Electromagnetic Field Theory

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ELECTROMAGNETIC FIELD THEORY

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Preface

The current book is an outgrowth of the lecture notes that I prepared for the four-credit course Electrodynamics that was introduced in the Uppsala University curriculum in 1992, to become the five-credit course Classical Electrodynamics in 1997. To some extent, parts of these notes were based on lecture notes prepared, in Swedish, by BENGT LUNDBORG who created, developed and taught the earlier, two-credit course Electromagnetic Radiation at our faculty.

Intended primarily as a textbook for physics students at the advanced undergraduate or beginning graduate level, it is hoped that the present book may be useful for research workers too. It provides a thorough treatment of the theory of electrodynamics, mainly from a classical field theoretical point of view, and includes such things as formal electrostatics and magnetostatics and their unification into electrodynamics, the electromagnetic potentials, gauge transformations, covariant formulation of classical electrodynamics, force, momentum and energy of the electromagnetic field, radiation and scattering phenomena, electromagnetic waves and their propagation in vacuum and in media, and covariant Lagrangian/Hamiltonian field theoretical methods for electromagnetic fields, particles and interactions. The aim has been to write a book that can serve both as an advanced text in Classical Electrodynamics and as a preparation for studies in Quantum Electrodynamics and related subjects.

In an attempt to encourage participation by other scientists and students in the authoring of this book, and to ensure its quality and scope to make it useful in higher university education anywhere in the world, it was produced within a World-Wide Web (WWW) project. This turned out to be a rather successful move. By making an electronic version of the book freely down-loadable on the net, comments have been only received from fellow Internet physicists around the world and from WWW 'hit' statistics it seems that the book serves as a frequently used Internet resource. This way it is hoped that it will be particularly useful for students and researchers working under financial or other circumstances that make it difficult to procure a printed copy of the book.

Thanks are due not only to Bengt Lundborg for providing the inspiration to write this book, but also to professor CHRISTER WAHLBERG and professor GÖRAN FÄLDT, Uppsala University, and professor YAKOV ISTOMIN, Lebedev Institute, Moscow, for interesting discussions on electrodynamics and relativity in general and on this book in particular. Comments from former graduate students MATTIAS WALDENVIK, TOBIA CAROZZI and ROGER KARLSSON as well as ANDERS ERIKSSON, all at the Swedish Institute of Space Physics in Uppsala and who all have participated in the teaching,

on the material covered in the course and in this book are gratefully acknowledged. Thanks are also due to my long-term space physics colleague HELMUT KOPKA of the Max-Planck-Institut für Aeronomie, Lindau, Germany, who not only taught me about the practical aspects of the of high-power radio wave transmitters and transmission lines, but also about the more delicate aspects of typesetting a book in TEX and LATEX. I am particularly indebted to Academician professor VITALIY LAZAREVICH GINZBURG, 2003 Nobel Laureate in Physics, for his many fascinating and very elucidating lectures, comments and historical footnotes on electromagnetic radiation while cruising on the Volga river at our joint Russian-Swedish summer schools during the 1990s and for numerous private discussions.

Finally, I would like to thank all students and Internet users who have downloaded and commented on the book during its life on the World-Wide Web.

Uppsala, Sweden BO THIDÉ
January, 2004

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To the memory of professor
LEV MIKHAILOVICH ERUKHIMOV (1936–1997)
dear friend, great physicist, poet
and a truly remarkable man.

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CHAPTER 1

Classical Electrodynamics

Classical electrodynamics deals with electric and magnetic fields and interactions caused by *macroscopic* distributions of electric charges and currents. This means that the concepts of localised electric charges and currents assume the validity of certain mathematical limiting processes in which it is considered possible for the charge and current distributions to be localised in infinitesimally small volumes of space. Clearly, this is in contradiction to electromagnetism on a truly *microscopic* scale, where charges and currents have to be treated as spatially extended objects and quantum corrections must be included. However, the limiting processes used will yield results which are correct on small as well as large *macroscopic* scales.

It took the genius of JAMES CLERK MAXWELL to unify electricity and magnetism into a super-theory, *electromagnetism* or *classical electrodynamics* (CED), and to realise that optics is a subfield of this super-theory. Early in the 20th century, Nobel laureate HENDRIK ANTOON LORENTZ took the electrodynamics theory further to the microscopic scale and also laid the foundation for the special theory of relativity, formulated by Nobel laureate ALBERT EINSTEIN in 1905. In the 1930s PAUL A. M. DIRAC expanded electrodynamics to a more symmetric form, including magnetic as well as electric charges. With his relativistic quantum mechanics, he also paved the way for the development of *quantum electrodynamics* (QED) for which RICHARD P. FEYNMAN, JULIAN SCHWINGER, and SIN-ITIRO TOMONAGA in 1965 received their Nobel prizes. Around the same time, physicists such as Nobel laureates SHELDON GLASHOW, ABDUS SALAM, and STEVEN WEINBERG managed to unify electrodynamics with the weak interaction theory to yet another super-theory, *electroweak theory*. The modern theory of strong interactions, *quantum chromodynamics* (QCD), is influenced by QED.

In this chapter we start with the force interactions in classical electrostatics and classical magnetostatics and introduce the static electric and magnetic fields and find two uncoupled systems of equations for them. Then we see how the conservation of electric charge and its relation to electric current leads to the dynamic connection between electricity and magnetism and how the two can be unified into one 'super-theory', classical electrodynamics, described by one system of coupled dynamic field equations—the Maxwell equations.

At the end of the chapter we study Dirac's symmetrised form of Maxwell's equations by introducing (hypothetical) magnetic charges and magnetic currents into the theory. While not identified unambiguously in experiments yet, magnetic charges and currents make the theory much more appealing for instance by allowing for duality transformations in a most natural way.

1.1 Electrostatics

The theory which describes physical phenomena related to the interaction between stationary electric charges or charge distributions in space with stationary boundaries is called *electrostatics*. For a long time electrostatics, under the name *electricity*, was considered an independent physical theory of its own, alongside other physical theories such as magnetism, mechanics, optics and thermodynamics.[1]

1.1.1 Coulomb's law

It has been found experimentally that in classical electrostatics the interaction between stationary, electrically charged bodies can be described in terms of a mechanical force. Let us consider the simple case described by Figure 1.1 on page 3. Let $\mathbf{F}$ denote the force acting on a electrically charged particle with charge q located at $\mathbf{x}$, due to the presence of a charge q' located at $\mathbf{x}'$. According to *Coulomb's law* this force is, in vacuum, given by the expression

$$\mathbf{F}(\mathbf{x}) = \frac{qq'}{4\pi\varepsilon_0}\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = -\frac{qq'}{4\pi\varepsilon_0}\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = \frac{qq'}{4\pi\varepsilon_0}\boldsymbol{\nabla}'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \tag{1.1}$$

where in the last step Formula (F.71) on page 161 was used. In SI units, which we shall use throughout, the force $\mathbf{F}$ is measured in Newton (N), the electric charges q and q' in Coulomb (C) [= Ampère-seconds (As)], and the length $|\mathbf{x}-\mathbf{x}'|$ in metres (m). The constant $\varepsilon_0 = 10^7/(4\pi c^2) \approx 8.8542 \times 10^{-12}$ Farad per metre (F/m) is the

[1] The physicist and philosopher Pierre Duhem (1861–1916) once wrote:

> 'The whole theory of electrostatics constitutes a group of abstract ideas and general propositions, formulated in the clear and concise language of geometry and algebra, and connected with one another by the rules of strict logic. This whole fully satisfies the reason of a French physicist and his taste for clarity, simplicity and order. . . .'

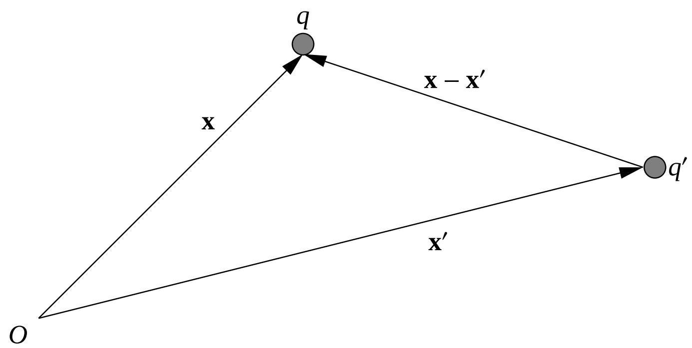

FIGURE 1.1: Coulomb's law describes how a static electric charge q, located at a point $\mathbf{x}$ relative to the origin O, experiences an electrostatic force from a static electric charge q' located at $\mathbf{x}'$.

vacuum permittivity and $c \approx 2.9979 \times 10^8$ m/s is the speed of light in vacuum. In CGS units $\varepsilon_0 = 1/(4\pi)$ and the force is measured in dyne, electric charge in statcoulomb, and length in centimetres (cm).

1.1.2 The electrostatic field

Instead of describing the electrostatic interaction in terms of a 'force action at a distance', it turns out that it is for most purposes more useful to introduce the concept of a field and to describe the electrostatic interaction in terms of a static vectorial *electric field* $\mathbf{E}^{\text{stat}}$ defined by the limiting process

$$\mathbf{E}^{\text{stat}} \stackrel{\text{def}}{\equiv} \lim_{q \to 0} \frac{\mathbf{F}}{q} \tag{1.2}$$

where $\mathbf{F}$ is the electrostatic force, as defined in Equation (1.1) on the preceding page, from a net electric charge q' on the test particle with a small electric net electric charge q. Since the purpose of the limiting process is to assure that the test charge q does not distort the field set up by q', the expression for $\mathbf{E}^{\text{stat}}$ does not depend explicitly on q but only on the charge q' and the relative radius vector $\mathbf{x} - \mathbf{x}'$. This means that we can say that any net electric charge produces an electric field in the space that surrounds it, regardless of the existence of a second charge anywhere in this space.[2]

[2] In the preface to the first edition of the first volume of his book *A Treatise on Electricity and Magnetism*, first published in 1873, James Clerk Maxwell describes this in the following, almost poetic, manner [9]:

> 'For instance, Faraday, in his mind's eye, saw lines of force traversing all space where the mathematicians saw centres of force attracting at a distance: Faraday saw a medium where they saw nothing but distance: Faraday sought the seat of the phenomena in real actions

Using (1.1) and Equation (1.2) on the preceding page, and Formula (F.70) on page 160, we find that the electrostatic field $\mathbf{E}^{\text{stat}}$ at the *field point* $\mathbf{x}$ (also known as the *observation point*), due to a field-producing electric charge q' at the *source point* $\mathbf{x}'$, is given by

$$\mathbf{E}^{\text{stat}}(\mathbf{x}) = \frac{q'}{4\pi\varepsilon_0}\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = -\frac{q'}{4\pi\varepsilon_0}\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = \frac{q'}{4\pi\varepsilon_0}\boldsymbol{\nabla}'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \tag{1.3}$$

In the presence of several field producing discrete electric charges q'_i, located at the points $\mathbf{x}'_i$, $i = 1, 2, 3, \ldots$, respectively, in an otherwise empty space, the assumption of linearity of vacuum[3] allows us to superimpose their individual electrostatic fields into a total electrostatic field

$$\mathbf{E}^{\text{stat}}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\sum_i q'_i \frac{\mathbf{x}-\mathbf{x}'_i}{\left|\mathbf{x}-\mathbf{x}'_i\right|^3} \tag{1.4}$$

If the discrete electric charges are small and numerous enough, we introduce the *electric charge density* ρ, measured in C/m^3 in SI units, located at $\mathbf{x}'$ within a volume V' of limited extent and replace summation with integration over this volume. This allows us to describe the total field as

$$\begin{aligned}\mathbf{E}^{\text{stat}}(\mathbf{x}) &= \frac{1}{4\pi\varepsilon_0}\int_{V'} \mathrm{d}^3x'\,\rho(\mathbf{x}')\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = -\frac{1}{4\pi\varepsilon_0}\int_{V'} \mathrm{d}^3x'\,\rho(\mathbf{x}')\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \\ &= -\frac{1}{4\pi\varepsilon_0}\boldsymbol{\nabla}\int_{V'} \mathrm{d}^3x'\,\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\end{aligned} \tag{1.5}$$

where we used Formula (F.70) on page 160 and the fact that $\rho(\mathbf{x}')$ does not depend on the unprimed (field point) coordinates on which $\boldsymbol{\nabla}$ operates.

We emphasise that under the assumption of linear superposition, Equation (1.5) above is valid for an arbitrary distribution of electric charges, including discrete charges, in which case ρ is expressed in terms of Dirac delta distributions:

$$\rho(\mathbf{x}') = \sum_i q'_i\,\delta(\mathbf{x}'-\mathbf{x}'_i) \tag{1.6}$$

as illustrated in Figure 1.2 on the facing page. Inserting this expression into expression (1.5) above we recover expression (1.4).

Taking the divergence of the general $\mathbf{E}^{\text{stat}}$ expression for an arbitrary electric charge distribution, Equation (1.5) above, and using the representation of the Dirac

going on in the medium, they were satisfied that they had found it in a power of action at a distance impressed on the electric fluids.'

[3] In fact, vacuum exhibits a *quantum mechanical nonlinearity* due to *vacuum polarisation effects* manifesting themselves in the momentary creation and annihilation of electron-positron pairs, but classically this nonlinearity is negligible.

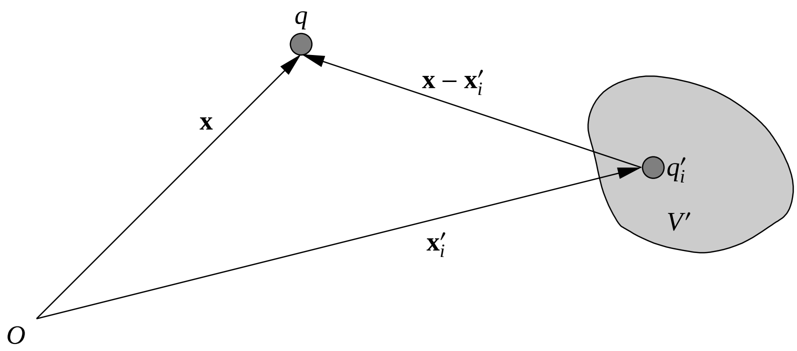

FIGURE 1.2: Coulomb's law for a distribution of individual charges $\mathbf{x}'_i$ localised within a volume V' of limited extent.

delta distribution, Formula (F.73) on page 161, we find that

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{E}^{\text{stat}}(\mathbf{x}) &= \boldsymbol{\nabla}\cdot\frac{1}{4\pi\varepsilon_0}\int_{V'}\mathrm{d}^3x'\,\rho(\mathbf{x}')\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\\ &= -\frac{1}{4\pi\varepsilon_0}\int_{V'}\mathrm{d}^3x'\,\rho(\mathbf{x}')\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\\ &= -\frac{1}{4\pi\varepsilon_0}\int_{V'}\mathrm{d}^3x'\,\rho(\mathbf{x}')\,\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\\ &= \frac{1}{\varepsilon_0}\int_{V'}\mathrm{d}^3x'\,\rho(\mathbf{x}')\,\delta(\mathbf{x}-\mathbf{x}') = \frac{\rho(\mathbf{x})}{\varepsilon_0}\end{aligned} \tag{1.7}$$

which is the differential form of *Gauss's law of electrostatics*.

Since, according to Formula (F.62) on page 160, $\boldsymbol{\nabla}\times[\boldsymbol{\nabla}\alpha(\mathbf{x})]\equiv\mathbf{0}$ for any 3D $\mathbb{R}^3$ scalar field $\alpha(\mathbf{x})$, we immediately find that in electrostatics

$$\boldsymbol{\nabla}\times\mathbf{E}^{\text{stat}}(\mathbf{x}) = -\frac{1}{4\pi\varepsilon_0}\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\int_{V'}\mathrm{d}^3x'\,\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right) = \mathbf{0} \tag{1.8}$$

i.e., that $\mathbf{E}^{\text{stat}}$ is an *irrotational* field.

To summarise, electrostatics can be described in terms of two vector partial differential equations

$$\boldsymbol{\nabla}\cdot\mathbf{E}^{\text{stat}}(\mathbf{x}) = \frac{\rho(\mathbf{x})}{\varepsilon_0} \tag{1.9a}$$

$$\boldsymbol{\nabla}\times\mathbf{E}^{\text{stat}}(\mathbf{x}) = \mathbf{0} \tag{1.9b}$$

representing four scalar partial differential equations.

1.2 Magnetostatics

While electrostatics deals with static electric charges, *magnetostatics* deals with stationary electric currents, *i.e.*, electric charges moving with constant speeds, and the interaction between these currents. Here we shall discuss this theory in some detail.

1.2.1 Ampère's law

Experiments on the interaction between two small loops of electric current have shown that they interact via a mechanical force, much the same way that electric charges interact. In Figure 1.3 on the facing page, let $\mathbf{F}$ denote such a force acting on a small loop C, with tangential line element $\mathrm{d}\mathbf{l}$, located at $\mathbf{x}$ and carrying a current I in the direction of $\mathrm{d}\mathbf{l}$, due to the presence of a small loop C', with tangential line element $\mathrm{d}\mathbf{l}'$, located at $\mathbf{x}'$ and carrying a current I' in the direction of $\mathrm{d}\mathbf{l}'$. According to *Ampère's law* this force is, in vacuum, given by the expression

$$\begin{aligned}\mathbf{F}(\mathbf{x}) &= \frac{\mu_0 II'}{4\pi}\oint_C \mathrm{d}\mathbf{l}\times\oint_{C'}\mathrm{d}\mathbf{l}'\times\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\\ &= -\frac{\mu_0 II'}{4\pi}\oint_C \mathrm{d}\mathbf{l}\times\oint_{C'}\mathrm{d}\mathbf{l}'\times\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\end{aligned} \tag{1.10}$$

In SI units, $\mu_0 = 4\pi\times10^{-7} \approx 1.2566\times10^{-6}$ H/m is the *vacuum permeability*. From the definition of ε_0 and μ_0 (in SI units) we observe that

$$\varepsilon_0\mu_0 = \frac{10^7}{4\pi c^2}\ (\mathrm{F/m})\ \times 4\pi\times10^{-7}\ (\mathrm{H/m})\ = \frac{1}{c^2}\ (\mathrm{s^2/m^2}) \tag{1.11}$$

which is a most useful relation.

At first glance, Equation (1.10) above may appear unsymmetric in terms of the loops and therefore to be a force law which is in contradiction with Newton's third law. However, by applying the vector triple product 'bac-cab' Formula (F.51) on page 160, we can rewrite (1.10) as

$$\begin{aligned}\mathbf{F}(\mathbf{x}) = &-\frac{\mu_0 II'}{4\pi}\oint_{C'}\mathrm{d}\mathbf{l}'\oint_C \mathrm{d}\mathbf{l}\cdot\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\\ &-\frac{\mu_0 II'}{4\pi}\oint_C\oint_{C'}\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\,\mathrm{d}\mathbf{l}\cdot\mathrm{d}\mathbf{l}'\end{aligned} \tag{1.12}$$

Since the integrand in the first integral is an exact differential, this integral vanishes and we can rewrite the force expression, Equation (1.10) above, in the following symmetric way

$$\mathbf{F}(\mathbf{x}) = -\frac{\mu_0 II'}{4\pi}\oint_C\oint_{C'}\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\,\mathrm{d}\mathbf{l}\cdot\mathrm{d}\mathbf{l}' \tag{1.13}$$

which clearly exhibits the expected symmetry in terms of loops C and C'.

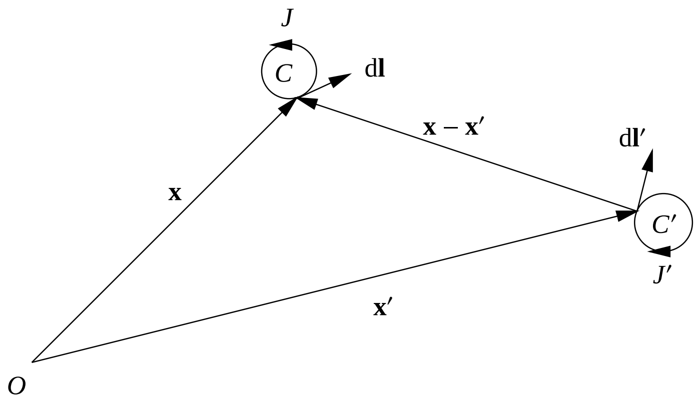

FIGURE 1.3: Ampère's law describes how a small loop C, carrying a static electric current I through its tangential line element $\mathrm{d}\mathbf{l}$ located at $\mathbf{x}$, experiences a magnetostatic force from a small loop C', carrying a static electric current I' through the tangential line element $\mathrm{d}\mathbf{l}'$ located at $\mathbf{x}'$. The loops can have arbitrary shapes as long as they are simple and closed.

1.2.2 The magnetostatic field

In analogy with the electrostatic case, we may attribute the magnetostatic interaction to a static vectorial *magnetic field* $\mathbf{B}^{\text{stat}}$. It turns out that the elemental $\mathbf{B}^{\text{stat}}$ can be defined as

$$\mathrm{d}\mathbf{B}^{\text{stat}}(\mathbf{x}) \stackrel{\text{def}}{\equiv} \frac{\mu_0 I'}{4\pi}\mathrm{d}\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \tag{1.14}$$

which expresses the small element $\mathrm{d}\mathbf{B}^{\text{stat}}(\mathbf{x})$ of the static magnetic field set up at the field point $\mathbf{x}$ by a small line element $\mathrm{d}\mathbf{l}'$ of stationary current I' at the source point $\mathbf{x}'$. The SI unit for the magnetic field, sometimes called the *magnetic flux density* or *magnetic induction*, is Tesla (T).

If we generalise expression (1.14) to an integrated steady state *electric current density* $\mathbf{j}(\mathbf{x})$, measured in A/m^2 in SI units, we obtain *Biot-Savart's law*:

$$\begin{aligned}\mathbf{B}^{\text{stat}}(\mathbf{x}) &= \frac{\mu_0}{4\pi}\int_{V'} \mathrm{d}^3x'\, \mathbf{j}(\mathbf{x}') \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = -\frac{\mu_0}{4\pi}\int_{V'} \mathrm{d}^3x'\, \mathbf{j}(\mathbf{x}') \times \boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \\ &= \frac{\mu_0}{4\pi}\boldsymbol{\nabla} \times \int_{V'} \mathrm{d}^3x'\, \frac{\mathbf{j}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\end{aligned} \tag{1.15}$$

where we used Formula (F.70) on page 160, Formula (F.57) on page 160, and the fact that $\mathbf{j}(\mathbf{x}')$ does not depend on the unprimed coordinates on which $\boldsymbol{\nabla}$ operates. Comparing Equation (1.5) on page 4 with Equation (1.15), we see that there exists a close analogy between the expressions for $\mathbf{E}^{\text{stat}}$ and $\mathbf{B}^{\text{stat}}$ but that they differ in their vectorial characteristics. With this definition of $\mathbf{B}^{\text{stat}}$, Equation (1.10) on page 6 may we written

$$\mathbf{F}(\mathbf{x}) = I \oint_C \mathrm{d}\mathbf{l} \times \mathbf{B}^{\text{stat}}(\mathbf{x}) \tag{1.16}$$

In order to assess the properties of $\mathbf{B}^{\text{stat}}$, we determine its divergence and curl. Taking the divergence of both sides of Equation (1.15) on the preceding page and utilising Formula (F.63) on page 160, we obtain

$$\boldsymbol{\nabla} \cdot \mathbf{B}^{\text{stat}}(\mathbf{x}) = \frac{\mu_0}{4\pi} \boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \times \int_{V'} \mathrm{d}^3x' \, \frac{\mathbf{j}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \right) = 0 \tag{1.17}$$

since, according to Formula (F.63) on page 160, $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{a})$ vanishes for any vector field $\mathbf{a}(\mathbf{x})$.

Applying the operator 'bac-cab' rule, Formula (F.64) on page 160, the curl of Equation (1.15) on the preceding page can be written

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{B}^{\text{stat}}(\mathbf{x}) &= \frac{\mu_0}{4\pi} \boldsymbol{\nabla} \times \left(\boldsymbol{\nabla} \times \int_{V'} \mathrm{d}^3x' \, \frac{\mathbf{j}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \right) = \\ &= -\frac{\mu_0}{4\pi} \int_{V'} \mathrm{d}^3x' \, \mathbf{j}(\mathbf{x}') \, \nabla^2 \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) + \frac{\mu_0}{4\pi} \int_{V'} \mathrm{d}^3x' \, [\mathbf{j}(\mathbf{x}') \cdot \boldsymbol{\nabla}'] \, \boldsymbol{\nabla}' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \end{aligned} \tag{1.18}$$

In the first of the two integrals on the right hand side, we use the representation of the Dirac delta function given in Formula (F.73) on page 161, and integrate the second one by parts, by utilising Formula (F.56) on page 160 as follows:

$$\begin{aligned} &\int_{V'} \mathrm{d}^3x' \, [\mathbf{j}(\mathbf{x}') \cdot \boldsymbol{\nabla}'] \boldsymbol{\nabla}' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \\ &= \hat{\mathbf{x}}_k \int_{V'} \mathrm{d}^3x' \, \boldsymbol{\nabla}' \cdot \left\{ \mathbf{j}(\mathbf{x}') \left[\frac{\partial}{\partial x'_k} \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \right] \right\} - \int_{V'} \mathrm{d}^3x' \, \left[\boldsymbol{\nabla}' \cdot \mathbf{j}(\mathbf{x}') \right] \boldsymbol{\nabla}' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \\ &= \hat{\mathbf{x}}_k \int_S \mathrm{d}\mathbf{S} \cdot \mathbf{j}(\mathbf{x}') \frac{\partial}{\partial x'_k} \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) - \int_{V'} \mathrm{d}^3x' \, \left[\boldsymbol{\nabla}' \cdot \mathbf{j}(\mathbf{x}') \right] \boldsymbol{\nabla}' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \end{aligned} \tag{1.19}$$

Then we note that the first integral in the result, obtained by applying Gauss's theorem, vanishes when integrated over a large sphere far away from the localised source $\mathbf{j}(\mathbf{x}')$, and that the second integral vanishes because $\boldsymbol{\nabla} \cdot \mathbf{j} = 0$ for stationary currents (no charge accumulation in space). The net result is simply

$$\boldsymbol{\nabla} \times \mathbf{B}^{\text{stat}}(\mathbf{x}) = \mu_0 \int_{V'} \mathrm{d}^3x' \, \mathbf{j}(\mathbf{x}') \delta(\mathbf{x} - \mathbf{x}') = \mu_0 \mathbf{j}(\mathbf{x}) \tag{1.20}$$

1.3 Electrodynamics

As we saw in the previous sections, the laws of electrostatics and magnetostatics can be summarised in two pairs of time-independent, uncoupled vector partial differential equations, namely the *equations of classical electrostatics*

$$\nabla \cdot \mathbf{E}^{\text{stat}}(\mathbf{x}) = \frac{\rho(\mathbf{x})}{\varepsilon_0} \tag{1.21a}$$

$$\nabla \times \mathbf{E}^{\text{stat}}(\mathbf{x}) = \mathbf{0} \tag{1.21b}$$

and the *equations of classical magnetostatics*

$$\nabla \cdot \mathbf{B}^{\text{stat}}(\mathbf{x}) = 0 \tag{1.22a}$$

$$\nabla \times \mathbf{B}^{\text{stat}}(\mathbf{x}) = \mu_0 \mathbf{j}(\mathbf{x}) \tag{1.22b}$$

Since there is nothing *a priori* which connects $\mathbf{E}^{\text{stat}}$ directly with $\mathbf{B}^{\text{stat}}$, we must consider classical electrostatics and classical magnetostatics as two independent theories.

However, when we include time-dependence, these theories are unified into one theory, *classical electrodynamics*. This unification of the theories of electricity and magnetism is motivated by two empirically established facts:

1. Electric charge is a conserved quantity and electric current is a transport of electric charge. This fact manifests itself in the equation of continuity and, as a consequence, in Maxwell's displacement current.

2. A change in the magnetic flux through a loop will induce an EMF electric field in the loop. This is the celebrated Faraday's law of induction.

1.3.1 Equation of continuity for electric charge

Let $\mathbf{j}(t, \mathbf{x})$ denote the time-dependent electric current density. In the simplest case it can be defined as $\mathbf{j} = \mathbf{v}\rho$ where $\mathbf{v}$ is the velocity of the electric charge density ρ. In general, $\mathbf{j}$ has to be defined in statistical mechanical terms as $\mathbf{j}(t, \mathbf{x}) = \sum_\alpha q_\alpha \int \mathrm{d}^3v\, \mathbf{v} f_\alpha(t, \mathbf{x}, \mathbf{v})$ where $f_\alpha(t, \mathbf{x}, \mathbf{v})$ is the (normalised) distribution function for particle species α with electric charge q_α.

The *electric charge conservation law* can be formulated in the *equation of continuity*

$$\frac{\partial \rho(t, \mathbf{x})}{\partial t} + \nabla \cdot \mathbf{j}(t, \mathbf{x}) = 0 \tag{1.23}$$

which states that the time rate of change of electric charge $\rho(t, \mathbf{x})$ is balanced by a divergence in the electric current density $\mathbf{j}(t, \mathbf{x})$.

1.3.2 Maxwell's displacement current

We recall from the derivation of Equation (1.20) on page 8 that there we used the fact that in magnetostatics $\nabla \cdot \mathbf{j}(\mathbf{x}) = 0$. In the case of non-stationary sources and fields, we must, in accordance with the continuity Equation (1.23) on the preceding page, set $\nabla \cdot \mathbf{j}(t, \mathbf{x}) = -\partial\rho(t, \mathbf{x})/\partial t$. Doing so, and formally repeating the steps in the derivation of Equation (1.20) on page 8, we would obtain the formal result

$$\begin{aligned}\nabla \times \mathbf{B}(t, \mathbf{x}) &= \mu_0 \int_{V'} \mathrm{d}^3x'\ \mathbf{j}(t, \mathbf{x}')\delta(\mathbf{x} - \mathbf{x}') + \frac{\mu_0}{4\pi}\frac{\partial}{\partial t}\int_{V'} \mathrm{d}^3x'\ \rho(t, \mathbf{x}')\nabla'\left(\frac{1}{|\mathbf{x} - \mathbf{x}'|}\right) \\ &= \mu_0 \mathbf{j}(t, \mathbf{x}) + \mu_0 \frac{\partial}{\partial t}\varepsilon_0 \mathbf{E}(t, \mathbf{x})\end{aligned} \tag{1.24}$$

where, in the last step, we have assumed that a generalisation of Equation (1.5) on page 4 to time-varying fields allows us to make the identification[4]

$$\begin{aligned}\frac{1}{4\pi\varepsilon_0}\frac{\partial}{\partial t}\int_{V'} \mathrm{d}^3x'\ \rho(t, \mathbf{x}')\nabla'\left(\frac{1}{|\mathbf{x} - \mathbf{x}'|}\right) &= \frac{\partial}{\partial t}\left[-\frac{1}{4\pi\varepsilon_0}\int_{V'} \mathrm{d}^3x'\ \rho(t, \mathbf{x}')\nabla\left(\frac{1}{|\mathbf{x} - \mathbf{x}'|}\right)\right] \\ = \frac{\partial}{\partial t}\left[-\frac{1}{4\pi\varepsilon_0}\nabla\int_{V'} \mathrm{d}^3x'\ \frac{\rho(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\right] &= \frac{\partial}{\partial t}\mathbf{E}(t, \mathbf{x})\end{aligned} \tag{1.25}$$

The result is Maxwell's source equation for the **B** field

$$\nabla \times \mathbf{B}(t, \mathbf{x}) = \mu_0 \left(\mathbf{j}(t, \mathbf{x}) + \frac{\partial}{\partial t}\varepsilon_0 \mathbf{E}(t, \mathbf{x})\right) \tag{1.26}$$

where the last term $\partial\varepsilon_0\mathbf{E}(t, \mathbf{x})/\partial t$ is the famous *displacement current*. This term was introduced, in a stroke of genius, by Maxwell[8] in order to make the right hand side of this equation divergence free when $\mathbf{j}(t, \mathbf{x})$ is assumed to represent the density of the total electric current, which can be split up in 'ordinary' conduction currents, polarisation currents and magnetisation currents. The displacement current is an extra term which behaves like a current density flowing in vacuum. As we shall see later, its existence has far-reaching physical consequences as it predicts the existence of electromagnetic radiation that can carry energy and momentum over very long distances, even in vacuum.

1.3.3 Electromotive force

If an electric field $\mathbf{E}(t, \mathbf{x})$ is applied to a conducting medium, a current density $\mathbf{j}(t, \mathbf{x})$ will be produced in this medium. There exist also hydrodynamical and chemical processes which can create currents. Under certain physical conditions, and for certain

[4]Later, we will need to consider this generalisation and formal identification further.

materials, one can sometimes assume a linear relationship between the electric current density $\mathbf{j}$ and $\mathbf{E}$, called *Ohm's law*:

$$\mathbf{j}(t,\mathbf{x}) = \sigma\mathbf{E}(t,\mathbf{x}) \tag{1.27}$$

where σ is the *electric conductivity* (S/m). In the most general cases, for instance in an anisotropic conductor, σ is a tensor.

We can view Ohm's law, Equation (1.27) above, as the first term in a Taylor expansion of the law $\mathbf{j}[\mathbf{E}(t,\mathbf{x})]$. This general law incorporates *non-linear effects* such as frequency mixing. Examples of media which are highly non-linear are semiconductors and plasma. We draw the attention to the fact that even in cases when the linear relation between $\mathbf{E}$ and $\mathbf{j}$ is a good approximation, we still have to use Ohm's law with care. The conductivity σ is, in general, time-dependent (*temporal dispersive media*) but then it is often the case that Equation (1.27) is valid for each individual Fourier component of the field.

If the current is caused by an applied electric field $\mathbf{E}(t,\mathbf{x})$, this electric field will exert work on the charges in the medium and, unless the medium is super-conducting, there will be some energy loss. The rate at which this energy is expended is $\mathbf{j}\cdot\mathbf{E}$ per unit volume. If $\mathbf{E}$ is irrotational (conservative), $\mathbf{j}$ will decay away with time. Stationary currents therefore require that an electric field which corresponds to an *electromotive force (EMF)* is present. In the presence of such a field $\mathbf{E}^{\text{EMF}}$, Ohm's law, Equation (1.27) above, takes the form

$$\mathbf{j} = \sigma(\mathbf{E}^{\text{stat}} + \mathbf{E}^{\text{EMF}}) \tag{1.28}$$

The electromotive force is defined as

$$\mathcal{E} = \oint_C \mathrm{d}\mathbf{l}\cdot(\mathbf{E}^{\text{stat}} + \mathbf{E}^{\text{EMF}}) \tag{1.29}$$

where $\mathrm{d}\mathbf{l}$ is a tangential line element of the closed loop C.

1.3.4 Faraday's law of induction

In Subsection 1.1.2 we derived the differential equations for the electrostatic field. In particular, on page 5 we derived Equation (1.8) which states that $\boldsymbol{\nabla}\times\mathbf{E}^{\text{stat}}(\mathbf{x}) = \mathbf{0}$ and thus that $\mathbf{E}^{\text{stat}}$ is a *conservative field* (it can be expressed as a gradient of a scalar field). This implies that the closed line integral of $\mathbf{E}^{\text{stat}}$ in Equation (1.29) above vanishes and that this equation becomes

$$\mathcal{E} = \oint_C \mathrm{d}\mathbf{l}\cdot\mathbf{E}^{\text{EMF}} \tag{1.30}$$

It has been established experimentally that a nonconservative EMF field is produced in a closed circuit C if the magnetic flux through this circuit varies with time.

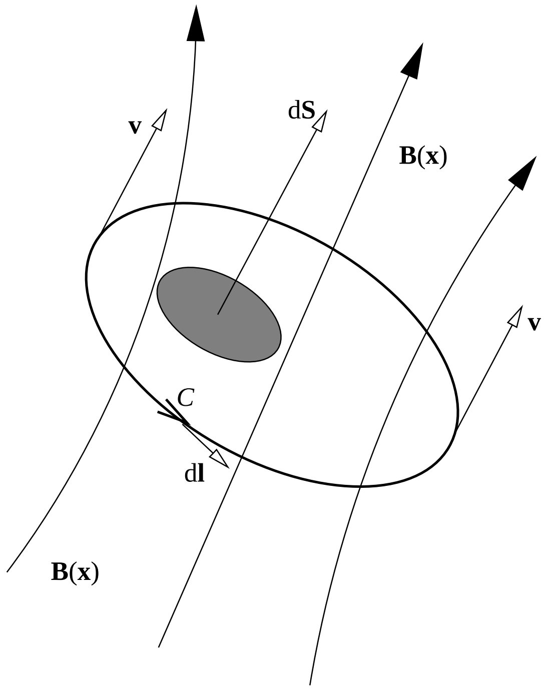

FIGURE 1.4: A loop C which moves with velocity $\mathbf{v}$ in a spatially varying magnetic field $\mathbf{B}(\mathbf{x})$ will sense a varying magnetic flux during the motion.

This is formulated in *Faraday's law* which, in Maxwell's generalised form, reads

$$\begin{aligned}\mathcal{E}(t,\mathbf{x}) &= \oint_C \mathrm{d}\mathbf{l}\cdot\mathbf{E}(t,\mathbf{x}) = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi_\mathrm{m}(t,\mathbf{x}) \\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\int_S \mathrm{d}\mathbf{S}\cdot\mathbf{B}(t,\mathbf{x}) = -\int_S \mathrm{d}\mathbf{S}\cdot\frac{\partial}{\partial t}\mathbf{B}(t,\mathbf{x})\end{aligned} \tag{1.31}$$

where Φ_m is the *magnetic flux* and S is the surface encircled by C which can be interpreted as a generic stationary 'loop' and not necessarily as a conducting circuit. Application of Stokes' theorem on this integral equation, transforms it into the differential equation

$$\boldsymbol{\nabla}\times\mathbf{E}(t,\mathbf{x}) = -\frac{\partial}{\partial t}\mathbf{B}(t,\mathbf{x}) \tag{1.32}$$

which is valid for arbitrary variations in the fields and constitutes the Maxwell equation which explicitly connects electricity with magnetism.

Any change of the magnetic flux Φ_m will induce an EMF. Let us therefore consider the case, illustrated if Figure 1.4, that the 'loop' is moved in such a way that it links a magnetic field which varies during the movement. The *convective derivative* is evaluated according to the well-known operator formula

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} \tag{1.33}$$

which follows immediately from the rules of differentiation of an arbitrary differentiable function $f(t, \mathbf{x}(t))$. Applying this rule to Faraday's law, Equation (1.31) on the preceding page, we obtain

$$\mathcal{E}(t, \mathbf{x}) = -\frac{\mathrm{d}}{\mathrm{d}t}\int_S \mathrm{d}\mathbf{S}\cdot\mathbf{B} = -\int_S \mathrm{d}\mathbf{S}\cdot\frac{\partial \mathbf{B}}{\partial t} - \int_S \mathrm{d}\mathbf{S}\cdot(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{B} \tag{1.34}$$

During spatial differentiation $\mathbf{v}$ is to be considered as constant, and Equation (1.17) on page 8 holds also for time-varying fields:

$$\boldsymbol{\nabla}\cdot\mathbf{B}(t, \mathbf{x}) = 0 \tag{1.35}$$

(it is one of Maxwell's equations) so that, according to Formula (F.59) on page 160,

$$\boldsymbol{\nabla}\times(\mathbf{B}\times\mathbf{v}) = (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{B} \tag{1.36}$$

allowing us to rewrite Equation (1.34) in the following way:

$$\begin{aligned}\mathcal{E}(t, \mathbf{x}) &= \oint_C \mathrm{d}\mathbf{l}\cdot\mathbf{E}^{\mathrm{EMF}} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_S \mathrm{d}\mathbf{S}\cdot\mathbf{B} \\ &= -\int_S \mathrm{d}\mathbf{S}\cdot\frac{\partial \mathbf{B}}{\partial t} - \int_S \mathrm{d}\mathbf{S}\cdot\boldsymbol{\nabla}\times(\mathbf{B}\times\mathbf{v})\end{aligned} \tag{1.37}$$

With Stokes' theorem applied to the last integral, we finally get

$$\mathcal{E}(t, \mathbf{x}) = \oint_C \mathrm{d}\mathbf{l}\cdot\mathbf{E}^{\mathrm{EMF}} = -\int_S \mathrm{d}\mathbf{S}\cdot\frac{\partial \mathbf{B}}{\partial t} - \oint_C \mathrm{d}\mathbf{l}\cdot(\mathbf{B}\times\mathbf{v}) \tag{1.38}$$

or, rearranging the terms,

$$\oint_C \mathrm{d}\mathbf{l}\cdot(\mathbf{E}^{\mathrm{EMF}} - \mathbf{v}\times\mathbf{B}) = -\int_S \mathrm{d}\mathbf{S}\cdot\frac{\partial \mathbf{B}}{\partial t} \tag{1.39}$$

where $\mathbf{E}^{\mathrm{EMF}}$ is the field which is induced in the 'loop', *i.e.*, in the *moving* system. The use of Stokes' theorem 'backwards' on Equation (1.39) above yields

$$\boldsymbol{\nabla}\times(\mathbf{E}^{\mathrm{EMF}} - \mathbf{v}\times\mathbf{B}) = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.40}$$

In the *fixed* system, an observer measures the electric field

$$\mathbf{E} = \mathbf{E}^{\mathrm{EMF}} - \mathbf{v}\times\mathbf{B} \tag{1.41}$$

Hence, a moving observer measures the following *Lorentz force* on a charge q

$$q\mathbf{E}^{\mathrm{EMF}} = q\mathbf{E} + q(\mathbf{v} \times \mathbf{B}) \tag{1.42}$$

corresponding to an 'effective' electric field in the 'loop' (moving observer)

$$\mathbf{E}^{\mathrm{EMF}} = \mathbf{E} + \mathbf{v} \times \mathbf{B} \tag{1.43}$$

Hence, we can conclude that for a *stationary* observer, the Maxwell equation

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.44}$$

is indeed valid even if the 'loop' is moving.

1.3.5 Maxwell's microscopic equations

We are now able to collect the results from the above considerations and formulate the equations of classical electrodynamics valid for arbitrary variations in time and space of the coupled electric and magnetic fields $\mathbf{E}(t, \mathbf{x})$ and $\mathbf{B}(t, \mathbf{x})$. The equations are

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{1.45a}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.45b}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{1.45c}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{j}(t, \mathbf{x}) \tag{1.45d}$$

In these equations $\rho(t, \mathbf{x})$ represents the total, possibly both time and space dependent, electric charge, *i.e.*, free as well as induced (polarisation) charges, and $\mathbf{j}(t, \mathbf{x})$ represents the total, possibly both time and space dependent, electric current, *i.e.*, conduction currents (motion of free charges) as well as all atomistic (polarisation, magnetisation) currents. As they stand, the equations therefore incorporate the classical interaction between all electric charges and currents in the system and are called *Maxwell's microscopic equations*. Another name often used for them is the *Maxwell-Lorentz equations*. Together with the appropriate *constitutive relations*, which relate ρ and $\mathbf{j}$ to the fields, and the initial and boundary conditions pertinent to the physical situation at hand, they form a system of well-posed partial differential equations which completely determine $\mathbf{E}$ and $\mathbf{B}$.

1.3.6 Maxwell's macroscopic equations

The microscopic field equations (1.45) provide a correct classical picture for arbitrary field and source distributions, including both microscopic and macroscopic scales.

However, for macroscopic substances it is sometimes convenient to introduce new derived fields which represent the electric and magnetic fields in which, in an average sense, the material properties of the substances are already included. These fields are the *electric displacement* $\mathbf{D}$ and the *magnetising field* $\mathbf{H}$. In the most general case, these derived fields are complicated nonlocal, nonlinear functionals of the primary fields $\mathbf{E}$ and $\mathbf{B}$:

$$\mathbf{D} = \mathbf{D}[t, \mathbf{x}; \mathbf{E}, \mathbf{B}] \tag{1.46a}$$

$$\mathbf{H} = \mathbf{H}[t, \mathbf{x}; \mathbf{E}, \mathbf{B}] \tag{1.46b}$$

Under certain conditions, for instance for very low field strengths, we may assume that the response of a substance to the fields is linear so that

$$\mathbf{D} = \varepsilon \mathbf{E} \tag{1.47}$$

$$\mathbf{H} = \mu^{-1} \mathbf{B} \tag{1.48}$$

i.e., that the derived fields are linearly proportional to the primary fields and that the electric displacement (magnetising field) is only dependent on the electric (magnetic) field.

The field equations expressed in terms of the derived field quantities $\mathbf{D}$ and $\mathbf{H}$ are

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho(t, \mathbf{x}) \tag{1.49a}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.49b}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{1.49c}$$

$$\boldsymbol{\nabla} \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j}(t, \mathbf{x}) \tag{1.49d}$$

and are called *Maxwell's macroscopic equations*. We will study them in more detail in Chapter 6.

1.4 Electromagnetic duality

If we look more closely at the microscopic Maxwell equations (1.45), we see that they exhibit a certain, albeit not a complete, symmetry. Let us follow Dirac and make the *ad hoc* assumption that there exist *magnetic monopoles* represented by a *magnetic charge density*, which we denote by $\rho^{\mathrm{m}} = \rho^{\mathrm{m}}(t, \mathbf{x})$, and a *magnetic current density*, which we denote by $\mathbf{j}^{\mathrm{m}} = \mathbf{j}^{\mathrm{m}}(t, \mathbf{x})$. With these new quantities included in the theory, and with the electric charge density denoted ρ^{e} and the electric current density denoted $\mathbf{j}^{\mathrm{e}}$, the Maxwell equations will be symmetrised into the following four coupled, vector, partial differential equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho^{\mathrm{e}}}{\varepsilon_0} \tag{1.50a}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} - \mu_0 \mathbf{j}^{\mathrm{m}} \tag{1.50b}$$

$$\nabla \cdot \mathbf{B} = \mu_0 \rho^{\mathrm{m}} \tag{1.50c}$$

$$\nabla \times \mathbf{B} = \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{j}^{\mathrm{e}} \tag{1.50d}$$

We shall call these equations *Dirac's symmetrised Maxwell equations* or the *electromagnetodynamic equations*.

Taking the divergence of (1.50b), we find that

$$\nabla \cdot (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}) - \mu_0 \nabla \cdot \mathbf{j}^{\mathrm{m}} \equiv 0 \tag{1.51}$$

where we used the fact that, according to Formula (F.63) on page 160, the divergence of a curl always vanishes. Using (1.50c) to rewrite this relation, we obtain the *equation of continuity for magnetic monopoles*

$$\frac{\partial \rho^{\mathrm{m}}}{\partial t} + \nabla \cdot \mathbf{j}^{\mathrm{m}} = 0 \tag{1.52}$$

which has the same form as that for the electric monopoles (electric charges) and currents, Equation (1.23) on page 9.

We notice that the new Equations (1.50) exhibit the following symmetry (recall that $\varepsilon_0 \mu_0 = 1/c^2$):

$$\mathbf{E} \to c\mathbf{B} \tag{1.53a}$$

$$c\mathbf{B} \to -\mathbf{E} \tag{1.53b}$$

$$c\rho^{\mathrm{e}} \to \rho^{\mathrm{m}} \tag{1.53c}$$

$$\rho^{\mathrm{m}} \to -c\rho^{\mathrm{e}} \tag{1.53d}$$

$$c\mathbf{j}^{\mathrm{e}} \to \mathbf{j}^{\mathrm{m}} \tag{1.53e}$$

$$\mathbf{j}^{\mathrm{m}} \to -c\mathbf{j}^{\mathrm{e}} \tag{1.53f}$$

which is a particular case ($\theta = \pi/2$) of the general *duality transformation* (depicted by the *Hodge star operator*)

$$^{\star}\mathbf{E} = \mathbf{E}\cos\theta + c\mathbf{B}\sin\theta \tag{1.54a}$$

$$c\,^{\star}\mathbf{B} = -\mathbf{E}\sin\theta + c\mathbf{B}\cos\theta \tag{1.54b}$$

$$c\,^{\star}\rho^{\mathrm{e}} = c\rho^{\mathrm{e}}\cos\theta + \rho^{\mathrm{m}}\sin\theta \tag{1.54c}$$

$$^{\star}\rho^{\mathrm{m}} = -c\rho^{\mathrm{e}}\sin\theta + \rho^{\mathrm{m}}\cos\theta \tag{1.54d}$$

$$c\,^{\star}\mathbf{j}^{\mathrm{e}} = c\mathbf{j}^{\mathrm{e}}\cos\theta + \mathbf{j}^{\mathrm{m}}\sin\theta \tag{1.54e}$$

$$^{\star}\mathbf{j}^{\mathrm{m}} = -c\mathbf{j}^{\mathrm{e}}\sin\theta + \mathbf{j}^{\mathrm{m}}\cos\theta \tag{1.54f}$$

which leaves the symmetrised Maxwell equations, and hence the physics they describe (often referred to as *electromagnetodynamics*), invariant. Since $\mathbf{E}$ and $\mathbf{j}^{\mathrm{e}}$ are (true or polar) vectors, $\mathbf{B}$ a pseudovector (axial vector), ρ^{e} a (true) scalar, then ρ^{m} and θ, which behaves as a *mixing angle* in a two-dimensional 'charge space', must be pseudoscalars and $\mathbf{j}^{\mathrm{m}}$ a pseudovector.

▷FARADAY'S LAW AS A CONSEQUENCE OF CONSERVATION OF MAGNETIC CHARGE——— EXAMPLE 1.1

Postulate 1.1 (Indestructibility of magnetic charge). *Magnetic charge exists and is indestructible in the same way that electric charge exists and is indestructible. In other words we* postulate *that there exists an equation of continuity for magnetic charges:*

$$\frac{\partial\rho^{\mathrm{m}}(t,\mathbf{x})}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{j}^{\mathrm{m}}(t,\mathbf{x}) = 0$$

Use this postulate and Dirac's symmetrised form of Maxwell's equations to derive Faraday's law.

The assumption of the existence of magnetic charges suggests a Coulomb-like law for magnetic fields:

$$\begin{aligned}\mathbf{B}^{\mathrm{stat}}(\mathbf{x}) &= \frac{\mu_0}{4\pi}\int_{V'}\mathrm{d}^3x'\,\rho^{\mathrm{m}}(\mathbf{x}')\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = -\frac{\mu_0}{4\pi}\int_{V'}\mathrm{d}^3x'\,\rho^{\mathrm{m}}(\mathbf{x}')\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\\ &= -\frac{\mu_0}{4\pi}\boldsymbol{\nabla}\int_{V'}\mathrm{d}^3x'\,\frac{\rho^{\mathrm{m}}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\end{aligned} \tag{1.55}$$

[*cf.* Equation (1.5) on page 4 for $\mathbf{E}^{\mathrm{stat}}$] and, if magnetic currents exist, a Biot-Savart-like law for electric fields [*cf.* Equation (1.15) on page 7 for $\mathbf{B}^{\mathrm{stat}}$]:

$$\begin{aligned}\mathbf{E}^{\mathrm{stat}}(\mathbf{x}) &= -\frac{\mu_0}{4\pi}\int_{V'}\mathrm{d}^3x'\,\mathbf{j}^{\mathrm{m}}(\mathbf{x}')\times\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = \frac{\mu_0}{4\pi}\int_{V'}\mathrm{d}^3x'\,\mathbf{j}^{\mathrm{m}}(\mathbf{x}')\times\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\\ &= -\frac{\mu_0}{4\pi}\boldsymbol{\nabla}\times\int_{V'}\mathrm{d}^3x'\,\frac{\mathbf{j}^{\mathrm{m}}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\end{aligned} \tag{1.56}$$

Taking the curl of the latter and using the operator 'bac-cab' rule, Formula (F.59) on page 160, we find that

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{E}^{\mathrm{stat}}(\mathbf{x}) &= -\frac{\mu_0}{4\pi}\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\int_{V'}\mathrm{d}^3x'\,\frac{\mathbf{j}^{\mathrm{m}}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right) =\\ &= -\frac{\mu_0}{4\pi}\int_{V'}\mathrm{d}^3x'\,\mathbf{j}^{\mathrm{m}}(\mathbf{x}')\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) + \frac{\mu_0}{4\pi}\int_{V'}\mathrm{d}^3x'\,[\mathbf{j}^{\mathrm{m}}(\mathbf{x}')\cdot\boldsymbol{\nabla}']\boldsymbol{\nabla}'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\end{aligned} \tag{1.57}$$

Comparing with Equation (1.18) on page 8 for $\mathbf{E}^{\mathrm{stat}}$ and the evaluation of the integrals there, we obtain

$$\boldsymbol{\nabla}\times\mathbf{E}^{\mathrm{stat}}(\mathbf{x}) = -\mu_0\int_{V'}\mathrm{d}^3x'\,\mathbf{j}^{\mathrm{m}}(\mathbf{x}')\,\delta(\mathbf{x}-\mathbf{x}') = -\mu_0\mathbf{j}^{\mathrm{m}}(\mathbf{x}) \tag{1.58}$$

We assume that Formula (1.56) above is valid also for time-varying magnetic currents. Then, with the use of the representation of the Dirac delta function, Equation (F.73) on page 161, the equation of continuity for magnetic charge, Equation (1.52) on the preceding page, and the assumption of the generalisation of Equation (1.55) to time-dependent magnetic charge distributions, we obtain, formally,

$$\begin{aligned}\nabla\times\mathbf{E}(t,\mathbf{x}) &= -\mu_0\int_{V'}\mathrm{d}^3x'\;\mathbf{j}^{\mathrm{m}}(t,\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}') - \frac{\mu_0}{4\pi}\frac{\partial}{\partial t}\int_{V'}\mathrm{d}^3x'\;\rho^{\mathrm{m}}(t,\mathbf{x}')\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\\ &= -\mu_0\mathbf{j}^{\mathrm{m}}(t,\mathbf{x}) - \frac{\partial}{\partial t}\mathbf{B}(t,\mathbf{x})\end{aligned} \tag{1.59}$$

[*cf.* Equation (1.24) on page 10] which we recognise as Equation (1.50b) on page 16. A transformation of this electromagnetodynamic result by rotating into the 'electric realm' of charge space, thereby letting $\mathbf{j}^{\mathrm{m}}$ tend to zero, yields the electrodynamic Equation (1.50b) on page 16, *i.e.*, the Faraday law in the ordinary Maxwell equations. This process also provides an alternative interpretation of the term $\partial\mathbf{B}/\partial t$ as a *magnetic displacement current*, dual to the *electric displacement current* [cf. Equation (1.26) on page 10].

By postulating the indestructibility of a hypothetical magnetic charge, we have thereby been able to replace Faraday's experimental results on electromotive forces and induction in loops as a foundation for the Maxwell equations by a more appealing one.

◁ END OF EXAMPLE 1.1

EXAMPLE 1.2 ▷ DUALITY OF THE ELECTROMAGNETODYNAMIC EQUATIONS

Show that the symmetric, electromagnetodynamic form of Maxwell's equations (Dirac's symmetrised Maxwell equations), Equations (1.50) on page 16, are invariant under the duality transformation (1.54).

Explicit application of the transformation yields

$$\begin{aligned}\nabla\cdot{}^{\star}\mathbf{E} &= \nabla\cdot(\mathbf{E}\cos\theta + c\mathbf{B}\sin\theta) = \frac{\rho^{\mathrm{e}}}{\varepsilon_0}\cos\theta + c\mu_0\rho^{\mathrm{m}}\sin\theta\\ &= \frac{1}{\varepsilon_0}\left(\rho^{\mathrm{e}}\cos\theta + \frac{1}{c}\rho^{\mathrm{m}}\sin\theta\right) = \frac{{}^{\star}\rho^{\mathrm{e}}}{\varepsilon_0}\end{aligned} \tag{1.60}$$

$$\begin{aligned}\nabla\times{}^{\star}\mathbf{E} + \frac{\partial{}^{\star}\mathbf{B}}{\partial t} &= \nabla\times(\mathbf{E}\cos\theta + c\mathbf{B}\sin\theta) + \frac{\partial}{\partial t}\left(-\frac{1}{c}\mathbf{E}\sin\theta + \mathbf{B}\cos\theta\right)\\ &= -\mu_0\mathbf{j}^{\mathrm{m}}\cos\theta - \frac{\partial\mathbf{B}}{\partial t}\cos\theta + c\mu_0\mathbf{j}^{\mathrm{e}}\sin\theta + \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\sin\theta\\ &\quad - \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\sin\theta + \frac{\partial\mathbf{B}}{\partial t}\cos\theta = -\mu_0\mathbf{j}^{\mathrm{m}}\cos\theta + c\mu_0\mathbf{j}^{\mathrm{e}}\sin\theta\\ &= -\mu_0(-c\mathbf{j}^{\mathrm{e}}\sin\theta + \mathbf{j}^{\mathrm{m}}\cos\theta) = -\mu_0{}^{\star}\mathbf{j}^{\mathrm{m}}\end{aligned} \tag{1.61}$$

$$\begin{aligned}\nabla\cdot{}^{\star}\mathbf{B} &= \nabla\cdot(-\frac{1}{c}\mathbf{E}\sin\theta + \mathbf{B}\cos\theta) = -\frac{\rho^{\mathrm{e}}}{c\varepsilon_0}\sin\theta + \mu_0\rho^{\mathrm{m}}\cos\theta\\ &= \mu_0\left(-c\rho^{\mathrm{e}}\sin\theta + \rho^{\mathrm{m}}\cos\theta\right) = \mu_0{}^{\star}\rho^{\mathrm{m}}\end{aligned} \tag{1.62}$$

$$
\begin{aligned}
\nabla \times {}^{\star}\mathbf{B} - \frac{1}{c^2}\frac{\partial^{\star}\mathbf{E}}{\partial t} &= \nabla \times (-\frac{1}{c}\mathbf{E}\sin\theta + \mathbf{B}\cos\theta) - \frac{1}{c^2}\frac{\partial}{\partial t}(\mathbf{E}\cos\theta + c\mathbf{B}\sin\theta) \\
&= \frac{1}{c}\mu_0\mathbf{j}^{\mathrm{m}}\sin\theta + \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\cos\theta + \mu_0\mathbf{j}^{\mathrm{e}}\cos\theta + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}\cos\theta \\
&\quad - \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}\cos\theta - \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\sin\theta \\
&= \mu_0\left(\frac{1}{c}\mathbf{j}^{\mathrm{m}}\sin\theta + \mathbf{j}^{\mathrm{e}}\cos\theta\right) = \mu_0{}^{\star}\mathbf{j}^{\mathrm{e}}
\end{aligned}
\tag{1.63}
$$

QED ■

◁ END OF EXAMPLE 1.2

▷ DIRAC'S SYMMETRISED MAXWELL EQUATIONS FOR A FIXED MIXING ANGLE — EXAMPLE 1.3

Show that for a fixed mixing angle θ such that

$$\rho^{\mathrm{m}} = c\rho^{\mathrm{e}}\tan\theta \tag{1.64a}$$

$$\mathbf{j}^{\mathrm{m}} = c\mathbf{j}^{\mathrm{e}}\tan\theta \tag{1.64b}$$

the symmetrised Maxwell equations reduce to the usual Maxwell equations.

Explicit application of the fixed mixing angle conditions on the duality transformation (1.54) on page 16 yields

$$
\begin{aligned}
{}^{\star}\rho^{\mathrm{e}} &= \rho^{\mathrm{e}}\cos\theta + \frac{1}{c}\rho^{\mathrm{m}}\sin\theta = \rho^{\mathrm{e}}\cos\theta + \frac{1}{c}c\rho^{\mathrm{e}}\tan\theta\sin\theta \\
&= \frac{1}{\cos\theta}(\rho^{\mathrm{e}}\cos^2\theta + \rho^{\mathrm{e}}\sin^2\theta) = \frac{1}{\cos\theta}\rho^{\mathrm{e}}
\end{aligned}
\tag{1.65a}
$$

$${}^{\star}\rho^{\mathrm{m}} = -c\rho^{\mathrm{e}}\sin\theta + c\rho^{\mathrm{e}}\tan\theta\cos\theta = -c\rho^{\mathrm{e}}\sin\theta + c\rho^{\mathrm{e}}\sin\theta = 0 \tag{1.65b}$$

$${}^{\star}\mathbf{j}^{\mathrm{e}} = \mathbf{j}^{\mathrm{e}}\cos\theta + \mathbf{j}^{\mathrm{e}}\tan\theta\sin\theta = \frac{1}{\cos\theta}(\mathbf{j}^{\mathrm{e}}\cos^2\theta + \mathbf{j}^{\mathrm{e}}\sin^2\theta) = \frac{1}{\cos\theta}\mathbf{j}^{\mathrm{e}} \tag{1.65c}$$

$${}^{\star}\mathbf{j}^{\mathrm{m}} = -c\mathbf{j}^{\mathrm{e}}\sin\theta + c\mathbf{j}^{\mathrm{e}}\tan\theta\cos\theta = -c\mathbf{j}^{\mathrm{e}}\sin\theta + c\mathbf{j}^{\mathrm{e}}\sin\theta = \mathbf{0} \tag{1.65d}$$

Hence, a fixed mixing angle, or, equivalently, a fixed ratio between the electric and magnetic charges/currents, 'hides' the magnetic monopole influence (ρ^{m} and $\mathbf{j}^{\mathrm{m}}$) on the dynamic equations.

We notice that the inverse of the transformation given by Equation (1.54) on page 16 yields

$$\mathbf{E} = {}^{\star}\mathbf{E}\cos\theta - c{}^{\star}\mathbf{B}\sin\theta \tag{1.66}$$

This means that

$$\nabla\cdot\mathbf{E} = \nabla\cdot{}^{\star}\mathbf{E}\cos\theta - c\nabla\cdot{}^{\star}\mathbf{B}\sin\theta \tag{1.67}$$

Furthermore, from the expressions for the transformed charges and currents above, we find that

$$\nabla\cdot{}^{\star}\mathbf{E} = \frac{{}^{\star}\rho^{\mathrm{e}}}{\varepsilon_0} = \frac{1}{\cos\theta}\frac{\rho^{\mathrm{e}}}{\varepsilon_0} \tag{1.68}$$

and

$$\nabla \cdot {}^{\star}\mathbf{B} = \mu_0 {}^{\star}\rho^{\mathrm{m}} = 0 \tag{1.69}$$

so that

$$\nabla \cdot \mathbf{E} = \frac{1}{\cos\theta}\frac{\rho^{\mathrm{e}}}{\varepsilon_0}\cos\theta - 0 = \frac{\rho^{\mathrm{e}}}{\varepsilon_0} \tag{1.70}$$

and so on for the other equations. QED ■

◁ END OF EXAMPLE 1.3

The invariance of Dirac's symmetrised Maxwell equations under the similarity transformation means that the amount of magnetic monopole density ρ^{m} is irrelevant for the physics as long as the ratio $\rho^{\mathrm{m}}/\rho^{\mathrm{e}} = \tan\theta$ is kept constant. So whether we assume that the particles are only electrically charged or have also a magnetic charge with a given, fixed ratio between the two types of charges is a matter of convention, as long as we assume that this fraction is *the same for all particles*. Such particles are referred to as *dyons* [14]. By varying the mixing angle θ we can change the fraction of magnetic monopoles at will without changing the laws of electrodynamics. For $\theta = 0$ we recover the usual Maxwell electrodynamics as we know it.[5]

EXAMPLE 1.4 ▷COMPLEX FIELD SIX-VECTOR FORMALISM

The *complex field six-vector*

$$\mathbf{G}(t,\mathbf{x}) = \mathbf{E}(t,\mathbf{x}) + \mathrm{i}c\mathbf{B}(t,\mathbf{x}) \tag{1.71}$$

where $\mathbf{E},\mathbf{B} \in \mathbb{R}^3$ and hence $\mathbf{G} \in \mathbb{C}^3$, has a number of interesting properties:

1. The inner product of $\mathbf{G}$ with itself

$$\mathbf{G}\cdot\mathbf{G} = (\mathbf{E} + \mathrm{i}c\mathbf{B})\cdot(\mathbf{E} + \mathrm{i}c\mathbf{B}) = E^2 - c^2B^2 + 2\mathrm{i}c\mathbf{E}\cdot\mathbf{B} \tag{1.72}$$

is conserved. *I.e.*,

$$E^2 - c^2B^2 = Const \tag{1.73a}$$

$$\mathbf{E}\cdot\mathbf{B} = Const \tag{1.73b}$$

as we shall see later.

2. The inner product of $\mathbf{G}$ with the complex conjugate of itself

$$\mathbf{G}\cdot\mathbf{G}^* = (\mathbf{E} + \mathrm{i}c\mathbf{B})\cdot(\mathbf{E} - \mathrm{i}c\mathbf{B}) = E^2 + c^2B^2 \tag{1.74}$$

is proportional to the electromagnetic field energy.

3. As with any vector, the cross product of $\mathbf{G}$ with itself vanishes:

[5] As Julian Schwinger (1918–1994) put it [15]:

> '…there are strong theoretical reasons to believe that magnetic charge exists in nature, and may have played an important role in the development of the universe. Searches for magnetic charge continue at the present time, emphasizing that electromagnetism is very far from being a closed object'.

$$
\begin{aligned}
\mathbf{G}\times\mathbf{G} &= (\mathbf{E}+\mathrm{i}c\mathbf{B})\times(\mathbf{E}+\mathrm{i}c\mathbf{B}) \\
&= \mathbf{E}\times\mathbf{E} - c^2\mathbf{B}\times\mathbf{B} + \mathrm{i}c(\mathbf{E}\times\mathbf{B}) + \mathrm{i}c(\mathbf{B}\times\mathbf{E}) \\
&= \mathbf{0}+\mathbf{0}+\mathrm{i}c(\mathbf{E}\times\mathbf{B}) - \mathrm{i}c(\mathbf{E}\times\mathbf{B}) = \mathbf{0}
\end{aligned} \tag{1.75}
$$

4. The cross product of **G** with the complex conjugate of itself

$$
\begin{aligned}
\mathbf{G}\times\mathbf{G}^* &= (\mathbf{E}+\mathrm{i}c\mathbf{B})\times(\mathbf{E}-\mathrm{i}c\mathbf{B}) \\
&= \mathbf{E}\times\mathbf{E} + c^2\mathbf{B}\times\mathbf{B} - \mathrm{i}c(\mathbf{E}\times\mathbf{B}) + \mathrm{i}c(\mathbf{B}\times\mathbf{E}) \\
&= \mathbf{0}+\mathbf{0}-\mathrm{i}c(\mathbf{E}\times\mathbf{B}) - \mathrm{i}c(\mathbf{E}\times\mathbf{B}) = -2\mathrm{i}c(\mathbf{E}\times\mathbf{B})
\end{aligned} \tag{1.76}
$$

is proportional to the electromagnetic power flux.

◁ END OF EXAMPLE 1.4

▷DUALITY EXPRESSED IN THE COMPLEX FIELD SIX-VECTOR — EXAMPLE 1.5

Expressed in the complex field vector, introduced in Example 1.4 on the facing page, the duality transformation Equations (1.54) on page 16 become

$$
\begin{aligned}
{}^\star\mathbf{G} &= {}^\star\mathbf{E} + \mathrm{i}c{}^\star\mathbf{B} = \mathbf{E}\cos\theta + c\mathbf{B}\sin\theta - \mathrm{i}\mathbf{E}\sin\theta + \mathrm{i}c\mathbf{B}\cos\theta \\
&= \mathbf{E}(\cos\theta - \mathrm{i}\sin\theta) + \mathrm{i}c\mathbf{B}(\cos\theta - \mathrm{i}\sin\theta) = e^{-\mathrm{i}\theta}(\mathbf{E}+\mathrm{i}c\mathbf{B}) = e^{-\mathrm{i}\theta}\mathbf{G}
\end{aligned} \tag{1.77}
$$

from which it is easy to see that

$$
{}^\star\mathbf{G}\cdot{}^\star\mathbf{G}^* = \left|{}^\star G\right|^2 = e^{-i\theta}\mathbf{G}\cdot e^{i\theta}\mathbf{G}^* = |G|^2 \tag{1.78}
$$

while

$$
{}^\star\mathbf{G}\cdot{}^\star\mathbf{G} = e^{-2\mathrm{i}\theta}\mathbf{G}\cdot\mathbf{G} \tag{1.79}
$$

Furthermore, assuming that $\theta = \theta(t, \mathbf{x})$, we see that the spatial and temporal differentiation of ${}^\star\mathbf{G}$ leads to

$$
\partial_t{}^\star\mathbf{G} \equiv \frac{\partial{}^\star\mathbf{G}}{\partial t} = -\mathrm{i}(\partial_t\theta)e^{-\mathrm{i}\theta}\mathbf{G} + e^{-\mathrm{i}\theta}\partial_t\mathbf{G} \tag{1.80a}
$$

$$
\boldsymbol{\partial}\cdot{}^\star\mathbf{G} \equiv \boldsymbol{\nabla}\cdot{}^\star\mathbf{G} = -\mathrm{i}e^{-\mathrm{i}\theta}\boldsymbol{\nabla}\theta\cdot\mathbf{G} + e^{-\mathrm{i}\theta}\boldsymbol{\nabla}\cdot\mathbf{G} \tag{1.80b}
$$

$$
\boldsymbol{\partial}\times{}^\star\mathbf{G} \equiv \boldsymbol{\nabla}\times{}^\star\mathbf{G} = -\mathrm{i}e^{-\mathrm{i}\theta}\boldsymbol{\nabla}\theta\times\mathbf{G} + e^{-\mathrm{i}\theta}\boldsymbol{\nabla}\times\mathbf{G} \tag{1.80c}
$$

which means that $\partial_t{}^\star\mathbf{G}$ transforms as ${}^\star\mathbf{G}$ itself only if θ is time-independent, and that $\boldsymbol{\nabla}\cdot{}^\star\mathbf{G}$ and $\boldsymbol{\nabla}\times{}^\star\mathbf{G}$ transform as ${}^\star\mathbf{G}$ itself only if θ is space-independent.

◁ END OF EXAMPLE 1.5

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CHAPTER 2

Electromagnetic Waves

In this chapter we investigate the dynamical properties of the electromagnetic field by deriving a set of equations which are alternatives to the Maxwell equations. It turns out that these alternative equations are wave equations, indicating that electromagnetic waves are natural and common manifestations of electrodynamics.

Maxwell's microscopic equations [*cf.* Equations (1.45) on page 14] are

$$\nabla \cdot \mathbf{E} = \frac{\rho(t, \mathbf{x})}{\varepsilon_0} \qquad \text{(Coulomb's/Gauss's law)} \tag{2.1a}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(Faraday's law)} \tag{2.1b}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(No free magnetic charges)} \tag{2.1c}$$

$$\nabla \times \mathbf{B} = \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{j}(t, \mathbf{x}) \qquad \text{(Ampère's/Maxwell's law)} \tag{2.1d}$$

and can be viewed as an axiomatic basis for classical electrodynamics. In particular, these equations are well suited for calculating the electric and magnetic fields **E** and **B** from given, prescribed charge distributions $\rho(t, \mathbf{x})$ and current distributions $\mathbf{j}(t, \mathbf{x})$ of arbitrary time- and space-dependent form.

However, as is well known from the theory of differential equations, these four first order, coupled partial differential vector equations can be rewritten as two uncoupled, second order partial equations, one for **E** and one for **B**. We shall derive these second order equations which, as we shall see are *wave equations*, and then discuss the implications of them. We shall also show how the **B** wave field can be easily calculated from the solution of the **E** wave equation.

2.1 The wave equations

We restrict ourselves to derive the wave equations for the electric field vector **E** and the magnetic field vector **B** in a volume with no net charge, $\rho = 0$, and no electromotive force $\mathbf{E}^{\text{EMF}} = \mathbf{0}$.

2.1.1 The wave equation for E

In order to derive the wave equation for **E** we take the curl of (2.1b) and using (2.1d), to obtain

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) = -\frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \mathbf{B}) = -\mu_0 \frac{\partial}{\partial t}\left(\mathbf{j} + \varepsilon_0 \frac{\partial}{\partial t}\mathbf{E}\right) \tag{2.2}$$

According to the operator triple product 'bac-cab' rule Equation (F.64) on page 160

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \tag{2.3}$$

Furthermore, since $\rho = 0$, Equation (2.1a) on the preceding page yields

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 0 \tag{2.4}$$

and since $\mathbf{E}^{\text{EMF}} = \mathbf{0}$, Ohm's law, Equation (1.28) on page 11, yields

$$\mathbf{j} = \sigma \mathbf{E} \tag{2.5}$$

we find that Equation (2.2) can be rewritten

$$\nabla^2 \mathbf{E} - \mu_0 \frac{\partial}{\partial t}\left(\sigma \mathbf{E} + \varepsilon_0 \frac{\partial}{\partial t}\mathbf{E}\right) = \mathbf{0} \tag{2.6}$$

or, also using Equation (1.11) on page 6 and rearranging,

$$\nabla^2 \mathbf{E} - \mu_0 \sigma \frac{\partial \mathbf{E}}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \mathbf{0} \tag{2.7}$$

which is the *homogeneous wave equation* for **E**.

2.1.2 The wave equation for B

The wave equation for **B** is derived in much the same way as the wave equation for **E**. Take the curl of (2.1d) and use Ohm's law $\mathbf{j} = \sigma \mathbf{E}$ to obtain

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{B}) = \mu_0 \boldsymbol{\nabla} \times \mathbf{j} + \varepsilon_0 \mu_0 \frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \mathbf{E}) = \mu_0 \sigma \boldsymbol{\nabla} \times \mathbf{E} + \varepsilon_0 \mu_0 \frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \mathbf{E}) \tag{2.8}$$

which, with the use of Equation (F.64) on page 160 and Equation (2.1c) on page 25 can be rewritten

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{B}) - \nabla^2\mathbf{B} = -\mu_0\sigma\frac{\partial\mathbf{B}}{\partial t} - \varepsilon_0\mu_0\frac{\partial^2}{\partial t^2}\mathbf{B} \tag{2.9}$$

Using the fact that, according to (2.1c), $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ for any medium and rearranging, we can rewrite this equation as

$$\nabla^2\mathbf{B} - \mu_0\sigma\frac{\partial\mathbf{B}}{\partial t} - \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} = \mathbf{0} \tag{2.10}$$

This is the wave equation for the magnetic field. We notice that it is of exactly the same form as the wave equation for the electric field, Equation (2.7) on the facing page.

2.1.3 The time-independent wave equation for E

We now look for a solution to any of the wave equations in the form of a *time-harmonic wave*. As is clear from the above, it suffices to consider only the **E** field, since the results for the **B** field follow trivially. We therefore make the following *Fourier component* Ansatz

$$\mathbf{E} = \mathbf{E}_0(\mathbf{x})e^{-\mathrm{i}\omega t} \tag{2.11}$$

and insert this into Equation (2.7) on the preceding page. This yields

$$\begin{aligned}
\nabla^2\mathbf{E}_0(\mathbf{x})e^{-\mathrm{i}\omega t} - \mu_0\sigma\frac{\partial}{\partial t}\mathbf{E}_0(\mathbf{x})e^{-\mathrm{i}\omega t} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}_0(\mathbf{x})e^{-\mathrm{i}\omega t}&\\
= \nabla^2\mathbf{E} - \mu_0\sigma(-\mathrm{i}\omega)\mathbf{E}_0(\mathbf{x})e^{-\mathrm{i}\omega t} - \frac{1}{c^2}(-\mathrm{i}\omega)^2\mathbf{E}_0(\mathbf{x})e^{-\mathrm{i}\omega t}&\\
= \nabla^2\mathbf{E} - \mu_0\sigma(-\mathrm{i}\omega)\mathbf{E} - \frac{1}{c^2}(-\mathrm{i}\omega)^2\mathbf{E} =&\\
= \nabla^2\mathbf{E} + \frac{\omega^2}{c^2}\left(1 + \mathrm{i}\frac{\sigma}{\varepsilon_0\omega}\right)\mathbf{E} = \mathbf{0}&
\end{aligned} \tag{2.12}$$

Introducing the *relaxation time* $\tau = \varepsilon_0/\sigma$ of the medium in question we can rewrite this equation as

$$\nabla^2\mathbf{E} + \frac{\omega^2}{c^2}\left(1 + \frac{\mathrm{i}}{\tau\omega}\right)\mathbf{E} = \mathbf{0} \tag{2.13}$$

In the limit of long τ, Equation (2.13) tends to

$$\nabla^2\mathbf{E} + \frac{\omega^2}{c^2}\mathbf{E} = \mathbf{0} \tag{2.14}$$

which is a *time-independent wave equation* for **E**, representing weakly damped propagating waves. In the short τ limit we have instead

$$\nabla^2\mathbf{E} + \mathrm{i}\omega\mu_0\sigma\mathbf{E} = \mathbf{0} \tag{2.15}$$

which is a *time-independent diffusion equation* for **E**.

For most metals $\tau \sim 10^{-14}$ s, which means that the diffusion picture is good for all frequencies lower than optical frequencies. Hence, in metallic conductors, the propagation term $\partial^2\mathbf{E}/c^2\partial t^2$ is negligible even for VHF, UHF, and SHF signals. Alternatively, we may say that the displacement current $\varepsilon_0\partial\mathbf{E}/\partial t$ is negligible relative to the conduction current $\mathbf{j} = \sigma\mathbf{E}$.

If we introduce the *vacuum wave number*

$$k = \frac{\omega}{c} \tag{2.16}$$

we can write, using the fact that $c = 1/\sqrt{\varepsilon_0\mu_0}$ according to Equation (1.11) on page 6,

$$\frac{1}{\tau\omega} = \frac{\sigma}{\varepsilon_0\omega} = \frac{\sigma}{\varepsilon_0}\frac{1}{ck} = \frac{\sigma}{k}\sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{\sigma}{k}R_0 \tag{2.17}$$

where in the last step we introduced the *characteristic impedance* for vacuum

$$R_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 376.7\,\Omega \tag{2.18}$$

EXAMPLE 2.1 ▷WAVE EQUATIONS IN ELECTROMAGNETODYNAMICS

Derive the wave equation for the **E** field described by the electromagnetodynamic equations (Dirac's symmetrised Maxwell equations) [*cf.* Equations (1.50) on page 16]

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho^{\mathrm{e}}}{\varepsilon_0} \tag{2.19a}$$

$$\boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} - \mu_0\mathbf{j}^{\mathrm{m}} \tag{2.19b}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} = \mu_0\rho^{\mathrm{m}} \tag{2.19c}$$

$$\boldsymbol{\nabla}\times\mathbf{B} = \varepsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t} + \mu_0\mathbf{j}^{\mathrm{e}} \tag{2.19d}$$

under the assumption of vanishing net electric and magnetic charge densities and in the absence of electromotive and magnetomotive forces. Interpret this equation physically.

Taking the curl of (2.19b) and using (2.19d), and assuming, for symmetry reasons, that there exists a linear relation between the *magnetic* current density $\mathbf{j}^{\mathrm{m}}$ and the magnetic field **B** (the analogue of Ohm's law for *electric* currents, $\mathbf{j}^{\mathrm{e}} = \sigma^{\mathrm{e}}\mathbf{E}$)

$$\mathbf{j}^{\mathrm{m}} = \sigma^{\mathrm{m}}\mathbf{B} \tag{2.20}$$

one finds, noting that $\varepsilon_0\mu_0 = 1/c^2$, that

$$
\begin{aligned}
\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E}) &= -\mu_0\boldsymbol{\nabla}\times\mathbf{j}^{\mathrm{m}} - \frac{\partial}{\partial t}(\boldsymbol{\nabla}\times\mathbf{B}) = -\mu_0\sigma^{\mathrm{m}}\boldsymbol{\nabla}\times\mathbf{B} - \frac{\partial}{\partial t}\left(\mu_0\mathbf{j}^{\mathrm{e}} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}\right) \\
&= -\mu_0\sigma^{\mathrm{m}}\left(\mu_0\sigma^{\mathrm{e}}\mathbf{E} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}\right) - \mu_0\sigma^{\mathrm{e}}\frac{\partial\mathbf{E}}{\partial t} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}
\end{aligned}
\tag{2.21}
$$

Using the vector operator identity $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E}) - \nabla^2\mathbf{E}$, and the fact that $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$ for a vanishing net electric charge, we can rewrite the wave equation as

$$
\nabla^2\mathbf{E} - \mu_0\left(\sigma^{\mathrm{e}} + \frac{\sigma^{\mathrm{m}}}{c^2}\right)\frac{\partial\mathbf{E}}{\partial t} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} - \mu_0^2\sigma^{\mathrm{m}}\sigma^{\mathrm{e}}\mathbf{E} = \mathbf{0}
\tag{2.22}
$$

This is the homogeneous electromagnetodynamic wave equation for $\mathbf{E}$ we were after.

Compared to the ordinary electrodynamic wave equation for $\mathbf{E}$, Equation (2.7) on page 26, we see that we pick up extra terms. In order to understand what these extra terms mean physically, we analyse the time-independent wave equation for a single Fourier component. Then our wave equation becomes

$$
\begin{aligned}
&\nabla^2\mathbf{E} + \mathrm{i}\omega\mu_0\left(\sigma^{\mathrm{e}} + \frac{\sigma^{\mathrm{m}}}{c^2}\right)\mathbf{E} + \frac{\omega^2}{c^2}\mathbf{E} - \mu_0^2\sigma^{\mathrm{m}}\sigma^{\mathrm{e}}\mathbf{E} \\
&\quad = \nabla^2\mathbf{E} + \frac{\omega^2}{c^2}\left[\left(1 - \frac{1}{\omega^2}\frac{\mu_0}{\varepsilon_0}\sigma^{\mathrm{m}}\sigma^{\mathrm{e}}\right) + \mathrm{i}\frac{\sigma^{\mathrm{e}} + \sigma^{\mathrm{m}}/c^2}{\varepsilon_0\omega}\right]\mathbf{E} = \mathbf{0}
\end{aligned}
\tag{2.23}
$$

Realising that, according to Formula (2.18) on the preceding page, μ_0/ε_0 is the square of the vacuum radiation resistance R_0, and rearranging a bit, we obtain the time-independent wave equation in Dirac's symmetrised electrodynamics

$$
\nabla^2\mathbf{E} + \frac{\omega^2}{c^2}\left(1 - \frac{R_0^2}{\omega^2}\sigma^{\mathrm{m}}\sigma^{\mathrm{e}}\right)\left(1 + \mathrm{i}\frac{\sigma^{\mathrm{e}} + \sigma^{\mathrm{m}}/c^2}{\varepsilon_0\omega\left(1 - \frac{R_0^2}{\omega^2}\sigma^{\mathrm{m}}\sigma^{\mathrm{e}}\right)}\right)\mathbf{E} = \mathbf{0}
\tag{2.24}
$$

From this equation we conclude that the existence of magnetic charges (magnetic monopoles), and non-vanishing electric and magnetic conductivities would lead to a shift in the effective wave number of the wave. Furthermore, even if the electric conductivity σ^{e} vanishes, the imaginary term does not necessarily vanish and the wave might therefore experience damping (or growth) according as σ^{m} is positive (or negative). This would happen in a hypotethetical medium which is a perfect insulator for electric currents but which can carry magnetic currents.

Finally, we note that in the particular case that $\omega = R_0\sqrt{\sigma^{\mathrm{m}}\sigma^{\mathrm{e}}}$, the wave equation becomes a (time-independent) diffusion equation

$$
\nabla^2\mathbf{E} + \mathrm{i}\omega\mu_0\left(\sigma^{\mathrm{e}} + \frac{\sigma^{\mathrm{m}}}{c^2}\right)\mathbf{E} = \mathbf{0}
\tag{2.25}
$$

and, hence, no waves exist at all!

◁ END OF EXAMPLE 2.1

2.2 Plane waves

Consider now the case where all fields depend only on the distance ζ to a given plane with unit normal $\hat{\boldsymbol{n}}$. Then the *del* operator becomes

$$\boldsymbol{\nabla} = \hat{\boldsymbol{n}} \frac{\partial}{\partial \zeta} \tag{2.26}$$

and Maxwell's equations attain the form

$$\hat{\boldsymbol{n}} \cdot \frac{\partial \mathbf{E}}{\partial \zeta} = 0 \tag{2.27a}$$

$$\hat{\boldsymbol{n}} \times \frac{\partial \mathbf{E}}{\partial \zeta} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.27b}$$

$$\hat{\boldsymbol{n}} \cdot \frac{\partial \mathbf{B}}{\partial \zeta} = 0 \tag{2.27c}$$

$$\hat{\boldsymbol{n}} \times \frac{\partial \mathbf{B}}{\partial \zeta} = \mu_0 \mathbf{j}(t, \mathbf{x}) + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \sigma \mathbf{E} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \tag{2.27d}$$

Scalar multiplying (2.27d) by $\hat{\boldsymbol{n}}$, we find that

$$0 = \hat{\boldsymbol{n}} \cdot \left(\hat{\boldsymbol{n}} \times \frac{\partial \mathbf{B}}{\partial \zeta} \right) = \hat{\boldsymbol{n}} \cdot \left(\mu_0 \sigma + \varepsilon_0 \mu_0 \frac{\partial}{\partial t} \right) \mathbf{E} \tag{2.28}$$

which simplifies to the first-order ordinary differential equation for the normal component E_n of the electric field

$$\frac{\mathrm{d}E_n}{\mathrm{d}t} + \frac{\sigma}{\varepsilon_0} E_n = 0 \tag{2.29}$$

with the solution

$$E_n = E_{n_0} e^{-\sigma t/\varepsilon_0} = E_{n_0} e^{-t/\tau} \tag{2.30}$$

This, together with (2.27a), shows that the *longitudinal component* of $\mathbf{E}$, *i.e.*, the component which is perpendicular to the plane surface is independent of ζ and has a time dependence which exhibits an exponential decay, with a decrement given by the relaxation time τ in the medium.

Scalar multiplying (2.27b) by $\hat{\boldsymbol{n}}$, we similarly find that

$$0 = \hat{\boldsymbol{n}} \cdot \left(\hat{\boldsymbol{n}} \times \frac{\partial \mathbf{E}}{\partial \zeta} \right) = -\hat{\boldsymbol{n}} \cdot \frac{\partial \mathbf{B}}{\partial t} \tag{2.31}$$

or

$$\hat{\boldsymbol{n}} \cdot \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{2.32}$$

From this, and (2.27c), we conclude that the only longitudinal component of $\mathbf{B}$ must be constant in both time and space. In other words, the only non-static solution must consist of *transverse components*.

2.2.1 Telegrapher's equation

In analogy with Equation (2.7) on page 26, we can easily derive the equation

$$\frac{\partial^2 \mathbf{E}}{\partial \zeta^2} - \mu_0 \sigma \frac{\partial \mathbf{E}}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mathbf{0} \tag{2.33}$$

This equation, which describes the propagation of plane waves in a conducting medium, is called the *telegrapher's equation*. If the medium is an insulator so that $\sigma = 0$, then the equation takes the form of the *one-dimensional wave equation*

$$\frac{\partial^2 \mathbf{E}}{\partial \zeta^2} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mathbf{0} \tag{2.34}$$

As is well known, each component of this equation has a solution which can be written

$$E_i = f(\zeta - ct) + g(\zeta + ct), \quad i = 1, 2, 3 \tag{2.35}$$

where f and g are arbitrary (non-pathological) functions of their respective arguments. This general solution represents perturbations which propagate along ζ, where the f perturbation propagates in the positive ζ direction and the g perturbation propagates in the negative ζ direction.

If we assume that our electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ are time-harmonic, *i.e.*, that they can each be represented by a Fourier component proportional to $\exp\{-\mathrm{i}\omega t\}$, the solution of Equation (2.34) becomes

$$\mathbf{E} = \mathbf{E}_0 e^{-\mathrm{i}(\omega t \pm k\zeta)} = \mathbf{E}_0 e^{\mathrm{i}(\mp k\zeta - \omega t)} \tag{2.36}$$

By introducing the *wave vector*

$$\mathbf{k} = k\hat{\boldsymbol{n}} = \frac{\omega}{c}\hat{\boldsymbol{n}} = \frac{\omega}{c}\hat{\boldsymbol{k}} \tag{2.37}$$

this solution can be written as

$$\mathbf{E} = \mathbf{E}_0 e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x} - \omega t)} \tag{2.38}$$

Let us consider the lower sign in front of $k\zeta$ in the exponent in (2.36). This corresponds to a wave which propagates in the direction of increasing ζ. Inserting this solution into Equation (2.27b) on the preceding page, gives

$$\hat{\boldsymbol{n}} \times \frac{\partial \mathbf{E}}{\partial \zeta} = \mathrm{i}\omega \mathbf{B} = \mathrm{i}k\hat{\boldsymbol{n}} \times \mathbf{E} \tag{2.39}$$

or, solving for $\mathbf{B}$,

$$\mathbf{B} = \frac{k}{\omega}\hat{\boldsymbol{n}} \times \mathbf{E} = \frac{1}{\omega}\mathbf{k} \times \mathbf{E} = \frac{1}{c}\hat{\boldsymbol{k}} \times \mathbf{E} = \sqrt{\varepsilon_0\mu_0}\,\hat{\boldsymbol{n}} \times \mathbf{E} \tag{2.40}$$

Hence, to each transverse component of $\mathbf{E}$, there exists an associated magnetic field given by Equation (2.40). If $\mathbf{E}$ and/or $\mathbf{B}$ has a direction in space which is constant in time, we have a *plane polarised wave* (or *linearly polarised wave*).

2.2.2 Waves in conductive media

Assuming that our medium has a finite conductivity σ, and making the time-harmonic wave Ansatz in Equation (2.33) on the previous page, we find that the *time-independent telegrapher's equation* can be written

$$\frac{\partial^2 \mathbf{E}}{\partial \zeta^2} + \varepsilon_0\mu_0\omega^2\mathbf{E} + \mathrm{i}\mu_0\sigma\omega\mathbf{E} = \frac{\partial^2 \mathbf{E}}{\partial \zeta^2} + K^2\mathbf{E} = \mathbf{0} \tag{2.41}$$

where

$$K^2 = \varepsilon_0\mu_0\omega^2\left(1 + \mathrm{i}\frac{\sigma}{\varepsilon_0\omega}\right) = \frac{\omega^2}{c^2}\left(1 + \mathrm{i}\frac{\sigma}{\varepsilon_0\omega}\right) = k^2\left(1 + \mathrm{i}\frac{\sigma}{\varepsilon_0\omega}\right) \tag{2.42}$$

where, in the last step, Equation (2.16) on page 28 was used to introduce the wave number k. Taking the square root of this expression, we obtain

$$K = k\sqrt{1 + \mathrm{i}\frac{\sigma}{\varepsilon_0\omega}} = \alpha + \mathrm{i}\beta \tag{2.43}$$

Squaring, one finds that

$$k^2\left(1 + \mathrm{i}\frac{\sigma}{\varepsilon_0\omega}\right) = (\alpha^2 - \beta^2) + 2\mathrm{i}\alpha\beta \tag{2.44}$$

or

$$\beta^2 = \alpha^2 - k^2 \tag{2.45}$$

$$\alpha\beta = \frac{k^2\sigma}{2\varepsilon_0\omega} \tag{2.46}$$

Squaring the latter and combining with the former, one obtains the second order algebraic equation (in α^2)

$$\alpha^2(\alpha^2 - k^2) = \frac{k^4\sigma^2}{4\varepsilon_0^2\omega^2} \tag{2.47}$$

which can be easily solved and one finds that

$$\alpha = k\sqrt{\frac{\sqrt{1 + \left(\frac{\sigma}{\varepsilon_0\omega}\right)^2} + 1}{2}} \tag{2.48a}$$

$$\beta = k\sqrt{\frac{\sqrt{1 + \left(\frac{\sigma}{\varepsilon_0\omega}\right)^2} - 1}{2}} \tag{2.48b}$$

As a consequence, the solution of the time-independent telegrapher's equation, Equation (2.41) on the preceding page, can be written

$$\mathbf{E} = \mathbf{E}_0 e^{-\beta\zeta} e^{i(\alpha\zeta-\omega t)} \tag{2.49}$$

With the aid of Equation (2.40) on page 31 we can calculate the associated magnetic field, and find that it is given by

$$\mathbf{B} = \frac{1}{\omega} K \hat{\boldsymbol{k}} \times \mathbf{E} = \frac{1}{\omega} (\hat{\boldsymbol{k}} \times \mathbf{E})(\alpha + i\beta) = \frac{1}{\omega} (\hat{\boldsymbol{k}} \times \mathbf{E}) \, |A| \, e^{i\gamma} \tag{2.50}$$

where we have, in the last step, rewritten $\alpha+i\beta$ in the amplitude-phase form $|A| \exp\{i\gamma\}$. From the above, we immediately see that **E**, and consequently also **B**, is damped, and that **E** and **B** in the wave are out of phase.

In the limit $\varepsilon_0\omega \ll \sigma$, we can approximate K as follows:

$$\begin{aligned} K &= k\left(1 + i\frac{\sigma}{\varepsilon_0\omega}\right)^{\frac{1}{2}} = k\left[i\frac{\sigma}{\varepsilon_0\omega}\left(1 - i\frac{\varepsilon_0\omega}{\sigma}\right)\right]^{\frac{1}{2}} \approx k(1+i)\sqrt{\frac{\sigma}{2\varepsilon_0\omega}} \\ &= \sqrt{\varepsilon_0\mu_0}\,\omega(1+i)\sqrt{\frac{\sigma}{2\varepsilon_0\omega}} = (1+i)\sqrt{\frac{\mu_0\sigma\omega}{2}} \end{aligned} \tag{2.51}$$

In this limit we dind that when the wave impinges perpendicularly upon the medium, the fields are given, *inside* the medium, by

$$\mathbf{E}' = \mathbf{E}_0 \exp\left\{-\sqrt{\frac{\mu_0\sigma\omega}{2}}\zeta\right\} \exp\left\{i\left(\sqrt{\frac{\mu_0\sigma\omega}{2}}\zeta - \omega t\right)\right\} \tag{2.52a}$$

$$\mathbf{B}' = (1+i)\sqrt{\frac{\mu_0\sigma}{2\omega}}\,(\hat{\boldsymbol{n}} \times \mathbf{E}') \tag{2.52b}$$

Hence, both fields fall off by a factor $1/e$ at a distance

$$\delta = \sqrt{\frac{2}{\mu_0\sigma\omega}} \tag{2.53}$$

This distance δ is called the *skin depth*.

2.3 Observables and averages

In the above we have used *complex notation* quite extensively. This is for mathematical convenience only. For instance, in this notation differentiations are almost trivial to perform. However, every *physical measurable* quantity is always real valued. *I.e.*, '$\mathbf{E}_{\text{physical}} = \text{Re}\{\mathbf{E}_{\text{mathematical}}\}$'. It is particularly important to remember this when one works with products of physical quantities. For instance, if we have two physical vectors **F** and **G** which both are time-harmonic, *i.e.*, can be represented by Fourier components proportional to $\exp\{-i\omega t\}$, then we must make the following interpretation

$$\mathbf{F}(t,\mathbf{x})\cdot\mathbf{G}(t,\mathbf{x}) = \mathrm{Re}\,\{\mathbf{F}\}\cdot\mathrm{Re}\,\{\mathbf{G}\} = \mathrm{Re}\left\{\mathbf{F}_0(\mathbf{x})\,e^{-i\omega t}\right\}\cdot\mathrm{Re}\left\{\mathbf{G}_0(\mathbf{x})\,e^{-i\omega t}\right\} \tag{2.54}$$

Furthermore, letting * denote complex conjugate, we can express the real part of the complex vector **F** as

$$\mathrm{Re}\,\{\mathbf{F}\} = \mathrm{Re}\left\{\mathbf{F}_0(\mathbf{x})\,e^{-i\omega t}\right\} = \frac{1}{2}[\mathbf{F}_0(\mathbf{x})\,e^{-i\omega t} + \mathbf{F}_0^*(\mathbf{x})\,e^{i\omega t}] \tag{2.55}$$

and similarly for **G**. Hence, the physically acceptable interpretation of the scalar product of two complex vectors, representing physical observables, is

$$\begin{aligned}
\mathbf{F}(t,\mathbf{x})\cdot\mathbf{G}(t,\mathbf{x}) &= \mathrm{Re}\left\{\mathbf{F}_0(\mathbf{x})\,e^{-i\omega t}\right\}\cdot\mathrm{Re}\left\{\mathbf{G}_0(\mathbf{x})\,e^{-i\omega t}\right\}\\
&= \frac{1}{2}[\mathbf{F}_0(\mathbf{x})\,e^{-i\omega t} + \mathbf{F}_0^*(\mathbf{x})\,e^{i\omega t}]\cdot\frac{1}{2}[\mathbf{G}_0(\mathbf{x})\,e^{-i\omega t} + \mathbf{G}_0^*(\mathbf{x})\,e^{i\omega t}]\\
&= \frac{1}{4}\left(\mathbf{F}_0\cdot\mathbf{G}_0^* + \mathbf{F}_0^*\cdot\mathbf{G}_0 + \mathbf{F}_0\cdot\mathbf{G}_0\,e^{-2i\omega t} + \mathbf{F}_0^*\cdot\mathbf{G}_0^*\,e^{2i\omega t}\right)\\
&= \frac{1}{2}\mathrm{Re}\left\{\mathbf{F}_0\cdot\mathbf{G}_0^* + \mathbf{F}_0\cdot\mathbf{G}_0\,e^{-2i\omega t}\right\}\\
&= \frac{1}{2}\mathrm{Re}\left\{\mathbf{F}_0\,e^{-i\omega t}\cdot\mathbf{G}_0^*\,e^{i\omega t} + \mathbf{F}_0\cdot\mathbf{G}_0\,e^{-2i\omega t}\right\}\\
&= \frac{1}{2}\mathrm{Re}\left\{\mathbf{F}(t,\mathbf{x})\cdot\mathbf{G}^*(t,\mathbf{x}) + \mathbf{F}_0\cdot\mathbf{G}_0\,e^{-2i\omega t}\right\}
\end{aligned} \tag{2.56}$$

Often in physics, we measure temporal averages ($\langle\,\rangle$) of our physical observables. If so, we see that the average of the product of the two physical quantities represented by **F** and **G** can be expressed as

$$\langle\mathbf{F}\cdot\mathbf{G}\rangle \equiv \langle\mathrm{Re}\,\{\mathbf{F}\}\cdot\mathrm{Re}\,\{\mathbf{G}\}\rangle = \frac{1}{2}\mathrm{Re}\,\{\mathbf{F}\cdot\mathbf{G}^*\} = \frac{1}{2}\mathrm{Re}\,\{\mathbf{F}^*\cdot\mathbf{G}\} \tag{2.57}$$

since the temporal average of the oscillating function $\exp\{-2i\omega t\}$ vanishes.

2.4 Bibliography

[1] J. D. JACKSON, *Classical Electrodynamics*, third ed., John Wiley & Sons, Inc., New York, NY …, 1999, ISBN 0-471-30932-X.

[2] W. K. H. PANOFSKY AND M. PHILLIPS, *Classical Electricity and Magnetism*, second ed., Addison-Wesley Publishing Company, Inc., Reading, MA …, 1962, ISBN 0-201-05702-6.

CHAPTER 3

Electromagnetic Potentials

Instead of expressing the laws of electrodynamics in terms of electric and magnetic fields, it turns out that it is often more convenient to express the theory in terms of potentials. This is particularly true for problems related to radiation. In this chapter we will introduce and study the properties of such potentials and shall find that they exhibit some remarkable properties which elucidate the fundamental apects of electromagnetism and lead naturally to the special theory of relativity.

3.1 The electrostatic scalar potential

As we saw in Equation (1.8) on page 5, the electrostatic field $\mathbf{E}^{\mathrm{stat}}(\mathbf{x})$ is irrotational. Hence, it may be expressed in terms of the gradient of a scalar field. If we denote this scalar field by $-\phi^{\mathrm{stat}}(\mathbf{x})$, we get

$$\mathbf{E}^{\mathrm{stat}}(\mathbf{x}) = -\boldsymbol{\nabla}\phi^{\mathrm{stat}}(\mathbf{x}) \tag{3.1}$$

Taking the divergence of this and using Equation (1.7) on page 5, we obtain *Poisson's equation*

$$\nabla^2\phi^{\mathrm{stat}}(\mathbf{x}) = -\boldsymbol{\nabla}\cdot\mathbf{E}^{\mathrm{stat}}(\mathbf{x}) = -\frac{\rho(\mathbf{x})}{\varepsilon_0} \tag{3.2}$$

A comparison with the definition of $\mathbf{E}^{\mathrm{stat}}$, namely Equation (1.5) on page 4, shows that this equation has the solution

$$\phi^{\mathrm{stat}}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}^3x' + \alpha \tag{3.3}$$

where the integration is taken over all source points $\mathbf{x}'$ at which the charge density $\rho(\mathbf{x}')$ is non-zero and α is an arbitrary quantity which has a vanishing gradient. An example of such a quantity is a scalar constant. The scalar function $\phi^{\text{stat}}(\mathbf{x})$ in Equation (3.3) on the preceding page is called the *electrostatic scalar potential*.

3.2 The magnetostatic vector potential

Consider the equations of magnetostatics (1.22) on page 9. From Equation (F.63) on page 160 we know that any 3D vector $\mathbf{a}$ has the property that $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{a}) \equiv 0$ and in the derivation of Equation (1.17) on page 8 in magnetostatics we found that $\boldsymbol{\nabla} \cdot \mathbf{B}^{\text{stat}}(\mathbf{x}) = 0$. We therefore realise that we can always write

$$\mathbf{B}^{\text{stat}}(\mathbf{x}) = \boldsymbol{\nabla} \times \mathbf{A}^{\text{stat}}(\mathbf{x}) \tag{3.4}$$

where $\mathbf{A}^{\text{stat}}(\mathbf{x})$ is called the *magnetostatic vector potential*.

We saw above that the electrostatic potential (as any scalar potential) is not unique: we may, without changing the physics, add to it a quantity whose spatial gradient vanishes. A similar arbitrariness is true also for the magnetostatic vector potential.

In the magnetostatic case, we may start from Biot-Savart's law as expressed by Equation (1.15) on page 7. Identifying this expression with Equation (3.4) allows us to define the static vector potential as

$$\mathbf{A}^{\text{stat}}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{j}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, \mathrm{d}^3x' + \mathbf{a}(\mathbf{x}) \tag{3.5}$$

where $\mathbf{a}(\mathbf{x})$ is an arbitrary vector field whose curl vanishes. From Equation (F.62) on page 160 we know that such a vector can always be written as the gradient of a scalar field.

3.3 The electrodynamic potentials

Let us now generalise the static analysis above to the electrodynamic case, *i.e.*, the case with temporal and spatial dependent sources $\rho(t, \mathbf{x})$ and $\mathbf{j}(t, \mathbf{x})$, and corresponding fields $\mathbf{E}(t, \mathbf{x})$ and $\mathbf{B}(t, \mathbf{x})$, as described by Maxwell's equations (1.45) on page 14. In other words, let us study the *electrodynamic potentials* $\phi(t, \mathbf{x})$ and $\mathbf{A}(t, \mathbf{x})$.

From Equation (1.45c) on page 14 we note that also in electrodynamics the homogeneous equation $\boldsymbol{\nabla} \cdot \mathbf{B}(t, \mathbf{x}) = 0$ remains valid. Because of this divergence-free nature of the time- and space-dependent magnetic field, we can express it as the curl of an *electromagnetic vector potential*:

$$\mathbf{B}(t, \mathbf{x}) = \boldsymbol{\nabla} \times \mathbf{A}(t, \mathbf{x}) \tag{3.6}$$

Inserting this expression into the other homogeneous Maxwell equation (1.32) on page 12, we obtain

$$\nabla \times \mathbf{E}(t, \mathbf{x}) = -\frac{\partial}{\partial t} [\nabla \times \mathbf{A}(t, \mathbf{x})] = -\nabla \times \frac{\partial}{\partial t} \mathbf{A}(t, \mathbf{x}) \tag{3.7}$$

or, rearranging the terms,

$$\nabla \times \left(\mathbf{E}(t, \mathbf{x}) + \frac{\partial}{\partial t} \mathbf{A}(t, \mathbf{x}) \right) = \mathbf{0} \tag{3.8}$$

As before we utilise the vanishing curl of a vector expression to write this vector expression as the gradient of a scalar function. If, in analogy with the electrostatic case, we introduce the *electromagnetic scalar potential* function $-\phi(t, \mathbf{x})$, Equation (3.8) becomes equivalent to

$$\mathbf{E}(t, \mathbf{x}) + \frac{\partial}{\partial t} \mathbf{A}(t, \mathbf{x}) = -\nabla \phi(t, \mathbf{x}) \tag{3.9}$$

This means that in electrodynamics, $\mathbf{E}(t, \mathbf{x})$ can be calculated from the formula

$$\mathbf{E}(t, \mathbf{x}) = -\nabla \phi(t, \mathbf{x}) - \frac{\partial}{\partial t} \mathbf{A}(t, \mathbf{x}) \tag{3.10}$$

and $\mathbf{B}(t, \mathbf{x})$ from Equation (3.6) on the preceding page. Hence, it is a matter of taste whether we want to express the laws of electrodynamics in terms of the potentials $\phi(t, \mathbf{x})$ and $\mathbf{A}(t, \mathbf{x})$, or in terms of the fields $\mathbf{E}(t, \mathbf{x})$ and $\mathbf{B}(t, \mathbf{x})$. However, there exists an important difference between the two approaches: in classical electrodynamics the only directly observable quantities are the fields themselves (and quantities derived from them) and not the potentials. On the other hand, the treatment becomes significantly simpler if we use the potentials in our calculations and then, at the final stage, use Equation (3.6) on the facing page and Equation (3.10) above to calculate the fields or physical quantities expressed in the fields.

Inserting (3.10) and (3.6) on the facing page into Maxwell's equations (1.45) on page 14 we obtain, after some simple algebra and the use of Equation (1.11) on page 6, the *general inhomogeneous wave equations*

$$\nabla^2 \phi = -\frac{\rho(t, \mathbf{x})}{\varepsilon_0} - \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) \tag{3.11a}$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla(\nabla \cdot \mathbf{A}) = -\mu_0 \mathbf{j}(t, \mathbf{x}) + \frac{1}{c^2} \nabla \frac{\partial \phi}{\partial t} \tag{3.11b}$$

which can be rewritten in the following, more symmetric, form

$$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = \frac{\rho(t, \mathbf{x})}{\varepsilon_0} + \frac{\partial}{\partial t} \left(\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} \right) \tag{3.12a}$$

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \mu_0 \mathbf{j}(t, \mathbf{x}) - \nabla \left(\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} \right) \tag{3.12b}$$

These two second order, coupled, partial differential equations, representing in all four scalar equations (one for ϕ and one each for the three components A_i, $i = 1, 2, 3$ of $\mathbf{A}$) are completely equivalent to the formulation of electrodynamics in terms of Maxwell's equations, which represent eight scalar first-order, coupled, partial differential equations.

As they stand, Equations (3.11) on the preceding page and Equations (3.12) on the previous page look complicated and may seem to be of limited use. However, if we write Equation (3.6) on page 36 in the form $\boldsymbol{\nabla} \times \mathbf{A}(t, \mathbf{x}) = \mathbf{B}(t, \mathbf{x})$ we can consider this as a specification of $\boldsymbol{\nabla} \times \mathbf{A}$. But we know from *Helmholtz' theorem* that in order to determine the (spatial) behaviour of $\mathbf{A}$ completely, we must also specify $\boldsymbol{\nabla} \cdot \mathbf{A}$. Since this divergence does not enter the derivation above, *we are free to choose* $\boldsymbol{\nabla} \cdot \mathbf{A}$ *in whatever way we like and still obtain the same physical results*!

3.3.1 Lorenz-Lorentz gauge

If we choose $\boldsymbol{\nabla} \cdot \mathbf{A}$ to fulfil the so called *Lorenz-Lorentz gauge condition*[1]

$$\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0 \tag{3.13}$$

the coupled inhomegeneous wave Equation (3.12) on page 37 simplify into the following set of *uncoupled inhomogeneous wave equations*:

$$\Box^2 \phi \overset{\text{def}}{\equiv} \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \phi = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = \frac{\rho(t, \mathbf{x})}{\varepsilon_0} \tag{3.14a}$$

$$\Box^2 \mathbf{A} \overset{\text{def}}{\equiv} \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \mathbf{A} = \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \mu_0 \mathbf{j}(t, \mathbf{x}) \tag{3.14b}$$

where $\Box^2$ is the *d'Alembert operator* discussed in Example 13.5 on page 175. Each of these four scalar equations is an *inhomogeneous wave equation* of the following generic form:

$$\Box^2 \Psi(t, \mathbf{x}) = f(t, \mathbf{x}) \tag{3.15}$$

where Ψ is a shorthand for either ϕ or one of the components A_i of the vector potential $\mathbf{A}$, and f is the pertinent generic source component, $\rho(t, \mathbf{x})/\varepsilon_0$ or $\mu_0 j_i(t, \mathbf{x})$, respectively.

[1] In fact, the Dutch physicist Hendrik Antoon Lorentz, who in 1903 demonstrated the covariance of Maxwell's equations, was not the original discoverer of this condition. It had been discovered by the Danish physicist Ludvig V. Lorenz already in 1867 [5]. In the literature, this fact has sometimes been overlooked and the condition was earlier referred to as the *Lorentz gauge condition*.

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We assume that our sources are well-behaved enough in time t so that the *Fourier transform* pair for the generic source function

$$\mathcal{F}^{-1}[f_\omega(\mathbf{x})] \stackrel{\text{def}}{\equiv} f(t,\mathbf{x}) = \int_{-\infty}^{\infty} \mathrm{d}\omega\, f_\omega(\mathbf{x})\, e^{-\mathrm{i}\omega t} \tag{3.16a}$$

$$\mathcal{F}[f(t,\mathbf{x})] \stackrel{\text{def}}{\equiv} f_\omega(\mathbf{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t\, f(t,\mathbf{x})\, e^{\mathrm{i}\omega t} \tag{3.16b}$$

exists, and that the same is true for the generic potential component:

$$\Psi(t,\mathbf{x}) = \int_{-\infty}^{\infty} \mathrm{d}\omega\, \Psi_\omega(\mathbf{x})\, e^{-\mathrm{i}\omega t} \tag{3.17a}$$

$$\Psi_\omega(\mathbf{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t\, \Psi(t,\mathbf{x})\, e^{\mathrm{i}\omega t} \tag{3.17b}$$

Inserting the Fourier representations (3.16a) and (3.17a) into Equation (3.15) on the facing page, and using the vacuum dispersion relation for electromagnetic waves

$$\omega = ck \tag{3.18}$$

the generic 3D inhomogeneous wave equation, Equation (3.15) on the preceding page, turns into

$$\nabla^2 \Psi_\omega(\mathbf{x}) + k^2 \Psi_\omega(\mathbf{x}) = -f_\omega(\mathbf{x}) \tag{3.19}$$

which is a 3D *inhomogeneous time-independent wave equation*, often called the 3D *inhomogeneous Helmholtz equation*.

As postulated by *Huygen's principle*, each point on a wave front acts as a point source for spherical waves which form a new wave from a superposition of the individual waves from each of the point sources on the old wave front. The solution of (3.19) can therefore be expressed as a superposition of solutions of an equation where the source term has been replaced by a point source:

$$\nabla^2 G(\mathbf{x},\mathbf{x}') + k^2 G(\mathbf{x},\mathbf{x}') = -\delta(\mathbf{x}-\mathbf{x}') \tag{3.20}$$

and the solution of Equation (3.19) which corresponds to the frequency ω is given by the superposition

$$\Psi_\omega(\mathbf{x}) = \int_{V'} \mathrm{d}^3x'\, f_\omega(\mathbf{x}') G(\mathbf{x},\mathbf{x}') \tag{3.21}$$

The function $G(\mathbf{x},\mathbf{x}')$ is called the *Green function* or the *propagator*.

In Equation (3.20) above, the Dirac generalised function $\delta(\mathbf{x}-\mathbf{x}')$, which represents the point source, depends only on $\mathbf{x}-\mathbf{x}'$ and there is no angular dependence in the equation. Hence, the solution can only be dependent on $r = |\mathbf{x}-\mathbf{x}'|$ and not on the direction of $\mathbf{x}-\mathbf{x}'$. If we interpret r as the radial coordinate in a spherically polar coordinate system, and recall the expression for the Laplace operator in such a coordinate system, Equation (3.20) on the preceding page becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}(rG) + k^2(rG) = -r\delta(r) \tag{3.22}$$

Away from $r = |\mathbf{x} - \mathbf{x}'| = 0$, *i.e.*, away from the source point $\mathbf{x}'$, this equation takes the form

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}(rG) + k^2(rG) = 0 \tag{3.23}$$

with the well-known general solution

$$G = C^+ \frac{e^{\mathrm{i}kr}}{r} + C^- \frac{e^{-\mathrm{i}kr}}{r} \equiv C^+ \frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} + C^- \frac{e^{-\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \tag{3.24}$$

where $C^{\pm}$ are constants.

In order to evaluate the constants $C^{\pm}$, we insert the general solution, Equation (3.24) above, into Equation (3.20) on the preceding page and integrate over a small volume around $r = |\mathbf{x} - \mathbf{x}'| = 0$. Since

$$G(\left|\mathbf{x}-\mathbf{x}'\right|) \sim C^+ \frac{1}{|\mathbf{x}-\mathbf{x}'|} + C^- \frac{1}{|\mathbf{x}-\mathbf{x}'|}, \qquad \left|\mathbf{x}-\mathbf{x}'\right| \to 0 \tag{3.25}$$

The volume integrated Equation (3.20) on the previous page can under this assumption be approximated by

$$\begin{aligned}\left(C^+ + C^-\right) \int_{V'} \mathrm{d}^3x'\; \nabla^2 \left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) + k^2 \left(C^+ + C^-\right) \int_{V'} \mathrm{d}^3x'\; \frac{1}{|\mathbf{x}-\mathbf{x}'|} \\ = -\int_{V'} \mathrm{d}^3x'\; \delta(\left|\mathbf{x}-\mathbf{x}'\right|)\end{aligned} \tag{3.26}$$

In virtue of the fact that the volume element d^3x' in spherical polar coordinates is proportional to $|\mathbf{x} - \mathbf{x}'|^2$, the second integral vanishes when $|\mathbf{x} - \mathbf{x}'| \to 0$. Furthermore, from Equation (F.73) on page 161, we find that the integrand in the first integral can be written as $-4\pi\delta(|\mathbf{x} - \mathbf{x}'|)$ and, hence, that

$$C^+ + C^- = \frac{1}{4\pi} \tag{3.27}$$

Insertion of the general solution Equation (3.24) into Equation (3.21) on the previous page gives

$$\Psi_\omega(\mathbf{x}) = C^+ \int_{V'} \mathrm{d}^3x'\; f_\omega(\mathbf{x}') \frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} + C^- \int_{V'} \mathrm{d}^3x'\; f_\omega(\mathbf{x}') \frac{e^{-\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \tag{3.28}$$

The inverse Fourier transform of this back to the t domain is obtained by inserting the above expression for $\Psi_\omega(\mathbf{x})$ into Equation (3.17a) on page 39:

$$\begin{aligned}\Psi(t,\mathbf{x}) &= C^+ \int_{V'} \mathrm{d}^3x' \int_{-\infty}^{\infty} \mathrm{d}\omega\; f_\omega(\mathbf{x}') \frac{\exp\left[-\mathrm{i}\omega\left(t - \frac{k|\mathbf{x}-\mathbf{x}'|}{\omega}\right)\right]}{|\mathbf{x}-\mathbf{x}'|} \\ &+ C^- \int_{V'} \mathrm{d}^3x' \int_{-\infty}^{\infty} \mathrm{d}\omega\; f_\omega(\mathbf{x}') \frac{\exp\left[-\mathrm{i}\omega\left(t + \frac{k|\mathbf{x}-\mathbf{x}'|}{\omega}\right)\right]}{|\mathbf{x}-\mathbf{x}'|}\end{aligned} \tag{3.29}$$

If we introduce the *retarded time* t'_{ret} and the *advanced time* t'_{adv} in the following way [using the fact that in vacuum $k/\omega = 1/c$, according to Equation (3.18) on page 39]:

$$t'_{\mathrm{ret}} = t'_{\mathrm{ret}}(t, \left|\mathbf{x}-\mathbf{x}'\right|) = t - \frac{k\,|\mathbf{x}-\mathbf{x}'|}{\omega} = t - \frac{|\mathbf{x}-\mathbf{x}'|}{c} \tag{3.30a}$$

$$t'_{\mathrm{adv}} = t'_{\mathrm{adv}}(t, \left|\mathbf{x}-\mathbf{x}'\right|) = t + \frac{k\,|\mathbf{x}-\mathbf{x}'|}{\omega} = t + \frac{|\mathbf{x}-\mathbf{x}'|}{c} \tag{3.30b}$$

and use Equation (3.16a) on page 39, we obtain

$$\Psi(t,\mathbf{x}) = C^+ \int_{V'} \mathrm{d}^3x'\; \frac{f(t'_{\mathrm{ret}},\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + C^- \int_{V'} \mathrm{d}^3x'\; \frac{f(t'_{\mathrm{adv}},\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \tag{3.31}$$

This is a solution to the generic inhomogeneous wave equation for the potential components Equation (3.15) on page 38. We note that the solution at time t at the field point $\mathbf{x}$ is dependent on the behaviour at other times t' of the source at $\mathbf{x}'$ and that both retarded and advanced t' are mathematically acceptable solutions. However, if we assume that causality requires that the potential at $(t,\mathbf{x})$ is set up by the source at an earlier time, *i.e.*, at $(t'_{\mathrm{ret}},\mathbf{x}')$, we must in Equation (3.31) set $C^- = 0$ and therefore, according to Equation (3.27) on the preceding page, $C^+ = 1/(4\pi)$.[2]

The retarded potentials

From the above discussion on the solution of the inhomogeneous wave equations in the Lorenz-Lorentz gauge we conclude that, under the assumption of causality, the electrodynamic potentials in vacuum can be written

$$\phi(t,\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \mathrm{d}^3x'\; \frac{\rho(t'_{\mathrm{ret}},\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \tag{3.32a}$$

$$\mathbf{A}(t,\mathbf{x}) = \frac{\mu_0}{4\pi} \int_{V'} \mathrm{d}^3x'\; \frac{\mathbf{j}(t'_{\mathrm{ret}},\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \tag{3.32b}$$

Since these *retarded potentials* were obtained as solutions to the Lorenz-Lorentz equations (3.14) on page 38 they are valid in the Lorenz-Lorentz gauge but may be gauge transformed according to the scheme described in subsection 3.3.3 on the next page. As they stand, we shall use them frequently in the following.

[2] In fact, inspired by a discussion by Paul A. M. Dirac, John A. Wheeler and Richard P. Feynman derived in 1945 a fully self-consistent electrodynamics using both the retarded and the advanced potentials [7]; see also [3].

3.3.2 Coulomb gauge

In *Coulomb gauge*, often employed in *quantum electrodynamics*, one chooses $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$ so that Equations (3.11) on page 37 or Equations (3.12) on page 37 become

$$\nabla^2 \phi = -\frac{\rho(t, \mathbf{x})}{\varepsilon_0} \tag{3.33}$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{j}(t, \mathbf{x}) + \frac{1}{c^2} \boldsymbol{\nabla} \frac{\partial \phi}{\partial t} \tag{3.34}$$

The first of these two is the *time-dependent Poisson's equation* which, in analogy with Equation (3.3) on page 35, has the solution

$$\phi(t, \mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \mathrm{d}^3x' \, \frac{\rho(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + \alpha \tag{3.35}$$

where α has vanishing gradient. We note that in the scalar potential expression the charge density source is evaluated at time t. The retardation (and advancement) effects occur only in the vector potential, which is the solution of the inhomogeneous wave equation

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{j} + \frac{\mu_0}{4\pi} \boldsymbol{\nabla} \frac{\partial}{\partial t} \int_{V'} \mathrm{d}^3x' \, \frac{\rho(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \tag{3.36}$$

Other useful gauges are

- The *temporal gauge*, also known as the *Hamilton gauge*, defined by $\phi = 0$.
- The *axial gauge*, defined by $A_3 = 0$.

3.3.3 Gauge transformations

We saw in Section 3.1 on page 35 and in Section 3.2 on page 36 that in electrostatics and magnetostatics we have a certain *mathematical* degree of freedom, up to terms of vanishing gradients and curls, to pick suitable forms for the potentials and still get the same *physical* result. In fact, the way the electromagnetic scalar potential $\phi(t, \mathbf{x})$ and the vector potential $\mathbf{A}(t, \mathbf{x})$ are related to the physically observables gives leeway for similar 'manipulation' of them also in electrodynamics. If we transform $\phi(t, \mathbf{x})$ and $\mathbf{A}(t, \mathbf{x})$ *simultaneously* into new ones $\phi'(t, \mathbf{x})$ and $\mathbf{A}'(t, \mathbf{x})$ according to the mapping scheme

$$\phi(t, \mathbf{x}) \mapsto \phi'(t, \mathbf{x}) = \phi(t, \mathbf{x}) + \frac{\partial \Gamma(t, \mathbf{x})}{\partial t} \tag{3.37a}$$

$$\mathbf{A}(t, \mathbf{x}) \mapsto \mathbf{A}'(t, \mathbf{x}) = \mathbf{A}(t, \mathbf{x}) - \boldsymbol{\nabla}\Gamma(t, \mathbf{x}) \tag{3.37b}$$

where $\Gamma(t, \mathbf{x})$ is an arbitrary, differentiable scalar function called the *gauge function*, and insert the transformed potentials into Equation (3.10) on page 37 for the electric field and into Equation (3.6) on page 36 for the magnetic field, we obtain the transformed fields

$$\mathbf{E}' = -\boldsymbol{\nabla}\phi' - \frac{\partial \mathbf{A}'}{\partial t} = -\boldsymbol{\nabla}\phi - \frac{\partial(\boldsymbol{\nabla}\Gamma)}{\partial t} - \frac{\partial \mathbf{A}}{\partial t} + \frac{\partial(\boldsymbol{\nabla}\Gamma)}{\partial t} = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t} \tag{3.38a}$$

$$\mathbf{B}' = \boldsymbol{\nabla} \times \mathbf{A}' = \boldsymbol{\nabla} \times \mathbf{A} - \boldsymbol{\nabla} \times (\boldsymbol{\nabla}\Gamma) = \boldsymbol{\nabla} \times \mathbf{A} \tag{3.38b}$$

where, once again Equation (F.62) on page 160 was used. We see that the fields are unaffected by the gauge transformation (3.37). A transformation of the potentials ϕ and $\mathbf{A}$ which leaves the fields, and hence Maxwell's equations, invariant is called a *gauge transformation*. A physical law which does not change under a gauge transformation is said to be *gauge invariant*. By definition, the fields themselves are, of course, gauge invariant.

The potentials $\phi(t, \mathbf{x})$ and $\mathbf{A}(t, \mathbf{x})$ calculated from (3.11a) on page 37, with an arbitrary choice of $\boldsymbol{\nabla} \cdot \mathbf{A}$, can be further gauge transformed according to (3.37) on the preceding page. If, in particular, we choose $\boldsymbol{\nabla} \cdot \mathbf{A}$ according to the Lorenz-Lorentz condition, Equation (3.13) on page 38, and apply the gauge transformation (3.37) on the resulting Lorenz-Lorentz potential equations (3.14) on page 38, these equations will be transformed into

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2\phi + \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial^2 \Gamma}{\partial t^2} - \nabla^2\Gamma\right) = \frac{\rho(t, \mathbf{x})}{\varepsilon_0} \tag{3.39a}$$

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2\mathbf{A} - \boldsymbol{\nabla}\left(\frac{1}{c^2}\frac{\partial^2 \Gamma}{\partial t^2} - \nabla^2\Gamma\right) = \mu_0 \mathbf{j}(t, \mathbf{x}) \tag{3.39b}$$

We notice that if we require that the gauge function $\Gamma(t, \mathbf{x})$ itself be restricted to fulfil the wave equation

$$\frac{1}{c^2}\frac{\partial^2 \Gamma}{\partial t^2} - \nabla^2\Gamma = 0 \tag{3.40}$$

these transformed Lorenz-Lorentz equations will keep their original form. The set of potentials which have been gauge transformed according to Equation (3.37) on the facing page with a gauge function $\Gamma(t, \mathbf{x})$ restricted to fulfil Equation (3.40) above, or, in other words, those gauge transformed potentials for which the Lorenz-Lorentz equations (3.14) are invariant, comprises the *Lorenz-Lorentz gauge*.

The process of choosing a particular gauge condition is referred to as *gauge fixing*.

▷ELECTROMAGNETODYNAMIC POTENTIALS — EXAMPLE 3.1

In Dirac's symmetrised form of electrodynamics (electromagnetodynamics), Maxwell's equations are replaced by [see also Equations (1.50) on page 16]:

$$\nabla \cdot \mathbf{E} = \frac{\rho^{\mathrm{e}}}{\varepsilon_0} \tag{3.41a}$$

$$\nabla \times \mathbf{E} = -\mu_0 \mathbf{j}^{\mathrm{m}} - \frac{\partial \mathbf{B}}{\partial t} \tag{3.41b}$$

$$\nabla \cdot \mathbf{B} = \mu_0 \rho^{\mathrm{m}} \tag{3.41c}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}^{\mathrm{e}} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \tag{3.41d}$$

In this theory, one derives the inhomogeneous wave equations for the usual 'electric' scalar and vector potentials $(\phi^{\mathrm{e}}, \mathbf{A}^{\mathrm{e}})$ and their 'magnetic' counterparts $(\phi^{\mathrm{m}}, \mathbf{A}^{\mathrm{m}})$ by assuming that the potentials are related to the fields in the following symmetrised form:

$$\mathbf{E} = -\nabla \phi^{\mathrm{e}}(t, \mathbf{x}) - \frac{\partial}{\partial t} \mathbf{A}^{\mathrm{e}}(t, \mathbf{x}) - \nabla \times \mathbf{A}^{\mathrm{m}} \tag{3.42a}$$

$$\mathbf{B} = -\frac{1}{c^2} \nabla \phi^{\mathrm{m}}(t, \mathbf{x}) - \frac{1}{c^2} \frac{\partial}{\partial t} \mathbf{A}^{\mathrm{m}}(t, \mathbf{x}) + \nabla \times \mathbf{A}^{\mathrm{e}} \tag{3.42b}$$

In the absence of magnetic charges, or, equivalenty for $\phi^{\mathrm{m}} \equiv 0$ and $\mathbf{A}^{\mathrm{m}} \equiv \mathbf{0}$, these formulae reduce to the usual Maxwell theory Formula (3.10) on page 37 and Formula (3.6) on page 36, respectively, as they should.

Inserting the symmetrised expressions (3.42) above into Equations (3.41), one obtains [*cf.*, Equations (3.11a) on page 37]

$$\nabla^2 \phi^{\mathrm{e}} + \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}^{\mathrm{e}}) = -\frac{\rho^{\mathrm{e}}(t, \mathbf{x})}{\varepsilon_0} \tag{3.43a}$$

$$\nabla^2 \phi^{\mathrm{m}} + \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}^{\mathrm{m}}) = -\frac{\rho^{\mathrm{m}}(t, \mathbf{x})}{\varepsilon_0} \tag{3.43b}$$

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{A}^{\mathrm{e}}}{\partial t^2} - \nabla^2 \mathbf{A}^{\mathrm{e}} + \nabla \left(\nabla \cdot \mathbf{A}^{\mathrm{e}} + \frac{1}{c^2} \frac{\partial \phi^{\mathrm{e}}}{\partial t} \right) = \mu_0 \mathbf{j}^{\mathrm{e}}(t, \mathbf{x}) \tag{3.43c}$$

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{A}^{\mathrm{m}}}{\partial t^2} - \nabla^2 \mathbf{A}^{\mathrm{m}} + \nabla \left(\nabla \cdot \mathbf{A}^{\mathrm{m}} + \frac{1}{c^2} \frac{\partial \phi^{\mathrm{m}}}{\partial t} \right) = \mu_0 \mathbf{j}^{\mathrm{m}}(t, \mathbf{x}) \tag{3.43d}$$

By choosing the conditions on the vector potentials according to the Lorenz-Lorentz prescripton [*cf.*, Equation (3.13) on page 38]

$$\nabla \cdot \mathbf{A}^{\mathrm{e}} + \frac{1}{c^2} \frac{\partial}{\partial t} \phi^{\mathrm{e}} = 0 \tag{3.44}$$

$$\nabla \cdot \mathbf{A}^{\mathrm{m}} + \frac{1}{c^2} \frac{\partial}{\partial t} \phi^{\mathrm{m}} = 0 \tag{3.45}$$

these coupled wave equations simplify to

$$\frac{1}{c^2} \frac{\partial^2 \phi^{\mathrm{e}}}{\partial t^2} - \nabla^2 \phi^{\mathrm{e}} = \frac{\rho^{\mathrm{e}}(t, \mathbf{x})}{\varepsilon_0} \tag{3.46a}$$

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{A}^{\mathrm{e}}}{\partial t^2} - \nabla^2 \mathbf{A}^{\mathrm{e}} = \mu_0 \mathbf{j}^{\mathrm{e}}(t, \mathbf{x}) \tag{3.46b}$$

$$\frac{1}{c^2} \frac{\partial^2 \phi^{\mathrm{m}}}{\partial t^2} - \nabla^2 \phi^{\mathrm{m}} = \frac{\rho^{\mathrm{m}}(t, \mathbf{x})}{\varepsilon_0} \tag{3.46c}$$

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{A}^{\mathrm{m}}}{\partial t^2} - \nabla^2 \mathbf{A}^{\mathrm{m}} = \mu_0 \mathbf{j}^{\mathrm{m}}(t, \mathbf{x}) \tag{3.46d}$$

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exhibiting once again, the striking properties of Dirac's symmetrised Maxwell theory.

◁ END OF EXAMPLE 3.1

3.4 Bibliography

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CHAPTER 4

Relativistic Electrodynamics

We saw in Chapter 3 how the derivation of the electrodynamic potentials led, in a most natural way, to the introduction of a characteristic, finite speed of propagation in vacuum that equals the speed of light $c = 1/\sqrt{\varepsilon_0\mu_0}$ and which can be considered as a constant of nature. To take this finite speed of propagation of information into account, and to ensure that our laws of physics be independent of any specific coordinate frame, requires a treatment of electrodynamics in a relativistically covariant (coordinate independent) form. This is the object of this chapter.

4.1 The special theory of relativity

An *inertial system*, or *inertial reference frame*, is a system of reference, or rigid coordinate system, in which the *law of inertia* (*Galileo's law*, *Newton's first law*) holds. In other words, an inertial system is a system in which free bodies move uniformly and do not experience any acceleration. The *special theory of relativity*[1] describes

[1] *The Special Theory of Relativity*, by the American physicist and philosopher David Bohm, opens with the following paragraph [4]:

> 'The theory of relativity is not merely a scientific development of great importance in its own right. It is even more significant as the first stage of a radical change in our basic concepts, which began in physics, and which is spreading into other fields of science, and indeed, even into a great deal of thinking outside of science. For as is well known, the modern trend is away from the notion of sure 'absolute' truth, (*i.e.*, one which holds independently of all conditions, contexts, degrees, and types of approximation *etc.*.) and toward the idea that a given concept has significance only in relation to suitable broader forms of reference, within which that concept can be given its full meaning.'

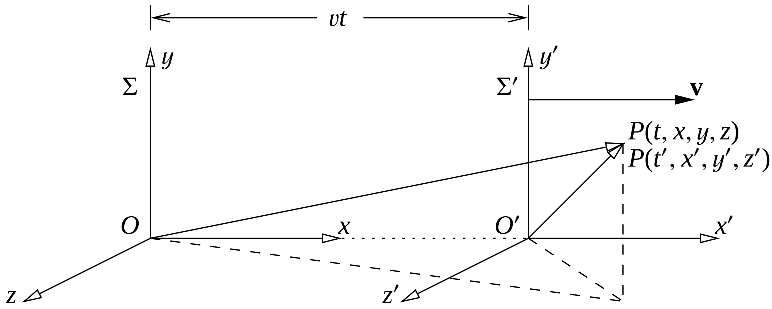

FIGURE 4.1: Two inertial systems Σ and Σ' in relative motion with velocity $\mathbf{v}$ along the $x = x'$ axis. At time $t = t' = 0$ the origin O' of Σ' coincided with the origin O of Σ. At time t, the inertial system Σ' has been translated a distance vt along the x axis in Σ. An event represented by $P(t, x, y, z)$ in Σ is represented by $P(t', x', y', z')$ in Σ'.

how physical processes are interrelated when observed in different inertial systems in uniform, rectilinear motion relative to each other and is based on two postulates:

Postulate 4.1 (*Relativity principle*; Poincaré, 1905). *All laws of physics (except the laws of gravitation) are independent of the uniform translational motion of the system on which they operate.*

Postulate 4.2 (Einstein, 1905). *The velocity of light in empty space is independent of the motion of the source that emits the light.*

A consequence of the first postulate is that all geometrical objects (vectors, tensors) in an equation describing a physical process must transform in a *covariant* manner, *i.e.*, in the same way.

4.1.1 The Lorentz transformation

Let us consider two three-dimensional inertial systems Σ and Σ' in vacuum which are in rectilinear motion relative to each other in such a way that Σ' moves with constant velocity $\mathbf{v}$ along the x axis of the Σ system. The times and the spatial coordinates as measured in the two systems are t and (x, y, z), and t' and (x', y', z'), respectively. At time $t = t' = 0$ the origins O and O' and the x and x' axes of the two inertial systems coincide and at a later time t they have the relative location as depicted in Figure 4.1, referred to as the *standard configuration*.

For convenience, let us introduce the two quantities

$$\beta = \frac{v}{c} \tag{4.1}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{4.2}$$

where $v = |\mathbf{v}|$. In the following, we shall make frequent use of these shorthand notations.

As shown by Einstein, the two postulates of special relativity require that the spatial coordinates and times as measured by an observer in Σ and Σ', respectively, are connected by the following transformation:

$$ct' = \gamma(ct - x\beta) \tag{4.3a}$$

$$x' = \gamma(x - vt) \tag{4.3b}$$

$$y' = y \tag{4.3c}$$

$$z' = z \tag{4.3d}$$

Taking the difference between the square of (4.3a) and the square of (4.3b) we find that

$$\begin{aligned} c^2t'^2 - x'^2 &= \gamma^2 \left(c^2t^2 - 2xc\beta t + x^2\beta^2 - x^2 + 2xvt - v^2t^2\right) \\ &= \frac{1}{1 - \frac{v^2}{c^2}} \left[c^2t^2 \left(1 - \frac{v^2}{c^2}\right) - x^2 \left(1 - \frac{v^2}{c^2}\right)\right] \\ &= c^2t^2 - x^2 \end{aligned} \tag{4.4}$$

From Equations (4.3) we see that the y and z coordinates are unaffected by the translational motion of the inertial system Σ' along the x axis of system Σ. Using this fact, we find that we can generalise the result in Equation (4.4) above to

$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2 \tag{4.5}$$

which means that if a light wave is transmitted from the coinciding origins O and O' at time $t = t' = 0$ it will arrive at an observer at (x, y, z) at time t in Σ and an observer at (x', y', z') at time t' in Σ' in such a way that both observers conclude that the speed (spatial distance divided by time) of light in vacuum is c. Hence, the speed of light in Σ and Σ' is the same. A linear coordinate transformation which has this property is called a (homogeneous) *Lorentz transformation*.

4.1.2 Lorentz space

Let us introduce an ordered quadruple of real numbers, enumerated with the help of upper indices $\mu = 0, 1, 2, 3$, where the zeroth component is ct (c is the speed of light and t is time), and the remaining components are the components of the ordinary $\mathbb{R}^3$ radius vector $\mathbf{x}$ defined in Equation (M.1) on page 164:

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z) \equiv (ct, \mathbf{x}) \tag{4.6}$$

We want to interpret this quadruple x^μ as (the component form of) a *radius four-vector* in a real, linear, *four-dimensional vector space*.[2] We require that this four-dimensional space be a *Riemannian space*, *i.e.*, a metric space where a 'distance' and a scalar product are defined. In this space we therefore define a *metric tensor*, also known as the *fundamental tensor*, which we denote by $g_{\mu\nu}$.

Radius four-vector in contravariant and covariant form

The radius four-vector $x^\mu = (x^0, x^1, x^2, x^3) = (ct, \mathbf{x})$, as defined in Equation (4.6) above, is, by definition, the prototype of a *contravariant vector* (or, more accurately, a vector in *contravariant component form*). To every such vector there exists a *dual vector*. The vector dual to x^μ is the *covariant vector* x_μ, obtained as

$$x_\mu = g_{\mu\nu} x^\nu \tag{4.7}$$

where the upper index μ in x^μ is summed over and is therefore a *dummy index* and may be replaced by another dummy index ν This summation process is an example of *index contraction* and is often referred to as *index lowering*.

Scalar product and norm

The scalar product of x^μ with itself in a Riemannian space is defined as

$$g_{\mu\nu} x^\nu x^\mu = x_\mu x^\mu \tag{4.8}$$

This scalar product acts as an invariant 'distance', or *norm*, in this space.

To describe the physical property of Lorentz transformation invariance, described by Equation (4.5) on the preceding page, in mathematical language it is convienient to to perceive it as the manifestation of the conservation of the norm in a 4D Riemannian space. Then the explicit expression for the scalar product of x^μ with itself in this space must be

$$x_\mu x^\mu = c^2 t^2 - x^2 - y^2 - z^2 \tag{4.9}$$

[2]The British mathematician and philosopher Alfred North Whitehead writes in his book *The Concept of Nature* [13]:

> 'I regret that it has been necessary for me in this lecture to administer a large dose of four-dimensional geometry. I do not apologise, because I am really not responsible for the fact that nature in its most fundamental aspect is four-dimensional. Things are what they are. . . .'

We notice that our space will have an *indefinite norm* which means that we deal with a *non-Euclidean space*. We call the four-dimensional space (or *space-time*) with this property *Lorentz space* and denote it $\mathbb{L}^4$. A corresponding real, linear 4D space with a *positive definite norm* which is conserved during ordinary rotations is a *Euclidean vector space*. We denote such a space $\mathbb{R}^4$.

Metric tensor

By choosing the metric tensor in $\mathbb{L}^4$ as

$$g_{\mu\nu} = \begin{cases} 1 & \text{if } \mu = \nu = 0 \\ -1 & \text{if } \mu = \nu = i = j = 1, 2, 3 \\ 0 & \text{if } \mu \neq \nu \end{cases} \tag{4.10}$$

or, in matrix notation,

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{4.11}$$

i.e., a matrix with a main diagonal that has the sign sequence, or *signature*, $\{+, -, -, -\}$, the index lowering operation in our chosen flat 4D space becomes nearly trivial:

$$x_\mu = g_{\mu\nu} x^\nu = (ct, -\mathbf{x}) \tag{4.12}$$

Using matrix algebra, this can be written

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{pmatrix} \tag{4.13}$$

Hence, if the metric tensor is defined according to expression (4.10) above the covariant radius four-vector x_μ is obtained from the contravariant radius four-vector x^μ simply by changing the sign of the last three components. These components are referred to as the *space components*; the zeroth component is referred to as the *time component*.

As we see, for this particular choice of metric, the scalar product of x^μ with itself becomes

$$x_\mu x^\mu = (ct, \mathbf{x}) \cdot (ct, -\mathbf{x}) = c^2 t^2 - x^2 - y^2 - z^2 \tag{4.14}$$

which indeed is the desired Lorentz transformation invariance as required by Equation (4.9) on the facing page. Without changing the physics, one can alternatively choose a signature $\{-,+,+,+\}$. The latter has the advantage that the transition from 3D to 4D becomes smooth, while it will introduce some annoying minus signs in the theory. In current physics literature, the signature $\{+,-,-,-\}$ seems to be the most commonly used one.

The $\mathbb{L}^4$ metric tensor Equation (4.10) on the previous page has a number of interesting properties: firstly, we see that this tensor has a trace $\mathrm{Tr}\big(g_{\mu\nu}\big) = -2$ whereas in $\mathbb{R}^4$, as in any vector space with definite norm, the trace equals the space dimensionality. Secondly, we find, after trivial algebra, that the following relations between the contravariant, covariant and mixed forms of the metric tensor hold:

$$g_{\mu\nu} = g_{\nu\mu} \tag{4.15a}$$

$$g^{\mu\nu} = g_{\mu\nu} \tag{4.15b}$$

$$g_{\nu\kappa} g^{\kappa\mu} = g_{\nu}^{\mu} = \delta_{\nu}^{\mu} \tag{4.15c}$$

$$g^{\nu\kappa} g_{\kappa\mu} = g_{\mu}^{\nu} = \delta_{\mu}^{\nu} \tag{4.15d}$$

Here we have introduced the 4D version of the Kronecker delta δ_{ν}^{μ}, a mixed four-tensor of rank 2 which fulfils

$$\delta_{\nu}^{\mu} = \delta_{\mu}^{\nu} = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases} \tag{4.16}$$

Invariant line element and proper time

The *differential distance* $\mathrm{d}s$ between the two points x^μ and $x^\mu + \mathrm{d}x^\mu$ in $\mathbb{L}^4$ can be calculated from the *Riemannian metric*, given by the *quadratic differential form*

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\nu \mathrm{d}x^\mu = \mathrm{d}x_\mu \mathrm{d}x^\mu = (\mathrm{d}x^0)^2 - (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2 \tag{4.17}$$

where the metric tensor is as in Equation (4.10) on the preceding page. As we see, this form is *indefinite* as expected for a non-Euclidean space. The square root of this expression is the *invariant line element*

$$\begin{aligned} \mathrm{d}s &= c\,\mathrm{d}t \sqrt{1 - \frac{1}{c^2}\left[\left(\frac{\mathrm{d}x^1}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}x^2}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}x^3}{\mathrm{d}t}\right)^2\right]} \\ &= c\,\mathrm{d}t \sqrt{1 - \frac{1}{c^2}\left[(v_x)^2 + (v_y)^2 + (v_z)^2\right]} = c\,\mathrm{d}t \sqrt{1 - \frac{v^2}{c^2}} \\ &= c\,\mathrm{d}t \sqrt{1 - \beta^2} = c\frac{\mathrm{d}t}{\gamma} = c\,\mathrm{d}\tau \end{aligned} \tag{4.18}$$

where we introduced

$$\mathrm{d}\tau = \mathrm{d}t/\gamma \tag{4.19}$$

Since $d\tau$ measures the time when no spatial changes are present, it is called the *proper time*.

Expressing the property of the Lorentz transformation described by Equations (4.5) on page 49 in terms of the differential interval ds and comparing with Equation (4.17) on the facing page, we find that

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \tag{4.20}$$

is invariant during a Lorentz transformation. Conversely, we may say that every coordinate transformation which preserves this differential interval is a Lorentz transformation.

If in some inertial system

$$dx^2 + dy^2 + dz^2 < c^2 dt^2 \tag{4.21}$$

ds is a *time-like interval*, but if

$$dx^2 + dy^2 + dz^2 > c^2 dt^2 \tag{4.22}$$

ds is a *space-like interval*, whereas

$$dx^2 + dy^2 + dz^2 = c^2 dt^2 \tag{4.23}$$

is a *light-like interval*; we may also say that in this case we are on the *light cone*. A vector which has a light-like interval is called a *null vector*. The time-like, space-like or light-like aspects of an interval ds are invariant under a Lorentz transformation. *I.e.*, it is not possible to change a time-like interval into a space-like one or *vice versa* via a Lorentz transformation.

Four-vector fields

Any quantity which relative to any coordinate system has a quadruple of real numbers and transforms in the same way as the radius four-vector x^μ does, is called a *four-vector*. In analogy with the notation for the radius four-vector we introduce the notation $a^\mu = (a^0, \mathbf{a})$ for a general *contravariant four-vector field* in $\mathbb{L}^4$ and find that the 'lowering of index' rule, Formula (4.7) on page 50, for such an arbitrary four-vector yields the dual *covariant four-vector field*

$$a_\mu(x^\kappa) = g_{\mu\nu} a^\nu(x^\kappa) = (a^0(x^\kappa), -\mathbf{a}(x^\kappa)) \tag{4.24}$$

The scalar product between this four-vector field and another one $b^\mu(x^\kappa)$ is

$$g_{\mu\nu} a^\nu(x^\kappa) b^\mu(x^\kappa) = (a^0, -\mathbf{a}) \cdot (b^0, \mathbf{b}) = a^0 b^0 - \mathbf{a} \cdot \mathbf{b} \tag{4.25}$$

which is a *scalar field*, *i.e.*, an invariant scalar quantity $\alpha(x^\kappa)$ which depends on time and space, as described by $x^\kappa = (ct, x, y, z)$.

The Lorentz transformation matrix

Introducing the transformation matrix

$$\left(\Lambda^{\mu}{}_{\nu}\right) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.26}$$

the linear Lorentz transformation (4.3) on page 49, *i.e.*, the coordinate transformation $x^{\mu} \to x'^{\mu} = x'^{\mu}(x^0, x^1, x^2, x^3)$, from one inertial system Σ to another inertial system Σ' in the standard configuration, can be written

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{4.27}$$

The Lorentz group

It is easy to show, by means of direct algebra, that two successive Lorentz transformations of the type in Equation (4.27) above, and defined by the speed parameters β_1 and β_2, respectively, correspond to a single transformation with speed parameter

$$\beta = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} \tag{4.28}$$

This means that the nonempty set of Lorentz transformations constitutes a *closed algebraic structure* with a binary operation which is *associative*. Furthermore, one can show that this set possesses at least one *identity element* and at least one *inverse element*. In other words, this set of Lorentz transformations constitutes a *mathematical group*. However tempting, we shall not make any further use of *group theory*.

4.1.3 Minkowski space

Specifying a point $x^{\mu} = (x^0, x^1, x^2, x^3)$ in 4D space-time is a way of saying that 'something takes place at a certain time $t = x^0/c$ and at a certain place $(x, y, z) = (x^1, x^2, x^3)$'. Such a point is therefore called an *event*. The trajectory for an event as a function of time and space is called a *world line*. For instance, the world line for a light ray which propagates in vacuum is the trajectory $x^0 = x^1$.

Introducing

$$X^0 = \mathrm{i}x^0 = \mathrm{i}ct \tag{4.29a}$$

$$X^1 = x^1 \tag{4.29b}$$

$$X^2 = x^2 \tag{4.29c}$$

$$X^3 = x^3 \tag{4.29d}$$

$$\mathrm{d}S = \mathrm{i}\,\mathrm{d}s \tag{4.29e}$$

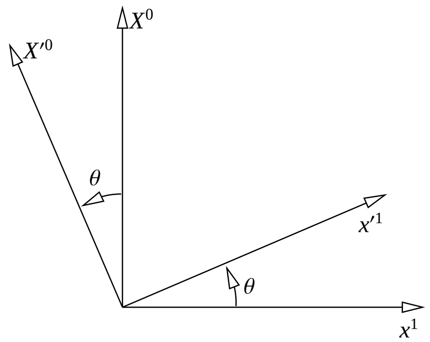

FIGURE 4.2: Minkowski space can be considered an ordinary Euclidean space where a Lorentz transformation from $(x^1, X^0 = \mathrm{i}ct)$ to $(x'^1, X'^0 = \mathrm{i}ct')$ corresponds to an ordinary rotation through an angle θ. This rotation leaves the Euclidean distance $\left(x^1\right)^2 + \left(X^0\right)^2 = x^2 - c^2t^2$ invariant.

where $\mathrm{i} = \sqrt{-1}$, we see that Equation (4.17) on page 52 transforms into

$$\mathrm{d}S^2 = (\mathrm{d}X^0)^2 + (\mathrm{d}X^1)^2 + (\mathrm{d}X^2)^2 + (\mathrm{d}X^3)^2 \tag{4.30}$$

i.e., into a 4D differential form which is *positive definite* just as is ordinary 3D *Euclidean space* $\mathbb{R}^3$. We shall call the 4D Euclidean space constructed in this way the *Minkowski space* $\mathbb{M}^4$.[3]

As before, it suffices to consider the simplified case where the relative motion between Σ and Σ' is along the x axes. Then

$$\mathrm{d}S^2 = (\mathrm{d}X^0)^2 + (\mathrm{d}X^1)^2 = (\mathrm{d}X^0)^2 + (\mathrm{d}x^1)^2 \tag{4.31}$$

and we consider the X^0 and $X^1 = x^1$ axes as orthogonal axes in a Euclidean space. As in all Euclidean spaces, every interval is invariant under a rotation of the X^0x^1 plane through an angle θ into $X'^0x'^1$:

$$X'^0 = -x^1 \sin\theta + X^0 \cos\theta \tag{4.32a}$$

$$x'^1 = x^1 \cos\theta + X^0 \sin\theta \tag{4.32b}$$

See Figure 4.2.

If we introduce the angle $\varphi = -\mathrm{i}\theta$, often called the *rapidity* or the *Lorentz boost parameter*, and transform back to the original space and time variables by using Equation (4.29) on the facing page backwards, we obtain

$$ct' = -x \sinh\varphi + ct \cosh\varphi \tag{4.33a}$$

$$x' = x \cosh\varphi - ct \sinh\varphi \tag{4.33b}$$

[3]The fact that our Riemannian space can be transformed in this way into an Euclidean one means that it is, strictly speaking, a *pseudo-Riemannian space*.

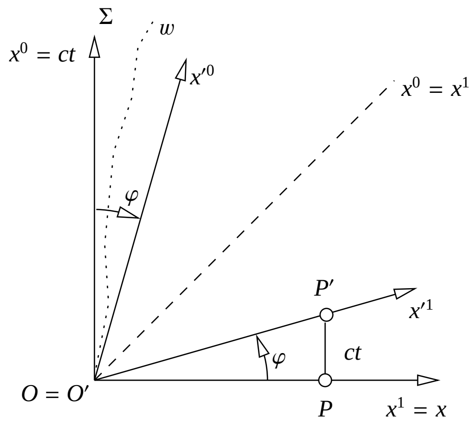

FIGURE 4.3: Minkowski diagram depicting geometrically the transformation (4.33) from the unprimed system to the primed system. Here w denotes the world line for an event and the line $x^0 = x^1 \Leftrightarrow x = ct$ the world line for a light ray in vacuum. Note that the event P is simultaneous with all points on the x^1 axis ($t = 0$), including the origin O. The event P', which is simultaneous with all points on the x' axis, including $O' = O$, to an observer at rest in the primed system, is not simultaneous with O in the unprimed system but occurs there at time $|P - P'|/c$.

which are identical to the transformation equations (4.3) on page 49 if we let

$$\sinh\varphi = \gamma\beta \tag{4.34a}$$

$$\cosh\varphi = \gamma \tag{4.34b}$$

$$\tanh\varphi = \beta \tag{4.34c}$$

It is therefore possible to envisage the Lorentz transformation as an 'ordinary' rotation in the 4D Euclidean space $\mathbb{M}^4$. Such a rotation in $\mathbb{M}^4$ corresponds to a coordinate change in $\mathbb{L}^4$ as depicted in Figure 4.3. Equation (4.28) on page 54 for successive Lorentz transformation then corresponds to the tanh addition formula

$$\tanh(\varphi_1 + \varphi_2) = \frac{\tanh\varphi_1 + \tanh\varphi_2}{1 + \tanh\varphi_1 \tanh\varphi_2} \tag{4.35}$$

The use of ict and $\mathbb{M}^4$, which leads to the interpretation of the Lorentz transformation as an 'ordinary' rotation, may, at best, be illustrative, but is not very physical. Besides, if we leave the flat $\mathbb{L}^4$ space and enter the curved space of general relativity, the 'ict' trick will turn out to be an impasse. Let us therefore immediately return to $\mathbb{L}^4$ where all components are real valued.

4.2 Covariant classical mechanics

The invariance of the differential 'distance' ds in $\mathbb{L}^4$, and the associated differential proper time dτ [see Equation (4.18) on page 52] allows us to define the *four-velocity*

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \gamma(c, \mathbf{v}) = \left(\frac{c}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}} \right) = (u^0, \mathbf{u}) \tag{4.36}$$

which, when multiplied with the scalar invariant m_0 yields the *four-momentum*

$$p^\mu = m_0 \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = m_0\gamma(c, \mathbf{v}) = \left(\frac{m_0 c}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{m_0\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}} \right) = (p^0, \mathbf{p}) \tag{4.37}$$

From this we see that we can write

$$\mathbf{p} = m\mathbf{v} \tag{4.38}$$

where

$$m = \gamma m_0 = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \tag{4.39}$$

We can interpret this such that the Lorentz covariance implies that the mass-like term in the ordinary 3D linear momentum is not invariant. A better way to look at this is that $\mathbf{p} = m\mathbf{v} = \gamma m_0 \mathbf{v}$ is the covariantly correct expression for the kinetic three-momentum.

Multiplying the zeroth (time) component of the four-momentum p^μ with the scalar invariant c, we obtain

$$cp^0 = \gamma m_0 c^2 = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} = mc^2 \tag{4.40}$$

Since this component has the dimension of energy and is the result of a covariant description of the motion of a particle with its kinetic momentum described by the spatial components of the four-momentum, Equation (4.37) above, we interpret cp^0 as the total energy E. Hence,

$$cp^\mu = (cp^0, c\mathbf{p}) = (E, c\mathbf{p}) \tag{4.41}$$

Scalar multiplying this four-vector with itself, we obtain

$$\begin{aligned} cp_\mu cp^\mu &= c^2 g_{\mu\nu} p^\nu p^\mu = c^2[(p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2] \\ &= (E, -c\mathbf{p}) \cdot (E, c\mathbf{p}) = E^2 - c^2\mathbf{p}^2 \\ &= \frac{(m_0 c^2)^2}{1-\frac{v^2}{c^2}} \left(1 - \frac{v^2}{c^2}\right) = (m_0 c^2)^2 \end{aligned} \tag{4.42}$$

Since this is an invariant, this equation holds in any inertial frame, particularly in the frame where $\mathbf{p} = \mathbf{0}$ and there we have

$$E = m_0 c^2 \tag{4.43}$$

This is probably the most famous formula in physics history.

4.3 Covariant classical electrodynamics

Let us consired a charge density which in its rest inertial system is denoted by ρ_0. The four-vector (in contravariant component form)

$$j^\mu = \rho_0 \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \rho_0 u^\mu = \rho_0 \gamma(c, \mathbf{v}) = (\rho c, \rho \mathbf{v}) \tag{4.44}$$

where we introduced

$$\rho = \gamma \rho_0 \tag{4.45}$$

is called the *four-current*.

The contravariant form of the four-del operator $\partial^\mu = \partial/\partial x_\mu$ is defined in Equation (M.71) on page 175 and its covariant counterpart $\partial_\mu = \partial/\partial x^\mu$ in Equation (M.72) on page 175, respectively. As is shown in Example 13.5 on page 175, the *d'Alembert operator* is the scalar product of the four-del with itself:

$$\Box^2 = \partial^\mu \partial_\mu = \partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \tag{4.46}$$

Since it has the characteristics of a four-scalar, the d'Alembert operator is invariant and, hence, the homogeneous wave equation $\Box^2 f(t, \mathbf{x}) = 0$ is Lorentz covariant.

4.3.1 The four-potential

If we introduce the *four-potential*

$$A^\mu = \left(\frac{\phi}{c}, \mathbf{A}\right) \tag{4.47}$$

where ϕ is the scalar potential and $\mathbf{A}$ the vector potential, defined in Section 3.3 on page 36, we can write the uncoupled inhomogeneous wave equations, Equations (3.14) on page 38, in the following compact (and covariant) way:

$$\Box^2 A^\mu = \mu_0 j^\mu \tag{4.48}$$

With the help of the above, we can formulate our electrodynamic equations covariantly. For instance, the covariant form of the *equation of continuity*, Equation (1.23) on page 9 is

$$\partial_\mu j^\mu = 0 \tag{4.49}$$

and the *Lorenz-Lorentz gauge condition*, Equation (3.13) on page 38, can be written

$$\partial_\mu A^\mu = 0 \tag{4.50}$$

The gauge transformations (3.37) on page 42 in covariant form are

$$A^\mu \mapsto A'^\mu = A^\mu + \partial^\mu \Gamma(x^\nu) \tag{4.51}$$

If only one dimension Lorentz contracts (for instance, due to relative motion along the x direction), a 3D spatial volume element transforms according to

$$\mathrm{d}V = \mathrm{d}^3x = \frac{1}{\gamma}\mathrm{d}V_0 = \mathrm{d}V_0\sqrt{1-\beta^2} = \mathrm{d}V_0\sqrt{1-\frac{v^2}{c^2}} \tag{4.52}$$

where $\mathrm{d}V_0$ denotes the volume element as measured in the rest system, then from Equation (4.45) on the facing page we see that

$$\rho \mathrm{d}V = \rho_0 \mathrm{d}V_0 \tag{4.53}$$

i.e., the charge in a given volume is conserved. We can therefore conclude that the elementary charge is a universal constant.

4.3.2 The Liénard-Wiechert potentials

Let us now solve the the inhomogeneous wave equations (3.14) on page 38 in vacuum for the case of a well-localised charge q' at a source point defined by the radius four-vector $x'^\mu = (x'^0 = ct', x'^1, x'^2, x'^3)$. The field point (observation point) is denoted by the radius four-vector $x^\mu = (x^0 = ct, x^1, x^2, x^3)$.

In the rest system we know that the solution is simply

$$(A^\mu)_0 = \left(\frac{\phi}{c}, \mathbf{A}\right)_{\mathbf{v}=\mathbf{0}} = \left(\frac{q'}{4\pi\varepsilon_0}\frac{1}{c\,|\mathbf{x}-\mathbf{x}'|_0}, \mathbf{0}\right) \tag{4.54}$$

where $|\mathbf{x}-\mathbf{x}'|_0$ is the usual distance from the source point to the field point, evaluated in the rest system (signified by the index '0').

Let us introduce the relative radius four-vector between the source point and the field point:

$$R^\mu = x^\mu - x'^\mu = (c(t-t'), \mathbf{x}-\mathbf{x}') \tag{4.55}$$

Scalar multiplying this relative four-vector with itself, we obtain

$$R^\mu R_\mu = (c(t-t'), \mathbf{x}-\mathbf{x}') \cdot (c(t-t'), -(\mathbf{x}-\mathbf{x}')) = c^2(t-t')^2 - \left|\mathbf{x}-\mathbf{x}'\right|^2 \quad (4.56)$$

We know that in vacuum the signal (field) from the charge q' at x'^μ propagates to x^μ with the speed of light c so that

$$\left|\mathbf{x}-\mathbf{x}'\right| = c(t-t') \quad (4.57)$$

Inserting this into Equation (4.56), we see that

$$R^\mu R_\mu = 0 \quad (4.58)$$

or that Equation (4.55) on the previous page can be written

$$R^\mu = (\left|\mathbf{x}-\mathbf{x}'\right|, \mathbf{x}-\mathbf{x}') \quad (4.59)$$

Now we want to find the correspondence to the rest system solution, Equation (4.54) on the preceding page, in an arbitrary inertial system. We note from Equation (4.36) on page 57 that in the rest system

$$(u^\mu)_0 = \left(\frac{c}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right)_{\mathbf{v}=\mathbf{0}} = (c, \mathbf{0}) \quad (4.60)$$

and

$$(R^\mu)_0 = (\left|\mathbf{x}-\mathbf{x}'\right|, \mathbf{x}-\mathbf{x}')_0 = (\left|\mathbf{x}-\mathbf{x}'\right|_0, (\mathbf{x}-\mathbf{x}')_0) \quad (4.61)$$

As all scalar products, $u^\mu R_\mu$ is invariant, which means that we can evaluate it in any inertial system and it will have the same value in all other inertial systems. If we evaluate it in the rest system the result is:

$$\begin{aligned} u^\mu R_\mu &= \left(u^\mu R_\mu\right)_0 = (u^\mu)_0 (R_\mu)_0 \\ &= (c, \mathbf{0}) \cdot (\left|\mathbf{x}-\mathbf{x}'\right|_0, -(\mathbf{x}-\mathbf{x}')_0) = c\left|\mathbf{x}-\mathbf{x}'\right|_0 \end{aligned} \quad (4.62)$$

We therefore see that the expression

$$A^\mu = \frac{q'}{4\pi\varepsilon_0} \frac{u^\mu}{c u^\nu R_\nu} \quad (4.63)$$

subject to the condition $R^\mu R_\mu = 0$ has the proper transformation properties (proper tensor form) and reduces, in the rest system, to the solution Equation (4.54) on the preceding page. It is therefore the correct solution, valid in any inertial system.

According to Equation (4.36) on page 57 and Equation (4.59)

$$u^\nu R_\nu = \gamma(c, \mathbf{v}) \cdot \left(\left|\mathbf{x}-\mathbf{x}'\right|, -(\mathbf{x}-\mathbf{x}')\right) = \gamma\left(c\left|\mathbf{x}-\mathbf{x}'\right| - \mathbf{v}\cdot(\mathbf{x}-\mathbf{x}')\right) \quad (4.64)$$

Generalising expression (4.1) on page 49 to vector form:

$$\boldsymbol{\beta} = \beta\hat{\mathbf{v}} \stackrel{\text{def}}{\equiv} \frac{\mathbf{v}}{c} \tag{4.65}$$

and introducing

$$s \stackrel{\text{def}}{\equiv} \left|\mathbf{x}-\mathbf{x}'\right| - \frac{\mathbf{v}\cdot(\mathbf{x}-\mathbf{x}')}{c} \equiv \left|\mathbf{x}-\mathbf{x}'\right| - \boldsymbol{\beta}\cdot(\mathbf{x}-\mathbf{x}') \tag{4.66}$$

we can write

$$u^{\nu}R_{\nu} = \gamma cs \tag{4.67}$$

and

$$\frac{u^{\mu}}{cu^{\nu}R_{\nu}} = \left(\frac{1}{cs}, \frac{\mathbf{v}}{c^2 s}\right) \tag{4.68}$$

from which we see that the solution (4.63) can be written

$$A^{\mu}(x^{\kappa}) = \frac{q'}{4\pi\varepsilon_0}\left(\frac{1}{cs}, \frac{\mathbf{v}}{c^2 s}\right) = \left(\frac{\phi}{c}, \mathbf{A}\right) \tag{4.69}$$

where in the last step the definition of the four-potential, Equation (4.47) on page 58, was used. Writing the solution in the ordinary 3D-way, we conclude that for a very localised charge volume, moving relative an observer with a velocity $\mathbf{v}$, the scalar and vector potentials are given by the expressions

$$\phi(t,\mathbf{x}) = \frac{q'}{4\pi\varepsilon_0}\frac{1}{s} = \frac{q'}{4\pi\varepsilon_0}\frac{1}{|\mathbf{x}-\mathbf{x}'| - \boldsymbol{\beta}\cdot(\mathbf{x}-\mathbf{x}')} \tag{4.70a}$$

$$\mathbf{A}(t,\mathbf{x}) = \frac{q'}{4\pi\varepsilon_0 c^2}\frac{\mathbf{v}}{s} = \frac{q'}{4\pi\varepsilon_0 c^2}\frac{\mathbf{v}}{|\mathbf{x}-\mathbf{x}'| - \boldsymbol{\beta}\cdot(\mathbf{x}-\mathbf{x}')} \tag{4.70b}$$

These potentials are called the *Liénard-Wiechert potentials*.

4.3.3 The electromagnetic field tensor

Consider a vectorial (cross) product $\mathbf{c}$ between two ordinary vectors $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{c} = \mathbf{a}\times\mathbf{b} = \epsilon_{ijk}a_i b_j \hat{\mathbf{x}}_k = (a_2b_3 - a_3b_2)\hat{\mathbf{x}}_1 + (a_3b_1 - a_1b_3)\hat{\mathbf{x}}_2 + (a_1b_2 - a_2b_1)\hat{\mathbf{x}}_3 \tag{4.71}$$

We notice that the kth component of the vector $\mathbf{c}$ can be represented as

$$c_k = a_i b_j - a_j b_i = c_{ij} = -c_{ji}, \quad i,j \neq k \tag{4.72}$$

In other words, the *pseudovector* $\mathbf{c} = \mathbf{a}\times\mathbf{b}$ can be considered as an *antisymmetric tensor* of rank two.

The same is true for the curl operator $\boldsymbol{\nabla}\times$. For instance, the Maxwell equation

$$\boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{4.73}$$

can in this tensor notation be written

$$\frac{\partial E_j}{\partial x^i} - \frac{\partial E_i}{\partial x^j} = -\frac{\partial B_{ij}}{\partial t} \tag{4.74}$$

We know from Chapter 3 that the fields can be derived from the electromagnetic potentials in the following way:

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} \tag{4.75a}$$

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t} \tag{4.75b}$$

In component form, this can be written

$$B_{ij} = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} = \partial_i A_j - \partial_j A_i \tag{4.76a}$$

$$E_i = -\frac{\partial \phi}{\partial x^i} - \frac{\partial A_i}{\partial t} = -\partial_i \phi - \partial_t A_i \tag{4.76b}$$

From this, we notice the clear difference between the *axial vector* (pseudovector) $\mathbf{B}$ and the *polar vector* ('ordinary vector') $\mathbf{E}$.

Our goal is to express the electric and magnetic fields in a tensor form where the components are functions of the covariant form of the four-potential, Equation (4.47) on page 58:

$$A^\mu = \left(\frac{\phi}{c}, \mathbf{A}\right) \tag{4.77}$$

Inspection of (4.77) and Equation (4.76) makes it natural to define the four-tensor

$$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{4.78}$$

This anti-symmetric (*skew-symmetric*), four-tensor of rank 2 is called the *electromagnetic field tensor*. In matrix representation, the *contravariant field tensor* can be written

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{4.79}$$

The *covariant field tensor* is obtained from the contravariant field tensor in the usual manner by index lowering:

$$F_{\mu\nu} = g_{\mu\kappa} g_{\nu\lambda} F^{\kappa\lambda} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{4.80}$$

In matrix representation

$$\left(F_{\mu\nu}\right) = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{4.81}$$

It is perhaps interesting to note that the field tensor is a sort of four-dimensional curl of the four-potential vector A^μ.

That the two Maxwell source equations can be written

$$\partial_\nu F^{\nu\mu} = \mu_0 j^\mu \tag{4.82}$$

is immediately observed by explicitly setting $\mu = 0$ in this covariant equation and using the matrix representation Formula (4.79) on the preceding page for the covariant component form of the electromagnetic field tensor $F^{\mu\nu}$, to obtain

$$\begin{aligned} &\frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{10}}{\partial x^1} + \frac{\partial F^{20}}{\partial x^2} + \frac{\partial F^{30}}{\partial x^3} = 0 + \frac{1}{c}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) \\ &\quad = \frac{1}{c}\boldsymbol{\nabla}\cdot\mathbf{E} = \mu_0 j^0 = \mu_0 c\rho \end{aligned} \tag{4.83}$$

or, equivalently,

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \mu_0 c^2 \rho = \frac{\rho}{\varepsilon_0} \tag{4.84}$$

which is the Maxwell source equation for the electric field, Equation (1.45a) on page 14.

For $\mu = 1$, Equation (4.83) above yields

$$\frac{\partial F^{01}}{\partial x^0} + \frac{\partial F^{11}}{\partial x^1} + \frac{\partial F^{21}}{\partial x^2} + \frac{\partial F^{31}}{\partial x^3} = -\frac{1}{c^2}\frac{\partial E_x}{\partial t} + 0 - \frac{\partial B_z}{\partial y} + \frac{\partial B_y}{\partial z} = \mu_0 j^1 = \mu_0 \rho v_x \tag{4.85}$$

or, using $\varepsilon_0\mu_0 = 1/c^2$,

$$\frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y} - \varepsilon_0\mu_0\frac{\partial E_x}{\partial t} = \mu_0 j_x \tag{4.86}$$

and similarly for $\mu = 2, 3$. In summary, in three-vector form, we can write the result as

$$\boldsymbol{\nabla}\times\mathbf{B} - \varepsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t} = \mu_0\mathbf{j}(t, \mathbf{x}) \tag{4.87}$$

which is the Maxwell source equation for the magnetic field, Equation (1.45d) on page 14.

The two Maxwell field equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{4.88}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{4.89}$$

correspond to (no summation!)

$$\partial_\kappa F_{\mu\nu} + \partial_\mu F_{\nu\kappa} + \partial_\nu F_{\kappa\mu} = 0 \tag{4.90}$$

Hence, Equation (4.82) on the previous page and Equation (4.90) constitute Maxwell's equations in four-dimensional formalism.

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CHAPTER 5

Electromagnetic Fields and Particles

In previous chapters, we calculated the electromagnetic fields and potentials from arbitrary, but prescribed distributions of charges and currents. In this chapter we study the general problem of interaction between electric and magnetic fields and electrically charged particles. The analysis is based on Lagrangian and Hamiltonian methods, is fully covariant, and yields results which are relativistically correct.

5.1 Charged particles in an electromagnetic field

We first establish a relativistically correct theory describing the motion of charged particles in prescribed electric and magnetic fields. From these equations we may then calculate the charged particle dynamics in the most general case.

5.1.1 Covariant equations of motion

We will show that for our problem we can derive the correct equations of motion by using in four-dimensional $\mathbb{L}^4$ a function with similiar properties as a Lagrange function in 3D and then apply a variational principle. We will also show that we can find a Hamiltonian-type function in 4D and solve the corresponding Hamilton-type equations to obtain the correct covariant formulation of classical electrodynamics.

Lagrange formalism

Let us now introduce a generalised action

$$S_4 = \int L_4(x^\mu, u^\mu)\,\mathrm{d}\tau \tag{5.1}$$

where $\mathrm{d}\tau$ is the proper time defined via Equation (4.18) on page 52, and L_4 acts as a kind of generalisation to the common 3D Lagrangian so that the variational principle

$$\delta S_4 = \delta \int_{\tau_0}^{\tau_1} L_4(x^\mu, u^\mu)\,\mathrm{d}\tau = 0 \tag{5.2}$$

with fixed endpoints τ_0, τ_1 is fulfilled. We require that L_4 is a scalar invariant which does not contain higher than the second power of the four-velocity u^μ in order that the equations of motion be linear.

According to Formula (M.98) on page 180 the ordinary 3D Lagrangian is the difference between the kinetic and potential energies. A free particle has only kinetic energy. If the particle mass is m_0 then in 3D the kinetic energy is $m_0 v^2/2$. This suggests that in 4D the Lagrangian for a free particle should be

$$L_4^{\text{free}} = \frac{1}{2} m_0 u^\mu u_\mu \tag{5.3}$$

For an interaction with the electromagnetic field we can introduce the interaction with the help of the four-potential given by Equation (4.77) on page 62 in the following way

$$L_4 = \frac{1}{2} m_0 u^\mu u_\mu + q u_\mu A^\mu(x^\nu) \tag{5.4}$$

We call this the *four-Lagrangian* and shall now show how this function, together with the variation principle, Formula (5.2), yields covariant results which are physically correct.

The variation principle (5.2) with the 4D Lagrangian (5.4) inserted, leads to

$$\begin{aligned}
\delta S_4 &= \delta \int_{\tau_0}^{\tau_1} \left(\frac{m_0}{2} u^\mu u_\mu + q u^\mu A_\mu \right) \mathrm{d}\tau \\
&= \int_{\tau_0}^{\tau_1} \left[\frac{m_0}{2} \frac{\partial (u^\mu u_\mu)}{\partial u^\mu} \delta u^\mu + q \left(A_\mu \delta u^\mu + u^\mu \frac{\partial A_\mu}{\partial x^\nu} \delta x^\nu \right) \right] \mathrm{d}\tau \\
&= \int_{\tau_0}^{\tau_1} \left[m_0 u_\mu \delta u^\mu + q \left(A_\mu \delta u^\mu + u^\mu \partial_\nu A_\mu \delta x^\nu \right) \right] \mathrm{d}\tau = 0
\end{aligned} \tag{5.5}$$

According to Equation (4.36) on page 57, the four-velocity is

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \tag{5.6}$$

which means that we can write the variation of u^μ as a total derivative with respect to τ:

$$\delta u^\mu = \delta\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}}{\mathrm{d}\tau}(\delta x^\mu) \tag{5.7}$$

Inserting this into the first two terms in the last integral in Equation (5.5) on the facing page, we obtain

$$\delta S_4 = \int_{\tau_0}^{\tau_1}\left(m_0 u_\mu \frac{\mathrm{d}}{\mathrm{d}\tau}(\delta x^\mu) + qA_\mu \frac{\mathrm{d}}{\mathrm{d}\tau}(\delta x^\mu) + qu^\mu \partial_\nu A_\mu \delta x^\nu\right) \mathrm{d}\tau \tag{5.8}$$

Partial integration in the two first terms in the right hand member of (5.8) gives

$$\delta S_4 = \int_{\tau_0}^{\tau_1}\left(-m_0 \frac{\mathrm{d}u_\mu}{\mathrm{d}\tau}\delta x^\mu - q\frac{\mathrm{d}A_\mu}{\mathrm{d}\tau}\delta x^\mu + qu^\mu \partial_\nu A_\mu \delta x^\nu\right) \mathrm{d}\tau \tag{5.9}$$

where the integrated parts do not contribute since the variations at the endpoints vanish. A change of irrelevant summation index from μ to ν in the first two terms of the right hand member of (5.9) yields, after moving the ensuing common factor δx^ν outside the partenthesis, the following expression:

$$\delta S_4 = \int_{\tau_0}^{\tau_1}\left(-m_0 \frac{\mathrm{d}u_\nu}{\mathrm{d}\tau} - q\frac{\mathrm{d}A_\nu}{\mathrm{d}\tau} + qu^\mu \partial_\nu A_\mu\right) \delta x^\nu \,\mathrm{d}\tau \tag{5.10}$$

Applying well-known rules of differentiation and the expression (4.36) for the four-velocity, we can express $\mathrm{d}A_\nu/\mathrm{d}\tau$ as follows:

$$\frac{\mathrm{d}A_\nu}{\mathrm{d}\tau} = \frac{\partial A_\nu}{\partial x^\mu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \partial_\mu A_\nu u^\mu \tag{5.11}$$

By inserting this expression (5.11) into the second term in right-hand member of Equation (5.10) above, and noting the common factor qu^μ of the resulting term and the last term, we obtain the final variational principle expression

$$\delta S_4 = \int_{\tau_0}^{\tau_1}\left[-m_0 \frac{\mathrm{d}u_\nu}{\mathrm{d}\tau} + qu^\mu\left(\partial_\nu A_\mu - \partial_\mu A_\nu\right)\right] \delta x^\nu \,\mathrm{d}\tau \tag{5.12}$$

Since, according to the variational principle, this expression shall vanish and δx^ν is arbitrary between the fixed end points τ_0 and τ_1, the expression inside $[\;]$ in the integrand in the right hand member of Equation (5.12) must vanish. In other words, we have found an equation of motion for a charged particle in a prescribed electromagnetic field:

$$m_0 \frac{\mathrm{d}u_\nu}{\mathrm{d}\tau} = qu^\mu\left(\partial_\nu A_\mu - \partial_\mu A_\nu\right) \tag{5.13}$$

With the help of Equation (4.78) on page 62 we can express this equation in terms of the electromagnetic field tensor in the following way:

$$m_0 \frac{\mathrm{d}u_\nu}{\mathrm{d}\tau} = qu^\mu F_{\nu\mu} \tag{5.14}$$

This is the sought-for covariant equation of motion for a particle in an electromagnetic field. It is often referred to as the *Minkowski equation*. As the reader can easily verify, the spatial part of this 4-vector equation is the covariant (relativistically correct) expression for the *Newton-Lorentz force equation*.

Hamiltonian formalism

The usual *Hamilton equations* for a 3D space are given by Equation (M.103) on page 180 in Appendix M. These six first-order partial differential equations are

$$\frac{\partial H}{\partial p_i} = \frac{\mathrm{d}q_i}{\mathrm{d}t} \tag{5.15a}$$

$$\frac{\partial H}{\partial q_i} = -\frac{\mathrm{d}p_i}{\mathrm{d}t} \tag{5.15b}$$

where $H(p_i, q_i, t) = p_i \dot{q}_i - L(q_i, \dot{q}_i, t)$ is the ordinary 3D Hamiltonian, q_i is a *generalised coordinate* and p_i is its *canonically conjugate momentum*.

We seek a similar set of equations in 4D space. To this end we introduce a *canonically conjugate four-momentum p^μ* in an analogous way as the ordinary 3D conjugate momentum:

$$p^\mu = \frac{\partial L_4}{\partial u_\mu} \tag{5.16}$$

and utilise the four-velocity u^μ, as given by Equation (4.36) on page 57, to define the *four-Hamiltonian*

$$H_4 = p^\mu u_\mu - L_4 \tag{5.17}$$

With the help of these, the radius four-vector x^μ, considered as the *generalised four-coordinate*, and the invariant line element $\mathrm{d}s$, defined in Equation (4.18) on page 52, we introduce the following eight partial differential equations:

$$\frac{\partial H_4}{\partial p^\mu} = \frac{\mathrm{d}x_\mu}{\mathrm{d}\tau} \tag{5.18a}$$

$$\frac{\partial H_4}{\partial x^\mu} = -\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau} \tag{5.18b}$$

which form the *four-dimensional Hamilton equations*.

Our strategy now is to use Equation (5.16) above and Equations (5.18) to derive an explicit algebraic expression for the canonically conjugate momentum four-vector. According to Equation (4.41) on page 57, c times a four-momentum has a zeroth (time) component which we can identify with the total energy. Hence we require that the component p^0 of the conjugate four-momentum vector defined according to Equation (5.16) be identical to the ordinary 3D Hamiltonian H divided by c and hence that this cp^0 solves the Hamilton equations, Equations (5.15) on the preceding page. This later consistency check is left as an exercise to the reader.

Using the definition of H_4, Equation (5.17) on the facing page, and the expression for L_4, Equation (5.4) on page 68, we obtain

$$H_4 = p^\mu u_\mu - L_4 = p^\mu u_\mu - \frac{1}{2} m_0 u^\mu u_\mu - q u_\mu A^\mu(x^\nu) \tag{5.19}$$

Furthermore, from the definition (5.16) of the canonically conjugate four-momentum p^μ, we see that

$$p^\mu = \frac{\partial L_4}{\partial u_\mu} = \frac{\partial}{\partial u_\mu}\left(\frac{1}{2} m_0 u^\mu u_\mu + q u_\mu A^\mu(x^\nu)\right) = m_0 u^\mu + q A^\mu \tag{5.20}$$

Inserting this into (5.19), we obtain

$$H_4 = m_0 u^\mu u_\mu + q A^\mu u_\mu - \frac{1}{2} m_0 u^\mu u_\mu - q u^\mu A_\mu(x^\nu) = \frac{1}{2} m_0 u^\mu u_\mu \tag{5.21}$$

Since the four-velocity scalar-multiplied by itself is $u^\mu u_\mu = c^2$, we clearly see from Equation (5.21) that H_4 is indeed a scalar invariant, whose value is simply

$$H_4 = \frac{m_0 c^2}{2} \tag{5.22}$$

However, at the same time (5.20) provides the algebraic relationship

$$u^\mu = \frac{1}{m_0}\left(p^\mu - q A^\mu\right) \tag{5.23}$$

and if this is used in (5.21) to eliminate u^μ, one gets

$$\begin{aligned} H_4 &= \frac{m_0}{2}\left(\frac{1}{m_0}\left(p^\mu - qA^\mu\right)\frac{1}{m_0}\left(p_\mu - qA_\mu\right)\right) \\ &= \frac{1}{2m_0}\left(p^\mu - qA^\mu\right)\left(p_\mu - qA_\mu\right) \\ &= \frac{1}{2m_0}\left(p^\mu p_\mu - 2qA^\mu p_\mu + q^2 A^\mu A_\mu\right) \end{aligned} \tag{5.24}$$

That this four-Hamiltonian yields the correct covariant equation of motion can be seen by inserting it into the four-dimensional Hamilton's equations (5.18) and using the relation (5.23):

$$\begin{aligned} \frac{\partial H_4}{\partial x^\mu} &= -\frac{q}{m_0}\left(p^\nu - qA^\nu\right)\frac{\partial A_\nu}{\partial x^\mu} \\ &= -\frac{q}{m_0} m_0 u^\nu \frac{\partial A_\nu}{\partial x^\mu} \\ &= -q u^\nu \frac{\partial A_\nu}{\partial x^\mu} \\ &= -\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau} = -m_0 \frac{\mathrm{d}u_\mu}{\mathrm{d}\tau} - q\frac{\partial A_\mu}{\partial x^\nu} u^\nu \end{aligned} \tag{5.25}$$

where in the last step Equation (5.20) on the previous page was used. Rearranging terms, and using Equation (4.79) on page 62, we obtain

$$m_0 \frac{\mathrm{d}u_\mu}{\mathrm{d}\tau} = qu^\nu \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) = qu^\nu F_{\mu\nu} \tag{5.26}$$

which is identical to the covariant equation of motion Equation (5.14) on page 69. We can then safely conclude that the Hamiltonian in question is correct.

Recalling expression (4.47) on page 58 and representing the canonically conjugate four-momentum as $p^\mu = (p^0, \mathbf{p})$, we obtain the following scalar products:

$$p^\mu p_\mu = (p^0)^2 - (\mathbf{p})^2 \tag{5.27a}$$

$$A^\mu p_\mu = \frac{1}{c}\phi p^0 - (\mathbf{p}\cdot\mathbf{A}) \tag{5.27b}$$

$$A^\mu A_\mu = \frac{1}{c^2}\phi^2 - (\mathbf{A})^2 \tag{5.27c}$$

Inserting these explicit expressions into Equation (5.24) on the previous page, and using the fact that for H_4 is equal to the scalar value $m_0c^2/2$, as derived in Equation (5.22) on the preceding page, we obtain the equation

$$\frac{m_0c^2}{2} = \frac{1}{2m_0}\left[(p^0)^2 - (\mathbf{p})^2 - \frac{2}{c}q\phi p^0 + 2q(\mathbf{p}\cdot\mathbf{A}) + \frac{q^2}{c^2}\phi^2 - q^2(\mathbf{A})^2\right] \tag{5.28}$$

which is the second order algebraic equation in p^0:

$$(p^0)^2 - \frac{2q}{c}\phi p^0 - \underbrace{\left[(\mathbf{p})^2 - 2q\mathbf{p}\cdot\mathbf{A} + q^2(\mathbf{A})^2\right]}_{(\mathbf{p}-q\mathbf{A})^2} + \frac{q^2}{c^2}\phi^2 - m_0^2c^2 = 0 \tag{5.29}$$

with two possible solutions

$$p^0 = \frac{q}{c}\phi \pm \sqrt{(\mathbf{p}-q\mathbf{A})^2 + m_0^2c^2} \tag{5.30}$$

Since the zeroth component (time component) p^0 of a four-momentum vector p^μ multiplied by c represents the energy [*cf.* Equation (4.41) on page 57], the positive solution in Equation (5.30) must be identified with the ordinary Hamilton function H divided by c. Consequently,

$$H \equiv cp^0 = q\phi + c\sqrt{(\mathbf{p}-q\mathbf{A})^2 + m_0^2c^2} \tag{5.31}$$

is the ordinary 3D Hamilton function for a charged particle moving in scalar and vector potentials associated with prescribed electric and magnetic fields.

The ordinary Lagrange and Hamilton functions L and H are related to each other by the 3D transformation [*cf.* the 4D transformation (5.17) between L_4 and H_4]

$$L = \mathbf{p}\cdot\mathbf{v} - H \tag{5.32}$$

Using the explicit expressions (Equation (5.31) on the preceding page) and (Equation (5.32) on the facing page), we obtain the explicit expression for the ordinary 3D Lagrange function

$$L = \mathbf{p} \cdot \mathbf{v} - q\phi - c\sqrt{(\mathbf{p} - q\mathbf{A})^2 + m_0^2 c^2} \tag{5.33}$$

and if we make the identification

$$\mathbf{p} - q\mathbf{A} = \frac{m_0 \mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}} = m\mathbf{v} \tag{5.34}$$

where the quantity $m\mathbf{v}$ is the usual *kinetic momentum*, we can rewrite this expression for the ordinary Lagrangian as follows:

$$\begin{aligned} L &= q\mathbf{A} \cdot \mathbf{v} + mv^2 - q\phi - c\sqrt{m^2 v^2 + m_0^2 c^2} \\ &= mv^2 - q(\phi - \mathbf{A} \cdot \mathbf{v}) - mc^2 = -q\phi + q\mathbf{A} \cdot \mathbf{v} - m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} \end{aligned} \tag{5.35}$$

What we have obtained is the relativstically correct (covariant) expression for the Lagrangian describing the motion of a charged particle in scalar and vector potentials associated with prescribed electric and magnetic fields.

5.2 Covariant field theory

So far, we have considered two classes of problems. Either we have calculated the fields from given, prescribed distributions of charges and currents, or we have derived the equations of motion for charged particles in given, prescribed fields. Let us now put the fields and the particles on an equal footing and present a theoretical description which treats the fields, the particles, and their interactions in a unified way. This involves transition to a field picture with an infinite number of degrees of freedom. We shall first consider a simple mechanical problem whose solution is well known. Then, drawing inferences from this model problem, we apply a similar view on the electromagnetic problem.

5.2.1 Lagrange-Hamilton formalism for fields and interactions

Consider the situation, illustrated in Figure 5.1 on the next page, with N identical mass points, each with mass m and connected to its neighbour along a one-dimensional straight line, which we choose to be the x axis, by identical ideal springs with spring

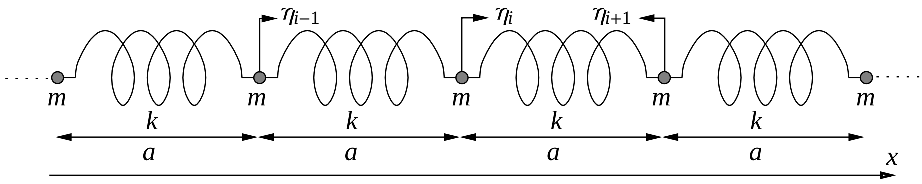

FIGURE 5.1: A one-dimensional chain consisting of N discrete, identical mass points m, connected to their neighbours with identical, ideal springs with spring constants k. The equilibrium distance between the neighbouring mass points is a and $\eta_{i-1}(t)$, $\eta_i(t)$, $\eta_{i+1}(t)$ are the instantaneous deviations, along the x axis, of positions of the $(i-1)$th, ith, and $(i+1)$th mass point, respectively.

constants k. At equilibrium the mass points are at rest, distributed evenly with a distance a to their two nearest neighbours. After perturbation, the motion of mass point i will be a one-dimensional oscillatory motion along $\hat{\boldsymbol{x}}$. Let us denote the deviation for mass point i from its equilibrium position by $\eta_i(t)\hat{\boldsymbol{x}}$.

The solution to this mechanical problem can be obtained if we can find a *Lagrangian* (*Lagrange function*) L which satisfies the variational equation

$$\delta \int L(\eta_i, \dot{\eta}_i, t)\,\mathrm{d}t = 0 \tag{5.36}$$

According to Equation (M.98) on page 180, the Lagrangian is $L = T - V$ where T denotes the *kinetic energy* and V the *potential energy* of a classical mechanical system with *conservative forces*. In our case the Lagrangian is

$$L = \frac{1}{2}\sum_{i=1}^{N}\left[m\dot{\eta}_i^2 - k(\eta_{i+1} - \eta_i)^2\right] \tag{5.37}$$

Let us write the Lagrangian, as given by Equation (5.37), in the following way:

$$L = \sum_{i=1}^{N} a\mathscr{L}_i \tag{5.38}$$

Here,

$$\mathscr{L}_i = \frac{1}{2}\left[\frac{m}{a}\dot{\eta}_i^2 - ka\left(\frac{\eta_{i+1} - \eta_i}{a}\right)^2\right] \tag{5.39}$$

is the so called linear *Lagrange density*. If we now let $N \to \infty$ and, at the same time, let the springs become infinitesimally short according to the following scheme:

$$a \to \mathrm{d}x \tag{5.40a}$$

$$\frac{m}{a} \to \frac{\mathrm{d}m}{\mathrm{d}x} = \mu \qquad \textit{linear mass density} \tag{5.40b}$$

$$ka \to Y \qquad \textit{Young's modulus} \tag{5.40c}$$

$$\frac{\eta_{i+1} - \eta_i}{a} \to \frac{\partial\eta}{\partial x} \tag{5.40d}$$

we obtain

$$L = \int \mathscr{L}\,\mathrm{d}x \tag{5.41}$$

where

$$\mathscr{L}\left(\eta, \frac{\partial\eta}{\partial t}, \frac{\partial\eta}{\partial x}, t\right) = \frac{1}{2}\left[\mu\left(\frac{\partial\eta}{\partial t}\right)^2 - Y\left(\frac{\partial\eta}{\partial x}\right)^2\right] \tag{5.42}$$

Notice how we made a transition from a discrete description, in which the mass points were identified by a discrete integer variable $i = 1, 2, \ldots, N$, to a continuous description, where the infinitesimal mass points were instead identified by a continuous real parameter x, namely their position along $\hat{\boldsymbol{x}}$.

A consequence of this transition is that the number of degrees of freedom for the system went from the finite number N to infinity! Another consequence is that $\mathscr{L}$ has now become dependent also on the partial derivative with respect to x of the 'field coordinate' η. But, as we shall see, the transition is well worth the price because it allows us to treat all fields, be it classical scalar or vectorial fields, or wave functions, spinors and other fields that appear in quantum physics, on an equal footing.

Under the assumption of time independence and fixed endpoints, the variation principle (5.36) on the preceding page yields:

$$\begin{aligned}
&\delta \int L\,dt \\
&\quad = \delta \iint \mathscr{L}\left(\eta, \frac{\partial\eta}{\partial t}, \frac{\partial\eta}{\partial x}\right) \mathrm{d}x\,\mathrm{d}t \\
&\quad = \iint \left[\frac{\partial\mathscr{L}}{\partial\eta}\delta\eta + \frac{\partial\mathscr{L}}{\partial\left(\frac{\partial\eta}{\partial t}\right)}\,\delta\left(\frac{\partial\eta}{\partial t}\right) + \frac{\partial\mathscr{L}}{\partial\left(\frac{\partial\eta}{\partial x}\right)}\,\delta\left(\frac{\partial\eta}{\partial x}\right)\right] \mathrm{d}x\,\mathrm{d}t \\
&\quad = 0
\end{aligned} \tag{5.43}$$

The last integral can be integrated by parts. This results in the expression

$$\iint \left[\frac{\partial\mathscr{L}}{\partial\eta} - \frac{\partial}{\partial t}\left(\frac{\partial\mathscr{L}}{\partial\left(\frac{\partial\eta}{\partial t}\right)}\right) - \frac{\partial}{\partial x}\left(\frac{\partial\mathscr{L}}{\partial\left(\frac{\partial\eta}{\partial x}\right)}\right)\right] \delta\eta\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{5.44}$$

where the variation is arbitrary (and the endpoints fixed). This means that the integrand itself must vanish. If we introduce the *functional derivative*

$$\frac{\delta \mathscr{L}}{\delta \eta} = \frac{\partial \mathscr{L}}{\partial \eta} - \frac{\partial}{\partial x}\left(\frac{\partial \mathscr{L}}{\partial\left(\frac{\partial \eta}{\partial x}\right)}\right) \tag{5.45}$$

we can express this as

$$\frac{\delta \mathscr{L}}{\delta \eta} - \frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial\left(\frac{\partial \eta}{\partial t}\right)}\right) = 0 \tag{5.46}$$

which is the one-dimensional *Euler-Lagrange equation*.

Inserting the linear mass point chain Lagrangian density, Equation (5.42) on the preceding page, into Equation (5.46) above, we obtain the equation of motion for our one-dimensional linear mechanical structure. It is:

$$\mu\frac{\partial^2}{\partial t^2}\eta - Y\frac{\partial^2}{\partial x^2}\eta = \left(\frac{\mu}{Y}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\eta = 0 \tag{5.47}$$

i.e., the one-dimensional wave equation for compression waves which propagate with phase speed $v_\phi = \sqrt{Y/\mu}$ along the linear structure.

A generalisation of the above 1D results to a three-dimensional continuum is straightforward. For this 3D case we get the variational principle

$$\begin{aligned}\delta\int L\,dt &= \delta\iint \mathscr{L}\,\mathrm{d}^3x\,dt \\ &= \delta\int \mathscr{L}\left(\eta, \frac{\partial\eta}{\partial x^\mu}\right)\mathrm{d}^4x \\ &= \iint\left[\frac{\partial \mathscr{L}}{\partial \eta} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathscr{L}}{\partial\left(\frac{\partial \eta}{\partial x^\mu}\right)}\right)\right]\delta\eta\,\mathrm{d}^4x \\ &= 0\end{aligned} \tag{5.48}$$

where the variation $\delta\eta$ is arbitrary and the endpoints are fixed. This means that the integrand itself must vanish:

$$\frac{\partial \mathscr{L}}{\partial \eta} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathscr{L}}{\partial\left(\frac{\partial \eta}{\partial x^\mu}\right)}\right) = 0 \tag{5.49}$$

This constitutes the four-dimensional *Euler-Lagrange equations*.

Introducing the *three-dimensional functional derivative*

$$\frac{\delta \mathscr{L}}{\delta \eta} = \frac{\partial \mathscr{L}}{\partial \eta} - \frac{\partial}{\partial x^i}\left(\frac{\partial \mathscr{L}}{\partial\left(\frac{\partial \eta}{\partial x^i}\right)}\right) \tag{5.50}$$

we can express this as

$$\frac{\delta \mathscr{L}}{\delta \eta} - \frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial\left(\frac{\partial \eta}{\partial t}\right)}\right) = 0 \tag{5.51}$$

In analogy with particle mechanics (finite number of degrees of freedom), we may introduce the *canonically conjugate momentum density*

$$\pi(x^\mu) = \pi(t, \mathbf{x}) = \frac{\partial \mathscr{L}}{\partial\left(\frac{\partial \eta}{\partial t}\right)} \tag{5.52}$$

and define the *Hamilton density*

$$\mathscr{H}\left(\pi, \eta, \frac{\partial \eta}{\partial x^i}; t\right) = \pi \frac{\partial \eta}{\partial t} - \mathscr{L}\left(\eta, \frac{\partial \eta}{\partial t}, \frac{\partial \eta}{\partial x^i}\right) \tag{5.53}$$

If, as usual, we differentiate this expression and identify terms, we obtain the following *Hamilton density equations*

$$\frac{\partial \mathscr{H}}{\partial \pi} = \frac{\partial \eta}{\partial t} \tag{5.54a}$$

$$\frac{\delta \mathscr{H}}{\delta \eta} = -\frac{\partial \pi}{\partial t} \tag{5.54b}$$

The Hamilton density functions are in many ways similar to the ordinary Hamilton functions and lead to similar results.

The electromagnetic field

Above, when we described the mechanical field, we used a scalar field $\eta(t, \mathbf{x})$. If we want to describe the electromagnetic field in terms of a Lagrange density $\mathscr{L}$ and Euler-Lagrange equations, it comes natural to express $\mathscr{L}$ in terms of the four-potential $A^\mu(x^\kappa)$.

The entire system of particles and fields consists of a mechanical part, a field part and an interaction part. We therefore assume that the total Lagrange density $\mathscr{L}^{\text{tot}}$ for this system can be expressed as

$$\mathscr{L}^{\text{tot}} = \mathscr{L}^{\text{mech}} + \mathscr{L}^{\text{inter}} + \mathscr{L}^{\text{field}} \tag{5.55}$$

where the mechanical part has to do with the particle motion (kinetic energy). It is given by L_4/V where L_4 is given by Equation (5.3) on page 68 and V is the volume. Expressed in the *rest mass density* ϱ_0, the *mechanical Lagrange density* can be written

$$\mathscr{L}^{\text{mech}} = \frac{1}{2}\varrho_0 u^\mu u_\mu \tag{5.56}$$

The $\mathscr{L}^{\text{inter}}$ part describes the interaction between the charged particles and the external electromagnetic field. A convenient expression for this *interaction Lagrange density* is

$$\mathscr{L}^{\text{inter}} = j^{\mu} A_{\mu} \tag{5.57}$$

For the field part $\mathscr{L}^{\text{field}}$ we choose the difference between magnetic and electric energy density (in analogy with the difference between kinetic and potential energy in a mechanical field). Using the field tensor, we express this *field Lagrange density* as

$$\mathscr{L}^{\text{field}} = \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} \tag{5.58}$$

so that the total Lagrangian density can be written

$$\mathscr{L}^{\text{tot}} = \frac{1}{2}\varrho_0 u^{\mu} u_{\mu} + j^{\mu} A_{\mu} + \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} \tag{5.59}$$

From this we can calculate all physical quantities.

EXAMPLE 5.1 ▷FIELD ENERGY DIFFERENCE EXPRESSED IN THE FIELD TENSOR

Show, by explicit calculation, that

$$\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} = \frac{1}{2}\left(\frac{B^2}{\mu_0} - \varepsilon_0 E^2\right) \tag{5.60}$$

i.e., the difference between the magnetic and electric field energy densities.

From Formula (4.79) on page 62 we recall that

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{5.61}$$

and from Formula (4.81) on page 63 that

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{5.62}$$

where μ denotes the row number and ν the column number. Then, Einstein summation and direct substitution yields

$$
\begin{aligned}
F^{\mu\nu}F_{\mu\nu} &= F^{00}F_{00} + F^{01}F_{01} + F^{02}F_{02} + F^{03}F_{03} \\
&\quad + F^{10}F_{10} + F^{11}F_{11} + F^{12}F_{12} + F^{13}F_{13} \\
&\quad + F^{20}F_{20} + F^{21}F_{21} + F^{22}F_{22} + F^{23}F_{23} \\
&\quad + F^{30}F_{30} + F^{31}F_{31} + F^{32}F_{32} + F^{33}F_{33} \\
&= 0 - E_x^2/c^2 - E_y^2/c^2 - E_z^2/c^2 \\
&\quad - E_x^2/c^2 + 0 + B_z^2 + B_y^2 \\
&\quad - E_y^2/c^2 + B_z^2 + 0 + B_x^2 \\
&\quad - E_z^2/c^2 + B_y^2 + B_x^2 + 0 \\
&= -2E_x^2/c^2 - 2E_y^2/c^2 - 2E_z^2/c^2 + 2B_x^2 + 2B_y^2 + 2B_z^2 \\
&= -2E^2/c^2 + 2B^2 = 2(B^2 - E^2/c^2)
\end{aligned}
\tag{5.63}
$$

or

$$
\frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}\left(\frac{B^2}{\mu_0} - \frac{1}{c^2\mu_0}E^2\right) = \frac{1}{2}\left(\frac{B^2}{\mu_0} - \varepsilon_0 E^2\right) \tag{5.64}
$$

where, in the last step, the identity $\varepsilon_0\mu_0 = 1/c^2$ was used. QED ■

◁ END OF EXAMPLE 5.1

Using $\mathscr{L}^{\text{tot}}$ in the 3D Euler-Lagrange equations, Equation (5.49) on page 76 (with η replaced by A_ν), we can derive the dynamics for the whole system. For instance, the electromagnetic part of the Lagrangian density

$$
\mathscr{L}^{\text{EM}} = \mathscr{L}^{\text{inter}} + \mathscr{L}^{\text{field}} = j^\nu A_\nu + \frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu} \tag{5.65}
$$

inserted into the Euler-Lagrange equations, expression (5.49) on page 76, yields two of Maxwell's equations. To see this, we note from Equation (5.65) above and the results in Example 5.1 that

$$
\frac{\partial \mathscr{L}^{\text{EM}}}{\partial A_\nu} = j^\nu \tag{5.66}
$$

Furthermore,

$$
\begin{aligned}
\partial_\mu\left[\frac{\partial \mathscr{L}^{\text{EM}}}{\partial(\partial_\mu A_\nu)}\right] &= \frac{1}{4\mu_0}\partial_\mu\left[\frac{\partial}{\partial(\partial_\mu A_\nu)}\left(F^{\kappa\lambda}F_{\kappa\lambda}\right)\right] \\
&= \frac{1}{4\mu_0}\partial_\mu\left\{\frac{\partial}{\partial(\partial_\mu A_\nu)}\left[(\partial^\kappa A^\lambda - \partial^\lambda A^\kappa)(\partial_\kappa A_\lambda - \partial_\lambda A_\kappa)\right]\right\} \\
&= \frac{1}{4\mu_0}\partial_\mu\left\{\frac{\partial}{\partial(\partial_\mu A_\nu)}\left[\partial^\kappa A^\lambda\partial_\kappa A_\lambda - \partial^\kappa A^\lambda\partial_\lambda A_\kappa \right.\right. \\
&\qquad \left.\left. - \partial^\lambda A^\kappa\partial_\kappa A_\lambda + \partial^\lambda A^\kappa\partial_\lambda A_\kappa\right]\right\} \\
&= \frac{1}{2\mu_0}\partial_\mu\left[\frac{\partial}{\partial(\partial_\mu A_\nu)}\left(\partial^\kappa A^\lambda\partial_\kappa A_\lambda - \partial^\kappa A^\lambda\partial_\lambda A_\kappa\right)\right]
\end{aligned}
\tag{5.67}
$$

But

$$\begin{aligned}\frac{\partial}{\partial(\partial_\mu A_\nu)}\left(\partial^\kappa A^\lambda \partial_\kappa A_\lambda\right) &= \partial^\kappa A^\lambda \frac{\partial}{\partial(\partial_\mu A_\nu)}\partial_\kappa A_\lambda + \partial_\kappa A_\lambda \frac{\partial}{\partial(\partial_\mu A_\nu)}\partial^\kappa A^\lambda \\ &= \partial^\kappa A^\lambda \frac{\partial}{\partial(\partial_\mu A_\nu)}\partial_\kappa A_\lambda + \partial_\kappa A_\lambda \frac{\partial}{\partial(\partial_\mu A_\nu)} g^{\kappa\alpha}\partial_\alpha g^{\lambda\beta} A_\beta \\ &= \partial^\kappa A^\lambda \frac{\partial}{\partial(\partial_\mu A_\nu)}\partial_\kappa A_\lambda + g^{\kappa\alpha} g^{\lambda\beta}\partial_\kappa A_\lambda \frac{\partial}{\partial(\partial_\mu A_\nu)}\partial_\alpha A_\beta \\ &= \partial^\kappa A^\lambda \frac{\partial}{\partial(\partial_\mu A_\nu)}\partial_\kappa A_\lambda + \partial^\alpha A^\beta \frac{\partial}{\partial(\partial_\mu A_\nu)}\partial_\alpha A_\beta \\ &= 2\partial^\mu A^\nu\end{aligned} \tag{5.68}$$

Similarly,

$$\frac{\partial}{\partial(\partial_\mu A_\nu)}\left(\partial^\kappa A^\lambda \partial_\lambda A_\kappa\right) = 2\partial^\nu A^\mu \tag{5.69}$$

so that

$$\partial_\mu \left[\frac{\partial \mathscr{L}^{\mathrm{EM}}}{\partial(\partial_\mu A_\nu)}\right] = \frac{1}{\mu_0}\partial_\mu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = \frac{1}{\mu_0}\partial_\mu F^{\mu\nu} \tag{5.70}$$

This means that the Euler-Lagrange equations, expression (5.49) on page 76, for the Lagrangian density $\mathscr{L}^{\mathrm{EM}}$ and with A_ν as the field quantity become

$$\frac{\partial \mathscr{L}^{\mathrm{EM}}}{\partial A_\nu} - \partial_\mu\left[\frac{\partial \mathscr{L}^{\mathrm{EM}}}{\partial(\partial_\mu A_\nu)}\right] = j^\nu - \frac{1}{\mu_0}\partial_\mu F^{\mu\nu} = 0 \tag{5.71}$$

or

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu \tag{5.72}$$

which, according to Equation (4.82) on page 63, is the covariant version of Maxwell's source equations.

Other fields

In general, the dynamic equations for most any fields, and not only electromagnetic ones, can be derived from a Lagrangian density together with a variational principle (the Euler-Lagrange equations). Both linear and non-linear fields are studied with this technique. As a simple example, consider a real, scalar field η which has the following Lagrange density:

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu\eta\partial^\mu\eta - m^2\eta^2\right) \tag{5.73}$$

Insertion into the 1D Euler-Lagrange equation, Equation (5.46) on page 76, yields the dynamic equation

$$(\Box^2 - m^2)\eta = 0 \tag{5.74}$$

with the solution

$$\eta = e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)}\frac{e^{-m|\mathbf{x}|}}{|\mathbf{x}|} \tag{5.75}$$

which describes the *Yukawa meson field* for a scalar meson with mass m. With

$$\pi = \frac{1}{c^2}\frac{\partial\eta}{\partial t} \tag{5.76}$$

we obtain the Hamilton density

$$\mathscr{H} = \frac{1}{2}\left[c^2\pi^2 + (\boldsymbol{\nabla}\eta)^2 + m^2\eta^2\right] \tag{5.77}$$

which is positive definite.

Another Lagrangian density which has attracted quite some interest is the *Proca Lagrangian*

$$\mathscr{L}^{\mathrm{EM}} = \mathscr{L}^{\mathrm{inter}} + \mathscr{L}^{\mathrm{field}} = j^\nu A_\nu + \frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu} + m^2A^\mu A_\mu \tag{5.78}$$

which leads to the dynamic equation

$$\partial_\mu F^{\mu\nu} - m^2A^\nu = \mu_0 j^\nu \tag{5.79}$$

This equation describes an electromagnetic field with a mass, or, in other words, *massive photons*. If massive photons would exist, large-scale magnetic fields, including those of the earth and galactic spiral arms, would be significantly modified to yield measurable discrepances from their usual form. Space experiments of this kind onboard satellites have led to stringent upper bounds on the photon mass. If the photon really has a mass, it will have an impact on electrodynamics as well as on cosmology and astrophysics.

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CHAPTER 6

Electromagnetic Fields and Matter

The microscopic Maxwell equations (1.45) derived in Chapter 1 are valid on all scales where a classical description is good. However, when macroscopic matter is present, it is sometimes convenient to use the corresponding macroscopic Maxwell equations (in a statistical sense) in which auxiliary, derived fields are introduced in order to incorporate effects of macroscopic matter when this is immersed fully or partially in an electromagnetic field.

6.1 Electric polarisation and displacement

In certain cases, for instance in engineering applications, it may be convenient to separate the influence of an external electric field on free charges on the one hand and on neutral matter in bulk on the other. This view, which, as we shall see, has certain limitations, leads to the introduction of (di)electric polarisation and magnetisation which, in turn, justifies the introduction of two help quantities, the *electric displacement vector* $\mathbf{D}$ and the *magnetising field* $\mathbf{H}$.

6.1.1 Electric multipole moments

The electrostatic properties of a spatial volume containing electric charges and located near a point $\mathbf{x}_0$ can be characterized in terms of the *total charge* or *electric monopole moment*

$$q = \int_{V'} \mathrm{d}^3x' \, \rho(\mathbf{x}') \tag{6.1}$$

where the ρ is the charge density introduced in Equation (1.7) on page 5, the *electric dipole moment vector*

$$\mathbf{p}(\mathbf{x}_0) = \int_{V'} \mathrm{d}^3x' \; (\mathbf{x}' - \mathbf{x}_0)\,\rho(\mathbf{x}') \tag{6.2}$$

with components p_i, $i = 1, 2, 3$, the *electric quadrupole moment tensor*

$$\mathbf{Q}(\mathbf{x}_0) = \int_{V'} \mathrm{d}^3x' \; (\mathbf{x}' - \mathbf{x}_0)(\mathbf{x}' - \mathbf{x}_0)\,\rho(\mathbf{x}') \tag{6.3}$$

with components Q_{ij}, $i, j = 1, 2, 3$, and higher order electric moments.

In particular, the electrostatic potential Equation (3.3) on page 35 from a charge distribution located near $\mathbf{x}_0$ can be Taylor expanded in the following way:

$$\begin{aligned}\phi^{\mathrm{stat}}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\Bigg[&\frac{q}{|\mathbf{x}-\mathbf{x}_0|} + \frac{1}{|\mathbf{x}-\mathbf{x}_0|^2}\, p_i \frac{(\mathbf{x}-\mathbf{x}_0)_i}{|\mathbf{x}-\mathbf{x}_0|} \\ &+ \frac{1}{|\mathbf{x}-\mathbf{x}_0|^3} Q_{ij}\left(\frac{3}{2}\frac{(\mathbf{x}-\mathbf{x}_0)_i}{|\mathbf{x}-\mathbf{x}_0|}\frac{(\mathbf{x}-\mathbf{x}_0)_j}{|\mathbf{x}-\mathbf{x}_0|} - \frac{1}{2}\delta_{ij}\right) + \ldots\Bigg]\end{aligned} \tag{6.4}$$

where Einstein's summation convention over i and j is implied. As can be seen from this expression, only the first few terms are important if the field point (observation point) is far away from $\mathbf{x}_0$.

For a normal medium, the major contributions to the electrostatic interactions come from the net charge and the lowest order electric multipole moments induced by the polarisation due to an applied electric field. Particularly important is the dipole moment. Let $\mathbf{P}$ denote the electric dipole moment density (electric dipole moment per unit volume; unit: $\mathrm{C/m^2}$), also known as the *electric polarisation*, in some medium. In analogy with the second term in the expansion Equation (6.4) above, the electric potential from this volume distribution $\mathbf{P}(\mathbf{x}')$ of electric dipole moments $\mathbf{p}$ at the source point $\mathbf{x}'$ can be written

$$\begin{aligned}\phi_{\mathbf{p}}(\mathbf{x}) &= \frac{1}{4\pi\varepsilon_0}\int_{V'} \mathrm{d}^3x' \; \mathbf{P}(\mathbf{x}') \cdot \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = -\frac{1}{4\pi\varepsilon_0}\int_{V'} \mathrm{d}^3x' \; \mathbf{P}(\mathbf{x}') \cdot \boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \\ &= \frac{1}{4\pi\varepsilon_0}\int_{V'} \mathrm{d}^3x' \; \mathbf{P}(\mathbf{x}') \cdot \boldsymbol{\nabla}'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\end{aligned} \tag{6.5}$$

Using the expression Equation (M.85) on page 177 and applying the divergence theorem, we can rewrite this expression for the potential as follows:

$$\begin{aligned}\phi_{\mathbf{p}}(\mathbf{x}) &= \frac{1}{4\pi\varepsilon_0}\left[\int_{V'} \mathrm{d}^3x' \; \boldsymbol{\nabla}' \cdot \left(\frac{\mathbf{P}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right) - \int_{V'} \mathrm{d}^3x' \; \frac{\boldsymbol{\nabla}' \cdot \mathbf{P}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right] \\ &= \frac{1}{4\pi\varepsilon_0}\left[\oint_{S'} \mathrm{d}^2x' \; \frac{\mathbf{P}(\mathbf{x}') \cdot \hat{\mathbf{n}}}{|\mathbf{x}-\mathbf{x}'|} - \int_{V'} \mathrm{d}^3x' \; \frac{\boldsymbol{\nabla}' \cdot \mathbf{P}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right]\end{aligned} \tag{6.6}$$

where the first term, which describes the effects of the induced, non-cancelling dipole moment on the surface of the volume, can be neglected, unless there is a discontinuity in $\mathbf{P} \cdot \hat{\boldsymbol{n}}$ at the surface. Doing so, we find that the contribution from the electric dipole moments to the potential is given by

$$\phi_{\mathbf{p}} = \frac{1}{4\pi\varepsilon_0} \int_{V'} \mathrm{d}^3x' \, \frac{-\boldsymbol{\nabla}' \cdot \mathbf{P}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \tag{6.7}$$

Comparing this expression with expression Equation (3.3) on page 35 for the electrostatic potential from a static charge distribution ρ, we see that $-\boldsymbol{\nabla} \cdot \mathbf{P}(\mathbf{x})$ has the characteristics of a charge density and that, to the lowest order, the effective charge density becomes $\rho(\mathbf{x}) - \boldsymbol{\nabla} \cdot \mathbf{P}(\mathbf{x})$, in which the second term is a polarisation term.

The version of Equation (1.7) on page 5 where free, 'true' charges and bound, *polarisation charges* are separated thus becomes

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho^{\text{true}}(\mathbf{x}) - \boldsymbol{\nabla} \cdot \mathbf{P}(\mathbf{x})}{\varepsilon_0} \tag{6.8}$$

Rewriting this equation, and at the same time introducing the *electric displacement vector* (C/m^2)

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \tag{6.9}$$

we obtain

$$\boldsymbol{\nabla} \cdot (\varepsilon_0 \mathbf{E} + \mathbf{P}) = \boldsymbol{\nabla} \cdot \mathbf{D} = \rho^{\text{true}}(\mathbf{x}) \tag{6.10}$$

where ρ^{true} is the 'true' charge density in the medium. This is one of Maxwell's equations and is valid also for time varying fields. By introducing the notation $\rho^{\text{pol}} = -\boldsymbol{\nabla} \cdot \mathbf{P}$ for the 'polarised' charge density in the medium, and $\rho^{\text{total}} = \rho^{\text{true}} + \rho^{\text{pol}}$ for the 'total' charge density, we can write down the following alternative version of Maxwell's equation (6.22a) on page 87

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho^{\text{total}}(\mathbf{x})}{\varepsilon_0} \tag{6.11}$$

Often, for low enough field strengths $|\mathbf{E}|$, the linear and isotropic relationship between $\mathbf{P}$ and $\mathbf{E}$

$$\mathbf{P} = \varepsilon_0 \chi \mathbf{E} \tag{6.12}$$

is a good approximation. The quantity χ is the *electric susceptibility* which is material dependent. For electromagnetically anisotropic media such as a magnetised plasma or a birefringent crystal, the susceptibility is a tensor. In general, the relationship is not of a simple linear form as in Equation (6.12) above but non-linear terms are important. In such a situation the principle of superposition is no longer valid and non-linear effects such as frequency conversion and mixing can be expected.

Inserting the approximation (6.12) into Equation (6.9) on the preceding page, we can write the latter

$$\mathbf{D} = \varepsilon\mathbf{E} \tag{6.13}$$

where, approximately,

$$\varepsilon = \varepsilon_0(1 + \chi) \tag{6.14}$$

6.2 Magnetisation and the magnetising field

An analysis of the properties of stationary magnetic media and the associated currents shows that three such types of currents exist:

1. In analogy with 'true' charges for the electric case, we may have 'true' currents $\mathbf{j}^{\text{true}}$, *i.e.*, a physical transport of true charges.
2. In analogy with electric polarisation $\mathbf{P}$ there may be a form of charge transport associated with the changes of the polarisation with time. Such currents, induced by an external field, are called *polarisation currents* and are identified with $\partial\mathbf{P}/\partial t$.
3. There may also be intrinsic currents of a microscopic, often atomic, nature that are inaccessible to direct observation, but which may produce net effects at discontinuities and boundaries. These *magnetisation currents* are denoted $\mathbf{j}^{\text{M}}$.

No magnetic monopoles have been observed yet. So there is no correspondence in the magnetic case to the electric monopole moment (6.1). The lowest order magnetic moment, corresponding to the electric dipole moment (6.2), is the *magnetic dipole moment*

$$\mathbf{m} = \frac{1}{2}\int_{V'} \mathrm{d}^3x' \, (\mathbf{x}' - \mathbf{x}_0) \times \mathbf{j}(\mathbf{x}') \tag{6.15}$$

For a distribution of magnetic dipole moments in a volume, we may describe this volume in terms of the *magnetisation*, or magnetic dipole moment per unit volume, $\mathbf{M}$. Via the definition of the vector potential one can show that the magnetisation current and the magnetisation is simply related:

$$\mathbf{j}^{\text{M}} = \boldsymbol{\nabla} \times \mathbf{M} \tag{6.16}$$

In a stationary medium we therefore have a total current which is (approximately) the sum of the three currents enumerated above:

$$\mathbf{j}^{\text{total}} = \mathbf{j}^{\text{true}} + \frac{\partial\mathbf{P}}{\partial t} + \boldsymbol{\nabla} \times \mathbf{M} \tag{6.17}$$

One might then, erroneously, be led to think that

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{j}^{\text{true}} + \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M} \right) \qquad \textbf{(INCORRECT)}$$

Moving the term $\nabla \times \mathbf{M}$ to the left hand side and introducing the *magnetising field* (*magnetic field intensity*, *Ampère-turn density*) as

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \tag{6.18}$$

and using the definition for **D**, Equation (6.9) on page 85, we can write this incorrect equation in the following form

$$\nabla \times \mathbf{H} = \mathbf{j}^{\text{true}} + \frac{\partial \mathbf{P}}{\partial t} = \mathbf{j}^{\text{true}} + \frac{\partial \mathbf{D}}{\partial t} - \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{6.19}$$

As we see, in this simplistic view, we would pick up a term which makes the equation inconsistent; the divergence of the left hand side vanishes while the divergence of the right hand side does not. Maxwell realised this and to overcome this inconsistency he was forced to add his famous displacement current term which precisely compensates for the last term in the right hand side. In Chapter 1, we discussed an alternative way, based on the postulate of conservation of electric charge, to introduce the displacement current.

We may, in analogy with the electric case, introduce a *magnetic susceptibility* for the medium. Denoting it χ_{m}, we can write

$$\mathbf{H} = \frac{\mathbf{B}}{\mu} \tag{6.20}$$

where, approximately,

$$\mu = \mu_0(1 + \chi_{\text{m}}) \tag{6.21}$$

Maxwell's equations expressed in terms of the derived field quantities **D** and **H** are

$$\nabla \cdot \mathbf{D} = \rho(t, \mathbf{x}) \tag{6.22a}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{6.22b}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{6.22c}$$

$$\nabla \times \mathbf{H} = \mathbf{j}(t, \mathbf{x}) + \frac{\partial}{\partial t}\mathbf{D} \tag{6.22d}$$

and are called *Maxwell's macroscopic equations*. These equations are convenient to use in certain simple cases. Together with the boundary conditions and the constitutive relations, they describe uniquely (but only approximately!) the properties of the electric and magnetic fields in matter.

6.3 Energy and momentum

We shall use Maxwell's macroscopic equations in the following considerations on the energy and momentum of the electromagnetic field and its interaction with matter.

6.3.1 The energy theorem in Maxwell's theory

Scalar multiplying (6.22c) by **H**, (6.22d) by **E** and subtracting, we obtain

$$\begin{aligned}\mathbf{H}\cdot(\boldsymbol{\nabla}\times\mathbf{E})-\mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{H}) &= \boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{H})\\ &= -\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}-\mathbf{E}\cdot\mathbf{j}-\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} = -\frac{1}{2}\frac{\partial}{\partial t}(\mathbf{H}\cdot\mathbf{B}+\mathbf{E}\cdot\mathbf{D})-\mathbf{j}\cdot\mathbf{E}\end{aligned} \tag{6.23}$$

Integration over the entire volume V and using Gauss's theorem (the divergence theorem), we obtain

$$-\frac{\partial}{\partial t}\int_{V'}\mathrm{d}^3x'\,\frac{1}{2}(\mathbf{H}\cdot\mathbf{B}+\mathbf{E}\cdot\mathbf{D}) = \int_{V'}\mathrm{d}^3x'\,\mathbf{j}\cdot\mathbf{E}+\oint_{S'}\mathrm{d}^2x'\,(\mathbf{E}\times\mathbf{H})\cdot\hat{\boldsymbol{n}} \tag{6.24}$$

We assume the validity of Ohm's law so that in the presence of an electromotive force field, we make the linear approximation Equation (1.28) on page 11:

$$\mathbf{j}=\sigma(\mathbf{E}+\mathbf{E}^{\mathrm{EMF}}) \tag{6.25}$$

which means that

$$\int_{V'}\mathrm{d}^3x'\,\mathbf{j}\cdot\mathbf{E} = \int_{V'}\mathrm{d}^3x'\,\frac{j^2}{\sigma}-\int_{V'}\mathrm{d}^3x'\,\mathbf{j}\cdot\mathbf{E}^{\mathrm{EMF}} \tag{6.26}$$

Inserting this into Equation (6.24) above, one obtains

$$\underbrace{\int_{V'}\mathrm{d}^3x'\,\mathbf{j}\cdot\mathbf{E}^{\mathrm{EMF}}}_{\text{Applied electric power}} = \underbrace{\int_{V'}\mathrm{d}^3x'\,\frac{j^2}{\sigma}}_{\text{Joule heat}}+\frac{\partial}{\partial t}\underbrace{\int_{V'}\mathrm{d}^3x'\,\frac{1}{2}(\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B})}_{\text{Field energy}}+\underbrace{\oint_{S'}\mathrm{d}^2x'\,(\mathbf{E}\times\mathbf{H})\cdot\hat{\boldsymbol{n}}}_{\text{Radiated power}} \tag{6.27}$$

which is the *energy theorem in Maxwell's theory* also known as *Poynting's theorem*.

It is convenient to introduce the following quantities:

$$U_{\mathrm{e}}=\frac{1}{2}\int_{V'}\mathrm{d}^3x'\,\mathbf{E}\cdot\mathbf{D} \tag{6.28}$$

$$U_{\mathrm{m}}=\frac{1}{2}\int_{V'}\mathrm{d}^3x'\,\mathbf{H}\cdot\mathbf{B} \tag{6.29}$$

$$\mathbf{S}=\mathbf{E}\times\mathbf{H} \tag{6.30}$$

where U_{e} is the *electric field energy*, U_{m} is the *magnetic field energy*, both measured in J, and **S** is the *Poynting vector* (*power flux*), measured in $\mathrm{W/m^2}$.

6.3.2 The momentum theorem in Maxwell's theory

Let us now investigate the momentum balance (force actions) in the case that a field interacts with matter in a non-relativistic way. For this purpose we consider the force density given by the *Lorentz force* per unit volume $\rho\mathbf{E} + \mathbf{j} \times \mathbf{B}$. Using Maxwell's equations (6.22) and symmetrising, we obtain

$$
\begin{aligned}
\rho\mathbf{E} + \mathbf{j} \times \mathbf{B} &= (\nabla \cdot \mathbf{D})\mathbf{E} + \left(\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t}\right) \times \mathbf{B} \\
&= \mathbf{E}(\nabla \cdot \mathbf{D}) + (\nabla \times \mathbf{H}) \times \mathbf{B} - \frac{\partial \mathbf{D}}{\partial t} \times \mathbf{B} \\
&= \mathbf{E}(\nabla \cdot \mathbf{D}) - \mathbf{B} \times (\nabla \times \mathbf{H}) \\
&\quad - \frac{\partial}{\partial t}(\mathbf{D} \times \mathbf{B}) + \mathbf{D} \times \frac{\partial \mathbf{B}}{\partial t} \\
&= \mathbf{E}(\nabla \cdot \mathbf{D}) - \mathbf{B} \times (\nabla \times \mathbf{H}) \\
&\quad - \frac{\partial}{\partial t}(\mathbf{D} \times \mathbf{B}) - \mathbf{D} \times (\nabla \times \mathbf{E}) + \mathbf{H}(\underbrace{\nabla \cdot \mathbf{B}}_{=0}) \\
&= [\mathbf{E}(\nabla \cdot \mathbf{D}) - \mathbf{D} \times (\nabla \times \mathbf{E})] + [\mathbf{H}(\nabla \cdot \mathbf{B}) - \mathbf{B} \times (\nabla \times \mathbf{H})] \\
&\quad - \frac{\partial}{\partial t}(\mathbf{D} \times \mathbf{B})
\end{aligned}
\tag{6.31}
$$

One verifies easily that the ith vector components of the two terms in square brackets in the right hand member of (6.31) can be expressed as

$$
[\mathbf{E}(\nabla \cdot \mathbf{D}) - \mathbf{D} \times (\nabla \times \mathbf{E})]_i = \frac{1}{2}\left(\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial x_i} - \mathbf{D} \cdot \frac{\partial \mathbf{E}}{\partial x_i}\right) + \frac{\partial}{\partial x_j}\left(E_i D_j - \frac{1}{2}\mathbf{E} \cdot \mathbf{D}\,\delta_{ij}\right)
\tag{6.32}
$$

and

$$
[\mathbf{H}(\nabla \cdot \mathbf{B}) - \mathbf{B} \times (\nabla \times \mathbf{H})]_i = \frac{1}{2}\left(\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial x_i} - \mathbf{B} \cdot \frac{\partial \mathbf{H}}{\partial x_i}\right) + \frac{\partial}{\partial x_j}\left(H_i B_j - \frac{1}{2}\mathbf{B} \cdot \mathbf{H}\,\delta_{ij}\right)
\tag{6.33}
$$

respectively.

Using these two expressions in the ith component of Equation (6.31) and reshuffling terms, we get

$$
\begin{aligned}
&(\rho\mathbf{E} + \mathbf{j} \times \mathbf{B})_i - \frac{1}{2}\left[\left(\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial x_i} - \mathbf{D} \cdot \frac{\partial \mathbf{E}}{\partial x_i}\right) + \left(\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial x_i} - \mathbf{B} \cdot \frac{\partial \mathbf{H}}{\partial x_i}\right)\right] + \frac{\partial}{\partial t}(\mathbf{D} \times \mathbf{B})_i \\
&= \frac{\partial}{\partial x_j}\left(E_i D_j - \frac{1}{2}\mathbf{E} \cdot \mathbf{D}\,\delta_{ij} + H_i B_j - \frac{1}{2}\mathbf{H} \cdot \mathbf{B}\,\delta_{ij}\right)
\end{aligned}
\tag{6.34}
$$

Introducing the *electric volume force* $\mathbf{F}_{\text{ev}}$ via its ith component

$$(\mathbf{F}_{\text{ev}})_i = (\rho\mathbf{E} + \mathbf{j} \times \mathbf{B})_i - \frac{1}{2}\left[\left(\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial x_i} - \mathbf{D} \cdot \frac{\partial \mathbf{E}}{\partial x_i}\right) + \left(\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial x_i} - \mathbf{B} \cdot \frac{\partial \mathbf{H}}{\partial x_i}\right)\right] \tag{6.35}$$

and the *Maxwell stress tensor* **T** with components

$$T_{ij} = E_i D_j - \frac{1}{2}\mathbf{E} \cdot \mathbf{D}\,\delta_{ij} + H_i B_j - \frac{1}{2}\mathbf{H} \cdot \mathbf{B}\,\delta_{ij} \tag{6.36}$$

we finally obtain the force equation

$$\left[\mathbf{F}_{\text{ev}} + \frac{\partial}{\partial t}(\mathbf{D} \times \mathbf{B})\right]_i = \frac{\partial T_{ij}}{\partial x_j} = (\nabla \cdot \mathbf{T})_i \tag{6.37}$$

If we introduce the *relative electric permittivity* κ and the *relative magnetic permeability* κ_{m} as

$$\mathbf{D} = \kappa\varepsilon_0\mathbf{E} = \varepsilon\mathbf{E} \tag{6.38}$$

$$\mathbf{B} = \kappa_{\text{m}}\mu_0\mathbf{H} = \mu\mathbf{H} \tag{6.39}$$

we can rewrite (6.37) as

$$\frac{\partial T_{ij}}{\partial x_j} = \left(\mathbf{F}_{\text{ev}} + \frac{\kappa\kappa_{\text{m}}}{c^2}\frac{\partial \mathbf{S}}{\partial t}\right)_i \tag{6.40}$$

where **S** is the Poynting vector defined in Equation (6.28) on page 88. Integration over the entire volume V yields

$$\underbrace{\int_{V'} \mathrm{d}^3x'\;\mathbf{F}_{\text{ev}}}_{\text{Force on the matter}} + \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int_{V'} \mathrm{d}^3x'\;\frac{\kappa\kappa_{\text{m}}}{c^2}\mathbf{S}}_{\text{Field momentum}} = \underbrace{\oint_{S'} \mathrm{d}^2x'\;\mathbf{T}_{\hat{\boldsymbol{n}}}}_{\text{Maxwell stress}} \tag{6.41}$$

which expresses the balance between the force on the matter, the rate of change of the electromagnetic field momentum and the Maxwell stress. This equation is called the *momentum theorem in Maxwell's theory*.

In vacuum (6.41) becomes

$$\int_{V'} \mathrm{d}^3x'\;\rho(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{V'} \mathrm{d}^3x'\;\mathbf{S} = \oint_{S'} \mathrm{d}^2x'\;\mathbf{T}_{\hat{\boldsymbol{n}}} \tag{6.42}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}^{\text{mech}} + \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}^{\text{field}} = \oint_{S'} \mathrm{d}^2x'\;\mathbf{T}_{\hat{\boldsymbol{n}}} \tag{6.43}$$

6.4 Bibliography

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CHAPTER 7

Electromagnetic Fields from Arbitrary Source Distributions

While, in principle, the electric and magnetic fields can be calculated from the Maxwell equations in Chapter 1, or even from the wave equations in Chapter 2, it is often physically more lucid to calculate them from the electromagnetic potentials derived in Chapter 3. In this chapter we will derive the electric and magnetic fields from the potentials.

We recall that in order to find the solution (3.31) for the generic inhomogeneous wave equation (3.15) on page 38 we presupposed the existence of a Fourier transform pair (3.16a) on page 39 for the generic source term

$$f(t, \mathbf{x}) = \int_{-\infty}^{\infty} \mathrm{d}\omega\, f_\omega(\mathbf{x})\, e^{-\mathrm{i}\omega t} \tag{7.1a}$$

$$f_\omega(\mathbf{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t\, f(t, \mathbf{x})\, e^{\mathrm{i}\omega t} \tag{7.1b}$$

That such transform pairs exist is true for most physical variables which are neither strictly monotonically increasing nor strictly monotonically decreasing with time. For charge and current densities varying in time we can therefore, without loss of generality, work with individual Fourier components $\rho_\omega(\mathbf{x})$ and $\mathbf{j}_\omega(\mathbf{x})$, respectively. Strictly speaking, the existence of a single Fourier component assumes a *monochromatic* source (*i.e.*, a source containing only one single frequency component), which in turn requires that the electric and magnetic fields exist for infinitely long times. However, by taking the proper limits, we may still use this approach even for sources and fields of finite duration.

This is the method we shall utilise in this chapter in order to derive the electric and magnetic fields in vacuum from arbitrary given charge densities $\rho(t, \mathbf{x})$ and current densities $\mathbf{j}(t,\mathbf{x})$, defined by the temporal Fourier transform pairs

$$\rho(t,\mathbf{x}) = \int_{-\infty}^{\infty} \mathrm{d}\omega\, \rho_\omega(\mathbf{x})\, e^{-\mathrm{i}\omega t} \tag{7.2a}$$

$$\rho_\omega(\mathbf{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t\, \rho(t,\mathbf{x})\, e^{\mathrm{i}\omega t} \tag{7.2b}$$

and

$$\mathbf{j}(t,\mathbf{x}) = \int_{-\infty}^{\infty} \mathrm{d}\omega\, \mathbf{j}_\omega(\mathbf{x})\, e^{-\mathrm{i}\omega t} \tag{7.3a}$$

$$\mathbf{j}_\omega(\mathbf{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t\, \mathbf{j}(t,\mathbf{x})\, e^{\mathrm{i}\omega t} \tag{7.3b}$$

under the assumption that only *retarded* potentials produce physically acceptable solutions.

The temporal Fourier transform pair for the retarded scalar potential can then be written

$$\phi(t,\mathbf{x}) = \int_{-\infty}^{\infty} \mathrm{d}\omega\, \phi_\omega(\mathbf{x})\, e^{-\mathrm{i}\omega t} \tag{7.4a}$$

$$\phi_\omega(\mathbf{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t\, \phi(t,\mathbf{x})\, e^{\mathrm{i}\omega t} = \frac{1}{4\pi\varepsilon_0} \int_{V'} \mathrm{d}^3x'\, \rho_\omega(\mathbf{x}') \frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \tag{7.4b}$$

where in the last step, we made use of the explicit expression for the temporal Fourier transform of the generic potential component $\Psi_\omega(\mathbf{x})$, Equation (3.28) on page 40. Similarly, the following Fourier transform pair for the vector potential must exist:

$$\mathbf{A}(t,\mathbf{x}) = \int_{-\infty}^{\infty} \mathrm{d}\omega\, \mathbf{A}_\omega(\mathbf{x})\, e^{-\mathrm{i}\omega t} \tag{7.5a}$$

$$\mathbf{A}_\omega(\mathbf{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t\, \mathbf{A}(t,\mathbf{x})\, e^{\mathrm{i}\omega t} = \frac{\mu_0}{4\pi} \int_{V'} \mathrm{d}^3x'\, \mathbf{j}_\omega(\mathbf{x}') \frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \tag{7.5b}$$

Clearly, we must require that

$$\mathbf{A}_\omega = \mathbf{A}^*_{-\omega}, \qquad \phi_\omega = \phi^*_{-\omega} \tag{7.6}$$

in order that all physical quantities be real. Similar transform pairs and requirements of real-valuedness exist for the fields themselves.

In the limit that the sources can be considered monochromatic containing only one single frequency ω_0, we have the much simpler expressions

$$\rho(t,\mathbf{x}) = \rho_0(\mathbf{x}) e^{-\mathrm{i}\omega_0 t} \tag{7.7a}$$

$$\mathbf{j}(t,\mathbf{x}) = \mathbf{j}_0(\mathbf{x}) e^{-\mathrm{i}\omega_0 t} \tag{7.7b}$$

$$\phi(t,\mathbf{x}) = \phi_0(\mathbf{x}) e^{-\mathrm{i}\omega_0 t} \tag{7.7c}$$

$$\mathbf{A}(t,\mathbf{x}) = \mathbf{A}_0(\mathbf{x}) e^{-\mathrm{i}\omega_0 t} \tag{7.7d}$$

where again the real-valuedness of all these quantities is implied. As discussed above, we can safely assume that all formulae derived for a general temporal Fourier representation of the source (general distribution of frequencies in the source) are valid for these simple limiting cases. We note that in this context, we can make the formal identification $\rho_\omega = \rho_0\delta(\omega-\omega_0)$, $\mathbf{j}_\omega = \mathbf{j}_0\delta(\omega-\omega_0)$ *etc.*, and that we therefore, without any loss of stringence, let ρ_0 mean the same as the Fourier amplitude ρ_ω and so on.

7.1 The magnetic field

Let us now compute the magnetic field from the vector potential, defined by Equation (7.5a) and Equation (7.5b) on the preceding page, and Formula (3.6) on page 36:

$$\mathbf{B}(t,\mathbf{x}) = \boldsymbol{\nabla}\times\mathbf{A}(t,\mathbf{x}) \tag{7.8}$$

The calculations are much simplified if we work in ω space and, at the final stage, inverse Fourier transform back to ordinary t space. We are working in the Lorentz gauge and note that in ω space the Lorentz condition, Equation (3.13) on page 38, takes the form

$$\boldsymbol{\nabla}\cdot\mathbf{A}_\omega - \mathrm{i}\frac{k}{c}\phi_\omega = 0 \tag{7.9}$$

which provides a relation between (the Fourier transforms of) the vector and scalar potentials.

Using the Fourier transformed version of Equation (7.8) and Equation (7.5b) on the preceding page, we obtain

$$\mathbf{B}_\omega(\mathbf{x}) = \boldsymbol{\nabla}\times\mathbf{A}_\omega(\mathbf{x}) = \frac{\mu_0}{4\pi}\boldsymbol{\nabla}\times\int_{V'}\mathrm{d}^3x'\;\mathbf{j}_\omega(\mathbf{x}')\frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \tag{7.10}$$

Utilising Formula (F.57) on page 160 and recalling that $\mathbf{j}_\omega(\mathbf{x}')$ does not depend on $\mathbf{x}$, we can rewrite this as

$$\begin{aligned}
\mathbf{B}_\omega(\mathbf{x}) &= -\frac{\mu_0}{4\pi}\int_{V'}\mathrm{d}^3x'\;\mathbf{j}_\omega(\mathbf{x}')\times\left[\boldsymbol{\nabla}\left(\frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\right)\right] \\
&= -\frac{\mu_0}{4\pi}\left[\int_{V'}\mathrm{d}^3x'\;\mathbf{j}_\omega(\mathbf{x}')\times\left(-\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\right)e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}\right. \\
&\qquad\left.+\int_{V'}\mathrm{d}^3x'\;\mathbf{j}_\omega(\mathbf{x}')\times\left(\mathrm{i}k\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}\right)\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right] \\
&= \frac{\mu_0}{4\pi}\left[\int_{V'}\mathrm{d}^3x'\;\frac{\mathbf{j}_\omega(\mathbf{x}')e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\right. \\
&\qquad\left.+\int_{V'}\mathrm{d}^3x'\;\frac{(-\mathrm{i}k)\mathbf{j}_\omega(\mathbf{x}')e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^2}\right]
\end{aligned} \tag{7.11}$$

From this expression for the magnetic field in the frequency (ω) domain, we obtain the total magnetic field in the temporal (t) domain by taking the inverse Fourier transform (using the identity $-\mathrm{i}k = -\mathrm{i}\omega/c$):

$$
\begin{aligned}
\mathbf{B}(t,\mathbf{x}) &= \int_{-\infty}^{\infty} \mathrm{d}\omega\, \mathbf{B}_\omega(\mathbf{x})\, e^{-\mathrm{i}\omega t} \\
&= \frac{\mu_0}{4\pi}\left\{ \int_{V'} \mathrm{d}^3x' \, \frac{\left[\int_{-\infty}^{\infty} \mathrm{d}\omega\, \mathbf{j}_\omega(\mathbf{x}') e^{\mathrm{i}(k|\mathbf{x}-\mathbf{x}'|-\omega t)}\right] \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} \right. \\
&\quad \left. + \frac{1}{c}\int_{V'} \mathrm{d}^3x' \, \frac{\left[\int_{-\infty}^{\infty} \mathrm{d}\omega\, (-\mathrm{i}\omega)\mathbf{j}_\omega(\mathbf{x}') e^{\mathrm{i}(k|\mathbf{x}-\mathbf{x}'|-\omega t)}\right] \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^2} \right\} \\
&= \underbrace{\frac{\mu_0}{4\pi}\int_{V'} \mathrm{d}^3x' \, \frac{\mathbf{j}(t'_{\mathrm{ret}},\mathbf{x}') \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}}_{\text{Induction field}} + \underbrace{\frac{\mu_0}{4\pi c}\int_{V'} \mathrm{d}^3x' \, \frac{\dot{\mathbf{j}}(t'_{\mathrm{ret}},\mathbf{x}') \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^2}}_{\text{Radiation field}}
\end{aligned}
\tag{7.12}
$$

where

$$
\dot{\mathbf{j}}(t'_{\mathrm{ret}},\mathbf{x}') \stackrel{\text{def}}{\equiv} \left(\frac{\partial \mathbf{j}}{\partial t}\right)_{t=t'_{\mathrm{ret}}}
\tag{7.13}
$$

The first term, the *induction field*, dominates near the current source but falls off rapidly with distance from it, is the electrodynamic version of the Biot-Savart law in electrostatics, Formula (1.15) on page 7. The second term, the *radiation field* or the *far field*, dominates at large distances and represents energy that is transported out to infinity. Note how the spatial derivatives ($\boldsymbol{\nabla}$) gave rise to a time derivative ($\dot{}$)!

7.2 The electric field

In order to calculate the electric field, we use the temporally Fourier transformed version of Formula (3.10) on page 37, inserting Equations (7.4b) and (7.5b) as the explicit expressions for the Fourier transforms of ϕ and $\mathbf{A}$:

$$
\begin{aligned}
\mathbf{E}_\omega(\mathbf{x}) &= -\boldsymbol{\nabla}\phi_\omega(\mathbf{x}) + \mathrm{i}\omega\mathbf{A}_\omega(\mathbf{x}) \\
&= -\frac{1}{4\pi\varepsilon_0}\boldsymbol{\nabla}\int_{V'} \mathrm{d}^3x' \, \rho_\omega(\mathbf{x}')\frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} + \frac{\mathrm{i}\mu_0\omega}{4\pi}\int_{V'} \mathrm{d}^3x' \, \mathbf{j}_\omega(\mathbf{x}')\frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \\
&= \frac{1}{4\pi\varepsilon_0}\left[\int_{V'} \mathrm{d}^3x' \, \frac{\rho_\omega(\mathbf{x}')e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} \right. \\
&\quad \left. - \mathrm{i}k\int_{V'} \mathrm{d}^3x' \left(\frac{\rho_\omega(\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} - \frac{\mathbf{j}_\omega(\mathbf{x}')}{c}\right)\frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\right]
\end{aligned}
\tag{7.14}
$$

Using the Fourier transform of the continuity Equation (1.23) on page 9

$$
\boldsymbol{\nabla}' \cdot \mathbf{j}_\omega(\mathbf{x}') - \mathrm{i}\omega\rho_\omega(\mathbf{x}') = 0
\tag{7.15}
$$

we see that we can express ρ_ω in terms of $\mathbf{j}_\omega$ as follows

$$\rho_\omega(\mathbf{x}') = -\frac{\mathrm{i}}{\omega}\boldsymbol{\nabla}'\cdot\mathbf{j}_\omega(\mathbf{x}') \tag{7.16}$$

Doing so in the last term of Equation (7.14) on the facing page, and also using the fact that $k = \omega/c$, we can rewrite this Equation as

$$\begin{aligned}\mathbf{E}_\omega(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\Bigg[&\int_{V'}\mathrm{d}^3x'\,\frac{\rho_\omega(\mathbf{x}')e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\\ &-\frac{1}{c}\underbrace{\int_{V'}\mathrm{d}^3x'\left(\frac{[\boldsymbol{\nabla}'\cdot\mathbf{j}_\omega(\mathbf{x}')](\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}-\mathrm{i}k\mathbf{j}_\omega(\mathbf{x}')\right)\frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}}_{\mathbf{I}_\omega}\Bigg]\end{aligned} \tag{7.17}$$

The last vector-valued integral can be further rewritten in the following way:

$$\begin{aligned}\mathbf{I}_\omega &= \int_{V'}\mathrm{d}^3x'\left(\frac{[\boldsymbol{\nabla}'\cdot\mathbf{j}_\omega(\mathbf{x}')](\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}-\mathrm{i}k\mathbf{j}_\omega(\mathbf{x}')\right)\frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\\ &= \int_{V'}\mathrm{d}^3x'\left(\frac{\partial j_{\omega m}}{\partial x'_m}\frac{x_l-x'_l}{|\mathbf{x}-\mathbf{x}'|}-\mathrm{i}k\,j_{\omega l}(\mathbf{x}')\right)\hat{\mathbf{x}}_l\,\frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\end{aligned} \tag{7.18}$$

But, since

$$\begin{aligned}\frac{\partial}{\partial x'_m}\left(j_{\omega m}\frac{x_l-x'_l}{|\mathbf{x}-\mathbf{x}'|^2}e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}\right) &= \left(\frac{\partial j_{\omega m}}{\partial x'_m}\right)\frac{x_l-x'_l}{|\mathbf{x}-\mathbf{x}'|^2}e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}\\ &+ j_{\omega m}\frac{\partial}{\partial x'_m}\left(\frac{x_l-x'_l}{|\mathbf{x}-\mathbf{x}'|^2}e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}\right)\end{aligned} \tag{7.19}$$

we can rewrite $\mathbf{I}_\omega$ as

$$\begin{aligned}\mathbf{I}_\omega = &-\int_{V'}\mathrm{d}^3x'\left[j_{\omega m}\frac{\partial}{\partial x'_m}\left(\frac{x_l-x'_l}{|\mathbf{x}-\mathbf{x}'|^2}\hat{\mathbf{x}}_l\,e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}\right)+\mathrm{i}k\mathbf{j}_\omega\frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\right]\\ &+\int_{V'}\mathrm{d}^3x'\,\frac{\partial}{\partial x'_m}\left(j_{\omega m}\frac{x_l-x'_l}{|\mathbf{x}-\mathbf{x}'|^2}\hat{\mathbf{x}}_l\,e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}\right)\end{aligned} \tag{7.20}$$

where, according to Gauss's theorem, the last term vanishes if $\mathbf{j}_\omega$ is assumed to be limited and tends to zero at large distances. Further evaluation of the derivative in the first term makes it possible to write

$$\begin{aligned}\mathbf{I}_\omega = &-\int_{V'}\mathrm{d}^3x'\left(-\mathbf{j}_\omega\frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|^2}+\frac{2}{|\mathbf{x}-\mathbf{x}'|^4}\left[\mathbf{j}_\omega\cdot(\mathbf{x}-\mathbf{x}')\right](\mathbf{x}-\mathbf{x}')e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}\right)\\ &-\mathrm{i}k\int_{V'}\mathrm{d}^3x'\left(-\frac{\left[\mathbf{j}_\omega\cdot(\mathbf{x}-\mathbf{x}')\right](\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}+\mathbf{j}_\omega\frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\right)\end{aligned} \tag{7.21}$$

Using the triple product 'bac-cab' Formula (F.51) on page 160 backwards, and inserting the resulting expression for $\mathbf{I}_\omega$ into Equation (7.17) on the preceding page, we arrive at the following final expression for the Fourier transform of the total $\mathbf{E}$ field:

$$
\begin{aligned}
\mathbf{E}_\omega(\mathbf{x}) &= -\frac{1}{4\pi\varepsilon_0}\boldsymbol{\nabla}\int_{V'}\mathrm{d}^3x'\,\rho_\omega(\mathbf{x}')\frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} + \frac{\mathrm{i}\mu_0\omega}{4\pi}\int_{V'}\mathrm{d}^3x'\,\mathbf{j}_\omega(\mathbf{x}')\frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \\
&= \frac{1}{4\pi\varepsilon_0}\left[\int_{V'}\mathrm{d}^3x'\,\frac{\rho_\omega(\mathbf{x}')e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\right. \\
&\quad + \frac{1}{c}\int_{V'}\mathrm{d}^3x'\,\frac{[\mathbf{j}_\omega(\mathbf{x}')e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}\cdot(\mathbf{x}-\mathbf{x}')](\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^4} \\
&\quad + \frac{1}{c}\int_{V'}\mathrm{d}^3x'\,\frac{[\mathbf{j}_\omega(\mathbf{x}')e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}\times(\mathbf{x}-\mathbf{x}')]\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^4} \\
&\quad \left. - \frac{\mathrm{i}k}{c}\int_{V'}\mathrm{d}^3x'\,\frac{[\mathbf{j}_\omega(\mathbf{x}')e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}\times(\mathbf{x}-\mathbf{x}')]\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\right]
\end{aligned}
\tag{7.22}
$$

Taking the inverse Fourier transform of Equation (7.22) above, once again using the vacuum relation $\omega = kc$, we find, at last, the expression in time domain for the total electric field:

$$
\begin{aligned}
\mathbf{E}(t,\mathbf{x}) &= \int_{-\infty}^{\infty}\mathrm{d}\omega\,\mathbf{E}_\omega(\mathbf{x})\,e^{-\mathrm{i}\omega t} \\
&= \underbrace{\frac{1}{4\pi\varepsilon_0}\int_{V'}\mathrm{d}^3x'\,\frac{\rho(t'_{\mathrm{ret}},\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}}_{\text{Retarded Coulomb field}} \\
&\quad + \underbrace{\frac{1}{4\pi\varepsilon_0 c}\int_{V'}\mathrm{d}^3x'\,\frac{[\mathbf{j}(t'_{\mathrm{ret}},\mathbf{x}')\cdot(\mathbf{x}-\mathbf{x}')](\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^4}}_{\text{Intermediate field}} \\
&\quad + \underbrace{\frac{1}{4\pi\varepsilon_0 c}\int_{V'}\mathrm{d}^3x'\,\frac{[\mathbf{j}(t'_{\mathrm{ret}},\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')]\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^4}}_{\text{Intermediate field}} \\
&\quad + \underbrace{\frac{1}{4\pi\varepsilon_0 c^2}\int_{V'}\mathrm{d}^3x'\,\frac{[\dot{\mathbf{j}}(t'_{\mathrm{ret}},\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')]\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}}_{\text{Radiation field}}
\end{aligned}
\tag{7.23}
$$

Here, the first term represents the *retarded Coulomb field* and the last term represents the *radiation field* which carries energy over very large distances. The other two terms represent an *intermediate field* which contributes only in the *near zone* and must be taken into account there.

With this we have achieved our goal of finding closed-form analytic expressions for the electric and magnetic fields when the sources of the fields are completely arbitrary, prescribed distributions of charges and currents. The only assumption made is that the advanced potentials have been discarded; recall the discussion following Equation (3.31) on page 41 in Chapter 3.

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7.3 The radiation fields

In this section we study electromagnetic radiation, *i.e.*, the part of the electric and magnetic fields, calculated above, which are capable of carrying energy and momentum over large distances. We shall therefore make the assumption that the observer is located in the *far zone*, *i.e.*, very far away from the source region(s). The fields which are dominating in this zone are by definition the *radiation fields*.

From Equation (7.12) on page 96 and Equation (7.23) on the facing page, which give the total electric and magnetic fields, we obtain

$$\mathbf{B}^{\text{rad}}(t,\mathbf{x}) = \int_{-\infty}^{\infty} \mathrm{d}\omega\, \mathbf{B}^{\text{rad}}{}_{\omega}(\mathbf{x})\, e^{-\mathrm{i}\omega t} = \frac{\mu_0}{4\pi c}\int_{V'} \mathrm{d}^3x'\, \frac{\dot{\mathbf{j}}(t'_{\text{ret}},\mathbf{x}') \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^2} \tag{7.24a}$$

$$\begin{aligned}\mathbf{E}^{\text{rad}}(t,\mathbf{x}) &= \int_{-\infty}^{\infty} \mathrm{d}\omega\, \mathbf{E}^{\text{rad}}{}_{\omega}(\mathbf{x})\, e^{-\mathrm{i}\omega t} \\ &= \frac{1}{4\pi\varepsilon_0 c^2}\int_{V'} \mathrm{d}^3x'\, \frac{[\dot{\mathbf{j}}(t'_{\text{ret}},\mathbf{x}') \times (\mathbf{x}-\mathbf{x}')] \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\end{aligned} \tag{7.24b}$$

where

$$\dot{\mathbf{j}}(t'_{\text{ret}},\mathbf{x}') \stackrel{\text{def}}{\equiv} \left(\frac{\partial \mathbf{j}}{\partial t}\right)_{t=t'_{\text{ret}}} \tag{7.25}$$

Instead of studying the fields in the time domain, we can often make a spectrum analysis into the frequency domain and study each Fourier component separately. A superposition of all these components and a transformation back to the time domain will then yield the complete solution.

The Fourier representation of the radiation fields Equation (7.24a) above and Equation (7.24b) were included in Equation (7.11) on page 95 and Equation (7.22) on the facing page, respectively and are explicitly given by

$$\begin{aligned}\mathbf{B}^{\text{rad}}_{\omega}(\mathbf{x}) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}t\, \mathbf{B}^{\text{rad}}(t,\mathbf{x})\, e^{\mathrm{i}\omega t} \\ &= -\mathrm{i}\frac{k\mu_0}{4\pi}\int_{V'} \mathrm{d}^3x'\, \frac{\mathbf{j}_{\omega}(\mathbf{x}') \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^2}\, e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|} \\ &= -\mathrm{i}\frac{\mu_0}{4\pi}\int_{V'} \mathrm{d}^3x'\, \frac{\mathbf{j}_{\omega}(\mathbf{x}') \times \mathbf{k}}{|\mathbf{x}-\mathbf{x}'|}\, e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}\end{aligned} \tag{7.26a}$$

$$\begin{aligned}\mathbf{E}^{\text{rad}}_{\omega}(\mathbf{x}) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}t\, \mathbf{E}^{\text{rad}}(t,\mathbf{x})\, e^{\mathrm{i}\omega t} \\ &= -\mathrm{i}\frac{k}{4\pi\varepsilon_0 c}\int_{V'} \mathrm{d}^3x'\, \frac{[\mathbf{j}_{\omega}(\mathbf{x}') \times (\mathbf{x}-\mathbf{x}')] \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\, e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|} \\ &= -\mathrm{i}\frac{1}{4\pi\varepsilon_0 c}\int_{V'} \mathrm{d}^3x'\, \frac{[\mathbf{j}_{\omega}(\mathbf{x}') \times \mathbf{k}] \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^2}\, e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}\end{aligned} \tag{7.26b}$$

where we used the fact that $\mathbf{k} = k\hat{\boldsymbol{k}} = k(\mathbf{x}-\mathbf{x}')/\,|\mathbf{x}-\mathbf{x}`|$.

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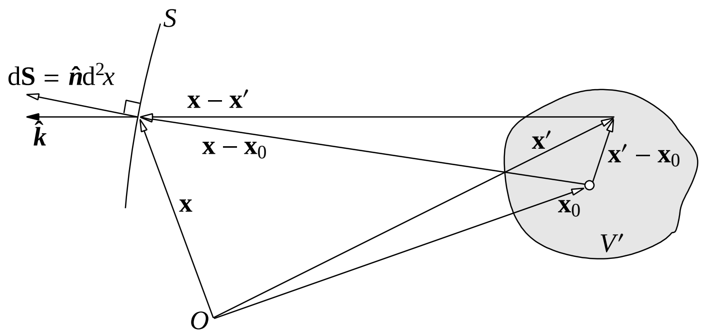

FIGURE 7.1: Relation between the surface normal and the $\mathbf{k}$ vector for radiation generated at source points $\mathbf{x}'$ near the point $\mathbf{x}_0$ in the source volume V. At distances much larger than the extent of V, the unit vector $\hat{\boldsymbol{n}}$, normal to the surface S which has its centre at $\mathbf{x}_0$, and the unit vector $\hat{\boldsymbol{k}}$ of the radiation $\mathbf{k}$ vector from $\mathbf{x}'$ are nearly coincident.

If the source is located inside a volume V near $\mathbf{x}_0$ and has such a limited spatial extent that $\max|\mathbf{x}' - \mathbf{x}_0| \ll |\mathbf{x} - \mathbf{x}'|$, and the integration surface S, centred on $\mathbf{x}_0$, has a large enough radius $|\mathbf{x} - \mathbf{x}_0| \gg \max|\mathbf{x}' - \mathbf{x}_0|$, we see from Figure 7.1 that we can approximate

$$
\begin{aligned}
k\left|\mathbf{x} - \mathbf{x}'\right| &\equiv \mathbf{k}\cdot(\mathbf{x} - \mathbf{x}') \equiv \mathbf{k}\cdot(\mathbf{x} - \mathbf{x}_0) - \mathbf{k}\cdot(\mathbf{x}' - \mathbf{x}_0) \\
&\approx k\left|\mathbf{x} - \mathbf{x}_0\right| - \mathbf{k}\cdot(\mathbf{x}' - \mathbf{x}_0)
\end{aligned}
\tag{7.27}
$$

Recalling from Formula (F.45) and Formula (F.46) on page 159 that

$$
\mathrm{d}S = |\mathbf{x} - \mathbf{x}_0|^2\,\mathrm{d}\Omega = |\mathbf{x} - \mathbf{x}_0|^2\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi
$$

and noting from Figure 7.1 that $\hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{n}}$ are nearly parallel, we see that we can approximate.

$$
\frac{\hat{\boldsymbol{k}}\cdot\mathrm{d}\mathbf{S}}{|\mathbf{x} - \mathbf{x}_0|^2} = \frac{\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{n}}}{|\mathbf{x} - \mathbf{x}_0|^2}\,\mathrm{d}S \approx \mathrm{d}\Omega
\tag{7.28}
$$

Both these approximations will be used in the following.

Within approximation (7.27) the expressions (7.26a) and (7.26b) for the radiation

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fields can be approximated as

$$\begin{aligned}\mathbf{B}_\omega^{\mathrm{rad}}(\mathbf{x}) &\approx -\mathrm{i}\frac{\mu_0}{4\pi}\, e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}\int_{V'}\mathrm{d}^3x'\,\frac{\mathbf{j}_\omega(\mathbf{x}')\times\mathbf{k}}{|\mathbf{x}-\mathbf{x}'|}\,e^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}_0)}\\ &\approx -\mathrm{i}\frac{\mu_0}{4\pi}\,\frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}\int_{V'}\mathrm{d}^3x'\,[\mathbf{j}_\omega(\mathbf{x}')\times\mathbf{k}]\,e^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}_0)}\end{aligned} \tag{7.29a}$$

$$\begin{aligned}\mathbf{E}_\omega^{\mathrm{rad}}(\mathbf{x}) &\approx -\mathrm{i}\frac{1}{4\pi\varepsilon_0 c}\, e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}\int_{V'}\mathrm{d}^3x'\,\frac{[\mathbf{j}_\omega(\mathbf{x}')\times\mathbf{k}]\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^2}\,e^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}_0)}\\ &\approx \mathrm{i}\frac{1}{4\pi\varepsilon_0 c}\,\frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}\,\frac{(\mathbf{x}-\mathbf{x}_0)}{|\mathbf{x}-\mathbf{x}_0|}\times\int_{V'}\mathrm{d}^3x'\,[\mathbf{j}_\omega(\mathbf{x}')\times\mathbf{k}]\,e^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}_0)}\end{aligned} \tag{7.29b}$$

I.e., if $\max|\mathbf{x}'-\mathbf{x}_0| \ll |\mathbf{x}-\mathbf{x}'|$, then the fields can be approximated as *spherical waves* multiplied by dimensional and angular factors, with integrals over points in the source volume only.

7.4 Radiated energy

Let us consider the energy that is carried in the radiation fields $\mathbf{B}^{\mathrm{rad}}$, Equation (7.26a), and $\mathbf{E}^{\mathrm{rad}}$, Equation (7.26b) on page 99. We have to treat signals with limited lifetime and hence finite frequency bandwidth differently from monochromatic signals.

7.4.1 Monochromatic signals

If the source is strictly monochromatic, we can obtain the temporal average of the radiated power P directly, simply by averaging over one period so that

$$\begin{aligned}\langle\mathbf{S}\rangle = \langle\mathbf{E}\times\mathbf{H}\rangle &= \frac{1}{2\mu_0}\mathrm{Re}\,\{\mathbf{E}\times\mathbf{B}^*\} = \frac{1}{2\mu_0}\mathrm{Re}\left\{\mathbf{E}_\omega e^{-\mathrm{i}\omega t}\times(\mathbf{B}_\omega e^{-\mathrm{i}\omega t})^*\right\}\\ &= \frac{1}{2\mu_0}\mathrm{Re}\left\{\mathbf{E}_\omega\times\mathbf{B}_\omega^*\,e^{-\mathrm{i}\omega t}e^{\mathrm{i}\omega t}\right\} = \frac{1}{2\mu_0}\mathrm{Re}\left\{\mathbf{E}_\omega\times\mathbf{B}_\omega^*\right\}\end{aligned} \tag{7.30}$$

Using the far-field approximations (7.29a) and (7.29b) and the fact that $1/c = \sqrt{\varepsilon_0\mu_0}$ and $R_0 = \sqrt{\mu_0/\varepsilon_0}$ according to the definition (2.18) on page 28, we obtain

$$\langle\mathbf{S}\rangle = \frac{1}{32\pi^2}R_0\,\frac{1}{|\mathbf{x}-\mathbf{x}_0|^2}\left|\int_{V'}\mathrm{d}^3x'\,(\mathbf{j}_\omega\times\mathbf{k})e^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}_0)}\right|^2\frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|} \tag{7.31}$$

or, making use of (7.28) on the facing page,

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{1}{32\pi^2}R_0\left|\int_{V'}\mathrm{d}^3x'\,(\mathbf{j}_\omega\times\mathbf{k})e^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}_0)}\right|^2 \tag{7.32}$$

which is the radiated power per unit solid angle.

7.4.2 Finite bandwidth signals

A signal with finite pulse width in time (t) domain has a certain spread in frequency (ω) domain. To calculate the total radiated energy we need to integrate over the whole bandwidth. The total energy transmitted through a unit area is the time integral of the Poynting vector:

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{d}t\, \mathbf{S}(t) &= \int_{-\infty}^{\infty} \mathrm{d}t\, (\mathbf{E} \times \mathbf{H}) \\ &= \int_{-\infty}^{\infty} \mathrm{d}\omega \int_{-\infty}^{\infty} \mathrm{d}\omega' \int_{-\infty}^{\infty} \mathrm{d}t\, (\mathbf{E}_\omega \times \mathbf{H}_{\omega'})\, e^{-\mathrm{i}(\omega+\omega')t}\end{aligned} \tag{7.33}$$

If we carry out the temporal integration first and use the fact that

$$\int_{-\infty}^{\infty} \mathrm{d}t\, e^{-\mathrm{i}(\omega+\omega')t} = 2\pi\delta(\omega + \omega') \tag{7.34}$$

Equation (7.33) above can be written [*cf. Parseval's identity*]

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{d}t\, \mathbf{S}(t) &= 2\pi \int_{-\infty}^{\infty} \mathrm{d}\omega\, (\mathbf{E}_\omega \times \mathbf{H}_{-\omega}) \\ &= 2\pi \left(\int_{0}^{\infty} (\mathbf{E}_\omega \times \mathbf{H}_{-\omega})\, \mathrm{d}\omega + \int_{-\infty}^{0} (\mathbf{E}_\omega \times \mathbf{H}_{-\omega})\, \mathrm{d}\omega \right) \\ &= 2\pi \left(\int_{0}^{\infty} (\mathbf{E}_\omega \times \mathbf{H}_{-\omega})\, \mathrm{d}\omega - \int_{0}^{-\infty} (\mathbf{E}_\omega \times \mathbf{H}_{-\omega})\, \mathrm{d}\omega \right) \\ &= 2\pi \left(\int_{0}^{\infty} (\mathbf{E}_\omega \times \mathbf{H}_{-\omega})\, \mathrm{d}\omega + \int_{0}^{\infty} (\mathbf{E}_{-\omega} \times \mathbf{H}_{\omega})\, \mathrm{d}\omega \right) \\ &= \frac{2\pi}{\mu_0} \int_{0}^{\infty} (\mathbf{E}_\omega \times \mathbf{B}_{-\omega} + \mathbf{E}_{-\omega} \times \mathbf{B}_{\omega})\, \mathrm{d}\omega \\ &= \frac{2\pi}{\mu_0} \int_{0}^{\infty} (\mathbf{E}_\omega \times \mathbf{B}^*_{\omega} + \mathbf{E}^*_{\omega} \times \mathbf{B}_{\omega})\, \mathrm{d}\omega\end{aligned} \tag{7.35}$$

where the last step follows from physical requirement of real-valuedness of $\mathbf{E}_\omega$ and $\mathbf{B}_\omega$. We insert the Fourier transforms of the field components which dominate at large distances, *i.e.*, the radiation fields (7.26a) and (7.26b). The result, after integration over the area S of a large sphere which encloses the source, is

$$U = \frac{1}{4\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}} \oint_S \mathrm{d}^2x\, \hat{\boldsymbol{n}} \cdot \int_0^\infty \mathrm{d}\omega \left| \int_{V'} \mathrm{d}^3x'\, \frac{\mathbf{j}_\omega \times \mathbf{k}}{|\mathbf{x} - \mathbf{x}'|} e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|} \right|^2 \hat{\boldsymbol{k}} \tag{7.36}$$

Inserting the approximations (7.27) and (7.28) into Equation (7.36) and also introducing

$$U = \int_0^\infty U_\omega\, \mathrm{d}\omega \tag{7.37}$$

and recalling the definition (2.18) on page 28 for the vacuum resistance R_0 we obtain

$$\frac{\mathrm{d}U_\omega}{\mathrm{d}\Omega}\,\mathrm{d}\omega \approx \frac{1}{4\pi}R_0 \left| \int_{V'} \mathrm{d}^3x'\; (\mathbf{j}_\omega \times \mathbf{k}) e^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}_0)} \right|^2 \mathrm{d}\omega \tag{7.38}$$

which, at large distances, is a good approximation to the energy that is radiated per unit solid angle $\mathrm{d}\Omega$ in a frequency band $\mathrm{d}\omega$. It is important to notice that Formula (7.38) includes only source coordinates. This means that the amount of energy that is being radiated is independent on the distance to the source (as long as it is large).

7.5 Bibliography

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CHAPTER 8

Electromagnetic Radiation and Radiating Systems

In Chapter 3 we were able to derive general expressions for the scalar and vector potentials from which we then, in Chapter 7, calculated the total electric and magnetic fields from arbitrary distributions of charge and current sources. The only limitation in the calculation of the fields was that the advanced potentials were discarded.

Thus, one can, at least in principle, calculate the radiated fields, Poynting flux and energy for an arbitrary current density Fourier component and then add these Fourier components together to construct the complete electromagnetic field at any time at any point in space. However, in practice, it is often difficult to evaluate the source integrals unless the current has a simple distribution in space. In the general case, one has to resort to approximations. We shall consider both these situations.

8.1 Radiation from extended sources

Certain radiation systems have a geometry which is one-dimensional, symmetric or in any other way simple enough that a direct calculation of the radiated fields and energy is possible. This is for instance the case when the current flows in one direction in space only and is limited in extent. An example of this is a linear antenna.

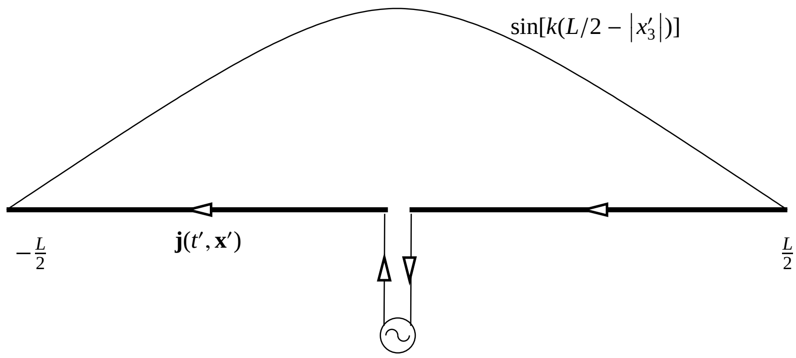

FIGURE 8.1: A linear antenna used for transmission. The current in the feeder and the antenna wire is set up by the EMF of the generator (the transmitter). At the ends of the wire, the current is reflected back with a 180° phase shift to produce a antenna current in the form of a standing wave.

8.1.1 Radiation from a one-dimensional current distribution

Let us apply Equation (7.32) on page 101 to calculate the power from a linear, transmitting antenna, fed across a small gap at its centre with a monochromatic source. The antenna is a straight, thin conductor of length L which carries a one-dimensional time-varying current so that it produces electromagnetic radiation.

We assume that the conductor resistance and the energy loss due to the electromagnetic radiation are negligible. The charges in this thin wire are set in motion due to the EMF of the generator (transmitter) to produce an antenna current which is the source of the EM radiation. Since we can assume that the antenna wire is infinitely thin, the current must vanish at the end points $-L/2$ and $L/2$. Furthermore, for a monochromatic signal, the current is sinusoidal and is reflected at the ends of the antenna wire and undergoes there a phase shift of π radians. The combined effect of this is that the antenna current forms a *standing wave* as indicated in Figure 8.1

For a Fourier component ω_0 the standing wave current density can be written as $\mathbf{j}(t', \mathbf{x}') = \mathbf{j}_0(\mathbf{x}') \exp\{-\mathrm{i}\omega_0 t'\}$ [*cf.* Equations (7.7) on page 94] where

$$\mathbf{j}_0(\mathbf{x}') = I_0 \delta(x'_1)\delta(x'_2) \frac{\sin[k(L/2 - |x'_3|)]}{\sin(kL/2)} \hat{\boldsymbol{x}}_3 \tag{8.1}$$

where the current amplitude I_0 is a constant (measured in A).

In order to evaluate Formula (7.32) on page 101 with the explicit monochromatic current (8.1) inserted, we use a spherical polar coordinate system as in Figure 8.2 on

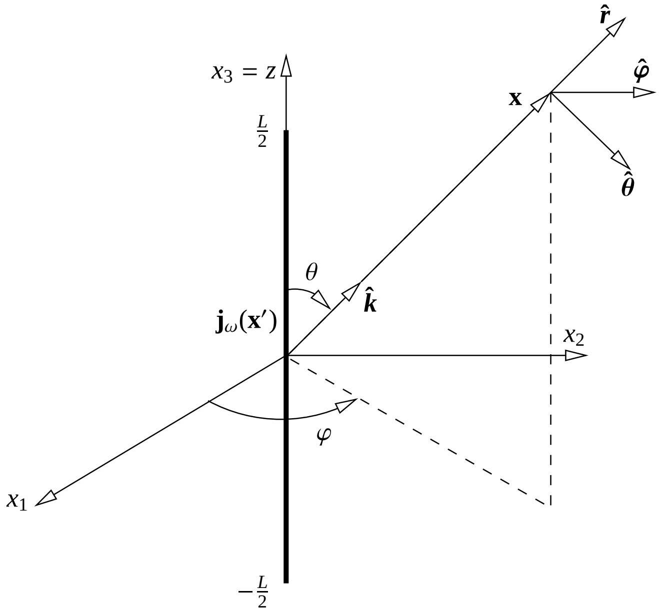

FIGURE 8.2: We choose a spherical polar coordinate system ($r = |\mathbf{x}|, \theta, \varphi$) and orient it so that the linear antenna axis (and thus the antenna current density $\mathbf{j}_\omega$) is along the polar axis with the feed point at the origin.

the facing page to evaluate the source integral

$$\begin{aligned}
&\left| \int_{V'} \mathrm{d}^3x' \; \mathbf{j}_0 \times \mathbf{k}\, e^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}_0)} \right|^2 \\
&\quad = \left| \int_{-L/2}^{L/2} I_0 \frac{\sin[k(L/2 - |x_3'|)]}{\sin(kL/2)} k \sin\theta e^{-\mathrm{i}kx_3'\cos\theta} e^{\mathrm{i}kx_0\cos\theta}\, \mathrm{d}x_3' \right|^2 \\
&\quad = I_0^2 \frac{k^2 \sin^2\theta}{\sin^2(kL/2)} \left| e^{\mathrm{i}kx_0\cos\theta} \right|^2 \left| 2 \int_0^{L/2} \sin[k(L/2 - x_3')] \cos(kx_3'\cos\theta)\, \mathrm{d}x_3' \right|^2 \\
&\quad = 4I_0^2 \left(\frac{\cos[(kL/2)\cos\theta] - \cos(kL/2)}{\sin\theta \sin(kL/2)} \right)^2
\end{aligned} \tag{8.2}$$

Inserting this expression and $\mathrm{d}\Omega = 2\pi \sin\theta\, \mathrm{d}\theta$ into Formula (7.32) on page 101 and integrating over θ, we find that the total radiated power from the antenna is

$$P(L) = R_0 I_0^2 \frac{1}{4\pi} \int_0^\pi \left(\frac{\cos[(kL/2)\cos\theta] - \cos(kL/2)}{\sin\theta \sin(kL/2)} \right)^2 \sin\theta\, \mathrm{d}\theta \tag{8.3}$$

One can show that

$$\lim_{kL\to 0} P(L) = \frac{\pi}{12}\left(\frac{L}{\lambda}\right)^2 R_0 I_0^2 \tag{8.4}$$

where λ is the vacuum wavelength.

The quantity

$$R^{\text{rad}}(L) = \frac{P(L)}{I_{\text{eff}}^2} = \frac{P(L)}{\frac{1}{2}I_0^2} = R_0\frac{\pi}{6}\left(\frac{L}{\lambda}\right)^2 \approx 197\left(\frac{L}{\lambda}\right)^2\ \Omega \tag{8.5}$$

is called the *radiation resistance*. For the technologically important case of a half-wave antenna, *i.e.*, for $L = \lambda/2$ or $kL = \pi$, Formula (8.3) on the preceding page reduces to

$$P(\lambda/2) = R_0 I_0^2 \frac{1}{4\pi}\int_0^{\pi} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\,\mathrm{d}\theta \tag{8.6}$$

The integral in (8.6) can always be evaluated numerically. But, it can in fact also be evaluated analytically as follows:

$$\begin{aligned}
\int_0^{\pi} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\,\mathrm{d}\theta &= [\cos\theta \to u] = \int_{-1}^{1} \frac{\cos^2\left(\frac{\pi}{2}u\right)}{1-u^2}\,\mathrm{d}u = \\
&\left[\cos^2\left(\frac{\pi}{2}u\right) = \frac{1+\cos(\pi u)}{2}\right] \\
&= \frac{1}{2}\int_{-1}^{1} \frac{1+\cos(\pi u)}{(1+u)(1-u)}\,\mathrm{d}u \\
&= \frac{1}{4}\int_{-1}^{1} \frac{1+\cos(\pi u)}{(1+u)}\,\mathrm{d}u + \frac{1}{4}\int_{-1}^{1} \frac{1+\cos(\pi u)}{(1-u)}\,\mathrm{d}u \\
&= \frac{1}{2}\int_{-1}^{1} \frac{1+\cos(\pi u)}{(1+u)}\,\mathrm{d}u = \left[1+u \to \frac{v}{\pi}\right] \\
&= \frac{1}{2}\int_0^{2\pi} \frac{1-\cos v}{v}\,\mathrm{d}v = \frac{1}{2}[\gamma + \ln 2\pi - \mathrm{Ci}(2\pi)] \\
&\approx 1.22
\end{aligned} \tag{8.7}$$

where in the last step the *Euler-Mascheroni constant* $\gamma = 0.5772\ldots$ and the *cosine integral* $\mathrm{Ci}(x)$ were introduced. Inserting this into the expression Equation (8.6) above we obtain the value $R^{\text{rad}}(\lambda/2) \approx 73\ \Omega$.

8.1.2 Radiation from a two-dimensional current distribution

As an example of a two-dimensional current distribution we consider a circular loop antenna and calculate the radiated fields from such an antenna. We choose the Cartesian

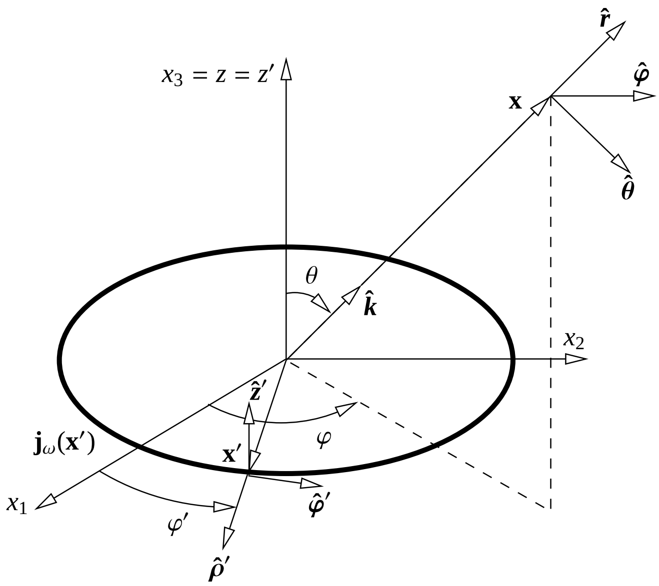

FIGURE 8.3: For the loop antenna the spherical coordinate system (r, θ, φ) describes the field point $\mathbf{x}$ (the radiation field) and the cylindrical coordinate system (ρ', φ', z') describes the source point $\mathbf{x}'$ (the antenna current).

coordinate system $x_1x_2x_3$ with its origin at the centre of the loop as in Figure 8.3 on the facing page

According to Equation (7.29a) on page 101 the Fourier component of the radiation part of the magnetic field generated by an extended, monochromatic current source is

$$\mathbf{B}^{\text{rad}}_\omega = \frac{-\mathrm{i}\mu_0 e^{\mathrm{i}k|\mathbf{x}|}}{4\pi\,|\mathbf{x}|} \int_{V'} \mathrm{d}^3x'\; e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}'}\, \mathbf{j}_\omega \times \mathbf{k} \tag{8.8}$$

In our case the generator produces a single frequency ω and we feed the antenna across a small gap where the loop crosses the positive x_1 axis. The circumference of the loop is chosen to be exactly one wavelength $\lambda = 2\pi c/\omega$. This means that the antenna current oscillates in the form of a sinusoidal standing current wave around the circular loop with a Fourier amplitude

$$\mathbf{j}_\omega = I_0 \cos\varphi' \delta(\rho' - a)\delta(z')\hat{\boldsymbol{\varphi}}' \tag{8.9}$$

For the spherical coordinate system of the field point, we recall from subsection F.4.1 on page 159 that the following relations between the base vectors hold:

$$\hat{\mathbf{r}} = \sin\theta\cos\varphi\hat{\mathbf{x}}_1 + \sin\theta\sin\varphi\hat{\mathbf{x}}_2 + \cos\theta\hat{\mathbf{x}}_3$$
$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\varphi\hat{\mathbf{x}}_1 + \cos\theta\sin\varphi\hat{\mathbf{x}}_2 - \sin\theta\hat{\mathbf{x}}_3$$
$$\hat{\boldsymbol{\varphi}} = -\sin\varphi\hat{\mathbf{x}}_1 + \cos\varphi\hat{\mathbf{x}}_2$$

and

$$\hat{\boldsymbol{x}}_1 = \sin\theta\cos\varphi\hat{\mathbf{r}} + \cos\theta\cos\varphi\hat{\boldsymbol{\theta}} - \sin\varphi\hat{\boldsymbol{\varphi}}$$
$$\hat{\boldsymbol{x}}_2 = \sin\theta\sin\varphi\hat{\mathbf{r}} + \cos\theta\sin\varphi\hat{\boldsymbol{\theta}} + \cos\varphi\hat{\boldsymbol{\varphi}}$$
$$\hat{\boldsymbol{x}}_3 = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}$$

With the use of the above transformations and trigonometric identities, we obtain for the cylindrical coordinate system which describes the source:

$$\begin{aligned}\hat{\boldsymbol{\rho}}' &= \cos\varphi'\hat{\boldsymbol{x}}_1 + \sin\varphi'\hat{\boldsymbol{x}}_2 \\ &= \sin\theta\cos(\varphi'-\varphi)\hat{\mathbf{r}} + \cos\theta\cos(\varphi'-\varphi)\hat{\boldsymbol{\theta}} + \sin(\varphi'-\varphi)\hat{\boldsymbol{\varphi}}\end{aligned} \tag{8.10}$$

$$\begin{aligned}\hat{\boldsymbol{\varphi}}' &= -\sin\varphi'\hat{\boldsymbol{x}}_1 + \cos\varphi'\hat{\boldsymbol{x}}_2 \\ &= -\sin\theta\sin(\varphi'-\varphi)\hat{\mathbf{r}} - \cos\theta\sin(\varphi'-\varphi)\hat{\boldsymbol{\theta}} + \cos(\varphi'-\varphi)\hat{\boldsymbol{\varphi}}\end{aligned} \tag{8.11}$$

$$\hat{\mathbf{z}}' = \hat{\boldsymbol{x}}_3 = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}} \tag{8.12}$$

This choice of coordinate systems means that $\mathbf{k} = k\hat{\mathbf{r}}$ and $\mathbf{x}' = a\hat{\boldsymbol{\rho}}'$ so that

$$\mathbf{k}\cdot\mathbf{x}' = ka\sin\theta\cos(\varphi'-\varphi) \tag{8.13}$$

and

$$\hat{\boldsymbol{\varphi}}' \times \mathbf{k} = k[\cos(\varphi'-\varphi)\hat{\boldsymbol{\theta}} + \cos\theta\sin(\varphi'-\varphi)\hat{\boldsymbol{\varphi}}] \tag{8.14}$$

With these expressions inserted and $\mathrm{d}^3x' = \rho'\mathrm{d}\rho'\mathrm{d}\varphi'\mathrm{d}z'$, the source integral becomes

$$\begin{aligned}\int_{V'}\mathrm{d}^3x'\, e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}'}\,\mathbf{j}_\omega\times\mathbf{k} &= a\int_0^{2\pi}\mathrm{d}\varphi'\, e^{-\mathrm{i}ka\sin\theta\cos(\varphi'-\varphi)} I_0\cos\varphi'\,\hat{\boldsymbol{\varphi}}\times\mathbf{k} \\ &= I_0ak\int_0^{2\pi} e^{-\mathrm{i}ka\sin\theta\cos(\varphi'-\varphi)}\cos(\varphi'-\varphi)\cos\varphi'\,\mathrm{d}\varphi'\,\hat{\boldsymbol{\theta}} \\ &\quad + I_0ak\,\cos\theta\int_0^{2\pi} e^{-\mathrm{i}ka\sin\theta\cos(\varphi'-\varphi)}\sin(\varphi'-\varphi)\cos\varphi'\,\mathrm{d}\varphi'\,\hat{\boldsymbol{\varphi}}\end{aligned} \tag{8.15}$$

Utilising the periodicity of the integrands over the integration interval $[0, 2\pi]$, introducing the auxiliary integration variable $\varphi'' = \varphi' - \varphi$, and utilising standard trigonometric identities, the first integral in the RHS of (8.15) can be rewritten

$$\begin{aligned}&\int_0^{2\pi} e^{-ika\sin\theta\cos\varphi''}\cos\varphi''\cos(\varphi''+\varphi)\,\mathrm{d}\varphi'' \\ &= \cos\varphi\int_0^{2\pi} e^{-ika\sin\theta\cos\varphi''}\cos^2\varphi''\,\mathrm{d}\varphi'' + \text{a vanishing integral} \\ &= \cos\varphi\int_0^{2\pi} e^{-ika\sin\theta\cos\varphi''}\left(\frac{1}{2}+\frac{1}{2}\cos 2\varphi''\right)\mathrm{d}\varphi'' \\ &= \frac{1}{2}\cos\varphi\int_0^{2\pi} e^{-ika\sin\theta\cos\varphi''}\,\mathrm{d}\varphi'' \\ &\quad + \frac{1}{2}\cos\varphi\int_0^{2\pi} e^{-ika\sin\theta\cos\varphi''}\cos 2\varphi''\,\mathrm{d}\varphi''\end{aligned} \tag{8.16}$$

Analogously, the second integral in the RHS of (8.15) can be rewritten

$$\begin{aligned}&\int_0^{2\pi} e^{-ika\sin\theta\cos\varphi''}\sin\varphi''\cos(\varphi''+\varphi)\,\mathrm{d}\varphi''\\&\quad=\frac{1}{2}\sin\varphi\int_0^{2\pi} e^{-ika\sin\theta\cos\varphi''}\,\mathrm{d}\varphi''\\&\qquad-\frac{1}{2}\sin\varphi\int_0^{2\pi} e^{-ika\sin\theta\cos\varphi''}\cos 2\varphi''\,\mathrm{d}\varphi''\end{aligned}\tag{8.17}$$

As is well-known from the theory of *Bessel functions*,

$$\begin{aligned}&J_n(-\xi)=(-1)^n J_n(\xi)\\&J_n(-\xi)=\frac{\mathrm{i}^{-n}}{\pi}\int_0^{\pi} e^{-\mathrm{i}\xi\cos\varphi}\cos n\varphi\,\mathrm{d}\varphi=\frac{\mathrm{i}^{-n}}{2\pi}\int_0^{2\pi} e^{-\mathrm{i}\xi\cos\varphi}\cos n\varphi\,\mathrm{d}\varphi\end{aligned}\tag{8.18}$$

which means that

$$\begin{aligned}&\int_0^{2\pi} e^{-ika\sin\theta\cos\varphi''}\,\mathrm{d}\varphi''=2\pi J_0(ka\sin\theta)\\&\int_0^{2\pi} e^{-ika\sin\theta\cos\varphi''}\cos 2\varphi''\,\mathrm{d}\varphi''=-2\pi J_2(ka\sin\theta)\end{aligned}\tag{8.19}$$

Putting everything together, we find that

$$\begin{aligned}&\int_{V'}\mathrm{d}^3x'\,e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}'}\,\mathbf{j}_\omega\times\mathbf{k}=\mathcal{I}_\theta\,\hat{\boldsymbol{\theta}}+\mathcal{I}_\varphi\,\hat{\boldsymbol{\varphi}}\\&\quad=I_0ak\pi\cos\varphi\,[J_0(ka\sin\theta)-J_2(ka\sin\theta)]\,\hat{\boldsymbol{\theta}}\\&\qquad+I_0ak\pi\cos\theta\sin\varphi\,[J_0(ka\sin\theta)+J_2(ka\sin\theta)]\,\hat{\boldsymbol{\varphi}}\end{aligned}\tag{8.20}$$

so that, in spherical coordinates where $|\mathbf{x}|=r$,

$$\mathbf{B}^{\mathrm{rad}}_\omega(\mathbf{x})=\frac{-\mathrm{i}\mu_0 e^{ikr}}{4\pi r}\left(\mathcal{I}_\theta\,\hat{\boldsymbol{\theta}}+\mathcal{I}_\varphi\,\hat{\boldsymbol{\varphi}}\right)\tag{8.21}$$

To obtain the desired physical magnetic field in the radiation (far) zone we must Fourier transform back to t space and take the real part and evaluate it at the retarded time:

$$\begin{aligned}\mathbf{B}^{\mathrm{rad}}(t,\mathbf{x})&=\mathrm{Re}\left\{\frac{-\mathrm{i}\mu_0 e^{(\mathrm{i}kr-\omega t')}}{4\pi r}\left(\mathcal{I}_\theta\,\hat{\boldsymbol{\theta}}+\mathcal{I}_\varphi\,\hat{\boldsymbol{\varphi}}\right)\right\}\\&=\frac{\mu_0}{4\pi r}\sin(kr-\omega t')\left(\mathcal{I}_\theta\,\hat{\boldsymbol{\theta}}+\mathcal{I}_\varphi\,\hat{\boldsymbol{\varphi}}\right)\\&=\frac{I_0ak\mu_0}{4r}\sin(kr-\omega t')\Big(\cos\varphi\,[J_0(ka\sin\theta)-J_2(ka\sin\theta)]\,\hat{\boldsymbol{\theta}}\\&\qquad+\cos\theta\sin\varphi\,[J_0(ka\sin\theta)+J_2(ka\sin\theta)]\,\hat{\boldsymbol{\varphi}}\Big)\end{aligned}\tag{8.22}$$

From this expression for the radiated **B** field, we can obtain the radiated **E** field with the help of Maxwell's equations.

8.2 Multipole radiation

In the general case, and when we are interested in evaluating the radiation far from the source volume, we can introduce an approximation which leads to a *multipole expansion* where individual terms can be evaluated analytically. We shall use *Hertz' method* to obtain this expansion.

8.2.1 The Hertz potential

Let us consider the equation of continuity, which, according to expression (1.23) on page 9, can be written

$$\frac{\partial \rho(t,\mathbf{x})}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j}(t,\mathbf{x}) = 0 \tag{8.23}$$

In Section 6.1.1 we introduced the electric polarisation $\mathbf{P}$ such that $\boldsymbol{\nabla} \cdot \mathbf{P} = -\rho^{\text{pol}}$, the polarisation charge density. If we introduce a vector field $\boldsymbol{\pi}(t,\mathbf{x})$ such that

$$\boldsymbol{\nabla} \cdot \boldsymbol{\pi} = -\rho^{\text{true}} \tag{8.24a}$$

$$\frac{\partial \boldsymbol{\pi}}{\partial t} = \mathbf{j}^{\text{true}} \tag{8.24b}$$

and compare with Equation (8.23), we see that $\boldsymbol{\pi}(t,\mathbf{x})$ satisfies this equation of continuity. Furthermore, if we compare with the electric polarisation [*cf.* Equation (6.9) on page 85], we see that the quantity $\boldsymbol{\pi}$ is related to the 'true' charges in the same way as $\mathbf{P}$ is related to polarised charge, namely as a dipole moment density. The quantity $\boldsymbol{\pi}$ is referred to as the *polarisation vector* since, formally, it treats also the 'true' (free) charges as polarisation charges so that

$$\boldsymbol{\nabla} \cdot \ E = \frac{\rho^{\text{true}} + \rho^{\text{pol}}}{\varepsilon_0} = \frac{-\boldsymbol{\nabla} \cdot \boldsymbol{\pi} - \boldsymbol{\nabla} \cdot \mathbf{P}}{\varepsilon_0} \tag{8.25}$$

We introduce a further potential $\boldsymbol{\Pi}^{\text{e}}$ with the following property

$$\boldsymbol{\nabla} \cdot \boldsymbol{\Pi}^{\text{e}} = -\phi \tag{8.26a}$$

$$\frac{1}{c^2}\frac{\partial \boldsymbol{\Pi}^{\text{e}}}{\partial t} = \mathbf{A} \tag{8.26b}$$

where ϕ and $\mathbf{A}$ are the electromagnetic scalar and vector potentials, respectively. As we see, $\boldsymbol{\Pi}^{\text{e}}$ acts as a '*super-potential*' in the sense that it is a potential from which we can obtain other potentials. It is called the *Hertz' vector* or *polarisation potential*. Requiring that the scalar and vector potentials ϕ and $\mathbf{A}$, respectively, fulfil their inhomogeneous wave equations, one finds, using (8.24) and (8.26), that Hertz' vector must satisfy the inhomogeneous wave equation

$$\Box^2 \boldsymbol{\Pi}^{\text{e}} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{\Pi}^{\text{e}} - \nabla^2 \boldsymbol{\Pi}^{\text{e}} = \frac{\boldsymbol{\pi}}{\varepsilon_0} \tag{8.27}$$

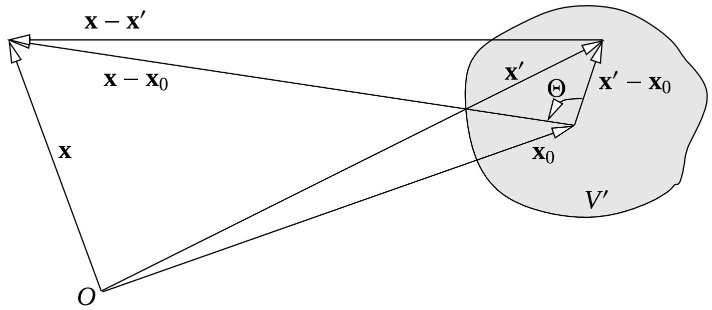

FIGURE 8.4: Geometry of a typical multipole radiation problem where the field point $\mathbf{x}$ is located some distance away from the finite source volume V' centred around $\mathbf{x}_0$. If $k\,|\mathbf{x}' - \mathbf{x}_0| \ll 1 \ll k\,|\mathbf{x} - \mathbf{x}_0|$, then the radiation at $\mathbf{x}$ is well approximated by a few terms in the multipole expansion.

This equation is of the same type as Equation (3.15) on page 38, and has therefore the retarded solution

$$\mathbf{\Pi}^{\mathrm{e}}(t,\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \mathrm{d}^3x'\, \frac{\boldsymbol{\pi}(t'_{\mathrm{ret}},\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \tag{8.28}$$

with Fourier components

$$\mathbf{\Pi}^{\mathrm{e}}_{\omega}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \mathrm{d}^3x'\, \frac{\boldsymbol{\pi}_{\omega}(\mathbf{x}')e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \tag{8.29}$$

If we introduce the *help vector* $\mathbf{C}$ such that

$$\mathbf{C} = \boldsymbol{\nabla} \times \mathbf{\Pi}^{\mathrm{e}} \tag{8.30}$$

we see that we can calculate the magnetic and electric fields, respectively, as follows

$$\mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{C}}{\partial t} \tag{8.31a}$$

$$\mathbf{E} = \boldsymbol{\nabla} \times \mathbf{C} \tag{8.31b}$$

Clearly, the last equation is valid only outside the source volume, where $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$. Since we are mainly interested in the fields in the far zone, a long distance from the source region, this is no essential limitation.

Assume that the source region is a limited volume around some central point $\mathbf{x}_0$ far away from the field (observation) point $\mathbf{x}$ illustrated in Figure 8.4. Under these assumptions, we can expand the Hertz' vector, expression (8.28), due to the presence of non-vanishing $\boldsymbol{\pi}(t'_{\mathrm{ret}},\mathbf{x}')$ in the vicinity of $\mathbf{x}_0$, in a formal series. For this purpose we recall from *potential theory* that

$$\begin{aligned}\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} &\equiv \frac{e^{ik|(\mathbf{x}-\mathbf{x}_0)-(\mathbf{x}'-\mathbf{x}_0)|}}{|(\mathbf{x}-\mathbf{x}_0)-(\mathbf{x}'-\mathbf{x}_0)|} \\ &= \mathrm{i}k\sum_{n=0}^{\infty}(2n+1)P_n(\cos\Theta)\,j_n(k\left|\mathbf{x}'-\mathbf{x}_0\right|)h_n^{(1)}(k\left|\mathbf{x}-\mathbf{x}_0\right|)\end{aligned} \tag{8.32}$$

where (see Figure 8.4 on the previous page)

$\dfrac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}$ is a *Green function*

Θ is the angle between $\mathbf{x}'-\mathbf{x}_0$ and $\mathbf{x}-\mathbf{x}_0$

$P_n(\cos\Theta)$ is the *Legendre polynomial* of order n

$j_n(k\left|\mathbf{x}'-\mathbf{x}_0\right|)$ is the *spherical Bessel function of the first kind* of order n

$h_n^{(1)}(k\left|\mathbf{x}-\mathbf{x}_0\right|)$ is the *spherical Hankel function of the first kind* of order n

According to the addition theorem for Legendre polynomials

$$P_n(\cos\Theta) = \sum_{m=-n}^{n}(-1)^m P_n^m(\cos\theta)P_n^{-m}(\cos\theta')e^{\mathrm{i}m(\varphi-\varphi')} \tag{8.33}$$

where P_n^m is an *associated Legendre polynomial* and, in spherical polar coordinates,

$$\mathbf{x}'-\mathbf{x}_0 = (\left|\mathbf{x}'-\mathbf{x}_0\right|, \theta', \phi') \tag{8.34a}$$

$$\mathbf{x}-\mathbf{x}_0 = (\left|\mathbf{x}-\mathbf{x}_0\right|, \theta, \phi) \tag{8.34b}$$

Inserting Equation (8.32), together with Formula (8.33) above, into Equation (8.29) on the preceding page, we can in a formally exact way expand the Fourier component of the Hertz' vector as

$$\begin{aligned}\mathbf{\Pi}_\omega^{\mathrm{e}} = &\frac{\mathrm{i}k}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(2n+1)(-1)^m h_n^{(1)}(k\left|\mathbf{x}-\mathbf{x}_0\right|)\,P_n^m(\cos\theta)\,e^{\mathrm{i}m\varphi} \\ &\times\int_{V'}\mathrm{d}^3x'\,\boldsymbol{\pi}_\omega(\mathbf{x}')\,j_n(k\left|\mathbf{x}'-\mathbf{x}_0\right|)\,P_n^{-m}(\cos\theta')\,e^{-\mathrm{i}m\varphi'}\end{aligned} \tag{8.35}$$

We notice that there is no dependence on $\mathbf{x}-\mathbf{x}_0$ inside the integral; the integrand is only dependent on the relative source vector $\mathbf{x}'-\mathbf{x}_0$.

We are interested in the case where the field point is many wavelengths away from the well-localised sources, *i.e.*, when the following inequalities

$$k\left|\mathbf{x}'-\mathbf{x}_0\right| \ll 1 \ll k\left|\mathbf{x}-\mathbf{x}_0\right| \tag{8.36}$$

hold. Then we may to a good approximation replace $h_n^{(1)}$ with the first term in its asymptotic expansion:

$$h_n^{(1)}(k\left|\mathbf{x}-\mathbf{x}_0\right|) \approx (-\mathrm{i})^{n+1}\frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{k\left|\mathbf{x}-\mathbf{x}_0\right|} \tag{8.37}$$

and replace j_n with the first term in its power series expansion:

$$j_n(k\left|\mathbf{x}'-\mathbf{x}_0\right|) \approx \frac{2^n n!}{(2n+1)!}\left(k\left|\mathbf{x}'-\mathbf{x}_0\right|\right)^n \tag{8.38}$$

Inserting these expansions into Equation (8.35) on the preceding page, we obtain the *multipole expansion* of the Fourier component of the Hertz' vector

$$\mathbf{\Pi}^{\mathrm{e}}_{\omega} \approx \sum_{n=0}^{\infty} \mathbf{\Pi}^{\mathrm{e}(n)}_{\omega} \tag{8.39a}$$

where

$$\mathbf{\Pi}^{\mathrm{e}(n)}_{\omega} = (-\mathrm{i})^n \frac{1}{4\pi\varepsilon_0}\frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}\frac{2^n n!}{(2n)!}\int_{V'}\mathrm{d}^3x'\,\boldsymbol{\pi}_{\omega}(\mathbf{x}')\,(k\left|\mathbf{x}'-\mathbf{x}_0\right|)^n P_n(\cos\Theta) \tag{8.39b}$$

This expression is approximately correct only if certain care is exercised; if many $\mathbf{\Pi}^{\mathrm{e}(n)}_{\omega}$ terms are needed for an accurate result, the expansions of the spherical Hankel and Bessel functions used above may not be consistent and must be replaced by more accurate expressions. Taking the inverse Fourier transform of $\mathbf{\Pi}^{\mathrm{e}}_{\omega}$ will yield the Hertz' vector in time domain, which inserted into Equation (8.30) on page 113 will yield $\mathbf{C}$. The resulting expression can then in turn be inserted into Equation (8.31) on page 113 in order to obtain the radiation fields.

For a linear source distribution along the polar axis, $\Theta = \theta$ in expression (8.39b), and $P_n(\cos\theta)$ gives the angular distribution of the radiation. In the general case, however, the angular distribution must be computed with the help of Formula (8.33) on the preceding page. Let us now study the lowest order contributions to the expansion of Hertz' vector.

8.2.2 Electric dipole radiation

Choosing $n = 0$ in expression (8.39b), we obtain

$$\mathbf{\Pi}^{\mathrm{e}(0)}_{\omega} = \frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{4\pi\varepsilon_0\,|\mathbf{x}-\mathbf{x}_0|}\int_{V'}\mathrm{d}^3x'\,\boldsymbol{\pi}_{\omega}(\mathbf{x}') = \frac{1}{4\pi\varepsilon_0}\frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}\,\mathbf{p}_{\omega} \tag{8.40}$$

Since $\boldsymbol{\pi}$ represents a dipole moment density for the 'true' charges (in the same vein as P does so for the polarised charges), $\mathbf{p}_{\omega} = \int_{V'}\mathrm{d}^3x'\,\boldsymbol{\pi}_{\omega}(\mathbf{x}')$ is the Fourier component of the *electric dipole moment*

$$\mathbf{p}(t,\mathbf{x}_0) = \int_{V'}\mathrm{d}^3x'\,\boldsymbol{\pi}(t',\mathbf{x}') = \int_{V'}\mathrm{d}^3x'\,(\mathbf{x}'-\mathbf{x}_0)\rho(t',\mathbf{x}') \tag{8.41}$$

[*cf.* Equation (6.2) on page 84 which describes the static dipole moment]. If a spherical coordinate system is chosen with its polar axis along $\mathbf{p}_{\omega}$ as in Figure 8.5 on the

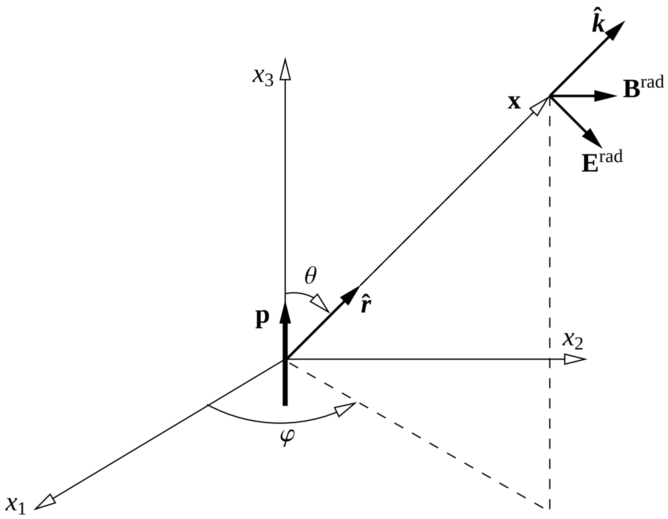

FIGURE 8.5: If a spherical polar coordinate system (r, θ, φ) is chosen such that the electric dipole moment $\mathbf{p}$ (and thus its Fourier transform $\mathbf{p}_\omega$) is located at the origin and directed along the polar axis, the calculations are simplified.

following page, the components of $\mathbf{\Pi}_\omega^{e(0)}$ are

$$\Pi_r^e \stackrel{\text{def}}{\equiv} \mathbf{\Pi}_\omega^{e(0)} \cdot \hat{\boldsymbol{r}} = \frac{1}{4\pi\varepsilon_0} \frac{e^{ik|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|} p_\omega \cos\theta \tag{8.42a}$$

$$\Pi_\theta^e \stackrel{\text{def}}{\equiv} \mathbf{\Pi}_\omega^{e(0)} \cdot \hat{\boldsymbol{\theta}} = -\frac{1}{4\pi\varepsilon_0} \frac{e^{ik|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|} p_\omega \sin\theta \tag{8.42b}$$

$$\Pi_\varphi^e \stackrel{\text{def}}{\equiv} \mathbf{\Pi}_\omega^{e(0)} \cdot \hat{\boldsymbol{\varphi}} = 0 \tag{8.42c}$$

Evaluating Formula (8.30) on page 113 for the help vector $\mathbf{C}$, with the spherically polar components (8.42) of $\mathbf{\Pi}_\omega^{e(0)}$ inserted, we obtain

$$\mathbf{C}_\omega = C_{\omega,\varphi}^{(0)} \hat{\boldsymbol{\varphi}} = \frac{1}{4\pi\varepsilon_0} \left(\frac{1}{|\mathbf{x}-\mathbf{x}_0|} - ik \right) \frac{e^{ik|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|} p_\omega \sin\theta \, \hat{\boldsymbol{\varphi}} \tag{8.43}$$

Applying this to Equation (8.31) on page 113, we obtain directly the Fourier components of the fields

$$\mathbf{B}_\omega = -i \frac{\omega\mu_0}{4\pi} \left(\frac{1}{|\mathbf{x}-\mathbf{x}_0|} - ik \right) \frac{e^{ik|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|} p_\omega \sin\theta \, \hat{\boldsymbol{\varphi}} \tag{8.44a}$$

$$\begin{aligned} \mathbf{E}_\omega = \frac{1}{4\pi\varepsilon_0} \Bigg[& 2 \left(\frac{1}{|\mathbf{x}-\mathbf{x}_0|^2} - \frac{ik}{|\mathbf{x}-\mathbf{x}_0|} \right) \cos\theta \frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|} \\ & + \left(\frac{1}{|\mathbf{x}-\mathbf{x}_0|^2} - \frac{ik}{|\mathbf{x}-\mathbf{x}_0|} - k^2 \right) \sin\theta \, \hat{\boldsymbol{\theta}} \Bigg] \frac{e^{ik|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|} p_\omega \end{aligned} \tag{8.44b}$$

Keeping only those parts of the fields which dominate at large distances (the radiation fields) and recalling that the wave vector $\mathbf{k} = k(\mathbf{x} - \mathbf{x}_0)/\left|\mathbf{x} - \mathbf{x}_0\right|$ where $k = \omega/c$, we can now write down the Fourier components of the radiation parts of the magnetic and electric fields from the dipole:

$$\mathbf{B}_\omega^{\text{rad}} = -\frac{\omega\mu_0}{4\pi}\frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}\,p_\omega k\sin\theta\,\hat{\boldsymbol{\varphi}} = -\frac{\omega\mu_0}{4\pi}\frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}\,(\mathbf{p}_\omega\times\mathbf{k}) \tag{8.45a}$$

$$\mathbf{E}_\omega^{\text{rad}} = -\frac{1}{4\pi\varepsilon_0}\frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}\,p_\omega k^2\sin\theta\,\hat{\boldsymbol{\theta}} = -\frac{1}{4\pi\varepsilon_0}\frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}\,[(\mathbf{p}_\omega\times\mathbf{k})\times\mathbf{k}] \tag{8.45b}$$

These fields constitute the *electric dipole radiation*, also known as *E1 radiation*.

8.2.3 Magnetic dipole radiation

The next term in the expression (8.39b) on page 115 for the expansion of the Fourier transform of the Hertz' vector is for $n = 1$:

$$\begin{aligned}\boldsymbol{\Pi}_\omega^{\mathrm{e}(1)} &= -\mathrm{i}\frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{4\pi\varepsilon_0\,|\mathbf{x}-\mathbf{x}_0|}\int_{V'}\mathrm{d}^3x'\;k\left|\mathbf{x}'-\mathbf{x}_0\right|\boldsymbol{\pi}_\omega(\mathbf{x}')\cos\Theta\\ &= -\mathrm{i}k\frac{1}{4\pi\varepsilon_0}\frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|^2}\int_{V'}\mathrm{d}^3x'\;[(\mathbf{x}-\mathbf{x}_0)\cdot(\mathbf{x}'-\mathbf{x}_0)]\,\boldsymbol{\pi}_\omega(\mathbf{x}')\end{aligned} \tag{8.46}$$

Here, the term $[(\mathbf{x}-\mathbf{x}_0)\cdot(\mathbf{x}'-\mathbf{x}_0)]\,\boldsymbol{\pi}_\omega(\mathbf{x}')$ can be rewritten

$$[(\mathbf{x}-\mathbf{x}_0)\cdot(\mathbf{x}'-\mathbf{x}_0)]\,\boldsymbol{\pi}_\omega(\mathbf{x}') = (x_i - x_{0,i})(x'_i - x_{0,i})\,\boldsymbol{\pi}_\omega(\mathbf{x}') \tag{8.47}$$

and introducing

$$\eta_i = x_i - x_{0,i} \tag{8.48a}$$

$$\eta'_i = x'_i - x_{0,i} \tag{8.48b}$$

the jth component of the integrand in $\boldsymbol{\Pi}_\omega^{\mathrm{e}\,(1)}$ can be broken up into

$$\begin{aligned}\{[(\mathbf{x}-\mathbf{x}_0)\cdot(\mathbf{x}'-\mathbf{x}_0)]\,\boldsymbol{\pi}_\omega(\mathbf{x}')\}_j &= \frac{1}{2}\eta_i\left(\pi_{\omega,j}\eta'_i + \pi_{\omega,i}\eta'_j\right)\\ &\quad+\frac{1}{2}\eta_i\left(\pi_{\omega,j}\eta'_i - \pi_{\omega,i}\eta'_j\right)\end{aligned} \tag{8.49}$$

i.e., as the sum of two parts, the first being symmetric and the second antisymmetric in the indices i, j. We note that the antisymmetric part can be written as

$$\begin{aligned}\frac{1}{2}\eta_i\left(\pi_{\omega,j}\eta'_i - \pi_{\omega,i}\eta'_j\right) &= \frac{1}{2}[\pi_{\omega,j}(\eta_i\eta'_i) - \eta'_j(\eta_i\pi_{\omega,i})]\\ &= \frac{1}{2}[\boldsymbol{\pi}_\omega(\boldsymbol{\eta}\cdot\boldsymbol{\eta}') - \boldsymbol{\eta}'(\boldsymbol{\eta}\cdot\boldsymbol{\pi}_\omega)]_j\\ &= \frac{1}{2}\left\{(\mathbf{x}-\mathbf{x}_0)\times[\boldsymbol{\pi}_\omega\times(\mathbf{x}'-\mathbf{x}_0)]\right\}_j\end{aligned} \tag{8.50}$$

The utilisation of Equations (8.24) on page 112, and the fact that we are considering a single Fourier component,

$$\boldsymbol{\pi}(t, \mathbf{x}) = \boldsymbol{\pi}_\omega e^{-\mathrm{i}\omega t} \tag{8.51}$$

allow us to express $\boldsymbol{\pi}_\omega$ in $\mathbf{j}_\omega$ as

$$\boldsymbol{\pi}_\omega = \mathrm{i}\,\frac{\mathbf{j}_\omega}{\omega} \tag{8.52}$$

Hence, we can write the antisymmetric part of the integral in Formula (8.46) on the previous page as

$$\begin{aligned}
&\frac{1}{2}(\mathbf{x}-\mathbf{x}_0)\times\int_{V'}\mathrm{d}^3x'\,\boldsymbol{\pi}_\omega(\mathbf{x}')\times(\mathbf{x}'-\mathbf{x}_0)\\
&\qquad=\mathrm{i}\frac{1}{2\omega}(\mathbf{x}-\mathbf{x}_0)\times\int_{V'}\mathrm{d}^3x'\,\mathbf{j}_\omega(\mathbf{x}')\times(\mathbf{x}'-\mathbf{x}_0)\\
&\qquad=-\mathrm{i}\frac{1}{\omega}(\mathbf{x}-\mathbf{x}_0)\times\mathbf{m}_\omega
\end{aligned} \tag{8.53}$$

where we introduced the Fourier transform of the *magnetic dipole moment*

$$\mathbf{m}_\omega = \frac{1}{2}\int_{V'}\mathrm{d}^3x'\,(\mathbf{x}'-\mathbf{x}_0)\times\mathbf{j}_\omega(\mathbf{x}') \tag{8.54}$$

The final result is that the antisymmetric, magnetic dipole, part of $\boldsymbol{\Pi}^{\mathrm{e}(1)}_\omega$ can be written

$$\boldsymbol{\Pi}^{\mathrm{e,antisym}(1)}_\omega = -\frac{k}{4\pi\varepsilon_0\omega}\frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|^2}(\mathbf{x}-\mathbf{x}_0)\times\mathbf{m}_\omega \tag{8.55}$$

In analogy with the electric dipole case, we insert this expression into Equation (8.30) on page 113 to evaluate **C**, with which Equations (8.31) on page 113 then gives the **B** and **E** fields. Discarding, as before, all terms belonging to the near fields and transition fields and keeping only the terms that dominate at large distances, we obtain

$$\mathbf{B}^{\mathrm{rad}}_\omega(\mathbf{x}) = -\frac{\mu_0}{4\pi}\frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}(\mathbf{m}_\omega\times\mathbf{k})\times\mathbf{k} \tag{8.56a}$$

$$\mathbf{E}^{\mathrm{rad}}_\omega(\mathbf{x}) = \frac{k}{4\pi\varepsilon_0 c}\frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}\mathbf{m}_\omega\times\mathbf{k} \tag{8.56b}$$

which are the fields of the *magnetic dipole radiation* (*M1 radiation*).

8.2.4 Electric quadrupole radiation

The symmetric part $\boldsymbol{\Pi}^{\mathrm{e,sym}(1)}_\omega$ of the $n = 1$ contribution in the Equation (8.39b) on page 115 for the expansion of the Hertz' vector can be expressed in terms of the *electric quadrupole tensor*, which is defined in accordance with Equation (6.3) on page 84:

$$\mathbf{Q}(t, \mathbf{x}_0) = \int_{V'} \mathrm{d}^3x' \, (\mathbf{x}' - \mathbf{x}_0)(\mathbf{x}' - \mathbf{x}_0)\rho(t, \mathbf{x}') \tag{8.57}$$

Again we use this expression in Equation (8.30) on page 113 to calculate the fields via Equations (8.31) on page 113. Tedious, but fairly straightforward algebra (which we will not present here), yields the resulting fields. The radiation components of the fields in the far field zone (wave zone) are given by

$$\mathbf{B}_\omega^{\mathrm{rad}}(\mathbf{x}) = \frac{\mathrm{i}\mu_0\omega}{8\pi} \frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x} - \mathbf{x}_0|} (\mathbf{k} \cdot \mathbf{Q}_\omega) \times \mathbf{k} \tag{8.58a}$$

$$\mathbf{E}_\omega^{\mathrm{rad}}(\mathbf{x}) = \frac{\mathrm{i}}{8\pi\varepsilon_0} \frac{e^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x} - \mathbf{x}_0|} [(\mathbf{k} \cdot \mathbf{Q}_\omega) \times \mathbf{k}] \times \mathbf{k} \tag{8.58b}$$

This type of radiation is called *electric quadrupole radiation* or *E2 radiation*.

8.3 Radiation from a localised charge in arbitrary motion

The derivation of the radiation fields for the case of the source moving relative to the observer is considerably more complicated than the stationary cases studied above. In order to handle this non-stationary situation, we use the retarded potentials (3.32) on page 41 in Chapter 3

$$\phi(t, \mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \mathrm{d}^3x' \, \frac{\rho(t'_{\mathrm{ret}}, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \tag{8.59a}$$

$$\mathbf{A}(t, \mathbf{x}) = \frac{\mu_0}{4\pi} \int_{V'} \mathrm{d}^3x' \, \frac{\mathbf{j}(t'_{\mathrm{ret}}, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \tag{8.59b}$$

and consider a source region with such a limited spatial extent that the charges and currents are well localised. Specifically, we consider a charge q', for instance an electron, which, classically, can be thought of as a localised, unstructured and rigid 'charge distribution' with a small, finite radius. The part of this 'charge distribution' $\mathrm{d}q'$ which we are considering is located in $\mathrm{d}V' = \mathrm{d}^3x'$ in the sphere in Figure 8.6 on the next page. Since we assume that the electron (or any other other similar electric charge) moves with a velocity $\mathbf{v}$ whose direction is arbitrary and whose magnitude can be almost comparable to the speed of light, we cannot say that the charge and current to be used in (8.59) is $\int_V \rho(t'_{\mathrm{ret}}, \mathbf{x}')\,\mathrm{d}^3x'$ and $\int_V \mathbf{v}\rho(t'_{\mathrm{ret}}, \mathbf{x}')\,\mathrm{d}^3x'$, respectively, because in the finite time interval during which the observed signal is generated, part of the charge distribution will 'leak' out of the volume element d^3x'.

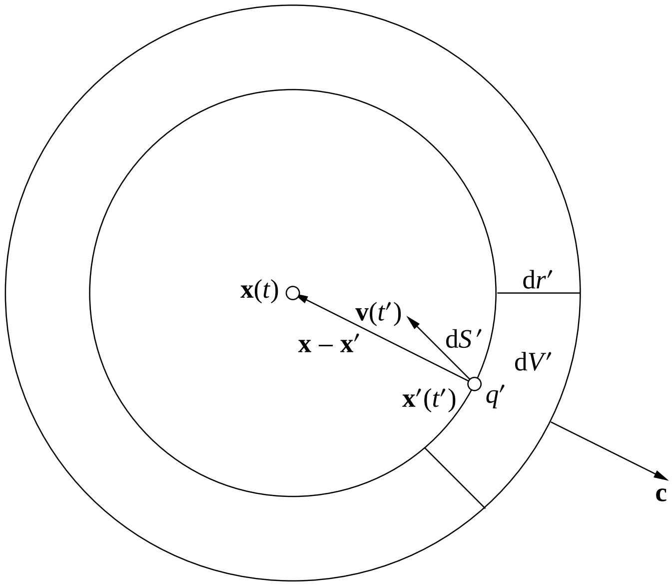

Figure 8.6: Signals which are observed at the field point $\mathbf{x}$ at time t were generated at source points $\mathbf{x}'(t')$ on a sphere, centred on $\mathbf{x}$ and expanding, as time increases, with the velocity $\mathbf{c}$ outward from the centre. The source charge element moves with an arbitrary velocity $\mathbf{v}$ and gives rise to a source 'leakage' out of the source volume $\mathrm{d}V' = \mathrm{d}^3x'$.

8.3.1 The Liénard-Wiechert potentials

The charge distribution in Figure 8.6 on page 120 which contributes to the field at $\mathbf{x}(t)$ is located at $\mathbf{x}'(t')$ on a sphere with radius $r = |\mathbf{x} - \mathbf{x}'| = c(t - t')$. The radius interval of this sphere from which radiation is received at the field point $\mathbf{x}$ during the time interval $(t', t' + \mathrm{d}t')$ is $(r', r' + \mathrm{d}r')$ and the net amount of charge in this radial interval is

$$\mathrm{d}q' = \rho(t'_{\mathrm{ret}}, \mathbf{x}')\,\mathrm{d}S'\,\mathrm{d}r' - \rho(t'_{\mathrm{ret}}, \mathbf{x}')\frac{(\mathbf{x} - \mathbf{x}')\cdot\mathbf{v}}{|\mathbf{x} - \mathbf{x}'|}\,\mathrm{d}S'\,\mathrm{d}t' \tag{8.60}$$

where the last term represents the amount of 'source leakage' due to the fact that the charge distribution moves with velocity $\mathbf{v}(t') = \mathrm{d}\mathbf{x}/\mathrm{d}t'$. Since $\mathrm{d}t' = \mathrm{d}r'/c$ and $\mathrm{d}S'\,\mathrm{d}r' = \mathrm{d}^3x'$ we can rewrite the expression for the net charge as

$$\begin{aligned}\mathrm{d}q' &= \rho(t'_{\mathrm{ret}}, \mathbf{x}')\,\mathrm{d}^3x' - \rho(t'_{\mathrm{ret}}, \mathbf{x}')\frac{(\mathbf{x} - \mathbf{x}')\cdot\mathbf{v}}{c\,|\mathbf{x} - \mathbf{x}'|}\,\mathrm{d}^3x' \\ &= \rho(t'_{\mathrm{ret}}, \mathbf{x}')\left(1 - \frac{(\mathbf{x} - \mathbf{x}')\cdot\mathbf{v}}{c\,|\mathbf{x} - \mathbf{x}'|}\right)\mathrm{d}^3x'\end{aligned} \tag{8.61}$$

or

$$\rho(t'_{\text{ret}}, \mathbf{x}')\,\mathrm{d}^3x' = \frac{\mathrm{d}q'}{1 - \frac{(\mathbf{x}-\mathbf{x}')\cdot\mathbf{v}}{c|\mathbf{x}-\mathbf{x}'|}} \tag{8.62}$$

which leads to the expression

$$\frac{\rho(t'_{\text{ret}}, \mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}^3x' = \frac{\mathrm{d}q'}{|\mathbf{x}-\mathbf{x}'| - \frac{(\mathbf{x}-\mathbf{x}')\cdot\mathbf{v}}{c}} \tag{8.63}$$

This is the expression to be used in the Formulae (8.59) on page 119 for the retarded potentials. The result is (recall that $\mathbf{j} = \rho\mathbf{v}$)

$$\phi(t, \mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\mathrm{d}q'}{|\mathbf{x}-\mathbf{x}'| - \frac{(\mathbf{x}-\mathbf{x}')\cdot\mathbf{v}}{c}} \tag{8.64a}$$

$$\mathbf{A}(t, \mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{v}\,\mathrm{d}q'}{|\mathbf{x}-\mathbf{x}'| - \frac{(\mathbf{x}-\mathbf{x}')\cdot\mathbf{v}}{c}} \tag{8.64b}$$

For a sufficiently small and well localised charge distribution we can, assuming that the integrands do not change sign in the integration volume, use the mean value theorem and the fact that $\int_V \mathrm{d}q' = q'$ to evaluate these expressions to become

$$\phi(t, \mathbf{x}) = \frac{q'}{4\pi\varepsilon_0}\frac{1}{|\mathbf{x}-\mathbf{x}'| - \frac{(\mathbf{x}-\mathbf{x}')\cdot\mathbf{v}}{c}} = \frac{q'}{4\pi\varepsilon_0}\frac{1}{s} \tag{8.65a}$$

$$\mathbf{A}(t, \mathbf{x}) = \frac{q'}{4\pi\varepsilon_0 c^2}\frac{\mathbf{v}}{|\mathbf{x}-\mathbf{x}'| - \frac{(\mathbf{x}-\mathbf{x}')\cdot\mathbf{v}}{c}} = \frac{q'}{4\pi\varepsilon_0 c^2}\frac{\mathbf{v}}{s} = \frac{\mathbf{v}}{c^2}\phi(t, \mathbf{x}) \tag{8.65b}$$

where

$$s = s(t', \mathbf{x}) = \left|\mathbf{x} - \mathbf{x}'(t')\right| - \frac{(\mathbf{x} - \mathbf{x}'(t'))\cdot\mathbf{v}(t')}{c} \tag{8.66a}$$

$$= \left|\mathbf{x} - \mathbf{x}'(t')\right|\left(1 - \frac{\mathbf{x} - \mathbf{x}'(t')}{|\mathbf{x} - \mathbf{x}'(t')|}\cdot\frac{\mathbf{v}(t')}{c}\right) \tag{8.66b}$$

$$= (\mathbf{x} - \mathbf{x}'(t'))\cdot\left(\frac{\mathbf{x} - \mathbf{x}'(t')}{|\mathbf{x} - \mathbf{x}'(t')|} - \frac{\mathbf{v}(t')}{c}\right) \tag{8.66c}$$

is the *retarded relative distance*. The potentials (8.65) are precisely the *Liénard-Wiechert potentials* which we derived in Section 4.3.2 on page 59 by using a covariant formalism.

It is important to realise that in the complicated derivation presented here, the observer is in a coordinate system which has an 'absolute' meaning and the velocity $\mathbf{v}$ is that of the particle, whereas in the covariant derivation two frames of equal standing were moving relative to each other with $\mathbf{v}$. Expressed in the four-potential, Equation (4.47) on page 58, the Liénard-Wiechert potentials become

$$A^\mu(x^\kappa) = \frac{q'}{4\pi\varepsilon_0}\left(\frac{1}{cs}, \frac{\mathbf{v}}{c^2 s}\right) = \left(\frac{\phi}{c}, \mathbf{A}\right) \tag{8.67}$$

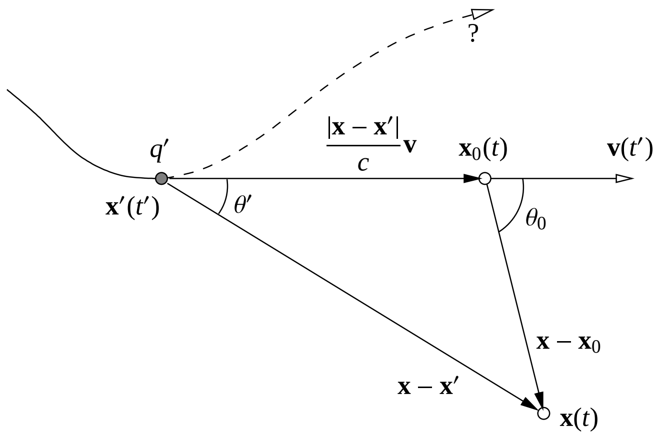

FIGURE 8.7: Signals which are observed at the field point $\mathbf{x}$ at time t were generated at the source point $\mathbf{x}'(t')$. After time t' the particle, which moves with nonuniform velocity, has followed a yet unknown trajectory. Extrapolating tangentially the trajectory from $\mathbf{x}'(t')$, based on the velocity $\mathbf{v}(t')$, defines the *virtual simultaneous coordinate* $\mathbf{x}_0(t)$.

The Liénard-Wiechert potentials are applicable to all problems where a spatially localised charge emits electromagnetic radiation, and we shall now study such emission problems. The electric and magnetic fields are calculated from the potentials in the usual way:

$$\mathbf{B}(t,\mathbf{x}) = \boldsymbol{\nabla}\times\mathbf{A}(t,\mathbf{x}) \tag{8.68a}$$

$$\mathbf{E}(t,\mathbf{x}) = -\boldsymbol{\nabla}\phi(t,\mathbf{x}) - \frac{\partial\mathbf{A}(t,\mathbf{x})}{\partial t} \tag{8.68b}$$

8.3.2 Radiation from an accelerated point charge

Consider a localised charge q' and assume that its trajectory is known experimentally as a function of *retarded time*

$$\mathbf{x}' = \mathbf{x}'(t') \tag{8.69}$$

(in the interest of simplifying our notation, we drop the subscript 'ret' on t' from now on). This means that we know the trajectory of the charge q', *i.e.*, $\mathbf{x}'$, for all times up to the time t' at which a signal was emitted in order to precisely arrive at the field point $\mathbf{x}$ at time t. Because of the finite speed of propagation of the fields, the trajectory at times later than t' is not (yet) known.

The retarded velocity and acceleration at time t' are given by

$$\mathbf{v}(t') = \frac{d\mathbf{x}'}{dt'} \tag{8.70a}$$

$$\mathbf{a}(t') = \dot{\mathbf{v}}(t') = \frac{d\mathbf{v}}{dt'} = \frac{d^2\mathbf{x}'}{dt'^2} \tag{8.70b}$$

As for the charge coordinate $\mathbf{x}'$ itself, we have in general no knowledge of the velocity and acceleration at times later than t', in particular not at the time of observation t. If we choose the field point $\mathbf{x}$ as fixed, application of (8.70) to the relative vector $\mathbf{x} - \mathbf{x}'$ yields

$$\frac{d}{dt'}(\mathbf{x} - \mathbf{x}'(t')) = -\mathbf{v}(t') \tag{8.71a}$$

$$\frac{d^2}{dt'^2}(\mathbf{x} - \mathbf{x}'(t')) = -\dot{\mathbf{v}}(t') \tag{8.71b}$$

The retarded time t' can, at least in principle, be calculated from the implicit relation

$$t' = t'(t, \mathbf{x}) = t - \frac{|\mathbf{x} - \mathbf{x}'(t')|}{c} \tag{8.72}$$

and we shall see later how this relation can be taken into account in the calculations.

According to Formulae (8.68) on the preceding page the electric and magnetic fields are determined via differentiation of the retarded potentials at the observation time t and at the observation point $\mathbf{x}$. In these formulae the unprimed $\boldsymbol{\nabla}$, *i.e.*, the spatial derivative differentiation operator $\boldsymbol{\nabla} = \hat{x}_i \partial/\partial x_i$ means that we differentiate with respect to the coordinates $\mathbf{x} = (x_1, x_2, x_3)$ while keeping t fixed, and the unprimed time derivative operator $\partial/\partial t$ means that we differentiate with respect to t while keeping $\mathbf{x}$ fixed. But the Liénard-Wiechert potentials ϕ and $\mathbf{A}$, Equations (8.65) on page 121, are expressed in the charge velocity $\mathbf{v}(t')$ given by Equation (8.70a) above and the retarded relative distance $s(t', \mathbf{x})$ given by Equation (8.66) on page 121. This means that the expressions for the potentials ϕ and $\mathbf{A}$ contain terms which are expressed explicitly in t', which in turn is expressed implicitly in t via Equation (8.72) above. Despite this complication it is possible, as we shall see below, to determine the electric and magnetic fields and associated quantities at the time of observation t. To this end, we need to investigate carefully the action of differentiation on the potentials.

The differential operator method

We introduce the convention that a differential operator embraced by parentheses with an index $\mathbf{x}$ or t means that the operator in question is applied at constant $\mathbf{x}$ and t, respectively. With this convention, we find that

$$\left(\frac{\partial}{\partial t'}\right)_{\mathbf{x}} \left|\mathbf{x} - \mathbf{x}'(t')\right| = \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} \cdot \left(\frac{\partial}{\partial t'}\right)_{\mathbf{x}} \left(\mathbf{x} - \mathbf{x}'(t')\right) = -\frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}(t')}{|\mathbf{x} - \mathbf{x}'|} \tag{8.73}$$

Furthermore, by applying the operator $(\partial/\partial t)_{\mathbf{x}}$ to Equation (8.72) on the preceding page we find that

$$\begin{aligned}\left(\frac{\partial t'}{\partial t}\right)_{\mathbf{x}} &= 1 - \left(\frac{\partial}{\partial t}\right)_{\mathbf{x}} \frac{|\mathbf{x} - \mathbf{x}'(t'(t,\mathbf{x}))|}{c} \\ &= 1 - \left[\left(\frac{\partial}{\partial t'}\right)_{\mathbf{x}} \frac{|\mathbf{x} - \mathbf{x}'|}{c}\right] \left(\frac{\partial t'}{\partial t}\right)_{\mathbf{x}} \\ &= 1 + \frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}(t')}{c\,|\mathbf{x} - \mathbf{x}'|} \left(\frac{\partial t'}{\partial t}\right)_{\mathbf{x}}\end{aligned} \tag{8.74}$$

This is an algebraic equation in $(\partial t'/\partial t)_{\mathbf{x}}$ which we can solve to obtain

$$\left(\frac{\partial t'}{\partial t}\right)_{\mathbf{x}} = \frac{|\mathbf{x} - \mathbf{x}'|}{|\mathbf{x} - \mathbf{x}'| - (\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}(t')/c} = \frac{|\mathbf{x} - \mathbf{x}'|}{s} \tag{8.75}$$

where $s = s(t', \mathbf{x})$ is the retarded relative distance given by Equation (8.66) on page 121. Making use of Equation (8.75), we obtain the following useful operator identity

$$\left(\frac{\partial}{\partial t}\right)_{\mathbf{x}} = \left(\frac{\partial t'}{\partial t}\right)_{\mathbf{x}} \left(\frac{\partial}{\partial t'}\right)_{\mathbf{x}} = \frac{|\mathbf{x} - \mathbf{x}'|}{s} \left(\frac{\partial}{\partial t'}\right)_{\mathbf{x}} \tag{8.76}$$

Likewise, by applying $(\boldsymbol{\nabla})_t$ to Equation (8.72) on the previous page we obtain

$$\begin{aligned}(\boldsymbol{\nabla})_t\, t' &= -(\boldsymbol{\nabla})_t \frac{|\mathbf{x} - \mathbf{x}'(t'(t,\mathbf{x}))|}{c} = -\frac{\mathbf{x} - \mathbf{x}'}{c\,|\mathbf{x} - \mathbf{x}'|} \cdot (\boldsymbol{\nabla})_t (\mathbf{x} - \mathbf{x}') \\ &= -\frac{\mathbf{x} - \mathbf{x}'}{c\,|\mathbf{x} - \mathbf{x}'|} + \frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}(t')}{c\,|\mathbf{x} - \mathbf{x}'|} (\boldsymbol{\nabla})_t\, t'\end{aligned} \tag{8.77}$$

This is an algebraic equation in $(\boldsymbol{\nabla})_t\, t'$ with the solution

$$(\boldsymbol{\nabla})_t t' = -\frac{\mathbf{x} - \mathbf{x}'}{cs} \tag{8.78}$$

which gives the following operator relation when $(\boldsymbol{\nabla})_t$ is acting on an arbitrary function of t' and $\mathbf{x}$:

$$(\boldsymbol{\nabla})_t = \left[(\boldsymbol{\nabla})_t t'\right] \left(\frac{\partial}{\partial t'}\right)_{\mathbf{x}} + (\boldsymbol{\nabla})_{t'} = -\frac{\mathbf{x} - \mathbf{x}'}{cs} \left(\frac{\partial}{\partial t'}\right)_{\mathbf{x}} + (\boldsymbol{\nabla})_{t'} \tag{8.79}$$

With the help of the rules (8.79) and (8.76) we are now able to replace t by t' in the operations which we need to perform. We find, for instance, that

$$\begin{aligned}\boldsymbol{\nabla}\phi \equiv (\boldsymbol{\nabla}\phi)_t &= \boldsymbol{\nabla} \left(\frac{1}{4\pi\varepsilon_0} \frac{q'}{s}\right) \\ &= -\frac{q'}{4\pi\varepsilon_0 s^2} \left[\frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} - \frac{\mathbf{v}(t')}{c} - \frac{\mathbf{x} - \mathbf{x}'}{cs} \left(\frac{\partial s}{\partial t'}\right)_{\mathbf{x}}\right]\end{aligned} \tag{8.80a}$$

$$\begin{aligned}\frac{\partial \mathbf{A}}{\partial t} \equiv \left(\frac{\partial \mathbf{A}}{\partial t}\right)_{\mathbf{x}} &= \frac{\partial}{\partial t} \left(\frac{\mu_0}{4\pi} \frac{q'\mathbf{v}(t')}{s}\right)_{\mathbf{x}} \\ &= \frac{q'}{4\pi\varepsilon_0 c^2 s^3} \left[\left|\mathbf{x} - \mathbf{x}'\right| s\dot{\mathbf{v}}(t') - \left|\mathbf{x} - \mathbf{x}'\right| \mathbf{v}(t') \left(\frac{\partial s}{\partial t'}\right)_{\mathbf{x}}\right]\end{aligned} \tag{8.80b}$$

Utilising these relations in the calculation of the **E** field from the Liénard-Wiechert potentials, Equations (8.65) on page 121, we obtain

$$\begin{aligned}\mathbf{E}(t,\mathbf{x}) &= -\boldsymbol{\nabla}\phi(t,\mathbf{x}) - \frac{\partial}{\partial t}\mathbf{A}(t,\mathbf{x}) \\ &= \frac{q'}{4\pi\varepsilon_0 s^2(t',\mathbf{x})}\left[\frac{(\mathbf{x}-\mathbf{x}'(t')) - |\mathbf{x}-\mathbf{x}'(t')|\,\mathbf{v}(t')/c}{|\mathbf{x}-\mathbf{x}'(t')|}\right. \\ &\qquad \left. - \frac{(\mathbf{x}-\mathbf{x}'(t')) - |\mathbf{x}-\mathbf{x}'(t')|\,\mathbf{v}(t')/c}{cs(t',\mathbf{x})}\left(\frac{\partial s(t',\mathbf{x})}{\partial t'}\right)_{\mathbf{x}} - \frac{|\mathbf{x}-\mathbf{x}'(t')|\,\dot{\mathbf{v}}(t')}{c^2}\right]\end{aligned} \tag{8.81}$$

Starting from expression (8.66a) on page 121 for the retarded relative distance $s(t',\mathbf{x})$, we see that we can evaluate $(\partial s/\partial t')_{\mathbf{x}}$ in the following way

$$\begin{aligned}\left(\frac{\partial s}{\partial t'}\right)_{\mathbf{x}} &= \left(\frac{\partial}{\partial t'}\right)_{\mathbf{x}}\left(\left|\mathbf{x}-\mathbf{x}'\right| - \frac{(\mathbf{x}-\mathbf{x}')\cdot\mathbf{v}(t')}{c}\right) \\ &= \frac{\partial}{\partial t'}\left|\mathbf{x}-\mathbf{x}'(t')\right| - \frac{1}{c}\left(\frac{\partial\big(\mathbf{x}-\mathbf{x}'(t')\big)}{\partial t'}\cdot\mathbf{v}(t') + (\mathbf{x}-\mathbf{x}'(t'))\cdot\frac{\partial\mathbf{v}(t')}{\partial t'}\right) \\ &= -\frac{(\mathbf{x}-\mathbf{x}')\cdot\mathbf{v}(t')}{|\mathbf{x}-\mathbf{x}'|} + \frac{v^2(t')}{c} - \frac{(\mathbf{x}-\mathbf{x}')\cdot\dot{\mathbf{v}}(t')}{c}\end{aligned} \tag{8.82}$$

where Equation (8.73) on page 123 and Equations (8.70) on page 123, respectively, were used. Hence, the electric field generated by an arbitrarily moving charged particle at $\mathbf{x}'(t')$ is given by the expression

$$\begin{aligned}\mathbf{E}(t,\mathbf{x}) &= \underbrace{\frac{q'}{4\pi\varepsilon_0 s^3(t',\mathbf{x})}\left((\mathbf{x}-\mathbf{x}'(t')) - \frac{|\mathbf{x}-\mathbf{x}'(t')|\,\mathbf{v}(t')}{c}\right)\left(1-\frac{v^2(t')}{c^2}\right)}_{\text{Coulomb field when } v\to 0} \\ &\quad + \underbrace{\frac{q'}{4\pi\varepsilon_0 s^3(t',\mathbf{x})}\left\{\frac{\mathbf{x}-\mathbf{x}'(t')}{c^2}\times\left[\left((\mathbf{x}-\mathbf{x}'(t')) - \frac{|\mathbf{x}-\mathbf{x}'(t')|\,\mathbf{v}(t')}{c}\right)\times\dot{\mathbf{v}}(t')\right]\right\}}_{\text{Radiation (acceleration) field}}\end{aligned} \tag{8.83}$$

The first part of the field, the *velocity field*, tends to the ordinary Coulomb field when $v \to 0$ and does not contribute to the radiation. The second part of the field, the *acceleration field*, is radiated into the far zone and is therefore also called the *radiation field*.

From Figure 8.7 on page 122 we see that the position the charged particle would have had if at t' all external forces would have been switched off so that the trajectory from then on would have been a straight line in the direction of the tangent at $\mathbf{x}'(t')$ is $\mathbf{x}_0(t)$, the *virtual simultaneous coordinate*. During the arbitrary motion, we interpret

$\mathbf{x}-\mathbf{x}_0$ as the coordinate of the field point $\mathbf{x}$ relative to the virtual simultaneous coordinate $\mathbf{x}_0(t)$. Since the time it takes for a signal to propagate (in the assumed vacuum) from $\mathbf{x}'(t')$ to $\mathbf{x}$ is $|\mathbf{x}-\mathbf{x}'|/c$, this relative vector is given by

$$\mathbf{x}-\mathbf{x}_0(t) = \mathbf{x}-\mathbf{x}'(t') - \frac{|\mathbf{x}-\mathbf{x}'(t')|\,\mathbf{v}(t')}{c} \tag{8.84}$$

This allows us to rewrite Equation (8.83) on the previous page in the following way

$$\mathbf{E}(t,\mathbf{x}) = \frac{q'}{4\pi\varepsilon_0 s^3}\left[(\mathbf{x}-\mathbf{x}_0)\left(1-\frac{v^2}{c^2}\right) + (\mathbf{x}-\mathbf{x}')\times\frac{(\mathbf{x}-\mathbf{x}_0)\times\dot{\mathbf{v}}}{c^2}\right] \tag{8.85}$$

In a similar manner we can compute the magnetic field:

$$\begin{aligned}\mathbf{B}(t,\mathbf{x}) &= \boldsymbol{\nabla}\times\mathbf{A}(t,\mathbf{x}) \equiv (\boldsymbol{\nabla})_t\times\mathbf{A} = (\boldsymbol{\nabla})_{t'}\times\mathbf{A} - \frac{\mathbf{x}-\mathbf{x}'}{cs}\times\left(\frac{\partial}{\partial t'}\right)_{\mathbf{x}}\mathbf{A}\\ &= -\frac{q'}{4\pi\varepsilon_0 c^2 s^2}\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\times\mathbf{v} - \frac{\mathbf{x}-\mathbf{x}'}{c\,|\mathbf{x}-\mathbf{x}'|}\times\left(\frac{\partial\mathbf{A}}{\partial t}\right)_{\mathbf{x}}\end{aligned} \tag{8.86}$$

where we made use of Equation (8.65) on page 121 and Formula (8.76) on page 124. But, according to (8.80a),

$$\frac{\mathbf{x}-\mathbf{x}'}{c\,|\mathbf{x}-\mathbf{x}'|}\times(\boldsymbol{\nabla})_t\phi = \frac{q'}{4\pi\varepsilon_0 c^2 s^2}\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\times\mathbf{v} \tag{8.87}$$

so that

$$\begin{aligned}\mathbf{B}(t,\mathbf{x}) &= \frac{\mathbf{x}-\mathbf{x}'}{c\,|\mathbf{x}-\mathbf{x}'|}\times\left[-(\boldsymbol{\nabla}\phi)_t - \left(\frac{\partial\mathbf{A}}{\partial t}\right)_{\mathbf{x}}\right]\\ &= \frac{\mathbf{x}-\mathbf{x}'}{c\,|\mathbf{x}-\mathbf{x}'|}\times\mathbf{E}(t,\mathbf{x})\end{aligned} \tag{8.88}$$

The radiation part of the electric field is obtained from the acceleration field in Formula (8.83) on the preceding page as

$$\begin{aligned}\mathbf{E}^{\mathrm{rad}}(t,\mathbf{x}) &= \lim_{|\mathbf{x}-\mathbf{x}'|\to\infty}\mathbf{E}(t,\mathbf{x})\\ &= \frac{q'}{4\pi\varepsilon_0 c^2 s^3}(\mathbf{x}-\mathbf{x}')\times\left[\left((\mathbf{x}-\mathbf{x}') - \frac{|\mathbf{x}-\mathbf{x}'|\,\mathbf{v}}{c}\right)\times\dot{\mathbf{v}}\right]\\ &= \frac{q'}{4\pi\varepsilon_0 c^2 s^3}(\mathbf{x}-\mathbf{x}')\times[(\mathbf{x}-\mathbf{x}_0)\times\dot{\mathbf{v}}]\end{aligned} \tag{8.89}$$

where in the last step we again used Formula (8.84) above. Using this formula and Formula (8.88), the radiation part of the magnetic field can be written

$$\mathbf{B}^{\mathrm{rad}}(t,\mathbf{x}) = \frac{\mathbf{x}-\mathbf{x}'}{c\,|\mathbf{x}-\mathbf{x}'|}\times\mathbf{E}^{\mathrm{rad}}(t,\mathbf{x}) \tag{8.90}$$

The direct method

An alternative to the differential operator transformation technique just described is to try to express all quantities in the potentials directly in t and $\mathbf{x}$. An example of such a quantity is the retarded relative distance $s(t', \mathbf{x})$. According to Equation (8.66) on page 121, the square of this retarded relative distance can be written

$$s^2(t', \mathbf{x}) = \left|\mathbf{x} - \mathbf{x}'(t')\right|^2 - 2\left|\mathbf{x} - \mathbf{x}'(t')\right| \frac{(\mathbf{x} - \mathbf{x}'(t')) \cdot \mathbf{v}(t')}{c} + \left(\frac{(\mathbf{x} - \mathbf{x}'(t')) \cdot \mathbf{v}(t')}{c}\right)^2 \tag{8.91}$$

If we use the following handy identity

$$\begin{aligned}
&\left(\frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}}{c}\right)^2 + \left(\frac{(\mathbf{x} - \mathbf{x}') \times \mathbf{v}}{c}\right)^2 \\
&= \frac{|\mathbf{x} - \mathbf{x}'|^2 v^2}{c^2} \cos^2\theta' + \frac{|\mathbf{x} - \mathbf{x}'|^2 v^2}{c^2} \sin^2\theta' \\
&= \frac{|\mathbf{x} - \mathbf{x}'|^2 v^2}{c^2} (\cos^2\theta' + \sin^2\theta') = \frac{|\mathbf{x} - \mathbf{x}'|^2 v^2}{c^2}
\end{aligned} \tag{8.92}$$

we find that

$$\left(\frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}}{c}\right)^2 = \frac{|\mathbf{x} - \mathbf{x}'|^2 v^2}{c^2} - \left(\frac{(\mathbf{x} - \mathbf{x}') \times \mathbf{v}}{c}\right)^2 \tag{8.93}$$

Furthermore, from Equation (8.84) on the facing page, we obtain the following identity:

$$(\mathbf{x} - \mathbf{x}'(t')) \times \mathbf{v} = (\mathbf{x} - \mathbf{x}_0(t)) \times \mathbf{v} \tag{8.94}$$

which, when inserted into Equation (8.93) above, yields the relation

$$\left(\frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}}{c}\right)^2 = \frac{|\mathbf{x} - \mathbf{x}'|^2 v^2}{c^2} - \left(\frac{(\mathbf{x} - \mathbf{x}_0) \times \mathbf{v}}{c}\right)^2 \tag{8.95}$$

Inserting the above into expression (8.91) for s^2, this expression becomes

$$\begin{aligned}
s^2 &= \left|\mathbf{x} - \mathbf{x}'\right|^2 - 2\left|\mathbf{x} - \mathbf{x}'\right| \frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}}{c} + \frac{|\mathbf{x} - \mathbf{x}'|^2 v^2}{c^2} - \left(\frac{(\mathbf{x} - \mathbf{x}_0) \times \mathbf{v}}{c}\right)^2 \\
&= \left((\mathbf{x} - \mathbf{x}') - \frac{|\mathbf{x} - \mathbf{x}'|\,\mathbf{v}}{c}\right)^2 - \left(\frac{(\mathbf{x} - \mathbf{x}_0) \times \mathbf{v}}{c}\right)^2 \\
&= (\mathbf{x} - \mathbf{x}_0)^2 - \left(\frac{(\mathbf{x} - \mathbf{x}_0) \times \mathbf{v}}{c}\right)^2 \\
&\equiv |\mathbf{x} - \mathbf{x}_0(t)|^2 - \left(\frac{(\mathbf{x} - \mathbf{x}_0(t)) \times \mathbf{v}(t')}{c}\right)^2
\end{aligned} \tag{8.96}$$

where in the penultimate step we used Equation (8.84) on page 126.

What we have just demonstrated is that if the particle velocity at time t can be calculated or projected from its value at the retarded time t', the retarded distance s in the Liénard-Wiechert potentials (8.65) can be expressed in terms of the virtual simultaneous coordinate $\mathbf{x}_0(t)$, *viz.*, the point at which the particle will have arrived at time t, *i.e.*, when we obtain the first knowledge of its existence at the source point $\mathbf{x}'$ at the retarded time t', and in the field coordinate $\mathbf{x} = \mathbf{x}(t)$, where we make our observations. We have, in other words, shown that all quantities in the definition of s, and hence s itself, can, when the motion of the charge is somehow known, be expressed in terms of the time t alone. *I.e.*, in this special case we are able to express the retarded relative distance as $s = s(t, \mathbf{x})$ and we do not have to involve the retarded time t' or any transformed differential operators in our calculations.

Taking the square root of both sides of Equation (8.96) on the preceding page, we obtain the following alternative final expressions for the retarded relative distance s in terms of the charge's virtual simultaneous coordinate $\mathbf{x}_0(t)$:

$$s(t,\mathbf{x}) = \sqrt{|\mathbf{x}-\mathbf{x}_0|^2 - \left(\frac{(\mathbf{x}-\mathbf{x}_0)\times\mathbf{v}}{c}\right)^2} \tag{8.97a}$$

$$= |\mathbf{x}-\mathbf{x}_0| \sqrt{1-\frac{v^2}{c^2}\sin^2\theta_0} \tag{8.97b}$$

$$= \sqrt{|\mathbf{x}-\mathbf{x}_0|^2 \left(1-\frac{v^2}{c^2}\right) + \left(\frac{(\mathbf{x}-\mathbf{x}_0)\cdot\mathbf{v}}{c}\right)^2} \tag{8.97c}$$

Using Equation (8.97c) above and standard vector analytic formulae, we obtain

$$\begin{aligned}\boldsymbol{\nabla} s^2 &= \boldsymbol{\nabla}\left[|\mathbf{x}-\mathbf{x}_0|^2 \left(1-\frac{v^2}{c^2}\right) + \left(\frac{(\mathbf{x}-\mathbf{x}_0)\cdot\mathbf{v}}{c}\right)^2\right] \\ &= 2\left[(\mathbf{x}-\mathbf{x}_0)\left(1-\frac{v^2}{c^2}\right) + \frac{\mathbf{v}\mathbf{v}}{c^2}\cdot(\mathbf{x}-\mathbf{x}_0)\right] \\ &= 2\left[(\mathbf{x}-\mathbf{x}_0) + \frac{\mathbf{v}}{c}\times\left(\frac{\mathbf{v}}{c}\times(\mathbf{x}-\mathbf{x}_0)\right)\right]\end{aligned} \tag{8.98}$$

which we shall use in the following example of a uniformly, unaccelerated motion of the charge.

EXAMPLE 8.1 ▷THE FIELDS FROM A UNIFORMLY MOVING CHARGE

In the special case of uniform motion, the localised charge moves in a field-free, isolated space and we know that it will not be affected by any external forces. It will therefore move uniformly in a straight line with the constant velocity $\mathbf{v}$. This gives us the possibility to extrapolate its position at the observation time, $\mathbf{x}'(t)$, from its position at the retarded time, $\mathbf{x}'(t')$. Since the particle is not accelerated, $\dot{\mathbf{v}} \equiv \mathbf{0}$, the virtual simultaneous coordinate $\mathbf{x}_0$ will be identical to the actual *simultaneous coordinate* of the particle at time t, *i.e.*, $\mathbf{x}_0(t) = \mathbf{x}'(t)$. As depicted in Figure 8.7 on page 122, the angle between $\mathbf{x}-\mathbf{x}_0$ and $\mathbf{v}$ is θ_0 while then angle between $\mathbf{x}-\mathbf{x}'$

and **v** is θ'.

We note that in the case of uniform velocity **v**, time and space derivatives are closely related in the following way when they operate on functions of $\mathbf{x}(t)$ [*cf.* Equation (1.33) on page 13]:

$$\frac{\partial}{\partial t} \to -\mathbf{v}\cdot\boldsymbol{\nabla} \tag{8.99}$$

Hence, the **E** and **B** fields can be obtained from Formulae (8.68) on page 122, with the potentials given by Equations (8.65) on page 121 as follows:

$$\begin{aligned}
\mathbf{E} &= -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t} = -\boldsymbol{\nabla}\phi - \frac{1}{c^2}\frac{\partial \mathbf{v}\phi}{\partial t} = -\boldsymbol{\nabla}\phi - \frac{\mathbf{v}}{c^2}\frac{\partial\phi}{\partial t}\\
&= -\boldsymbol{\nabla}\phi + \frac{\mathbf{v}}{c}\left(\frac{\mathbf{v}}{c}\cdot\boldsymbol{\nabla}\phi\right) = -\left(1 - \frac{\mathbf{vv}}{c^2}\cdot\right)\boldsymbol{\nabla}\phi\\
&= \left(\frac{\mathbf{vv}}{c^2} - \mathbf{1}\right)\cdot\boldsymbol{\nabla}\phi
\end{aligned} \tag{8.100a}$$

$$\begin{aligned}
\mathbf{B} &= \boldsymbol{\nabla}\times\mathbf{A} = \boldsymbol{\nabla}\times\left(\frac{\mathbf{v}}{c^2}\phi\right) = \boldsymbol{\nabla}\phi\times\frac{\mathbf{v}}{c^2} = -\frac{\mathbf{v}}{c^2}\times\boldsymbol{\nabla}\phi\\
&= \frac{\mathbf{v}}{c^2}\times\left[\left(\frac{\mathbf{v}}{c}\cdot\boldsymbol{\nabla}\phi\right)\frac{\mathbf{v}}{c} - \boldsymbol{\nabla}\phi\right] = \frac{\mathbf{v}}{c^2}\times\left(\frac{\mathbf{vv}}{c^2} - \mathbf{1}\right)\cdot\boldsymbol{\nabla}\phi\\
&= \frac{\mathbf{v}}{c^2}\times\mathbf{E}
\end{aligned} \tag{8.100b}$$

Here $\mathbf{1} = \hat{\mathbf{x}}_i\hat{\mathbf{x}}_i$ is the unit dyad and we used the fact that $\mathbf{v}\times\mathbf{v} \equiv 0$. What remains is just to express $\boldsymbol{\nabla}\phi$ in quantities evaluated at t and $\mathbf{x}$.

From Equation (8.65a) on page 121 and Equation (8.98) on the preceding page we find that

$$\begin{aligned}
\boldsymbol{\nabla}\phi &= \frac{q'}{4\pi\varepsilon_0}\boldsymbol{\nabla}\left(\frac{1}{s}\right) = -\frac{q'}{8\pi\varepsilon_0 s^3}\boldsymbol{\nabla}s^2\\
&= -\frac{q'}{4\pi\varepsilon_0 s^3}\left[(\mathbf{x}-\mathbf{x}_0) + \frac{\mathbf{v}}{c}\times\left(\frac{\mathbf{v}}{c}\times(\mathbf{x}-\mathbf{x}_0)\right)\right]
\end{aligned} \tag{8.101}$$

When this expression for $\boldsymbol{\nabla}\phi$ is inserted into Equation (8.100a), the following result

$$\begin{aligned}
\mathbf{E}(t,\mathbf{x}) &= \left(\frac{\mathbf{vv}}{c^2} - \mathbf{1}\right)\cdot\boldsymbol{\nabla}\phi = -\frac{q'}{8\pi\varepsilon_0 s^3}\left(\frac{\mathbf{vv}}{c^2} - \mathbf{1}\right)\cdot\boldsymbol{\nabla}s^2\\
&= \frac{q'}{4\pi\varepsilon_0 s^3}\left\{(\mathbf{x}-\mathbf{x}_0) + \frac{\mathbf{v}}{c}\times\left(\frac{\mathbf{v}}{c}\times(\mathbf{x}-\mathbf{x}_0)\right)\right.\\
&\qquad\left. - \frac{\mathbf{v}}{c}\left(\frac{\mathbf{v}}{c}\cdot(\mathbf{x}-\mathbf{x}_0)\right) - \frac{\mathbf{vv}}{c^2}\cdot\left[\frac{\mathbf{v}}{c}\times\left(\frac{\mathbf{v}}{c}\times(\mathbf{x}-\mathbf{x}_0)\right)\right]\right\}\\
&= \frac{q'}{4\pi\varepsilon_0 s^3}\left[(\mathbf{x}-\mathbf{x}_0) + \frac{\mathbf{v}}{c}\left(\frac{\mathbf{v}}{c}\cdot(\mathbf{x}-\mathbf{x}_0)\right) - (\mathbf{x}-\mathbf{x}_0)\frac{v^2}{c^2}\right.\\
&\qquad\left. - \frac{\mathbf{v}}{c}\left(\frac{\mathbf{v}}{c}\cdot(\mathbf{x}-\mathbf{x}_0)\right)\right]\\
&= \frac{q'}{4\pi\varepsilon_0 s^3}(\mathbf{x}-\mathbf{x}_0)\left(1 - \frac{v^2}{c^2}\right)
\end{aligned} \tag{8.102}$$

follows. Of course, the same result also follows from Equation (8.85) on page 126 with $\dot{\mathbf{v}} \equiv \mathbf{0}$ inserted.

From Equation (8.102) we conclude that **E** is directed along the vector from the simultaneous coordinate $\mathbf{x}_0(t)$ to the field (observation) coordinate $\mathbf{x}(t)$. In a similar way, the magnetic field can be calculated and one finds that

$$\mathbf{B}(t,\mathbf{x}) = \frac{\mu_0 q'}{4\pi s^3}\left(1-\frac{v^2}{c^2}\right)\mathbf{v}\times(\mathbf{x}-\mathbf{x}_0) = \frac{1}{c^2}\mathbf{v}\times\mathbf{E} \tag{8.103}$$

From these explicit formulae for the $\mathbf{E}$ and $\mathbf{B}$ fields we can discern the following cases:

1. $v \to 0 \Rightarrow \mathbf{E}$ goes over into the Coulomb field $\mathbf{E}^{\text{Coulomb}}$
2. $v \to 0 \Rightarrow \mathbf{B}$ goes over into the Biot-Savart field
3. $v \to c \Rightarrow \mathbf{E}$ becomes dependent on θ_0
4. $v \to c, \sin\theta_0 \approx 0 \Rightarrow \mathbf{E} \to (1-v^2/c^2)\mathbf{E}^{\text{Coulomb}}$
5. $v \to c, \sin\theta_0 \approx 1 \Rightarrow \mathbf{E} \to (1-v^2/c^2)^{-1/2}\mathbf{E}^{\text{Coulomb}}$

◁ END OF EXAMPLE 8.1

▷ THE CONVECTION POTENTIAL AND THE CONVECTION FORCE — EXAMPLE 8.2

Let us consider in more detail the treatment of the radiation from a uniformly moving rigid charge distribution.

If we return to the original definition of the potentials and the inhomogeneous wave equation, Formula (3.15) on page 38, for a generic potential component $\Psi(t,\mathbf{x})$ and a generic source component $f(t,\mathbf{x})$,

$$\Box^2\Psi(t,\mathbf{x}) = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)\Psi(t,\mathbf{x}) = f(t,\mathbf{x}) \tag{8.104}$$

we find that under the assumption that $\mathbf{v} = v\hat{\mathbf{x}}_1$, this equation can be written

$$\left(1-\frac{v^2}{c^2}\right)\frac{\partial^2\Psi}{\partial x_1^2}+\frac{\partial^2\Psi}{\partial x_2^2}+\frac{\partial^2\Psi}{\partial x_3^2} = -f(\mathbf{x}) \tag{8.105}$$

i.e., in a time-independent form. Transforming

$$\xi_1 = \frac{x_1}{\sqrt{1-v^2/c^2}} \tag{8.106a}$$

$$\xi_2 = x_2 \tag{8.106b}$$

$$\xi_3 = x_3 \tag{8.106c}$$

and introducing the vectorial nabla operator in ξ space, $\nabla_\xi \stackrel{\text{def}}{\equiv} (\partial/\partial\xi_1, \partial/\partial\xi_2, \partial/\partial\xi_3)$, the time-independent equation (8.105) reduces to an ordinary *Poisson equation*

$$\nabla^2_\xi \Psi(\boldsymbol{\xi}) = -f(\sqrt{1 - v^2/c^2}\,\xi_1, \xi_2, \xi_3) \equiv -f(\boldsymbol{\xi}) \tag{8.107}$$

in this space. This equation has the well-known Coulomb potential solution

$$\Psi(\boldsymbol{\xi}) = \frac{1}{4\pi} \int_V \frac{f(\boldsymbol{\xi}')}{|\boldsymbol{\xi} - \boldsymbol{\xi}'|} \,\mathrm{d}^3\xi' \tag{8.108}$$

After inverse transformation back to the original coordinates, this becomes

$$\Psi(\mathbf{x}) = \frac{1}{4\pi} \int_V \frac{f(\mathbf{x}')}{s} \,\mathrm{d}^3x' \tag{8.109}$$

where, in the denominator,

$$s = \left[(x_1 - x_1')^2 + \left(1 - \frac{v^2}{c^2}\right)[(x_2 - x_2')^2 + (x_3 - x_3')^2]\right]^{\frac{1}{2}} \tag{8.110}$$

Applying this to the explicit scalar and vector potential components, realising that for a rigid charge distribution ρ moving with velocity $\mathbf{v}$ the current is given by $\mathbf{j} = \rho\mathbf{v}$, we obtain

$$\phi(t, \mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\mathbf{x}')}{s} \,\mathrm{d}^3x' \tag{8.111a}$$

$$\mathbf{A}(t, \mathbf{x}) = \frac{1}{4\pi\varepsilon_0 c^2} \int_V \frac{\mathbf{v}\rho(\mathbf{x}')}{s} \,\mathrm{d}^3x' = \frac{\mathbf{v}}{c^2}\phi(t, \mathbf{x}) \tag{8.111b}$$

For a localised charge where $\int \rho \,\mathrm{d}^3x' = q'$, these expressions reduce to

$$\phi(t, \mathbf{x}) = \frac{q'}{4\pi\varepsilon_0 s} \tag{8.112a}$$

$$\mathbf{A}(t, \mathbf{x}) = \frac{q'\mathbf{v}}{4\pi\varepsilon_0 c^2 s} \tag{8.112b}$$

which we recognise as the *Liénard-Wiechert potentials*; *cf.* Equations (8.65) on page 121. We notice, however, that the derivation here, based on a mathematical technique which in fact is a *Lorentz transformation*, is of more general validity than the one leading to Equations (8.65) on page 121.

Let us now consider the action of the fields produced from a moving, rigid charge distribution represented by q' moving with velocity $\mathbf{v}$, on a charged particle q, also moving with velocity $\mathbf{v}$. This force is given by the *Lorentz force*

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{8.113}$$

With the help of Equation (8.103) on the facing page and Equations (8.111) above, and the fact that $\partial_t = -\mathbf{v} \cdot \boldsymbol{\nabla}$ [*cf.* Formula (8.99) on page 129], we can rewrite expression (8.113) above as

$$\mathbf{F} = q\left[\mathbf{E} + \mathbf{v} \times \left(\frac{\mathbf{v}}{c^2} \times \mathbf{E}\right)\right] = q\left[\left(\frac{\mathbf{v}}{c} \cdot \boldsymbol{\nabla}\phi\right)\frac{\mathbf{v}}{c} - \boldsymbol{\nabla}\phi - \frac{\mathbf{v}}{c} \times \left(\frac{\mathbf{v}}{c} \times \boldsymbol{\nabla}\phi\right)\right] \tag{8.114}$$

Applying the 'bac-cab' rule, Formula (F.51) on page 160, on the last term yields

$$\frac{\mathbf{v}}{c} \times \left(\frac{\mathbf{v}}{c} \times \boldsymbol{\nabla}\phi\right) = \left(\frac{\mathbf{v}}{c} \cdot \boldsymbol{\nabla}\phi\right)\frac{\mathbf{v}}{c} - \frac{v^2}{c^2}\boldsymbol{\nabla}\phi \tag{8.115}$$

which means that we can write

$$\mathbf{F} = -q\boldsymbol{\nabla}\psi \tag{8.116}$$

where

$$\psi = \left(1 - \frac{v^2}{c^2}\right)\phi \tag{8.117}$$

The scalar function ψ is called the *convection potential* or the *Heaviside potential*. When the rigid charge distribution is well localised so that we can use the potentials (8.112) the convection potential becomes

$$\psi = \left(1 - \frac{v^2}{c^2}\right)\frac{q'}{4\pi\varepsilon_0 s} \tag{8.118}$$

The convection potential from a point charge is constant on flattened ellipsoids of revolution, defined through Equation (8.110) on the previous page as

$$\begin{aligned}&\left(\frac{x_1 - x_1'}{\sqrt{1 - v^2/c^2}}\right)^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2\\&= \gamma^2(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2 = \mathit{Const}\end{aligned} \tag{8.119}$$

These Heaviside ellipsoids are equipotential surfaces, and since the force is proportional to the gradient of ψ, which means that it is perpendicular to the ellipsoid surface, the force between two charges is in general *not* directed along the line which connects the charges. A consequence of this is that a system consisting of two co-moving charges connected with a rigid bar, will experience a torque. This is the idea behind the Trouton-Noble experiment, aimed at measuring the *absolute speed* of the earth or the galaxy. The negative outcome of this experiment is explained by the special theory of relativity which postulates that mechanical laws follow the same rules as electromagnetic laws, so that a compensating torque appears due to mechanical stresses within the charge-bar system.

◁ END OF EXAMPLE 8.2

Radiation for small velocities

If the charge moves at such low speeds that $v/c \ll 1$, Formula (8.66) on page 121 simplifies to

$$s = \left|\mathbf{x} - \mathbf{x}'\right| - \frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}}{c} \approx \left|\mathbf{x} - \mathbf{x}'\right|, \quad v \ll c \tag{8.120}$$

and Formula (8.84) on page 126

$$\mathbf{x} - \mathbf{x}_0 = (\mathbf{x} - \mathbf{x}') - \frac{|\mathbf{x} - \mathbf{x}'|\,\mathbf{v}}{c} \approx \mathbf{x} - \mathbf{x}', \quad v \ll c \tag{8.121}$$

so that the radiation field Equation (8.89) on page 126 can be approximated by

$$\mathbf{E}^{\mathrm{rad}}(t, \mathbf{x}) = \frac{q'}{4\pi\varepsilon_0 c^2\,|\mathbf{x} - \mathbf{x}'|^3}\,(\mathbf{x} - \mathbf{x}') \times [(\mathbf{x} - \mathbf{x}') \times \dot{\mathbf{v}}], \quad v \ll c \tag{8.122}$$

from which we obtain, with the use of Formula (8.88) on page 126, the magnetic field

$$\mathbf{B}^{\text{rad}}(t,\mathbf{x}) = \frac{q'}{4\pi\varepsilon_0 c^3\,|\mathbf{x}-\mathbf{x}'|^2}[\dot{\mathbf{v}}\times(\mathbf{x}-\mathbf{x}')], \quad v \ll c \tag{8.123}$$

It is interesting to note the close correspondence which exists between the nonrelativistic fields (8.122) and (8.123) and the electric dipole field Equations (8.45) on page 117 if we introduce

$$\mathbf{p} = q'\mathbf{x}'(t') \tag{8.124}$$

and at the same time make the transitions

$$q'\dot{\mathbf{v}} = \ddot{\mathbf{p}} \to -\omega^2\mathbf{p}_\omega \tag{8.125a}$$

$$\mathbf{x}-\mathbf{x}' = \mathbf{x}-\mathbf{x}_0 \tag{8.125b}$$

The power flux in the far zone is described by the Poynting vector as a function of $\mathbf{E}^{\text{rad}}$ and $\mathbf{B}^{\text{rad}}$. We use the close correspondence with the dipole case to find that it becomes

$$\mathbf{S} = \frac{\mu_0 q'^2(\dot{\mathbf{v}})^2}{16\pi^2 c\,|\mathbf{x}-\mathbf{x}'|^2}\sin^2\theta\,\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} \tag{8.126}$$

where θ is the angle between $\dot{\mathbf{v}}$ and $\mathbf{x}-\mathbf{x}_0$. The total radiated power (integrated over a closed spherical surface) becomes

$$P = \frac{\mu_0 q'^2(\dot{\mathbf{v}})^2}{6\pi c} = \frac{q'^2\dot{v}^2}{6\pi\varepsilon_0 c^3} \tag{8.127}$$

which is the *Larmor formula for radiated power* from an accelerated charge. Note that here we are treating a charge with $v \ll c$ but otherwise *totally unspecified motion* while we compare with formulae derived for a *stationary oscillating dipole*. The electric and magnetic fields, Equation (8.122) on the preceding page and Equation (8.123), respectively, and the expressions for the Poynting flux and power derived from them, are here *instantaneous* values, dependent on the instantaneous position of the charge at $\mathbf{x}'(t')$. The angular distribution is that which is 'frozen' to the point from which the energy is radiated.

8.3.3 Bremsstrahlung

An important special case of radiation is when the velocity $\mathbf{v}$ and the acceleration $\dot{\mathbf{v}}$ are collinear (parallel or anti-parallel) so that $\mathbf{v}\times\dot{\mathbf{v}} = \mathbf{0}$. This condition (for an arbitrary magnitude of $\mathbf{v}$) inserted into expression (8.89) on page 126 for the radiation field, yields

$$\mathbf{E}^{\text{rad}}(t,\mathbf{x}) = \frac{q'}{4\pi\varepsilon_0 c^2 s^3}(\mathbf{x}-\mathbf{x}')\times[(\mathbf{x}-\mathbf{x}')\times\dot{\mathbf{v}}], \quad \mathbf{v}\parallel\dot{\mathbf{v}} \tag{8.128}$$

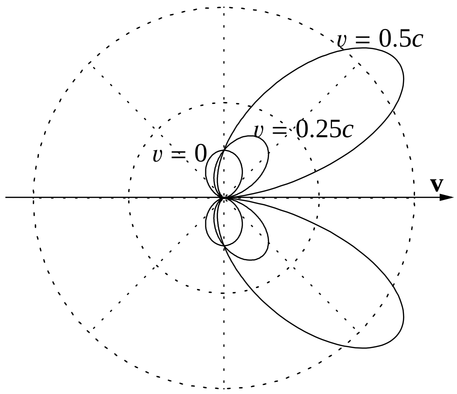

FIGURE 8.8: Polar diagram of the energy loss angular distribution factor $\sin^2\theta/(1 - v\cos\theta/c)^5$ during bremsstrahlung for particle speeds $v = 0$, $v = 0.25c$, and $v = 0.5c$.

from which we obtain, with the use of Formula (8.88) on page 126, the magnetic field

$$\mathbf{B}^{\text{rad}}(t, \mathbf{x}) = \frac{q' \, |\mathbf{x} - \mathbf{x}'|}{4\pi\varepsilon_0 c^3 s^3} [\dot{\mathbf{v}} \times (\mathbf{x} - \mathbf{x}')], \quad \mathbf{v} \parallel \dot{\mathbf{v}} \tag{8.129}$$

The difference between this case and the previous case of $v \ll c$ is that the approximate expression (8.120) on page 132 for s is no longer valid; we must instead use the correct expression (8.66) on page 121. The angular distribution of the power flux (Poynting vector) therefore becomes

$$\mathbf{S} = \frac{\mu_0 q'^2 \dot{v}^2}{16\pi^2 c \, |\mathbf{x} - \mathbf{x}'|^2} \frac{\sin^2\theta}{\left(1 - \frac{v}{c}\cos\theta\right)^6} \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} \tag{8.130}$$

It is interesting to note that the magnitudes of the electric and magnetic fields are the same whether $\mathbf{v}$ and $\dot{\mathbf{v}}$ are parallel or anti-parallel.

We must be careful when we compute the energy ($\mathbf{S}$ integrated over time). The Poynting vector is related to the time t when it is measured and to a *fixed* surface in space. The radiated power into a solid angle element $\mathrm{d}\Omega$, measured relative to the particle's retarded position, is given by the formula

$$\frac{\mathrm{d}U^{\text{rad}}(\theta)}{\mathrm{d}t} \,\mathrm{d}\Omega = \mathbf{S} \cdot (\mathbf{x} - \mathbf{x}') \left|\mathbf{x} - \mathbf{x}'\right| \,\mathrm{d}\Omega = \frac{\mu_0 q'^2 \dot{v}^2}{16\pi^2 c} \frac{\sin^2\theta}{\left(1 - \frac{v}{c}\cos\theta\right)^6} \,\mathrm{d}\Omega \tag{8.131}$$

On the other hand, the radiation loss due to radiation from the charge at retarded time t' :

$$\frac{\mathrm{d}U^{\text{rad}}}{\mathrm{d}t'} \,\mathrm{d}\Omega = \frac{\mathrm{d}U^{\text{rad}}}{\mathrm{d}t} \left(\frac{\partial t}{\partial t'}\right)_{\mathbf{x}} \mathrm{d}\Omega \tag{8.132}$$

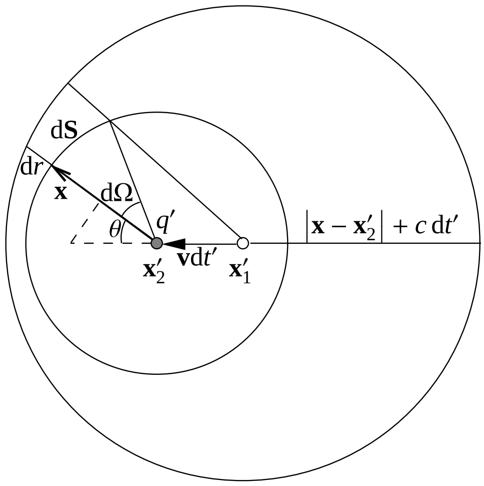

FIGURE 8.9: Location of radiation between two spheres as the charge moves with velocity **v** from $\mathbf{x}'_1$ to $\mathbf{x}'_2$ during the time interval $(t', t' + \mathrm{d}t')$. The observation point (field point) is at the fixed location **x**.

Using Formula (8.76) on page 124, we obtain

$$\frac{\mathrm{d}U^{\mathrm{rad}}}{\mathrm{d}t'}\,\mathrm{d}\Omega = \frac{\mathrm{d}U^{\mathrm{rad}}}{\mathrm{d}t}\frac{s}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}\Omega = \mathbf{S}\cdot(\mathbf{x}-\mathbf{x}')s\,\mathrm{d}\Omega \tag{8.133}$$

Inserting Equation (8.130) on the facing page for **S** into (8.133), we obtain the explicit expression for the energy loss due to radiation evaluated at the retarded time

$$\frac{\mathrm{d}U^{\mathrm{rad}}(\theta)}{\mathrm{d}t'}\,\mathrm{d}\Omega = \frac{\mu_0 q'^2 \dot{v}^2}{16\pi^2 c}\frac{\sin^2\theta}{\left(1-\frac{v}{c}\cos\theta\right)^5}\,\mathrm{d}\Omega \tag{8.134}$$

The angular factors of this expression, for three different particle speeds, are plotted in Figure 8.8 on the preceding page.

Comparing expression (8.131) on the facing page with expression (8.134) above, we see that they differ by a factor $1 - v\cos\theta/c$ which comes from the extra factor $s/|\mathbf{x}-\mathbf{x}'|$ introduced in (8.133). Let us explain this in geometrical terms.

During the interval $(t', t' + \mathrm{d}t')$ and within the solid angle element $\mathrm{d}\Omega$ the particle radiates an energy $[\mathrm{d}U^{\mathrm{rad}}(\theta)/\mathrm{d}t']\,\mathrm{d}t'\mathrm{d}\Omega$. As shown in 8.9 this energy is at time t located between two spheres, one outer with its origin at $\mathbf{x}'_1(t')$ and radius $c(t - t')$, and one inner with its origin at $\mathbf{x}'_2(t'+\mathrm{d}t') = \mathbf{x}'_1(t')+\mathbf{v}\,\mathrm{d}t'$ and radius $c[t-(t'+\mathrm{d}t')] = c(t-t'-\mathrm{d}t')$.

From Figure 8.9 we see that the volume element subtending the solid angle element

$$\mathrm{d}\Omega = \frac{\mathrm{d}S}{\left|\mathbf{x}-\mathbf{x}'_2\right|^2} \tag{8.135}$$

is

$$d^3x = dS\,dr = \left|\mathbf{x} - \mathbf{x}_2'\right|^2\,d\Omega\,dr \tag{8.136}$$

Here, dr denotes the differential distance between the two spheres and can be evaluated in the following way

$$\begin{aligned} dr &= \left|\mathbf{x} - \mathbf{x}_2'\right| + c\,dt' - \left|\mathbf{x} - \mathbf{x}_2'\right| - \underbrace{\frac{\mathbf{x} - \mathbf{x}_2'}{\left|\mathbf{x} - \mathbf{x}_2'\right|} \cdot \mathbf{v}}_{v\cos\theta}\,dt' \\ &= \left(c - \frac{\mathbf{x} - \mathbf{x}_2'}{\left|\mathbf{x} - \mathbf{x}_2'\right|} \cdot \mathbf{v}\right) dt' = \frac{cs}{\left|\mathbf{x} - \mathbf{x}_2'\right|}\,dt' \end{aligned} \tag{8.137}$$

where Formula (8.66) on page 121 was used in the last step. Hence, the volume element under consideration is

$$d^3x = dS\,dr = \frac{s}{\left|\mathbf{x} - \mathbf{x}_2'\right|}\,dS\,c\,dt' \tag{8.138}$$

We see that the energy which is radiated per unit solid angle during the time interval $(t', t' + dt')$ is located in a volume element whose size is θ dependent. This explains the difference between expression (8.131) on page 134 and expression (8.134) on the preceding page.

Let the radiated energy, integrated over Ω, be denoted $\tilde{U}^{\text{rad}}$. After tedious, but relatively straightforward integration of Formula (8.134) on the previous page, one obtains

$$\frac{d\tilde{U}^{\text{rad}}}{dt'} = \frac{\mu_0 q'^2 \dot{v}^2}{6\pi c} \frac{1}{\left(1 - \frac{v^2}{c^2}\right)^3} = \frac{2}{3} \frac{q'^2 \dot{v}^2}{4\pi\varepsilon_0 c^3} \left(1 - \frac{v^2}{c^2}\right)^{-3} \tag{8.139}$$

If we know $\mathbf{v}(t')$, we can integrate this expression over t' and obtain the total energy radiated during the acceleration or deceleration of the particle. This way we obtain a classical picture of *bremsstrahlung* (*braking radiation*). Often, an atomistic treatment is required for an acceptable result.

▷BREMSSTRAHLUNG FOR LOW SPEEDS AND SHORT ACCELERATION TIMES ——— EXAMPLE 8.3

Calculate the bremsstrahlung when a charged particle, moving at a non-relativistic speed, is accelerated or decelerated during an infinitely short time interval.

We approximate the velocity change at time $t' = t_0$ by a delta function:

$$\dot{\mathbf{v}}(t') = \Delta\mathbf{v}\,\delta(t' - t_0) \tag{8.140}$$

which means that

$$\Delta\mathbf{v}(t_0) = \int_{-\infty}^{\infty} \dot{\mathbf{v}}\,dt' \tag{8.141}$$

Also, we assume $v/c \ll 1$ so that, according to Formula (8.66) on page 121,

$$s \approx |\mathbf{x} - \mathbf{x}'| \tag{8.142}$$

and, according to Formula (8.84) on page 126,

$$\mathbf{x} - \mathbf{x}_0 \approx \mathbf{x} - \mathbf{x}' \tag{8.143}$$

From the general expression (8.88) on page 126 we conclude that $\mathbf{E} \perp \mathbf{B}$ and that it suffices to consider $E \equiv \left|\mathbf{E}^{\text{rad}}\right|$. According to the 'bremsstrahlung expression' for $\mathbf{E}^{\text{rad}}$, Equation (8.128) on page 133,

$$E = \frac{q' \sin\theta}{4\pi\varepsilon_0 c^2 \, |\mathbf{x} - \mathbf{x}'|} \Delta v \, \delta(t' - t_0) \tag{8.144}$$

In this simple case $B \equiv \left|\mathbf{B}^{\text{rad}}\right|$ is given by

$$B = \frac{E}{c} \tag{8.145}$$

Fourier transforming expression (8.144) for E is trivial, yielding

$$E_\omega = \frac{q' \sin\theta}{8\pi^2\varepsilon_0 c^2 \, |\mathbf{x} - \mathbf{x}'|} \Delta v \, e^{\mathrm{i}\omega t_0} \tag{8.146}$$

We note that the magnitude of this Fourier component is independent of ω. This is a consequence of the infinitely short 'impulsive step' $\delta(t' - t_0)$ in the time domain which produces an infinite spectrum in the frequency domain.

The total radiation energy is given by the expression

$$\begin{aligned} \tilde{U}^{\text{rad}} &= \int \frac{\mathrm{d}\tilde{U}^{\text{rad}}}{\mathrm{d}t'} \, \mathrm{d}t' = \int_{-\infty}^{\infty} \int_S \left(\mathbf{E} \times \frac{\mathbf{B}}{\mu_0} \right) \cdot \mathrm{d}\mathbf{S} \, \mathrm{d}t' \\ &= \frac{1}{\mu_0} \int_S \int_{-\infty}^{\infty} E B \, \mathrm{d}t' \, \mathrm{d}^2x' = \frac{1}{\mu_0 c} \int_S \int_{-\infty}^{\infty} E^2 \, \mathrm{d}t' \, \mathrm{d}^2x' \\ &= \varepsilon_0 c \int_S \int_{-\infty}^{\infty} E^2 \, \mathrm{d}t' \, \mathrm{d}^2x' \end{aligned} \tag{8.147}$$

According to *Parseval's identity* [*cf.* Equation (7.35) on page 102] the following equality holds:

$$\int_{-\infty}^{\infty} E^2 \, \mathrm{d}t' = 4\pi \int_0^{\infty} |E_\omega|^2 \, \mathrm{d}\omega \tag{8.148}$$

which means that the radiated energy in the frequency interval $(\omega, \omega + \mathrm{d}\omega)$ is

$$\tilde{U}^{\text{rad}}_\omega \mathrm{d}\omega = 4\pi\varepsilon_0 c \left(\int_S |E_\omega|^2 \, \mathrm{d}^2x \right) \mathrm{d}\omega \tag{8.149}$$

For our infinite spectrum, Equation (8.146), we obtain

$$\begin{aligned} \tilde{U}^{\text{rad}}_\omega \mathrm{d}\omega &= \frac{q'^2 (\Delta v)^2}{16\pi^3 \varepsilon_0 c^3} \int_S \frac{\sin^2\theta}{|\mathbf{x} - \mathbf{x}'|^2} \, \mathrm{d}^2x \, \mathrm{d}\omega \\ &= \frac{q'^2 (\Delta v)^2}{16\pi^3 \varepsilon_0 c^3} \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \sin^2\theta \, \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\omega \\ &= \frac{q'^2}{3\pi\varepsilon_0 c} \left(\frac{\Delta v}{c} \right)^2 \frac{\mathrm{d}\omega}{2\pi} \end{aligned} \tag{8.150}$$

We see that the energy spectrum $\tilde{U}^{\mathrm{rad}}_{\omega}$ is independent of frequency ω. This means that if we would integrate it over all frequencies $\omega \in [0, \infty)$, a divergent integral would result.

In reality, all spectra have finite widths, with an upper *cutoff* limit set by the quantum condition

$$\hbar\omega_{\max} = \frac{1}{2}m(\Delta v)^2 \tag{8.151}$$

which expresses that the highest possible frequency $\omega_{\max}$ in the spectrum is that for which all kinetic energy difference has gone into one single *field quantum* (*photon*) with energy $\hbar\omega_{\max}$. If we adopt the picture that the total energy is quantised in terms of N_ω photons radiated during the process, we find that

$$\frac{\tilde{U}^{\mathrm{rad}}_{\omega}\,\mathrm{d}\omega}{\hbar\omega} = \mathrm{d}N_\omega \tag{8.152}$$

or, for an electron where $q' = -|e|$, where e is the elementary charge,

$$\mathrm{d}N_\omega = \frac{e^2}{4\pi\varepsilon_0\hbar c}\,\frac{2}{3\pi}\left(\frac{\Delta v}{c}\right)^2\frac{\mathrm{d}\omega}{\omega} \approx \frac{1}{137}\,\frac{2}{3\pi}\left(\frac{\Delta v}{c}\right)^2\frac{\mathrm{d}\omega}{\omega} \tag{8.153}$$

where we used the value of the *fine structure constant* $\alpha = e^2/(4\pi\varepsilon_0\hbar c) \approx 1/137$.

Even if the number of photons becomes infinite when $\omega \to 0$, these photons have negligible energies so that the total radiated energy is still finite.

◁ END OF EXAMPLE 8.3

8.3.4 Cyclotron and synchrotron radiation

Formula (8.88) and Formula (8.89) on page 126 for the magnetic field and the radiation part of the electric field are general, valid for any kind of motion of the localised charge. A very important special case is circular motion, *i.e.*, the case $\mathbf{v} \perp \dot{\mathbf{v}}$.

With the charged particle orbiting in the x_1x_2 plane as in Figure 8.10 on the facing page, an orbit radius a, and an angular frequency ω_0, we obtain

$$\varphi(t') = \omega_0 t' \tag{8.154a}$$
$$\mathbf{x}'(t') = a[\hat{\mathbf{x}}_1\cos\varphi(t') + \hat{\mathbf{x}}_2\sin\varphi(t')] \tag{8.154b}$$
$$\mathbf{v}(t') = \dot{\mathbf{x}}'(t') = a\omega_0[-\hat{\mathbf{x}}_1\sin\varphi(t') + \hat{\mathbf{x}}_2\cos\varphi(t')] \tag{8.154c}$$
$$v = |\mathbf{v}| = a\omega_0 \tag{8.154d}$$
$$\dot{\mathbf{v}}(t') = \ddot{\mathbf{x}}'(t') = -a\omega_0^2[\hat{\mathbf{x}}_1\cos\varphi(t') + \hat{\mathbf{x}}_2\sin\varphi(t')] \tag{8.154e}$$
$$\dot{v} = |\dot{\mathbf{v}}| = a\omega_0^2 \tag{8.154f}$$

Because of the rotational symmetry we can, without loss of generality, rotate our coordinate system around the x_3 axis so the relative vector $\mathbf{x} - \mathbf{x}'$ from the source point to an arbitrary field point always lies in the x_2x_3 plane, *i.e.*,

$$\mathbf{x} - \mathbf{x}' = \left|\mathbf{x} - \mathbf{x}'\right|(\hat{\mathbf{x}}_2\sin\alpha + \hat{\mathbf{x}}_3\cos\alpha) \tag{8.155}$$

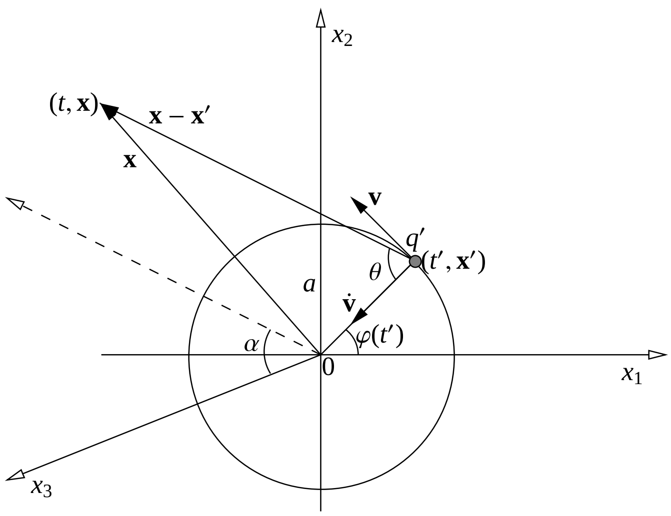

FIGURE 8.10: Coordinate system for the radiation from a charged particle at $\mathbf{x}'(t')$ in circular motion with velocity $\mathbf{v}(t')$ along the tangent and constant acceleration $\dot{\mathbf{v}}(t')$ toward the origin. The $x_1 x_2$ axes are chosen so that the relative field point vector $\mathbf{x} - \mathbf{x}'$ makes an angle α with the x_3 axis which is normal to the plane of the orbital motion. The radius of the orbit is a.

where α is the angle between $\mathbf{x} - \mathbf{x}'$ and the normal to the plane of the particle orbit (see Figure 8.10). From the above expressions we obtain

$$(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v} = \left|\mathbf{x} - \mathbf{x}'\right| v \sin\alpha \cos\varphi \tag{8.156a}$$

$$(\mathbf{x} - \mathbf{x}') \cdot \dot{\mathbf{v}} = -\left|\mathbf{x} - \mathbf{x}'\right| \dot{v} \sin\alpha \sin\varphi = \left|\mathbf{x} - \mathbf{x}'\right| \dot{v} \cos\theta \tag{8.156b}$$

where in the last step we simply used the definition of a scalar product and the fact that the angle between $\dot{\mathbf{v}}$ and $\mathbf{x} - \mathbf{x}'$ is θ.

The power flux is given by the Poynting vector, which, with the help of Formula (8.88) on page 126, can be written

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) = \frac{1}{c\mu_0} |\mathbf{E}|^2 \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} \tag{8.157}$$

Inserting this into Equation (8.133) on page 135, we obtain

$$\frac{\mathrm{d}U^{\mathrm{rad}}(\alpha, \varphi)}{\mathrm{d}t'} = \frac{|\mathbf{x} - \mathbf{x}'|\, s}{c\mu_0} |\mathbf{E}|^2 \tag{8.158}$$

where the retarded distance s is given by expression (8.66) on page 121. With the radiation part of the electric field, expression (8.89) on page 126, inserted, and using

(8.156a) and (8.156b) on the previous page, one finds, after some algebra, that

$$\frac{\mathrm{d}U^{\mathrm{rad}}(\alpha,\varphi)}{\mathrm{d}t'} = \frac{\mu_0 q'^2 \dot{v}^2}{16\pi^2 c} \frac{\left(1-\frac{v}{c}\sin\alpha\cos\varphi\right)^2 - \left(1-\frac{v^2}{c^2}\right)\sin^2\alpha\sin^2\varphi}{\left(1-\frac{v}{c}\sin\alpha\cos\varphi\right)^5} \tag{8.159}$$

The angles θ and φ vary in time during the rotation, so that θ refers to a *moving* coordinate system. But we can parametrise the solid angle $\mathrm{d}\Omega$ in the angle φ and the (fixed) angle α so that $\mathrm{d}\Omega = \sin\alpha\,\mathrm{d}\alpha\,\mathrm{d}\varphi$. Integration of Equation (8.159) over this $\mathrm{d}\Omega$ gives, after some cumbersome algebra, the angular integrated expression

$$\frac{\mathrm{d}\tilde{U}^{\mathrm{rad}}}{\mathrm{d}t'} = \frac{\mu_0 q'^2 \dot{v}^2}{6\pi c} \frac{1}{\left(1-\frac{v^2}{c^2}\right)^2} \tag{8.160}$$

In Equation (8.159) above, two limits are particularly interesting:

1. $v/c \ll 1$ which corresponds to *cyclotron radiation*.
2. $v/c \lesssim 1$ which corresponds to *synchrotron radiation*.

Cyclotron radiation

For a non-relativistic speed $v \ll c$, Equation (8.159) reduces to

$$\frac{\mathrm{d}U^{\mathrm{rad}}(\alpha,\varphi)}{\mathrm{d}t'} = \frac{\mu_0 q'^2 \dot{v}^2}{16\pi^2 c} \left(1-\sin^2\alpha\sin^2\varphi\right) \tag{8.161}$$

But, according to Equation (8.156b) on the previous page

$$\sin^2\alpha\sin^2\varphi = \cos^2\theta \tag{8.162}$$

where θ is defined in Figure 8.10 on the preceding page. This means that we can write

$$\frac{\mathrm{d}U^{\mathrm{rad}}(\theta)}{\mathrm{d}t'} = \frac{\mu_0 q'^2 \dot{v}^2}{16\pi^2 c} \left(1-\cos^2\theta\right) = \frac{\mu_0 q'^2 \dot{v}^2}{16\pi^2 c} \sin^2\theta \tag{8.163}$$

Consequently, a fixed observer near the orbit plane will observe cyclotron radiation twice per revolution in the form of two equally broad pulses of radiation with alternating polarisation.

Synchrotron radiation

When the particle is relativistic, $v \lesssim c$, the denominator in Equation (8.159) above becomes very small if $\sin\alpha\cos\varphi \approx 1$, which defines the forward direction of the particle motion ($\alpha \approx \pi/2,\ \varphi \approx 0$). Equation (8.159) then becomes

$$\frac{\mathrm{d}U^{\mathrm{rad}}(\pi/2,0)}{\mathrm{d}t'} = \frac{\mu_0 q'^2 \dot{v}^2}{16\pi^2 c} \frac{1}{\left(1-\frac{v}{c}\right)^3} \tag{8.164}$$

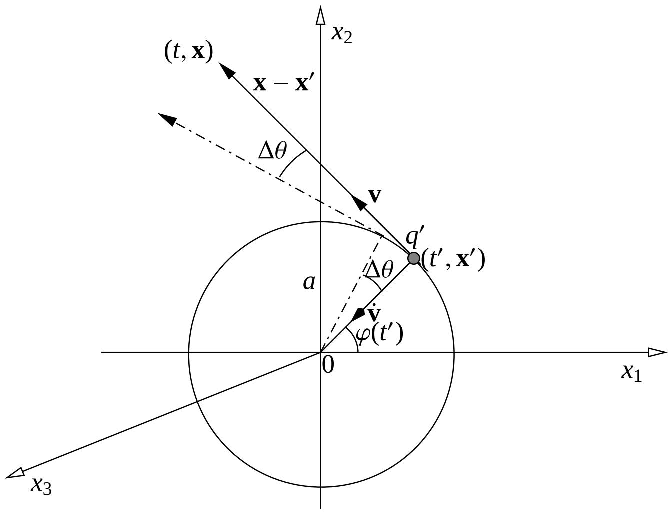

FIGURE 8.11: When the observation point is in the plane of the particle orbit, *i.e.*, $\alpha = \pi/2$ the lobe width is given by $\Delta\theta$.

which means that an observer near the orbit plane sees a very strong pulse followed, half an orbit period later, by a much weaker pulse.

The two cases represented by Equation (8.163) on the preceding page and Equation (8.164) on the facing page are very important results since they can be used to determine the characteristics of the particle motion both in particle accelerators and in astrophysical objects where a direct measurement of particle velocities are impossible.

In the orbit plane ($\alpha = \pi/2$), Equation (8.159) on the preceding page gives

$$\frac{\mathrm{d}U^{\mathrm{rad}}(\pi/2, \varphi)}{\mathrm{d}t'} = \frac{\mu_0 q'^2 \dot{v}^2}{16\pi^2 c} \frac{\left(1 - \frac{v}{c}\cos\varphi\right)^2 - \left(1 - \frac{v^2}{c^2}\right)\sin^2\varphi}{\left(1 - \frac{v}{c}\cos\varphi\right)^5} \tag{8.165}$$

which vanishes for angles φ_0 such that

$$\cos\varphi_0 = \frac{v}{c} \tag{8.166a}$$

$$\sin\varphi_0 = \sqrt{1 - \frac{v^2}{c^2}} \tag{8.166b}$$

Hence, the angle φ_0 is a measure of the *synchrotron radiation lobe width* $\Delta\theta$; see Figure 8.11. For ultra-relativistic particles, defined by

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \gg 1, \quad \sqrt{1 - \frac{v^2}{c^2}} \ll 1, \tag{8.167}$$

one can approximate

$$\varphi_0 \approx \sin\varphi_0 = \sqrt{1-\frac{v^2}{c^2}} = \frac{1}{\gamma} \tag{8.168}$$

Hence, synchrotron radiation from ultra-relativistic charges is characterized by a radiation lobe width which is approximately

$$\Delta\theta \approx \frac{1}{\gamma} \tag{8.169}$$

This angular interval is swept by the charge during the time interval

$$\Delta t' = \frac{\Delta\theta}{\omega_0} \tag{8.170}$$

during which the particle moves a length interval

$$\Delta l = v\Delta t' = v\frac{\Delta\theta}{\omega_0} \tag{8.171}$$

in the direction toward the observer who therefore measures a pulse width of length

$$\begin{aligned}\Delta t &= \Delta t' - \frac{\Delta l}{c} = \Delta t' - \frac{v\Delta t'}{c} = \left(1-\frac{v}{c}\right)\Delta t' = \left(1-\frac{v}{c}\right)\frac{\Delta\theta}{\omega_0} \approx \left(1-\frac{v}{c}\right)\frac{1}{\gamma\omega_0} \\ &= \frac{\left(1-\frac{v}{c}\right)\left(1+\frac{v}{c}\right)}{\underbrace{1+\frac{v}{c}}_{\approx 2}}\frac{1}{\gamma\omega_0} \approx \underbrace{\left(1-\frac{v^2}{c^2}\right)}_{1/\gamma^2}\frac{1}{2\gamma\omega_0} = \frac{1}{2\gamma^3}\frac{1}{\omega_0}\end{aligned} \tag{8.172}$$

Typically, the spectral width of a pulse of length Δt is $\Delta\omega \lesssim 1/\Delta t$. In the ultra-relativistic synchrotron case one can therefore expect frequency components up to

$$\omega_{\max} \approx \frac{1}{\Delta t} = 2\gamma^3\omega_0 \tag{8.173}$$

A spectral analysis of the radiation pulse will therefore exhibit a (broadened) line spectrum of Fourier components $n\omega_0$ from $n = 1$ up to $n \approx 2\gamma^3$.

When many charged particles, N say, contribute to the radiation, we can have three different situations depending on the relative phases of the radiation fields from the individual particles:

1. All N radiating particles are spatially much closer to each other than a typical wavelength. Then the relative phase differences of the individual electric and magnetic fields radiated are negligible and the total radiated fields from all individual particles will add up to become N times that from one particle. This means that the power radiated from the N particles will be N^2 higher than for a single charged particle. This is called *coherent radiation*.

2. The charged particles are perfectly evenly distributed in the orbit. In this case the phases of the radiation fields cause a complete cancellation of the fields themselves. No radiation escapes.

3. The charged particles are somewhat unevenly distributed in the orbit. This happens for an open ring current, carried initially by evenly distributed charged particles, which is subject to thermal fluctuations. From statistical mechanics we know that this happens for all open systems and that the particle densities exhibit fluctuations of order $\sqrt{N}$. This means that out of the N particles, $\sqrt{N}$ will exhibit deviation from perfect randomness—and thereby perfect radiation field cancellation—and give rise to net radiation fields which are proportional to $\sqrt{N}$. As a result, the radiated power will be proportional to N, and we speak about *incoherent radiation*. Examples of this can be found both in earthly laboratories and under cosmic conditions.

Radiation in the general case

We recall that the general expression for the radiation **E** field from a moving charge concentration is given by expression (8.89) on page 126. This expression in Equation (8.158) on page 139 yields the general formula

$$\frac{\mathrm{d}U^{\mathrm{rad}}(\theta,\varphi)}{\mathrm{d}t'} = \frac{\mu_0 q'^2\,|\mathbf{x}-\mathbf{x}'|}{16\pi^2 c s^5}\left\{(\mathbf{x}-\mathbf{x}')\times\left[\left((\mathbf{x}-\mathbf{x}') - \frac{|\mathbf{x}-\mathbf{x}'|\,\mathbf{v}}{c}\right)\times\dot{\mathbf{v}}\right]\right\}^2 \tag{8.174}$$

Integration over the solid angle Ω gives the totally radiated power as

$$\frac{\mathrm{d}\tilde{U}^{\mathrm{rad}}}{\mathrm{d}t'} = \frac{\mu_0 q'^2 \dot{v}^2}{6\pi c}\,\frac{1-\frac{v^2}{c^2}\sin^2\psi}{\left(1-\frac{v^2}{c^2}\right)^3} \tag{8.175}$$

where ψ is the angle between **v** and $\dot{\mathbf{v}}$.

If **v** is collinear with $\dot{\mathbf{v}}$, then $\sin\psi = 0$, we get *bremsstrahlung*. For $\mathbf{v}\perp\dot{\mathbf{v}}$, $\sin\psi = 1$, which corresponds to *cyclotron radiation* or *synchrotron radiation*.

Virtual photons

Let us consider a charge q' moving with constant, high velocity $\mathbf{v}(t')$ along the $\mathbf{x}_1$ axis. According to Formula (8.102) on page 129 and Figure 8.12 on the following page, the perpendicular component along the $\mathbf{x}_3$ axis of the electric field from this moving charge is

$$E_\perp = E_3 = \frac{q'}{4\pi\varepsilon_0 s^3}\left(1-\frac{v^2}{c^2}\right)(\mathbf{x}-\mathbf{x}_0)\cdot\hat{\boldsymbol{x}}_3 \tag{8.176}$$

$$\tag{8.177}$$

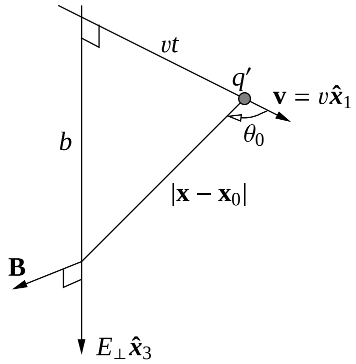

FIGURE 8.12: The perpendicular field of a charge q' moving with velocity $\mathbf{v} = v\hat{\mathbf{x}}$ is $E_\perp \hat{\mathbf{z}}$.

Utilising expression (8.97a) on page 128 and simple geometrical relations, we can rewrite this as

$$E_\perp = \frac{q'}{4\pi\varepsilon_0} \frac{b}{\gamma^2 \left[(vt)^2 + b^2/\gamma^2\right]^{3/2}} \tag{8.178}$$

This represents a contracted field, approaching the field of a plane wave. The passage of this field 'pulse' corresponds to a frequency distribution of the field energy. Fourier transforming, we obtain

$$E_{\omega,\perp} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t\, E_\perp(t)\, e^{\mathrm{i}\omega t} = \frac{q'}{4\pi^2 \varepsilon_0 b v} \left[\left(\frac{b\omega}{v\gamma}\right) K_1 \left(\frac{b\omega}{v\gamma}\right) \right] \tag{8.179}$$

Here, K_1 is the *Kelvin function* (Bessel function of the second kind with imaginary argument) which behaves in such a way for small and large arguments that

$$E_{\omega,\perp} \sim \frac{q'}{4\pi^2 \varepsilon_0 b v}, \quad b\omega \ll v\gamma \tag{8.180a}$$

$$E_{\omega,\perp} \sim 0, \quad b\omega \gg v\gamma \tag{8.180b}$$

showing that the 'pulse' length is of the order $b/(v\gamma)$.

Due to the equipartition of the field energy into the electric and magnetic fields, the total field energy can be written

$$\tilde{U} = \varepsilon_0 \int_V E_\perp^2\, \mathrm{d}^3x = \varepsilon_0 \int_{b_{\min}}^{b_{\max}} \int_{-\infty}^{\infty} E_\perp^2\, v\mathrm{d}t\, 2\pi b\, \mathrm{d}b \tag{8.181}$$

where the volume integration is over the plane perpendicular to $\mathbf{v}$. With the use of *Parseval's identity* for Fourier transforms, Formula (7.35) on page 102, we can rewrite this as

$$
\begin{aligned}
\tilde{U} &= \int_0^\infty \tilde{U}_\omega \,\mathrm{d}\omega = 4\pi\varepsilon_0 v \int_{b_{\min}}^{b_{\max}} \int_0^\infty \left| E_{\omega,\perp} \right|^2 \,\mathrm{d}\omega\, 2\pi b \,\mathrm{d}b \\
&\approx \frac{q'^2}{2\pi^2\varepsilon_0 v} \int_0^\infty \int_{b_{\min}}^{v\gamma/\omega} \frac{\mathrm{d}b}{b} \,\mathrm{d}\omega
\end{aligned}
\tag{8.182}
$$

from which we conclude that

$$
\tilde{U}_\omega \approx \frac{q'^2}{2\pi^2\varepsilon_0 v} \ln\left(\frac{v\gamma}{b_{\min}\omega}\right) \tag{8.183}
$$

where an explicit value of $b_{\min}$ can be calculated in quantum theory only.

As in the case of bremsstrahlung, it is intriguing to quantise the energy into photons [*cf.* Equation (8.152) on page 138]. Then we find that

$$
N_\omega \,\mathrm{d}\omega \approx \frac{2\alpha}{\pi} \ln\left(\frac{c\gamma}{b_{\min}\omega}\right) \frac{\mathrm{d}\omega}{\omega} \tag{8.184}
$$

where $\alpha = e^2/(4\pi\varepsilon_0\hbar c) \approx 1/137$ is the *fine structure constant*.

Let us consider the interaction of two (classical) electrons, 1 and 2. The result of this interaction is that they change their linear momenta from $\mathbf{p}_1$ to $\mathbf{p}'_1$ and $\mathbf{p}_2$ to $\mathbf{p}'_2$, respectively. Heisenberg's uncertainty principle gives $b_{\min} \sim \hbar / \left|\mathbf{p}_1 - \mathbf{p}'_1\right|$ so that the number of photons exchanged in the process is of the order

$$
N_\omega \,\mathrm{d}\omega \approx \frac{2\alpha}{\pi} \ln\left(\frac{c\gamma}{\hbar\omega} \left|\mathbf{p}_1 - \mathbf{p}'_1\right|\right) \frac{\mathrm{d}\omega}{\omega} \tag{8.185}
$$

Since this change in momentum corresponds to a change in energy $\hbar\omega = E_1 - E'_1$ and $E_1 = m_0\gamma c^2$, we see that

$$
N_\omega \,\mathrm{d}\omega \approx \frac{2\alpha}{\pi} \ln\left(\frac{E_1}{m_0 c^2} \frac{\left|c\mathbf{p}_1 - c\mathbf{p}'_1\right|}{E_1 - E'_1}\right) \frac{\mathrm{d}\omega}{\omega} \tag{8.186}
$$

a formula which gives a reasonable semi-classical account of a photon-induced electron-electron interaction process. In quantum theory, including only the lowest order contributions, this process is known as *Møller scattering*. A diagrammatic representation of (a semi-classical approximation of) this process is given in Figure 8.13 on the following page.

8.3.5 Radiation from charges moving in matter

When electromagnetic radiation is propagating through matter, new phenomena may appear which are (at least classically) not present in vacuum. As mentioned earlier, one can under certain simplifying assumptions include, to some extent, the influence

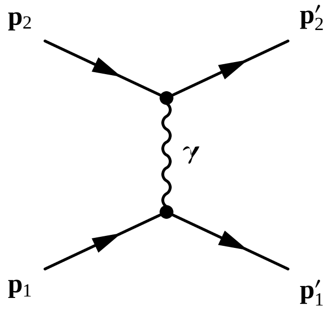

FIGURE 8.13: Diagrammatic representation of the semi-classical electron-electron interaction (Møller scattering).

from matter on the electromagnetic fields by introducing new, derived field quantities **D** and **H** according to

$$\mathbf{D} = \varepsilon(t, \mathbf{x})\mathbf{E} = \kappa\varepsilon_0\mathbf{E} \tag{8.187}$$

$$\mathbf{B} = \mu(t, \mathbf{x})\mathbf{H} = \kappa_{\mathrm{m}}\mu_0\mathbf{H} \tag{8.188}$$

Expressed in terms of these derived field quantities, the Maxwell equations, often called *macroscopic Maxwell equations*, take the form

$$\nabla \cdot \mathbf{D} = \rho(t, \mathbf{x}) \tag{8.189a}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{8.189b}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{8.189c}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j}(t, \mathbf{x}) \tag{8.189d}$$

Assuming for simplicity that the *electric permittivity* ε and the *magnetic permeability* μ, and hence the *relative permittivity* κ and the *relative permeability* κ_{m} all have fixed values, independent on time and space, for each type of material we consider, we can derive the general *telegrapher's equation* [*cf.* Equation (2.33) on page 31]

$$\frac{\partial^2 \mathbf{E}}{\partial \zeta^2} - \sigma\mu\frac{\partial \mathbf{E}}{\partial t} - \varepsilon\mu\frac{\partial^2 \mathbf{E}}{\partial t^2} = \mathbf{0} \tag{8.190}$$

describing (1D) wave propagation in a material medium.

In Chapter 2 we concluded that the existence of a finite conductivity, manifesting itself in a *collisional interaction* between the charge carriers, causes the waves to decay exponentially with time and space. Let us therefore assume that in our medium $\sigma = 0$ so that the wave equation simplifies to

$$\frac{\partial^2 \mathbf{E}}{\partial \zeta^2} - \varepsilon\mu\frac{\partial^2 \mathbf{E}}{\partial t^2} = \mathbf{0} \tag{8.191}$$

If we introduce the *phase velocity* in the medium as

$$v_\varphi = \frac{1}{\sqrt{\varepsilon\mu}} = \frac{1}{\sqrt{\kappa\varepsilon_0\kappa_\mathrm{m}\mu_0}} = \frac{c}{\sqrt{\kappa\kappa_\mathrm{m}}} \tag{8.192}$$

where, according to Equation (1.11) on page 6, $c = 1/\sqrt{\varepsilon_0\mu_0}$ is the speed of light, *i.e.*, the phase speed of electromagnetic waves in vacuum, then the general solution to each component of Equation (8.191) on the preceding page

$$E_i = f(\zeta - v_\varphi t) + g(\zeta + v_\varphi t), \quad i = 1, 2, 3 \tag{8.193}$$

The ratio of the phase speed in vacuum and in the medium

$$\frac{c}{v_\varphi} = \sqrt{\kappa\kappa_\mathrm{m}} = c\sqrt{\varepsilon\mu} \stackrel{\mathrm{def}}{\equiv} n \tag{8.194}$$

is called the *refractive index* of the medium. In general n is a function of both time and space as are the quantities ε, μ, κ, and κ_m themselves. If, in addition, the medium is *anisotropic* or *birefringent*, all these quantities are rank-two tensor fields. Under our simplifying assumptions, in each medium we consider $n = Const$ for each frequency component of the fields.

Associated with the phase speed of a medium for a wave of a given frequency ω we have a *wave vector*, defined as

$$\mathbf{k} \stackrel{\mathrm{def}}{\equiv} k\hat{\boldsymbol{k}} = k\hat{\mathbf{v}}_\varphi = \frac{\omega}{v_\varphi}\frac{\mathbf{v}_\varphi}{v_\varphi} \tag{8.195}$$

As in the vacuum case discussed in Chapter 2, assuming that $\mathbf{E}$ is time-harmonic, *i.e.*, can be represented by a Fourier component proportional to $\exp\{-\mathrm{i}\omega t\}$, the solution of Equation (8.191) can be written

$$\mathbf{E} = \mathbf{E}_0 e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{8.196}$$

where now $\mathbf{k}$ is the wave vector *in the medium* given by Equation (8.195). With these definitions, the vacuum formula for the associated magnetic field, Equation (2.40) on page 31,

$$\mathbf{B} = \sqrt{\varepsilon\mu}\,\hat{\boldsymbol{k}}\times\mathbf{E} = \frac{1}{v_\varphi}\hat{\boldsymbol{k}}\times\mathbf{E} = \frac{1}{\omega}\mathbf{k}\times\mathbf{E} \tag{8.197}$$

is valid also in a material medium (assuming, as mentioned, that n has a fixed constant scalar value). A consequence of a $\kappa \neq 1$ is that the electric field will, in general, have a longitudinal component.

It is important to notice that depending on the electric and magnetic properties of a medium, and, hence, on the value of the refractive index n, the phase speed in the medium can be smaller or larger than the speed of light:

$$v_\varphi = \frac{c}{n} = \frac{\omega}{k} \tag{8.198}$$

where, in the last step, we used Equation (8.195) on the preceding page.

If the medium has a refractive index which, as is usually the case, dependent on frequency ω, we say that the medium is *dispersive*. Because in this case also $\mathbf{k}(\omega)$ and $\omega(\mathbf{k})$, so that the *group velocity*

$$v_{\mathrm{g}} = \frac{\partial \omega}{\partial k} \tag{8.199}$$

has a unique value for each frequency component, and is different from v_{φ}. Except in regions of *anomalous dispersion*, v_{φ} is always smaller than c. In a gas of free charges, such as a *plasma*, the refractive index is given by the expression

$$n^2(\omega) = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2} \tag{8.200}$$

where

$$\omega_{\mathrm{p}}^2 = \sum_{\sigma} \frac{N_{\sigma} q_{\sigma}^2}{\varepsilon_0 m_{\sigma}} \tag{8.201}$$

is the square of the *plasma frequency* ω_{p}. Here m_{σ} and N_{σ} denote the mass and number density, respectively, of charged particle species σ. In an inhomogeneous plasma, $N_{\sigma} = N_{\sigma}(\mathbf{x})$ so that the refractive index and also the phase and group velocities are space dependent. As can be easily seen, for each given frequency, the phase and group velocities in a plasma are different from each other. If the frequency ω is such that it coincides with ω_{p} at some point in the medium, then at that point $v_{\varphi} \to \infty$ while $v_{\mathrm{g}} \to 0$ and the wave Fourier component at ω is reflected there.

Vavilov-Čerenkov radiation

As we saw in Subsection 8.1, a charge in uniform, rectilinear motion *in vacuum* does not give rise to any radiation; see in particular Equation (8.100a) on page 129. Let us now consider a charge in uniform, rectilinear motion *in a medium* with electric properties which are different from those of a (classical) vacuum. Specifically, consider a medium where

$$\varepsilon = Const > \varepsilon_0 \tag{8.202a}$$

$$\mu = \mu_0 \tag{8.202b}$$

This implies that in this medium the phase speed is

$$v_{\varphi} = \frac{c}{n} = \frac{1}{\sqrt{\varepsilon \mu_0}} < c \tag{8.203}$$

Hence, in this particular medium, the speed of propagation of (the phase planes of) electromagnetic waves is less than the speed of light in vacuum, which we know is an absolute limit for the motion of anything, including particles. A medium of this kind has the interesting property that particles, entering into the medium at high speeds $|\mathbf{v}|$, which, of course, are below the phase speed *in vacuum*, can experience that the particle speeds are *higher* than the phase speed *in the medium*. This is the basis for the *Vavilov-Čerenkov radiation* that we shall now study.

If we recall the general derivation, in the vacuum case, of the retarded (and advanced) potentials in Chapter 3 and the Liénard-Wiechert potentials, Equations (8.65) on page 121, we realise that we obtain the latter in the medium by a simple formal replacement $c \to c/n$ in the expression (8.66) on page 121 for s. Hence, the Liénard-Wiechert potentials in a medium characterized by a refractive index n, are

$$\phi(t, \mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \frac{q'}{\left| |\mathbf{x} - \mathbf{x}'| - n\frac{(\mathbf{x}-\mathbf{x}')\cdot\mathbf{v}}{c} \right|} = \frac{1}{4\pi\varepsilon_0} \frac{q'}{s} \tag{8.204a}$$

$$\mathbf{A}(t, \mathbf{x}) = \frac{1}{4\pi\varepsilon_0 c^2} \frac{q'\mathbf{v}}{\left| |\mathbf{x} - \mathbf{x}'| - n\frac{(\mathbf{x}-\mathbf{x}')\cdot\mathbf{v}}{c} \right|} = \frac{1}{4\pi\varepsilon_0 c^2} \frac{q'\mathbf{v}}{s} \tag{8.204b}$$

where now

$$s = \left| \left|\mathbf{x} - \mathbf{x}'\right| - n\frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}}{c} \right| \tag{8.205}$$

The need for the absolute value of the expression for s is obvious in the case when $v/c \geq 1/n$ because then the second term can be larger than the first term; if $v/c \ll 1/n$ we recover the well-known vacuum case but with modified phase speed. We also note that the retarded and advanced times in the medium are [*cf.* Equation (3.30) on page 41]

$$t'_{\text{ret}} = t'_{\text{ret}}(t, \left|\mathbf{x} - \mathbf{x}'\right|) = t - \frac{k\left|\mathbf{x} - \mathbf{x}'\right|}{\omega} = t - \frac{\left|\mathbf{x} - \mathbf{x}'\right| n}{c} \tag{8.206a}$$

$$t'_{\text{adv}} = t'_{\text{adv}}(t, \left|\mathbf{x} - \mathbf{x}'\right|) = t + \frac{k\left|\mathbf{x} - \mathbf{x}'\right|}{\omega} = t + \frac{\left|\mathbf{x} - \mathbf{x}'\right| n}{c} \tag{8.206b}$$

so that the usual time interval $t - t'$ between the time measured at the point of observation and the retarded time *in a medium* becomes

$$t - t' = \frac{\left|\mathbf{x} - \mathbf{x}'\right| n}{c} \tag{8.207}$$

For $v/c \geq 1/n$, the retarded distance s, and therefore the denominators in Equations (8.204) above vanish when

$$n(\mathbf{x} - \mathbf{x}') \cdot \frac{\mathbf{v}}{c} = \left|\mathbf{x} - \mathbf{x}'\right| \frac{nv}{c} \cos\theta_{\text{c}} = \left|\mathbf{x} - \mathbf{x}'\right| \tag{8.208}$$

or, equivalently, when

$$\cos\theta_{\text{c}} = \frac{c}{nv} \tag{8.209}$$

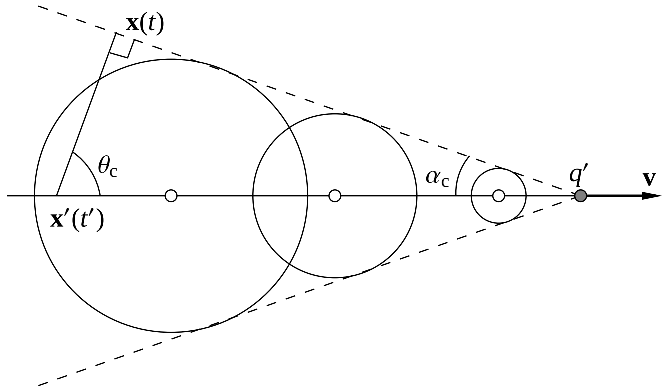

FIGURE 8.14: Instantaneous picture of the expanding field spheres from a point charge moving with constant speed $v/c > 1/n$ in a medium where $n > 1$. This generates a Vavilov-Čerenkov shock wave in the form of a cone.

In the direction defined by this angle θ_c, the potentials become singular. During the time interval $t - t'$ given by expression (8.207) on the preceding page, the field exists within a sphere of radius $|\mathbf{x} - \mathbf{x}'|$ around the particle while the particle moves a distance

$$l = v(t - t') \tag{8.210}$$

along the direction of $\mathbf{v}$.

In the direction θ_c where the potentials are singular, all field spheres are tangent to a straight cone with its apex at the instantaneous position of the particle and with the apex half angle α_c defined according to

$$\sin\alpha_c = \cos\theta_c = \frac{c}{nv} \tag{8.211}$$

This cone of potential singularities and field sphere circumferences propagates with speed c/n in the form of a *shock front*, called *Vavilov-Čerenkov radiation*.[1] The

[1] The first systematic exploration of this radiation was made by P. A. Čerenkov in 1934, who was then a post-graduate student in S. I. Vavilov's research group at the Lebedev Institute in Moscow. Vavilov wrote a manuscript with the experimental findings, put Čerenkov as the author, and submitted it to *Nature*. In the manuscript, Vavilov explained the results in terms of radioactive particles creating Compton electrons which gave rise to the radiation (which was the correct interpretation), but the paper was rejected. The paper was then sent to *Physical Review* and was, after some controversy with the American editors who claimed the results to be wrong, eventually published in 1937. In the same year, I. E. Tamm and I. M. Frank published the theory for the effect ('the singing electron'). In fact, predictions of a similar effect had been made as early as 1888 by Heaviside, and by Sommerfeld in his 1904 paper 'Radiating body moving with velocity of light'. On May 8, 1937, Sommerfeld sent a letter to Tamm via Austria, saying that he was surprised that his old 1904 ideas were now becoming interesting. Tamm, Frank and Čerenkov received the Nobel Prize in 1958 'for the discovery and the interpretation of the Čerenkov effect' [V. L. Ginzburg, *private communication*].
The first observation of this type of radiation was reported by Marie Curie in 1910, but she never pursued the exploration of it [8].

Vavilov-Čerenkov cone is similar in nature to the *Mach cone* in acoustics.

In order to make some quantitative estimates of this radiation, we note that we can describe the motion of each charged particle q' as a current density:

$$\mathbf{j} = q'\mathbf{v}\,\delta(\mathbf{x}' - \mathbf{v}t') = q'v\,\delta(x' - vt')\delta(y')\delta(z')\hat{\boldsymbol{x}}_1 \tag{8.212}$$

which has the trivial Fourier transform

$$\mathbf{j}_\omega = \frac{q'}{2\pi}\,e^{\mathrm{i}\omega x'/v}\,\delta(y')\delta(z')\hat{\boldsymbol{x}}_1 \tag{8.213}$$

This Fourier component can be used in the formulae derived for a linear current in Subsection 8.1.1 if only we make the replacements

$$\varepsilon_0 \to \varepsilon = n^2\varepsilon_0 \tag{8.214a}$$

$$k \to \frac{n\omega}{c} \tag{8.214b}$$

In this manner, using $\mathbf{j}_\omega$ from Equation (8.213) above, the resulting Fourier transforms of the Vavilov-Čerenkov magnetic and electric radiation fields can be calculated from the expressions (7.11) on page 95) and (7.22) on page 98, respectively.

The total energy content is then obtained from Equation (7.35) on page 102 (integrated over a closed sphere at large distances). For a Fourier component one obtains [*cf.* Equation (7.38) on page 103]

$$\begin{aligned} U_\omega^{\mathrm{rad}}\mathrm{d}\Omega &\approx \frac{1}{4\pi\varepsilon_0 nc}\left|\int_V (\mathbf{j}_\omega \times \mathbf{k})e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}'}\,\mathrm{d}^3x'\right|^2 \mathrm{d}\Omega \\ &= \frac{q'^2 n\omega^2}{16\pi^3\varepsilon_0 c^3}\left|\int_{-\infty}^{\infty}\exp\left[\mathrm{i}\left(\frac{\omega x'}{v} - kx'\cos\theta\right)\right]\mathrm{d}x'\right|^2 \sin^2\theta\,\mathrm{d}\Omega \end{aligned} \tag{8.215}$$

where θ is the angle between the direction of motion, $\hat{\boldsymbol{x}}'_1$, and the direction to the observer, $\hat{\boldsymbol{k}}$. The integral in (8.215) is singular of a 'Dirac delta type'. If we limit the spatial extent of the motion of the particle to the closed interval $[-X, X]$ on the x' axis we can evaluate the integral to obtain

$$U_\omega^{\mathrm{rad}}\mathrm{d}\Omega = \frac{q'^2 n\omega^2\sin^2\theta}{4\pi^3\varepsilon_0 c^3}\,\frac{\sin^2\left[\left(1 - \frac{nv}{c}\cos\theta\right)\frac{X\omega}{v}\right]}{\left[\left(1 - \frac{nv}{c}\cos\theta\right)\frac{\omega}{v}\right]^2}\mathrm{d}\Omega \tag{8.216}$$

which has a maximum in the direction θ_c as expected. The magnitude of this maximum grows and its width narrows as $X \to \infty$. The integration of (8.216) over Ω therefore picks up the main contributions from $\theta \approx \theta_\mathrm{c}$. Consequently, we can set $\sin^2\theta \approx \sin^2\theta_\mathrm{c}$ and the result of the integration is

$$\begin{aligned} \tilde{U}_\omega^{\mathrm{rad}} &= 2\pi\int_0^\pi U_\omega^{\mathrm{rad}}(\theta)\sin\theta\,\mathrm{d}\theta = \lceil\cos\theta = -\xi\rfloor = 2\pi\int_{-1}^{1} U_\omega^{\mathrm{rad}}(\xi)\,\mathrm{d}\xi \\ &\approx \frac{q'^2 n\omega^2\sin^2\theta_\mathrm{c}}{2\pi^2\varepsilon_0 c^3}\int_{-1}^{1}\frac{\sin^2\left[\left(1 + \frac{nv\xi}{c}\right)\frac{X\omega}{v}\right]}{\left[\left(1 + \frac{nv\xi}{c}\right)\frac{\omega}{v}\right]^2}\,\mathrm{d}\xi \end{aligned} \tag{8.217}$$

The integrand in (8.217) is strongly peaked near $\xi = -c/(nv)$, or, equivalently, near $\cos\theta_c = c/(nv)$. This means that the integrand function is practically zero outside the integration interval $\xi \in [-1, 1]$. Consequently, one may extend the ξ integration interval to $(-\infty, \infty)$ without introducing too much an error. Via yet another variable substitution we can therefore approximate

$$\sin^2\theta_c \int_{-1}^{1} \frac{\sin^2\left[\left(1+\frac{nv\xi}{c}\right)\frac{X\omega}{v}\right]}{\left[\left(1+\frac{nv\xi}{c}\right)\frac{\omega}{v}\right]^2} \, \mathrm{d}\xi \approx \left(1-\frac{c^2}{n^2v^2}\right)\frac{cX}{\omega n}\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}\,\mathrm{d}x = \frac{cX\pi}{\omega n}\left(1-\frac{c^2}{n^2v^2}\right) \tag{8.218}$$

leading to the final approximate result for the total energy loss in the frequency interval $(\omega, \omega + \mathrm{d}\omega)$

$$\tilde{U}^{\mathrm{rad}}_{\omega}\mathrm{d}\omega = \frac{q'^2 X}{2\pi\varepsilon_0 c^2}\left(1-\frac{c^2}{n^2v^2}\right)\omega\,\mathrm{d}\omega \tag{8.219}$$

As mentioned earlier, the refractive index is usually frequency dependent. Realising this, we find that the radiation energy per frequency unit and *per unit length* is

$$\frac{\tilde{U}^{\mathrm{rad}}_{\omega}\mathrm{d}\omega}{2X} = \frac{q'^2\omega}{4\pi\varepsilon_0 c^2}\left(1-\frac{c^2}{n^2(\omega)v^2}\right)\mathrm{d}\omega \tag{8.220}$$

This result was derived under the assumption that $v/c > 1/n(\omega)$, *i.e.*, under the condition that the expression inside the parentheses in the right hand side is positive. For all media it is true that $n(\omega) \to 1$ when $\omega \to \infty$, so there exist always a highest frequency for which we can obtain Vavilov-Čerenkov radiation from a fast charge in a medium. Our derivation above for a fixed value of n is valid for each individual Fourier component.

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APPENDIX F

Formulae

F.1 The electromagnetic field

F.1.1 Maxwell's equations

$$\nabla \cdot \mathbf{D} = \rho \tag{F.1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{F.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} \tag{F.3}$$

$$\nabla \times \mathbf{H} = \mathbf{j} + \frac{\partial}{\partial t}\mathbf{D} \tag{F.4}$$

Constitutive relations

$$\mathbf{D} = \varepsilon \mathbf{E} \tag{F.5}$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu} \tag{F.6}$$

$$\mathbf{j} = \sigma \mathbf{E} \tag{F.7}$$

$$\mathbf{P} = \varepsilon_0 \chi \mathbf{E} \tag{F.8}$$

F.1.2 Fields and potentials

Vector and scalar potentials

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{F.9}$$

$$\mathbf{E} = -\nabla\phi - \frac{\partial}{\partial t}\mathbf{A} \tag{F.10}$$

The Lorenz-Lorentz gauge condition in vacuum

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\phi = 0 \tag{F.11}$$

F.1.3 Force and energy

Poynting's vector

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \tag{F.12}$$

Maxwell's stress tensor

$$T_{ij} = E_i D_j + H_i B_j - \frac{1}{2}\delta_{ij}\,(E_k D_k + H_k B_k) \tag{F.13}$$

F.2 Electromagnetic radiation

F.2.1 Relationship between the field vectors in a plane wave

$$\mathbf{B} = \frac{\hat{\boldsymbol{k}} \times \mathbf{E}}{c} \tag{F.14}$$

F.2.2 The far fields from an extended source distribution

$$\mathbf{B}_\omega^{\text{rad}}(\mathbf{x}) = \frac{-\mathrm{i}\mu_0}{4\pi}\frac{e^{\mathrm{i}k|\mathbf{x}|}}{|\mathbf{x}|}\int_{V'} \mathrm{d}^3x'\, e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}'}\, \mathbf{j}_\omega \times \mathbf{k} \tag{F.15}$$

$$\mathbf{E}_\omega^{\text{rad}}(\mathbf{x}) = \frac{\mathrm{i}}{4\pi\varepsilon_0 c}\frac{e^{\mathrm{i}k|\mathbf{x}|}}{|\mathbf{x}|}\hat{\mathbf{x}} \times \int_{V'} \mathrm{d}^3x'\, e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}'}\, \mathbf{j}_\omega \times \mathbf{k} \tag{F.16}$$

F.2.3 The far fields from an electric dipole

$$\mathbf{B}_\omega^{\text{rad}}(\mathbf{x}) = -\frac{\omega\mu_0}{4\pi}\frac{e^{\mathrm{i}k|\mathbf{x}|}}{|\mathbf{x}|}\mathbf{p}_\omega \times \mathbf{k} \tag{F.17}$$

$$\mathbf{E}_\omega^{\text{rad}}(\mathbf{x}) = -\frac{1}{4\pi\varepsilon_0}\frac{e^{\mathrm{i}k|\mathbf{x}|}}{|\mathbf{x}|}\,(\mathbf{p}_\omega \times \mathbf{k}) \times \mathbf{k} \tag{F.18}$$

F.2.4 The far fields from a magnetic dipole

$$\mathbf{B}_\omega^{\rm rad}(\mathbf{x}) = -\frac{\mu_0}{4\pi}\frac{e^{\mathrm{i}k|\mathbf{x}|}}{|\mathbf{x}|}\,(\mathbf{m}_\omega \times \mathbf{k}) \times \mathbf{k} \tag{F.19}$$

$$\mathbf{E}_\omega^{\rm rad}(\mathbf{x}) = \frac{k}{4\pi\varepsilon_0 c}\frac{e^{\mathrm{i}k|\mathbf{x}|}}{|\mathbf{x}|}\,\mathbf{m}_\omega \times \mathbf{k} \tag{F.20}$$

F.2.5 The far fields from an electric quadrupole

$$\mathbf{B}_\omega^{\rm rad}(\mathbf{x}) = \frac{\mathrm{i}\mu_0\omega}{8\pi}\frac{e^{\mathrm{i}k|\mathbf{x}|}}{|\mathbf{x}|}\,(\mathbf{k}\cdot\mathbf{Q}_\omega) \times \mathbf{k} \tag{F.21}$$

$$\mathbf{E}_\omega^{\rm rad}(\mathbf{x}) = \frac{\mathrm{i}}{8\pi\varepsilon_0}\frac{e^{\mathrm{i}k|\mathbf{x}|}}{|\mathbf{x}|}\,[(\mathbf{k}\cdot\mathbf{Q}_\omega) \times \mathbf{k}] \times \mathbf{k} \tag{F.22}$$

F.2.6 The fields from a point charge in arbitrary motion

$$\mathbf{E}(t,\mathbf{x}) = \frac{q}{4\pi\varepsilon_0 s^3}\left[(\mathbf{x}-\mathbf{x}_0)\left(1-\frac{v^2}{c^2}\right) + (\mathbf{x}-\mathbf{x}') \times \frac{(\mathbf{x}-\mathbf{x}_0)\times\dot{\mathbf{v}}}{c^2}\right] \tag{F.23}$$

$$\mathbf{B}(t,\mathbf{x}) = (\mathbf{x}-\mathbf{x}') \times \frac{\mathbf{E}(t,\mathbf{x})}{c|\mathbf{x}-\mathbf{x}'|} \tag{F.24}$$

$$s = \left|\mathbf{x}-\mathbf{x}'\right| - (\mathbf{x}-\mathbf{x}')\cdot\frac{\mathbf{v}}{c} \tag{F.25}$$

$$\mathbf{x}-\mathbf{x}_0 = (\mathbf{x}-\mathbf{x}') - |\mathbf{x}-\mathbf{x}'|\frac{\mathbf{v}}{c} \tag{F.26}$$

$$\left(\frac{\partial t'}{\partial t}\right)_{\mathbf{x}} = \frac{|\mathbf{x}-\mathbf{x}'|}{s} \tag{F.27}$$

F.3 Special relativity

F.3.1 Metric tensor

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{F.28}$$

F.3.2 Covariant and contravariant four-vectors

$$v_\mu = g_{\mu\nu} v^\nu \tag{F.29}$$

F.3.3 Lorentz transformation of a four-vector

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{F.30}$$

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{F.31}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{F.32}$$

$$\beta = \frac{v}{c} \tag{F.33}$$

F.3.4 Invariant line element

$$\mathrm{d}s = c\,\frac{\mathrm{d}t}{\gamma} = c\,\mathrm{d}\tau \tag{F.34}$$

F.3.5 Four-velocity

$$u^{\mu} = \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau} = \gamma(c, \mathbf{v}) \tag{F.35}$$

F.3.6 Four-momentum

$$p^{\mu} = m_0 u^{\mu} = \left(\frac{E}{c}, \mathbf{p}\right) \tag{F.36}$$

F.3.7 Four-current density

$$j^{\mu} = \rho_0 u^{\mu} \tag{F.37}$$

F.3.8 Four-potential

$$A^{\mu} = \left(\frac{\phi}{c}, \mathbf{A}\right) \tag{F.38}$$

F.3.9 Field tensor

$$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{F.39}$$

F.4 Vector relations

Let $\mathbf{x}$ be the radius vector (coordinate vector) from the origin to the point $(x_1, x_2, x_3) \equiv (x, y, z)$ and let $|\mathbf{x}|$ denote the magnitude ('length') of $\mathbf{x}$. Let further $\alpha(\mathbf{x}), \beta(\mathbf{x}), \ldots$ be arbitrary scalar fields and $\mathbf{a}(\mathbf{x}), \mathbf{b}(\mathbf{x}), \mathbf{c}(\mathbf{x}), \mathbf{d}(\mathbf{x}), \ldots$ arbitrary vector fields.

The differential vector operator $\boldsymbol{\nabla}$ is in Cartesian coordinates given by

$$\boldsymbol{\nabla} \equiv \sum_{i=1}^{3} \hat{\boldsymbol{x}}_i \frac{\partial}{\partial x_i} \stackrel{\text{def}}{\equiv} \hat{\boldsymbol{x}}_i \frac{\partial}{\partial x_i} \stackrel{\text{def}}{\equiv} \boldsymbol{\partial} \tag{F.40}$$

where $\hat{\boldsymbol{x}}_i$, $i = 1, 2, 3$ is the ith unit vector and $\hat{\boldsymbol{x}}_1 \equiv \hat{\boldsymbol{x}}$, $\hat{\boldsymbol{x}}_2 \equiv \hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{x}}_3 \equiv \hat{\boldsymbol{z}}$. In component (tensor) notation $\boldsymbol{\nabla}$ can be written

$$\nabla_i = \partial_i = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \tag{F.41}$$

F.4.1 Spherical polar coordinates

Base vectors

$$\hat{\boldsymbol{r}} = \sin\theta\cos\varphi\,\hat{\boldsymbol{x}}_1 + \sin\theta\sin\varphi\,\hat{\boldsymbol{x}}_2 + \cos\theta\,\hat{\boldsymbol{x}}_3 \tag{F.42a}$$

$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\varphi\,\hat{\boldsymbol{x}}_1 + \cos\theta\sin\varphi\,\hat{\boldsymbol{x}}_2 - \sin\theta\,\hat{\boldsymbol{x}}_3 \tag{F.42b}$$

$$\hat{\boldsymbol{\varphi}} = -\sin\varphi\,\hat{\boldsymbol{x}}_1 + \cos\varphi\,\hat{\boldsymbol{x}}_2 \tag{F.42c}$$

$$\hat{\boldsymbol{x}}_1 = \sin\theta\cos\varphi\,\hat{\boldsymbol{r}} + \cos\theta\cos\varphi\,\hat{\boldsymbol{\theta}} - \sin\varphi\,\hat{\boldsymbol{\varphi}} \tag{F.43a}$$

$$\hat{\boldsymbol{x}}_2 = \sin\theta\sin\varphi\,\hat{\boldsymbol{r}} + \cos\theta\sin\varphi\,\hat{\boldsymbol{\theta}} + \cos\varphi\,\hat{\boldsymbol{\varphi}} \tag{F.43b}$$

$$\hat{\boldsymbol{x}}_3 = \cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \tag{F.43c}$$

Directed line element

$$\mathrm{d}x\,\hat{\boldsymbol{x}} = \mathrm{d}\mathbf{l} = \mathrm{d}r\,\hat{\boldsymbol{r}} + r\,\mathrm{d}\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,\mathrm{d}\varphi\,\hat{\boldsymbol{\varphi}} \tag{F.44}$$

Solid angle element

$$\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \tag{F.45}$$

Directed area element

$$\mathrm{d}^2x\,\hat{\boldsymbol{n}} = \mathrm{d}\mathbf{S} = \mathrm{d}S\,\hat{\boldsymbol{r}} = r^2\mathrm{d}\Omega\,\hat{\boldsymbol{r}} \tag{F.46}$$

Volume element

$$\mathrm{d}^3x = \mathrm{d}V = \mathrm{d}r\mathrm{d}S = r^2\mathrm{d}r\,\mathrm{d}\Omega \tag{F.47}$$

F.4.2 Vector formulae

General vector algebraic identities

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} = \delta_{ij}a_ib_j = ab\cos\theta \tag{F.48}$$

$$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a} = \epsilon_{ijk}a_jb_k\hat{\mathbf{x}}_i \tag{F.49}$$

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} \tag{F.50}$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b}) \tag{F.51}$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{b}\times(\mathbf{c}\times\mathbf{a}) + \mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = \mathbf{0} \tag{F.52}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \mathbf{a}\cdot[\mathbf{b}\times(\mathbf{c}\times\mathbf{d})] = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) \tag{F.53}$$

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\times\mathbf{b}\cdot\mathbf{d})\mathbf{c} - (\mathbf{a}\times\mathbf{b}\cdot\mathbf{c})\mathbf{d} \tag{F.54}$$

General vector analytic identities

$$\boldsymbol{\nabla}(\alpha\beta) = \alpha\boldsymbol{\nabla}\beta + \beta\boldsymbol{\nabla}\alpha \tag{F.55}$$

$$\boldsymbol{\nabla}\cdot(\alpha\mathbf{a}) = \mathbf{a}\cdot\boldsymbol{\nabla}\alpha + \alpha\boldsymbol{\nabla}\cdot\mathbf{a} \tag{F.56}$$

$$\boldsymbol{\nabla}\times(\alpha\mathbf{a}) = \alpha\boldsymbol{\nabla}\times\mathbf{a} - \mathbf{a}\times\boldsymbol{\nabla}\alpha \tag{F.57}$$

$$\boldsymbol{\nabla}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\boldsymbol{\nabla}\times\mathbf{a}) - \mathbf{a}\cdot(\boldsymbol{\nabla}\times\mathbf{b}) \tag{F.58}$$

$$\boldsymbol{\nabla}\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\boldsymbol{\nabla}\cdot\mathbf{b}) - \mathbf{b}(\boldsymbol{\nabla}\cdot\mathbf{a}) + (\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{a} - (\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{b} \tag{F.59}$$

$$\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\times(\boldsymbol{\nabla}\times\mathbf{b}) + \mathbf{b}\times(\boldsymbol{\nabla}\times\mathbf{a}) + (\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{a} + (\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{b} \tag{F.60}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\alpha = \nabla^2\alpha \tag{F.61}$$

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\alpha = \mathbf{0} \tag{F.62}$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{a}) = 0 \tag{F.63}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{a}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{a}) - \nabla^2\mathbf{a} \tag{F.64}$$

Special identities

In the following $\mathbf{x} = x_i\hat{\mathbf{x}}_i$ and $\mathbf{x}' = x'_i\hat{\mathbf{x}}_i$ are radius vectors, $\mathbf{k}$ an arbitrary *constant* vector, $\mathbf{a} = \mathbf{a}(\mathbf{x})$ an arbitrary vector field, $\boldsymbol{\nabla} \equiv \frac{\partial}{\partial x_i}\hat{\mathbf{x}}_i$, and $\boldsymbol{\nabla}' \equiv \frac{\partial}{\partial x'_i}\hat{\mathbf{x}}_i$.

$$\boldsymbol{\nabla}\cdot\mathbf{x} = 3 \tag{F.65}$$

$$\boldsymbol{\nabla}\times\mathbf{x} = \mathbf{0} \tag{F.66}$$

$$\boldsymbol{\nabla}(\mathbf{k}\cdot\mathbf{x}) = \mathbf{k} \tag{F.67}$$

$$\boldsymbol{\nabla}|\mathbf{x}| = \frac{\mathbf{x}}{|\mathbf{x}|} \tag{F.68}$$

$$\boldsymbol{\nabla}\left(|\mathbf{x}-\mathbf{x}'|\right) = \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} = -\boldsymbol{\nabla}'\left(|\mathbf{x}-\mathbf{x}'|\right) \tag{F.69}$$

$$\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}|}\right) = -\frac{\mathbf{x}}{|\mathbf{x}|^3} \tag{F.70}$$

$$\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = -\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = -\boldsymbol{\nabla}'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \tag{F.71}$$

$$\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{x}}{|\mathbf{x}|^3}\right) = -\nabla^2\left(\frac{1}{|\mathbf{x}|}\right) = 4\pi\delta(\mathbf{x}) \tag{F.72}$$

$$\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\right) = -\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = 4\pi\delta(\mathbf{x}-\mathbf{x}') \tag{F.73}$$

$$\boldsymbol{\nabla}\left(\frac{\mathbf{k}}{|\mathbf{x}|}\right) = \mathbf{k}\cdot\left[\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}|}\right)\right] = -\frac{\mathbf{k}\cdot\mathbf{x}}{|\mathbf{x}|^3} \tag{F.74}$$

$$\boldsymbol{\nabla}\times\left[\mathbf{k}\times\left(\frac{\mathbf{x}}{|\mathbf{x}|^3}\right)\right] = -\boldsymbol{\nabla}\left(\frac{\mathbf{k}\cdot\mathbf{x}}{|\mathbf{x}|^3}\right) \text{ if } |\mathbf{x}| \neq 0 \tag{F.75}$$

$$\nabla^2\left(\frac{\mathbf{k}}{|\mathbf{x}|}\right) = \mathbf{k}\nabla^2\left(\frac{1}{|\mathbf{x}|}\right) = -4\pi\mathbf{k}\delta(\mathbf{x}) \tag{F.76}$$

$$\boldsymbol{\nabla}\times(\mathbf{k}\times\mathbf{a}) = \mathbf{k}(\boldsymbol{\nabla}\cdot\mathbf{a}) + \mathbf{k}\times(\boldsymbol{\nabla}\times\mathbf{a}) - \boldsymbol{\nabla}(\mathbf{k}\cdot\mathbf{a}) \tag{F.77}$$

Integral relations

Let $V(S)$ be the volume bounded by the closed surface $S(V)$. Denote the 3-dimensional volume element by $\mathrm{d}^3x(\equiv \mathrm{d}V)$ and the surface element, directed along the outward pointing surface normal unit vector $\hat{\boldsymbol{n}}$, by $\mathrm{d}\mathbf{S}(\equiv \mathrm{d}^2x\,\hat{\boldsymbol{n}})$. Then

$$\int_V (\boldsymbol{\nabla}\cdot\mathbf{a})\,\mathrm{d}^3x = \oint_S \mathrm{d}\mathbf{S}\cdot\mathbf{a} \tag{F.78}$$

$$\int_V (\boldsymbol{\nabla}\alpha)\,\mathrm{d}^3x = \oint_S \mathrm{d}\mathbf{S}\,\alpha \tag{F.79}$$

$$\int_V (\boldsymbol{\nabla}\times\mathbf{a})\,\mathrm{d}^3x = \oint_S \mathrm{d}\mathbf{S}\times\mathbf{a} \tag{F.80}$$

If $S(C)$ is an open surface bounded by the contour $C(S)$, whose line element is $\mathrm{d}\mathbf{l}$, then

$$\oint_C \alpha\,\mathrm{d}\mathbf{l} = \int_S \mathrm{d}\mathbf{S}\times\boldsymbol{\nabla}\alpha \tag{F.81}$$

$$\oint_C \mathbf{a}\cdot\mathrm{d}\mathbf{l} = \int_S \mathrm{d}\mathbf{S}\cdot(\boldsymbol{\nabla}\times\mathbf{a}) \tag{F.82}$$

F.5 Bibliography

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APPENDIX M

Mathematical Methods

M.1 Scalars, vectors and tensors

Every physical observable can be described by a geometric object. We will describe the observables in classical electrodynamics mathematically in terms of scalars, pseudoscalars, vectors, pseudovectors, tensors or pseudotensors and will not exploit differential forms to any significant degree.

A *scalar* describes a scalar quantity which may or may not be constant in time and/or space. A *vector* describes some kind of physical motion due to vection and a *tensor* describes the motion or deformation due to some form of tension. However, generalisations to more abstract notions of these quantities are commonplace. The difference between a scalar, vector and tensor and a *pseudoscalar*, *pseudovector* and a *pseudotensor* is that the latter behave differently under such coordinate transformations which cannot be reduced to pure rotations.

Throughout we adopt the convention that Latin indices $i, j, k, l, \dots$ run over the range $1, 2, 3$ to denote vector or tensor components in the real Euclidean three-dimensional (3D) configuration space $\mathbb{R}^3$, and Greek indices $\mu, \nu, \kappa, \lambda, \dots$, which are used in four-dimensional (4D) space, run over the range $0, 1, 2, 3$.

M.1.1 Vectors

Radius vector

A vector can be represented mathematically in a number of different ways. One suitable representation is in terms of an ordered N-tuple, or *row vector*, of the coordinates x_N where N is the dimensionality of the space under consideration. The most basic vector is the *radius vector* which is the vector from the origin to the point of interest. Its N-tuple representation simply enumerates the coordinates which describe this point. In this sense, the radius vector from the origin to a point is synonymous with the coordinates of the point itself.

In the 3D Euclidean space $\mathbb{R}^3$, we have $N = 3$ and the radius vector can be represented by the triplet (x_1, x_2, x_3) of coordinates x_i, $i = 1, 2, 3$. The coordinates x_i are scalar quantities which describe the position along the unit base vectors $\hat{\boldsymbol{x}}_i$ which span $\mathbb{R}^3$. Therefore a representation of the radius vector in $\mathbb{R}^3$ is

$$\mathbf{x} = \sum_{i=1}^{3} x_i \hat{\boldsymbol{x}}_i \stackrel{\text{def}}{\equiv} x_i \hat{\boldsymbol{x}}_i \tag{M.1}$$

where we have introduced *Einstein's summation convention* (EΣ) which states that a repeated index in a term implies summation over the range of the index in question. Whenever possible and convenient we shall in the following always assume EΣ and suppress explicit summation in our formulae. Typographically, we represent a vector in 3D Euclidean space $\mathbb{R}^3$ by a boldface letter or symbol in a Roman font.

Alternatively, we may describe the radius vector in *component notation* as follows:

$$x_i \stackrel{\text{def}}{\equiv} (x_1, x_2, x_3) \equiv (x, y, z) \tag{M.2}$$

This component notation is particularly useful in 4D space where we can represent the radius vector either in its *contravariant component form*

$$x^\mu \stackrel{\text{def}}{\equiv} (x^0, x^1, x^2, x^3) \tag{M.3}$$

or its *covariant component form*

$$x_\mu \stackrel{\text{def}}{\equiv} (x_0, x_1, x_2, x_3) \tag{M.4}$$

The relation between the covariant and contravariant forms is determined by the *metric tensor* (also known as the *fundamental tensor*) whose actual form is dictated by the properties of the vector space in question. The dual representation of vectors in contravariant and covariant forms is most convenient when we work in a non-Euclidean vector space with an indefinite *metric*. An example is *Lorentz space* $\mathbb{L}^4$ which is a 4D *Riemannian space* utilised to formulate the special theory of relativity.

We note that for a change of coordinates $x^\mu \to x'^\mu = x'^\mu(x^0, x^1, x^2, x^3)$, due to a transformation from a system Σ to another system Σ', the differential radius vector $\mathrm{d}x^\mu$ transforms as

$$\mathrm{d}x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} \mathrm{d}x^\nu \tag{M.5}$$

which follows trivially from the rules of differentiation of x'^μ considered as functions of four variables x^ν.

M.1.2 Fields

A *field* is a physical entity which depends on one or more continuous parameters. Such a parameter can be viewed as a ‘continuous index’ which enumerates the ‘coordinates’ of the field. In particular, in a field which depends on the usual radius vector $\mathbf{x}$ of $\mathbb{R}^3$, each point in this space can be considered as one degree of freedom so that a field is a representation of a physical entity which has an infinite number of degrees of freedom.

Scalar fields

We denote an arbitrary *scalar field* in $\mathbb{R}^3$ by

$$\alpha(\mathbf{x}) = \alpha(x_1, x_2, x_3) \stackrel{\text{def}}{\equiv} \alpha(x_i) \tag{M.6}$$

This field describes how the scalar quantity α varies continuously in 3D $\mathbb{R}^3$ space.

In 4D, a *four-scalar* field is denoted

$$\alpha(x^0, x^1, x^2, x^3) \stackrel{\text{def}}{\equiv} \alpha(x^\mu) \tag{M.7}$$

which indicates that the four-scalar α depends on all four coordinates spanning this space. Since a four-scalar has the same value at a given point regardless of coordinate system, it is also called an *invariant*.

Analogous to the transformation rule, Equation (M.5) on the facing page, for the differential $\mathrm{d}x^\mu$, the transformation rule for the differential operator $\partial/\partial x^\mu$ under a transformation $x^\mu \to x'^\mu$ becomes

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} \tag{M.8}$$

which, again, follows trivially from the rules of differentiation.

Vector fields

We can represent an arbitrary vector field $\mathbf{a}(\mathbf{x})$ in $\mathbb{R}^3$ as follows:

$$\mathbf{a}(\mathbf{x}) = a_i(\mathbf{x})\hat{\mathbf{x}}_i \tag{M.9}$$

In component notation this same vector can be represented as

$$a_i(\mathbf{x}) = (a_1(\mathbf{x}), a_2(\mathbf{x}), a_3(\mathbf{x})) = a_i(x_j) \tag{M.10}$$

In 4D, an arbitrary *four-vector* field in contravariant component form can be represented as

$$a^\mu(x^\nu) = (a^0(x^\nu), a^1(x^\nu), a^2(x^\nu), a^3(x^\nu)) \tag{M.11}$$

or, in *covariant* component form, as

$$a_\mu(x^\nu) = (a_0(x^\nu), a_1(x^\nu), a_2(x^\nu), a_3(x^\nu)) \tag{M.12}$$

where x^ν is the radius four-vector. Again, the relation between a^μ and a_μ is determined by the metric of the physical 4D system under consideration.

Whether an arbitrary N-tuple fulfils the requirement of being an (N-dimensional) contravariant vector or not, depends on its transformation properties during a change of coordinates. For instance, in 4D an assemblage $y^\mu = (y^0, y^1, y^2, y^3)$ constitutes a *contravariant four-vector* (or the contravariant components of a four-vector) if and only if, during a transformation from a system Σ with coordinates x^μ to a system Σ' with coordinates x'^μ, it transforms to the new system according to the rule

$$y'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} y^\nu \tag{M.13}$$

i.e., in the same way as the differential coordinate element dx^μ transforms according to Equation (M.5) on page 164.

The analogous requirement for a *covariant four-vector* is that it transforms, during the change from Σ to Σ', according to the rule

$$y'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} y_\nu \tag{M.14}$$

i.e., in the same way as the differential operator $\partial/\partial x^\mu$ transforms according to Equation (M.8) on the preceding page.

Tensor fields

We denote an arbitrary *tensor field* in $\mathbb{R}^3$ by $\mathsf{A}(\mathbf{x})$. This tensor field can be represented in a number of ways, for instance in the following *matrix form*:

$$\left(\mathsf{A}_{ij}(x_k)\right) \stackrel{\text{def}}{\equiv} \begin{pmatrix} \mathsf{A}_{11}(\mathbf{x}) & \mathsf{A}_{12}(\mathbf{x}) & \mathsf{A}_{13}(\mathbf{x}) \\ \mathsf{A}_{21}(\mathbf{x}) & \mathsf{A}_{22}(\mathbf{x}) & \mathsf{A}_{23}(\mathbf{x}) \\ \mathsf{A}_{31}(\mathbf{x}) & \mathsf{A}_{32}(\mathbf{x}) & \mathsf{A}_{33}(\mathbf{x}) \end{pmatrix} \tag{M.15}$$

Strictly speaking, the tensor field described here is a tensor of *rank* two.

A particularly simple rank-two tensor in $\mathbb{R}^3$ is the 3D *Kronecker delta* symbol δ_{ij}, with the following properties:

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \tag{M.16}$$

The 3D Kronecker delta has the following matrix representation

$$(\delta_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{M.17}$$

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Another common and useful tensor is the fully antisymmetric tensor of rank 3, also known as the *Levi-Civita tensor*

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } i, j, k \text{ is an } \textit{even} \text{ permutation of 1,2,3} \\ 0 & \text{if at least two of } i, j, k \text{ are equal} \\ -1 & \text{if } i, j, k \text{ is an } \textit{odd} \text{ permutation of 1,2,3} \end{cases} \tag{M.18}$$

with the following further property

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \tag{M.19}$$

In fact, tensors may have any rank n. In this picture a scalar is considered to be a tensor of rank $n = 0$ and a vector a tensor of rank $n = 1$. Consequently, the notation where a vector (tensor) is represented in its component form is called the *tensor notation*. A tensor of rank $n = 2$ may be represented by a two-dimensional array or matrix whereas higher rank tensors are best represented in their component forms (tensor notation).

▷TENSORS IN 3D SPACE — EXAMPLE 13.1

Consider a tetrahedron-like volume element V of a solid, fluid, or gaseous body, whose atomistic structure is irrelevant for the present analysis; figure M.1 on the next page indicates how this volume may look like. Let $\mathrm{d}\mathbf{S} = \mathrm{d}^2x\,\hat{\boldsymbol{n}}$ be the directed surface element of this volume element and let the vector $\mathbf{T}_{\hat{\boldsymbol{n}}}\,\mathrm{d}^2x$ be the force that matter, lying on the side of d^2x toward which the unit normal vector $\hat{\boldsymbol{n}}$ points, acts on matter which lies on the opposite side of d^2x. This force concept is meaningful only if the forces are short-range enough that they can be assumed to act only in the surface proper. According to Newton's third law, this surface force fulfils

$$\mathbf{T}_{-\hat{\boldsymbol{n}}} = -\mathbf{T}_{\hat{\boldsymbol{n}}} \tag{M.20}$$

Using (M.20) and Newton's second law, we find that the matter of mass m, which at a given instant is located in V obeys the equation of motion

$$\mathbf{T}_{\hat{\boldsymbol{n}}}\,\mathrm{d}^2x - \cos\theta_1\mathbf{T}_{\hat{\mathbf{x}}_1}\,\mathrm{d}^2x - \cos\theta_2\mathbf{T}_{\hat{\mathbf{x}}_2}\,\mathrm{d}^2x - \cos\theta_3\mathbf{T}_{\hat{\mathbf{x}}_3}\,\mathrm{d}^2x + \mathbf{F}_{\text{ext}} = m\mathbf{a} \tag{M.21}$$

where $\mathbf{F}_{\text{ext}}$ is the external force and $\mathbf{a}$ is the acceleration of the volume element. In other words

$$\mathbf{T}_{\hat{\boldsymbol{n}}} = n_1\mathbf{T}_{\hat{\mathbf{x}}_1} + n_2\mathbf{T}_{\hat{\mathbf{x}}_2} + n_3\mathbf{T}_{\hat{\mathbf{x}}_3} + \frac{m}{\mathrm{d}^2x}\left(\mathbf{a} - \frac{\mathbf{F}_{\text{ext}}}{m}\right) \tag{M.22}$$

Since both $\mathbf{a}$ and $\mathbf{F}_{\text{ext}}/m$ remain finite whereas $m/\mathrm{d}^2x \to 0$ as $V \to 0$, one finds that in this limit

$$\mathbf{T}_{\hat{\boldsymbol{n}}} = \sum_{i=1}^{3} n_i\mathbf{T}_{\hat{\mathbf{x}}_i} \equiv n_i\mathbf{T}_{\hat{\mathbf{x}}_i} \tag{M.23}$$

From the above derivation it is clear that Equation (M.23) is valid not only in equilibrium but also when the matter in V is in motion.

Introducing the notation

$$T_{ij} = \left(\mathbf{T}_{\hat{\mathbf{x}}_i}\right)_j \tag{M.24}$$

for the jth component of the vector $\mathbf{T}_{\hat{\mathbf{x}}_i}$, we can write Equation (M.23) above in component

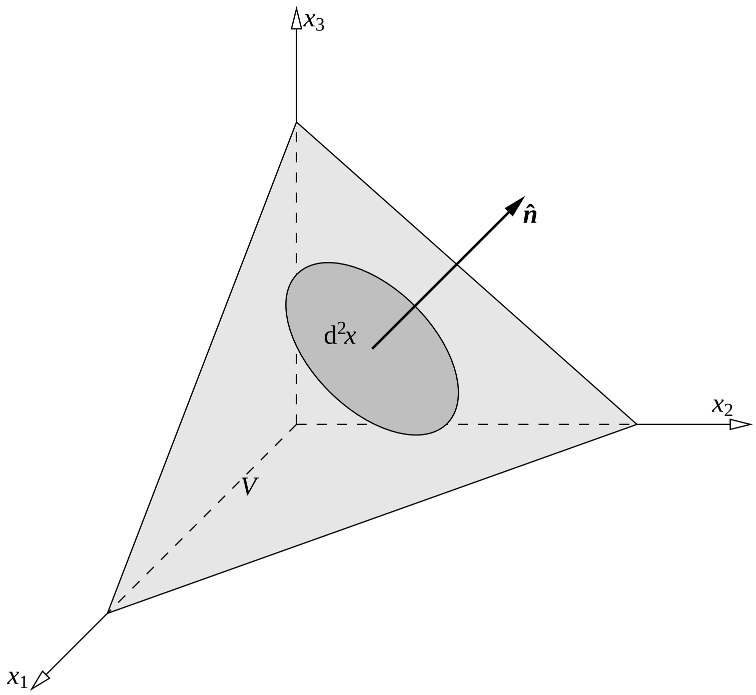

FIGURE M.1: Terahedron-like volume element V containing matter.

form as follows

$$T_{\hat{\mathbf{n}}j} = (\mathbf{T}_{\hat{\mathbf{n}}})_j = \sum_{i=1}^{3} n_i T_{ij} \equiv n_i T_{ij} \tag{M.25}$$

Using Equation (M.25), we find that the component of the vector $\mathbf{T}_{\hat{\mathbf{n}}}$ in the direction of an arbitrary unit vector $\hat{\mathbf{m}}$ is

$$\begin{aligned} T_{\hat{\mathbf{n}}\hat{\mathbf{m}}} &= \mathbf{T}_{\hat{\mathbf{n}}} \cdot \hat{\mathbf{m}} \\ &= \sum_{j=1}^{3} T_{\hat{\mathbf{n}}j} m_j = \sum_{j=1}^{3} \left(\sum_{i=1}^{3} n_i T_{ij} \right) m_j \equiv n_i T_{ij} m_j = \hat{\mathbf{n}} \cdot \mathbf{T} \cdot \hat{\mathbf{m}} \end{aligned} \tag{M.26}$$

Hence, the jth component of the vector $\mathbf{T}_{\hat{\mathbf{x}}_i}$, here denoted T_{ij}, can be interpreted as the ijth component of a tensor $\mathbf{T}$. Note that $T_{\hat{\mathbf{n}}\hat{\mathbf{m}}}$ is independent of the particular coordinate system used in the derivation.

We shall now show how one can use the momentum law (force equation) to derive the equation of motion for an arbitrary element of mass in the body. To this end we consider a part V of the body. If the external force density (force per unit volume) is denoted by $\mathbf{f}$ and the velocity for a mass element dm is denoted by $\mathbf{v}$, we obtain

$$\frac{d}{dt} \int_V \mathbf{v}\, dm = \int_V \mathbf{f}\, d^3x + \int_S \mathbf{T}_{\hat{\mathbf{n}}}\, d^2x \tag{M.27}$$

The jth component of this equation can be written

$$\int_V \frac{\mathrm{d}}{\mathrm{d}t} v_j \, \mathrm{d}m = \int_V f_j \, \mathrm{d}^3x + \int_S T_{\hat{\mathbf{n}}j} \, \mathrm{d}^2x = \int_V f_j \, \mathrm{d}^3x + \int_S n_i T_{ij} \, \mathrm{d}^2x \tag{M.28}$$

where, in the last step, Equation (M.25) on the preceding page was used. Setting $\mathrm{d}m = \rho \, \mathrm{d}^3x$ and using the divergence theorem on the last term, we can rewrite the result as

$$\int_V \rho \frac{\mathrm{d}}{\mathrm{d}t} v_j \, \mathrm{d}^3x = \int_V f_j \, \mathrm{d}^3x + \int_V \frac{\partial T_{ij}}{\partial x_i} \, \mathrm{d}^3x \tag{M.29}$$

Since this formula is valid for any arbitrary volume, we must require that

$$\rho \frac{\mathrm{d}}{\mathrm{d}t} v_j - f_j - \frac{\partial T_{ij}}{\partial x_i} = 0 \tag{M.30}$$

or, equivalently

$$\rho \frac{\partial v_j}{\partial t} + \rho \mathbf{v} \cdot \boldsymbol{\nabla} v_j - f_j - \frac{\partial T_{ij}}{\partial x_i} = 0 \tag{M.31}$$

Note that $\partial v_j / \partial t$ is the rate of change with time of the velocity component v_j at a *fixed* point $\mathbf{x} = (x_1, x_1, x_3)$.

◁ END OF EXAMPLE 13.1

In 4D, we have three forms of *four-tensor fields* of rank n. We speak of

- a *contravariant four-tensor field*, denoted $A^{\mu_1 \mu_2 \ldots \mu_n}(x^\nu)$,
- a *covariant four-tensor field*, denoted $A_{\mu_1 \mu_2 \ldots \mu_n}(x^\nu)$,
- a *mixed four-tensor field*, denoted $A^{\mu_1 \mu_2 \ldots \mu_k}_{\mu_{k+1} \ldots \mu_n}(x^\nu)$.

The 4D *metric tensor* (*fundamental tensor*) mentioned above is a particularly important four-tensor of rank 2. In covariant component form we shall denote it $g_{\mu\nu}$. This metric tensor determines the relation between an arbitrary contravariant four-vector a^μ and its covariant counterpart a_μ according to the following rule:

$$a_\mu(x^\kappa) \stackrel{\text{def}}{\equiv} g_{\mu\nu} a^\nu(x^\kappa) \tag{M.32}$$

This rule is often called *lowering of index*. The *raising of index* analogue of the index lowering rule is:

$$a^\mu(x^\kappa) \stackrel{\text{def}}{\equiv} g^{\mu\nu} a_\nu(x^\kappa) \tag{M.33}$$

More generally, the following lowering and raising rules hold for arbitrary rank n mixed tensor fields:

$$g_{\mu_k \nu_k} A^{\nu_1 \nu_2 \ldots \nu_{k-1} \nu_k}_{\nu_{k+1} \nu_{k+2} \ldots \nu_n}(x^\kappa) = A^{\nu_1 \nu_2 \ldots \nu_{k-1}}_{\mu_k \nu_{k+1} \ldots \nu_n}(x^\kappa) \tag{M.34}$$

$$g^{\mu_k \nu_k} A^{\nu_1 \nu_2 \ldots \nu_{k-1}}_{\nu_k \nu_{k+1} \ldots \nu_n}(x^\kappa) = A^{\nu_1 \nu_2 \ldots \nu_{k-1} \mu_k}_{\nu_{k+1} \nu_{k+2} \ldots \nu_n}(x^\kappa) \tag{M.35}$$

Successive lowering and raising of more than one index is achieved by a repeated application of this rule. For example, a dual application of the lowering operation on a rank 2 tensor in contravariant form yields

$$A_{\mu\nu} = g_{\mu\kappa} g_{\lambda\nu} A^{\kappa\lambda} \tag{M.36}$$

i.e., the same rank 2 tensor in covariant form. This operation is also known as a *tensor contraction*.

EXAMPLE 13.2 ▷CONTRAVARIANT AND COVARIANT VECTORS IN FLAT LORENTZ SPACE

The 4D Lorentz space $\mathbb{L}^4$ has a simple metric which can be described either by the metric tensor

$$g_{\mu\nu} = \begin{cases} 1 & \text{if } \mu = \nu = 0 \\ -1 & \text{if } \mu = \nu = i = j = 1, 2, 3 \\ 0 & \text{if } \mu \neq \nu \end{cases} \tag{M.37}$$

which, in matrix notation, is represented as

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{M.38}$$

i.e., a matrix with a main diagonal that has the sign sequence, or *signature*, $\{+, -, -, -\}$ or

$$g_{\mu\nu} = \begin{cases} -1 & \text{if } \mu = \nu = 0 \\ 1 & \text{if } \mu = \nu = i = j = 1, 2, 3 \\ 0 & \text{if } \mu \neq \nu \end{cases} \tag{M.39}$$

which, in matrix notation, is represented as

$$(g_{\mu\nu}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{M.40}$$

i.e., a matrix with signature $\{-, +, +, +\}$.

Consider an arbitrary contravariant four-vector a^ν in this space. In component form it can be written:

$$a^\nu \stackrel{\text{def}}{\equiv} (a^0, a^1, a^2, a^3) = (a^0, \mathbf{a}) \tag{M.41}$$

According to the index lowering rule, Equation (M.32) on the preceding page, we obtain the covariant version of this vector as

$$a_\mu \stackrel{\text{def}}{\equiv} (a_0, a_1, a_2, a_3) = g_{\mu\nu} a^\nu \tag{M.42}$$

In the $\{+, -, -, -\}$ metric we obtain

$$\mu = 0: \quad a_0 = 1 \cdot a^0 + 0 \cdot a^1 + 0 \cdot a^2 + 0 \cdot a^3 = a^0 \tag{M.43}$$
$$\mu = 1: \quad a_1 = 0 \cdot a^0 - 1 \cdot a^1 + 0 \cdot a^2 + 0 \cdot a^3 = -a^1 \tag{M.44}$$
$$\mu = 2: \quad a_2 = 0 \cdot a^0 + 0 \cdot a^1 - 1 \cdot a^2 + 0 \cdot a^3 = -a^2 \tag{M.45}$$
$$\mu = 3: \quad a_3 = 0 \cdot a^0 + 0 \cdot a^1 + 0 \cdot a^2 + 1 \cdot a^3 = -a^3 \tag{M.46}$$

or

$$a_\mu = (a_0, a_1, a_2, a_3) = (a^0, -a^1, -a^2, -a^3) = (a^0, -\mathbf{a}) \tag{M.47}$$

Radius 4-vector itself in $\mathbb{L}^4$ and in this metric is given by

$$\begin{aligned} x^\mu &= (x^0, x^1, x^2, x^3) = (x^0, x, y, z) = (x^0, \mathbf{x}) \\ x_\mu &= (x_0, x_1, x_2, x_3) = (x^0, -x^1, -x^2, -x^3) = (x^0, -\mathbf{x}) \end{aligned} \tag{M.48}$$

where $x^0 = ct$.

Analogously, using the $\{-, +, +, +\}$ metric we obtain

$$a_\mu = (a_0, a_1, a_2, a_3) = (-a^0, a^1, a^2, a^3) = (-a^0, \mathbf{a}) \tag{M.49}$$

◁ END OF EXAMPLE 13.2

M.1.3 Vector algebra

Scalar product

The *scalar product* (*dot product*, *inner product*) of two arbitrary 3D vectors **a** and **b** in ordinary $\mathbb{R}^3$ space is the scalar number

$$\mathbf{a} \cdot \mathbf{b} = a_i \hat{\mathbf{x}}_i \cdot b_j \hat{\mathbf{x}}_j = \hat{\mathbf{x}}_i \cdot \hat{\mathbf{x}}_j a_i b_j = \delta_{ij} a_i b_j = a_i b_i \tag{M.50}$$

where we used the fact that the scalar product $\hat{\mathbf{x}}_i \cdot \hat{\mathbf{x}}_j$ is a representation of the Kronecker delta δ_{ij} defined in Equation (M.16) on page 166. In Russian literature, the 3D scalar product is often denoted (**ab**). The scalar product of **a** in $\mathbb{R}^3$ with itself is

$$\mathbf{a} \cdot \mathbf{a} \stackrel{\text{def}}{\equiv} (\mathbf{a})^2 = |\mathbf{a}|^2 = (a_i)^2 = a^2 \tag{M.51}$$

and simlarly for **b**. This allows us to write

$$\mathbf{a} \cdot \mathbf{b} = ab \cos\theta \tag{M.52}$$

where θ is the angle between **a** and **b**.

In 4D space we define the scalar product of two arbitrary four-vectors a^μ and b^μ in the following way

$$a_\mu b^\mu = g_{\nu\mu} a^\nu b^\mu = a^\nu b_\nu = g^{\mu\nu} a_\mu b_\nu \tag{M.53}$$

where we made use of the index lowering and raising rules (M.32) and (M.33). The result is a four-scalar, *i.e.*, an invariant which is independent of in which 4D coordinate system it is measured.

The *quadratic differential form*

$$\mathrm{d}s^2 = g_{\mu\nu} \mathrm{d}x^\nu \mathrm{d}x^\mu = \mathrm{d}x_\mu \mathrm{d}x^\mu \tag{M.54}$$

i.e., the scalar product of the differential radius four-vector with itself, is an invariant called the *metric*. It is also the square of the *line element* ds which is the distance between neighbouring points with coordinates x^μ and $x^\mu + \mathrm{d}x^\mu$.

EXAMPLE 13.3 ▷INNER PRODUCTS IN COMPLEX VECTOR SPACE

A 3D *complex vector* $\mathbf{A}$ is a vector in $\mathbb{C}^3$ (or, if we like, in $\mathbb{R}^6$), expressed in terms of two real vectors $\mathbf{a}_\mathrm{R}$ and $\mathbf{a}_\mathrm{I}$ in $\mathbb{R}^3$ in the following way

$$\mathbf{A} \stackrel{\text{def}}{\equiv} \mathbf{a}_\mathrm{R} + \mathrm{i}\mathbf{a}_\mathrm{I} = a_\mathrm{R}\hat{\boldsymbol{a}}_\mathrm{R} + \mathrm{i}a_\mathrm{I}\hat{\boldsymbol{a}}_\mathrm{I} \stackrel{\text{def}}{\equiv} A\hat{\boldsymbol{A}} \in \mathbb{C}^3 \tag{M.55}$$

The inner product of $\mathbf{A}$ with itself may be defined as

$$\mathbf{A}^2 \stackrel{\text{def}}{\equiv} \mathbf{A}\cdot\mathbf{A} = a_\mathrm{R}^2 - a_\mathrm{I}^2 + 2\mathrm{i}\mathbf{a}_\mathrm{R}\cdot\mathbf{a}_\mathrm{I} \stackrel{\text{def}}{\equiv} A^2 \in \mathbb{C} \tag{M.56}$$

from which we find that

$$A = \sqrt{a_\mathrm{R}^2 - a_\mathrm{I}^2 + 2\mathrm{i}\mathbf{a}_\mathrm{R}\cdot\mathbf{a}_\mathrm{I}} \in \mathbb{C} \tag{M.57}$$

Using this in Equation (M.55), we see that we can interpret this so that the complex unit vector is

$$\begin{aligned}\hat{\boldsymbol{A}} = \frac{\mathbf{A}}{A} &= \frac{a_\mathrm{R}}{\sqrt{a_\mathrm{R}^2 - a_\mathrm{I}^2 + 2\mathrm{i}\mathbf{a}_\mathrm{R}\cdot\mathbf{a}_\mathrm{I}}}\hat{\boldsymbol{a}}_\mathrm{R} + \mathrm{i}\frac{a_\mathrm{I}}{\sqrt{a_\mathrm{R}^2 - a_\mathrm{I}^2 + 2\mathrm{i}\mathbf{a}_\mathrm{R}\cdot\mathbf{a}_\mathrm{I}}}\hat{\boldsymbol{a}}_\mathrm{I} \\ &= \frac{a_\mathrm{R}\sqrt{a_\mathrm{R}^2 - a_\mathrm{I}^2 - 2\mathrm{i}\mathbf{a}_\mathrm{R}\cdot\mathbf{a}_\mathrm{I}}}{a_\mathrm{R}^2 + a_\mathrm{I}^2}\hat{\boldsymbol{a}}_\mathrm{R} + \mathrm{i}\frac{a_\mathrm{I}\sqrt{a_\mathrm{R}^2 - a_\mathrm{I}^2 - 2\mathrm{i}\mathbf{a}_\mathrm{R}\cdot\mathbf{a}_\mathrm{I}}}{a_\mathrm{R}^2 + a_\mathrm{I}^2}\hat{\boldsymbol{a}}_\mathrm{I} \in \mathbb{C}^3\end{aligned} \tag{M.58}$$

On the other hand, the definition of the scalar product in terms of the inner product of complex vector with its own complex conjugate yields

$$|\mathbf{A}|^2 \stackrel{\text{def}}{\equiv} \mathbf{A}\cdot\mathbf{A}^* = a_\mathrm{R}^2 + a_\mathrm{I}^2 = |A|^2 \tag{M.59}$$

with the help of which we can define the unit vector as

$$\begin{aligned}\hat{\boldsymbol{A}} = \frac{\mathbf{A}}{|A|} &= \frac{a_\mathrm{R}}{\sqrt{a_\mathrm{R}^2 + a_\mathrm{I}^2}}\hat{\boldsymbol{a}}_\mathrm{R} + \mathrm{i}\frac{a_\mathrm{I}}{\sqrt{a_\mathrm{R}^2 + a_\mathrm{I}^2}}\hat{\boldsymbol{a}}_\mathrm{I} \\ &= \frac{a_\mathrm{R}\sqrt{a_\mathrm{R}^2 + a_\mathrm{I}^2}}{a_\mathrm{R}^2 + a_\mathrm{I}^2}\hat{\boldsymbol{a}}_\mathrm{R} + \mathrm{i}\frac{a_\mathrm{I}\sqrt{a_\mathrm{R}^2 + a_\mathrm{I}^2}}{a_\mathrm{R}^2 + a_\mathrm{I}^2}\hat{\boldsymbol{a}}_\mathrm{I} \in \mathbb{C}^3\end{aligned} \tag{M.60}$$

◁ END OF EXAMPLE 13.3

▷SCALAR PRODUCT, NORM AND METRIC IN LORENTZ SPACE — EXAMPLE 13.4

In $\mathbb{L}^4$ the metric tensor attains a simple form [see Example 13.2 on page 170] and, hence, the scalar product in Equation (M.53) on the facing page can be evaluated almost trivially. For the $\{+,-,-,-\}$ signature it becomes

$$a_\mu b^\mu = (a_0, -\mathbf{a}) \cdot (b^0, \mathbf{b}) = a_0 b^0 - \mathbf{a} \cdot \mathbf{b} \tag{M.61}$$

The important scalar product of the $\mathbb{L}^4$ radius four-vector with itself becomes

$$\begin{aligned} x_\mu x^\mu &= (x_0, -\mathbf{x}) \cdot (x^0, \mathbf{x}) = (ct, -\mathbf{x}) \cdot (ct, \mathbf{x}) \\ &= (ct)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = s^2 \end{aligned} \tag{M.62}$$

which is the indefinite, real *norm* of $\mathbb{L}^4$. The $\mathbb{L}^4$ metric is the quadratic differential form

$$\mathrm{d}s^2 = \mathrm{d}x_\mu \mathrm{d}x^\mu = c^2(\mathrm{d}t)^2 - (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2 \tag{M.63}$$

◁ END OF EXAMPLE 13.4

Dyadic product

The *dyadic product* field $\mathsf{A}(\mathbf{x}) \equiv \mathbf{a}(\mathbf{x})\mathbf{b}(\mathbf{x})$ with two juxtaposed vector fields $\mathbf{a}(\mathbf{x})$ and $\mathbf{b}(\mathbf{x})$ is the *outer product* of $\mathbf{a}$ and $\mathbf{b}$. Operating on this dyad from the right and from the left with an inner product of an vector $\mathbf{c}$ one obtains

$$\mathsf{A} \cdot \mathbf{c} \overset{\text{def}}{\equiv} \mathbf{ab} \cdot \mathbf{c} \overset{\text{def}}{\equiv} \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) \tag{M.64a}$$

$$\mathbf{c} \cdot \mathsf{A} \overset{\text{def}}{\equiv} \mathbf{c} \cdot \mathbf{ab} \overset{\text{def}}{\equiv} (\mathbf{c} \cdot \mathbf{a})\mathbf{b} \tag{M.64b}$$

i.e., new vectors, proportional to $\mathbf{a}$ and $\mathbf{b}$, respectively. In mathematics, a dyadic product is often called *tensor product* and is frequently denoted $\mathbf{a} \otimes \mathbf{b}$.

In matrix notation the outer product of $\mathbf{a}$ and $\mathbf{b}$ is written

$$\left(\mathbf{ab}\right) = \begin{pmatrix} \hat{\mathbf{x}}_1 & \hat{\mathbf{x}}_2 & \hat{\mathbf{x}}_3 \end{pmatrix} \begin{pmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_1 b_2 & a_2 b_2 & a_2 b_3 \\ a_1 b_3 & a_3 b_2 & a_3 b_3 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}}_1 \\ \hat{\mathbf{x}}_2 \\ \hat{\mathbf{x}}_3 \end{pmatrix} \tag{M.65}$$

which means that we can represent the tensor $\mathsf{A}(\mathbf{x})$ in matrix form as

$$\left(\mathsf{A}_{ij}(x_k)\right) = \begin{pmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_1 b_2 & a_2 b_2 & a_2 b_3 \\ a_1 b_3 & a_3 b_2 & a_3 b_3 \end{pmatrix} \tag{M.66}$$

which we identify with expression (M.15) on page 166, *viz.* a tensor in matrix notation.

Vector product

The *vector product* or *cross product* of two arbitrary 3D vectors **a** and **b** in ordinary $\mathbb{R}^3$ space is the vector

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \epsilon_{ijk} a_j b_k \hat{\mathbf{x}}_i \tag{M.67}$$

Here ϵ_{ijk} is the Levi-Civita tensor defined in Equation (M.18) on page 167. Sometimes the 3D vector product of **a** and **b** is denoted $\mathbf{a} \wedge \mathbf{b}$ or, particularly in the Russian literature, $[\mathbf{ab}]$. Alternatively,

$$\mathbf{a} \times \mathbf{b} = ab \sin\theta \, \hat{\mathbf{e}} \tag{M.68}$$

where θ is the angle between **a** and **b** and $\hat{\mathbf{e}}$ is a unit vector perpendicular to the plane spanned by **a** and **b**.

A spatial reversal of the coordinate system $(x'_1, x'_2, x'_3) = (-x_1, -x_2, -x_3)$ changes sign of the components of the vectors **a** and **b** so that in the new coordinate system $\mathbf{a}' = -\mathbf{a}$ and $\mathbf{b}' = -\mathbf{b}$, which is to say that the direction of an ordinary vector is not dependent on the choice of directions of the coordinate axes. On the other hand, as is seen from Equation (M.67) above, the cross product vector **c** does not change sign. Therefore **a** (or **b**) is an example of a 'true' vector, or *polar vector*, whereas **c** is an example of an *axial vector*, or *pseudovector*.

A prototype for a pseudovector is the angular momentum vector $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ and hence the attribute 'axial'. Pseudovectors transform as ordinary vectors under translations and proper rotations, but reverse their sign relative to ordinary vectors for any coordinate change involving reflection. Tensors (of any rank) which transform analogously to pseudovectors are called *pseudotensors*. Scalars are tensors of rank zero, and zero-rank pseudotensors are therefore also called *pseudoscalars*, an example being the pseudoscalar $\hat{\mathbf{x}}_i \cdot (\hat{\mathbf{x}}_j \times \hat{\mathbf{x}}_k)$. This triple product is a representation of the ijk component of the Levi-Civita tensor ϵ_{ijk} which is a rank three pseudotensor.

M.1.4 Vector analysis

The *del* operator

In $\mathbb{R}^3$ the *del operator* is a *differential vector operator*, denoted in *Gibbs' notation* by $\boldsymbol{\nabla}$ and defined as

$$\boldsymbol{\nabla} \stackrel{\text{def}}{\equiv} \hat{\mathbf{x}}_i \frac{\partial}{\partial x_i} \stackrel{\text{def}}{\equiv} \boldsymbol{\partial} \tag{M.69}$$

where $\hat{\mathbf{x}}_i$ is the ith unit vector in a Cartesian coordinate system. Since the operator in itself has vectorial properties, we denote it with a boldface nabla. In 'component' notation we can write

$$\partial_i = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) \tag{M.70}$$

In 4D, the contravariant component representation of the *four-del operator* is defined by

$$\partial^\mu = \left(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) \tag{M.71}$$

whereas the covariant four-del operator is

$$\partial_\mu = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right) \tag{M.72}$$

We can use this four-del operator to express the transformation properties (M.13) and (M.14) on page 166 as

$$y'^\mu = \left(\partial_\nu x'^\mu\right) y^\nu \tag{M.73}$$

and

$$y'_\mu = \left(\partial'_\mu x^\nu\right) y_\nu \tag{M.74}$$

respectively.

▷THE FOUR-DEL OPERATOR IN LORENTZ SPACE —— EXAMPLE 13.5

In $\mathbb{L}^4$ the contravariant form of the four-del operator can be represented as

$$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\boldsymbol{\partial}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right) \tag{M.75}$$

and the covariant form as

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \boldsymbol{\partial}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right) \tag{M.76}$$

Taking the scalar product of these two, one obtains

$$\partial^\mu \partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = \Box^2 \tag{M.77}$$

which is the *d'Alembert operator*, sometimes denoted $\Box$, and sometimes defined with an opposite sign convention.

—— ◁ END OF EXAMPLE 13.5

With the help of the del operator we can define the gradient, divergence and curl of a tensor (in the generalised sense).

The gradient

The *gradient* of an $\mathbb{R}^3$ scalar field $\alpha(\mathbf{x})$, denoted $\boldsymbol{\nabla}\alpha(x)$, is an $\mathbb{R}^3$ vector field $\mathbf{a}(\mathbf{x})$:

$$\boldsymbol{\nabla}\alpha(\mathbf{x}) = \boldsymbol{\partial}\alpha(\mathbf{x}) = \hat{\mathbf{x}}_i\partial_i\alpha(\mathbf{x}) = \mathbf{a}(\mathbf{x}) \tag{M.78}$$

From this we see that the boldface notation for the nabla and del operators is very handy as it elucidates the 3D vectorial property of the gradient.

In 4D, the *four-gradient* is a covariant vector, formed as a derivative of a four-scalar field $\alpha(x^\mu)$, with the following component form:

$$\partial_\mu\alpha(x^\nu) = \frac{\partial\alpha(x^\nu)}{\partial x^\mu} \tag{M.79}$$

EXAMPLE 13.6 ▷GRADIENTS OF SCALAR FUNCTIONS OF RELATIVE DISTANCES IN 3D

Very often electrodynamic quantities are dependent on the relative distance in $\mathbb{R}^3$ between two vectors $\mathbf{x}$ and $\mathbf{x}'$, *i.e.*, on $|\mathbf{x}-\mathbf{x}'|$. In analogy with Equation (M.69) on page 174, we can define the primed del operator in the following way:

$$\boldsymbol{\nabla}' = \hat{\mathbf{x}}_i\frac{\partial}{\partial x'_i} = \boldsymbol{\partial}' \tag{M.80}$$

Using this, the unprimed version, Equation (M.69) on page 174, and elementary rules of differentiation, we obtain the following two very useful results:

$$\begin{aligned}\boldsymbol{\nabla}\left(|\mathbf{x}-\mathbf{x}'|\right) &= \hat{\mathbf{x}}_i\frac{\partial|\mathbf{x}-\mathbf{x}'|}{\partial x_i} = \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} = -\hat{\mathbf{x}}_i\frac{\partial|\mathbf{x}-\mathbf{x}'|}{\partial x'_i} \\ &= -\boldsymbol{\nabla}'\left(|\mathbf{x}-\mathbf{x}'|\right)\end{aligned} \tag{M.81}$$

and

$$\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = -\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = -\boldsymbol{\nabla}'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \tag{M.82}$$

◁ END OF EXAMPLE 13.6

The divergence

We define the 3D *divergence* of a vector field in $\mathbb{R}^3$ as

$$\nabla \cdot \mathbf{a}(\mathbf{x}) = \boldsymbol{\partial} \cdot \hat{\mathbf{x}}_j a_j(\mathbf{x}) = \delta_{ij}\partial_i a_j(\mathbf{x}) = \partial_i a_i(\mathbf{x}) = \frac{\partial a_i(\mathbf{x})}{\partial x_i} = \alpha(\mathbf{x}) \tag{M.83}$$

which, as indicated by the notation $\alpha(\mathbf{x})$, is a *scalar* field in $\mathbb{R}^3$. We may think of the divergence as a scalar product between a vectorial operator and a vector. As is the case for any scalar product, the result of a divergence operation is a scalar. Again we see that the boldface notation for the 3D del operator is very convenient.

The *four-divergence* of a four-vector a^μ is the following four-scalar:

$$\partial_\mu a^\mu(x^\nu) = \partial^\mu a_\mu(x^\nu) = \frac{\partial a^\mu(x^\nu)}{\partial x^\mu} \tag{M.84}$$

▷DIVERGENCE IN 3D —— EXAMPLE 13.7

For an arbitrary $\mathbb{R}^3$ vector field $\mathbf{a}(\mathbf{x}')$, the following relation holds:

$$\nabla' \cdot \left(\frac{\mathbf{a}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right) = \frac{\nabla' \cdot \mathbf{a}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \mathbf{a}(\mathbf{x}') \cdot \nabla' \left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \tag{M.85}$$

which demonstrates how the primed divergence, defined in terms of the primed del operator in Equation (M.80) on the preceding page, works.

—— ◁ END OF EXAMPLE 13.7

The Laplacian

The 3D *Laplace operator* or *Laplacian* can be described as the divergence of the gradient operator:

$$\nabla^2 = \Delta = \nabla \cdot \nabla = \frac{\partial}{\partial x_i}\hat{\mathbf{x}}_i \cdot \hat{\mathbf{x}}_j \frac{\partial}{\partial x_j} = \delta_{ij}\partial_i\partial_j = \partial_i^2 = \frac{\partial^2}{\partial x_i^2} \equiv \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} \tag{M.86}$$

The symbol ∇^2 is sometimes read *del squared*. If, for a scalar field $\alpha(\mathbf{x})$, $\nabla^2\alpha < 0$ at some point in 3D space, it is a sign of *concentration* of α at that point.

EXAMPLE 13.8 ▷THE LAPLACIAN AND THE DIRAC DELTA

A very useful formula in 3D $\mathbb{R}^3$ is

$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) = \nabla^2 \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) = -4\pi\delta(\mathbf{x} - \mathbf{x}') \tag{M.87}$$

where $\delta(\mathbf{x} - \mathbf{x}')$ is the 3D *Dirac delta* 'function'.

◁ END OF EXAMPLE 13.8

The curl

In $\mathbb{R}^3$ the *curl* of a vector field $\mathbf{a}(\mathbf{x})$, denoted $\boldsymbol{\nabla} \times \mathbf{a}(\mathbf{x})$, is another $\mathbb{R}^3$ vector field $\mathbf{b}(\mathbf{x})$ which can be defined in the following way:

$$\boldsymbol{\nabla} \times \mathbf{a}(\mathbf{x}) = \epsilon_{ijk}\hat{\mathbf{x}}_i \partial_j a_k(\mathbf{x}) = \epsilon_{ijk}\hat{\mathbf{x}}_i \frac{\partial a_k(\mathbf{x})}{\partial x_j} = \mathbf{b}(\mathbf{x}) \tag{M.88}$$

where use was made of the Levi-Civita tensor, introduced in Equation (M.18) on page 167.

The covariant 4D generalisation of the curl of a four-vector field $a^\mu(x^\nu)$ is the antisymmetric four-tensor field

$$G_{\mu\nu}(x^\kappa) = \partial_\mu a_\nu(x^\kappa) - \partial_\nu a_\mu(x^\kappa) = -G_{\nu\mu}(x^\kappa) \tag{M.89}$$

A vector with vanishing curl is said to be *irrotational*.

EXAMPLE 13.9 ▷THE CURL OF A GRADIENT

Using the definition of the $\mathbb{R}^3$ curl, Equation (M.88) above, and the gradient, Equation (M.78) on page 176, we see that

$$\boldsymbol{\nabla} \times [\boldsymbol{\nabla}\alpha(\mathbf{x})] = \epsilon_{ijk}\hat{\mathbf{x}}_i \partial_j \partial_k \alpha(\mathbf{x}) \tag{M.90}$$

which, due to the assumed well-behavedness of $\alpha(\mathbf{x})$, vanishes:

$$\begin{aligned}
\epsilon_{ijk}\hat{\mathbf{x}}_i \partial_j \partial_k \alpha(\mathbf{x}) &= \epsilon_{ijk} \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\alpha(\mathbf{x})\hat{\mathbf{x}}_i \\
&= \left(\frac{\partial^2}{\partial x_2 \partial x_3} - \frac{\partial^2}{\partial x_3 \partial x_2} \right) \alpha(\mathbf{x})\hat{\mathbf{x}}_1 \\
&\quad + \left(\frac{\partial^2}{\partial x_3 \partial x_1} - \frac{\partial^2}{\partial x_1 \partial x_3} \right) \alpha(\mathbf{x})\hat{\mathbf{x}}_2 \\
&\quad + \left(\frac{\partial^2}{\partial x_1 \partial x_2} - \frac{\partial^2}{\partial x_2 \partial x_1} \right) \alpha(\mathbf{x})\hat{\mathbf{x}}_3 \\
&\equiv \mathbf{0}
\end{aligned} \tag{M.91}$$

We thus find that

$$\nabla \times [\nabla \alpha(\mathbf{x})] \equiv \mathbf{0} \tag{M.92}$$

for any arbitrary, well-behaved $\mathbb{R}^3$ scalar field $\alpha(\mathbf{x})$.

In 4D we note that for any well-behaved four-scalar field $\alpha(x^\kappa)$

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\alpha(x^\kappa) \equiv 0 \tag{M.93}$$

so that the four-curl of a four-gradient vanishes just as does a curl of a gradient in $\mathbb{R}^3$.

Hence, a gradient is always irrotational.

◁ END OF EXAMPLE 13.9

▷THE DIVERGENCE OF A CURL — EXAMPLE 13.10

With the use of the definitions of the divergence (M.83) and the curl, Equation (M.88) on the preceding page, we find that

$$\nabla \cdot [\nabla \times \mathbf{a}(\mathbf{x})] = \partial_i [\nabla \times \mathbf{a}(\mathbf{x})]_i = \epsilon_{ijk} \partial_i \partial_j a_k(\mathbf{x}) \tag{M.94}$$

Using the definition for the Levi-Civita symbol, defined by Equation (M.18) on page 167, we find that, due to the assumed well-behavedness of $\mathbf{a}(\mathbf{x})$,

$$\begin{aligned}
\partial_i \epsilon_{ijk} \partial_j a_k(\mathbf{x}) &= \frac{\partial}{\partial x_i} \epsilon_{ijk} \frac{\partial}{\partial x_j} a_k \\
&= \left(\frac{\partial^2}{\partial x_2 \partial x_3} - \frac{\partial^2}{\partial x_3 \partial x_2} \right) a_1(\mathbf{x}) \\
&\quad + \left(\frac{\partial^2}{\partial x_3 \partial x_1} - \frac{\partial^2}{\partial x_1 \partial x_3} \right) a_2(\mathbf{x}) \\
&\quad + \left(\frac{\partial^2}{\partial x_1 \partial x_2} - \frac{\partial^2}{\partial x_2 \partial x_1} \right) a_3(\mathbf{x}) \\
&\equiv 0
\end{aligned} \tag{M.95}$$

i.e., that

$$\nabla \cdot [\nabla \times \mathbf{a}(\mathbf{x})] \equiv 0 \tag{M.96}$$

for any arbitrary, well-behaved $\mathbb{R}^3$ vector field $\mathbf{a}(\mathbf{x})$.

In 4D, the four-divergence of the four-curl is *not* zero, for

$$\partial^\nu G_{\mu\nu} = \partial^\mu \partial_\nu a^\nu(x^\kappa) - \Box^2 a^\mu(x^\kappa) \neq 0 \tag{M.97}$$

◁ END OF EXAMPLE 13.10

Numerous vector algebra and vector analysis formulae are given in Chapter F. Those which are not found there can often be easily derived by using the component forms of the vectors and tensors, together with the Kronecker and Levi-Civita tensors and their generalisations to higher ranks. A short but very useful reference in this respect is the article by A. Evett [3].

M.2 Analytical mechanics

M.2.1 Lagrange's equations

As is well known from elementary analytical mechanics, the *Lagrange function* or *Lagrangian* L is given by

$$L(q_i, \dot{q}_i, t) = L\left(q_i, \frac{\mathrm{d}q_i}{\mathrm{d}t}, t\right) = T - V \tag{M.98}$$

where q_i is the *generalised coordinate*, T the *kinetic energy* and V the *potential energy* of a mechanical system, The Lagrangian satisfies the *Lagrange equations*

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{M.99}$$

To the generalised coordinate q_i one defines a *canonically conjugate momentum* p_i according to

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{M.100}$$

and note from Equation (M.99) above that

$$\frac{\partial L}{\partial q_i} = \dot{p}_i \tag{M.101}$$

M.2.2 Hamilton's equations

From L, the *Hamiltonian* (*Hamilton function*) H can be defined via the *Legendre transformation*

$$H(p_i, q_i, t) = p_i\dot{q}_i - L(q_i, \dot{q}_i, t) \tag{M.102}$$

After differentiating the left and right hand sides of this definition and setting them equal we obtain

$$\frac{\partial H}{\partial p_i}\mathrm{d}p_i + \frac{\partial H}{\partial q_i}\mathrm{d}q_i + \frac{\partial H}{\partial t}\mathrm{d}t = \dot{q}_i\mathrm{d}p_i + p_i\mathrm{d}\dot{q}_i - \frac{\partial L}{\partial q_i}\mathrm{d}q_i - \frac{\partial L}{\partial \dot{q}_i}\mathrm{d}\dot{q}_i - \frac{\partial L}{\partial t}\mathrm{d}t \tag{M.103}$$

According to the definition of p_i, Equation (M.100) on the facing page, the second and fourth terms on the right hand side cancel. Furthermore, noting that according to Equation (M.101) on the preceding page the third term on the right hand side of Equation (M.103) on the facing page is equal to $-\dot{p}_i \mathrm{d}q_i$ and identifying terms, we obtain the *Hamilton equations*:

$$\frac{\partial H}{\partial p_i} = \dot{q}_i = \frac{\mathrm{d}q_i}{\mathrm{d}t} \tag{M.104a}$$

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i = -\frac{\mathrm{d}p_i}{\mathrm{d}t} \tag{M.104b}$$

M.3 Bibliography

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