SYMMETRIC POWERS AND TOPOLOGICAL TYPES(FINAL DRAFT)

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ABSTRACT. For a proper normal geometrically connected algebraic space over a separably closed field, Arnav Tripathy proved that there is a weak equivalence from the symmetric power of étale homotopy type of the algebraic space to the étale homotopy type of symmetric power of the algebraic space. By applying the qfh topology to the theory of topological types developed by the author, we provide a totally different proof that can be generalized to study coarse moduli spaces.

CONTENTS

1. INTRODUCTION

1.1. Motivation.

1.1.1. Let X be a pointed connected CW-complex. There is a canonical action of the symmetric group S_n of the n letters on the n-fold product X^n. The *nth symmetric power* $\mathrm{Sym}^n(X)$ is defined as the quotient space X^n/S_n. The classical Dold-Thom theorem [5, 6.10] states that for each $i > 0$ there is an isomorphism

$$H_i(X;\mathbb{Z}) \simeq \pi_i(\mathrm{Sym}^\infty(X))$$

where $\mathrm{Sym}^\infty(X)$ is the colimit of $\mathrm{Sym}^n(X)$ as n varies.

1.1.2. Recently, Arnav Tripathy showed [13] that the Dold-Thom theorem is still valid in the algebro-geometric world:

Theorem 1.1.3. ([13, Theorem 1]) *Let X be a proper, normal, noetherian, geometrically connected algebraic space over a separably closed field k. The natural map*

$$\mathrm{Sym}^n(h_{\mathrm{AM}}(X)) \to h_{\mathrm{AM}}(\mathrm{Sym}^n X)$$

of pro-objects in the homotopy category of simplicial sets is a weak equivalence. Here $h_{\mathrm{AM}}(-)$ denotes Artin-Mazur's étale homotopy type functor.

1.1.4. Formally, the theorem above says that Artin-Mazur's étale homotopy type functor commutes with the symmetric power functor. This type of formality fits better into the theory of topological types developed by the author [4]. Indeed, the topological types of algebraic stacks are defined by using model categories and derived functors, which generalizes the derived functor reformulation of étale homotopy type by Ilan Barnea and Tomer Schlank [2]. From this point of view, we expect the Dold-Thom theorem for étale homotopy types to follow formally from the machinery in [4].

1.2. **Statement of the main results.**

1.2.1. The main goal of this paper is to provide an alternate proof for the Dold-Thom theorem for étale homotopy types via the tools developed in [4]. We expect the generality of the idea of the proof to be applied to the study of algebraic stacks and their coarse moduli spaces.

1.2.2. As we use model category theory, we restate [13, Theorem 1] as following with the removal of the connected assumption:

The Dold-Thom Theorem for Topological types. (Theorem 5.0.4) *Let X be a geometrically normal and proper algebraic space over a separably closed field k. Then there is a canonical isomorphism*

$$\mathrm{Sym}^n(h_{ét}(X)) \xrightarrow{\sim} h_{ét}(\mathrm{Sym}^n(X)) \tag{1.2.2.1}$$

in $\boldsymbol{Ho}(\mathrm{pro}-\boldsymbol{SSet})$.

1.2.3. Firstly, we formally obtain the theorem by using the qfh topological types 2.2.10. Then show that the usual étale topological type for the symmetric power of algebraic space is nothing but the qfh topological type of the symmetric power. This comparison is mainly due to the cohomological comparison by Vladimir Voevodsky [14, 3.4.4]. We also use the computation on the fundamental group of the symmetric power by Indranil Biswas and Amit Hogadi [3, 1.2].

1.2.4. This formal approach is different from the work of Tripathy. He concretely analyzed the étale fundamental group of X, and used Deligne's work on the cohomology of $\mathrm{Sym}^n(X)$.

1.2.5. Let us explain in more detail how this new strategy works. Let X be a geometrically normal and proper algebraic space over a separably closed field k. The symmetric group S_n acts on the n-fold fiber product $(X/S)^n$ of X over S. The *nth symmetric power* $\mathrm{Sym}^n(X)$ exists as a GC quotient of the groupoid of algebraic spaces (see [11, 5.5] for details):

$$\underline{S_n} \times (X/S)^n \rightrightarrows (X/S)^n \longrightarrow \mathrm{Sym}^n(X/S) \tag{1.2.5.1}$$

Note that the S_n-action is not free and hence the GC quotient $\mathrm{Sym}^n(X)$ is not a sheaf quotient. So the canonical map

$$(X/S)^n \to \mathrm{Sym}^n(X/S)$$

is not an étale covering. Nonetheless, it is a covering with respect to the qfh topology [14, 3.1.2]. In particular, $\mathrm{Sym}^n(X)$ is a quotient sheaf with respect to the qfh topology. Namely, for the morphism of topoi 2.2.7

$$i = (i^*, i_*) : \mathrm{LFQ}(S)^\sim \to \mathrm{LF\acute{E}}(S)^\sim$$

the diagram (1.2.5.1) is pulled-back to a coequalizer in the category qfh sheaves 4.1.12. This is the reason why we prefer working with the qfh topology to the étale topology.

1.2.6. From that $\mathrm{Sym}^n(X)$ is a quotient qfh sheaf, one can formally obtain the Dold-Thom theorem for qfh topological types. Fix a locally noetherian scheme S. Consider the big qfh site $\mathrm{LFQ}(S)$ which is the full category of the category of schemes over S, whose objects are locally of finite type morphisms to S with coverings induced by coverings in the qfh topology on S ([14, 3.1.2]). Denote by $\mathrm{LFQ}(S)^\sim$ the associated topos. For any simplicial object $F_\bullet$ in the big qfh topos $\mathrm{LFQ}(S)^\sim$, the constant sheaf $\underline{S_n}$ associated to the symmetric group canonically acts on the n-fold product $F_\bullet^n$ of $F_\bullet$. We define the *nth symmetric power* $\mathrm{Sym}^n(F_\bullet)$ to be the coequalizer of the following diagram in the category $(\mathrm{LFQ}(S)^\sim)^{\Delta^{\mathrm{op}}}$ of simplicial qfh sheaves:

$$\underline{S_n} \times F_\bullet^n \rightrightarrows F_\bullet^n$$

where the two arrows are the $\underline{S_n}$-action and the projection onto $F_\bullet^n$.

1.2.7. The importance of the qfh topology is that as an immediate consequence of the definition, the symmetric power functor

$$\mathrm{Sym}^n : (\mathrm{LFQ}(S)^\sim)^{\Delta^{\mathrm{op}}} \to (\mathrm{LFQ}(S)^\sim)^{\Delta^{\mathrm{op}}}$$

preserves local weak equivalences as one can check at stalks 4.2.10. Note that for simplicial sets, we already know that the symmetric power functor preserves weak equivalences 4.1.5. Consider the connected component functor Π_{qfh} of the topos $\mathrm{LFQ}(S)^\sim$, which is a left adjoint of the constant sheaf functor. Then the following diagram commutes:

$$\begin{array}{ccc} (\mathrm{LFQ}(S)^\sim)^{\Delta^{\mathrm{op}}} & \xrightarrow{\mathrm{Sym}^n} & (\mathrm{LFQ}(S)^\sim)^{\Delta^{\mathrm{op}}} \\ \downarrow{\scriptstyle \Pi_{\mathrm{qfh}}} & & \downarrow{\scriptstyle \Pi_{\mathrm{qfh}}} \\ \mathbf{SSet} & \xrightarrow{\mathrm{Sym}^n} & \mathbf{SSet} \end{array}$$

By applying the Quillen derived functors to the diagram with respect to Barnea-Schlank's model category structure on $\mathrm{pro} - (\mathrm{LFQ}(S)^\sim)^{\Delta^{\mathrm{op}}}$ and Isaksen's model category structure on $\mathrm{pro} - \mathbf{SSet}$ (see [4, §2.3] for details), we see that the canonical map

$$\mathrm{Sym}^n(h_{\mathrm{qfh}}(F_\bullet)) \to h_{\mathrm{qfh}}(\mathrm{Sym}^n(F_\bullet))$$

is an isomorphism in the homotopy category of pro-simplicial sets 4.2.11 where h_{qfh} is the topological type functor using the qfh topology 2.2.10. Therefore, the Dold-Thom for qfh topological types is a formal consequence of the machinery in [4].

1.2.8. Finally, we compare the qfh topological types to the usual étale topological types. Consider a commutative diagram in $\mathbf{Ho}(\mathrm{pro} - \mathbf{SSet})$:

$$\begin{array}{ccc} \mathrm{Sym}^n(h_{\mathrm{qfh}}(X)) & \longrightarrow & h_{\mathrm{qfh}}(\mathrm{Sym}^n(X)) \\ \downarrow & & \downarrow \\ \mathrm{Sym}^n(h_{\mathrm{\acute{e}t}}(X)) & \longrightarrow & h_{\mathrm{\acute{e}t}}(\mathrm{Sym}^n(X)) \end{array}$$

We already know that the top map is an isomorphism. By the classical Dold-Thom, the left vertical map is an isomorphism because it can be reduced to the cohomological comparison 5.0.3. This comparison comes from the work of Voevodsky [14, 3.4.4] that the cohomology groups of schemes for the étale and qfh topologies coincide. On the other hand, it is already known by Biswas-Hogadi [3, 1.2] that the abelianization of the étale fundamental group of X is isomorphic to the étale fundamental group of $\mathrm{Sym}^n(X)$. This combined again with Voevodsky's cohomological comparison shows that the right vertical map is an isomorphism 5.0.2. Therefore, the bottom map is a weak equivalence. Namely, we obtain the Dold-Thom theorem for topological types.

1.3. Convention.

1.3.1. In what follows, for schemes, algebraic spaces, and algebraic stacks, we work over a fixed bases scheme S unless stated otherwise. Moreover, we assume that S is locally noetherian throughout the paper.

1.3.2. There could be some set-theoretical issue when working with the big topologies on schemes. Whenever this issue arises, we invoke [10, Tag 020M] so that we can assume the smallness on the sites LFQ(S) and LFÉ(S).

1.4. **Acknowledgements.** TO BE ADDED LATER

2. Topological types via qfh topology

In this section we define the topological types via qfh topology, and compare them to the usual étale topological types.

2.1. Qfh topology on schemes.

The qfh topology of schemes was developed by Vladimir Voevodsky [14] to study the homology of schemes. In this subsection we briefly review the qfh topology.

Definition 2.1.1.

(i) A continuous map $f : X \to Y$ of topological spaces is *submersive* if it is surjective and Y has the quotient topology, i.e., a subset $V \subset Y$ is open if and only if its preimage $f^{-1}V$ is open in X.
(ii) A morphism $f : X \to Y$ algebraic spaces is *submersive* if its associated map $|X| \to |Y|$ of topological spaces is submersive.
(iii) A morphism $f : X \to Y$ algebraic spaces is *universally submersive* if for every morphism $Z \to Y$ of algebraic spaces, its base change $X \times_Y Z \to Z$ is submersive.

Definition 2.1.2. (*[14, 3.1.2]*) Let X be a scheme. A collection $\{f_i : X_i \to X\}$ of morphisms of schemes is a *h covering* if it is a finite family of morphisms of finite type such that the morphism $\coprod f_i : \coprod X_i \to X$ is a universally submersive. If we furthermore require each f_i to be quasi-finite, then we call it a *qfh covering*.

Definition 2.1.3. Let S be a scheme. The *qfh site* on S, denoted by $(\mathbf{Sch}/S)_{\mathrm{qfh}}$, is the category of schemes over S with coverings induced by coverings in the qfh topology.

2.2. Topoloigcal types via qfh topology and its comparison to étale topology.

In this subsection we define the topological types of algebraic spaces via qfh topology, and compare it to the usual étale topological types ([4, 2.3.8]).

2.2.1. In order to study the étale homotopy types of algebraic spaces, we follow the theory of topological types in [4] where the homotopy theory of algebraic stacks was developed by using the machinery of Ilan Barnea and Tomer Schlank [2]. It not only extends the étale homotopy theory of schemes by Michael Artin and Barry Mazur [1], but also the étale topological theory of simplicial schemes by Eric Friedlander [6]. The main difference compared to these classical theories lies in the use of model category theory and the generalization to algebraic stacks. Furthermore, the theory is developed for general topoi so that it can be applied to different contexts including the qfh topoi which plays a key role in this paper.

2.2.2. Let us briefly review the notion of topological types of topoi. The reference is [4, §2.3]. Let T be a topos and consider the (2-categorical) unique morphism of topoi:

$$\Gamma = (\Gamma^*, \Gamma_*) : T \to \mathbf{Set}$$

Then a left adjoint L_{Γ^*} of Γ^* exists for the associated pro-categories of simplicial objects and it induces a left Quillen functor

$$L_{\Gamma^*} : \mathrm{pro} - T^{\Delta^{\mathrm{op}}} \to \mathrm{pro} - \mathbf{SSet}$$

with respect to Barnea-Schlank's model category structure on $\mathrm{pro} - T^{\Delta^{\mathrm{op}}}$ and Isaksen's model category structure on $\mathrm{pro} - \mathbf{SSet}$.

Definition 2.2.3. *([4, 2.3.8])* A *topological type* $h(T)$ of a topos T is the pro-simplicial set

$$\mathbf{L}L_{\Gamma^*}(*)$$

where $*$ is a final object of $T^{\Delta^{\mathrm{op}}}$ and $\mathbf{L}L_{\Gamma^*} : \mathbf{Ho}(\mathrm{pro} - T^{\Delta^{\mathrm{op}}}) \to \mathbf{Ho}(\mathrm{pro} - \mathbf{SSet})$ is the left derived functor of L_{Γ^*}. More generally, a *topological type* $h(F_\bullet)$ (or $h_T(F_\bullet)$ if we wish to make the reference to T explicit) of a simplicial object $F_\bullet$ in T is the pro-simplicial set

$$\mathbf{L}L_{\Gamma^*}(F_\bullet)$$

Definition 2.2.4. Let S be a locally noetherian scheme. A site $\mathrm{LFQ}(S)$ is the full category of the category of schemes over S, whose objects are locally of finite type morphisms to S with coverings induced by coverings in the qfh topology on S. Denote by $\mathrm{LFQ}(S)^\sim$ the associated topos.

Remark 2.2.5. When replacing the qfh coverings by the étale coverings, we recover the topos $\mathrm{LFÉ}(S)^\sim$ defined in [4, 3.1.2].

2.2.6. We fix a locally noetherian base scheme S throughout the rest of the paper.

2.2.7. To compare the étale and qfh topologies, note that there is a continuous functor

$$i : \mathrm{LFÉ}(S) \to \mathrm{LFQ}(S)$$

which commutes with finite limits, which induces a morphism of topoi

$$i = (i^*, i_*) : \mathrm{LFQ}(S)^\sim \to \mathrm{LFÉ}(S)^\sim \tag{2.2.7.1}$$

2.2.8. An algebraic space X/S is a sheaf on the big étale site on S, and is restricted to a sheaf on $\mathrm{LFÉ}(S)$. When pulled-back along the morphism i, we obtain a sheaf i^*X on $\mathrm{LFQ}(S)$.

2.2.9. The qfh topos $\mathrm{LFQ}(S)$ is locally connected in a sense that the constant sheaf functor admits a left adjoint, which is denoted by Π_{qfh} and called by the connected component functor. The proof for $\mathrm{LFÉ}(S)$ case works verbatim (see [4, 3.1.6]). So the functor $L_{\Gamma^*_{\mathrm{qfh}}}$ in the definition of the topological types can be identified with the connected component functor Π_{qfh} for the associated pro-categories.

Definition 2.2.10. Let X be a locally of finite type algebraic space over S. The *qfh-topological type* $h_{\mathrm{qfh}}(X)$ of X is the topological type of the qfh sheaf i^*X. Namely, it is the pro-simplicial set

$$\mathbf{L}\Pi_{\mathrm{qfh}}(i^*X)$$

2.2.11. Recall from [4, 3.2.2] that the topological type $h_{\text{ét}}(X)$ of the algebraic space X is the pro-simplicial set obtained by applying the previous definition to the topos $\mathrm{LF\acute{E}}(S)^{\sim}$. Therefore, for any algebraic space X that is locally of finite type over S, it follows from the functoriality of topological types [4, 2.3.31] that there is a canonical map between topological types:

$$h_{\mathrm{qfh}}(X) \to h_{\text{ét}}(X)$$

2.2.12. The following theorem shows a partial relationship between the usual étale topological type and the qfh topological type:

Theorem 2.2.13. ([14, 3.4.4], [9, Theorem 1]) *Let X be an algebraic space that is locally of finite type over S. Then the canonical map of topological types*

$$h_{qfh}(X) \to h_{\acute{e}t}(X)$$

induces an isomorphism

$$H^n(h_{\acute{e}t}(X), M) \xrightarrow{\sim} H^n(h_{qfh}(X), M)$$

for every $n \geq 0$ and every local coefficient system M of abelian groups. In particular, there is an isomorphism on the abelianization of fundamental groups

$$\pi_1^{\mathrm{ab}}(h_{qfh}(X)) \xrightarrow{\sim} \pi_1^{\mathrm{ab}}(h_{\acute{e}t}(X))$$

Proof. The statement on cohomology is the result of Voevodsky [14, 3.4.4]. For the abelianized fundamental groups, it suffices to show that for any abelian group G, the top horizontal map in the following commutative diagram is an isomorphism:

$$\begin{array}{ccc} \mathrm{Hom}_{\text{pro-groups}}(\pi_1(h_{\mathrm{qfh}}(X)), G) & \longrightarrow & \mathrm{Hom}_{\text{pro-groups}}(\pi_1(h_{\text{ét}}(X)), G) \\ \downarrow & & \downarrow \\ H^1(h_{\mathrm{qfh}}(X), G) & \longrightarrow & H^1(h_{\mathrm{qfh}}(X), G) \end{array}$$

The two vertical maps are isomorphisms by [4, 2.4.7] and the bottom map is an isomorphism again by Voevodsky [14, 3.4.4]. Therefore, the top map is also an isomorphism. □

3. Quotients by algebraic spaces

In this section we review the notion of geometric quotients of algebraic spaces. Then we prove that for the case of interest these quotients can be viewed as coequalizers of qfh sheaves.

3.1. **Geometric quotients.** In this subsection we summarize some results on geometric quotients following David Rydh [11].

Definition 3.1.1. (*[11, 2.2]*) Let $s, t : R \rightrightarrows X$ be a groupoid of algebraic spaces over S, and $q : X \to Y$ be a morphism of algebraic spaces over S. A morphism q is *equivariant* if $q \circ s = q \circ t$. If a property of q is stable under flat base change (resp. every base change) $Y' \to Y$, the property is *uniform* (resp. *universal*). For an equivariant q,

(i) q is a *Zariski quotient* if the diagram of associated topoloigcal spaces
$$|R| \rightrightarrows |X| \longrightarrow |Y|$$
is a coequalizer in the category of topological spaces.

(ii) q is a *constructible quotient* if the diagram of associated constructible topoloigcal spaces
$$|R|^{\text{cons}} \rightrightarrows |X|^{\text{cons}} \longrightarrow |Y|^{\text{cons}}$$
is a coequalizer in the category of topological spaces.

(iii) q is a *topological quotient* if it is both a universal Zariski quotient and a universal constructible quotient.

(iv) q is a *strongly topological quotient* if it is a topological quotient and $j_Y = (s,t) : R \to X \times_Y X$ is universally submersive.

(v) q is a *geometric quotient* if it is a topological quotient and if
$$\mathcal{O}_Y \longrightarrow q_*\mathcal{O}_X \rightrightarrows (q \circ s)_*\mathcal{O}_R$$
is an equalizer in the category sheaves on $(\mathbf{Sch}/S)_{\text{ét}}$.

(vi) q is a *strongly geometric quotient* if it is both a geometric quotient and a strongly topological quotient.

(vii) q is a *GC quotient* if it is a strongly geometric quotient that satisfies the descent condition for separated étale morphisms uniformly (see [11, 3.6] for details).

Definition 3.1.2. Let $s, t : R \rightrightarrows X$ be a groupoid of algebraic spaces over S.

(i) The groupoid is *finite locally free* if s, or equivalently t, is finitely locally free. That is, if s is affine and $s_*\mathcal{O}_R$ is a finite locally free $\mathcal{O}_X$-module.

(ii) The *stabilizer* is the base change of $j = (s,t) : R \to X \times_S X$ along the diagonal on X over S:
$$\begin{array}{ccc} j^{-1}(\Delta(X)) & \longrightarrow & X \\ \downarrow & & \downarrow \Delta \\ R & \xrightarrow{j} & X \times_S X \end{array}$$

(iii) The stabilizer is *finite* if the structure morphism is a finite morphism.

3.1.3. For a property $\mathcal{P}$ of morphism of schemes, we say that a groupoid has $\mathcal{P}$ if s, or equivalently t, has $\mathcal{P}$.

Theorem 3.1.4. ([11, 5.3]) *Let S be a locally noetherian scheme and let $s, t : R \rightrightarrows X$ be a finite locally free groupoid of algebraic spaces over S with finite stabilizer $j^{-1}(\Delta(X)) \to X$. Assume X is locally of finite type and separated over S. Then there exists a GC quotient $q : X \to X/R$ with the following properties:*

(i) q is integral and surjective,
(ii) X/R is locally of finite type and separated over S, and
(iii) The diagonal $j_{X/R} = (s,t) : R \to X \times_{X/R} X$ is proper and surjective.

Proof. The existence of GC quotient and that q is affine follow from [11, 5.3]. Since q is a GC quotient, q and $j_{X/R}$ are, in particular, universally submersive and so they are surjective. All the other properties follow from [11, 4.7]. □

Corollary 3.1.5. *Under the assumption of 3.1.4, the morphism $q : X \to X/R$ is a qfh covering. If we assume further that the groupoid $s,t : R \rightrightarrows X$ is affine, then $j_{X/R} : R \to X \times_{X/R} X$ is also a qfh covering.*

Proof. Since q is integral and locally of finite type, it is finite and thus is a qfh covering. If s, or equivalently t, is affine, then so is $j_{X/R}$ because q is separated. In this case, $j_{X/R}$ is proper and affine, and thus is finite. In particular, $j_{X/R}$ is a qfh covering. □

3.2. **Quotients as qfh sheaves.** In this subsection we describe the GC quotient of our interest as coequalizer in the category of qfh sheaves.

3.2.1. Recall from [11, 2.16] that for a flat and locally of finitely presented groupoid $s,t : R \rightrightarrows X$ of algebraic spaces over S with $j : R \to X \times_S X$ a monomorphism, there is a universal strongly geometric quotient $q : X \to X/R$ which is also the categorical quotient in the category of algebraic spaces. In fact, the diagram of algebraic spaces

$$R \underset{t}{\overset{s}{\rightrightarrows}} X \xrightarrow{q} X/R$$

is a coequalizer in the category of étale sheaves on S. For example, if a group scheme G/S that is flat and locally of finite presentation over S acts freely on an algebraic space X/S, then the quotient $q : X \to X/G$ is a coequalizer in the category of étale sheaves. However, if the action is not free, we cannot expect the quotient to be a coequalizer. Nonetheless, we show that that this is the case when using the qfh topology:

Theorem 3.2.2. *Under the assumption of 3.1.4, assume further that the groupoid $s,t : R \rightrightarrows X$ is affine. Then the diagram*

$$i^*R \rightrightarrows i^*X \longrightarrow i^*(X/R)$$

is a coequalizer in the category of qfh sheaves on S.

Proof. By the lemma below, it suffices to show that the pull-backs of q and $j_{X/R}$ are epimorphisms in the category of qfh sheaves. Note from 3.1.5 that both q and $j_{X/R}$ are qfh coverings. Hence, it is enough to show that the pull-back of a qfh covering of algebraic spaces is an epimorphism. So let $Y \to Z$ be a qfh cover of algebraic spaces. Choose an étale covering $V \to Y$ (resp. $W \to Z$) with V (resp. W) a scheme. Consider the diagram of étale sheaves:

$$\begin{array}{ccccc} V \times_Z W & \longrightarrow & Y \times_Z W & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ V & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

Since $W \to Z$ is already an epimorphism in the category of étale sheaves, its pull-back is also an epimorphism. So it reduces to show that the composition $V \times_Z W \to Y \times_Z W \to W$ is an epimorphism when pulled-back to $\mathrm{LFQ}(S)^{\sim}$. The composition is a qfh covering of schemes and it follows immediately that it pulls back to an epimorphism of qfh sheaves. □

Lemma 3.2.3. *Let $\mathcal{C}$ be a site with the associated topos T. Let*

$$F_1 \overset{a}{\underset{b}{\rightrightarrows}} F_2 \xrightarrow{c} F_3$$

be a diagram in T with $c \circ a = c \circ b$. Assume that c and the morphism $(a, b) : F_1 \to F_2 \times_{F_3} F_2$ are epimorphisms. Then the diagram is a coequalizer.

Proof. Given a morphism $d : F_2 \to G$ with $d \circ a = d \circ b$, we need to prove that there exists a unique dotted arrow filling in the diagram below:

$$\begin{array}{ccccc} F_1 & \overset{a}{\underset{b}{\rightrightarrows}} & F_2 & \xrightarrow{c} & F_3 \\ & & & \searrow^{d} & \vdots \\ & & & & G \end{array}$$

The uniqueness follows from the assumption that c is an epimorphism. For the existence, let us construct a morphism $f : F_3 \to G$. Consider a section x_3 of $F_3(U)$ for $U \in \mathcal{C}$. After refinement, we can lift it to sections of F_2. Then their images under d glue together to give a section of G because of the assumption that $F_1 \to F_2 \times_{F_3} F_2$ is an epimorphism. The same assumption also shows that the section of G is independent of the choice of lifts of x_3 to the sections of F_2. Hence, given a covering $\{U_i \to U\}$, there is a well-defined section of G, which we defined to be $f(x_3)$. Once more, the assumption on $F_1 \to F_2 \times_{F_3} F_2$ shows that the section $f(x_3)$ does not depend on the choice of coverings of U. Therefore, there is a well-defined morphism $f : F_3 \to G$. This finishes the proof because $f \circ c = d$ by the construction of f. □

4. Symmetric power

In this section we study symmetric powers in various contexts, and then prove the Dold-Thom theorem for qfh topological types. We also analyze the fundamental group of symmetric power.

4.1. **Symmetric power of algebraic spaces.**

4.1.1. Let X be a topological space. There is a canonical action of the symmetric group S_n on the n-fold product space X^n. The nth symmetric power $\mathrm{Sym}^n(X)$ of X is defined as the quotient space X^n/S_n. This quotient space behaves well for CW-complexes: For a map of CW-complexes, the induced map on the symmetric powers preserves homotopy weak equivalence. Since topological types are defined as pro-simplicial sets rather than pro-topological spaces, we restate this property in terms of simplicial sets for convenience.

4.1.2. Let $X_\bullet$ be a simplicial set. As a constant simplicial set, the symmetric group S_n canonically acts on the n-fold product $X_\bullet^n$ of $X_\bullet$. So there is a groupoid of simplicial sets

$$S_n \times X_\bullet^n \rightrightarrows X_\bullet^n \tag{4.1.2.1}$$

where the two maps are the S_n-action and the projection onto $X_\bullet^n$.

Definition 4.1.3. The *nth symmetric power* $\mathrm{Sym}^n(X_\bullet)$ of a simplicial set $X_\bullet$ is the coequalizer of the diagram (4.1.2.1) in the category of simplicial sets.

4.1.4. Concretely, $\mathrm{Sym}^n(X_\bullet)$ can be described as following: For a set X, there is a S_n-action on the n-fold product X^n of X. Then we can form the orbit space X^n/S_n. This construction is functorial in X, and so can be applied to $X_\bullet$ degree-wise. So we obtain a simplicial set whose degree m is the orbit space of X_m by S_n. This is isomorphic to the nth symmetric power $\mathrm{Sym}^n(X_\bullet)$ defined above.

Lemma 4.1.5. *For every $n \geq 0$, the nth symmetric power functor*

$$\mathrm{Sym}^n : \boldsymbol{SSet} \to \boldsymbol{SSet} : X_\bullet \mapsto \mathrm{Sym}^n(X_\bullet)$$

preserves weak equivalences of simplicial sets.

Proof. Let $X_\bullet$ be a simplicial set. Since the geometric realization functor preserves colimits and finite limits, we can identify $|\mathrm{Sym}^n(X_\bullet)|$ with $\mathrm{Sym}^n(|X_\bullet|)$ where the latter is usual symmetric power of the CW-complex. Then the statement follows from the well-known result that the nth symmetric power preserves a homotopy equivalence between CW-complexes. □

4.1.6. Now we discuss the nth symmetric power of algebraic spaces. In this case, a careful approach is necessary. The situation is not as simple as the case of simplicial sets where we take the categorical quotient. Even for schemes, it is not clear whether such a categorical quotient is representable by schemes. The well-known case is when X is a quasi-projective scheme over S. However, we deal with more general case where X is proper. In that case, although the categorical quotient may not be representable by scheme, it could be representable by algebraic spaces. So we begin with the symmetric powers of algebraic spaces.

4.1.7. Let X/S be an algebraic space. The constant group scheme $\underline{S_n}$ associated to the symmetric group S_n canonically acts on the n-fold product $(X/S)^n = \underbrace{X \times_S X \times_S \cdots \times_S X}_{n}$ of X over S. So there is a groupoid of algebraic spaces over S

$$\underline{S_n} \times (X/S)^n \rightrightarrows (X/S)^n \tag{4.1.7.1}$$

where the two arrows are the $\underline{S_n}$-action and the projection onto $(X/S)^n$.

Definition 4.1.8. Let X/S be an algebraic space. Its *nth symmetric power* $\mathrm{Sym}^n(X/S)$ is the GC quotient of the groupoid of algebraic spaces in (4.1.7.1), if exists.

4.1.9. In fact, the nth symmetric power of algebraic spaces exists under mild assumption:

Proposition 4.1.10. ([11, 5.5]) *Let X be a separated algebraic space over S. Then the nth symmetric power* $\mathrm{Sym}^n(X/S)$ *exists.*

Proof. See [11, 5.5]. □

4.1.11. For a separated algebraic space X/S, there is a diagram of algebraic spaces

$$\underline{S_n} \times (X/S)^n \rightrightarrows (X/S)^n \tag{4.1.11.1}$$

The following theorem is the reason why we want to work with the qfh topology instead of the étale topology. Although (4.1.11.1) is not a coequalizer diagram of étale sheaves on S, as indicated in 3.2.2, this is the case for when pulled-back to qfh sheaves provided that X is locally of finite type over S:

Theorem 4.1.12. *Let X be an algebraic space that is locally of finite type and separated over S. Then the pull-back diagram of (4.1.11.1)*

$$\underline{S_n} \times (i^*X)^n \rightrightarrows (i^*X)^n \longrightarrow i^*(\mathrm{Sym}^n(X/S))$$

is a coequalizer in the category $LFQ(S)^\sim$

Proof. Note that $\mathrm{Sym}^n(X/S)$ exists from 4.1.10. That X is locally of finite type over S implies that every condition on 3.1.4 is satisfied. Furthermore, $\underline{S_n} \to S$ is affine and thus the groupoid in (4.1.7.1) is also affine. So the assumption of 3.1.5 is satisfied. Therefore, we can apply 3.2.2 to conclude that the symmetric power is the coequalizer as a qfh sheaf. ☐

4.2. **Symmetric power and weak equivalence.** In this subsection we provide a formal proof of the Dold-Thom theorem for qfh topological types.

4.2.1. Let $F_\bullet$ be a simplicial object in the qfh topos $\mathrm{LFQ}(S)^\sim$. The constant group scheme $\underline{S_n}$ associated to the symmetric group S_n canonically acts on the n-fold product $F_\bullet^n$ of $F_\bullet$. By regarding $\underline{S_n}$ as a constant simplicial sheaf, there is a groupoid of simplicial sheaves on $\mathrm{LFQ}(S)$:

$$\underline{S_n} \times F_\bullet^n \rightrightarrows F_\bullet^n \tag{4.2.1.1}$$

where the two arrows are the $\underline{S_n}$-action and the projection onto $F_\bullet^n$.

Definition 4.2.2. The *nth symmetric power* $\mathrm{Sym}^n(F_\bullet)$ of a simplicial object $F_\bullet$ in $\mathrm{LFQ}(S)^\sim$ is the coequalizer of the diagram (4.2.1.1) in the category $(\mathrm{LFQ}(S)^\sim)^{\Delta^{\mathrm{op}}}$ of simplicial qfh sheaves on S.

4.2.3. This construction is purely categorical and so we can repeat the concrete construction in the case of simplicial sets 4.1.4. That is, $\mathrm{Sym}^n(F_\bullet)$ can be described as following: For a sheaf F, there is a $\underline{S_n}$-action on the n-fold product F^n of F. So there is a groupoid of sheaves on $\mathrm{LFQ}(S)$:

$$\underline{S_n} \times F^n \rightrightarrows F^n$$

where the two arrows are the $\underline{S_n}$-action and the projection onto $F_\bullet^n$. By taking the coequalizer, we get the quotient sheaf $F^n/\underline{S_n}$. This construction is functorial in F, and so can be applied to $F_\bullet$ degree-wise. So we obtain a simplicial sheaf whose degree m is the quotient sheaf of F_m by S_n. This is isomorphic to the nth symmetric power $\mathrm{Sym}^n(F_\bullet)$ defined above.

4.2.4. The symmetric power construction is functorial in $F_\bullet$ and so induces the symmetric power functor

$$\mathrm{Sym}^n : (\mathrm{LFQ}(S)^\sim)^{\Delta^{\mathrm{op}}} \to (\mathrm{LFQ}(S)^\sim)^{\Delta^{\mathrm{op}}} : F_\bullet \mapsto \mathrm{Sym}^n(F_\bullet)$$

4.2.5. For an algebraic space X/S, one can take its symmetric power as a GC quotient and pull it back to a qfh sheaf. Or one can first pull it back to a qfh sheaf and take the symmetric power in the sense of 4.2.2. These two approaches are equivalent:

Proposition 4.2.6. *Let X be a locally of finite type and separated algebraic space over S. Then there is a canonical isomorphism*

$$\mathrm{Sym}^n(i^*X) \xrightarrow{\sim} i^*(\mathrm{Sym}^n(X/S))$$

where i^*X is viewed as a constant simplicial qfh sheaf.

Proof. This is an immediate consequence of 4.1.12. □

4.2.7. Let T be a topos with enough points. Recall that a morphism between simplicial objects in T is a local weak equivalence if and only it induces weak equivalences of simplicial sets at stalks (see [8, p.64] for details). The topos $\mathrm{LFQ}(S)^\sim$ has enough points, and thus we can check the local weak equivalence at stalks:

Lemma 4.2.8. *The topos $LFQ(S)^\sim$ has enough points.*

Proof. The site $\mathrm{LFQ}(S)$ has all finite limits and every covering is a finite covering. Then the statement follows from Deligne [12, Proposition 9.0, Exposé VI]. □

4.2.9. As the symmetric power for qfh sheaves is defined to be the categorical quotient, we expect it to behave like the symmetric power of simplicial sets 4.1.5:

Theorem 4.2.10. *For each $n \geq 0$, the symmetric power functor*

$$\mathrm{Sym}^n : (LFQ(S)^\sim)^{\Delta^{\mathrm{op}}} \to (LFQ(S)^\sim)^{\Delta^{\mathrm{op}}}$$

preserves local weak equivalences.

Proof. Let $F_\bullet \to G_\bullet$ be a local weak equivalence of simplicial qfh sheaves. We check at stalks the local weak equivalence of $\mathrm{Sym}^n(F_\bullet) \to \mathrm{Sym}^n(G_\bullet)$. Consider the diagram of simplicial sets:

$$\begin{array}{ccc} \mathrm{Sym}^n(x^*F_\bullet) & \longrightarrow & \mathrm{Sym}^n(x^*G_\bullet) \\ \downarrow & & \downarrow \\ x^*(\mathrm{Sym}^n(F_\bullet)) & \longrightarrow & x^*(\mathrm{Sym}^n(G_\bullet)) \end{array}$$

For any point $x : \mathbf{Set} \to \mathrm{LFQ}(S)^\sim$, its pull-back preserves coequalizers and thus two vertical maps are isomorphisms. Now since $x^*F_\bullet \to i^*G_\bullet$ is a weak equivalence of simplicial sets, it follows from 4.1.5 that the top map is also a weak equivalence. Therefore, the bottom map is a weak equivalence, which completes the proof. □

Corollary 4.2.11. (The Dold-Thom theorem for qfh topological types) *Let $F_\bullet$ be a simplicial object in $LFQ(S)^\sim$. Then there is a canonical isomorphism*

$$\mathrm{Sym}^n(h_{qfh}(F_\bullet)) \xrightarrow{\sim} h_{qfh}(\mathrm{Sym}^n(F_\bullet))$$

in $\boldsymbol{Ho}(\mathrm{pro}-\boldsymbol{SSet})$. Furthermore, these two pro-simplicial sets are strictly weakly equivalent.

Proof. Recall from 4.2.2 that the symmetric power of simplicial qfh sheaves is defined by coequalizers. Since the connected component functor

$$\Pi_{\mathrm{qfh}} : \mathrm{LFQ}(S)^\sim \to \mathbf{Set}$$

commutes with colimits and non-empty finite products, the following diagram commutes:

$$\begin{array}{ccc} (\mathrm{LFQ}(S)^{\sim})^{\Delta^{\mathrm{op}}} & \xrightarrow{\mathrm{Sym}^n} & (\mathrm{LFQ}(S)^{\sim})^{\Delta^{\mathrm{op}}} \\ \downarrow{\scriptstyle \Pi_{\mathrm{qfh}}} & & \downarrow{\scriptstyle \Pi_{\mathrm{qfh}}} \\ \mathbf{SSet} & \xrightarrow{\mathrm{Sym}^n} & \mathbf{SSet} \end{array}$$

Since the top (resp. bottom) symmetric product functor preserves local weak equivalence (resp. weak equivalences) by 4.2.10 (resp. by 4.1.5), the statement follows by taking the left derived functors for the associated pro-categories. □

4.3. **Fundamental group of symmetric powers.** In this subsection we study the fundamental group of both the qfh and the étale topological types of symmetric powers.

4.3.1. For a topological space X, the map on fundamental groups

$$\pi_1(X) \to \pi_1(\mathrm{Sym}^n(X))$$

factors through the abelianized fundamental group $\pi_1^{\mathrm{ab}}(X)$. Moreover, the Dold-Thom theorem implies that $\pi_1(\mathrm{Sym}^n(X))$ is the first homology group $H_1(X;\mathbb{Z})$ of X with integer coefficient. Since the first homology group is the abelianization of fundamental group, it follows that there is an isomorphism

$$\pi_1^{\mathrm{ab}}(X) \xrightarrow{\sim} \pi_1(\mathrm{Sym}^n(X)) \tag{4.3.1.1}$$

4.3.2. As the symmetric power for qfh sheaves is defined to be categorical quotients, we expect a similar result for qfh topological types:

Corollary 4.3.3. *Let $F_\bullet$ be a simplicial object in LFQ$(S)^{\sim}$, the canonical map*

$$\pi_1(h_{qfh}(F_\bullet)) \to \pi_1(h_{qfh}(\mathrm{Sym}^n(F_\bullet))$$

factors through the abelianization $\pi_1^{\mathrm{ab}}(h(F_\bullet))$. Furthermore, there is a canonical isomorphism

$$\pi_1^{\mathrm{ab}}(h_{qfh}(F_\bullet)) \xrightarrow{\sim} \pi_1(h_{qfh}(\mathrm{Sym}^n(F_\bullet))$$

Proof. This is an immediate consequence of 4.2.11. Indeed, as $\mathrm{Sym}^n(h_{\mathrm{qfh}}(F_\bullet))$ is isomorphic to $h_{\mathrm{qfh}}(\mathrm{Sym}^n(F_\bullet))$, one can reduce to the case of simplicial sets where we already know the result by the Dold-Thom theorem. □

4.3.4. In fact, this result is still true for algebraic spaces by the work of Biswas-Hogadi [3, 1.2]: For an integral proper algebraic space over an algebraically closed field k, the canonical map (4.3.1.1) for étale topological types is an isomorphism. In 5.0.2 we use this result in the following form:

Theorem 4.3.5. ([3, 1.2]) *Let X be a geometrically normal and proper algebraic space over a separably closed field k. Then there is an canonical isomorphism*

$$\pi_1^{\mathrm{ab}}(h_{\acute{e}t}(X)) \xrightarrow{\sim} \pi_1(h_{\acute{e}t}(\mathrm{Sym}^n(X)))$$

Proof. Fix an algebraically closure $\overline{k}$ of k. Recall from [4, 4.1.16] that there is a strict weak equivalence

$$h_{\acute{\mathrm{e}}\mathrm{t}}(\overline{X}) \to h_{\acute{\mathrm{e}}\mathrm{t}}(X)$$

Also, recall from [11, 2.10] that the strongly geometric quotient is stable under flat base change. That is, the canonical map

$$\mathrm{Sym}^n(\overline{X}) \to \mathrm{Sym}^n(X) \otimes_k \overline{k}$$

is an isomorphism of algebraic spaces over $\overline{k}$. Again by [4, 4.1.16], there is a strict weak equivalence

$$h_{\text{ét}}(\mathrm{Sym}^n(\overline{X})) \to h_{\text{ét}}(\mathrm{Sym}^n(X))$$

Therefore, we may assume that k is algebraically closed. The n-fold product $(X/S)^n = X \times_S X \times_S \cdots \times_S X$ is normal ([10, Tag 06DG]) and thus so is $\mathrm{Sym}^n(X)$. We know that $\mathrm{Sym}^n(X)$ is locally of finite type and separated by 3.1.4. That X is quasi-compact implies $\mathrm{Sym}^n(X)$ is also quasi-compact. So $h_{\text{ét}}(\mathrm{Sym}^n(X))$ is profinite by [4, 5.3.4]. In particular, $\pi_1(h_{\text{ét}}(\mathrm{Sym}^n(X)))$ is profinite. This implies that the fundamental group of $h_{\text{ét}}(\mathrm{Sym}^n(X))$ is isomorphic to the fundamental group of $\mathrm{Sym}^n(X)$ in the sense of Noohi by [4, 3.5.2]. Similarly, $\pi_1(h_{\text{ét}}(X))$ is isomorphic to the fundamental group of X in the sense of Noohi. Then our statement follows from [3, 1.2]. □

5. The proof of the Dold-Thom theorem for topological types

In this section we compare the topological types of the symmetric power for the qfh and étale topologies, and prove the the Dold-Thom theorem for étale topological types.

5.0.1. In general, it may not be true that for an algebraic space its étale topological type is weakly equivalent to its qfh topological type. Nonetheless, this is the case when it comes to the symmetric powers:

Proposition 5.0.2. *Let X be a geometrically normal and proper algebraic space over a separably closed field k. Then the canonical map of topological types*

$$h_{qfh}(\mathrm{Sym}^n(X)) \to h_{\acute{e}t}(\mathrm{Sym}^n(X))$$

is a strict weak equivalence of pro-simplicial sets.

Proof. It follows from 2.2.13 that the map induces isomorphisms on cohomology groups for every local coefficient system of abelian groups. So it suffices to show that the map on fundamental groups is an isomorphism. Consider a commutative diagram of pro-groups:

$$\begin{array}{ccc} \pi_1^{\mathrm{ab}}(h_{\mathrm{qfh}}(X)) & \longrightarrow & \pi_1^{\mathrm{ab}}(h_{\text{ét}}(X)) \\ \downarrow & & \downarrow \\ \pi_1(h_{\mathrm{qfh}}(\mathrm{Sym}^n(X))) & \longrightarrow & \pi_1(h_{\text{ét}}(\mathrm{Sym}^n(X))) \end{array}$$

The left vertical map is an isomorphism by 4.3.3 and the top horizontal map is an isomorphism by 2.2.13. Also, it follows from 4.3.5 that the right vertical map is an isomorphism. Therefore the bottom map is an isomorphism as desired. □

Lemma 5.0.3. *Let $X \to Y$ be a morphism of pro-simplicial sets. Assume that*

$$H^n(Y; M) \to H^n(X; M)$$

is an isomorphism for all $n \geq 0$ and all abelian groups M. Then the induced map on symmetric powers

$$\mathrm{Sym}^n(X) \to \mathrm{Sym}^n(Y)$$

is a weak equivalence of pro-simplicial sets.

Proof. We prove that the map on symmetric powers induces isomorphisms on all homotopy groups. Recall from [7, 5.5] that the isomorphisms on cohomology groups with abelian coefficients imply that the map on homology pro-groups

$$H_n(X, \mathbb{Z}) \to H_n(Y, \mathbb{Z})$$

is an isomorphism for all $n \geq 0$. Then the statement follows from the classical Dold-Thom theorem. □

Theorem 5.0.4. *Let X be a geometrically normal and proper algebraic space over a separably closed field k. Then there is a canonical isomorphism*

$$\mathrm{Sym}^n(h_{ét}(X)) \xrightarrow{\sim} h_{ét}(\mathrm{Sym}^n(X))$$

in $\boldsymbol{Ho}(\mathrm{pro} - \boldsymbol{SSet})$.

Proof. Consider the commutative diagram of pro-simplicial sets in $\mathbf{Ho}(\mathrm{pro} - \mathbf{SSet})$:

$$\begin{array}{ccc} \mathrm{Sym}^n(h_{\mathrm{qfh}}(X)) & \longrightarrow & h_{\mathrm{qfh}}(\mathrm{Sym}^n(X)) \\ \downarrow & & \downarrow \\ \mathrm{Sym}^n(h_{\mathrm{\acute{e}t}}(X)) & \longrightarrow & h_{\mathrm{\acute{e}t}}(\mathrm{Sym}^n(X)) \end{array}$$

The top arrow is an isomorphism in the homotopy category of pro-simplicial sets by 4.2.11. Also, the right vertical map is a strict weak equivalence of pro-simplicial sets by 5.0.2. On the other hand, the canonical map of topological types

$$h_{\mathrm{qfh}}(X) \to h_{\mathrm{\acute{e}t}}(X)$$

satisfies the assumption of 5.0.3 due to 2.2.13. Hence the left vertical map is a weak equivalence of pro-simplicial sets. Therefore, the bottom map is an isomorphism in the homotopy category. □

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