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Set Input to State Stability for Multi-Agent Systems

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ABSTRACT

In this paper, we investigate set Input to State Stability (set-ISS) in the context of multi-agent systems, specifically when agent interaction is spatial in nature. We review the definition of Input-to-State (ISS) Lyapunov functions with respect to sets, from which we provide a structural formulation of set-ISS that accommodates the local and per-agent nature of interacting systems. We argue that such a non-global characterization of set-ISS results in intuitive studies of multi-agent systems subject to external disturbances, resulting in superior understanding of collective and asymptotic behaviors. For the validation of our propositions, we consider a decentralized control law to reach swarm aggregation towards a bounded region of space. We demonstrate that our set-ISS structure, coupled with the requirement of physical occupancy in spatial interaction, connects with the classical notions of set-ISS, and yields both a fundamentally simplified analysis and superior insight into agent behavior compared to previous work.

1 Introduction

Multi-agent systems have become an important focus of the control community in recent work, particularly given the theoretical spectrum of their analysis and the widely varying implications of their application in real-world environments. Examples of general multi-agent analyses include investigations into consensus processes [15], formation control [5] and distributed decision and control frameworks [4] , with applications ranging from target tracking [13], to environmental monitoring [10, 24].

We are concerned in this work with multi-agent systems that are both mobile and spatially interacting. That is, systems whose collaboration and objectives, e.g. information exchange and collective behavior, is directly related to configuration in space. There is a large body of work that investigates such systems. Examples range from generalized swarming and constrained interaction [25], to coverage or flocking behaviors [2, 23]. When analyzing the collective behavior of systems that interact spatially, it is typical to apply swarm energy functions together with Laplacian techniques and standard Lyapunov methods to arrive at bounds on cohesiveness or system aggregation. However, few works (which we highlight here) treat multi-agent systems through input-to-state stability (ISS) or its set-based counterpart set Input to State Stability (set-ISS), our primary focus in this work.

It is generally known that ISS and set-ISS are powerful tools in the analysis of the stability and robustness of control systems. Seminal works on ISS and set-ISS include [12] which provides a converse Lyapunov theory for set stability, and [11, 20, 21] which introduce ISS, extend the notion to non-compact sets, and generalize to arbitrary closed invariant sets, respectively. In the context of multi-agent systems, [14] adapts ISS notions to networked control systems. Mobile systems are considered in [18], where the authors investigate set stability in guaranteeing global target aggregation when there exist informed leaders in the network, inducing a priori knowledge of the set of convergence. Similarly, [19] views set-ISS for set tracking when network leaders, possibly non-stationary, define the desired set of aggregation over time. However, the application of set-ISS in spatially interacting systems, particularly without leader influence, is noticeably sparse. Further, the methods of works such as [18, 19] are specialized to the application of interest, while providing notions of set stability, they deviate from classical Lyapunov-based set-ISS approaches, making generalization beyond leaderless systems cumbersome. We thus aim to provide an intuitive tool for studying per-agent behavior, abstracting from the stacked vector nature of the system.

Our proposition is then to adapt and formalize the structural components of works such as [18,19,21] to study purely collaborative systems, i.e. those without leader influence. We first show how the classical ISS and set-ISS notions of [11, 20, 21] can be structured with proper choice of a set-ISS Lyapunov function and local input-to-state bounds, to yield set stability with input disturbances for leaderless multi-agent systems. Then, we demonstrate how our construction can be applied in the context of spatial interaction, by considering constraints on physical occupancy in the workspace, yielding insights into the tradeoffs of disturbance, collective behaviors, and the parameters of interaction. Our results show that when agents spatially interact under realistic conditions, the application of set-ISS can be formulated, unlike previous works such as [18, 19], in the classical manner of [11, 20, 21], effectively exploiting the known conditions of the system to vastly simplify analysis. Further, this formulation gives us physical intuition in spatially interacting systems, that relates the parameters of interaction with input disturbance analysis. This relationship could also be extended to spatially interacting systems with competing objectives, specifically by treating such objectives as systematic disturbances. While our formulations share certain mathematical machinery with works such as [18,19], we provide a fundamentally different goal, as opposed to having the convergent set implicit (through leader information), we aim to provide useful tools for the set-ISS characterization of *fully* collaborative multi-agent systems.

2 Preliminaries

Consider a system of n mobile agents operating in $\mathbb{R}^d$ and denote with $x_i \in \mathbb{R}^d$ the location of the agent i. The aggregate state of the system is given by the stacked vector $\mathbf{x} = [x_1, \ldots, x_n]^T \in \mathbb{R}^{nd}$. Assume each agent i has the following dynamics:

$$\dot{x}_i = f_i(\mathbf{x}, u_i) \tag{1}$$

where $f_i : \mathbb{R}^{nd} \times \mathbb{R}^m \to \mathbb{R}^d$ is a continuous map locally Lipschitz on x, and the local agent state x_i and the local input u_i are function of time $t \in \mathbb{R}^+$, with values $x_i(t) \in \mathbb{R}^d$ and $u_i(t) \in \mathbb{R}^m$. In particular, an input u_i is a measurable locally essentially bounded function $u_i : \mathbb{R} \to \mathbb{R}^m$. The dynamics of the entire system can be then written as:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \tag{2}$$

with $\mathbf{f} = [f_i(\mathbf{x}, u_1), \ldots, f_i(\mathbf{x}, u_n)]^T$ the stacked vector of agent dynamics and $\mathbf{u} = [u_1, \ldots, u_n]^T$ the stacked vector of exogenous inputs.

The interactions of the agents can be described by $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}(t)\}$, the time-varying graph encoding the network topology of the multi-agent system at time t, with $\mathcal{V} = \{1, \ldots, n\}$ the set of vertexes modeling the agents, and $\mathcal{E}(t) = \{(i, j) \in \mathcal{V} \times \mathcal{V}\}$ the set of edges representing the inter-agents interaction at time t. Agents with $(i, j) \in \mathcal{E}$ are called *neighbors* and the neighbor set for an agent i is denoted $\mathcal{N}_i = \{j \in \mathcal{V} \,|\, (i, j) \in \mathcal{E}\}$

Assume that we wish to study convergence of the agents towards a nonempty set $\mathcal{A} \in \mathbb{R}^d$ and denote with $\|x\|_\mathcal{A}$ the point to set distance[1] defined as:

$$\|x\|_\mathcal{A} = \text{dist}(x, \mathcal{A}) = \inf_{a \in \mathcal{A}} \{\|x - a\|\}. \tag{3}$$

where $\|x - a\|$ is the Euclidean distance. Further, consider the following norm definition describing the upper bound on the input disturbance over time [22]:

$$\|u_i\|_\infty = \sup \{\|u_i(t)\|, t \geq 0\} \tag{4}$$

In studying convergence to $\mathcal{A}$, we will further require the following notions of so-called $\mathcal{K}$ *functions*. First, a function $\phi : [0, a) \to [0, \infty]$ is said to be positive if $\phi(s) > 0$ for all $s > 0$ and $\phi(0) = 0$. A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is positive, strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$. An example of a class $\mathcal{K}_\infty$ function is $\alpha(r) = r^c$ with

[1]Note that, the computation of the distance of a point to a set often requires one to solve a constrained optimization problem as such a distance does not generally admit a closed form. Indeed, a closed form for this distance exists only in special cases, e.g., the distance of a point to a ball.

$c > 0$. Similarly, the continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$. An example of a class $\mathcal{KL}$ function is $\beta(r, c) = r^c e^{-s}$ with $c > 0$.

As will be required in the sequel, consider a function $V : \mathbb{R}^{nd} \to \mathbb{R}$, a locally Lipschitz function, and the composed function $\psi(t) = V(x(t))$ with $\mathbf{x}$ the state of the system (2). Usually the function $\psi(t)$ is not differentiable, however the composition of a locally Lipschitz function V and an absolutely continuous function $\mathbf{x}(t)$ is also absolutely continuous, and thus it is differentiable almost everywhere. Therefore, we can consider the *upper right-hand Dini derivative* as:

$$D^+\psi(t) = \lim \sup_{h \to 0^+} \frac{\psi(t+h)) - \psi(t)}{h} \tag{5}$$

Interestingly, if an absolutely continuous function $\psi(t)$ is defined on and interval $[t_1, t_2]$, it has the right Dini derivative $D^+\psi(t)$ nonpositive almost everywhere, that it is non-increasing on such interval as in the case of differentiable functions.

Note that, given the system (2) the upper right-hand Dini derivative with respect to the composite function $V(x(t))$ can be expressed as:

$$D^+V(\mathbf{x}) = \lim \sup_{h \to 0^+} \frac{V(\mathbf{x} + h\, f(\mathbf{x})) - V(\mathbf{x})}{h} \tag{6}$$

Consider the following special case for which the Lyapunov derivative admits an explicit expression, since the directional derivative can be written in a simple way.

Lemma 1 ([3]) *Let $V_i : \mathbb{R}^d \to \mathbb{R}$ with $i \in 1, \ldots, n$ be continuously differentiable, $V_i \in \mathcal{C}^1$, and let $V = \max_{i=1,\ldots,n}\{V_i(x_i)\}$. Denote as $I(x) = \{i : V(\mathbf{x}) = V_i(x_i)\}$ the set of indices where the maximum is reached, then:*

$$D^+V = \max_{i \in I(x)} \left\{ \dot{V}_i(x_i) \right\}. \tag{7}$$

The above property will prove fundamental to determining the set-ISS of a multi-agent system, specifically in terms of viewing the system through worst-case behaviors instead of studying the entire aggregate system, noting that the worst behavior continues to imply information about the remainder of the system.

3 Set Input to State Stability

In studying the ISS and set-ISS properties of a system, the following generalized system is considered (noting that in this summary of concepts, n is not the agent cardinality, but a general state dimension):

$$\dot{x} = f(x, u) \tag{8}$$

where $x \in \mathbb{R}^n$ is the system state, $f : \mathbb{R}^n \times \mathcal{D} \to \mathbb{R}^n$ is a continuous map locally Lipschitz on x and $u \in \mathbb{R}^m$ is the input, i.e., a locally essentially bounded measurable function of time $u : \mathbb{R}^+ \to \mathbb{R}^m$. Denote with $x(t, x_0, u)$ the trajectory of the system (8) with initial state $x_0 = x(0)$ and input u.

Let us now introduce the concept of asymptotic gain. This generalizes the idea of finite (linear) gain, classically used in input/output stability theory [22].

Definition 1 *The system* (8) *is said to have* $\mathcal{K}$ *asymptotic gain with respect to a set* $\mathcal{A} \in \mathbb{R}^n$ *if there exists a class* $\mathcal{K}$ *function* γ *such that:*

$$\lim_{t\to\infty} \|x(t, x_0, u)\|_{\mathcal{A}} \le \gamma(\|u\|_\infty) \tag{9}$$

uniformly[2] *on subsets of the form* $\{x \in \mathbb{R}^n,\ \|x\|_{\mathcal{A}} \le r\}$ *with* $r \ge 0$ *and* u.

Let us now introduce the concept of set stability. To this end, consider the "zero-input" or "undisturbed" system:

$$\dot{x} = f_0(x) = f(x, 0). \tag{10}$$

that is, system (8) under the assumption $u(t) = 0$, $\forall\, t \in \mathbb{R}^+$. The reader is referred to [12] for a more comprehensive description of the topic.[3]

Definition 2 *The system* (8) *is said to be globally asymptotically stable with respect to a closed set* $\mathcal{A} \subseteq \mathbb{R}^n$ *if and only if there exists a class* $\mathcal{KL}$ *function* β *such that, given any initial state* x_0*, the solution* $x(t, x_0, 0)$ *satisfies:*

$$\|x(t, x_0, 0)\|_{\mathcal{A}} \le \beta(\|x_0\|_{\mathcal{A}}, t), \qquad \forall\, t \ge 0. \tag{11}$$

An intuitive approach to study the stability of a system with respect to a set is by introducing the concept of Lyapunov function with respect to sets.

Definition 3 *A Lyapunov function for the system* (10) *with respect to a nonempty set* $\mathcal{A} \in \mathbb{R}^n$ *is a function* $V : \mathbb{R}^n \to \mathbb{R}^+$ *such that there exist* $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$[4] *over* $\mathbb{R}^n \setminus \mathcal{A}$ *where:*

$$-\alpha_1(\|x\|_{\mathcal{A}}) \le V(x) \le -\alpha_2(\|x\|_{\mathcal{A}}) \tag{12}$$

and

$$\nabla V(x) f(x, 0) \le -\alpha_3(\|x\|_{\mathcal{A}}) \tag{13}$$

for all $x \in \mathbb{R}^n$. *Then the system* (10) *is globally asymptotically stable with respect to the set* $\mathcal{A}$.

Note that, according to Definition 2 the set $\mathcal{A} \subset \mathbb{R}^n$ is said to be 0-invariant set for the system (10), as by definition any solution starting in $\mathcal{A}$ is defined for all $t \ge 0$ and stays in $\mathcal{A}$: $x(t, x_0, 0) \in \mathcal{A}$, $\forall t \ge 0$, $\forall\, x_0 \in \mathcal{A}$.

Let us now introduce the concept of set input-to-state (set ISS) stability.

Definition 4 *The system* (8) *is said to the set input-to-state stable with respect to a closed,* 0*-invariant set* $\mathcal{A} \subseteq \mathbb{R}^n$ *if and only if there exist a class* $\mathcal{KL}$ *function* β *and a class* $\mathcal{K}$ *function* γ *such that:*

$$\|x(t, x_0, u)\|_{\mathcal{A}} \le \beta(\|x_0\|_{\mathcal{A}}, t) + \gamma(\|u\|_\infty), \qquad \forall\, t \ge 0. \tag{14}$$

[2] As pointed out in [22], the uniformity requirement means that solutions exist for all initial states, controls, and times and for each real numbers r, ϵ, there is some $T = T(r, \epsilon)$ so that $\|x(t, x_0, u)\|_{\mathcal{A}} \le \epsilon + \gamma(\|u\|_\infty)$ for all u, all $\|x\|_{\mathcal{A}} \le r$. and all $t \ge T$.

[3] Note that, in this work for the sake of simplicity we do not mention the forward completeness assumption. As pointed out in [12] this is redundant for compact sets as (11) implies the boundedness of the solution. In addition, we assume parameters to be constant over time.

[4] An alternative definition would be to consider a continuous, positive definite function α_3 rather than $\alpha_3 \in \mathcal{K}_\infty$. However, as pointed out in [11] these two variants are equivalent, as it is always possible to build an ISS Lyapunov from the positive definite function for which a function $\alpha_3 \in \mathcal{K}_\infty$ exists.

which can be equivalently expressed as:

$$\lim_{t\to\infty} \|x(t, x_0, u)\|_{\mathcal{A}} \leq \gamma(\|u\|_\infty) \tag{15}$$

characterizing the ultimate asymptotic boundedness of the system with respect to the input disturbance, the representation we adopt in this work.

Note that, any function γ for which (14) holds (for some β) is an ISS-gain for the system (with respect to $\mathcal{A}$) such a γ is in particular an asymptotic gain.

A suitable approach to study the set ISS property of a system is by introducing the concept of ISS Lyapunov function with respect to a set $\mathcal{A} \subset \mathbb{R}^n$.

Definition 5 *A Lyapunov function for the system* (8) *with respect to a nonempty set* $\mathcal{A} \in \mathbb{R}^n$ *is a function* $V : \mathbb{R}^n \to \mathbb{R}^+$ *such that there exists* $\alpha_1, \alpha_2, \alpha_3, \chi \in \mathcal{K}_\infty$ *over* $\mathbb{R}^n \setminus \mathcal{A}$ *where:*

$$-\alpha_1(\|x\|_{\mathcal{A}}) \leq V(x) \leq -\alpha_2(\|x\|_{\mathcal{A}}) \tag{16}$$

and

$$\nabla V(x) f(x) \leq -\alpha_3(\|x\|_{\mathcal{A}}), \qquad \|x\|_{\mathcal{A}} > \chi(\|u\|) \tag{17}$$

for all $x \in \mathbb{R}^n$ *and all* $u \in \mathbb{R}^m$*. Then the system is set ISS with asymptotic gain* χ*.*

Remark 1 *The above* converse Lyapunov theorem *exists in generally equivalent forms where there are characterizations ranging from* V *being smooth [12], to Lipschitz [8], or even locally Lipschitz [16].*

4 Set ISS for Multi-Agent Systems

In this section, the set input-to-state stability results previously recalled are considered in the context of multi-agent systems. The objective is to structure the definition of the set ISS Lyapunov function in order to provide a useful tool to succinctly prove set stability and robustness against disturbances for typical applications in multi-agent systems such as swarm aggregation, as will be seen in Section 5. To begin let us provide a local variant of the classical set-ISS definition for each agent $i \in \mathcal{V}$:

$$\lim_{t\to\infty} \|x_i\|_{\mathcal{A}} \leq \max_{i\in\mathcal{V}}\{\chi(\|u_i\|_\infty)\}, \qquad \forall\, i \in \mathcal{V}. \tag{18}$$

with local[5] asymptotic gain χ.

We are now ready to state our main result, that is a characterization of the classical set-ISS Lyapunov theorem in the context of multi-agent systems.

Theorem 1 *Consider a multi-agent system where each agent* $i \in \mathcal{V}$ *has the dynamics given in* (1) *and assume the local Lyapunov function to be defined as* $V_i = \frac{1}{2}\|x_i\|^2_{\mathcal{A}}$*. Assume there exist* $\chi \in \mathcal{KL}$ *and* $\alpha_3 \in \mathcal{K}_\infty$ *such that:*

$$\dot{V}_i \leq -\alpha_3(\|x_i\|_{\mathcal{A}}), \quad \|x_i\|_{\mathcal{A}} \geq \chi(\|u_i\|_\infty), \quad i \in I(x) \tag{19}$$

[5]Note that we referred to χ as the local asymptotic gain rather than the $\max_{i\in\mathcal{V}}(\chi)$ itself to emphasize the fact that the agents share the same *structure* for the asymptotic bound on the state trajectory.

where we define the set of agents that reach the maximal distance from set $\mathcal{A}$ at time t,

$$I(x) = \underset{i \in \mathcal{V}}{\operatorname{argmax}} \left\{ \sqrt{V_i(t)} \right\}, \tag{20}$$

providing a worst-case characterization of system behavior[6]*. Then, the dynamics (1) is (locally) set ISS with respect to $\mathcal{A}$ with (local) asymptotic gain χ for each agent i.*

Proof: For brevity we highlight the most vital aspects of the relationship between our locally oriented structure and the general global result of Definition 5. Consider the following global metric:

$$\|x\|_{\mathcal{A}^n} = \max_{i \in \mathcal{V}} \{\|x_i\|_{\mathcal{A}}\} \tag{21}$$

and note that by choosing:

$$\alpha_1(\|x\|_{\mathcal{A}^n}) = \alpha_2(\|x\|_{\mathcal{A}^n}) = \frac{1}{2} \max_{i \in \mathcal{V}} \left\{\|x_i\|_{\mathcal{A}}^2\right\} \tag{22}$$

we have $\alpha_1(\|x\|_{\mathcal{A}^n}) = \alpha_2(\|x\|_{\mathcal{A}^n}) = V$, with the cartesian product $\mathcal{A}^n = \mathcal{A} \times \ldots \times \mathcal{A}$.

According to Lemma 1, it follows that the upper right Dini derivative can be computed as:

$$D^+V = \max_{i \in I(i)} \left\{\dot{V}_i\right\} \tag{23}$$

Furthermore, from (19) we obtain:

$$D^+V \leq -\alpha_3(\|x\|_{\mathcal{A}^n}), \quad \|x\|_{\mathcal{A}^n} \geq \chi(\|u\|_\infty) \tag{24}$$

which comes from the fact that:

$$\|x\|_{\mathcal{A}^n} \geq \chi(\|u\|_\infty) \quad \Longrightarrow \quad \|x_i\|_{\mathcal{A}} \geq \chi(\|u_i\|_\infty),\ i \in I(x) \tag{25}$$

which in turns implies that:

$$\dot{V}_i \leq -\alpha_3(\|x_i\|_{\mathcal{A}}),\ i \in I(x) \Longrightarrow D^+V \leq -\alpha_3(\|x\|_{\mathcal{A}^n}). \tag{26}$$

Therefore, it can be proven[7] that there exists a time instant $T > 0$ such that:

$$V(x(t)) \leq \alpha_2(\chi(\|u\|_\infty)), \quad \forall t \geq T, \tag{27}$$

from which we obtain:

$$\begin{aligned} \|x(t)\|_{\mathcal{A}^n} &\leq \alpha_1^{-1}(\alpha_2(\chi(\|u\|_\infty))) \\ &\leq \chi(\|u\|_\infty), \quad \forall t \geq T. \end{aligned} \tag{28}$$

At this point by recalling the norm definition given in (4) it follows:

$$\|x_i(t)\|_{\mathcal{A}} \leq \max_{i \in \mathcal{V}} \{\chi(\|u_i\|_\infty)\}, \quad \forall t \geq T \tag{29}$$

where it should be noticed that by construction:

$$\max_{i \in \mathcal{V}} \{\chi(\|u_i\|_\infty)\} \triangleq \chi\{\|u\|_\infty\}, \tag{30}$$

yielding the asymptotic trajectory boundedness given in (18).

[6] As opposed to a fully global viewpoint, which views all agent behavior in the aggregate.

[7] Proof is omitted due to space. See [9] and [1] for relevant methodologies.

Remark 2 *Let us summarize the implications of Theorem 1:*

- *It should be noticed that the connection of our localized formulation and the classical notions of set-ISS lies in the fact that the ISS gain is taken over the worst-case behavior of agents in the system (i.e.* $\max_{i\in\mathcal{V}}$*). Further, the fundamental coupling of dynamics in interacting systems renders such a construction intuitive, as one cannot hope to fully decouple agent behavior.*

- *For the proposed ISS Lyapunov function the* $\mathcal{K}$ *asymptotic gain is derived with respect to a single agent rather than the overall system. This is not a surprising result as in the context of multi-agent systems while referring to the disturbance affecting the system and preventing its convergence towards the set* $\mathcal{A}^n$*, we normally refer to the disturbance which prevents the convergence of each agent towards the set* $\mathcal{A}$*. In the next section where we consider as a case study the swarm aggregation of a multi-agent system towards a bounded region, i.e. the set* $\mathcal{A}$*, this concept will result clear.*

- *As pointed out in [11], the ISS Lyapunov function given in Definition 5 can be used to study the global asymptotic stability of the "zero-input" system* (10)*. Indeed, the same holds for the ISS Lyapunov function we propose in Theorem 1, in studying multi-agent systems with disturbance.*

4.1 Spatial Interaction and Physical Occupancy

Given the typical structure of collaborative multi-agent systems, that is:

$$\dot{x}_i = \sum_{j\in\mathcal{N}_i} w_{ij}(x_i, x_j)(x_j - x_i) \tag{31}$$

with time varying weight w_{ij}, in applying Theorem 1, we must generally evaluate the boundedness of the inner product $x_i^T(x_j - x_i)$, specifically in determining condition (19). Thus, we provide a geometric argument concerning the relationship between agents lying at a maximal distance from $\mathcal{A} \in \mathbb{R}^d$, e.g. $i \in I(x)$, and other agents in the system. In particular, the following result forms the basis of our physical intuition that relates set-ISS to spatially interacting systems:

Lemma 2 *Consider a pair of robots* (i, j) *with* i, $j \in \mathcal{V}$ *such that* $\|x_i\| \geq \|x_j\|$ *and assume each robot has a physical occupancy of* ϵ*, i.e.,* $\|x_i - x_j\| \geq 2\,\epsilon$*. For the sake of clarity, assume* $d = 2$*, although the concepts generalize for any* $d \in [1, 3]$*. It follows that:*

$$x_i^T\,(x_j - x_i) \leq -\kappa\,\|x_i\|^2 \tag{32}$$

with $\kappa \in \mathbb{R}^+$ *defined as:*

$$\kappa = 1 - \cos\left(2\,\sin^{-1}\left(\frac{\epsilon}{\mathcal{W}}\right)\right), \tag{33}$$

with $\mathcal{W}$ *a Euclidean measure of workspace boundedness.*

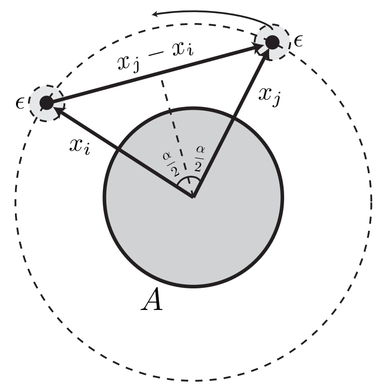

Figure 1: An illustration of the basic geometric concepts of Lemma 2, yielding a bound on the inner product $x_i^T(x_j - x_i)$ when $\|x_i\| \geq \|x_j\|, \forall\, i, j \in \mathcal{V}$. As agents i and j approach minimal occupancy $\|x_i - x_j\| = 2\epsilon$, the angle $\alpha \to 2\,\sin^{-1}(\epsilon/\|x_i\|)$, ultimately yielding a bound on $x_i^T(x_j - x_i)$.

Proof: In order to prove the lemma we provide a geometrical reasoning based on the physical occupancy assumption. In particular, we know that the inner product $x_i^T\,(x_j - x_i)$ can be written as $\|x_i\|^2 - \|x_i\|\,\|x_j\|\,\cos(\alpha)$. Since we are looking for a lower bound it follows that the worst case scenario is given by the case $\|x_i\| = \|x_j\|$ for which the following holds:

$$\|x_i\|^2 - \|x_i\|\,\|x_j\|\,\cos(\alpha) = \|x_i\|^2(1 - \cos(\alpha)).$$

Then, in order to compute the lower bound we need to compute the angle α for which the cosine is maximum according to the physical occupancy assumption. In particular for the case $\|x_i\| = \|x_j\|$, the minimum admissible angle α is given by the configuration for which the robots are at distance $2\,\epsilon$ from each other over a circle of radius $r = \|x_i\| = \mathcal{W}$. Therefore, by using the sine law we obtain $\alpha = 2\,\sin^{-1}\left(\frac{\epsilon}{\mathcal{W}}\right)$, and the thesis follows. See Figure 1 for a graphical representation of this argument in a 2-dimensional case. In other dimensions there will exist a κ, while not necessarily possessing the form (33), it *will* continue to be a function of the ratio $\epsilon/\mathcal{W}$.

Remark 3 *It is important to notice that κ is a function of the ratio of physical occupancy ϵ and the workspace scale $\mathcal{W}$. As we will see, κ is instrumental in the ISS gain of our system, and thus plays a key role in the tradeoff between the quality of asymptotic trajectory bounds, the magnitude of input disturbances, and the parameters of agent interaction.*

Now, assuming that a spatially interacting system meets the standards of Lemma 2, and particularly the intuition of Remark 3, we can intuitively evaluate the set-ISS properties of the system by applying Theorem 1. In the sequel we provide an illustrative example of our propositions.

5 Spatially Interacting Systems: An Illustrative Example

In this section we consider the swarm aggregation behavior originally proposed in [7] and successively extended for a distributed context in [6] as an illustrative example of Theorem

1 and Lemma 2. In particular, the following interaction modeling is considered for each agent i:

$$f_i(\mathbf{x}) = \sum_{j \in \mathcal{N}_i(t)} g(x_i - x_j) + u_i \tag{34}$$

where the interaction function $g(y)$ is defined as:

$$g(y) = -y \left[g_a(\|y\|) - g_r(\|y\|) \right], \quad y \in \mathbb{R}^d, \tag{35}$$

with the aggregative and repulsive terms defined as:

$$g_a(\|y\|) = a, \qquad g_r(\|y\|) = \frac{b}{\|y\|} \tag{36}$$

Briefly speaking, the following main results were proven in [7], [6], [17] for (34), when $u_i = 0$, $\forall\, i \in \mathcal{V}$:

- The swarm eventually reaches a steady state configuration.
- The swarm converges to a bounded region defined as:

  $$\tilde{\mathcal{A}} = \{x \in \mathbb{R}^{nd},\ \|x\| \le \frac{\bar{b}}{\bar{a}}\} \tag{37}$$

  with $\bar{a} = a\,\lambda_{2,\min}$ and $\bar{b} = \sqrt{n}\,(n-1)\,b$ where $\lambda_{2,\min}$ is the lower bound for the algebraic connectivity[8]. It is worth noting that these works derive localized swarm bounds per-agent (as we do in applying Theorem 1), however they do so by exploiting the relationship between the euclidean norm of the stacked vector, and that of a single agent, yielding looser bounds than we will demonstrate. Further, no effort is made to characterize input disturbances in these previous works.

Using the results of Theorem 1, we will now demonstrate an analysis of system (34) from the context of set-ISS. Considering now the system (34), we are interested in investigating how for each agent i the additive disturbance u_i might affect the convergence of each agent i towards the set $\mathcal{A}$. For the sake of the analysis, the following per agent coordinate transformation $e_i(t) = x_i(t) - \bar{x}$ is considered[9], i.e., the distance of the agent i from the instantaneous barycenter $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i(0)$. It follows that the dynamics of agent i given in (34) can be written as:

$$\dot{e}_i = f_i(\mathbf{e}) + u_i. \tag{38}$$

Let $\|e\|_{\mathcal{A}}$ be the point to set distance defined as:

$$\|e\|_{\mathcal{A}} = \begin{cases} \|e\| - r & \text{if } e \in \mathcal{R}^d \setminus \mathcal{A} \\ 0 & \text{otherwise} \end{cases}. \tag{39}$$

[8] A connectivity maintenance control law is required to enforce this property. The reader is referred to [17] and references therein for a comprehensive overview of this topic.

[9] Strictly speaking, this coordinates transformation makes the analysis independent from any particular choice of the reference frame as only relative distances are considered.

where the parameter r is the unknown radius of the hyperball of convergence to be determined.

According to Theorem 1 we consider the Lyapunov function $V(t) = \max_{i\in\mathcal{V}}\{V_i(x_i)\}$ where each $V_i : \mathbb{R}^d \to \mathbb{R}$ has the following structure:

$$V_i = \frac{1}{2}\|e_i\|_{\mathcal{A}}^2, \tag{40}$$

for which the time derivative is defined as:

$$\begin{aligned}\dot{V}_i &= \nabla_{\mathbf{e}}\left(\frac{1}{2}\|e_i(t)\|_{\mathcal{A}}^2\right)\dot{\mathbf{e}}(t)\\ &= \|e_i(t)\|_{\mathcal{A}}\,\nabla_{\mathbf{e}}\|e_i(t)\|_{\mathcal{A}}\,\dot{\mathbf{e}}(t)\end{aligned} \tag{41}$$

where the gradient $\nabla_{\mathbf{e}}\|e_i(t)\|_{\mathcal{A}}$ for any $\mathbf{e}(t) \in \mathcal{R}^{nd} \setminus \mathcal{A}^n$ is defined as:

$$\begin{aligned}\nabla_{\mathbf{e}}\|e_i(t)\|_{\mathcal{A}} &= \nabla_{\mathbf{e}}\left[\|e_i(t)\| - r\right] = \nabla_{\mathbf{e}}\|e_i(t)\|\\ &= \left[\frac{\partial\|e_i(t)\|}{\partial e_1}, \dots, \frac{\partial\|e_i(t)\|}{\partial e_i}, \dots, \frac{\partial\|e_i(t)\|}{\partial e_n}\right].\\ &= \left[0, \dots, \frac{e_i(t)}{\|e_i(t)\|}, \dots, 0\right]^T\end{aligned} \tag{42}$$

Therefore, by plugging (42) into (41) we obtain:

$$\dot{V}_i = \frac{\|e_i(t)\|_{\mathcal{A}}}{\|e_i(t)\|}e_i(t)^T\,\dot{e}_i(t) \tag{43}$$

Let us now assume the i-th robot to be such that $i \in I(x)$. In particular, the $\dot{V}_i$ can be further detailed as follows by plugging (38) into (43):

$$\begin{aligned}\dot{V}_i &= \frac{\|e_i(t)\|_{\mathcal{A}}}{\|e_i(t)\|}e_i(t)^T\,\left[f_i(\mathbf{e}) + u_i\right]\\ &= \underbrace{\frac{\|e_i(t)\|_{\mathcal{A}}}{\|e_i(t)\|}e_i(t)^T\,f_i(\mathbf{e})}_{\dot{V}_i^s} + \underbrace{\frac{\|e_i(t)\|_{\mathcal{A}}}{\|e_i(t)\|}e_i(t)^T\,u_i}_{\dot{V}_i^d}\end{aligned} \tag{44}$$

where we have partitioned $\dot{V}_i$ into systematic contribution $\dot{V}_i^s$ and disturbance $\dot{V}_i^d$. We first characterize $\dot{V}_i^s$ as follows:

$$\begin{aligned}\dot{V}_i^s &= \frac{\|e_i(t)\|_{\mathcal{A}}}{\|e_i(t)\|}e_i(t)^T\,f_i(\mathbf{e})\\ &= \frac{\|e_i(t)\|_{\mathcal{A}}}{\|e_i(t)\|}e_i(t)^T\sum_{j\in\mathcal{N}_i}\left[-(e_j - e_i)\,\left[a - \frac{b}{\|e_j - e_i\|}\right]\right]\\ &\le \frac{\|e_i(t)\|_{\mathcal{A}}}{\|e_i(t)\|}\Big[-|\mathcal{N}_i|\,\|e_i\|\big[-a\,\kappa\,\|e_i\| + b\big]\Big]\\ &\le \frac{\|e_i(t)\|_{\mathcal{A}}}{\|e_i(t)\|}\Big[-|\mathcal{N}_i|\,\|e_i\|\big[-a\,\kappa\,\|e_i\|_{\mathcal{A}}\|\big]\Big]\\ &\le -a\,\kappa\,\|e_i\|_{\mathcal{A}}^2\end{aligned} \tag{45}$$

where the fact $r = \frac{b}{\kappa a}$ and $|\mathcal{N}_i| \geq 1$ (by trivially assuming connectivity over time) has been used. Further, in bounding the inner product over the neighborhood $\mathcal{N}_i$, we have applied Lemma 2, as it must hold for $i \in \mathcal{I}(\mathcal{V})$, that $\|x_i\| \geq \|x_j\|$. Finally, as required for condition (19), the connectedness assumption ensures that there cannot exist an $i \in I(x)$ for which $\dot{V}_i = 0$. Note that, this provides a set stability characterization of the system, that is when zeroing the input.

Let us now simply characterize $\dot{V}_i^d$ through application of the Cauchy-Swartz inequality as follows:

$$\begin{aligned} \dot{V}_i^d &= \frac{\|e_i(t)\|_{\mathcal{A}}}{\|e_i(t)\|} e_i(t)^T u_i \\ &\leq \|e_i(t)\|_{\mathcal{A}} \|u_i\|_\infty \end{aligned} \tag{46}$$

At this point, by plugging (45) and (46) into (44) we have:

$$\begin{aligned} \dot{V}_i &= \dot{V}_i^s + \dot{V}_i^d \\ &\leq -a\,\kappa\,\|e_i\|_{\mathcal{A}}^2 + \|e_i(t)\|_{\mathcal{A}} \|u_i\|_\infty \\ &\leq -a\,\kappa\,(1-\theta)\|e_i\|_{\mathcal{A}}^2 \end{aligned} \tag{47}$$

with $0 < \theta < 1$, where the last inequality holds if:

$$\|e_i\|_{\mathcal{A}} > \frac{\|u_i\|_\infty}{\theta\,a\,\kappa} \tag{48}$$

Therefore, we can apply Theorem 1 with the following functions:

$$\begin{aligned} \alpha_3(\|x_i\|_{\mathcal{A}}) &= a\,\kappa\,(1-\theta)\|e_i\|_{\mathcal{A}}^2 \\ \chi(\|u_i\|_\infty) &= \frac{\|u_i\|_\infty}{\theta\,a\,\kappa} \end{aligned} \tag{49}$$

This implies that the system (1) is locally set ISS with respect to the set $\mathcal{A}$ defined as

$$\mathcal{A} = \left\{ x \in \mathbb{R}^d \,:\, \|x\| \leq r \right\}, \quad \text{with} \quad r = \frac{b}{a\,\kappa} \tag{50}$$

with local asymptotic gain $\chi(\|u_i\|_\infty) = \frac{\|u_i\|_\infty}{\theta\,a\,\kappa}$. Thus ensuring the boundedness condition given in (18) or equivalently the boundedness condition given in (15) for the stacked vector system with respect to the cartisian product $\mathcal{A}^n = \mathcal{A} \times \ldots \times \mathcal{A}$.

Remark 4 *We reiterate the role that our physical intuition plays in adapting set-ISS concepts to multi-agent systems with spatial interaction. In particular, contrary to the common notion that the difficulty of analysis scales with the fidelity of system modeling, our case study illustrates that by leveraging the* requirements *(not assumptions) of the system, we can arrive at simplified and clear analysis, and fundamentally better results.*

6 Conclusion

In this work, the set Input to State stability framework in the context of multi-agent systems has been investigated. Motivated by the typical analysis carried out in the context of multi-agent system to prove for instance formation control, or swarm aggregation or containment, we have reviewed the definition of ISS Lyapunov function with respect to sets in order to provide a handy tool to effectively prove set stability and robustness to external disturbances for typical applications. The swarm aggregation problem has been considered as a case study to prove the effectiveness of the proposed revised definition of ISS Lyapunov function. Future work will be focused on abstracting our results to studying the interplay of competing objectives in multi-agent systems.

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