REPRINTED FROM

NAVAL RESEARCH LOGISTICS QUARTERLY

JUNE 1980
VOL. 27, NO. 2

OFFICE OF NAVAL RESEARCH

NAVSO P-1278

A DYNAMIC INVENTORY SYSTEM WITH RECYCLING

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ABSTRACT

This paper deals with a periodic review inventory system in which a constant proportion of stock issued to meet demand each period feeds back into the inventory after a fixed number of periods. Various applications of the model are discussed, including blood bank management and the control of reparable item inventories. We assume that on hand inventory is subject to proportional decay. Demands in successive periods are assumed to be independent identically distributed random variables. The functional equation defining an optimal policy is formulated and a myopic base stock approximation is developed. This myopic policy is shown to be optimal for the case where the feedback delay is equal to one period. Both cost and ordering decision comparisons for optimal and myopic policies are carried out numerically for a delay time of two periods over a wide range of input parameter values.

INTRODUCTION

This paper deals with the analysis of inventory systems in which recycling occurs. The term recycling is used here to indicate that a fixed fraction of the stock used to satisfy demand returns to inventory after a fixed number of periods.

Feedback or recycling in inventory systems can occur in a number of different ways. Examples include systems where customers buy items with a rent/purchase option and return items that are not ultimately purchased. Another cause of recycling is the result of over-ordering stock. This occurs in hospital and regional blood banks since physicians requesting blood for their patients tend to over-order by a factor of two or three. A further application of recycling occurs in retail sales systems where a fixed fraction of stock purchased by customers may be returned, and subsequently mixed with existing inventory.

*Research supported by Grant ENG-77-07463 from the National Science Foundation

**Research supported by Grant 78-3494 from the Air Force Office of Scientific Research and Grant ENG 78-05928 from the National Science Foundation. Visiting, Department of Operations Research Stanford University, 1978-1979

The phenomenon of recycling can also be observed in reparable item inventories. An issue stock inventory is maintained to replace items in the field which are subject to failure. Failed items are returned for repair and after a fixed delay time (which includes the time required for transportation and repair), the repaired item is returned to the issue stock inventory. A fraction of those items that fail are condemned and leave the system forever. The reparable item system is pictured in Figure 1.

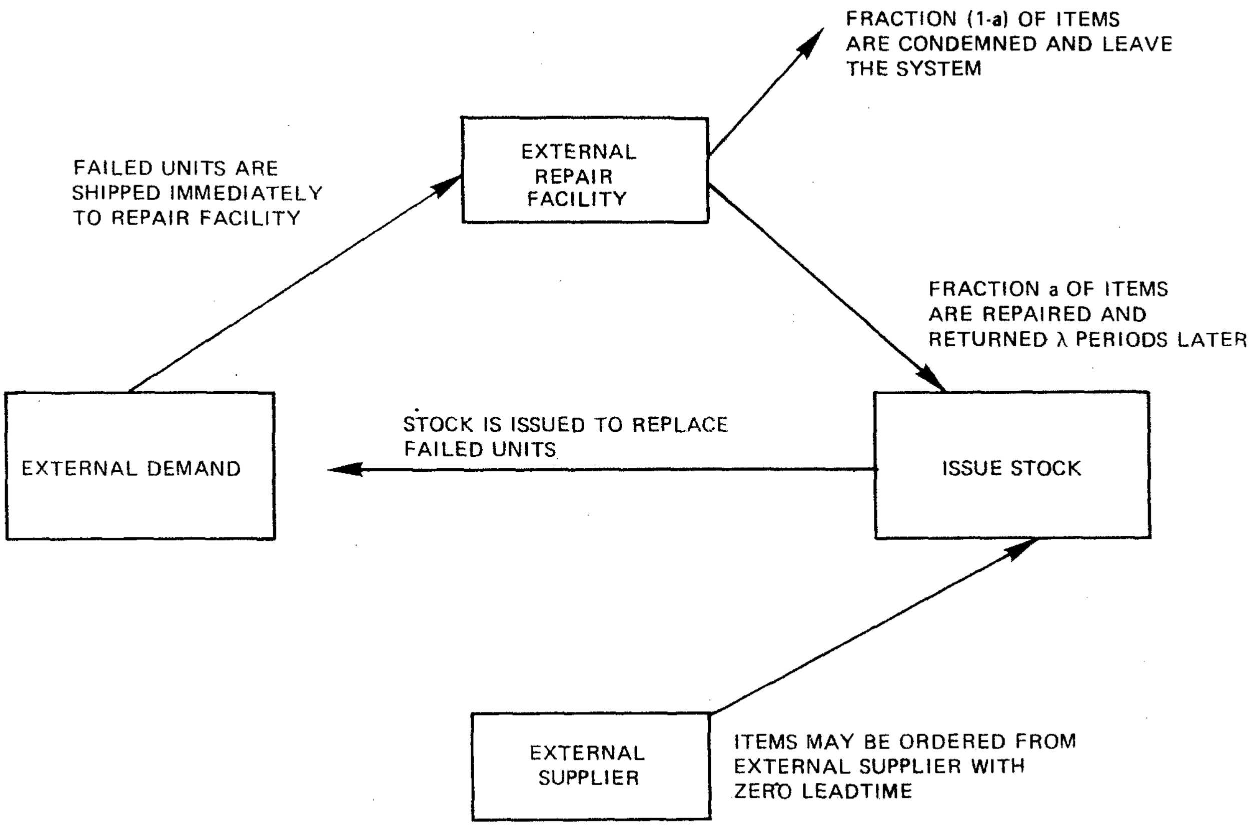

FIGURE 1.

Previous analyses of this class of inventory system have been restricted primarily to simulation studies of blood banks (Cohen and Pierskalla [2]), or systems where demand is deterministic and the proportion of stock recycling is treated as a random variable (Cohen, Nahmias and Pierskalla [3]). Related reparable item inventory models include Prawda and Wright [4] and Allen and D'Esopo [1]. This paper treats the case where demand (or failure) is stochastic and where the fraction of stock that feeds back into the system is fixed.

The paper begins with a description of notation and assumptions for a general model with arbitrary recycle lag and stochastic demand and the functional equation satisfied by the optimal order policy is formulated. A myopic approximation to the optimal order policy and conditions for its optimality are derived for the case of an arbitrary recycle period. The optimality of this policy for the case where the recycle period is equal to one is then demonstrated.

The final section reports on the results of a numerical analysis comparing optimal and myopic policies for a variety of cases where the recycle lag is equal to two periods. These results suggest that the myopic policy provides a very effective approximation to the optimal.

MODEL ASSUMPTIONS AND NOTATION

A periodic review inventory system with the following features is considered:

- Successive demands $\{D_i\}$ are independent and identically distributed random variables with known cumulative distribution function $F(\cdot)$ and density $f(\cdot)$.
- A fixed fraction, a, of stock issued to meet demand is returned after a delay of $\lambda \geqslant 1$ periods. The fraction, $1 - a$, is consumed.
- A fixed fraction, β, of stock on hand at the end of each review period survives, without decay, into the next period and the fraction, $1 - \beta$, is lost to decay.
- Excess demand is lost in each period.
- There is no leadtime for ordering. That is, orders are received in the period in which they are placed.
- Excess demand is lost (lost sales).

Time periods are numbered forward by integers n, $n = 1, 2, \ldots, T$, where T is the decision horizon for the problem.

The following variables describe the state of the system each period:

$\underset{\sim}{I}$ $= (I_1, \ldots, I_{\lambda-1})$ is the vector of stock quantities issued to meet demand in the previous $\lambda-1$ periods. Interpret I_i as the quantity issued exactly i periods previously.

u = starting inventory before ordering but after the arrival of recycled stock in the current period.

z = inventory on hand after ordering and after returns in the current period.

The state of the system at any point in time is described by the vector $(u,\underset{\sim}{I})$, and the decision variable (the order quantity), is given by $z - u$.

We will adopt the common conventions that the holding cost function $h(\cdot)$ and the shortage cost function $p(\cdot)$ are convex functions of ending stock in each period. The outdating cost is assumed to be θ per unit of stock that outdates at the end of each period, and the procurement cost is c per unit. If follows that the one period expected holding, shortage, and outdate cost function, say $L(z)$, is a convex function of the starting stock z, and is given by

$$L(z) = E\{h[(z - D)^+] + p[(D - z)^+] + \theta(1 - \beta)\ (z - D)^+\},$$
$$= \int_0^z h(z - t)f(t)dt + \int_z^\infty p(t - z)f(t)dt + \theta(1 - \beta)\int_0^z (z - t)f(t)dt.$$

Assuming that future costs are discounted by α where $0 < \alpha \leqslant 1$, it follows that the functional equations defining an optimal policy are given by:

$$C_n(u,\underset{\sim}{I}) = \min_{z \geqslant u} \Big\{ c(z-u) + L(z) + \alpha \int_0^z C_{n+1}(\beta(z-t) + aI_{\lambda-1}, t, I_1, \ldots, I_{\lambda-2}) f(t)\,dt + \alpha \int_z^\infty C_{n+1}(aI_{\lambda-1}, z, I_1, \ldots, I_{\lambda-2}) f(t)\,dt \Big\}$$

for $n \geqslant 1$ and $C_{T+1}(\cdot) = 0$. Note that this is equivalent to assuming that stock remaining at the end of the horizon is not salvaged. Other salvage assumptions are possible and will be considered in the next section.

Here $C_n(u,\underset{\sim}{I})$ has the interpretation as the minimum expected discounted cost at the start of period n when u is the starting stock after returns, and $\underset{\sim}{I}$ is the vector of previously issued stock.

The process dynamics are implied in the functional equations above. Let t be the realization of demand in period n. Then there are two cases:

(a) $t \leqslant z$. In this case $(-\beta)$ $(z-t)$ is lost due to decay and $\beta(z-t)$ transfers to the next period which combines with the stock which recycles in period $n+1$, $aI_{\lambda-1}$. Exactly t units are issued to meet the demand. In this case if $(u,\underset{\sim}{I})$ is the state vector in period n, it follows that $(\beta(z-t) + aI_{\lambda-1}, t, I_1, \ldots, I_{\lambda-2})$ is the state vector in period $n+1$.

(b) $t > z$. In this case ending stock in period n is zero and no stock decays or is transferred to the following period. Starting stock the next period consists only of the stock which recycles in period $n+1$, which is $aI_{\lambda-1}$. Since only z can be issued to meet demand, the state vector one period hence is $(aI_{\lambda-1}, z, I_1, \ldots, I_{\lambda-2})$.

The optimal policy is to order the max $(z_n(I) - u, 0)$ where $z_n(\underset{\sim}{I})$, the order to point, minimizes the bracketed term on the right hand side of the functional equation above. Computation of an optimal policy will be difficult due to the limitations of dynamic programming with vector valued state variables. However, for $\lambda = 1$ under reasonably general conditions the optimal policy can be shown to reduce to a single critical number in each period. In addition, a critical number approximation is derived for the case $\lambda > 1$.

A MYOPIC CRITICAL NUMBER APPROXIMATION

We will assume as above, that periods are numbered forwards and the planning horizon is exactly T periods where $\lambda \leqslant T \leqslant +\infty$. We ignore the case $T \leqslant \lambda$, as the feedback process will not be relevant, and the optimal policy reduces to the ordinary critical number order policy. The variables u_n and z_n are still to be interpreted as starting stock after returns before and after ordering respectively in period n. In addition, let $(I(1), \ldots, I(\lambda))$ represent stock issued in the final λ periods. That is, $I(1)$ is issued in period $T-\lambda+1$, $I(2)$ is issued in period $T-\lambda+2$, $\ldots$, and $I(\lambda)$ is issued in period T.

In order to construct the approximation we will need to assume that all stock remaining in the system at the end of the horizon can be salvaged at a return equal to the purchase cost of c per unit. This includes stock on hand at that time, (u_{T+1}), and the stock issued in the final λ periods of the horizon $I(1), \ldots, I(\lambda)$. In addition, we assume that the issued stock cannot be salvaged until it returns to inventory. Hence $aI(1)$ is salvaged in period $T+1$, $aI(2)$ in period $T+2$, $\ldots$, and $aI(\lambda)$ in period $T+\lambda$. (The salvage assumption was first used by Veinott [5].)

With these assumptions, the total discounted cost over T periods, say $TC(\underline{z})$, when following the ordering policy $\underline{z} = (z_1,\ldots,z_T)$ is

$$TC(\underline{z}) = E\left\{\sum_{n=1}^{T} \alpha^{n-1} \{c(z_n - u_n) + L(z_n)\}\right\} - \alpha^T c u_{T+1} - c \sum_{k=1}^{\lambda} \alpha^{T+k-1} a I(k).$$

The process dynamics imply that

$$u_n = \begin{cases} \beta(z_{n-1} - D_{n-1})^+ \text{ for } 2 \leqslant n \leqslant \lambda \\ \beta(z_{n-1} - D_{n-1})^+ + a \min(z_{n-\lambda}, D_{n-\lambda}) \text{ for } \lambda + 1 \leqslant n \leqslant T+1 \end{cases}$$

and $I(k) = \min(z_{T-\lambda+k}, D_{T-\lambda+k})$ for $1 \leqslant k \leqslant \lambda$.

By a rearrangement of terms, one can show that

$$TC(\underline{z}) = \sum_{n=1}^{T} \alpha^{n-1} W(z_n) - cu_1,$$

where

$$W(z_n) = E[c(z_n - \alpha\beta(z_n - D_n)^+ - \alpha^\lambda a \min(z_n, D_n)] + L(z_n) \text{ for } 1 \leqslant n \leqslant T.$$

We have the following result:

THEOREM 1: Assuming

1. All inventory on hand in periods $T+1, \ldots, T+\lambda$ can be salvaged in that period at a return of c per unit.
2. $p'(0) > (1 - \alpha^\lambda a)\, c$
3. $P\left\{D_n \geqslant \left(\dfrac{a+\beta-1}{\beta}\right) z^*\right\} = 1$ for $1 \leqslant n \leqslant T$
4. $u_1 < z^*$.

where z^* is the minimizing point of $W(z)$ and is the root of the equation $W'(z^*) = 0$, then the optimal policy is to order to z^* every period.

PROOF: Since $\sum_{n=1}^{T} W_n(z^*) \leqslant \sum_{n=1}^{T} W_n(z)$, it follows that z^* is the optimal order to point if z^* can be achieved. It will be possible to order to z^* in period n if and only if $z^* - u_n \geqslant 0$. Following the policy z^* in every period implies that

$$\begin{aligned} u_n &= \beta(z^* - D_{n-1})^+ + a \min(z^*, D_{n-\lambda}) \\ &\leqslant \beta(z^* - D_{n-1})^+ + az^* = \max(az^*, (a+\beta) z^* - \beta D_{n-1}). \end{aligned}$$

Clearly $az^* \leqslant z^*$. By assumption 3, $(a+\beta)\, z^* - \beta D_{n-1} \leqslant (a+\beta)\, z^* - (a+\beta-1)\, z^* = z^*$, hence, $u_n \leqslant z^*$.

Since $W(z)$ is convex in z, and by assumption 2, $W'(0) < 0$, we have that $z^* > 0$.

□

COROLLARY: For $\lambda = 1$, $P\left\{D_n \geqslant \left[\frac{a+\beta-1}{\beta}\right] z^*\right\} = 1$ for $1 \leqslant n \leqslant T$, and hence the myopic policy defined in Theorem 1 is optimal when only assumptions (1), (2) and (4) hold.

PROOF: For $\lambda = 1$ we have,

$$u_n = \beta(z^* - D_{n-1})^+ + a \min(z^*, D_{n-1}),$$

and it follows easily that $u_n \leqslant z^*$ for all realizations of D_{n-1}.

□

When all costs are linear, z^* will be given by,

$$(1) \qquad z^* = F^{-1}\left[\frac{p - c(1-\alpha^\lambda a)}{p + h + \theta(1-\beta) - \alpha c(\beta - \alpha^{\lambda-1} a)}\right].$$

Assumption 3 in Theorem 2 is somewhat tautological when $\lambda \geqslant 2$, since z^* depends on the distribution of D_n. When this assumption does not hold, it may not be possible to order to z^* in every period as it will not necessarily be true that $u_n \leqslant z^*$. However, since the expected cost function, $W(z)$, tends to be relatively flat in a neighborhood of the minimum, it seems reasonable to conjecture that when assumption 3 is not met, ordering $(z^* - u_n)^+$ in every period should give a good approximation.

In order to test this conjecture, numerical computations are performed for $\lambda = 2$ in the next section. Dynamic programming is used to compute the optimal stationary policy which is then compared to z^* for a variety of demand distributions and cost configurations.

NUMERICAL COMPARISONS FOR $\lambda = 2$

A series of runs were carried out for a variety of configurations of the system parameters to compare the effectiveness of the approximation to the optimal policy when the recycle delay was two periods. In order to reduce the number of different factors considered, the cost parameters (c,h,p) are combined into the single constant $m = (p-c)/(p+h)$ (which is motivated by the solution to the newsboy problem). For each demand distribution the following factors and levels are considered:

(1) cost ratio, $m \in \{.5, .75, .95\}$
(2) return fraction, $a \in \{.2, .5, .8\}$
(3) outdate cost, $\theta \in \{1, 2\}$
(4) outdate fraction, $\beta \in \{.2, .8\}$

These factor levels lead to a 36 case experiment. The discount factor α was fixed at .95, order cost c at 1 and holding cost h at .5. The required values of m were achieved by setting p at 2.5, 5.5 and 28.5, respectively.

The total 36 case experiment was run, for uniform, exponential and geometric distributions each with a mean value of five which resulted in a total of 108 cases. The output for a typical case is illustrated in Table I. The optimal solution was computed by standard value iteration techniques and the myopic policy was computed from (1). A 30 period horizon was selected to minimize transient effects. Convergence to the stationary optimal policy generally occurred in ten periods or less. We note, from Table I, that both the optimal solution and cost penalty for using the myopic approximation are relatively insensitive to changes in the initial inventory level. The maximum percent cost penalty for using the myopic policy is 0.3% in this case.

TABLE I—*Optimal and Approximate Policies for the Case* $m = .5$, $a = .2$, $\theta = 1$, $\beta = .8$, *Poisson Demand*

Initial Inventory I	Optimal Order Function ($z^*(I)$)	Myopic Order-up -to Level (z^*)	Average Cost per Period		% Difference in Cost
			using $z^*(I)$	using z^*	
0	6	5	3.232	3.240	0.2
1	6	5	3.227	3.237	0.3
2	6	5	3.225	3.236	0.3
3	5	5	3.211	3.218	0.2
4	5	5	3.205	3.212	0.2
5	6	5	3.200	3.208	0.3
6	6	5	3.196	3.205	0.3
7	6	5	3.193	1.204	0.3
8	5	5	3.179	3.186	0.2
9	5	5	3.173	3.181	0.3
10	6	5	3.169	3.176	0.2

Table II summarizes 36 runs selected from the set of 108 runs. The runs illustrated were selected by taking the worst case (that is, the largest cost error) over the three demand distributions for each case. Optimal and myopic policies and costs for just the single initial inventory level of five are indicated, since, as noted above, the results are not sensitive to the initial inventory level. Table II also indicates the maximum percent cost differences for each case taken over all initial inventory level values. Note that the maximum percent cost penalty ranges from 0.0% to 6.9% over all factor values. In addition, also note that 71% of all 108 cases had a maximum cost difference of less than 1% and that only 6.5% had a maximum cost difference of more than 5%.

It seems reasonable to conjecture that the myopic policy will also provide a good approximation for values of λ, the recycle delay parameter, larger than two as well. The approximation has the dual advantage of being both easy to compute and easy to implement. The model presented here is applicable to a variety of inventory problems where stock recycling is present, including blood bank inventory control and reparable item management.

ACKNOWLEDGMENT

We would like to acknowledge the assistance of Craig Uthe who wrote the program for the numerical comparisons.

TABLE II— *Worst Case Optimal and Myopic Cost Comparisons*

Run #	m	a	*theta*	*beta*	Demand CDF	Optimal z^* (5)	Optimal Average Cost at $I = 5$	Approx. Policy z^*	Average Cost of Approx. Policy	Maximum % Penalty
1	.50	.2	1	.2	Geometric	2	3.372	1	3.501	4.0
2	.75	.2	1	.2	Poisson	6	4.802	5	5.026	4.7
3	.95	.2	1	.2	Uniform	10	7.658	9	7.969	4.1
4	.50	.5	1	.2	Poisson	5	3.041	4	3.198	5.2
5	.75	.5	1	.2	Uniform	7	5.542	7	5.542	0.0
6	.95	.5	1	.2	Uniform	10	6.890	9	7.21	4.7
7	.50	.8	1	.2	Poisson	5	2.566	4	2.745	6.9
8	.75	.8	1	.2	Uniform	7	4.932	7	4.932	0.0
9	.95	.8	1	.2	Uniform	10	6.188	9	6.525	5.5
10	.50	.2	2	.2	Geometric	1	3.589	1	3.593	.1
11	.75	.2	2	.2	Poisson	5	5.394	5	5.394	0.0
12	.95	.2	2	.2	Poisson	8	7.980	7	8.403	5.3
13	.50	.5	2	.2	Uniform	3	4.649	3	4.649	0.0
14	.75	.5	2	.2	Uniform	7	6.603	6	6.639	0.1
15	.95	.5	2	.2	Poisson	8	7.236	7	7.655	5.9
16	.50	.8	2	.2	Geometric	2	3.095	1	3.188	3.7
17	.75	.8	2	.2	Poisson	6	4.101	5	4.193	2.2
18	.95	.8	2	.2	Poisson	8	6.557	7	7.003	6.7
19	.50	.2	1	.8	Poisson	6	3.200	5	3.208	0.4
20	.75	.2	1	.8	Geometric	6	3.652	5	3.728	2.1
21	.95	.2	1	.8	Geometric	9	4.608	9	4.609	0.0
22	.50	.5	1	.8	Uniform	7	3.161	6	3.217	1.9
23	.75	.5	1	.8	Uniform	9	3.666	8	3.696	0.8
24	.95	.5	1	.8	Poisson	9	3.655	9	3.656	0.0
25	.50	.8	1	.8	Geometric	3	2.162	4	2.185	2.6
26	.75	.8	1	.8	Geometric	6	2.903	6	2.926	1.4
27	.95	.8	1	.8	Geometric	9	3.883	9	3.885	0.2
28	.50	.2	2	.8	Uniform	6	4.090	6	4.093	0.1
29	.75	.2	2	.8	Poisson	7	3.944	6	4.030	2.3
30	.95	.2	2	.8	Poisson	9	4.828	8	5.060	4.8
31	.50	.5	2	.8	Uniform	6	3.417	6	3.427	0.4
32	.75	.5	2	.8	Poisson	7	3.208	6	3.323	3.6
33	.95	.5	2	.8	Poisson	9	4.086	8	4.312	5.5
34	.50	.8	2	.8	Poisson	6	2.070	5	2.092	1.3
35	.75	.8	2	.8	Poisson	7	2.629	6	2.699	2.1
36	.95	.8	2	.8	Poisson	9	3.499	8	3.639	3.9

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