, \ldots, X_n$ in p to obtain the polynomial $p[q_1, \ldots q_n]$ the subsequent diagram commutes

$$
\begin{array}{ccc}
R^m & \xrightarrow{(f_{q_1}, \ldots, f_{q_n})} & R^n \\
 & \searrow{\scriptstyle f_{p[q_1,\ldots,q_n]}} \quad \swarrow{\scriptstyle f_p} & \\
 & R &
\end{array}
\tag{3.1}
$$

If we use the familiar basic ring operations of the commutative ring R, then the maps $f_p : R^n \to R$ turn out to be the familiar polynomial functions on R induced by the respective polynomial p by substitution. *The slogan is that R is a ring iff we can give meaning to polynomial functions on R.*

If we replace any occurence of $\mathbb{Z}$ with an arbitrary but fixed commutative ring k in the preceding construction we obtain the Lawvere theory L_{CAlg_k} of comutative k-algebras.

Example 3.5 (Theory of C^∞-rings)**.** Let M be a smooth (paracompact, Hausdorff) manifold. What is the algebraic theory of rings of smooth functions[5] $C^\infty(M) := C^\infty(M, \mathbb{R})$? We know that $C^\infty(M)$ is a commutative $\mathbb{R}$-algebra. However, being an $\mathbb{R}$-algebra contains no structure and hence no information particular to the property of being 'smooth'.

How can we capture the property of being smooth in an algebraic theory? Apriori it is not clear what kind of algebraic axioms should determine smoothness. However, we can say something meaningful about what the n-ary operations should be. In the example of commutative rings we've seen that the n-ary operations are polynomials in n-variables over $\mathbb{Z}$, and that R is a ring iff we can give meaning to these formal

---

[5]Here 'smooth' means that the function $M \to \mathbb{R}$ has all partial derivatives of any degree, i.e., is differentiable arbitrarily often.

polynomials by actual polynomial functions on R. In the smooth case we consider the smooth functions $\mathbb{R}^n \to \mathbb{R}$ as the formal n-ary operations and define the Lawvere theory L_{C^∞} as follows. Objects are $\mathbb{R}$-vector spaces $\mathbb{R}^n$ for $n \in \mathbb{N}$ (with $\mathbb{R}^0$ being the trivial $\mathbb{R}$-vector space $\{0\}$) and morphisms $\mathbb{R}^m \to \mathbb{R}^n$ are the smooth functions. The product is given by $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$. The arity functor $I : \aleph_0^{op} \to L_{C^\infty}$ is the obvious one.

Since polynomial functions $\mathbb{R}^n \to \mathbb{R}^m$ are smooth we have a map of Lawvere theories $L_{\mathrm{CAlg}_\mathbb{R}} \to L_{C^\infty}$. This tells us that C^∞-rings are in particular commutative $\mathbb{R}$-algebras. Generalizing the observation made for commutative rings we would like to say that a C^∞-ring A is a commutative $\mathbb{R}$-algebra s.t. *any* smooth function $f : \mathbb{R}^n \to \mathbb{R}$ has an interpretation as an n-ary operation $A^n \to A$ which compose in the same way as the smooth functions do. To give meaning to this statement we need to define what we consider to be a model of a Lawvere theory first.

Definition 3.6 (Model of a Lawvere theory)**.** Let L be a Lawvere theory, and C a category with finite products. A *model* or *algebra* of L in C is a finite-product preserving functor $M : L \to C$. A homomorphisms between two models M and M' of L is a natural transformation $M \Rightarrow M'$.

Remark 3.7. The category $\mathrm{Mod}_L(C)$ of models of a Lawvere theory L in C is the full subcategory of $[L, C]$ of finite-product preserving functors. Given a map of Lawvere theories $F : L \to L'$ precomposition with F makes any model of L' a model of L. Indeed, the functor $F^* : [L', C] \to [L, C]$ restricts to a functor $F^* : \mathrm{Mod}_{L'}(C) \to \mathrm{Mod}_L(C)$.

Let's unwind the definition of a C-model M of a Lawvere theory L. As a category L is determined by the object $I(1)$ (the abstract sort) and the morphisms $I(n) \to I(1)$ for all $n \in \mathbb{N}$ (the n-ary operations) by the property that every object is a finite product of $I(1)$. M is a finite-product preserving functor, so $M(I(n)) \cong M(I(1)) \times \ldots \times M(I(1))$, and projections $I(n) \to I(1)$ are mapped to projections in C. Therefore, M is determined by $M(I(1))$, by the M-images of all the (non-trivial) n-ary operations $M(I(n)) \to M(I(1))$ and the requirement of being compatible with the compositions.

Conversely, to give a model of L in C amounts to give an object X and for every (non-trivial) n-ary operation $p : I(n) \to I(1)$ C-arrows $f_p : X^n \to X$ s.t. for any family of operations $q_i : I(n_i) \to I(1)$, $1 \le i \le n$, the subsequent diagram commutes

$$\begin{array}{ccc} X^m & \xrightarrow{\ (f_{q_1},\ldots,f_{q_n})\ } & X^n \\ & \searrow{\scriptstyle f_{p \circ (q_1,\ldots,q_n)}} \quad \swarrow{\scriptstyle f_p} & \\ & X & \end{array} \tag{3.2}$$

where $m = \sum_{i=1}^n k_i$. For $C = \mathrm{Set}$ this means to give a set X and for each n a family of n-ary operations which compose according to the formal rules as described by L. We arrive at a generalization of the observation we made for commutative rings as desired: *A set X is an L-model iff we can give meaning to the formal operations $I(n) \to I(1)$ by actual operations $X^n \to X$.*

Let $M, M' : L \to C$ be two C-models of L. A natural transformation $\alpha : M \Rightarrow M'$ is uniquely determined by its component $\alpha_{I(1)} : M(I(1)) \to M'(I(1))$. This is a consequence of the universal property of products, M, M' preserving finite products, and α commuting with the projections. Conversely, any arrow $h : M(I(1)) \to M'(I(1))$ s.t. for every operation $p : I(n) \to I(1)$ the following diagram commutes

$$
\begin{array}{ccc}
M(I(n)) & \xrightarrow{h \times \ldots \times h} & M'(I(n)) \\
\scriptstyle{M(p)} \downarrow & & \downarrow \scriptstyle{M'(p)} \\
M(I(1)) & \xrightarrow{h} & M'(I(1))
\end{array}
\tag{3.3}
$$

determines uniquely a morphism $\alpha^h : M \Rightarrow M'$ with $\alpha^h_{I(1)} = h$. For $C = \mathrm{Set}$ a morphism of Set-models of Lawvere theories is thus a map between the underlying sets compatible with all the n-ary operations.

Example 3.8 (Objects). A model M of the Lawvere theory L_i of equality in C is just an object X. Indeed, $\aleph_0$ is the free finite-coproduct completion of the terminal category 1, so $L_i = \aleph_0^{op}$ is the free finite-product completion of 1. Any finite-product preserving functor $F : L_i \to C$ is determined by $F(1)$ up to isomorphism, and any object of C induces a finite-product preserving functor $F_c : L_i \to C$ with $F_c(1) = c$.

Since there are no non-trivial operations in L_i, morphisms of models $M \Rightarrow M'$ can be identified with the arrows between the underlying objects $M(I(1)) \to M'(I(1))$. In fact, the evaluate-at-$I(1)$ functor yields an equivalence of categories $\mathrm{Mod}_{L_i}(C) \simeq C$. In terms of structures the algebraic theory of equality can hence be considered as a theory of objects.

Example 3.9 (Groups). Any group G gives us a Set-model of M_G of L_G with $M_G(\mathbb{Z}) \cong G$: take for M_G the hom-functor

$$
\hom(-, G) : \mathrm{Grp}^{op} \to \mathrm{Set}
$$

and restrict it to the subcategory L_G. As a hom-functor it is product preserving, and since $\mathbb{Z}$ is the free group on 1 generator, we have $\hom(\mathbb{Z}, G) \cong G$.

More generally any internal group (G, μ, u, S) in C induces a C-model of L_G. This follows from the fact that the binary operation of group multiplication $X_1 X_2 \in F_G(X_1, X_2)$, the 0-ary operation of identity element $e \in F_G(\emptyset) = \{e\}$, and the unary operation of inversion $X^{-1} \in F_G(X) = \mathbb{Z}$ generate all the other n-ary operations by composition for every n. To give a model of L_G in C, instead of giving arrows $f_p : G^n \to G$ for all n-ary operations $p \in F_G(n)$ s.t. the respective diagrams in (3.2) commute, it suffices to give arrows $f_{X_1 X_2} := \mu : G^2 \to G$, $f_e := u : 1 \to G$ and $f_{X^{-1}} := S : G \to G$ s.t. the following diagrams corresponding to the familiar group axioms of associativity, left and right unit law, and existence of inverses commute

$$G \times G \times G \xrightarrow{1_G \times \mu} G \times G \qquad\qquad G \cong 1 \times G \xrightarrow{u \times 1_G} G \xleftarrow{1_G \times u} 1 \times G \cong G$$

$$\mu \times 1_G \downarrow \qquad\qquad \downarrow \mu \qquad\qquad 1_G \searrow \quad \downarrow \mu \quad \swarrow 1_G$$

$$G \times G \xrightarrow{\mu} G \qquad\qquad\qquad\qquad G$$

$$G \times G \xrightarrow{1 \times S} G \times G$$

$$\Delta \nearrow \qquad\qquad \searrow \mu$$

$$G \xrightarrow{!_G} 1 =\!=\!=\!=\!= 1 \xrightarrow{u} G$$

$$\Delta \searrow \qquad\qquad \nearrow \mu$$

$$G \times G \xrightarrow{S \times 1} G \times G$$

i.e. that (G, μ, u, S) is an internal group in C. The commutativity of all the diagrams in (3.2) can be inferred from these. (Exercise :-)) Now consider any model $M : L_G \to C$. Amongst all the operations we have in particular $\mu := M(X_1 X_2)$, $u := M(e)$, $S := M(X^{-1})$, and the commutativity of the diagrams in (3.2) yields the commutativity of the defining diagrams of an internal group. So $(M(I(1)), \mu, u, S)$ is an internal group object in C. Both constructions can be lifted to functors which yield an equivalence of categories between the category of C-models of L_G and the category of internal groups in C.

$$\mathrm{Mod}_{L_G}(C) \simeq \mathrm{Grp}(C)$$

The concept of syntactic category L_G satisfies the two imposed requirements of (1) encoding all the operations of the theory of groups and (2) the separation of syntax and semantics. Hence we identify the algebraic theory of groups as such with the Lawvere theory L_G.

Example 3.10 (Commutative rings). For commutative rings and L_R we can make the same observations as for groups and L_G. Using the construction of L_R via finitely generated free commutative rings, that is, polynomial rings in finite variables, any commutative ring R yields via $\hom(-, R) : L_R \to \mathrm{Set}$ a Set-model of L_R. More generally we obtain an equivalence of the category $\mathrm{Mod}_{L_R}(C)$ of C-models of L_R and the category $\mathrm{CRng}(C)$ of internal commutative rings in C. This equivalence uses the characterization of L_R-models and the fact that the commutative diagrams corresponding to the standard axioms of a commutative ring imply the commutativity of all the diagrams in (3.2). We can thus identify the theory of commutative rings as such with the Lawvere theory L_R.

Remark 3.11. One might raise the objection that the construction of the syntactic categories L like L_G and L_R uses Set-models, and that we do not truly separate between syntax and semantics. This is, however, just a convenient presentation; the syntactic nature of L, namely encoding the operations and how they compose remains untouched by the choice of presentation. We could have constructed the syntactic categories also purely syntactically, as indicated above.

Example 3.12 (C^∞-rings)**.** The motivation behind the theory of C^∞-rings was to formulate an algebraic theory of smoothness. In particular, we wanted to capture the algebraic structure of $C^\infty(M)$ the ring of smooth functions on a manifold M. So is $C^\infty(M)$ a C^∞-ring? Unlike the case of groups and commutative rings we don't have explicit finite presentations of the theory at hand. Therefore we need to give the concrete operations for any smooth function $f \in C^\infty(\mathbb{R}^n)$. First note that for any open $U \subset \mathbb{R}^n$ the ring $C^\infty(U)$ is a C^∞-ring. We assign to any smooth $f : \mathbb{R}^m \to \mathbb{R}$ the operation

$$f_* : C^\infty(U)^m \to C^\infty(U), \qquad (g_1, \ldots, g_n) \mapsto f \circ (g_1, \ldots, g_n)$$

where $f \circ (g_1, \ldots, g_n) : U \to \mathbb{R}$ denotes the function $x \mapsto f(g_1(x), \ldots, g_n(x))$. One can check that this indeed makes all the respective diagrams in (3.2) commute.

A next important observation is that any quotient R/I of a C^∞-ring R by an ideal $I \subset R$ (in the ring theoretic sense) yields a C^∞-ring again. In particular any $C^\infty(\mathbb{R}^n)/I$ is a C^∞-ring. (This is a consequence of Hadamard's Lemma for smooth functions, [MR91, proposition 1.2])

Now we can turn to $C^\infty(M)$ for a manifold M. The Whitney embedding theorem allows us to consider M as a submanifold of some $\mathbb{R}^n$ and the ε-neighborhood theorem says that $M \subset \mathbb{R}^n$ is a smooth retract of an open subset $U \subset \mathbb{R}^n$, i.e., the inclusion $i : M \hookrightarrow U$ has a smooth left-inverse $r : U \to M$. Since $C^\infty(-, \mathbb{R}) :$ $\mathrm{Mfd}^{op} \to \mathrm{CRng}$ is a functor, we get that $C^\infty(M)$ is a retract, and in particular a quotient of the C^∞-ring $C^\infty(U)$. This proves $C^\infty(M)$ a C^∞-ring. In fact, a more detailed study of this argument reveals that $C^\infty(M)$ is even a finitely presented C^∞-ring. (See [MR91] proposition 1.1., corollary 2.2 and theorem 2.3)

Remark 3.13 (Why considering C^∞-rings? Algebraic Geometry of smooth manifolds and SDG). Why are we interested in such 'weird' algebraic theories like that of C^∞-rings? Duals of categories of C^∞-rings provide convenient categories in which we can embed the category Mfd of smooth manifolds (which is a category lacking a lot of nice properties and having ill-behaved limits). Let $\mathcal{C}^\infty$ denote the category of finitely generated C^∞-rings, then the functor

$$C^\infty(-, \mathbb{R}) : \mathrm{Mfd} \to (\mathcal{C}^\infty)^{op}$$

is full and faithful. It is not an equivalence, however: First $(\mathcal{C}^\infty)^{op}$ has all finite limits, whereas Mfd doesn't, and $(\mathcal{C}^\infty)^{op}$ contains objects like the ring of dual numbers $\mathbb{R}[\varepsilon] := \mathbb{R}[X]/(X^2)$, which is not the ring of smooth functions for any smooth maifold M. $\mathbb{R}[X]/(X^2)$ is a C^∞-ring, since $\mathbb{R}[X]/(X^2) \cong C^\infty(\mathbb{R})/(X^2)$ by *Hadamard's Lemma*. We can also give the operations explicitly: $\forall f \in C^\infty(\mathbb{R}^n)$

$$q_f : \mathbb{R}[\varepsilon]^n \to \mathbb{R}[\varepsilon], \qquad (a_i + b_i \varepsilon)_{1 \leq i \leq n} \mapsto f(a_1, \ldots, a_n) + \sum_{i=1}^{n} \partial_i f(a_1, \ldots, a_n) b_i \varepsilon$$

The point of view one adopts for $(\mathcal{C}^\infty)^{op}$ is that of formal smooth varieties. This viewpoint comes from Algebraic Geometry, where one considers duals of categories

of (finitely presented) k-algebras as formal varieties over k[6] For instance, the formal smooth variety $D \in \mathrm{ob}(C^\infty)^{op}$ corresponding to the ring $\mathbb{R}[\varepsilon]$ (i.e. $D = \mathbb{R}[\varepsilon]$ as objects) even though not a manifold carries the geometric meaning of an infinitesimal line segment; for C^∞-maps $C^\infty(M) \to \mathbb{R}[\varepsilon]$, i.e., maps $D \to M$ of formal smooth varieties correspond one-to-one to tangent vectors of the manifold M.

$(\mathcal{C}^\infty)^{op}$ is a convenient category to apply and use techniques of Algebraic Geometry to study smooth manifolds. (For this one should probably restrict $(\mathcal{C}^\infty)^{op}$ to the full subcategory of formal smooth varieties of finitely presented C^∞-rings.) For an approach to the Algebraic Geometry over C^∞-rings via schemes see [Joy10].

The most important use of C^∞-rings so far is to construct so called *well adapted* topos models of *Synthetic Differential Geometry* (SDG). SDG attempts to make the notion of infinitesimals mathematically rigorous. In contrast to *Non-Standard Analysis*, which has the same goal, SDG takes the approach from Algebraic Geometry via nilpotents. This approach turns out to be very close to the intuition used by differential geometers like Sophus Lie to reason about infinitesimals.

More precisely one postulates a $\mathbb{Q}$-algebra R, of which one thinks of as the algebraic model of the geometric (bipointed) affine line. The subset $D := \{d \in R \mid d^2 = 0\}$ is the algebraic model of the (pointed) infinitesimal line segment. (One can think of it as the intersection of the unit circle with the affine line tangent to it.) How can we characterize that D is an infinitesimal line segment? Combining the intuition that every infinitesimal curve should be a line segment together with the categorical point of view that the object is determined by arrows from it leads to the following answer: D is mapped by any map $f : D \to R$ in a linear affine way.

> **Axiom** 3.14 (Kock-Lawvere). *Let $f : D \to R$ be any map. Then there are unique $a, b \in R$ such that $f(d) = a + bd$ for all $d \in D$.*

One consequence of this axiom is that every map $f : R \to R$ is differentiable arbitrarily often in the following sense. To define $f' : R \to R$ at $x_0 \in R$ consider the map $D \to R, \quad d \mapsto f(x_0 + d)$, i.e. how f behaves in the (first-oder) infinitesimal neighbourhood of x_0. The Kock-Lawvere axiom gives us a unique $f'(x_0) \in R$ s.t.

$$f(x_0 + d) = f(x_0) + f'(x_0)d, \qquad \forall d \in D.$$

Within SDG one can develop a lot of results of Analysis in finite (and infinite dimensions) using the algebra of nilpotents very easily. Adding further Kock-Lawvere type axioms for coordinate spaces of infinitesimal geometric objects like infinitesimal squares, cubes, pencils of infinitesimal line segments with a common basepoint, etc. enables one to develop a lot of differential geometry in SDG, and in particular on spaces, which are much more general than manifolds; e.g. function spaces between manifolds, or the group of diffeomorphisms of a manifold.

So far we've been only talking about the theory; what about a model of the Kock-Lawvere axiom? The first result one can obtain very easily is that there is no

---

[6]This is due to the fact that for algebraically closed fields k we have Hilbert's Nullstellensatz, which can be read geometrically as saying that up to "infinitesimals" algebraic varieties can be distinguished by their rings of polynomial "coordinate" functions.

(non-trivial) model of SDG in any Boolean topos; in particular, there is no non-trivial model in Set. In other words, we don't have any model except for the trivial $\mathbb{Q}$-algebra $R = 0$ in classical logic. This is due to the Law of Excluded Middle, which allows us to construct a map $f : D \to R$ that takes the value 0 at 0 and the value 1 everywhere else. The Kock-Lawvere axiom can only hold for such an f iff $0 = 1$ in R. In 'intuitionistic' logic (more precisely Heyting logic) the Law of Excluded Middle doesn't hold, and the function f cannot be constructed.

Indeed, we can construct topos models for SDG using the category $\mathcal{C}^\infty$. The simplest one is the topos of presheaves on $(\mathcal{C}^\infty)^{op}$: $[\mathcal{C}^\infty, \mathrm{Set}]$. Composing the Yoneda-embedding y with the fully faithful embedding of $C^\infty(-, \mathbb{R}) : \mathrm{Mfd} \to (\mathcal{C}^\infty)^{op}$ gives us a fully faithful embedding

$$\iota : \mathrm{Mfd} \hookrightarrow [\mathcal{C}^\infty, \mathrm{Set}],$$

which preserves the "good" limits in Mfd, namely, pullbacks of smooth maps that are transversal, and the terminal object (M&R theorem 2.8). The real line $\mathbb{R}$ is mapped to the representable functor $R = \hom_{C^\infty}(C^\infty(\mathbb{R}), -)$, which is just the forgetful functor $C^\infty \to \mathrm{Set}$ by the universal property of the free C^∞-ring on one generator $C^\infty(\mathbb{R})$. Unpacking the definition of D in the internal logic of the topos amounts to form the obvious equalizer, which is computed pointwise; so $D(A) = \{a \in A \mid a^2 = 0\}$, for any finitely generated C^∞-ring A. Due the universal property of $\mathbb{R}[\varepsilon]$ this functor is seen to be representable as well: $D \cong \hom_{C^\infty}(\mathbb{R}[\varepsilon], -)$.

To show the Kock-Lawvere axiom being satisfied in $[\mathcal{C}^\infty, \mathrm{Set}]$ we can use *sheaf semantics* (see [MM92, section 4.7], and for this particular case [MR91, section II.2] for details). In arrow-theoretic form the Kock-Lawvere axiom holds in $[\mathcal{C}^\infty, \mathrm{Set}]$ iff for all finitely presented C^∞-rings A and all arrows $f : \hom_{C^\infty}(A, -) \to R^D$ there are arrows $a, b : \hom_{C^\infty}(A, -) \to R$ s.t. the subsequent diagram commutes

$$
\begin{array}{ccccc}
 & & R \times R \times D \xrightarrow{(+)\times 1_D} R \times D \hookrightarrow R \times R & & (3.4) \\
 & \nearrow^{(a,b)\times 1_D} & & & \searrow^{(\cdot)} \\
\hom_{C^\infty}(A, -) \times D & & & & R \\
 & \searrow_{f \times 1_D} & & \nearrow_{\mathrm{ev}} & \\
 & & R^D \times D & &
\end{array}
$$

and any other pair of arrows $a', b' : \hom_{C^\infty}(A, -) \to R$ satisfying that for any C^∞-ring B and homomorphism $h : A \to B$

$$R \times R \times D \xrightarrow{(+) \times 1_D} R \times D \hookrightarrow R \times R \qquad (3.5)$$

with diagonal arrow $(h^* \circ a', h^* \circ b') \times 1_D$ from $\hom_{C^\infty}(B,-) \times D$, arrow $(\cdot)$ from $R \times R$ to R, arrow $(f \circ h^*) \times 1_D$ from $\hom_{C^\infty}(B,-) \times D$ to $R^D \times D$, and arrow ev from $R^D \times D$ to R.

commutes, then $a' = a$ and $b' = b$. The first diagram is the interpretation of the existential statement and the second of the uniqueness assertion.

Since R and D are both representables, we can use Yoneda to translate the commutativity of the diagrams in statements about C^∞-rings. The natural transformation f corresponds to an $f \in R^D(A) = \text{Nat}(\hom_{C^\infty}(A,-) \times D, R)$; but applying Yoneda and representability of R and D once more we can compute $\text{Nat}(\hom_{C^\infty}(A,-) \times D, R)$ to be the ring $\hom_{C^\infty}(C^\infty(R), A[\varepsilon]) \cong A[\varepsilon]$. A is finitely generated, so $A \cong C^\infty(\mathbb{R}^n)/I$ and $A[\varepsilon] = (C^\infty(\mathbb{R}^n)/I)[X]/(X^2) \cong C^\infty(\mathbb{R}^n \times \mathbb{R})/(I, t^2)$, where t stands for the projection $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ to the last component. Hence $f \in R^D(A)$ can be identified with a class of smooth functions $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $(x, t) \mapsto f(x, t)$ modulo I in the first variable and modulo t^2 in the second. The natural transformations a, b correspond to elements of A, i.e., to classes of smooth functions $\mathbb{R}^n \to \mathbb{R}$ modulo I. Elements $d \in D(A)$ are classes of smooth functions $\mathbb{R}^n \to \mathbb{R}$ modulo I whose square is I.

Since the hom-functor maps coproducts to products in the first argument the functor $\hom_{C^\infty}(A,-) \times D$ is representable with representing object $A \otimes_\infty \mathbb{R}[\varepsilon] \cong A[\varepsilon]$. The natural isomorphism $\hom_{C^\infty}(A,-) \times D \cong \hom_{C^\infty}(A[\varepsilon],-)$ at stage $B \in (\mathcal{C}^\infty)^{op}$ sends a C^∞-homomorphisms $h : A[\varepsilon] \to B$ to $(h \circ i_A, h(\varepsilon))$, where $i_A : A \hookrightarrow A[\varepsilon]$ is the canonical inclusion $a \mapsto a + 0\varepsilon$. For $B = A[\varepsilon]$ the natural transformation $(a, b) \times 1_D$ corresponds under the Yoneda-isomorphism to the triple $(a + 0\varepsilon, b + 0\varepsilon, \varepsilon) \in A[\varepsilon] \times A[\varepsilon] \times D(A[\varepsilon])$, and ε is the class of smooth functions $t \mod t^2$. Similarly, the natural transformation $f \times 1_D$ corresponds to the pair $(f + 0\varepsilon, \varepsilon)$.

The ring structure of R is given pointwise; that is, the ring structure of $R(A[\varepsilon]) = A[\varepsilon]$ is that of $A[\varepsilon]$, so the composite of the upper path of the diagram (3.4) translates to the class of smooth functions

$$(x, t) \mapsto a(x) + b(x) \cdot t \mod (I, t^2)$$

For the lower path evaluation $\text{ev}_{A[\varepsilon]}(f + 0\varepsilon, \varepsilon) \in R(A[\varepsilon]) = A[\varepsilon]$ is the class $(x, t) \mapsto f(x, t) \mod (I, t^2)$, i.e. f. The commutativity of the diagram (3.4) is equivalent to saying that for each $f \in R^D(A)$ we can find $a, b \in A$ s.t.

$$f(x, t) = a(x) + b(x) \cdot t \mod (I, t^2) \qquad (3.6)$$

For the uniqueness assertion note that the commutativity of the diagram (3.5) just states that (3.6) is stable under C^∞-homomorphisms $h : A \to B$. Indeed, by the naturality of Yoneda the natural transformation $f \circ h^*$ corresponds to $h[\varepsilon](f)$

with f understood as the element in $R^D(A) = A[\varepsilon]$ and $h[\varepsilon]$ being the induced C^∞-homomorphism $A[\varepsilon] \to B[\varepsilon]$ by h (by functoriality of the construction of dual numbers). Similarly, $a \circ h^*$ and $b \circ h^*$ correspond to $h(a)$ and $h(b)$, respectively. Stability of (3.6) under h just means[7]

$$h[\varepsilon](f)(x,t) = h(a)(x) + h(b)(x) \cdot t \mod (I, t^2)$$

But (3.6) just uses the ring structure and is stable under any ring homomorphism. From the viewpoint of C^∞-rings the uniqueness assertion in (3.5) just says that the classes a and b in (3.6) are unique with this property.

Let $f \in R^D(A)$ be given and fix an arbitrary representant, also denoted by f. By Hadamard's Lemma there is a smooth function $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ s.t.

$$f(x,t) = f(x,0) + \partial_t f(x,0) \cdot t + g(x,t)t^2.$$

With $a := f(-,0) \mod I$, $b := \partial_t f(x,0) \mod I$ (3.6) is satisfied. As for the uniqueness for any a and b satisfying (3.6) there are smooth $u, v \in I$ and $w \in (I, t^2)$ s.t.

$$f(x,t) = a(x) + u(x) + (b(x) + v(x))t + w(x,t)t^2.$$

Taylor expansion in t at $t = 0$ parametrized by $x \in \mathbb{R}^n$ yields $a(x) + u(x) = f(x,0)$ and $b(x) + v(x) = \partial_t f(x,0)$. This shows the uniqueness.

So the Kock-Lawvere axiom does hold for R in $[\mathcal{C}^\infty, \mathrm{Set}]$. In fact, much stronger axioms are satisfied in this topos. Besides various Kock-Lawvere type axioms for "all" infinitesimal objects (properly defined using the structure of *Weil algebras*) another property of $[\mathcal{C}^\infty, \mathrm{Set}]$ is the satisfaction of the *Integration Axiom*, which states that any function $f : R \to R$ has a primitive:

> *Axiom* 3.15 (Integration). *Let $f : R \to R$ be any map. There is a unique $g : R \to R$ s.t. $g' = f$ and $g(0) = 0$.*

For more on SDG see [Koc09], [Lav96], [Bel09]; for the construction of topos models using C^∞-rings, see [MR91].

3.1. **Free Set-models of a Lawvere theory.** The construction of the syntactic category of a Lawvere theory of an algebraic theory (in the sense of universal algebra) in terms of finitely generated free Set-models is another instance of the Yoneda Lemma. Let L be a Lawvere theory. Consider $\mathrm{Mod}_L(\mathrm{Set}) \hookrightarrow [L, \mathrm{Set}]$. For the latter we have the Yoneda embedding

$$y : L^{op} \to [L, \mathrm{Set}], \qquad I(n) \mapsto \hom_L(I(n), -), \quad f \mapsto f^*$$

But the representables $\hom_L(I(n), -)$ preserves finite products, so y factors through $\mathrm{Mod}_L(\mathrm{Set})$. What kind of models of L are the representables?

---

[7]The notation is slightly misleading. Note that h is a map of classes, which doesn't need to be induced by an underlying C^∞-homomorphism between the free C^∞-rings. Therefore, $h(a)(x)$ has to be read as choosing an arbitrary representant of $h(a)$, etc.

Proposition 3.16. *The representable models $y(I(n)) = \hom_L(I(n), -)$ of L in* Set *are the free L-models on n generators.*

Proof. Let $M \in \mathrm{Mod}_L(\mathrm{Set})$. The Yoneda Lemma yields $\mathrm{Nat}(y(I(n)), -), M) \cong M(I(n))$. Since $M \circ I$ preserves finite products, we have

$$\mathrm{Nat}(y(I(n)), M) \cong M(I(n)) = M(I(1))^n$$

This reads that every morphism of L-models from $y(I(n))$ to any L-model M corresponds one-to-one to an n-tuple of elements of the undelying set $M(I(1))$ of M, and that the bijection is natural in M. In other words, $y(I(n))$ has the universal property of the free L-model on n generators. $\square$

The Yoneda embedding y induces an equivalence between L^{op} and the full subcategory generated by the finitely generated free L-models in Set. This is the abstract reason why the syntactic category L of a Lawvere theory L can be presented as the opposite of the category of finitely generated free L-models in Set we have used in all the examples except for C^∞-rings.

The restriction U_L of the evaluation-at-$I(1)$ functor $\mathrm{ev}_{I(1)} : [L, \mathrm{Set}] \to \mathrm{Set}$ to the full subcategory $\mathrm{Mod}_L(\mathrm{Set})$ maps each L-model to its underlying set. We can consider U_L as the "forgetful" functor from L-models to Set. Having identified the free L-models on n-generators as the representables we have

$$\mathrm{Nat}(y(I(n)), M) \cong M(I(1))^n = \mathrm{Set}(\{1, \ldots, n\}, U_L(M)) \qquad (3.7)$$

Is there a free L-model for any set $X \in \mathrm{Set}$, i.e., does U_L have a left adjoint F_L? Yes, for purely abstract reasons. Every set $X \in \mathrm{Set}$ is a *filtered colimit*[8] of its finite subsets. Equivalently, X is the filtered colimit of the canonical projection functor $P_X : (\aleph_0 \downarrow X) \to \mathrm{Set}$ restricted to its full subcategory $J(X)$ of injective maps. Here $\aleph_0$ is considered as a full subcategory of Set.

The finitely generated free L-models functor we have already is

$$y \circ I^{op} : \aleph_0 \to \mathrm{Mod}_L(\mathrm{Set}) \hookrightarrow [L, \mathrm{Set}]$$

This functor can be extended via the filtered-colimit construction to the whole category Set: for a set X define

$$F_L(X) := \varinjlim J(X) \xrightarrow{P_X} \aleph_0 \xrightarrow{y \circ I^{op}} \mathrm{Mod}_L(\mathrm{Set}) \ .$$

This is well-defined, since $[L, \mathrm{Set}]$ is cocomplete, and it is an L-model, since filtered colimits in Set commute with finite limits, in particular with finite products [Mac98, theorem IX.2.1]. Indeed, as limits and colimits are computed pointwise in $[L, \mathrm{Set}]$ filtered colimits of finite-product preserving functors are finite-product preserving.

Because of (3.7) the functor F_L is left adjoint to U_L

---

[8] A filtered colimit in a category C is a colimit of a functor $J \to C$, where the category J is filtered; i.e., every finite diagram in J has a cocone in J.

$$\text{Nat}(F_L(X), M) = \text{Nat}(\varinjlim_{m : I(n) \hookrightarrow X} y(I(n)), M)$$

$$= \varprojlim_{m : I(n) \hookrightarrow X} \text{Nat}(y(I(n)), M) \cong \varprojlim_{m : I(n) \hookrightarrow X} \text{Set}(\{m(1), \dots, m(n)\}, U_L(M))$$

$$= \text{Set}(\varinjlim_{m : I(n) \hookrightarrow X} m(I(n)), U_L(M))$$

$$\cong \text{Set}(X, M)$$

The construction of F_L in this way is one of the many extension-by-colimits (or limits) examples I mentioned in the supervisions. However, there is also a more concrete and hands on construction of $F_L(X)$ as the L-model with underlying set all the formal L-terms build from elements of X up to equality[9].

Indeed, the underlying set of an L-model M has to contain the image of every n-ary operation $M(q : I(n) \to I(1))$. Using this observation we can define $F_L(X)(I(1))$ as an apropriate set of formal L-terms

$$F_L(X)(I(1)) := \coprod_{n \in \mathbb{N}} \{q(x_1, \dots, x_n) \mid q \in L(I(n), I(1)), \ x_1, \dots, x_n \in X\} / \sim$$

The equivalence relation $\sim$ identifies $q(x_1, \dots, x_n)$ with $r(x_1, \dots, x_i, \dots \hat{x}_j \dots, x_n)$ (the hat stands for omitting the argument), whenever $x_i = x_j$ and the formal n-ary operation $q : I(n) \to I(1)$ coincides with $r \circ (\text{pr}_1, \dots, \text{pr}_i, \dots \text{pr}_i \dots \text{pr}_n)$ where the second pr_i is at the j-th position of the n-tuple, and $r : I(n-1) \to I(1)$. The realization $F_L(X)(q)$ of a formal n-ary operation q is the obvious map

$$(q_1(x_1^1, \dots, x_{k_1}^1), \dots, q_n(x_1^n, \dots, x_{k_n}^n)) \mapsto q \circ (q_1, \dots, q_n)(x_1^1, \dots, x_{k_1}^1, \dots, x_{k_n}^n)$$

It is straight forward (from the definition of $\sim$) that this makes (3.2) commute, and $F_L(X)$ to a free L-model generated by X.

Remark 3.17. The underlying set of the free L-model on n generators $y(I(n))$ is the set of formal operations $I(n) \to I(1)$. This seems to be in disagreement with $F_L(\{1, \dots, n\})$ constructed in the way above. However, using projections and respective diagonal maps ("doubling an argument") one can see that for every formal expression $r(n_1, \dots, n_k)$ for a k-ary operation r there is an n-ary operation q s.t. $q(1, \dots, n)$ is equivalent to $r(n_1, \dots, n_k)$.

Remark 3.18 (Coend formula for $F_L(X(I(1)))$). There is a conceptually more convenient representation of $F_L(X)(I(1))$ as a *coend* (see [Mac98, section IX.6] for a definition and elementary properties)

$$F_L(X)(I(1)) = \int^{n \in \aleph_0} L(I(n), I(1)) \times X^n \tag{3.8}$$

---

[9]Compare this with the viewpoint on elements of free groups, free monoids and free T-algebras of a monad T on Set as the formal operations of a theory.

More precisely, $F_L(X)(I(1))$ is the coend of the bifunctor

$$(y(I(1)) \circ I^{op}(-)) \times X^{(-)} : \aleph_0 \times \aleph_0^{op} \to \mathrm{Set},$$

where $X^{(-)}$ denots the unique product preserving functor $\aleph_0^{op} \to \mathrm{Set}$ (up to iso-morphism) that maps 1 to X. This coend-representation is much more useful for algebraic computations than the coproduct formula, due to the nice calculus of coends.

Remark 3.19 (Completeness and cocompleteness of $\mathrm{Mod}_L(\mathrm{Set})$). The category $[L, \mathrm{Set}]$ is complete, cocomplete and the limits and colimits are computed pointwise. Since finite limits commute with finite limits, finite limits of product preserving functors in $[L, \mathrm{Set}]$ give finite-product preserving functors again. The category $\mathrm{Mod}_L(\mathrm{Set})$ is therefore closed under small limits in $[L, \mathrm{Set}]$. Besides completeness this tells us that limits in $\mathrm{Mod}_L(\mathrm{Set})$ are computed pointwise. The same is true for filtered colimits.

To see that $\mathrm{Mod}_L(\mathrm{Set})$ has all small colimits, because of the filtered colimits, we only need to show the existence of finite coproducts [Mac98, theorem IX.1.1]. This can be done by representing L-models as quotients of free L-models by equivalence relations compatible with all the n-ary operations. However, this approach can also be used to show that $\mathrm{Mod}_L(\mathrm{Set})$ is a reflective subcategory of $[L, \mathrm{Set}]$: Given a functor $G : L \to \mathrm{Set}$, consider the free L-model $F_L(G(I(1)))$ and form the quotient by the equivalence relation generated by the identifications induced by $G(q)$ for any n-ary operation $q : I(n) \to I(1)$, i.e., $F_L(q)(x_1, \ldots, x_n) \sim G(q)y$ for $y \in G(I(n))$ and $x_i := G(\mathrm{pr}_i)y$ (see [Law63, theorem IV.1.1] and its proof for details.) Recall that a reflective subcategory of a category, which is cocomplete, is itself cocomplete.

3.2. **Syntax-Semantics duality.** In the previous section we've seen that the ab-stract reason behind the fact that a Lawvere theory can be recovered as the opposite category of its finitely generated free Set-models is the Yoneda Lemma; but is there a deeper conceptual reason? There is; we have a general *syntax-semantics ad-junction* which generalizes the equivalence of a Lawvere theory with the opposite category of its finitely generated Set-models. To be more precise we have a con-travariant syntax-semantics adjunction between the category of Lawvere theories and a certain category of categories over any category C with finite products. We shall sketch the construction of this adjunction, but leave out the technical details, which mainly have to deal with the size issues involved in this construction. (See [Law63] for the details in the case $C = \mathrm{Set}$.)

We have seen in remark 3.7 that for a fixed category C with finite products the assignment $L \mapsto \mathrm{Mod}_L(C)$ is functorial for morphisms of Lawvere theories. In fact, every category of models $\mathrm{Mod}_L(C)$ admits a 'forgetful' functor $U_L : \mathrm{Mod}_L(C) \to C$ by evaluating every model at $I(1)$. In other words, we map an L-model in C to its underlying object. Since morphisms $F : L \to L'$ of Lawvere theories commute with the arity functors, the induced morphisms $F^* : \mathrm{Mod}_{L'}(C) \to \mathrm{Mod}_L(C)$ commute with the 'forgetful' functors.

Let C be a category with finite products. We denote by $\mathrm{Sem}(C)$ a category of (certain) categories over C, i.e., objects are (certain) functors $U : D \to C$ and morphisms $F : U \to U'$ are functors $D \to D'$ s.t.

$$D \xrightarrow{\quad F \quad} D'$$

(diagram: $D \xrightarrow{F} D'$ with $U : D \to C$ and $U' : D' \to C$ commuting over C)

So we get a *semantics functor*

$$\mathcal{S} : \mathrm{Law}^{op} \to \mathrm{Sem}(C), \qquad C \mapsto (U_L : \mathrm{Mod}_L(C) \to C), \quad F \mapsto F^*$$

Does this functor have a left adjoint? We would need to extract a Lawvere theory from any 'forgetful' functor $U : D \to C$. In the case of $C = \mathrm{Set}$ and if U has a left adjoint F the natural candidate would be the opposite of the category of finitely generated L-models. For an arbitrary category C with finite products and functor U, however, U doesn't need to have a left adjoint; and even if that is the case, there doesn't need to exist a functor $\aleph_0 \to C$ like for $C = \mathrm{Set}$, in general.

The key observation to overcome these problems is that for $C = \mathrm{Set}$ the n-ary operations $I(n) \to I(1)$ of L correspond one-to-one to natural transformations $U^n \Rightarrow U$. Indeed, for $C = \mathrm{Set}$ Yoneda tells us that $U : \mathrm{Mod}_L(\mathrm{Set}) \to \mathrm{Set}$ is representable $U \cong \mathrm{NAT}(y(I(1)), -)$ [10]. Yoneda yields

$$\begin{aligned}
\mathrm{NAT}(U^n, U) &\cong \mathrm{NAT}(\mathrm{Nat}(y(I(1)), -)^n, \mathrm{Nat}(y(I(1)), -)) \\
&\cong \mathrm{NAT}(\mathrm{Nat}(n * y(I(1)), -), \mathrm{Nat}(y(I(1)), -)) \\
&\cong \mathrm{Nat}(y(I(1)), n * y(I(1))),
\end{aligned}$$

where $n * X$ denotes the n-th copower of X, i.e., the n-fold coproduct of X with itself. The functor $y \circ I^{op} = F_L : \aleph^0 \to \mathrm{Mod}_L(\mathrm{Set})$ preserves coproducts, since it is (isomorphic to) a restriction of the left adjoint $F : \mathrm{Set} \to \mathrm{Mod}_L(\mathrm{Set})$ to the full subcategory $\aleph_0$, which is closed under coproducts. So $n * y(I(1)) \cong y(n * I(1)) = y(I(n))$. Applying Yoneda a second time gives

$$\mathrm{Nat}(y(I(1)), n * y(I(1))) \cong \mathrm{Nat}(y(I(1)), y(I(n))) \cong \hom_L(I(n), I(1)),$$

hence $\mathrm{NAT}(U^n, U) \cong \hom_L(I(n), I(1))$ as claimed. In fact the same type of argument proves $\mathrm{NAT}(U^n, U^m) \cong \hom_L(I(n), I(m))$.

Remark 3.20. For a category C with finite products U_L is the evaluation functor $\mathrm{ev}_{I(1)}$, and $U_L^n \cong \mathrm{ev}_{I(n)}$. Using that evaluation is a functor $\mathrm{ev} : L \to [\mathrm{Mod}_L(C), C]$ we still get a map $\hom_L(I(n), I(m)) \to \mathrm{Nat}(U_L^n, U_L^m)$. However, on this level of generality, we cannot tell if this is an isomorphism or not. (Counterexamples?)

For any $U : D \to C$ in $\mathrm{Sem}(C)$ we can define the Lawvere theory $\mathcal{L}(U)$ as the full subcategory of $[\mathrm{Mod}_L(C), C]$ generated by the finite powers of U. (The arity functor is the unique finite-product preserving one induced by $1 \mapsto U$.) Every morphism $F : U \to U'$ in $\mathrm{Sem}(C)$ is just an ordinary functor $F : D \to D'$ s.t.

---

[10]Note that $\mathrm{NAT}(-, -)$ refers to the natural transformations in the functor category $[\mathrm{Mod}_L(\mathrm{Set}), \mathrm{Set}]$. Since $\mathrm{Mod}_L(\mathrm{Set})$ is a large category in the generic case, this is the main reason why one needs to pay attention to size issues here.

$U'F = U$. Therefore precomposition with F yields a product preserving functor $\mathcal{L}(U') \to \mathcal{L}(U')$ that commutes with the arity functors, and so a morphism of Lawvere theories. We obtain a *syntax functor*

$$\mathcal{L} : \mathrm{Sem}(C) \to \mathrm{Law}^{op}, \qquad U \mapsto \mathcal{L}(U), \quad F \mapsto F^*$$

Theorem 3.21 (Syntax-Semantics adjunction)**.** *For any category C with finite products the syntax functor is left adjoint to the semantics functor: $\mathcal{L} \dashv \mathcal{S}$. The unit $\eta : \mathcal{I}_{\mathrm{Sem}(C)} \Rightarrow \mathcal{S}\,\mathcal{L}$ is for each $U : D \to C$ the morphism induced by evaluation $\mathrm{ev} : D \to [[D,C],C]$*

$$\eta_U : D \to \mathrm{Mod}_{\mathcal{L}(U)}(C), \qquad d \mapsto \mathrm{ev}_d \,|\, \mathcal{L}(U), \quad f \mapsto \mathrm{ev}_f$$

The counit $\varepsilon : \mathcal{L}\mathcal{S} \Rightarrow \mathcal{I}_{\mathrm{Law}^{op}}$ is for each Lawvere theory L the morphism with underlying functor

$$\varepsilon_L : L \to \mathcal{L}(U_L), \qquad I(n) \mapsto \mathrm{ev}_{I(n)} = U_L^n, \quad f \mapsto \mathrm{ev}_f \,.[11]$$

Proof. We have $U_{\mathcal{L}(U)} \circ \eta_U = \mathrm{ev}_{I(1)} \circ \eta_U = U$, for $I(1) = U$; so η_U is indeed a morphism $U \to \mathcal{S}\,\mathcal{L}(U)$ in $\mathrm{Sem}(C)$. ε_L respects the arity functors of the Lawvere theories by construction of the arity functor in $\mathcal{L}(U_L)$. ε_L also preserves finite products, since the evaluation functor preserves finite products. (Note that $\mathrm{Mod}_{\mathcal{L}(U)}(C)$ has finite products, for C has them, and recall that they are computed pointwise.) This shows ε_L a morphism of Lawvere theories. ε and η are easily seen to be natural in L, respectively U.

For the adjunction we need to show the triangle equalities. The triangle identity $\varepsilon \mathcal{L} \cdot \mathcal{L} \eta = 1_{\mathcal{L}}$ amounts to show that for each functor $U : D \to C$ in $\mathrm{Sem}(C)$ the functor $\mathcal{L}\,\eta_U \varepsilon_{\mathcal{L}(U)}$ is the identity morphism of the Lawvere theory $\mathcal{L}(U)$. As an endomorphism of a Lawvere theory it has to be the identity on objects, for it has to commute with the arity functor $I : \aleph_0^{op} \to \mathcal{L}(U)$, which is a bijection on objects. It suffices to show that it is an identity on all the n-ary operations of $\mathcal{L}(U)$; but this is immediate from the action of η, ε and $\mathcal{L}$ on morphisms. Let $\theta : U^n \Rightarrow U$ be an n-ary operation, then

$$(\mathcal{L}(\eta_U) \circ \varepsilon_{\mathcal{L}(U)})(\theta) = \mathrm{ev}_\theta \cdot \eta_U$$

and the d-component of this natural transformation computes to

$$(\mathrm{ev}_\theta \cdot \eta_U)_d = \mathrm{ev}_\theta(\mathrm{ev}_d) = \mathrm{ev}_d(\theta) = \theta_d.$$

For the second triangle identity $\mathcal{S}\varepsilon \cdot \eta \mathcal{S} = 1_{\mathcal{S}}$ consider a Lawvere theory L, and the functor

$$\mathcal{S}(\varepsilon_L) \circ \eta_{U_L} : \mathrm{Mod}_L(C) \to \mathrm{Mod}_L(C).$$

For an L-model M in C we get

---

[11]The equality should be strictly speaking a natural isomorphism $\mathrm{ev}_{I(n)} \cong U_L^n$. The product structure on C is the primary one, and all the others are induced by it; but forcing the isomorphism to be an equality means to alter the product structure on C. Also, it seems more sensible to alter the notion of morphism of Lawvere theories from strictly commuting with arities to commuting with arities up to natural isomorphism.

$$(\mathcal{S}(\varepsilon_L) \circ \eta_{U_L})(M)(I(n)) = (\mathrm{ev}_M \circ \varepsilon_L)(I(n)) = \mathrm{ev}_M(\mathrm{ev}_{I(n)}) = \mathrm{ev}_{I(n)}(M) = M(I(n)).$$

The functor is thus an identity on objects; but because of the same double evaluation construction it is also an identity on natural transformations $\theta : M \Rightarrow M'$.

$\square$

Corollary 3.22 (Syntax-Semantics duality)**.** *For $C = \mathrm{Set}$ the syntax-semantics adjunction resticts to an equivalence between the category of Lawvere theories and the full replete reflective subcategory of $\mathrm{Sem}(\mathrm{Set})$ spanned by the categories of models of Lawvere theories in Set.*

Proof. From theorem 1.5 we know that the adjunction $\mathcal{L} \dashv \mathcal{S}$ restricts to an equivalence of full subcategories spanned by the objects for which the component of the unit and counit are an isomorphism, respectively.

The counit ε is an isomorphism for $C = \mathrm{Set}$. To see this note that the Yoneda lemma implies that the evaluation functor $\mathrm{ev} : L \to [[L, \mathrm{Set}], \mathrm{Set}]$ is fully faithful, hence ε_L is fully faithful. Since ε_L is always a bijection on objects, it is an isomorphism.

The monad $\mathcal{S}\mathcal{L}$ is therefore idempotent (see e.g. ES 3, ex 1). In particular, $\eta_{U_L} = \eta_{\mathcal{S}\,L}$ is an isomorphism. For any U in $\mathrm{Sem}(\mathrm{Set})$ isomorphic to some U_L for a Lawvere theory L the unit η_U is an isomorphism. So the full replete subcategory spanned by the models of Lawvere theories in Set is the other part of the equivalence. Because $\mathcal{S}\mathcal{L}$ is idempotent, this subcategory is reflective (cf. ES 3, ex 6). $\square$

3.3. Monads and Lawvere theories.
The primary motivation to introduce Lawvere theories in these notes was to make precise the intuition that monads 'are' algebraic theories. How are monads and Lawvere theories related?

In contrast to monads the distinguished feature of Lawvere theories was the context-independence, i.e, the separation of syntax and semantics. Monads, on the other hand, are context-dependent, since the models corresponding to the theory given by a monad T are its T-algebras, which are only defined on the category the monad is defined on. To relate monads and Lawvere theories we have to fix a base category C. The first obvious choice is $C = \mathrm{Set}$, but other choices are possible and sensible as well (locally finitely presentable categories, for example).

For a Lawvere theory L we have seen that the forgetful functor $U_L : \mathrm{Mod}_L(\mathrm{Set}) \to \mathrm{Set}$ has a left adjoint F_L. We can thus assign to any Lawvere theory L a monad $T_L = U_L F_L$ on Set.

Theorem 3.23. *Let L be a Lawvere theory. The forgetful functor $U_L : \mathrm{Mod}_L(\mathrm{Set}) \to \mathrm{Set}$ is monadic.*

Proof. We want to apply Beck's theorem. U_L has a left adjoint and reflects isomorphisms. We know that $\mathrm{Mod}_L(\mathrm{Set})$ is cocomplete (see remark 3.19). For monadicity it is sufficient to show that U_L preserves coequalizers of reflexive pairs. A coequalizer of a reflexive pair is a special type of a filtered colimit, which we know is computed pointwise in $\mathrm{Mod}_L(\mathrm{Set})$. U_L is the evaluation-at-$I(1)$ functor. As such it preserves all pointwise colimits in $\mathrm{Mod}_L(\mathrm{Set})$; in particular coequalizers of reflexive pairs. $\square$

Since $\mathrm{Mod}_L(\mathrm{Set})$ has coequalizers of reflexive pairs, U_L preserves them; since U_L also reflects ismorphims, it creates coequalizers of reflexive pairs. (In fact, U_L creates filtered colimits.) This result can be strengthened.

Proposition 3.24. U_L *creates coequalizers.*

Proof. Let (α, β) be a pair of natural transformations $M \Rightarrow N$ in $\mathrm{Mod}_L(\mathrm{Set})$. Recall that $U_L = \mathrm{ev}_{I(1)}$. A coequalizer of the pair $(\alpha_{I(1)}, \beta_{I(1)})$ in Set can be obtained as follows

$$M(I(1)) \overset{\alpha_{I(1)}}{\underset{\beta_{I(1)}}{\rightrightarrows}} N(I(1)) \overset{\pi}{\longrightarrow} N(I(1))/R$$

where R is the smallest equivalence relation on $N(I(1))$ containing the image of $\alpha_{I(1)} \times \beta_{I(1)}$ and π denotes the canonical quotient map.

We show that $Q := N(I(1))/R$ lifts to an L-model and π to an L-homomorphism. For any n-ary operation $p : I(n) \to I(1)$ we have

$$
\begin{array}{ccc}
M(I(n)) & \overset{h \times \ldots \times h}{\longrightarrow} & N(I(n)) \\
{\scriptstyle M(p)} \downarrow & & \downarrow {\scriptstyle N(p)} \\
M(I(1)) & \underset{h}{\longrightarrow} & N(I(1))
\end{array}
$$

for $h = \alpha_{I(1)}$ and $h = \beta_{I(1)}$. The operations

$$N(p) : N(I(1)) \times \ldots \times N(I(1)) \to N(I(1)$$

thus respect the relation $\alpha_{I(1)} \times \beta_{I(1)}(M(I(1))$ on $N(I(1))$, hence the equivalence relation R generated by it, and so descend to maps

$$Q(p) : Q \times \ldots \times Q \to Q$$

such that the subsequent diagram commutes

$$
\begin{array}{ccc}
N(I(1)) \times \ldots \times N(I(1)) & \overset{N(p)}{\longrightarrow} & N(I(1)) \\
{\scriptstyle \pi \times \ldots \times \pi} \downarrow & & \downarrow {\scriptstyle \pi} \\
Q \times \ldots \times Q & \underset{Q(p)}{\longrightarrow} & Q
\end{array}
$$

The maps $Q(p)$ are unique with this property[12]. As a consequence Q lifts to a product preserving functor $Q : L \to \mathrm{Set}$ and π lifts to an L-homomorphism (see the characterisation of L-models in terms of its underlying objects and L-homomorphisms in terms of the underlying arrows (3.2), (3.3)).

To verify the universal property of Q and π as the coequalizer of α and β in $\mathrm{Mod}_L(\mathrm{Set})$ consider an L-homomorphism $\gamma : N \to N'$ with $\gamma\alpha = \gamma\beta$. The universal property of $Q(I(1)) = N(I(1))/R$ in Set yields a unique map $\delta : Q(I(1)) \to$

---

[12]The abstract reason for quotients commuting with products in Set is that every quotient is the coequalizer of its kernel pair (i.e. the respective equivalence relation together with the projection maps). Any reflexive relation is a reflexive pair in Set, and its coequalizer a filtered colimit, which commutes with finite limits. R being the smallest equivalence relation generated by the image of $\alpha_{I(1)} \times \beta_{I(1)}$ means in the language of category theory that we take the kernel pair of the coequalizer of the pair $(\alpha_{I(1)}, \beta_{I(1)})$ in Set.

$N'(I(1))$ s.t. $\delta\pi_{I(1)} = \gamma_{I(1)}$. Since γ and π are L-homomorphisms, δ lifts to an L-homomorphism, too:

$$
\begin{aligned}
N'(p)(\delta \times \ldots \times \delta)\pi_{I(n)} &= N'(p)((\delta\pi_{I(1)}) \times \ldots \times (\delta\pi_{I(1)})) \\
&= N'(p)(\gamma_{I(1)} \times \ldots \times \gamma_{I(1)}) \\
&= N'(p)\gamma_{I(n)} \\
&= \gamma_{I(1)}N(p) \\
&= \delta\pi_{I(1)}N(p) \\
&= \delta Q(p)\pi_{I(n)},
\end{aligned}
$$

hence $N'(p)(\delta \times \ldots \times \delta) = \delta Q(p)$ since $\pi_{I(n)}$ is epi.

$\square$

The theorem says that the monad T_L assigned to a Lawvere theory L has an equivalent category of models, that is, the *semantics* of T_L and L is the same up to equivalence.

Remark 3.25. From the proof of the proposition it follows easily that U_L creates coequalizers of U_L-split pairs in the sense of [Mac98], i.e., that for any pair of L-homomorphisms α, β that has a split coequalizer π there is one and only one coequalizing arrow γ s.t. $U_L(\gamma) = \pi$ and this γ is moreover a coequalizer[13]. The comparison functor $K : \mathrm{Mod}_L(\mathrm{Set}) \to \mathrm{Set}^{T_L}$ is therefore not only an equivalence, but an *isomorphism* by [Mac98, theorem VI.7.1].

What about the functoriality of the assignment $L \mapsto T_L$? To study this question we have to define first what a morphism between monads is supposed to be. Recall that monads can be considered as internal monoids in the strict monoidal category $(\mathrm{End}(C), \circ, I_C)$. A natural choice of morphism $\theta : T \to T'$ is an internal monoid homomorphism. If we spell this out that amounts to a natural transformation $\theta : T \Rightarrow T'$ s.t. the subsequent diagrams are rendered commutative

$$
\begin{array}{ccc}
T^2 \xrightarrow{\theta^2} T'^2 & \quad & T \xrightarrow{\quad\theta\quad} T' \\
\mu \downarrow \quad \downarrow \mu' & & \eta \nwarrow \quad \nearrow \eta' \\
T \xrightarrow{\theta} T' & & I_C
\end{array}
$$

As regards the categories of Eilenberg-Moore algebras every such morphism induces a functor θ^*

$$
\begin{array}{ccc}
C^{T'} & \xrightarrow{\quad\theta^*\quad} & C^T \\
U^{T'} \searrow & & \swarrow U^T \\
& C &
\end{array}
$$

---

[13]Note that in contrast to the definition given in the lectures this definition of "creates coequalizers of ... " is not stable under equivalence of categories, and thus not a very sensible notion of creation of coequalizers.

and a natural transformation $\tau_\theta : F^T \Rightarrow \theta^* \circ F^{T'}$. (See [Mac98, exercise VI.2.3].) We obtain a (meta)category $\mathrm{Mnd}(C)$ of monads on C [14].

Let $F : L \to L'$ be a morphism of Lawvere theories. Recall that on $\aleph_0 \hookrightarrow \mathrm{Set}$ the free model functor F_L is just $y \circ I^{op}$ and hence $T_L(n) = U_L F_L(n) = L(I(n), I(1))$. For each $n \in \mathrm{ob}\,\aleph_0$ the arrow-component of the functor F yields a map

$$(\mathcal{M}\,F)_n := F_{I(n), I(1)} : L(I(n), I(1)) \to L'(I'(n), I'(1))$$

and hence maps $(\mathcal{M}\,F)_n : T_L(n) \to T_{L'}(n)$, natural in n. We use again that every set X is a canonical filtered colimit of objects in $\aleph_0$ to extend $\mathcal{M}\,F$ to a natural transformation $\mathcal{M}\,F : T_L \Rightarrow T_{L'}$. $\mathcal{M}\,F$ is well-defined because F_L, U_L and hence T_L (and $T_{L'}$) preserve filtered colimits. This construction is easily seen to be functorial.

Remark 3.26 (Using coends). Recall the representation (3.8) of $F_L(X)(I(1)) = T_L(X)$ as the coend

$$T_L(X) = \int^{n \in \aleph_0} L(I(n), I(1)) \times X^n$$

For every X The arrow-component of F yields a natural transformation of the bifunctors

$$\tau_F : L(I(-), I(1)) \times X^{(-)} \Rightarrow L'(I'(-), I'(1)) \times X^{(-)}$$

and hence a map $\int^{n \in \aleph_0} \tau_F(n, n) = \int^{n \in \aleph_0} F_{I(n), I(1)} \times 1_{X^n}$, which is obviously natural in X. This yields another way to obtain $\mathcal{M}\,F$

$$(\mathcal{M}\,F)_X = \int^{n \in \aleph_0} F_{I(n), I(1)} \times 1_{X^n}$$

It remains to check that this is a morphism of monads. Since T_L and I_{Set} preserve filtered colimits it is sufficient to check the commutativity of the respective diagrams for $\mathcal{M}\,F$ for $n \in \mathrm{ob}\,\aleph_0$. Let η_L denote the unit and μ_L the multiplication of the monad $T_L = U_L F_L$. The map

$$(\eta_L)_n : \{1, \dots n\} \to L(I(n), I(1))$$

sends k to the projection on the k-th component $\mathrm{pr}_k : I(n) \to I(1)$. F is a finite product preserving functor, as such maps projections to projections, and hence $(\mathcal{M}\,F)_n (\eta_L)_n = (\eta_{L'})_n$. Alternatively, one can see $(\mathcal{M}\,F)_X (\eta_L)_X = (\eta_{L'})_X$ from the coend representation of $\mathcal{M}\,F$ directly by using that the unit $(\eta_L)_X$ is the composite

$$X \xrightarrow{x \mapsto (1_{I(1)}, x)} L(I(1), I(1)) \times X \xrightarrow{\omega_1^X} \int^{n \in \aleph_0} L(I(n), I(1)) \times X^n$$

---

[14]Note that $\mathrm{Mnd}(C)$ doesn't have to have small hom-sets, or, if not assuming the Universe Axiom, doesn't need to have a model in (NBG) set theory. One particular example is $\mathrm{Mnd}(\mathrm{Set})$. We shall, however, ignore this size problem here and always speak of the category of monads.

where ω^X is the *universal wedge* of the coend. (See [Mac98, proposition IX.7.1].) The multiplication $(\mu_L)_n$ can be represented as follows. The composition in L induces a wedge

$$\kappa^n : L(I(-), I(1)) \times L(I(n), I(1))^{(-)} \to L(I(n), I(1))$$

with

$$\kappa_k^n : L(I(k), I(1)) \times L(I(n), I(1))^k \to L(I(n), I(1)), \quad (f, g_1, \ldots, g_k) \mapsto f \circ (g_1, \ldots, g_k)$$

By the universal property of $\omega^{L(I(n), I(1))}$ there is a unique map

$$(\mu_L)_n : T_L(L(I(n), I(1)) \to L(I(n), I(1))$$

s.t. $(\mu_L)_n \omega^{L(I(n), I(1))} = \kappa^n$. Because of the uniqueness $(\mu_L)_n$ is natural in n.

The equation $(\mathcal{M} F)_n (\mu_L)_n = (\mu_{L'})_n (\mathcal{M} F)_n^2$ is essentially a consequence of F commuting with composition. We can construct a wedge

$$\tau^n : L(I(-), I(1)) \times L(I(n), I(1))^{(-)} \to L'(I'(n), I'(1)) = T'n$$

as $F_{I(n), I(1)} \kappa^n$ and obtain a unique map $h : T^2 n \to T'n$, s.t. $h \omega^{L(I(n), I(1))} = \tau^n$. This h is $F_{I(n), I(1)} (\mu_L)_n = (\mathcal{M} F)_n (\mu_L)_n$. On the other hand, since F commutes with composition, we have also

$$(\tau^n)_k = (\kappa')_k^n \circ \left(F_{I(k), I(1)} \times \left(F_{I(n), I(1)} \right)^k \right),$$

where $(\kappa')^n$ denotes the respective wedge for $(\mu_{L'})_n$

$$(\kappa')^n : L'(I'(-), I'(1)) \times L'(I'(n), I'(1))^{(-)} \to L'(I'(n), I'(1))$$

By passing to the coends we get $h = (\mu'_L)_n (\mathcal{M} F)_n^2$, since

$$
\begin{aligned}
(\mathcal{M} F)_n^2 &= (\mathcal{M} F)_{T_L(n)} \circ T_L((\mathcal{M} F)_n) \\
&= \int^{k \in \aleph_0} F_{I(k), I(1)} \times 1_{L'(I'(n), I'(1))^k} \circ \int^{k \in \aleph_0} 1_{L(I(k), I(1))} \times \left(F_{I(n), I(1)} \right)^k \\
&= \int^{k \in \aleph_0} F_{I(k), I(1)} \times \left(F_{I(n), I(1)} \right)^k
\end{aligned}
$$

The assignment $L \mapsto T_L$ thus lifts to a functor

$$\mathcal{M} : \mathrm{Law} \to \mathrm{Mnd}(\mathrm{Set}), \qquad L \mapsto T_L, \quad F \mapsto \mathcal{M} F.$$

Theorem 3.27. *The functor $\mathcal{M}$ is fully faithful.*

Proof. Let L, L' be two Lawvere theories. Let $\theta : T_L \Rightarrow T_{L'}$ be a morphism of monads. We use the components $\theta_n : L(I(n), I(1)) \to L'(I'(n), I'(1))$ to define maps

$$F^\theta_{I(n),I(m)} : L(I(n), I(m)) \cong L(I(n), I(1))^m \to L'(I'(n), I'(1))^m \cong L'(I'(n), I'(m)),$$
$$(q_1, \ldots, q_m) \mapsto (\theta_n(q_1), \ldots, \theta_n(q_m))$$

Because of $\theta \mu_L = \mu_{L'} \theta^2$ (and the construction of μ_L and $\mu_{L'}$ from the wedges κ^n respectively $(\kappa')^n$) the maps $F^\theta_{I(n),I(m)}$ commute with the composition in L and L'. Because of $\theta \eta_L = \eta_{L'}$ the maps $F^\theta_{I(n),I(1)}$ map projections $\mathrm{pr}_k : I(n) \to I(1)$ to projections $\mathrm{pr}_k : I'(n) \to I'(1)$. With this every $F^\theta_{I(n),I(n)}$ preserves identities, and together with the object map

$$F^\theta_o : \mathrm{ob}\, L \to \mathrm{ob}\, L', \qquad I(n) \mapsto I'(n)$$

defines a finite-product preserving functor $F^\theta : L \to L'$ with $FI = I'$, hence a morphism of Lawvere theories. It is obvious from the construction that the map $\theta \mapsto F^\theta$ is a two-sided inverse to the arrow map $\mathcal{M}_{L,L'}$. This shows $\mathcal{M}$ to be fully faithful as asserted.

$\square$

Using the syntax-semantics adjunctions of the previous section we get a functor $\mathrm{Mnd}(\mathrm{Set}) \to \mathrm{Law}$. Recall that every morphism of monads $\theta : T \to T'$ induces a functor "by precomposition"

$$\theta^* : \mathrm{Set}^{T'} \to \mathrm{Set}^T, \qquad (X, h) \mapsto (X, h\theta_X), \quad f \mapsto f$$

Because of $U^T \theta^* = U^{T'}$ we get a contravariant functor

$$\mathrm{Mnd}(\mathrm{Set}) \to \mathrm{Sem}(\mathrm{Set})^{op}, \qquad T \mapsto U^T, \quad \theta \mapsto \theta^*$$

If we compose this functor with $\mathcal{L}^{op} : \mathrm{Sem}(\mathrm{Set})^{op} \to \mathrm{Law}$, we obtain a functor, which we denote by $\mathcal{L}$, too

$$\mathcal{L} : \mathrm{Mnd}(\mathrm{Set}) \to \mathrm{Law}, \qquad T \mapsto \mathcal{L}(U^T), \quad \theta \mapsto (\theta^*)^*$$

Theorem 3.28. *$\mathcal{L}$ is right adjoint to $\mathcal{M}$: $\mathcal{M} \dashv \mathcal{L}$.*

Proof. A natural bijection

$$\mathrm{hom}_{\mathrm{Mnd}(\mathrm{Set})}(\mathcal{M}(L), T) \cong \mathrm{hom}_{\mathrm{Law}}(L, \mathcal{L}(T))$$

can be obtained by generalizing the construction of $\theta \mapsto F^\theta$ in the proof that $\mathcal{M}$ is fully faithful.

(i) To begin with note that $U^T \cong \mathrm{hom}_T(F^T(1), -)$ is representable, and hence

$$\mathrm{hom}_{\mathcal{L}(T)}(I_T(n), I_T(m)) = \mathrm{Nat}((U^T)^n, (U^T)^m) \cong (U^T F^T(n))^m = (Tn)^m$$

by Yoneda and the adjunction $F^T \dashv U^T$, where I_T is the arity functor of $\mathcal{L}(T)$. In fact, these bijections can be seen as the arrow maps of an equivalence of the full subcategory $(\aleph_0)_T^{op}$ of Set_T generated by $\aleph_0$, and $\mathcal{L}(T)$

$$(\aleph_0)_T^{op} \to \mathcal{L}(T), \qquad n_T \mapsto (U^T)^n, \quad (f_T : m_T \to n_T) \mapsto (\mu_n Tf\ \eta_m)^*,$$

since the underlying map of $f_T : m_T \to n_T$ is the map $m \to Tn$, i.e., an element of $(Tn)^m$.

(ii) Let $\theta : T_L \Rightarrow T$ be a morphism of monads. We use the components $\theta_n : L(I(n), I(1)) \to Tn$ to define maps

$$F^\theta_{I(n),I(m)} : L(I(n), I(m)) \cong L(I(n), I(1))^m \to (Tn)^m \cong \hom_{\mathcal{L}(T)}(I_T(n), I_T(m)),$$
$$(q_1, \ldots, q_m) \mapsto (\theta_n(q_1), \ldots, \theta_n(q_m))$$

As a morphism of monads θ we have $\mu\theta^2 = \theta\mu_L$ and $\theta\eta_L = \eta$. Bearing the equivalence in mind mentioned above (as well as the the construction of μ_L and η_L) $F^\theta_{I(n),I(m)}$ can be seen to commute with composition and to preserve projections and identities. As in the proof that $\mathcal{M}$ is fully faithful we get a morphism of Lawvere theories $F^\theta : L \to \mathcal{L}(T)$.

(iii) Conversely, generalizing the construction of $\mathcal{M}F$ we can construct to any morphism $F : L \to \mathcal{L}(T)$ of Lawvere theories a morhism of monads $\theta^F : T_L \Rightarrow T$. Indeed, for any set X, F determines a wedge

$$\sigma^X : L(I(-), I(1)) \times X^{(-)} \to TX$$

with components

$$(\sigma^X)_n : L(I(n), I(1)) \times X^n \to TX, \qquad (q, (x_1, \ldots, x_n)) \mapsto (Fq)_{(TX,\mu_X)}(\eta_X(x_1), \ldots, \eta_X(x_n))$$

There is a unique map $\theta^F_X : T_L X \to TX$ s.t. $\theta^F_X \omega^X = \sigma^X$. Uniqueness shows θ^F_X natural in X.

Recall the representation of $(\eta_L)_X = \omega^X_1 \circ (x \mapsto 1_{I(1)}, 1_X)$, where ω^X is the universal wedge of $T_L X$. Evaluating σ^X_1 on $(1_{I(1)}, x)$ yields $\eta_X(x)$, and so $\theta^F \eta_L = \eta$.

To show $\theta^F \mu_L = \mu(\theta^F)^2$ we show $\theta^F_n (\mu_L)_n = \mu_n(\theta^F)^2_n$ for $n \in \mathrm{ob}\,\aleph_0$ first. Note that

$$\theta^F_n : L(I(n), I(1)) \to Tn, \qquad f \mapsto (Ff)_{(Tn,\mu_n)}(\eta_n(1), \ldots, \eta_n(n))$$

(This follows from $\omega^n_k(f, (n_1, \ldots, n_k)) = f \circ (\mathrm{pr}_{n_1}, \ldots, \mathrm{pr}_{n_k})$ with $\mathrm{pr}_{n_i} : I(n) \to I(1)$, $\theta^F_n \omega^n = \sigma^n$ and that F preserves finite products.) We construct a wedge

$$\tau^n : L(I(-), I(1)) \times L(I(n), I(1))^{(-)} \to Tn$$

as $\theta^F_n \kappa^n$ and obtain a unique map $h : T_L^2 n \to Tn$, s.t. $h\omega^{L(I(n),I(1))} = \tau^n$. This h is $\theta^F_n(\mu_L)_n$. On the other hand we can construct the same wedge τ^n as

$$\tau^n_k = \mu_n \circ \sigma^n_k \circ (1_{L(I(k),I(1))} \times (\theta^F_n)^k).$$

Indeed, let us denote the so constructed wedge by ρ^n, then

$$\rho_k^n(f, (g_1, \ldots, g_k)) = \mu_n \circ Ff_{(T^2n, \mu_{Tn})}$$
$$\circ (\eta_{Tn})^k \circ ((Fg_1)_{(Tn, \mu_n)}, \ldots, (Fg_k)_{(Tn, \mu_n)})(\eta_n(1), \ldots, \eta_n(n));$$

but $\mu_n : T^2n \to Tn$ is a a morphism $(T^2n, \mu_{Tn}) \to (Tn, \mu_n)$ by the associativity law for μ and Ff is a natural transformation $(U^T)^k \Rightarrow U^T$, so

$$\mu_n \circ Ff_{(T^2n, \mu_{Tn})} = Ff_{(Tn, \mu_n)} \circ (\mu_n)^k$$

Because of $\mu_n \eta_{Tn} = 1_{Tn}$ and since F is product preserving this yields

$$\rho_k^n(f, (g_1, \ldots, g_k)) = F(f \circ (g_1, \ldots, g_k))_{(Tn, \mu_n)}(\eta_n(1), \ldots, \eta_n(n))$$
$$= \tau_k^n(f, (g_1, \ldots, g_k))$$

as asserted. If we pass to the factorization though the universal wedge ω^n for ρ^n, we obtain

$$h = \mu_n \theta_{Tn}^F T_L \theta_n^F = \mu_n (\theta^F)_n^2.$$

We conclude $\theta_n^F(\mu_L)_n = \mu_n(\theta^F)_n^2$. To show that this holds for an arbitrary (small) set X we employ a colimit argument. Let $\iota : n \hookrightarrow X$ denote an inclusion of n as a finite subset of X. Consider the diagram

We have shown that the inner square commutes. The trapezia commute because of the naturality of μ, μ_L and θ^F. To show the outer square commutative we pass to the filtered colimit over the category $J(X)$ of all the inclusions ι

for every $n \in \operatorname{ob} \aleph_0$

$$
\begin{array}{ccc}
T_L^2 X & \xrightarrow{\;\;(\theta^F)_X^2\;\;} & T^2 X \\
\end{array}
$$

The diagonal maps are the universal ones coming from the colimit. The isomorphisms are due to T_L preserving filtered colimits (and $X = \varinjlim_{\iota \in J(X)} \operatorname{dom} \iota$). The inner square and the trapezia comutes because it does for every ι. Hence the outer square commutes.

(iv) It is straight forward to see that the mappings $F \mapsto \theta^F$ and $\theta \mapsto F^\theta$ are inverse to each other. Showing the naturality in T and L I leave as an exercise.

$\square$

Remark 3.29 (Equivalent definition of $\mathcal{L}$). Using the syntax-semantics correspondence for Lawvere theories, namely that a Lawvere theory L is (isomorphic to) the opposite of the category of finitely generated free L-models by Yoneda, one could define $\mathcal{L}(T)$ as the opposite of the full subcategory of Set^T generated by the finitely generated free T-algebras $F^T(n)$, $n \in \operatorname{ob} \aleph_0$ with the obvious arity functor. Since F^T might not be one-to-one on objects (see e.g. [Mac98, exercise VI.5.3]) we have to use the equivalent full subcategory $(\aleph_0)_T \hookrightarrow \operatorname{Set}_T$.

The functoriality of this construction is a consequence of the functoriality of the Kleisli construction. More explicitly, let $\theta : T \Rightarrow T'$ be a morphism of monads. Identifying the Kleisli categories with their equivalent full subcategories of Set^T generated by free T-algebras the induced functor θ_* is

$$
\theta_* : \operatorname{Set}_T \to \operatorname{Set}_{T'}, \qquad X_T \mapsto X_{T'}, \quad f_T \mapsto (\theta_{\operatorname{cod} f} f)_{T'}.
$$

It is clear from the definition that $\theta_* F_T(n) = F_{T'}$. Moreover θ_* preserves coproducts, so θ_*^{op} restricted to $\mathcal{L}(T)$ yields a morphism of Lawvere theories $\mathcal{L}(T) \to \mathcal{L}(T')$.

Remark 3.30 (Finite-product categories C). Why didn't we define $\mathcal{L}$ using the Kleisli category to begin with? Because the given definition of $\mathcal{L}$ generalizes to any finite-product category C. For any such C we can assign a Lawvere theory to any monad over C in a functorial way

$$
\mathcal{L} : \operatorname{Mnd}(C) \to \operatorname{Law}, \qquad T \mapsto \mathcal{L}(U^T), \quad \theta \mapsto (\theta^*)^*
$$

The construction of $\mathcal{M}$ does not generalize so obviously. For locally finitely presentable categories, however, one can show the forgetful functor U_L to have a left

adjoint F_L, and hence assign a monad $T_L = U_L F_L$ to any Lawvere theory L. I haven't checked the functoriality of this assignment in this case though.

Corollary 3.31. *The adjunction restricts to an equivalence of the category of Lawvere theories* Law *and the full (replete) reflective subcategory of* Mnd(Set) *of monads that preserve filtered colimits.*

Proof. As any adjunction also $\mathcal{M} \dashv \mathcal{L}$ restricts to an equivalence. We know that $\mathcal{M}$ is fully faithful already, so the unit of this adjunction is an isomorphism (dual of [Mac98, theorem IV.3.1]) and the induced monad on Mnd(Set) is idempotent. $\mathcal{M}$ is thus equivalent to a replete full subcategory of Mnd(Set). We have seen that each $\mathcal{M}(L) = T_L$ preserves filtered colimits. Since this is a property stable under isomorphism of monads we conclude the claim (see also the proof of theorem 3.22). $\square$

Example 3.32 (A monad that does not preserve filtered colimits)**.** The forgetful functor from the category $CHaus$ of compact Hausdorff spaces to Set has a left adjoint that assigns to any set X the Stone-Čech compactification βX of X equipped with the discrete topology. The induced monad T of this adjunction maps finite sets to finite sets (since a discrete topological space is compact iff its underlying set is finite). On the full subcategory category of finite sets the monad T is thus isomorphic to the identity monad. If T were to preserve filterd colimits, then, since every set X is the filtered colimit of its finite subsets, we would have $T \cong I_{\mathrm{Set}}$ as monads. But $\operatorname{card} T\mathbb{N} = 2^{\operatorname{card}\mathbb{R}} > \operatorname{card}\mathbb{N}$, so there is no bijection from $T\mathbb{N}$ to $\mathbb{N}$.

Remark 3.33 (Monads and infinitary Lawvere theories). As mentioned in the introduction the equivalence of the category of Lawvere theories and the category of filtered colimits-preserving monads has a generalization to all monads on Set. The notion of Lawvere theory has to be generalized to that of *infinitary Lawvere theory*, however. Infinitary Lawvere theories allow operations with arity of arbitrary cardinality, not just a finite one. One example is the theory of sup-lattices. Here the formation of suprema $\sup X$ for any subset X of the lattice are considered as operation of arity $\operatorname{card} X$.

Instead of using a skeleton of Set it is more practice-oriented to use the category Set as the domain of the arity functor instead of a skeleton for the definition of an infinitary Lawvere theory. An infinitary Lawvere theory is a category L (with small hom-sets) and small products together with a (small-)product preserving functor $I : \mathrm{Set}^{op} \to L$ that is essentially surjective, i.e., every object of L is isomorphic to some $I(X)$. Models of L in a category C with small products are (small-)product-preserving functors $M : L \to C$.

One can show that the forgetful functor $\mathrm{Mod}_L(\mathrm{Set}) \to \mathrm{Set}$ of an infinitary Lawvere theory L is monadic; in particular, to any infinitary Lawvere theory L we can associate a monad T_L. Conversely, to any monad T on Set we can associate the infinitary Lawvere theory Set_T^{op} with arity functor $F_T^{op} : \mathrm{Set}^{op} \to \mathrm{Set}_T^{op}$. The monad associated with Set_T^{op} is isomorphic to T and the Lawvere theory associated to T_L is equivalent to L.

Returning to the finitary case we have seen for a Lawvere theory L that, since U_L is monadic, the categories of models of L and that of T_L-algebras are isomorphic. For a monad T on Set and its corresponding Lawvere theory $\mathcal{L}(T)$ an equivalence

between the categories of models cannot exist, in general. The relationship is as follows.

Proposition 3.34. *We identify $\mathcal{L}(T)$ with the opposite of th full subcategory $(\aleph_0)_T$ of Set_T of finitely generated free T-algebras as in remark 3.29. There is a tensor-hom-adjunction*

$$\mathrm{Mod}_{\mathcal{L}(T)}(\mathrm{Set}) \underset{(-)\otimes\iota}{\overset{\hom_T(\iota,-)}{\rightleftarrows}} \mathrm{Set}^T ,$$

where $\iota : (\aleph_0)_T \to \mathrm{Set}^T$ is the restriction of the comparison functor $K_T : \mathrm{Set}_T \to \mathrm{Set}^T$. More explicitly

$$\hom_T(\iota, (X, h)) : \mathcal{L}(T) \to \mathrm{Set}, \qquad n_T \mapsto \hom_T(\iota(n_T), (X, h)), \quad f_T \mapsto \iota(f_T)^*,$$

$$M \otimes \iota = \int^{n\in\aleph_0} M(I(n)) * \iota(n_T) = \int^{n\in\aleph_0} (T(M(I(n)) \times n), \mu_{M(I(n))\times n}).$$

T is a monad that preserves filtered colimits iff this adjunction is an adjoint equivalence.

Proof. (Sketch) As Set^T is cocomplete, $M \otimes \iota$ is well-defined. To check that the given functors are adjoint is a good exercise in the end/coend calculus and is left to the reader.

We study the compostion of the tensor-hom adjunction with $F_{\mathcal{L}(T)} \dashv U_{\mathcal{L}(T)}$. There is a natural isomorphism $y(n_T) \otimes \iota \cong \iota(n_T)$. Recall that the arity functor of the Lawvere theory $(\aleph_0)_T^{op}$ is F_T^{op}, so $y \circ F_T$ is the free $\mathcal{L}(T)$-model functor restricted to $\aleph_0 \hookrightarrow \mathrm{Set}$; but $\iota \circ F_T = F^T|\aleph_0$, hence

$$(-) \otimes \iota \circ F_{\mathcal{L}(T)} \cong F^T,$$

since every set is a filtered colimit of its finite subsets, and all of the above functors preserve colimits.

$$U_{\mathcal{L}(T)} \circ \hom_T(\iota, -) \cong U^T$$

is a consequence of $\iota(1_T) = F^T(1)$, $U^T \cong \hom_T(F^T(1), -)$ and $U_{\mathcal{L}(T)} = \mathrm{ev}_{1_T}$. We get a morphism of monads

$$T_{\mathcal{L}(T)} \xrightarrow{U_{\mathcal{L}(T)}\eta^{ht}F_{\mathcal{L}(T)}} U_{\mathcal{L}(T)} \circ \hom_T(\iota, -) \circ (-) \otimes \iota \circ F_{\mathcal{L}(T)} \cong T.$$

If the tensor-hom-adjunction is an equivalence, $(-) \otimes \iota$ is fully faithful, the unit η^{ht} of the tensor-hom-adjunction is thus an isomorphism, and we have $T_{\mathcal{L}(T)} \cong T$. So T preserves filtered colimits, since $T_{\mathcal{L}(T)}$ does.

Suppose T preserves filtered colimits. Because of $y(n_T) \otimes \iota \cong \iota(n_T)$ we have that $\eta^{ht}_{y(n_T)}$ is an isomorphism, hence $T_{\mathcal{L}(T)}(n) \cong T(n)$, and finally $T_{\mathcal{L}(T)}(X) \cong T(X)$ for any set X due to the preservation of filtered colimits, i.e., $T_{\mathcal{L}(T)} \cong T$ as monads. Furthermore, we have

$$U^T \circ y(n_T) \otimes \iota = Tn \cong \hom_T(1_T, n_T) = U_{\mathcal{L}(T)}(y(n_T))$$

Every $\mathcal{L}(T)$-model is a filtered colimit of representables in $\mathrm{Mod}_{\mathcal{L}(T)}(\mathrm{Set})$. Tensoring $(-) \otimes \iota$ preserves colimits and since T preserves filtered colimits, so does U^T. We get

$$U^T \circ (-) \otimes \iota \cong U_{\mathcal{L}(T)}$$

This shows $(-) \otimes \iota$ to be isomorphic to the comparison functor. However, due to $T_{\mathcal{L}(T)} \cong T$ and $U_{\mathcal{L}(T)}$ being monadic, the comparison functor from $\mathrm{Mod}_{\mathcal{L}(T)} \to \mathrm{Set}^T$ is an equivalence. This shows the tensor-hom adjunction to be an equivalence. $\square$

Remark 3.35 (Monads as algebraic theories). If we consider the notion of (infinitary) algebraic theory to coincide with that of an (infinitary) Lawvere theory we can make the following observations concerning the relationship to monads.

(1) If we consider models and monads in Set only then monads and infinitary algebraic theories are equivalent notions, as well as are filtered-colimit-preserving monads and (finitary) algebraic theories.
(2) Algebraic theories can have models in any category C with finite products, and it may happen that an algebraic theory L has a non-trivial category of C-models, but that the forgetful functor U_L has not a left adjoint (example?)[15]
(3) To any monad T on a category with finite products we can associate a Lawvere theory $\mathcal{L}(T)$. However, the category $\mathrm{Mod}_{\mathcal{L}(T)}(C)$ doesn't need to be equivalent to C^T (example?)[16] On the other hand, there are monads over categories that don't have finite products.

This shows that monads and algebraic theories are apriori different notions. They can be related to each other whenever there is a common domain of definition, namely categories with finite products. (However, note that for any category C there is a finite-product-completion $\hat{C}$ with a fully faithful embedding $C \hookrightarrow \hat{C}$; any monad on C lifts to a monad on $\hat{C}$.) For some categories like $C = \mathrm{Set}$ both notions become equivalent.

Of course, even in situations where both notions are not equivalent it might still be helpful to think of a monad as an algebraic theory.

3.4. From finite product categories to monoidal categories: PROPs and operads.

In the supervisions I explained that monoid signatures have models in any *monoidal category*. Important examples are rings as internal monoids in the monoidal category $(\mathrm{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$ of abelian groups and R-algebras over a commutative ring R, which are internal monoids in the monoidal category $(\mathrm{Mod}_R, \otimes_R, R)$ of R-modules. On one hand this gives a nice unified conceptual viewpoint, on the

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[15]For Set one can construct such an example in the framework of ZFC+Universe, if one allows Lawvere theories to be large categories that don't have small hom-sets. In that case one can define a Lawvere theory of (small)-complete Boolean algebras. This theory has non-trivial models in Set, but there are no free complete Boolean algebras in Set, hence there is no monad over Set corresponding to this algebraic theory. However, this example is a rather forced one.

[16]A forced example is the compact Hausdorff space monad on Set; which is simply due to the fact that it doesn't preserve filtered colimits.

other hand it paves the way to a more unified treatment of algebraic theories and generalizations thereof.

Another important example we have encountered in the supervisions briefly was the relationship between groups and Hopf algebras. Asking about the correct notion of internal group in a symmetric monoidal category and by analyzing the necessary structure to define an internal group in a category with finite products we arrived at the notion of Hopf algebra. This relates the theory of groups and Hopf-algebras: an internal group in a category C with finite products is a Hopf algebra in the symmetric monoidal category $(C, \times, 1)$, where $\times$ denotes the cartesian product bifunctor and 1 the terminal object. To be able to capture this observation on a syntactic level we need to generalize Lawvere theories to the symmetric monoidal setting.

The key observation is that $\aleph_0^{op}$ is the category with finite products freely generated by one object, and that the syntactic category of a Lawvere theory L is the category with finite products freely generated by the generic model $I(1)$ of L (corresponding to the identity functor on L). Passing from cartesian products to a symmetric monoidal structure we replace $\aleph_0^{op}$ by the symmetric monoidal category $\mathbb{P}$ freely generated by one object (the *permutations category*). The objects of $\mathbb{P}$ are the natural numbers. There are only morphisms $n \to n$, namely the permutations of the set $\{1, \ldots, n\}$. One can consider $\mathbb{P}$ as a subcategory of $\aleph_0$ [17]. The symmetric monoidal structure comes from the coproduct of $\aleph_0$.

Definition 3.36. A *PROP* ("*PRO*duct and *P*ermutation category") is a symmetric monoidal category P together with a (strong) symmetric monoidal functor $I : \mathbb{P} \to P$, the arity functor. Morphisms of PROPs are symmetric monoidal functors commuting with the arity functors, and given a symmetric monoidal category S a *model* of P in S is a symmetric monoidal functor $P \to S$. *P-homomorphisms* are symmetric monoidal natural transformations between these functors. We obtain a category $\mathrm{Mod}_P(S)$.

With the notion of PROP we can consider the PROP P_H of Hopf-algebras. Note that this doesn't coincide with the Lawvere theory L_G of groups considered as a symmetric monoidal category. However, L_G is the PROP of cocommutative Hopf algebras [18]. There is a morphism of PROPs $P_H \to L_G$ and every symmetric monoidal functor $P_H \to C$ for a category C with finite products factors through this morphism, since every Hopf algebra in such a category C is necessarily cocommutative. Every Lawvere theory gives rise to a PROP.

A Lawvere theory L is determined by all the n-ary operations $I(n) \to I(1)$. This is *not* true for a PROP, in general. Due to the absence of projections and the universal property of products we can have operations $I(n) \to I(m)$ that are not tensor products of operations $I(n) \to I(1)$. One example is the comultiplication $\Delta : I(1) \to I(2)$ in P_H.

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[17]Generalizing from a cartesian product to a symmetric monoidal structure abstracts from the cartesian structure as a multiplication with unit and associativity up to natural isomorphism, and hence forgets about the projections and their universal property. Discarding the projections and all the composites involving them amounts to pass from $\aleph_0^{op}$ to the subcategory $\mathbb{P}^{op} \cong \mathbb{P}$.

[18]I haven't checked this!

Generalizing the data of n-ary operations and how they compose to the monoidal setting one arrives at the notion of a (symmetric) *operad* and the *algebra of an operad*. Every symmetric operad gives rise to PROP, but not every PROP is induced by an operad. Operads and PROPs arose as means of understanding the various algebraic structures encountered in Algebraic Topology. Operads and their algerbas in particular, have proven a very important algebraic notion in various fields of mathematics besides Algebraic Topology. (See, e.g., [MSS])

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