COMPOSITIONAL QUANTUM LOGIC

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ABSTRACT. Quantum logic aims to capture essential quantum mechanical structure in order-theoretic terms. The Achilles' heel of quantum logic is the absence of a canonical description of composite systems, given descriptions of their components. We introduce a framework in which order-theoretic structure comes with a primitive composition operation. The order is extracted from a generalisation of C*-algebra that applies to arbitrary dagger symmetric monoidal categories, which also provide the composition operation. In fact, our construction is entirely compositional, without any additional assumptions on limits or enrichment. Interpreted in the category of finite-dimensional Hilbert spaces, it yields the projection lattices of arbitrary finite-dimensional C*-algebras. Interestingly, there are models that falsify standardly assumed correspondences, most notably the correspondence between noncommutativity of the algebra and nondistributivity of the order.

1. INTRODUCTION

In 1936, Birkhoff and von Neumann questioned whether the full Hilbert space structure was needed to capture the essence of quantum mechanics [3]. They argued that the order-theoretic structure of the closed subspaces of state space, or equivalently, of the projections of the operator algebra of observables, may already tell the entire story. To be more precise, we need to consider an order together with an order-reversing involution on it, a so-called *orthocomplementation*, which can also be cast as an *orthogonality relation*. Support along those lines comes from Gleason's theorem [21], which characterises the Born rule in terms of order-theoretic structure. In turn, via Wigner's theorem [43], this fixes unitarity of the dynamics.

These developments prompted Mackey to formulate his programme for the mathematical foundations of quantum mechanics: the reconstruction of Hilbert space from operationally meaningful axioms on an order-theoretic structure [31]. In 1964, Piron "almost" completed that programme for the infinite-dimensional case [34, 35]. Full completion was achieved much more recently, by Solèr in 1995 [40].[1]

Birkhoff and von Neumann coined the term 'quantum logic', in light of the developments in algebraic logic which were also subject to an order-theoretic paradigm. In particular they observed that the distributive law for meets and joins, which is key to the deduction theorem in classical logic, fails to hold for the lattice of closed subspaces for a Hilbert space [3].

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[1]See also the survey [41], which provides a comprehensive overview of the entire reconstruction, drawing from the fundamental theorem of projective geometry. Reconstructions of quantum theory have recently seen a great revival [24, 6]. In contrast to the Piron-Solèr theorem, this more recent work is mainly restricted to the finite-dimensional case, and focuses on operational axioms concerning how (multiple) quantum and classical systems interact.

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This failure of distributivity and hence the absence of a deduction theorem resulted in rejection of the quantum 'logic' idea by a majority of logicians. However, while the name quantum logic was retained, many of its researchers also rejected the direct link to logic, and simply saw quantum logic as the study of the order-theoretic structure associated to quantum phenomena, as well as other structural paradigms that were proposed thereafter [20, 30].

The quantum logic paradigm. In the Mackey-Piron-Solèr reconstruction, the elements of the partially ordered set become the projections on the resulting Hilbert space, that is, the self-adjoint idempotents of the algebra of operators on the Hilbert space:

$$(1) \qquad\qquad p \circ p = p, \qquad\qquad p^\dagger = p.$$

Conversely, the ordering can be recovered from the composition structure on these projections:

$$(2) \qquad\qquad p \leq q \iff p \circ q = p,$$

and the orthogonality relation can be recovered from it, too:

$$(3) \qquad\qquad p \perp q \iff p \circ q = 0.$$

In fact, the reconstruction does not produce Hilbert space, but Hilbert space with superselection rules. That is, depending on the particular nature of the ordering that we start with, it either produces quantum theory or classical theory, or combinations thereof.

The presence of "quantumness" is famously heralded in order-theoretic terms by the failure of the distributive law, giving rise to the following comparison.

$$\frac{\text{classical}}{\text{quantum}} \simeq \frac{\text{distributive}}{\text{nondistributive}}$$

This translates as follows to the level of operator algebra.

$$\frac{\text{classical}}{\text{quantum}} \simeq \frac{\text{commutative}}{\text{noncommutative}}$$

Thus, the combination yields the following slogan.

$$(4) \qquad \frac{\text{distributive}}{\text{nondistributive}} \simeq \frac{\text{commutative}}{\text{noncommutative}}$$

This is indeed the case for the projection lattices of arbitrary von Neumann algebras: the projection lattice is distributive if and only if the algebra is commutative [36, Proposition 4.16], and has been a guiding thought within the quantum structures research community.

Categorical quantum mechanics. More recently, drawing on modern developments in logic and computer science, and mainly a branch called type-theory, Abramsky and Coecke introduced a radically different approach to quantum structures that has gained prominence, which takes *compositional structure* as the starting point [1]. Proof-of-concept was provided by the fact that many quantum information protocols which crucially rely on the description of compound quantum systems could be very succinctly derived at a high level of abstraction.

In what is now known as categorical quantum mechanics, composition of systems is treated as a primitive connective, typically as a so-called *dagger symmetric monoidal category*. Additional axioms may then be imposed on such categories to

capture the particular nature of quantum compoundness. In other words, a set of equations that axiomatise the Hilbert space tensor products is generalised to a broad range of theories. Importantly, at no point is an underlying vector-space like structure assumed.

In contrast to quantum logic, this approach led to an abstract language with high expressive power, that enabled one to address concretely posed problems in the area of quantum computing (see e.g. [4, 12, 18, 27]) and quantum foundations (see e.g. [9]), and that has even led to interesting connections between quantum structures and the structure of natural language [16, 8].

One of the key insights of this approach is the fact that many notions that are primitive in Hilbert space theory, and hence quantum theory, can actually be recovered in compositional terms. For example, given the pure operations of a theory, one can define mixed operations in purely compositional terms, which together give rise to a new dagger symmetric monoidal category [37]. We will refer to this construction, as (Selinger's) *CPM–construction*. While this construction applies to arbitrary dagger symmetric monoidal categories (as shown in [7, 10]), Selinger also assumed *compactness* [29], something that we will also do in this paper. These structures are called *dagger compact categories*.

Another example, also crucial to this paper, is the fact that orthonormal bases can be expressed purely in terms of certain so-called *dagger Frobenius algebras*, which only rely on dagger symmetric monoidal structure [15, 2]. In turn, these dagger Frobenius algebras enable one to define derived concepts such as stochastic maps. All of this still occurs within the language of dagger symmetric monoidal categories [14]. We will refer to this construction as the *Stoch–construction*. Similarly, finite-dimensional C*-algebras can also be realised as certain dagger Frobenius algebras, internal to the dagger compact category of finite-dimensional Hilbert spaces and linear maps, the tensor product, and the linear algebraic adjoint [42].

Recently [11], the authors proposed a construction, called the *CP*–construction*, that generalises this correspondence to certain dagger Frobenius algebras in arbitrary dagger compact caterories. At the same time, this construction unifies the CPM–construction and the Stoch-construction, starting from a given dagger compact category. The resulting structure is an abstract approach to classical-quantum interaction, with Selinger's CPM–fragment playing the role of the "purely quantum", and the abstract stochastic maps fragment playing the role of the "purely classical".[2]

Overview of this paper. In this paper, we take this framework of "generalised C*-algebras" as a starting point, and investigate the structure of the dagger idempotents. We will refer to these as in short as projections too, since these dagger idempotents provide the abstract counterpart to projections of concrete C*-algebras.

We show that, just as in the concrete case, one always obtains a partially ordered set with an orthogonality relation. However, equation (4) breaks down in general. More specifically, in the dagger compact category of sets and relations with the Cartesian product as tensor and the relational converse as the dagger, there are commutative algebras with nondistributive projection lattices.

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[2]There is an earlier unification of the CPM–construction and the Stoch-construction [38], into which our construction faithfully embeds, see [11]. However, this construction does not support the interpretation of "generalised C*-algebras" [11].

As mentioned above, the upshot of our approach is that it resolves a problem that rendered quantum logic useless for modern purposes: providing an order structure representing compound systems at an abstract level, given the ones describing the component systems. Since we start with a category with monoidal structure, of course composition for objects is built in from the start, and it canonically lifts to algebras thereon. Let us emphasise that our framework relies solely on dagger categorical and compositional structure: the (sequential) composition of morphisms, and the (parallel) tensor product of morphisms. This is a key improvement over previous work [22, 26, 23, 28] that combines order-theoretic and compositional structure.[3]

2. Background

For background on symmetric monoidal categories we refer to the existing literature on the subject [13]. In particular we will rely on their graphical representations, which are surveyed in [39].

Diagrams will be read from bottom to top. Wires represent the objects of the category, while boxes or dots or any other entity with incoming and outcoming wires – possibly none – represents a morphism, and their type is determined by the respective number of incoming and outgoing wires. The directions of arrows on wires represent *duals* of the compact structure.

Our main objects of study are symmetric Frobenius algebras, defined as follows. Let us emphasise that this is a larger class of Frobenius algebra than just the commutative ones, which previous works on categorical quantum mechanics have mainly considered.

Definition 2.1. Let $(\mathbf{C}, \otimes, I)$ be a symmetric monoidal category which carries a *dagger structure*, that is, an identity-on-objects contravariant involutive endofunctor $\dagger : \mathbf{C}^{op} \to \mathbf{C}$. A *Frobenius algebra* in $\mathbf{C}$ is an object A of $\mathbf{C}$ together with morphisms

$$\text{⬆} : A \otimes A \to A \qquad \text{⬆} : I \to A \qquad \text{⬇} : A \to A \otimes A \qquad \text{⬇} : A \to I$$

satisfying the following equations, called *associativity* (top), *coassociativity* (bottom), *(co)unitality*, and the *Frobenius condition*:

A Frobenius algebra is *symmetric* when the following equations hold:

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[3]The construction in [22] needs the rather strong extra assumption of dagger biproducts, while the construction in [26] requires the weaker assumption of dagger kernels. The intersection of both constructions can be made to work, provided one additionally assumes a weak form of additive enrichment [23].

A *dagger Frobenius algebra* is a Frobenius algebra that additionally satisfies the following equation:

$$\text{（diagram）} = \big(\,\text{（diagram）}\,\big)^{\dagger} \qquad\qquad \text{（diagram）} = \big(\text{（diagram）}\big)^{\dagger}$$

Symbolically, we denote the *multiplication of two points $p, q\colon I \to A$*, that is,

$$\text{（diagram）} \circ (p \otimes q)\colon I \to A \;,$$

as $p \cdot q$. Also, since the multiplication fixes its unit, and the dagger fixes the comultiplication given the multiplication, we will usually represent our algebras as $(A,\,\text{（diagram）})$.

Remark 2.2. In [11], rather than symmetry, the stronger condition of *normalisability* is used. As this condition implies symmetry for dagger Frobenius algebras [11, Theorem 2.6], the results in this paper apply unchanged to normalisable Frobenius algebras.

We write **FHilb** for the category of finite-dimensional Hilbert spaces and linear maps, with the tensor product as the monoidal structure, and linear adjoint as the dagger.

Theorem 2.3 ([42])**.** *Symmetric dagger Frobenius algebras in* **FHilb** *are in 1-to-1 correspondence with finite-dimensional C*-algebras.* □

Recall that **FHilb** is a *compact category* [29], that is, we can coherently pick a *compact structure* on each object as follows. If $\mathcal{H}$ is a Hilbert space and $\mathcal{H}^*$ is its conjugate space, the triple:

$$\left(\mathcal{H},\; \epsilon_{\mathcal{H}}\colon \mathbb{C} \to \mathcal{H}^* \otimes \mathcal{H} :: 1 \mapsto \sum_i |i\rangle \otimes |i\rangle,\; \eta_{\mathcal{H}}\colon \mathcal{H} \otimes \mathcal{H}^* \to \mathbb{C} :: |\psi\rangle \otimes |\phi\rangle \mapsto \langle\psi|\phi\rangle\right)$$

is a compact structure which can be shown to be independent of the choice of basis–see [13] for more details. We depict the maps $\epsilon_{\mathcal{H}}$ and $\eta_{\mathcal{H}}$ respectively as:

$$\text{（diagram）} \qquad\qquad\qquad \text{（diagram）}$$

and compactness means that they satisfy:

$$\text{（diagram）} = \text{（diagram）} \qquad\qquad \text{（diagram）} = \text{（diagram）}\,.$$

Each symmetric dagger Fobenius algebra also canonically induces a 'self-dual' compact structure. The cups and caps of this compact structure are given by:

$$\text{（diagram）} := \text{（diagram）} \qquad\qquad \text{（diagram）} := \text{（diagram）}\,,$$

and one easily verifies that it follows from the axioms of a symmetric Frobenius algebra that the required 'yanking' conditions hold:

$$\text{（diagram）} = \text{（diagram）} = \text{（diagram）}\,.$$

3. Abstract projections

A *projection* of a C*-algebra is a *-idempotent. In this section we will recast this definition in light of Theorem 2.3, that is, we will identify what these projections are when a C*-algebra is presented as a symmetric dagger Frobenius algebra in **FHilb**, as in [42].

We claim that the projections of a C*-algebra arise as points $p \colon I \to \mathcal{H}$ satisfying:

(5)

where the symmetric dagger Frobenius algebra is the one induced by Theorem 2.3. Note that the first condition is simply idempotence of -multiplication of points, and the second one is *self-conjugateness* with respect to the compact structure induced by the symmetric dagger Frobenius algebra. Symbolically, we denote this conjugate of p as p^*.

A C*-algebra is realised as a symmetric dagger Frobenius algebra as follows. Each finite dimensional C*-algebra decomposes as a direct sum of matrix algebras. These can then be represented as *endomorphism monoids* $\mathrm{End}(\mathcal{H})$ in **FHilb**, which are triples of the following form:

$$\left(\mathcal{H}^* \otimes \mathcal{H} \ , \ 1_{\mathcal{H}^*} \otimes \eta_{\mathcal{H}} \otimes 1_{\mathcal{H}} : (\mathcal{H}^* \otimes \mathcal{H}) \otimes (\mathcal{H}^* \otimes \mathcal{H}) \to \mathcal{H}^* \otimes \mathcal{H} \ , \ \epsilon_{\mathcal{H}} : \mathbb{C} \to \mathcal{H}^* \otimes \mathcal{H} \right) ,$$

Diagrammatically, for an endomorphism monoid the multiplication and its unit respectively are:

The elements $\rho : \mathbb{C}^n \to \mathbb{C}^n$ of the matrix algebra are then represented by underlying points:

By compactness, each point of type $\mathbb{C} \to (\mathbb{C}^n)^* \otimes \mathbb{C}^n$ is of this form. By Theorem 2.3 we know that all symmetric dagger Frobenius algebras in **FHilb** arise in this manner.

We can now verify the above stated claim on how the projections of a C*-algebra arise in this representation. For these points p_ρ the conditions of equation (5) respectively become:

that is, using again compactness, $\rho \circ \rho = \rho = \rho^\dagger$, i.e. *idempotence* and *self-adjointness*.

We can now generalise the definition of projection to points $p : I \to A$ to arbitrary symmetric dagger Frobenius algebras $(A, \ \overset{\uparrow}{\underset{\curlywedge}{\varphi}})$ in any dagger symmetric monoidal category.

Definition 3.1. A *projection* of a symmetric dagger Frobenius algebra $(A, \ \overset{\uparrow}{\underset{\curlywedge}{\varphi}})$ in a dagger symmetric monoidal category **C** is a morphism $p \colon I \to A$ satisfying equations (5).

The next section studies the structure of these generalised projections of abstract C*-algebras.

Before that, we compare abstract projections to *copyable points*. These played a key role for commutative abstract C*-algebras, because they correspond to the elements of an orthonormal basis that determines the algebra [15]. However, as we will now see, in the noncommutative case, there simply do not exist enough copyable points (whereas the projections do have interesting structure, as the next section shows). Recall that a point $x \colon I \to A$ is *copyable* when the following equation is satisfied.

(6)
$$ \text{(diagram)} = \text{(diagram)} $$

Lemma 3.2. *Copyable points of symmetric dagger Frobenius algebras are central.*

Proof. Graphically:

$$ \text{(diagram)} = \text{(diagram)} = \text{(diagram)} = \text{(diagram)} = \text{(diagram)} = \text{(diagram)} . $$

The middle equation follows from symmetry of $(A, \ \overset{\uparrow}{\underset{\curlywedge}{\varphi}})$. $\qquad\qquad\square$

Let us examine what this implies for the example of $A = (\mathbb{C}^n)^* \otimes \mathbb{C}^n$ in **FHilb** above. Equivalently, we may speak about n-by-n matrices, so that $\overset{\uparrow}{\underset{\curlywedge}{\varphi}}$ becomes actual matrix multiplication. Because it is well known that the central elements of matrix algebras are precisely the scalars, any copyable point is simply a scalar by the previous lemma. But substituting back into (6) shows that the only scalar satisfying this equation is 0 (unless $n = 1$). That is, no noncommutative symmetric dagger Frobenius algebra in **FHilb** can have nontrivial copyable points. This explains why we prefer to work with (abstract) projections.

4. Quantum logics for abstract C*-algebras

Definition 4.1. A *zero projection* of $(A, \ \overset{\uparrow}{\underset{\curlywedge}{\varphi}})$ is a projection $0 \colon I \to A$ satisfying
$$ 0 \cdot p = 0 $$
for all other projections $p \colon I \to A$ of $(A, \ \overset{\uparrow}{\underset{\curlywedge}{\varphi}})$.

We will assume that an algebra always has a zero projection.

Definition 4.2. An *orthogonality relation* is a binary relation satisfying the following axioms:
- *symmetry*: $a \perp b \iff b \perp a$;
- *antireflexivity above zero*: $a \perp a \implies a = 0$;

- *downward closure*: $a \leq a'$, $b \leq b'$, $a' \perp b' \implies a \perp b$.

Lemma 4.3. *We have*

Proof. First, note the following stardard equation for Frobenius algebras:

Then, the result follows from associativity:

$\square$

Lemma 4.4. *For projections we have:*
  (i) $(p \cdot q)^* = q^* \cdot p^*$;
  (ii) *If $p \cdot q$ is a projection, then $p \cdot q = q \cdot p$.*

Proof. (i) We have

where the middle equation follows from Lemma 4.3. (ii) If $p \cdot q = r$ then, by self-conjugateness of projections and (i), $q \cdot p = q^* \cdot p^* = (p \cdot q)^* = r^* = r = p \cdot q$. $\square$

Theorem 4.5. *In a dagger symmetric monoidal category, projections on a symmetric dagger Frobenius algebra with a zero projection are partially ordered and come with an orthogonality relation.*

Proof. The order is defined as $p \leq q \iff p \cdot q = p$. Reflexivity follows by the idempotence of projections. If $p \cdot q = p$ and $q \cdot p = q$ then by Lemma 4.4 (ii) we have $p = q$, so the order is anti-symmetric. If $p \cdot q = p$ and $q \cdot r = q$ then $p \cdot r = p \cdot q \cdot r = p \cdot q = p$, so the order is transitive.

Orthogonality is defined as $p \perp q \iff p \cdot q = 0$. Symmetry follows by Lemma 4.4 (ii) and anti-reflexivity above 0 by idempotence of projections. If $p \cdot p' = p$, $q \cdot q' = q$ and $p' \cdot q' = 0$ then $p \cdot q = p \cdot p' \cdot q' \cdot q = p \cdot 0 \cdot q = p \cdot 0 = 0$ where we twice relied on Lemma 4.4 (ii). $\square$

Remark 4.6. The zero projection guarantees that the partially ordered set has a bottom element.

Given a symmetric dagger Frobenius algebra $(A, \overset{\uparrow}{\underset{\curlywedge}{\circ}})$, we will denote the partial order and orthogonality of the previous theorem as $\mathrm{Proj}(A, \overset{\uparrow}{\underset{\curlywedge}{\circ}})$. The following two examples correspond to the "pure classical" and the "pure quantum" in the "concrete" case of **FHilb**.

Example 4.7. Commutative dagger special Frobenius algebras $(H, \;\text{⬚})$ in **FHilb** correspond to orthonormal bases of H [15]. For $\text{Proj}(H, \;\text{⬚})$, we obtain the atomistic Boolean algebra whose atoms are the 1-dimensional projections on the basis vectors.

Example 4.8. If H is a finite-dimensional Hilbert space with any chosen compact structure on it, then $L(H) = (H^* \otimes H, \;\text{⬚})$ is a symmetric dagger Frobenius algebra in **FHilb**. For $\text{Proj}(L(H))$ we obtain the usual projection lattice of projections $H \to H$, the paradigmatic example in [3].

Remark 4.9. In [11], it is shown that algebras of the form $(A^* \otimes A, \;\text{⬚})$ are those that realise Selinger's CPM–construction as a fragment of the encompassing CP*–construction. The commutative dagger special Frobenius algebras were the ones used to underpin abstract categories of stochastic maps in [14].

Proposition 4.10. *Let* $(A, \;\text{⬚})$ *be any symmetric dagger Frobenius algebra in any dagger symmetric monoidal category. For* $p, q \in \text{Proj}(A, \;\text{⬚})$, *the following are equivalent:*

(a) p *and* q *commute;*

(b) $p \cdot q \in \text{Proj}(A, \;\text{⬚})$;

(c) $p \cdot q$ *is the greatest lower bound of* p *and* q *in the partial order* $\text{Proj}(A, \;\text{⬚})$.

Proof. Unfold the definitions of Theorem 4.5. $\square$

In general, every commutative monoid of idempotents is a meet-semilattice with respect to the order $p \le q \iff p \cdot q = p$, and if it is furthermore finite, then it is even a (complete) lattice. As shown in [14], in this case the notion of an idempotent can be generalised to arbitrary types $A \to B$. Considered together for all algebras, this always yields a cartesian bicategory of relations in the sense of Carboni-Walters [5]. The conclusion we draw from the previous proposition is the following: considering noncommutative algebras obstructs the construction of the categorical operation of composition.

5. Composing quantum logics

Given two symmetric dagger Frobenius algebras we can define their *tensor* as follows.

$$(A, \;\text{⬚}) \otimes (B, \;\text{⬚}) := (A \otimes B, \;\text{⬚⬚})$$

It is easily seen to inherit the entire algebraic structure. So we can define a compositional structure on the corresponding partial orders with orthogonality as follows.

$$\text{Proj}(A, \;\text{⬚}) \otimes \text{Proj}(B, \;\text{⬚}) := \text{Proj}(A \otimes B, \;\text{⬚⬚}) \ .$$

By a *bi-order map* we mean a function of two variables that preserves the order in each argument separately when the other one is fixed (*cf.* bilinearity of the tensor product).

Theorem 5.1. *The following is a bi-order map.*

$$- \otimes -: \ \text{Proj}(A, \;\text{⬚}) \times \text{Proj}(B, \;\text{⬚}) \to \text{Proj}(A, \;\text{⬚}) \otimes \text{Proj}(B, \;\text{⬚})$$
$$(p, q) \mapsto p \otimes q$$

If the monoidal structure moreover preserves zeros, that is, if 0_A is a (necessarily unique) zero with respect to A then for all $q : I \to B$ we have that $0_A \otimes q$ is a zero with respect to $A \otimes B$, then the map $- \otimes -$ also preserves orthogonality in each component.

Proof. If $p \cdot p' = p$ then:

$$(p \otimes q) \cdot (p' \otimes q) = \;\;\vcenter{\hbox{[diagram]}}\;\; = \;\;\vcenter{\hbox{[diagram]}}\;\; = (p \cdot p') \otimes (q \cdot q) = p \otimes q \;.$$

If $p \cdot p' = 0$ then $(p \otimes q) \cdot (p' \otimes q) = (p \cdot p') \otimes (q \cdot q) = 0_A \otimes q = 0_{A \otimes B}.$ $\qquad\square$

Remark 5.2. The assumption of the existence of zero projections as well as the assumption of monoidal structure preserving zeros, are both comprehended by the single assumption of the existence of a "zero scalar", that is, a morphism $0_I : I \to I$ such that for any other morphisms $f, g : A \to B$ we have that $0_I \otimes f = 0_I \otimes g$. We can then define zero projections $0_A := \lambda_A \circ (0_I \otimes 1_A) \circ \lambda_A^{-1}$ where $\lambda_A : A \simeq I \otimes A$.

6. Commutativity versus distributivity

Having abstracted projection lattices to the setting of arbitrary dagger symmetric monoidal categories, we can now consider other models than Hilbert spaces.

We will be interested in the category **Rel** of sets and relations, where the monoidal structure is taken to be Cartesian product, and the dagger is given by relational converse. This setting will provide a counterexample to equation (4). Here, symmetric dagger Frobenius algebras were identified by Pavlovic (in the commutative case) and Heunen–Contreras–Cattaneo (in general) in [33] and [25], respectively. They are in 1-to-1 correspondence with small groupoids. As it turns out, even in the commutative case, groupoids may yield nondistributive projection lattices.

Proposition 6.1. *Let* **G** *be a groupoid, and* $(G, \vcenter{\hbox{[symbol]}})$ *the corresponding symmetric dagger Frobenius algebra in* **Rel***. Elements of* $\mathrm{Proj}(G, \vcenter{\hbox{[symbol]}})$ *are in 1-to-1 correspondence with subgroupoids of* **G***, i.e. subcategories of* **G** *that are groupoids themselves.*

Proof. This follows directly from [25, Theorem 16]. $\qquad\square$

It immediately follows that in **Rel**, like in **FHilb**, the abstract projection lattice is a complete lattice, even though we are not dealing with finite sets.

Corollary 6.2. *If* $(G, \vcenter{\hbox{[symbol]}})$ *is a symmetric dagger Frobenius algebra in* **Rel***, then* $\mathrm{Proj}(G, \vcenter{\hbox{[symbol]}})$ *forms a complete lattice.*

Proof. The collection of subgroupoids is closed under arbitrary intersections. $\qquad\square$

In fact, for our counterexample to equation (4), it suffices to consider groups (*i.e.* single-object groupoids). In this case abstract projections correspond to subgroups, and it is known precisely under which conditions the lattice of subgroupoids is distributive, thanks to the following classical theorem due to Ore. A group is *locally cyclic* when any finite subset of its elements generates a cyclic group.

Theorem 6.3. *The lattice of subgroups of a group G is distributive if and only if G is abelian and locally cyclic.*

Proof. See [32, Theorem 4].   □

Perhaps the simplest example of an abelian group that is not locally cyclic is $\mathbb{Z}_2 \times \mathbb{Z}_2$. It has three nontrivial subgroups, namely:

$$a := \mathbb{Z}_2 \times \{0\};$$
$$b := \{(0,0),(1,1)\};$$
$$c := \{0\} \times \mathbb{Z}_2.$$

But evidently distributivity breaks down: $a \wedge (b \vee c) = a \neq 0 = (a \wedge b) \vee (a \wedge c)$.

By Theorem 6.3, we know that the converse (distributive $\implies$ commutative) holds for groups, but what about for arbitrary groupoids. Consider the groupoid with two objects x, y and the only non-identity arrows $f : x \to y$ and $f^{-1} : y \to x$. The lattice of subgroupoids has the following Hasse diagram:
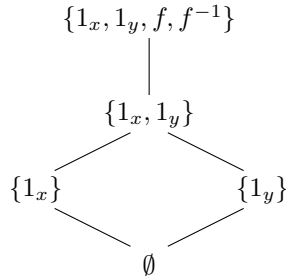
which is indeed distributive, but $f \circ f^{-1} \neq f^{-1} \circ f$. Thus we have proven the following corollary.

Corollary 6.4. *For symmetric dagger Frobenius algebras $(G, \overset{\uparrow}{\underset{\curlywedge}{\curlyvee}})$ in* **Rel***:*

$$(G, \overset{\uparrow}{\underset{\curlywedge}{\curlyvee}}) \text{ is commutative} \overset{\Longrightarrow}{\underset{\Longleftarrow}{}} \text{Proj}(G, \overset{\uparrow}{\underset{\curlywedge}{\curlyvee}}) \text{ is distributive.}$$

□

Let us finish by remarking on the copyable points in **Rel**. As in **FHilb**, they differ from the projections. But unlike in **FHilb**, where there are only trivial copyable points, copyable points in **Rel** are more interesting, for similar reasons as the above corollary.

Lemma 6.5. *If $(G, \overset{\uparrow}{\underset{\curlywedge}{\curlyvee}})$ is a symmetric dagger Frobenius algebra in* **Rel** *corresponding to a groupoid* **G***, then its copyable points correspond to the connected components of* **G***.*

Proof. A point x of G in **Rel** corresponds to a subset $X \subseteq \text{Mor}(\mathbf{G})$. Copyability now means precisely that

$$X^2 = \{(g, fg^{-1}) \mid f \in X, g \in \text{Mor}(\mathbf{G}), \text{dom}(f) = \text{dom}(g)\}.$$

Hence if $f \in X$, and $g \in \text{Mor}(\mathbf{G})$ has $\text{dom}(f) = \text{dom}(g)$, then also $g \in X$. Because **G** is a groupoid, this means that X is precisely (the set of morphisms of a) connected component of **G**.   □

7. Further work

From the point of view of traditional quantum logic, a number of questions arise, in particular about which categorical structure yields which order structure:

- when is the orthogonality relation an orthocomplementation?
- when do we obtain an orthoposet?
- when do we obtain an orthomodular poset?
- when is the partial order a (complete) lattice?
- when is this lattice Boolean, modular or orthomodular?

Conversely, what does the lattice structure say about the category? An important first step is the characterisation of dagger Frobenius algebras in more example categories besides **FHilb** and **Rel**.

There is a clear intuition of the comultiplication of the algebra being a "logical broadcasting operation" in the sense of [17]. A more general question then arises on the general operational significance of the partial ordering and orthogonality relation constructed in this paper.

One of the more recent compelling results which emerged from quantum logic is the Faure-Moore-Piron theorem [19] on the reconstruction of dynamics from the lattice structure together with the its operational interpretation. A key ingredient is the reliance on Galois adjoints. Does this construction have a counterpart within our framework, and its (to still be understood) operational significance?

References

[1] Samson Abramsky and Bob Coecke. A categorical semantics of quantum protocols. In *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science (LICS)*, pages 415–425. IEEE Computer Society, 2004. Extended version: arXiv:quant-ph/0402130.

[2] Samson Abramsky and Chris Heunen. H*-algebras and nonunital Frobenius algebras: first steps in infinite-dimensional categorical quantum mechanics. In Samson Abramsky and Michael Mislove, editors, *Clifford Lectures*, volume 71 of *Proceedings of Symposia in Applied Mathematics*, pages 1–24. American Mathematical Society, 2012.

[3] Garrett Birkhoff and John von Neumann. The logic of quantum mechanics. *Annals of Mathematics*, 37:823–843, 1936.

[4] Sergio Boixo and Chris Heunen. Entangled and sequential quantum protocols with dephasing. *Physical Review Letters*, 108:120402, 2012.

[5] Aurelio Carboni and Robert F. C. Walters. Cartesian bicategories I. *Journal of Pure and Applied Algebra*, 49:11–32, 1987.

[6] Giulio Chiribella, G. Mauro D'Ariano, and Paolo Perinotti. Informational derivation of quantum theory. *Physical Review A*, 84(1):012311, 2011.

[7] Bob Coecke. Axiomatic description of mixed states from Selinger's CPM-construction. *Electronic Notes in Theoretical Computer Science*, 210:3–13, 2008.

[8] Bob Coecke. The logic of quantum mechanics – take II. arXiv:1204.3458, 2012.

[9] Bob Coecke, Bill Edwards, and Robert W. Spekkens. Phase groups and the origin of non-locality for qubits. *Electronic Notes in Theoretical Computer Science*, 270(2):15–36, 2011. arXiv:1003.5005.

[10] Bob Coecke and Chris Heunen. Pictures of complete positivity in arbitrary dimension. *Quantum Phsyics and Logic, Electronic Proceedings in Theoretical Computer Science*, 95:27–35, 2011.

[11] Bob Coecke, Chris Heunen, and Aleks Kissinger. A category of classical and quantum channels. QPL'12, 2012.

[12] Bob Coecke and Aleks Kissinger. The compositional structure of multipartite quantum entanglement. In *Automata, Languages and Programming*, Lecture Notes in Computer Science, pages 297–308. Springer, 2010. Extended version: arXiv:1002.2540.

[13] Bob Coecke and Éric O. Paquette. Categories for the practicing physicist. In B. Coecke, editor, *New Structures for Physics*, Lecture Notes in Physics, pages 167–271. Springer, 2011. arXiv:0905.3010.

[14] Bob Coecke, Éric O. Paquette, and Duško Pavlović. Classical and quantum structuralism. In S. Gay and I. Mackie, editors, *Semantic Techniques in Quantum Computation*, pages 29–69. Cambridge University Press, 2010. arXiv:0904.1997.

[15] Bob Coecke, Duško Pavlović, and Jamie Vicary. A new description of orthogonal bases. *Mathematical Structures in Computer Science, to appear*, 2011. arXiv:quant-ph/0810.1037.

[16] Bob Coecke, Mehrnoosh Sadrzadeh, and Stephen Clark. Mathematical foundations for a compositional distributional model of meaning. *Linguistic Analysis*, 2010.

[17] Bob Coecke and Robert W. Spekkens. Picturing classical and quantum Bayesian inference. *Synthese*, 186:651–696, 2012. arXiv:1102.2368.

[18] Ross Duncan and Simon Perdrix. Rewriting measurement-based quantum computations with generalised flow. In *Proceedings of ICALP*, Lecture Notes in Computer Science, pages 285–296. Springer, 2010.

[19] Claude-Alain Faure, David J. Moore, and Constantin Piron. Deterministic evolutions and Schrödinger flows. *Helvetica Physica Acta*, 68(2):150–157, 1995.

[20] David J. Foulis and Charlie H. Randall. Operational statistics. I. Basic concepts. *Journal of Mathematical Physics*, 13(11):1667–1675, 1972.

[21] Andrew M. Gleason. Measures on the closed subspaces of a Hilbert space. *Journal of Mathematics and Mechanics*, 6:885–893, 1957.

[22] John Harding. A link between quantum logic and categorical quantum mechanics. *International Journal of Theoretical Physics*, 48(3):769–802, 2009.

[23] John Harding. Daggers, kernels, Baer *-semigroups, and orthomodularity. to appear in Journal of Philosophical Logic, 2010.

[24] Lucien Hardy. Quantum theory from five reasonable axioms. *arXiv:quant-ph/0101012*, 2001.

[25] Chris Heunen, Ivan Contreras, and Alberto S. Cattaneo. Relative frobenius algebras are groupoids. *Journal of Pure and Applied Algebra*, 217:114–124, 2012.

[26] Chris Heunen and Bart Jacobs. Quantum logic in dagger kernel categories. *Order*, 27(2):177–212, 2010.

[27] Clare Horsman. Quantum picturalism for topological cluster-state computing. *New Journal of Physics*, 13:095011, 2011. arXiv:1101.4722.

[28] Bart Jacobs. Orthomodular lattices, foulis semigroups and dagger kernel categories. *Logical Methods in Computer Science*, 6(2):1, 2010.

[29] G. Max Kelly and Miguel L. Laplaza. Coherence for compact closed categories. *Journal of Pure and Applied Algebra*, 19:193–213, 1980.

[30] Günther Ludwig. *An Axiomatic Basis of Quantum Mechanics. 1. Derivation of Hilbert Space.* Springer-Verlag, 1985.

[31] George W. Mackey. *The mathematical foundations of quantum mechanics.* W. A. Benjamin, New York, 1963.

[32] Oystein Ore. Structures and group theory II. *Duke Mathematical Journal*, 4(2):247–269, 1938.

[33] Duško Pavlovic. Quantum and classical structures in nondeterminstic computation. In *Proceedings of the 3rd International Symposium on Quantum Interaction*, QI '09, pages 143–157, Berlin, Heidelberg, 2009. Springer-Verlag.

[34] Constantin Piron. Axiomatique quantique. *Helvetia Physica Acta*, 37:439–468, 1964.

[35] Constantin Piron. *Foundations of quantum physics.* W. A. Benjamin, 1976.

[36] Miklós Rédei. *Quantum Logic in Algebraic Approach.* Kluwer, 1998.

[37] Peter Selinger. Dagger compact closed categories and completely positive maps. *Electronic Notes in Theoretical Computer Science*, 170:139–163, 2007.

[38] Peter Selinger. Idempotents in dagger categories (extended abstract). *Electronic Notes in Theoretical Computer Science*, 210:107–122, 2008.

[39] Peter Selinger. A survey of graphical languages for monoidal categories. In B. Coecke, editor, *New Structures for Physics*, Lecture Notes in Physics, pages 275–337. Springer-Verlag, 2011. arXiv:0908.3347.

[40] Maria Pia Solèr. Characterization of Hilbert spaces by orthomodular spaces. *Communications in Algebra*, 23(1):219–243, 1995.

[41] Isar Stubbe and Bart van Steirteghem. Propositional systems, Hilbert lattices and generalized Hilbert spaces. In D. Lehmann D. Gabbay and K. Engesser, editors, *Handbook Quantum Logic*, pages 477–524. Elsevier Publ., 2007. http://www.mat.uc.pt/∼isar/PDF/HilbertLatticesELSEVIER.pdf.

[42] Jamie Vicary. Categorical formulation of finite-dimensional quantum algebras. *Communications in Mathematical Physics*, 304(3):765–796, 2011.

[43] Eugene P. Wigner. *Gruppentheorie*. Friedrich Vieweg und Sohn, 1931.

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