Ontario

A Guide to Effective Instruction in Mathematics

Kindergarten to Grade 6

A Resource in Five Volumes
from the Ministry of Education

Volume Five
Teaching Basic Facts and Multidigit Computations

Every effort has been made in this publication to identify mathematics resources and tools (e.g., manipulatives) in generic terms. In cases where a particular product is used by teachers in schools across Ontario, that product is identified by its trade name, in the interests of clarity. Reference to particular products in no way implies an endorsement of those products by the Ministry of Education.

Une publication équivalente est disponible en français sous le titre suivant :
Guide d'enseignement efficace des mathématiques, de la maternelle à la 6e année.

Contents

Introduction

This is Volume Five of the five-volume reference guide *A Guide to Effective Instruction in Mathematics, Kindergarten to Grade 6*. This volume contains Chapter 10. Chapter 10 is devoted to the important subject of teaching basic facts and multidigit computations – the building blocks of students' computational proficiency. Effective instruction in this area is critical, as students' ability to perform operations accurately and with understanding will affect their achievement of expectations in all five strands of the curriculum. The chapter lays out the approaches and strategies that have proved most effective in helping students understand, learn, and consolidate their learning of the basic facts. Games and activities, with accompanying blackline masters, are also included here, to give teachers useful ideas for teaching the strategies to their students.* (See the Introduction of Volume One for a summary of the organization and contents of the complete five-volume guide.)

A list of suggested professional resources for teachers and administrators is included in Volume One. It is meant to provide useful suggestions, but should not be considered comprehensive. A glossary of terms used throughout the guide is also provided at the end of Volume One. References are listed at the end of each individual volume.

This guide contains a wide variety of forms and blackline masters, often provided in appendices, that teachers can use in the classroom. Electronic versions of all of these materials can be found at www.eworkshop.on.ca. These electronic forms and blackline masters are in a Word format that can be modified by teachers to accommodate the needs of their students.

Go to www.eworkshop.on.ca for electronic versions of the forms and blackline masters provided throughout this guide. They can be modified to meet classroom needs.

*Basic facts and multidigit computations are the focus of a group of expectations in the Number Sense and Numeration strand of the Ontario mathematics curriculum. The companion documents to this guide that focus on the Number Sense and Numeration strand also contain information and learning activities relating to basic facts and multidigit computations.

Locating Information Specific to Kindergarten, Primary, and Junior Students in This Guide

An important feature of this guide is the inclusion of grade-related information and examples that help clarify the principles articulated. Such information is identified in the margins of this guide by means of icons referring to the relevant grades – K for Kindergarten, Grades 1–3 for primary, Grades 4–6 for junior. Examples and other materials that are appropriate for use at more than one level or are applicable to more than one level are identified by the appropriate combination of icons.

K – Kindergarten
1–3 – Primary
4–6 – Junior

Look for the following icons in the margins of this document:

K **1-3** **4-6**

10. Approaches to Teaching Basic Facts and Multidigit Computations

Chapter Contents

Approaches to Teaching Basic Facts and Multidigit Computations

Introduction

This chapter focuses on the development of students' facility in using whole numbers in single-digit and multidigit computations. *Single-digit computations used to be called "basic facts", but many organizations now call them basic number combinations* (Kilpatrick, Swafford, & Findell, 2001). Basic facts include the addition, subtraction, multiplication, and division of numbers from 0 to 9. Multidigit computations include all combinations of numbers of two or more digits in addition, subtraction, multiplication, and division.

In the past, the emphasis in teaching was on memorization of the basic number facts, sometimes to the exclusion of establishing a firm foundation of number sense. Present research supports the importance of developing conceptual understanding for future success in mathematics (National Council of Teachers of Mathematics, 2000). Research also demonstrates that instruction that focuses on conceptual understanding improves numerical reasoning and procedural fluency and accuracy (National Research Council, 2001). Understanding the structures that underlie single-digit basic facts also gives students an effective tool for working with multidigit computations. However, it is often the case that instruction does not focus on the mathematical concepts inherent in multidigit computations. Instead, it focuses almost entirely on the procedures. Overreliance on memorized procedures prevents students from using mathematical reasoning. Students may persist with a regrouping procedure to solve 2000 – 50, come up with an answer such as 1050, and not be aware that one step in the procedure is missing and that the answer is wrong by a significant amount. Students who have developed reasoning skills through their work with single-digit numbers will be more likely to extend this reasoning into multidigit computations and to recognize that an answer that is almost 1000 less than the minuend cannot be right.

Learning the basic facts conceptually involves developing an understanding of the relationships between numbers (e.g., 7 is 3 less than 10 and 2 more than 5) and of how these relationships can be developed into strategies for doing the computations in a meaningful and logical manner. Students who learn the basic facts using a variety of strategies (making tens, using doubles) will be able to extend these strategies and their understanding of number to multidigit computations. In the primary grades, students begin to develop their understanding of the basic facts by making concrete representations of numbers (e.g., representing 4 with cubes). They use their developing sense of "more", "less", and "the same" to understand that 6 cubes is more than 4 cubes. In the next stage, they physically join quantities (e.g., 3 cubes joined to 4 cubes) and use their developing counting skills to count all the cubes to find a solution. Often at this stage, students have to "count all", meaning that they may count 3 cubes and then count 4 cubes, but they need to re-count the 3 cubes and the 4 cubes together to find a solution, rather than counting up from the first set of cubes. As their sense of number matures, students begin to recognize that it is quicker and easier to count on from the larger number.

At this same point in their development, it is helpful for students to begin to think in terms of anchors of 5 and 10. Their recognition of the relationship of all other single-digit numbers to 5, at first, and then to 5 and to 10 helps students to structure their understanding of number around these two important concepts. An important foundation of work with numbers is the use of 10 as an anchor. Students who use 10 as an anchor have a valuable strategy for computing basic facts. For example, they can reason 7 + 6 as 7 + 3 (of the 6) = 10 and 3 more makes 13. This strategy can later be generalized to multidigit computations as follows: 27 + 6 is the same as 27 + 3 (of the 6) = 30 and 3 more makes 33, and 27 + 36 is the same as 20 + 30 = 50, and 7 more makes 57, 3 more makes 60, and 3 more makes 63. The student who computes in this way is "making tens", which is an efficient way to do multidigit computations in written form and especially to do them mentally.

As noted earlier, instruction in multidigit problem solving has tended to involve the modelling of a mathematical procedure – an algorithm – by the teacher, with little attention to the underlying fundamental number concepts. The algorithm is presented as the "right way" to solve a problem, and students are not often encouraged to explore the many different ways in which computations can be solved. Liping Ma (1999), in her research on North American and Chinese elementary school teachers, found that, whereas the North American teachers focused their instruction on repeated practice using the standard algorithm, the Chinese teachers were more cognizant of the need to teach students the whole range of concepts that underpin the algorithm. This focus on developing understanding may account, to some extent, for the fact that Chinese students do significantly better than North American students on international mathematics tests.

Also central to developing a conceptual understanding of the basic facts is learning those facts in a problem-solving context. Young children can count, can model (that is, use concrete materials to represent) and solve problems, and can develop strategies for increasing their fluency with the facts. By solving problems, students develop a richer understanding of the relationship between problems and number facts (Baroody, 1998; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Kilpatrick, Swafford, & Findell, 2001; Reys, Lindquist, Lambdin, Suydam, & Smith, 2001). Using problem-solving contexts is just as important when solving multidigit computations. When students are given a problem, encouraged to develop and share a strategy with their peers, and encouraged to work flexibly with different strategies, they are further developing their understanding of the operations. When students are simply given a standard algorithm to use when solving multidigit computations, they often struggle to understand and memorize the procedure. This process tends to lead to ineffective use of the algorithm as well as to limited accuracy and understanding. Encouraging students to make sense of the problem and to develop their own strategies for solving it will lead to greater accuracy and computational fluency.

Because of the critical importance of problem-solving approaches for developing computational fluency, this chapter begins with a discussion of the topic.

Using the Problem-Solving Approach to Teach Basic Facts and Multidigit Computations

A comprehensive discussion of what problem solving looks like in the classroom is provided in Chapter 5: Problem-Solving, in Volume Two. That chapter emphasizes the importance of including, within a balanced mathematics program, both the instructional practice of teaching *through* problem solving and that of teaching *about* problem solving. Most mathematical learning in a rich mathematics program takes place through problem solving, and many of the problems posed are open-ended problems that allow for multiple problem-solving approaches. It is important that teachers become familiar with the contents of Chapter 5 so that they can understand the critical role that problem solving plays in all math instruction.

Providing students with problem-solving contexts that relate to the basic facts will allow them to develop a meaningful understanding of the operations. Using problems to introduce, practise, and consolidate the basic facts is one of the most effective strategies for helping students link the mathematical concepts to the abstract procedures. However, certain pitfalls, some connected with an overreliance on common problem-solving structures – for example, always using the "I have one marble; I get one more marble; how many marbles do I have now?" type of question – have been identified in the problems teachers pose to students. Research has shown that teachers tend to present students with problems of the standard "join problem" type, in which an amount is given at the outset, another amount is added to that starting amount, and the result is ascertained. This problem-solving structure is so prevalent that students will often respond automatically to any problem they are presented with as though it followed the same structure. In one well-known example, students were given the following problem: "The shepherd had 5 calves, 6 cows, and 60 sheep. How old is the shepherd?" With sheep-like conformity, the students added up the numbers and declared that the shepherd was 71 years old. Although the example may be unfairly based on a slightly tricky question, it still illustrates the point that students can become so accustomed to a particular problem structure that they no longer bother to use their reasoning skills (or to read carefully). Teachers need to become conscious of the structures of the problems they use with students.

Another concern connected with commonly used problem structures is that they do not allow for alternative ways of reasoning to find a solution. In other words, a problem structured as a subtraction problem is supposed to be solved using subtraction, even if it lends itself to being solved using addition. For example, a student may respond to the problem "Rosa had 26 books to sell in the school garage sale. She sold 23. How many books did she have left?" by simply adding up from 23 to 26 for an answer of 3. The teacher needs to recognize that the reasoning that is occurring is more important than strict adherence to a requirement that a "subtraction" problem has to be solved as such.

On the following pages are some examples of how different problem structures can be used effectively to help students learn basic facts of addition, subtraction, multiplication, and division as well as do multidigit computations with these operations.

PROBLEMS FOR BASIC FACTS OF ADDITION AND SUBTRACTION, AND FOR MULTIDIGIT COMPUTATIONS USING ADDITION AND SUBTRACTION

1-3

As noted earlier, problems can be highly effective in helping students link the mathematical concepts of basic facts and multidigit computations to the abstract procedures.

Very young children can use manipulatives to represent a problem such as "Megan has 5 marbles. She was given 3 more. How many marbles does she have now?" With instruction, students link the thinking in such a problem to the operation of addition, namely, $5 + 3 = 8$. Thus, they link their conceptual understanding of the problem with their procedural knowledge. It is important to note that although adults consider a problem such as "Megan had some marbles. She was given 3 more. Now she has 8. How many marbles did Megan start with?" a simple problem, young children do not. By modelling such a problem and then attaching an operation to the problem, teachers will help students to make connections between conceptual understanding and procedural fluency across problems that have varying degrees of complexity.

Types of Problems

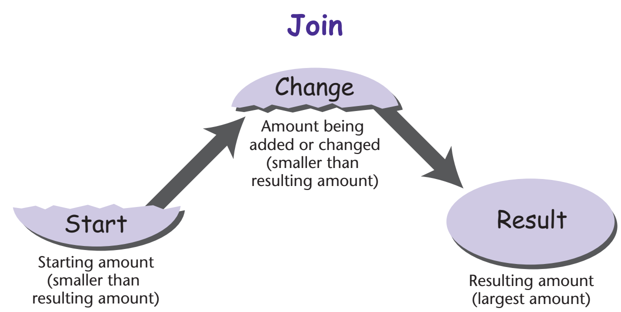
Students develop a strong understanding of operations (addition and subtraction) and of number relationships by solving problems. The types of problems shown next (with examples) can help students to envision addition and subtraction facts in various ways: as joining, separating, "part-part-whole", and comparing. Using problems to introduce the basic facts (and multidigit computations using addition and subtraction) compels students to use reasoning to find solutions and to make a solid connection between the facts and the various problem-solving scenarios they can represent.

Although the examples shown here include only single-digit facts, the same structures can be modified to include multidigit computations.

Join Problems

In join problems, the final amount is the largest.

- *Join: Result unknown*
  Jason had 6 candies. He bought 5 more.
  How many candies does Jason have now?
- *Join: Start unknown*
  Jason had some candies. He bought 5 more.
  Now he has 11. How many candies did Jason start with?
- *Join: Change unknown*
  Jason had 6 candies. He bought some more candies.
  Now he has 11. How many candies did Jason start with?

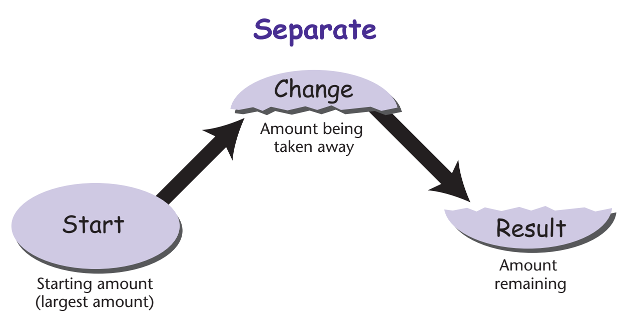


Kurt created 4 paintings on Monday. On Tuesday he created 3 more. How many paintings did Kurt create altogether?

Separate Problems

In separate problems, the first amount is the largest.

- *Separate: Result unknown*
  Nidhi had 15 dollars. She gave 5 dollars to her brother. How many dollars does Nidhi have now?
- *Separate: Change unknown*
  Nidhi had 15 dollars. She gave some to her brother. Now she has 10 dollars. How many dollars did Nidhi give to her brother?
- *Separate: Start unknown*
  Nidhi had some dollars. She gave 5 dollars to her brother. Now she has 10 dollars. How many dollars did Nidhi start with?

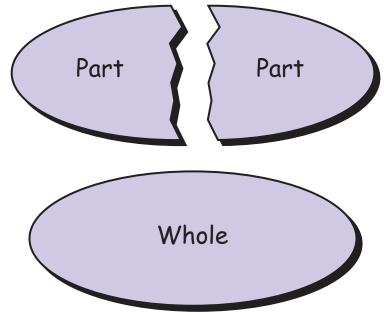


Mariclaire made 24 cupcakes for her classmates. She had 2 left after class. How many cupcakes were eaten?

Part-Part-Whole Problems

Part-part-whole problems contain two parts, which are combined into a whole.

- *Part-part-whole: Part unknown*
  Sana has 8 crayons. Three crayons are red. The rest are blue. How many blue crayons does Sana have?
- *Part-part-whole: Whole unknown*
  Sana has 3 red crayons and 5 blue crayons. How many crayons does Sana have?



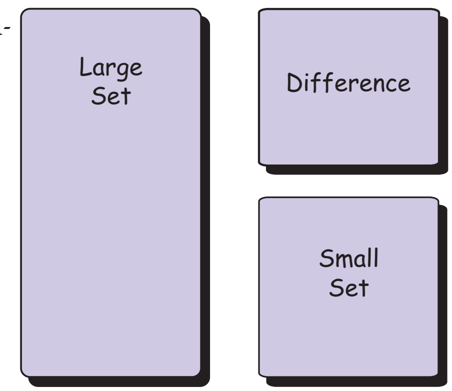
Part-Part-Whole

Gary has 14 pattern blocks. Ten blocks are squares. The rest are triangles. How many blocks are triangles?

Compare Problems

Compare problems involve the comparison of two quantities. The third quantity represents the difference.

- *Compare: Difference unknown*
  Judith has 7 dollars and Jean has 3 dollars.
  How many more dollars does Judith have than Jean?
  OR
  Judith has 7 dollars and Jean has 3 dollars.
  How many fewer dollars does Jean have than Judith?
- *Compare: Larger unknown*
  Judith has 4 more dollars than Jean. Jean has 3 dollars. How many dollars does Judith have?
  OR
  Jean has 4 fewer dollars than Judith. Jean has 3 dollars. How many dollars does Judith have?
- *Compare: Smaller unknown*
  Judith has 7 dollars and Jean has 4 dollars fewer than Judith.
  How many dollars does Jean have?
  OR
  Jean has 4 fewer dollars than Judith. Judith has 7 dollars.
  How many dollars does Jean have?



Compare

Mario has collected 15 rocks. Frank has collected 8 rocks. How many more rocks does Mario have than Frank?

PROBLEMS FOR BASIC FACTS OF MULTIPLICATION AND DIVISION, AND FOR MULTIDIGIT COMPUTATIONS USING MULTIPLICATION AND DIVISION

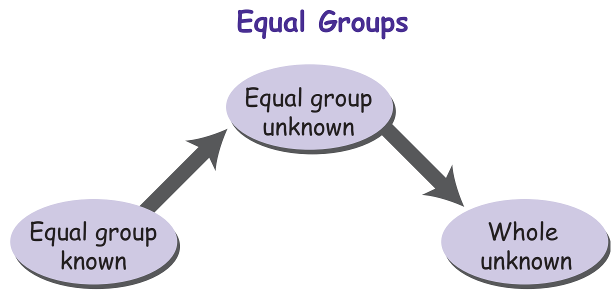
1-3 4-6

Students develop a strong understanding of operations (multiplication and division) and of number relationships through problems. The types of problems described below (with examples) help students to envision the multiplication and division facts in various ways: as equal group problems, as comparison problems, and as combination problems. Using problems in the introduction to the basic facts compels students to use reasoning to find the solutions and to make a solid connection between the facts and the various problem-solving scenarios they can represent.

Although the following examples include only single-digit facts, the problem structures shown can be modified to include multidigit computations.

Types of Problems

Equal Group Problems



- *Equal groups: Whole unknown (multiplication)*
  Wayne bought 5 books for his friends. The books were $2.00 each. How much money did Wayne pay for all the books?
- *Equal groups: Size of groups unknown (partitive division)*
  Wayne has 10 books. He wants to give them to 5 friends. How many books will each friend get?
- *Equal groups: Number of groups unknown (measurement division)*
  Wayne had 10 books. He put 2 books into each bag he had. How many bags did Wayne use?

Multiplicative Comparison Problems

- *Comparison: Product unknown (multiplication)*
  Mustapha has $2.00. Brian has 4 times as much. How much money does Brian have?
- *Comparison: Set size unknown (partition division)*
  Brian has $8.00. He has 4 times as much money as Mustapha. How much money does Mustapha have?
- *Comparison: Multiplier unknown (measurement division)*
  Brian has $8.00 and Mustapha has $2.00. How many times as much money does Brian have as Mustapha?

Combinations Problems

- *Combinations: Product unknown*
  Mustapha has 3 pairs of pants and 5 shirts. How many different outfits can Mustapha make?
- *Combinations: Size of set unknown*
  Mustapha has some new pants and shirts. He has a total of 15 different outfits. If he has 3 pairs of pants, how many shirts does Mustapha have?

Basic Facts: Addition and Subtraction, and Multiplication and Division

Students' facility in using basic math facts often has a significant effect on their confidence in themselves as mathematicians. This confidence can be diminished if teachers pay excessive attention to memorization and speed and spend too little time helping students to understand the *relationships and patterns* in the basic facts.

> *"For years, learning to compute has been viewed as a matter of following the teacher's directions and practicing until speedy execution is achieved.... More than just a means to produce answers, computation is increasingly seen as a window on the deep structure of the number system. Fortunately, research is demonstrating that both skilled performance and conceptual understanding are generated by the same kinds of activities. No tradeoffs are needed."*
>
> (Kilpatrick, Swafford, & Findell, 2001)

As noted earlier, research evidence suggests that the use of conceptual approaches in computation instruction results in improved achievement, good retention, and a reduction in the time students need to master computational skills. Furthermore, students make fewer typical kinds of errors when taught using conceptual approaches.

PRINCIPLES FOR TEACHING THE FACTS

- Most students can learn the basic facts accurately, although their speed may vary considerably.
- Students should learn the facts in a problem-solving context.
- Students should have many experiences modelling the facts using concrete and pictorial representations.
- Students should be encouraged to look for patterns and relationships between the operations and numbers in the facts.
- Students need strategies that help them reason their way to the solutions for the facts, rather than strategies for memorizing the facts.
- Students who learn basic facts without understanding them do not know when or how to use what they know. Such learning is often transitory.
- Students should not be compelled to memorize facts if they have limited strategies for solving facts. (It can be a waste of time, and it limits the opportunity for students to learn the whole range of fact strategies they will use throughout their elementary-school years.) Students who have a repertoire of strategies will be able to find an accurate answer, and over time their speed will naturally increase.

PRIOR LEARNING NECESSARY FOR DEVELOPING COMPUTATIONAL SENSE

1-3

Students need prior experience with exploring some of the "big ideas" of Number Sense and Numeration (in particular, counting and quantity). For example, young students who are learning to add and subtract need to know:

- how to count from 1 to 10;
- that each count should correspond to the objects being counted (one-to-one correspondence);
- that each number represents a network of connections between the quantities (such as counters), the symbol, the number name, and pictures (such as a number line);
- that numbers can be acted on and that such action represents a change in magnitude (3 fingers and 3 fingers will create a new quantity of 6 fingers);
- that the magnitude of numbers increases as students count on and decreases as they count back;
- that 5 and 10 are anchors for all numbers;
- "part-part-whole" concepts.

Students must understand some of the big ideas of Number Sense and Numeration (in particular, operational sense and quantity) before they can learn concepts of multiplication and division. In particular, students need to know how to:

- add and subtract (multiplication can be thought of as adding equal-sized groups, and division can be thought of as subtracting equal-sized groups);
- create groups or sets of equal size, and separate quantities into equal groups or sets.

See *A Guide to Effective Instruction in Mathematics, Kindergarten to Grade 3 – Number Sense and Numeration, 2003* (Ontario Ministry of Education, 2003a) for a detailed discussion of students' understanding of the big ideas of Number Sense and Numeration in Kindergarten and Grades 1, 2, and 3.

DEVELOPING COMPUTATIONAL SENSE

Many strategies can help students develop computational sense. Not all students use every strategy; some may use certain strategies (e.g., doubles) more frequently, and they may abandon some strategies as they find and develop more efficient ones of their own.

Students' development of computational sense goes through several stages. In each stage, students adopt or discard certain strategies. Students who are the most adept at computations usually adopt the most efficient strategies. Students who are not as successful often persist with inefficient strategies, such as counting, and may not recognize that more efficient strategies exist. Research indicates that students' computational sense is improved by exposure to a range of computational strategies through guided instruction by the teacher and shared learning opportunities with other students.

In the primary grades, students use objects or their fingers to model problems. As students gain experience, they begin to use more advanced counting strategies. Later, they develop other higher-level strategies, such as "making tens" or using "doubles", to solve addition and subtraction problems. As they learn more number facts, students become more adept at using their prior knowledge to determine new facts. For example, students who know that $6 + 6 = 12$ can use that knowledge to reason their way to the fact that $6 + 7$ must be 13, because 7 is 1 more than 6. A problem may occur if students have only their memory to rely on and have no strategies for double-checking whether their answers are correct. A student who gives 15 as the answer to the math fact $7 + 6$ may have memorized the fact, but then forgotten it. A student who learned the facts by using a strategy such as making tens can quickly double-check by thinking, "I know $7 + 3$ is 10, and that leaves 3 more in the 6, so the answer is 13." Or the student may use his or her knowledge of doubles and think, "I know $6 + 6$ is 12, and 7 is 1 more than 6, so the answer is 1 more than 12, or 13." 1-3

Students who have learned their facts by rote have no other strategies to rely on, so they may be prone to making simple errors.

The strategies many students use as they develop their computational fluency are set out below. Not all students use all these strategies, but the list will serve as a guideline for teachers who are trying to identify the strategies that students are using.

Direct Modelling and Counting

In the early stages, students usually need to model the facts directly by using their fingers or objects. They use their newly acquired counting skills to determine the answer to a problem such as the following: "Eleni has 3 apples and receives 2 more. How many does she have now?" Students may count out 3 counters, then count out 2 counters, and then have to count all the counters again to get the total of 5.

Students initially solve multiplication and division problems by using counters, tally marks, or drawings to represent the objects arranged in groups. They count the total number of objects to find the solution to the problem. For example, they may be given the following problem: "There are 3 bowls of apples. Each bowl contains 5 apples. How many apples are there altogether?" To solve the problem, students might draw a picture of the bowls of apples, and then count the apples to find the total amount.

Counting Strategies

As students gain experience with addition and subtraction, they make a transition from direct modelling to counting strategies. Although students may continue to use counters, their fingers, or tally marks, these objects are used to keep track of counts rather than to represent the problem. They may also use number lines, hundreds charts, or other devices to help them keep track of the counts.

Students make a considerable leap in their problem-solving effectiveness when they realize that they do not have to count all the objects for a question involving 3 + 2; they can simply count on from the larger number (3) to include the second number (2) being added.

Counting strategies for multiplication and division involve skip counting. Counting strategies work best when the problem involves a number such as 2 or 5, since most students learn the skip-counting sequences for these numbers.

Making Tens

Making tens is an efficient addition strategy that some students use (but many do not, if no instruction has been given). In making tens, students consolidate their understanding of the number combinations that make ten and their knowledge of the relatively easy calculation involved in adding 10 to single-digit numbers. To make tens, students think of each number fact in relation to an addition fact that includes 10. So, $9 + 7$ can be thought of as $9 + 1 = 10$, and 6 more makes 16. It is important to use ten frames when introducing basic facts to help students learn how to use making tens as a strategy.

Derived Facts

As students learn number facts, they use this knowledge to solve problems, but they sometimes need to revert to direct modelling and counting to support their thinking. Students learn certain number facts, such as doubles (e.g., $3 + 3$ and $6 + 6$), before others, and they can use such known facts to derive answers for unknown facts ($3 + 4$ is related to $3 + 3$ and $6 + 7$ is related to $6 + 6$).

It is important to remember that not all students will use every strategy, and those who do will not progress through them linearly.

- Students use manipulatives to directly model the numbers and the action in the facts, such as $8 + 3$, 8×2, $8 - 6$, or $8 \div 4$.
- Students use counting strategies (usually beginning with counting all and then moving to counting on) to solve fact questions.
- Students use derived facts to help them solve other facts (the student who forgets the answer to 6×7 uses a known fact such as 5×7 and then adds on one more 7).
- Students use their knowledge of the properties of the operations to help them solve facts (i.e., the commutativity principle of addition [$1 + 2 = 2 + 1$], the inverse relationship of subtraction to addition, and so on).
- Students develop strategies, such as using doubles.
- Students recognize patterns and relationships in number operations (such as part-part-whole relationships).
- Students consolidate and use a range of strategies to help them recall facts.

USING WORKSHEETS WITH A PURPOSE

Traditionally, teachers have used a variety of worksheets to help students practise their basic facts. In the past, some worksheets for primary students have looked like the following example. **1-3**

2 + 2 = ___	3 + 5 = ___	9 + 9 = ___
8 + 4 = ___	2 + 7 = ___	2 + 0 = ___
3 + 6 = ___	5 + 5 = ___	4 + 3 = ___
1 + 2 = ___	8 + 2 = ___	0 + 5 = ___
8 + 7 = ___	6 + 4 = ___	6 + 0 = ___

"Do not subject any student to fact drills unless the student has developed efficient strategies for the facts being practiced.… Short-term gains are almost certain to be lost over time. Practice prior to development of efficient methods is simply a waste of precious instruction time."

(Van de Walle, 2001, p. 144)

Closer examination of this worksheet shows that the facts given are not related; they are all different. Consequently, the worksheet does not provide an opportunity for students to practise any particular strategy. Instead, it requires them to rely on memorization of the facts – not on remembering the facts in a meaningful and connected way.

A teacher who has introduced a strategy such as doubles in addition (see page 24) should give students a wide range of opportunities to learn that strategy through problem solving, guided experiences, shared experiences, games, and so on. The teacher could use the following worksheet to provide students with practice using the strategy.

2 + 2 = ___	3 + 3 = ___	9 + 9 = ___
8 + 8 = ___	7 + 7 = ___	2 + 2 = ___
6 + 6 = ___	5 + 5 = ___	4 + 4 = ___
1 + 1 = ___	2 + 2 = ___	5 + 5 = ___
7 + 7 = ___	4 + 4 = ___	6 + 6 = ___
8 + 8 = ___	3 + 3 = ___	1 + 1 = ___

This worksheet was created with a purpose in mind. A subsequent worksheet might focus on questions such as 6 + 7 or 6 + 5, which would encourage students to use their known facts (the doubles) to help complete number sentences that illustrate the strategy of near-doubles (see page 24).

A worksheet for junior-level students might focus on a strategy for multiplication that involves using the distributive property. Once students recognize that questions such as 13 x 13 are often easier to answer by multiplying 10 x 13 and adding the **4-6** result to 3 x 13 (130 + 39 = 169), they might benefit from practice of the strategy by using a worksheet like the following:

13 x 13 = ___	13 x 130 = ___	11 x 160 = ___
13 x 14 = ___	18 x 11 = ___	24 x 12 = ___
11 x 13 = ___	18 x 12 = ___	26 x 11 = ___
16 x 11 = ___	23 x 13 = ___	230 x 11 = ___
13 x 15 = ___	11 x 130 = ___	42 x 11 = ___

When students have been exposed to a strategy and have had many opportunities to practise it, a drill with flashcards or a worksheet may be appropriate. The flashcards or worksheet, such as the one in the example provided above, should deal only with questions that apply to the strategy students have been exploring. Students will not have to rely on memory alone; they will have had many opportunities to understand the strategy and to make connections, and they will draw on those connections and on their understanding.

Classroom Idea to Support the Strategy

Have students use a worksheet such as The Big Race to rehearse specific strategies (see Appendix 10-2 for instructions; see also BLM1 in Appendix 10-3).

STRATEGY FLASH CARDS

In the past, flash cards were used for drill and practice of the basic facts. They usually consisted of a basic fact question on the front of the card and the answer on the back of the card. This practice encouraged the memorization of facts and did not emphasize strategies based on a student's understanding of the fact.

Strategy flash cards are created by writing the question on the front of the card and then laminating the card. Students can use "erasable" markers to record on the back of the flash card their answer and the strategy that they used to solve the question.

9 x 5 =

I know 10 x 5 is 50 so I took one 5 away to get 45

5 x 5 = 25 and 4 x 5 = 20 so 25 + 20 = 45

The flash cards should be large enough to allow for the recording of a number of student strategies. As students work with the flash cards, they discover that there are a number of successful strategies to choose from. The connections between these strategies will strengthen their understanding of the operations.

TIMED TESTS

Over the years, students have been given a variety of timed tests to demonstrate their learning, using worksheets similar to the one listing unrelated facts in the example near the top of page 17. This custom does not help students to consolidate their understanding. A time limit should not be placed on tests or worksheets when students are in the process of learning their basic facts, for the following reasons:

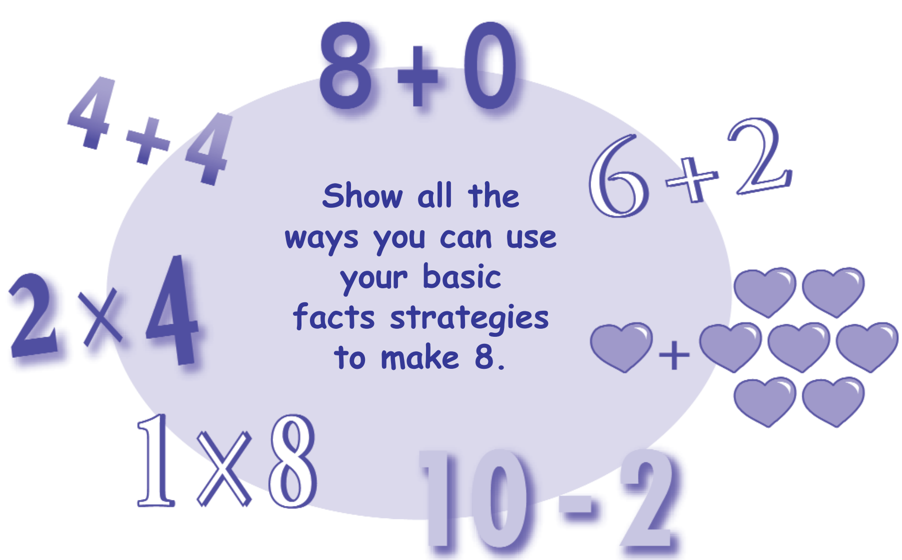
- A time limit discourages students from double-checking for accuracy.
- A time limit may intimidate students who cannot recall the facts quickly but who may be very accurate.
- Timed tests can create negative attitudes about mathematics in students who are not competitive.
- Timed tests do not provide a window into a student's thinking.
- Timed tests do not tell the teacher what strategies students are using.

"Teachers who use timed tests believe that the tests help children learn basic facts. This perspective makes no instructional sense.… Children who have difficulty with skills or who work more slowly run the risk of reinforcing wrong practices under pressure. Also, they can become fearful about, and negative toward, their mathematics learning."

(Burns, 1995, p. 408)

It is important for teachers to focus assessments not only on the answers students give, but also on the strategies they use to produce those answers and on their understanding of the underlying mathematical concepts and connections.

One method of assessing students' understanding is to ask them to show all the different ways they can use the strategies they know to make 8 (or another number) and have students record their answers. The teacher can use students' answers as evidence of their knowledge of, and their ability to apply, basic facts strategies.



Because it allows students to demonstrate their knowledge of the strategies in a wide variety of ways, this exercise can provide the teacher with a much clearer understanding of what students are thinking. In the example provided, the teacher would be able to ascertain that a student knows his or her doubles for addition and multiplication, the zero facts, and "one more than" and "two more than", and that he or she is able to make ten. This is a simple example of a way in which teachers can have students show what they know.

> *"Fact mastery relies significantly on how well students have constructed relationships of numbers and how well they understand the operations."*
>
> (Van de Walle, 2004, p. 156)

Basic Addition and Subtraction Facts

1–3

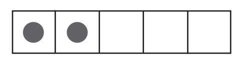
USING MODELS TO REPRESENT FACTS OF ADDITION AND SUBTRACTION

Using models can help students develop meaning for the operations and lessen the abstraction of the operations. Many models can help students make sense of the operations, including the following:

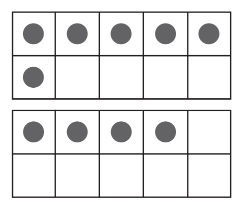
- movable objects such as counters or tiles;
- visual materials, such as pictures or arrays (an arrangement of rows and columns);
- five frames;
- ten frames;
- part-part-whole mats;
- two-colour tiles;
- number lines;
- hundreds charts.

Models can help students understand what the symbols in the operations represent. For example, in the problems shown on pages 9–10, students may use either an addition algorithm or a subtraction algorithm, depending on how they represent the problem. Teachers can help students match their understanding of the problems first with the manipulatives and then with the symbols as represented by the equal, plus, and minus signs.

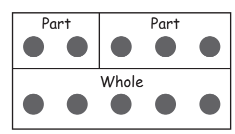
Array
$4 + 4 = 8$



Five frame
2 on a five frame



Ten frame
$6 + 4$ on a ten frame



Part-part-whole mat
$2 + 3 = 5$

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Hundreds chart

Models help students make connections between the models, the symbols, and the words. Below are different models of how students might represent the basic fact 6 + 3. All four representations are appropriate; however, as students gain automaticity with basic facts, they will move away from reliance on visual representations.

Pictures: ♥♥♥ ♥♥♥

♥♥♥

Number line:

Words: Six cookies and three more cookies makes nine cookies altogether.

Symbols: 6 + 3 = 9

STRATEGIES FOR BASIC FACTS OF ADDITION AND SUBTRACTION

Teachers can help students develop efficient strategies for fact retrieval by encouraging them to use reasoning skills and their observation of patterns and the relationships among numbers. The "facts" strategies described below build on students' prior knowledge of how to determine unknown facts. For instance, knowing the answer to 2 + 2 helps students work out the answer to 2 + 3 (since 3 is one more than 2, the answer must be one more than 2 + 2), and this is an important reasoning skill.

The strategies should not be taught outside a problem-solving context. Teachers should:

- choose good problems that encourage the use of these strategies;
- provide opportunities for students to concretely model the strategy;
- have students practise the strategies in context.

Students need many experiences with making ten and relating 10 to other numbers before they develop strategies for the basic facts. It is important for students to recognize 10 as an anchor number and to be able to identify relationships between it and the numbers 0 to 10. A solid understanding of how numbers relate to 10 (e.g., 8 is 2 less than 10 and 13 is 3 more than 10) will enable students to see patterns and number relationships when they are introduced to the basic facts strategies. Knowing how numbers relate to each other will be helpful for students when they learn strategies for composing or decomposing numbers. A ten frame is an important tool that allows students to create a concrete representation of the numbers from 0 to 10 and that provides them with the opportunity to find relationships between 10 and the other numbers as they fill part of the ten frame and can see the unfilled part. Students can work with counters and 1 or 2 ten frames to illustrate the numbers from 0 to 20 (two full ten frames). Using different-coloured counters will help students to identify part-part-whole relationships and to see how the numbers relate to 10. For example, having 8 blue counters and 6 red counters on 2 ten frames would allow students to see that 8 blue plus 2 red is 10, plus 4 more red is 14. This example gives students the opportunity to see the decomposition of the number 6 into 2 + 4, allowing for the friendlier addition sentences of 8 + 2 = 10, 10 + 4 = 14.

The facts strategies listed below are meant to help students who have prior understanding of foundational concepts such as the composition and decomposition of numbers and of how operations (such as addition) are used to act on numbers. The strategies are not listed in a particular order. Some students find certain strategies easier to use than others. Other students may ignore certain strategies altogether, because they have their own strategies or because they find it easier to memorize facts than to rely on a reasoning strategy. Nevertheless, the strategies outlined here can provide students with additional tools to help them become more efficient in their mental computations and gain a deeper understanding of numbers and the relationships between them. These strategies should not be presented to students as a series of procedures to be memorized but as tools to be used to help them with mental computation and problem solving.

Students build on their understanding as they are introduced to these strategies and provided with opportunities to rehearse and experiment with them. Activities and games offer interactive learning opportunities as they provide students with a venue for practising a strategy. As students are engaged with their teacher and peers in a game, they are using strategies and incorporating them into their repertoire. Suggestions for activities and games to support the rehearsal of the various strategies are provided following each strategy described below.

Note: *The instructions for each game or activity are given in Appendix 10-2, and the blackline masters follow, in Appendix 10-3.*

“One-More-Than” and “Two-More-Than” Facts

Facts such as 5 + 1 = 6, 5 + 2 = 7, 7 + 1 = 8, and 7 + 2 = 9 have an addend of 1 or 2. Because students can easily recall the next number (+ 1) or the number after that (+ 2), it is advisable to begin instruction with "one-more-than" and "two-more-than" facts.

One More Than, Two More Than

+	0	1	2	3	4	5	6	7	8	9
0	0	**1**	**2**	3	4	5	6	7	8	9
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
3	3	**4**	**5**	6	7	8	9	10	11	12
4	4	**5**	**6**	7	8	9	10	11	12	13
5	5	**6**	**7**	8	9	10	11	12	13	14
6	6	**7**	**8**	9	10	11	12	13	14	15
7	7	**8**	**9**	10	11	12	13	14	15	16
8	8	**9**	**10**	11	12	13	14	15	16	17
9	9	**10**	**11**	12	13	14	15	16	17	18

Classroom Idea to Support the Strategy

Let pairs of students play Domino Drop, Roll-O, and Elevator Ride to practise these strategies (see Appendix 10-2 for instructions; see also BLM2, BLM3, and BLM5 in Appendix 10-3).

Facts With Zero (1 + 0 = 1, 1 – 0 = 1)

These facts have zero as one of the addends. Students often overgeneralize the idea that answers to addition questions are always bigger than either of the addends and that answers to subtraction questions are always smaller than the minuend. Using flashcard and number cube games can reinforce the concept of zero.

Facts With Zero

+	0	1	2	3	4	5	6	7	8	9
0	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	**1**	2	3	4	5	6	7	8	9	10
2	**2**	3	4	5	6	7	8	9	10	11
3	**3**	4	5	6	7	8	9	10	11	12
4	**4**	5	6	7	8	9	10	11	12	13
5	**5**	6	7	8	9	10	11	12	13	14
6	**6**	7	8	9	10	11	12	13	14	15
7	**7**	8	9	10	11	12	13	14	15	16
8	**8**	9	10	11	12	13	14	15	16	17
9	**9**	10	11	12	13	14	15	16	17	18

Classroom Idea to Support the Strategy

Let pairs of students play Off to the Races and Race for Cover to practise these strategies (see Appendix 10-2 for instructions; see also BLM6 and BLM7 in Appendix 10-3).

Doubles (6 + 6, 2 + 2, etc.)

These facts have two addends that are the same. Students benefit from recognizing and learning the doubles. There are only ten double facts. Many doubles can be linked to memory devices:

3 double: An insect with 3 legs on one side and 3 on the other (6)

4 double: A spider with 4 legs on one side and 4 on the other (8)

5 double: The 5 fingers on one hand and the 5 on the other (10)

6 double: An egg carton with 6 egg holders on one side and 6 on the other (12)

7 double: The 7 days in one calendar week and the 7 days in the next week (14)

8 double: The 8 crayons in one row of the crayon box and the 8 in another row (16)

Doubles

+	0	1	2	3	4	5	6	7	8	9
0	**0**	1	2	3	4	5	6	7	8	9
1	1	**2**	3	4	5	6	7	8	9	10
2	2	3	**4**	5	6	7	8	9	10	11
3	3	4	5	**6**	7	8	9	10	11	12
4	4	5	6	7	**8**	9	10	11	12	13
5	5	6	7	8	9	**10**	11	12	13	14
6	6	7	8	9	10	11	**12**	13	14	15
7	7	8	9	10	11	12	13	**14**	15	16
8	8	9	10	11	12	13	14	15	**16**	17
9	9	10	11	12	13	14	15	16	17	**18**

Classroom Idea to Support the Strategy

Students can use many activities and games to practise this strategy: Concentrating on Doubles, Magic Doubles, Over-Easy Doubles, Last One to School, and Seeing Doubles (see Appendix 10-2 for instructions; see also BLM8, BLM9, BLM 10, and BLM11 in Appendix 10-3).

Near-Doubles or Doubles Plus One (5 + 4 can be thought of as 4 + 4 plus 1 more)

These facts have one addend that is one more than the other addend. Students can learn to recognize that addition facts in which one addend is one more than the other have the same answer as the double of the smaller addend plus one ($5 + 4 = 4 + 4 + 1$). Students need to know the doubles before they can attempt to use this strategy effectively.

Classroom Idea to Support the Strategy

Students can play Snappy Doubles, Find a Friendly Neighbour, and Spinning for Near-Doubles to practise these facts (see Appendix 10-2 for instructions; see also BLM14 in Appendix 10-3).

"Make-Ten" Facts (9 + 5 can be solved as 9 + 1 = 10 and then 10 + 4 = 14)

These facts have 8 or 9 as one addend. Students add 1 or 2 from the other addend to get 10 and then add the rest of the addend. Students benefit from having many experiences working with a ten frame so that the facts that add to 10 become automatic. They then benefit from experiences "making tens". For instance, in questions such as 8 + 5, students could make 10 with the 8 plus 2 from the 5 and then add the remaining 3 to make 13. Once these facts are consolidated, it becomes much easier for students to work with larger numbers, such as 18 + 5. They know that 18 + 2 makes 20, and they then add the remaining 3 to make 23.

Classroom Idea to Support the Strategy

Students can use games and activities like Flashy Fingers, Flip for 10 Plus, and Framing Ten and Up to practise this strategy (see Appendix 10-2 for instructions; see also BLM15 in Appendix 10-3).

Commutative Property (1 + 2 = 2 + 1)

Students who recognize the commutative property in addition can use this information to halve the number of facts they need to learn. Visual representations of facts such as 3 + 2 and 2 + 3 help students to see this relationship.

Classroom Idea to Support the Strategy

Play Triplets with the whole class (see Appendix 10-2 for instructions; see also BLM16 and BLM17 in Appendix 10-3).

Subtraction as Think-Addition for Facts With Sums to 10 (7 – 5 = 2 is the same as 5 + 2 = 7)

Students who know their addition facts can use them for the related subtraction facts (if 5 + 2 = 7 then 7 – 5 = 2). For example, "Julian had 6 marbles in his bag. George gave him some more. Then Julian had 14 marbles. How many marbles did George give Julian?" When presented with this problem, the student is thinking, "What number added to 6 makes 14?" He or she is using known addition facts to find the unknown quantity in the sentence.

Subtraction as Adding Up

Students sometimes have difficulty subtracting large numbers. It might be helpful for them to recognize that facts such as 17 – 13 can be solved by counting on from 13 to 17.

“One-Less-Than” and “Two-Less-Than” Facts

These facts have a subtrahend of 1 or 2 (e.g., 8 – 1, 4 – 2). Students can usually count back for the easier facts, such as any number – 0, – 1, or – 2.

Classroom Idea to Support the Strategy

Students can play Whichever Way Wins or the Dot Plate Game to rehearse this strategy (see Appendix 10-2 for instructions; see also BLM18 and BLM30 in Appendix 10-3).

Counting by Twos and Fives

Experiences counting by twos and fives, starting with any number – orally, on a hundreds chart, or using a calculator – can help students with the facts for + 2, + 5, and + 10.

“Make Ten” Extended

This strategy involves making tens with the decade numbers. For example, 17 + 5 can be thought of as 17 + 3 to make 20, and then the leftover 2 is added to the 20 to make 22.

Classroom Idea to Support the Strategy

Extend the game Framing Ten and Up to include number cards with decade numbers (e.g., 18, 19, 28, 29, 38, 39) and use 5 ten frames to play the game (see Appendix 10-2 for instructions; see also BLM15 in Appendix 10-3).

Back Down Through Ten

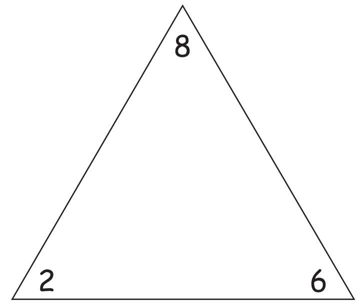
This strategy involves working backwards with 10 as a “bridge”. For example, 13 – 5 can be solved by subtracting 3 from 13 to get to 10, and then subtracting 2 to make 8.

PRACTICE AND CONSOLIDATION OF FACTS FOR ADDITION AND SUBTRACTION

Students need practice in the selection and use of computational strategies. To help them practise selecting and using strategies, the teacher should try the following approaches:

- Use problem solving as the route to practising the facts.
- Model problems (e.g., using counters) when needed.
- Use ten frames, number lines, hundreds charts, and calculators (e.g., to help with self-correction).
- Recognize that the level of strategy development for recalling the facts is rarely the same for all students, so drill activities that assume it is are not beneficial to many students.
- Use games, repetition of worthwhile activities or songs, and mnemonic devices to individualize strategy development.

- Ensure that any drill practice is focused on using strategies and not just on rote recall.
- Cluster facts and practice around strategies.
- Use strategy flash cards so that students can share the strategies they use to arrive at an answer.
- Have students make their own strategy list for the facts they find the hardest.
- Use triangular flashcards so that students can make the connection between the addition fact and its related subtraction fact. (Any drill activities of this nature should be short – 5 to 10 minutes – and should concentrate on a group of facts at a time. Keep the activities varied, interesting, and challenging.)



Put your finger over the 8 and ask, "What is 2 + 6?" Then, put your finger over the 2 and ask, "What is 8 – 6?"

Basic Multiplication and Division Facts

1-3 4-6

THINKING ABOUT MULTIPLICATION AND DIVISION

Learning basic facts is more complex than simply relying on memorization. It is important to realize that there are several different ways to think about these operations:

- *Multiplication* can be thought of as repeated addition, as an array, and as a collection of equal groups. Some properties and strategies that help with a conceptual understanding of multiplication are:
  - the identity property of whole number multiplication ($a \times 1$ is always a);
  - the zero property of whole number multiplication ($0 \times a$ is always 0);
  - the commutative property ($2 \times 3 = 3 \times 2$);
  - the distributive property ($4 \times 6 = 2 \times 6 + 2 \times 6$);
  - the associative property [$(2 \times 3) \times 4 = 2 \times (3 \times 4)$];
  - the inverse relationship with division.
- *Division* can be thought of as sharing, as making equal groups, or as repeated subtraction. Some properties and strategies that help with conceptual understanding of division are:
  - the identity property ($6 \div 1 = 6$);
  - the relationship of division to fractional sense (6 candies divided into 2 groups represents both $6 \div 2$ and the whole divided into 2 halves);
  - the inverse relationship with multiplication.

An understanding of the remainder in a division problem is often an area of difficulty. Consider, for example, the question, "If there are 3 students and 10 candies, how many candies does each student get?" Students who represent the problem concretely will not have difficulty, but students who create a division question such as

10 ÷ 3 = 3 R1 may have limited understanding of what the remainder of 1 represents – is it one more student or one more candy? The notion of a remainder is abstract, and students may not have any real sense of it (this is important to remember when students use calculators for division, too).

Basic Multiplication Facts

- Basic multiplication facts consist of all multiplication facts from 0 x 0 to 9 x 9.
- There are 100 basic multiplication facts.

Basic Division Facts

The multiplication grid can also be used for division.

- The division facts are the inverse of the multiplication facts, from 81 ÷ 9 to 1 ÷ 1.
- There are 90 division facts.
- They do not include any facts with 0 as a divisor, because a quantity cannot be divided into 0 groups or into groups of 0.

Multiplication Grid

x	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

USING MODELS TO REPRESENT FACTS OF MULTIPLICATION AND DIVISION

Using models can help students develop meaning for operations and lessen the abstraction of operations. Students can use various objects and materials to make models that will help them make sense of operations, including the following:

- movable objects such as counters, craft sticks, and tiles, and containers to put them in;
- multicubes or other interlocking cubes;
- visual materials such as pictures;
- arrays (an arrangement of rows and columns);
- base ten blocks;
- money (pennies, nickels, dimes, quarters, dollars);
- grid paper;
- number lines;
- hundreds charts.

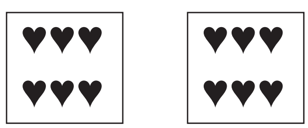
Models and representations can be used to help students understand what the symbols in operations mean. Students need to work with representations that help them see the relationships between multiplication and repeated addition, and between multiplication and division.

Arrays are powerful models for representing multiplication, division, and area. Many objects in real life can be arranged in arrays, and such arrays provide visual, concrete contexts for multiplication and division. The construction and understanding of an array as a model is an important concept for students to develop over time and with experience.

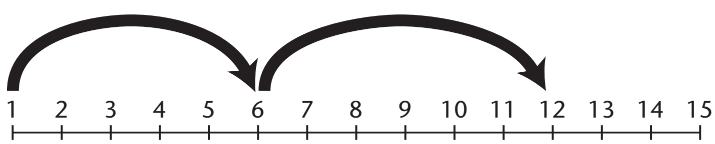
When students understand arrays, they are able to break factors into smaller, "friendlier" parts (e.g., $15 \times 12 = 10 \times 12 + 5 \times 12$). This ability helps develop an understanding of the distributive property as a strategy for multidigit multiplication.

Models help students make the connections between the models, the symbols, and the words. The examples that follow illustrate different models of how students might represent 2×6. All seven representations are appropriate; however, as students gain automaticity with basic facts, they will move away from reliance on visual representations. In the primary grades, only a few students will attain automaticity with basic multiplication facts. The emphasis is on understanding the concept and accessing the strategies for computations with multiplication.

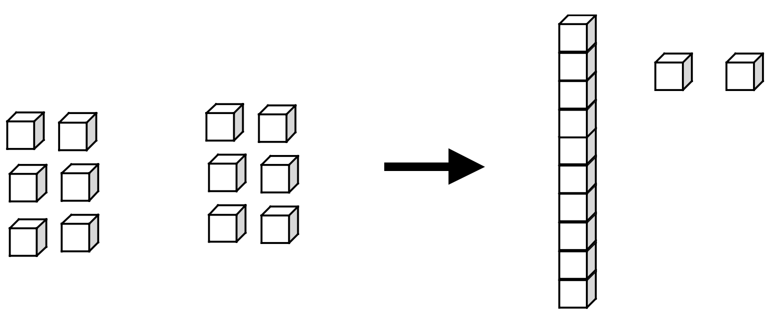
Pictures:



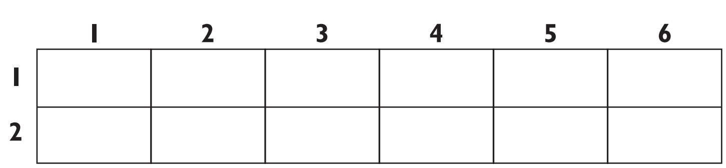
Number line:



Base ten blocks:

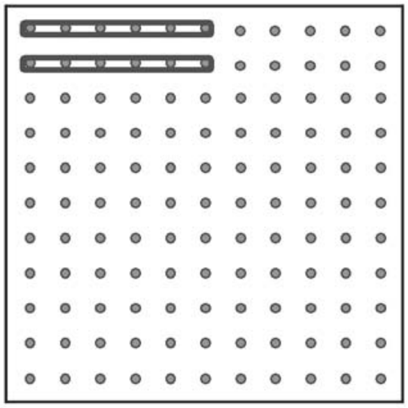


Arrays:



2 rows x 6 columns

Geoboard:



Words: Two boxes of six cookies makes twelve cookies altogether.

Symbols: $2 \times 6 = 12$

STRATEGIES FOR BASIC FACTS OF MULTIPLICATION AND DIVISION

Teachers help students develop efficient strategies for fact retrieval by encouraging them to use reasoning skills and to look for patterns and relationships between numbers. The "facts" strategies described on the following pages build on students' prior knowledge of how to determine unknown facts. For instance, knowing $2 \times 2 = 4$ helps students to know $3 \times 2 = 6$ (3×2 is one more 2 than 2×2), and this is an important reasoning skill.

The strategies should be taught within a problem-solving context. Teachers should:

- choose good problems that encourage the use of these strategies;
- provide opportunities for students to concretely model the strategy;
- have students practise the strategies in context.

Approaching basic facts instruction in this manner will help students develop a deeper understanding of the operations.

Before being introduced to the basic facts of multiplication and division, students need many experiences in using the operations in problem-solving situations and in modelling the relationships and actions inherent in these operations. They need to have developed an understanding of making equal groups and counting them, and of partitioning a whole into equal groups. Unitizing, or counting not only individual items but also groups, is a central concept for understanding multiplication and division. Students who know how to unitize are able to see the whole as the number of equal groups of a number of objects. Students also need to be able to skip count for different numbers (2, 3, 4, 5, and 10). The facts strategies are meant to help students who have prior understanding of these concepts. Students build on their understanding as they use the strategies.

The following strategies are not listed in a particular order. Some students prefer certain strategies. Other students may ignore certain strategies altogether, because they have their own strategies or because they find it easier to memorize facts than to rely on a reasoning strategy. Nevertheless, the strategies outlined here can provide students with additional tools to help them become more efficient in their mental computations and gain a deeper understanding of numbers and the relationships between them. These strategies should not be presented to students as a series of procedures to be memorized but as tools to be used to help them with mental computation and problem solving.

Commutative Property

Students benefit from experiences that help them to identify the commutative property of multiplication ($2 \times 4 = 4 \times 2$). Understanding the commutative property means that half the facts can be used to know the other half.

Commutative Property Grid

x	0	1	2	3	4	5	6	7	8	9
0	**0**	0	0	0	0	0	0	0	0	0
1	**0**	**1**	2	3	4	5	6	7	8	9
2	**0**	**2**	**4**	6	8	10	12	14	16	18
3	**0**	**3**	**6**	**9**	12	15	18	21	24	27
4	**0**	**4**	**8**	**12**	**16**	20	24	28	32	36
5	**0**	**5**	**10**	**15**	**20**	**25**	30	35	40	45
6	**0**	**6**	**12**	**18**	**24**	**30**	**36**	42	48	54
7	**0**	**7**	**14**	**21**	**28**	**35**	**42**	**49**	56	63
8	**0**	**8**	**16**	**24**	**32**	**40**	**48**	**56**	**64**	72
9	**0**	**9**	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**

Classroom Idea to Support the Strategy

Students can play Triplets for multiplication to practise this strategy (see Appendix 10-2 for instructions; see also BLM19 and BLM20 in Appendix 10-3).

Rules for Zero and One

Students benefit from experiences that help them to see the identity rule – one times any number has that number as its product ($1 \times 3 = 3$) – and the zero rule – zero times any number is zero. Experiences with problems, such as, "If I give you 3 handfuls of 0 cookies, how many cookies do you have?" are more helpful in understanding these rules than is rote practice.

Classroom Idea to Support the Strategy

In a shared learning experience, play Spinning for Zero to help students explore zero and its rules (see Appendix 10-2 for instructions; see also BLM21 and BLM22 in Appendix 10-3).

Doubles

The two times table should be linked to student's prior knowledge about the addition doubles. This strategy is particularly helpful because students who know the two times table well can relate these facts to the three times table. If 2 x 4 is 8, then 3 x 4 is 8 plus one more 4.

Double and Double Again (Van de Walle, 2001, p. 142)

Once students know the double facts (two times tables), they can apply this knowledge to the four times table. Any number times four is the same as double the answer to two times that number (4 x 6 is the same as 2 x 6 = 12, with the 12 doubled for an answer of 24).

Double and One More Set (Van de Walle, 2001, p. 143)

Again, if students know the double facts (two times table), they can use this information to figure out the three times table. If 2 x 4 is 8, then 3 x 4 is 8 plus one more 4 to make 12.

Five Facts

Students who have experience counting by fives and identifying the five fact patterns on a hundreds chart usually do not have difficulty learning the five facts. A wide variety of songs, poems, and storybooks based on fives are available that will support learning this strategy.

Add One More Set (Van de Walle, 2001, p. 142)

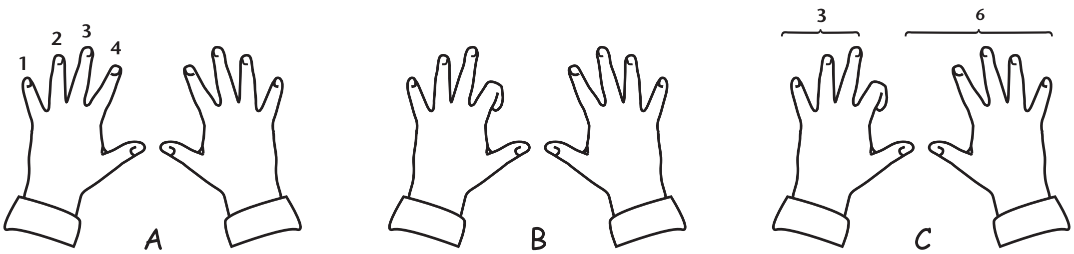
If a student knows 7 x 6 = 42, then he or she can figure out 8 x 6 by adding one more 6 to the 42 to make 48. This strategy is especially helpful with the six times table, which can be difficult. If the five times table is well retained, students figure out the six times table by simply adding one more of the factors in each case. For example, if 5 x 6 = 30, then 6 x 6 is simply 30 plus one more 6, or 36.

Nine Times Table

Students often learn the ten times table quite easily because they learn to count by tens at an early age. Once they know the ten times table, they can relate it to the nine times table by calculating the related ten times fact and then subtracting one more of the factors. For instance, 10 x 8 is 80, so 9 x 8 is 80 less one 8 or 72.

Teachers can help students recognize some of the patterns in the nine times table. For instance, all the digits in the products are equal to 9 if they are added (5 x 9 = 45; 4 + 5 = 9), and the tens in the product are always one less than the multiplier. For instance, in 6 x 9 = 54, the tens digit is one less than the multiplier of 6, and the sum of the two digits in the product (5 and 4) is 9. Students who conceptually understand the nine times table but who fail to retain the actual facts may benefit from learning the finger patterns, as shown below. Some students find the kinaesthetic method of remembering the nine times table helpful:

4 x 9



Students are asked to hold out their hands with their palms facing away from them. Their fingers (beginning at the left little finger) are numbered from 1 to 10 (A). The finger that represents the multiplier of nine is folded down. If multiplying 9 x 4, the fourth finger is folded down (B). The number of fingers to the left of the folded finger represents the tens digit. The number of fingers to the right of the folded finger represents the ones (C). So, in this instance, there are 3 fingers to the left of the four, representing 30, and 6 fingers to the right of the 4, representing 6, for a total answer of 36.

Half Then Double (Van de Walle, 2001, p. 143)

Suppose a student cannot recall a fact and the fact involves even factors. One factor can be halved, and if that is known, the product can be found and then doubled. For instance, if 6 x 8 cannot be recalled, halve one factor (3 x 8 = 24), and then double the product of 24 for an answer of 48. Students should investigate this strategy by using arrays to provide a visual representation. Sometimes, the strategies may seem cumbersome, but they provide students with a way to reason their way to a correct answer using known facts.

Inverse Relationship of Division and Multiplication

Students who know their basic multiplication facts should use this information to figure out the division facts. It is recommended that division and multiplication be taught simultaneously so that the relationships between the two are obvious. When students join 4 groups of 5 together for a total of 20, the teacher should encourage them to recognize that when the total is partitioned back to 4 groups, each group has 5.

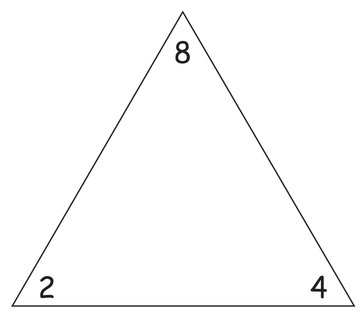
Learning the Division Facts in Sequence

A suggested sequence for teaching the division facts is to begin with the facts for divided by two, then proceed to divided by one, divided by five, divided by three, divided by four, divided by six, divided by seven, divided by eight, and divided by nine.

PRACTICE AND CONSOLIDATION OF FACTS FOR MULTIPLICATION AND DIVISION

Students need practice in the selection and use of strategies. To help them get the kind of practice that will benefit them most, the teacher should try the following approaches:

- Continue to use problem solving as the route to practising the facts.
- Continue to use models and manipulatives.
- Use ten frames, number lines, hundreds charts, and calculators (to help with self-correction).
- Recognize that the level of strategy development for recalling the facts is rarely the same for all students, so drill activities that assume it is are not beneficial to many students.
- Use games, repetition of worthwhile activities or songs, and mnemonic devices to individualize strategy development.
- Ensure that any drill practice is focused on using strategies, not just on rote recall.
- Cluster facts around strategies.
- Use multiplication grids and shade the areas that can be learned using the strategies set out above; students will find only six facts that really benefit from memorization: $6 \times 6 = 36$, $6 \times 7 = 42$, $7 \times 7 = 49$, $7 \times 8 = 56$, $8 \times 8 = 64$.
- Have students make their own strategy list for the facts they find the hardest.
- Use strategy flash cards so that students can share the strategies they use to find an answer.
- Use triangular flashcards so that students can make the connection between the multiplication fact and its related division fact.



Put your finger over the 8 and ask, "What is 2 x 4?" Then, put your finger over the 4 and ask, "What is 8 ÷ 2?"

Multidigit Whole Number Calculations

1-3 **4-6**

Algorithms are the step-by-step procedures for computing numbers in addition, subtraction, multiplication, and division. Simply put, an algorithm is "a recipe for computation" (Kilpatrick, Swafford, & Findell, 2001, p. 103).

"There is mounting evidence that students both in and out of school can construct methods for adding and subtracting multidigit numbers without explicit instruction."

(Carpenter, Franke, Jacobs, Fennema, & Empson, 1998, p. 4)

Many teachers learned only one way to solve multidigit computations – using the standard North American algorithm taught in schools. Solving a two-digit computation can be done in many ways, depending on the context of the problem, on whether it is being solved mentally or as a paper-and-pencil task, and on the numbers involved. When asked to solve a question such as 49 + 49, one can probably solve it most efficiently by thinking of 50 + 50 and then subtracting 2, rather than doing the extensive regrouping and carrying involved in the standard algorithm. Making change from a $10 bill for a $7.69 purchase also does not lend itself to the standard algorithm; instead, adding up from $7.69 by ones, tens, and then hundreds is a faster and more efficient way to make the calculation.

Similarly, when asked to mentally solve a problem such as 27 + 38, many students and adults try to use the standard algorithm (add the ones, carry the ten, add the tens, try to remember what was in the ones column, etc.) in their head. This method is sometimes effective, but not always. For instance, trying to do two-digit math mentally by using the standard algorithm is often quite hard, depending on the numbers involved. The student who has good number sense and knows how to compose and decompose numbers may solve a question such as 27 + 38 in any one of the following ways:

- "I can add 38 and 20 to make 58. I add 2 to get 60. The leftover 5 makes 65."
- "I can add 27 and 3 to make 30. I add the 30 to 35 (which is 38 minus the 3 I already used) and get an answer of 65."
- "I can combine 27 and 30 to make 57 and then add the remaining 8 to make 65."
- "I can add 30 plus 40 and take away 5 for an answer of 65."

In the above examples, the student is selecting a computational strategy based on reasoning and flexibility to create his or her own strategies for addition that are the most efficient for the situation. This reasoning indicates that the student has computational fluency and accuracy, as well as an understanding of place value and the composition and decomposition of numbers.

An important focus of computational fluency is on the development of such proficiency and flexibility in using a variety of strategies and algorithms. Students who can work flexibly with numbers, as in the above examples, are more likely to develop efficient strategies, accuracy, and a strong foundation for understanding a variety of algorithms. Substantial evidence shows that students benefit from many opportunities to develop their own ways of solving multidigit computations (Morrow & Kenney, 1998).

In the past, it was assumed by many that using the standard North American algorithm for an operation was the only way to perform that calculation. Cultural differences in how such calculations are done indicate that there are a number of efficient strategies and algorithms. *Many adults tend to believe that the algorithm they were taught is the only one possible. There are, in fact, several different ways to perform calculations efficiently*. Below are examples of different ways to solve 27 + 48 using a variety of strategies.

27 +48 75	27 +48 75	27 +48 15 60 75	$^{1}27$ +48 75	27 +48 60 10 5 75
Think 7+8=15; mentally add the 10 in 15 to 20 to make 30, and then add 40.	Think 7+8=15. Add 15 to the 20 to make 35. Add 35 to 40 to make 75.	Add the ones column, add the tens column, and then combine.	Record the 10 from the answer of 15 to the tens column and then add (the standard North American algorithm).	Add mentally by combining 20 and 40 to make 60. Add 8 and 2 to make another 10, then add the remaining 5 ones to make 75.

Each strategy contains different levels of abstraction. The levels are shown here to demonstrate the differences in how the strategies are used. Some steps are more explicit than others, while some steps are shortened to create a faster but more abstract route to the answer.

The standard North American algorithms were established to help make calculations fast. They incorporated many useful shortcuts, such as in the double-digit addition algorithm described above. These shortcuts are practical and useful for those who understand the algorithm, but for students who have not been taught the underlying concepts in the algorithm, memorizing the abstract algorithm is often the beginning of their belief that mathematics "doesn't make sense" and is simply dependent on memorizing rules and routines. Such memorization may seem useful in the short term, but it often fails students in the long term. Teachers know that many students have forgotten the subtraction or addition algorithm by September of the next year, and often the same cycle of memorizing algorithms without understanding them continues.

Marilyn Burns (2000, p. 144) gives the following anecdote about a teacher (Sandra) and her Grade 4 students' misconceptions about standard algorithms:

Students: First we x's one times nine. Next we x's the one by the one. Then we go down a line and put a zero under the nine. Next we x's the two by the nine and the two by the one. Now that we have two answers (one on top and one on bottom) we add them and get an answer of 399.

$$\begin{array}{r} 19 \\ \times\ 21 \\ \hline 19 \\ 380 \\ \hline 399 \end{array}$$

Sandra: Why did you go down a line? What do the two answers mean? Why did you add them?

Students: We changed lines because the rules are to change lines and the rules are also to put a zero in the beginning of the second line. The rules are that way because there is no way to get 399 with 380 on the same line as 19. We found this out in 3rd grade. We added them because that's the way we were taught and it gives a sensible answer.

Sandra: Now, can you explain each of the numbers in your answer?

Students: The 19 means one times 19 equals 19. The 38 means that two times 19 is 38. (Remember it's a rule to put a zero on the second line.) The 399 means 19 plus 380 equals 399. That's how we did the problem.

Sandra: Why is it a rule to put a zero on the second line? What's the purpose of the rule?

Students: Nobody in our group can remember why we put the zero on the second line.

Many students have misconceptions about the rules for algorithms. They memorize the rules without understanding what is happening to the numbers. Students in the above anecdote had no understanding that they were multiplying 20 x 19, not 2 x 19, and that the reason they needed a zero in the fourth row was because the answer was 380.

We know that students often come to a greater understanding of the standard algorithm if they are allowed to develop their own strategies and to connect their strategies to more efficient strategies, including the standard algorithm. As well, research has indicated that students in primary grades can do much more complex computations than we generally give them credit for – with two-digit and three-digit numbers, both mentally and in written form – if they are given the opportunity to learn multidigit calculations in a meaningful, sense-making way and have many experiences developing their own strategies for determining two-digit calculations (Carpenter et al., 1998; Kamii, 1985; Morrow & Kenney, 1998).

Students who are encouraged to use their own flexible strategies for computing multidigit numbers develop the following:

- **A better sense of number:** Students focus on the numbers rather than on the digits. For instance, in an addition problem such as 43 + 56, they recognize that the left-hand digits are not 4 + 5 but rather 40 and 50. Students who think of them as 4 + 5 are not using placement to determine value.
- **More flexibility in solving problems:** Instead of using only one way to answer a two-digit addition problem, students can answer a question such as 39 + 41 by combining the 1 with the 39 to make 40 and then adding the additional 40 to make an easier algorithm (40 + 40). In another problem, such as 17 + 55, they might change the 17 to 20, add 20 to 55 to make 75, and then subtract the extra 3 to make 72. Depending on the specific question, different ways of finding the answer may be appropriate, as long as the student uses reasoning.
- **A stronger understanding of place value:** In developing their own strategies, students will often use a left-to-right orientation rather than the traditional right-to-left orientation, which means the magnitude of the number as represented by the left digit stays foremost in their mind. For example, when adding 35 + 67, students will likely find it easier to add the tens of 30 and 60 (= 90) and then add the 5 + 5 to make another 10 (100) and then add the 2 left over for a total of 102.
- **Greater facility with mental calculations:** Students who are not dependent on the "cross out and borrow" method will find it easier to do mental computations. For example, when solving 90 – 87, students will count up from 87 to 90 rather than use an algorithm. Trying to keep track mentally of the "cross out and borrow" method is much more difficult for students (and adults). Students who can think of a question such as 49 + 49 as the same as 50 + 50 – 2 have a more efficient strategy for calculating an answer.
- **More ease in linking meaning to the symbols in the traditional algorithms when they are introduced:** Students who understand how to manoeuvre their way through numbers and calculations using methods that require them to reason their way to a solution recognize that mathematics makes sense. This reasoning can then be linked to the standard algorithms, and the points can be made that standard algorithms can be very useful and that they do make sense. It is easier for students to remember something that they understand.

TEACHING MULTIDIGIT COMPUTATIONS

It is very important that students have many opportunities from Kindergarten to Grade 3 to build prerequisite skills for working with computations by using, for example, part-part-whole concepts, anchors of 5 and 10, combinations to 10, and combinations to 20. In all grades, practice in composing and decomposing numbers into ones, tens, and sometimes hundreds will help students link these important concepts to multidigit computations.

The use of pictures, words (spoken and written), and numbers to communicate such computations helps students make the connections between concrete and symbolic representations of number operations. The premature introduction of abstract symbols (e.g., the equal sign), without linking such symbols to the concepts they represent (e.g., the equal sign represents equality or balance), is often the source of students' difficulty with computations. Rather than making sense of computations, students begin to see mathematics as a series of rules and routines with little applicability to day-to-day problem solving. The most important goal of teaching algorithms is to help students make sense of math and develop efficient and generalizable strategies.

The equal sign (=) represents equality or a balance between the left and right sides of an equation. It does not mean "the answer to".

Listed below are some of the important instructional points teachers should consider when they make decisions about teaching multidigit computations:

- **Problem-solving context:** All computations should be taught within a problem-solving context. Presenting students with real problems that involve multidigit computations provides a setting to help them further understand how to compose and decompose numbers. A problem such as the following provides students with a meaningful context as well as an incentive for finding a solution.

  "Today for art, we are going to be painting. We each need our own paintbrush. One box has 10 paintbrushes. How many more paintbrushes do we need?"

  Even young students (who do not know the standard algorithm) can devise methods to find the answer when given a problem involving 24 candies and 17 more candies.

- **Prior knowledge:** Students' understanding of and experiences in the earlier grades with the following should be linked to multidigit calculations:
  - part-part-whole concepts;
  - operations with single digits;
  - place value (e.g., that the 2 in 25 represents 2 tens or 20 ones);
  - how to represent single-digit computations using manipulatives, words, and pictures.

- **Mental computations:** Working problems out mentally is sometimes the conduit for students to develop more efficient ways of doing calculations using paper and pencil. Teachers can encourage students to use mental computations by showing computations horizontally (34 + 26) rather than vertically:

$$\begin{array}{r} 34 \\ +\ 26 \\ \hline \end{array}$$

  Students who take the time to think about a computation and how it can best be solved before they begin to do anything on paper are more likely to choose the most effective method for solving the problem. For instance, some thinking before doing the computation 19 + 21 may help a student recognize that an easy strategy would be to combine the 1 with 19 to make 20 and then add the remaining 20 for an answer of 40. When faced with a computational problem, students often – but not always – use flexible strategies mentally (with some recording to keep track of the solution).

- **Sharing:** It is imperative that students have the opportunity to share the strategies they use and to analyse the strategies used by others. To ensure that students have this opportunity, teachers should:
  - allow sufficient time for students to develop their own strategies;
  - allow students time to explore one another's strategies in pairs, in small groups, or with the whole class;
  - have students explain their strategies to their classmates.

- **Manipulatives:** Teachers should encourage students to use manipulatives to demonstrate the strategies they use. This kind of activity helps students remember what they did when it comes time to share with the class.

- **Observation:** By observing what students are doing, teachers become aware of the strategies students use. Some things to look for are:
  - whether students are aware that a digit in the tens place, such as the 2 in 28, represents 2 tens or 20 ones and not 2 ones (a common error students make when first using base ten materials);
  - whether students are able to count on and to do so with tens as well as with ones, as in this example: 25 + 35 is the same as 25, 35, 45, 55, and 5 more for 60;
  - whether students are able to use prior knowledge of the basic facts (single-digit computations) to help them with multidigit computations. For instance, "I know that 6 and 6 is 12, so 60 and 60 is 120."

Steps for Working Through a Multidigit Problem

The steps for working through a multidigit problem with students are outlined below. On the left is a description of what the teacher asks students to do, and on the right is an outline of some of the important points teachers need to consider.

What the Teacher Asks Students to Do	Important Points for the Teacher to Consider
Give students a computational question in a problem-solving context ("We need 100 people to sign up for the Skip-a-Thon. So far, 77 people have signed up. How many more people do we need to sign up?").	Encourage the use of horizontal formats such as 55 + 46 = or 92 – 45 = . This format encourages students to develop their own strategies rather than to rely on the traditional vertical arrangement. Some students may require the use of concrete or visual representations, even with large numbers. Vary the types of questions used for the problems (see the "Basic Facts" section for further information on problem types).
Encourage students to work in groups or pairs to solve the computations.	Students who have not yet worked in groups usually need experiences working in pairs before moving to group situations.
Prompt and question students to encourage them to reason their way to a solution.	Ask students who have developed a strategy, "How do you know this works?" A range of responses is possible, including these: "We need 100 people. We know that 77 plus 3 is 80. Then we need 20 more. And 20 + 3 is 23, so we need 23 more people for the Skip-a-Thon." "We need 100. We know that 77 plus 10 is 87, plus 10 is 97. And we need 3 more to get to 100. So the answer is 10 + 10 + 3, or 23."
Encourage students to share the strategies they used to find the solution. Ask students to describe how they found their answers.	Students can listen to one another and then think about how other students' strategies worked. They can also try out someone else's strategy to see how it works.
Encourage students to use manipulatives or drawings to represent their solutions. When they are asked to share their solution, they will be able to re-create it for the class by using the manipulatives. If appropriate, translate the models students present into algorithmic form as students explain what they did and why.	Encourage students to explain how they found a solution. Teachers need to be sensitive to students' experience of risk taking when they share their solutions. Students must be confident that they will be shown respect by their teacher and other students as they share. Orchestrate the sharing so that the important concepts are highlighted. Have students share their strategies with their group or the whole class. Post some examples on charts or the board. Pose another problem to students and have them use someone else's strategy to solve it.

Students will eventually use many strategies to calculate multidigit computations. The methods they use will depend on the context of the questions, on the numbers being used, and on students' personal preferences. As they work through the various methods, students will become more selective about which strategy works best for them. After multiple experiences, most students will adopt the methods that they find the easiest and the fastest and that are accurate. But the method students use on any given problem will vary depending on the numbers, just as the methods that adults use vary. A question such as 200 – 20 does not require a complicated method; simply counting backwards by tens is usually the quickest solution. However, a question such as 201 – 93 lends itself better to adding 7 to 93 to reach 100 and then adding the 101, for an answer of 108.

STUDENT-GENERATED COMPUTATIONAL STRATEGIES

Examples of Student-Generated Computational Strategies

The following are some examples of different ways that students might solve problems for all four operations. These are not the only methods for solving these problems. Many more exist and students need to have the opportunity to reason their way to other methods.

1-3

Addition	Subtraction
Add from left to right and combine 35 + 47 30 + 40 is 70 5 + 7 is 12 So, 35 + 47 = 70 + 12 Or 82	Subtract from left to right 89 – 26 80 – 20 is 60 9 – 6 is 3 So, 89 – 26 is 60 + 3, or 63
Count on by tens, then count on by ones 46 + 23 46…56…66… 67…68…69	Round one or both numbers, subtract, and adjust the answer 78 – 29 78 – 30 is 48 Add back 1 The answer is 49
Add tens first 37 + 26 37 + 20 = 57 57 + 6 = 63	Subtract tens first 93 – 28 93 – 20 is 73 73 – 8 is 65

Use an easier problem

48 + 27

> 50 + 27 = 77
> So, 48 + 27 is 2 less, or 75

Subtract the nearest multiple of ten and adjust the answer

83 – 28

> 83 – 30 is 53
> 30 is 2 more than 28
> So, 83 – 28 is 55

Transform the problem

39 + 57

> Make 39 one larger and
> 57 one smaller.
> 40 + 56 is 96

Use an easier problem

87 – 29

> 89 – 29 is 60
> So, 87 – 29 is 2 less, or 58

Adapted from the Waterloo County Board of Education, 1992, *Addition and Subtraction of Whole Numbers*, p. 23. Used with permission.

4-6

Multiplication

Use an easier problem

59 x 4

> 60 x 4 is 240
> But that is 4 too many
> The answer is 236

Multiply the left-most digit first, then the right-most digit, and add the products

47 x 6

> 40 x 6 is 240
> 7 x 6 is 42
> 240 + 42 is 282
> The answer is 282

Multiply, ignoring the zeros, then include them in the product

30 x 50

> 3 x 5 is 15
> There are two zeros in the factors
> So, the answer is 1500

Division

Work with a part at a time

426 ÷ 6

> 426 is 420 and 6
> 6 x 70 is 420, and 6 x 1 is 6
> The answer is 70 + 1, or 71

Divide, ignoring the zeros, then include them in the quotient

350 ÷ 5

> 35 ÷ 5 is 7
> There is one zero in the dividend
> So, 350 ÷ 5 is 70

Adapted from the Waterloo County Board of Education, 1993, *Multiplication and Division of Whole Numbers*, pp. 28–29. Used with permission.

Criteria for Student-Generated Computational Strategies

The examples in the table above give some sense of the different methods that students might use as they develop their own computational strategies. It is important to remember that the reason for using an algorithm is that it is an efficient and effective way to perform a calculation. Student-developed strategies are not necessarily efficient and effective. For example, a student-generated strategy may require more steps and time than is appropriate. However, students develop and use these strategies because they make sense to them. As students share and compare their strategies, they will become better at finding methods that are both efficient and effective. An *efficient* method is one that does not require a page of calculations and more than a reasonable amount of time to produce an answer. An *effective* method is one that works for all problems of a particular type (i.e., one that is generalizable to many problems using the same operation). Students need to learn to evaluate their own strategies and algorithms on the basis of the following criteria. Is the strategy:

- easy to understand?
- easy to duplicate and apply?
- easy to remember?
- easy to perform accurately?

With experience, students can learn to apply these criteria to their own strategies and develop ways for improving their efficiency and effectiveness.

Students' learning of multidigit computations can be a positive or negative introduction to the more complex and formal mathematics they will learn in higher grades. Multidigit computations are often the area of greatest confusion for students, and the one in which the links between the meaning of number operations and their relationship to solving problems is least obvious. As with all instruction, students benefit from a balance of guided, shared, and independent activities. Some suggestions for activities that may help students develop algorithmic understanding are provided in the following sections.

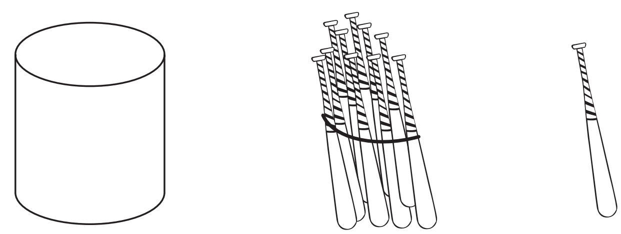
AN INVESTIGATIVE APPROACH TO WORKING WITH MULTIDIGIT COMPUTATIONS

Relevant and interesting investigations help students develop their facility with multidigit computations. Through investigations, students explore concepts and apply their understanding. Often, students are able to develop a stronger conceptual understanding of the topic under investigation. In the following sample investigation, students use a real-life context to explore place value in general and regrouping single units into tens and hundreds in particular. An explanation of the investigation for teachers and an outline of the problem are given in the accompanying box. A blackline master in Appendix 10-3 (BLM23) contains the student instructions for this sample investigation. BLM23a contains an order form.

The purpose of this investigation is to provide students with experiences that involve grouping ones into tens and tens into hundreds in different combinations. Students should work in pairs or a small group using the blackline masters.

Sample Investigation – Johnson Sports Store

Scenario: The Johnson Sports Store sells baseball bats to baseball leagues in Canada. The bats are sold as singles, in bundles of 10, and in canisters of 100 (10 bundles of 10). Students will work in groups to fill the orders for the bats. One craft stick represents 1 bat, a bundle of craft sticks (held together with an elastic band) represents a bundle of 10 bats, and 1 canister (a coffee canister or other container that will hold 10 bundles of ten craft sticks) will represent 100 bats.



Explain to students that the bats should be shipped as efficiently as possible. Define for students that *efficiency* means shipping bats in canisters or bundles whenever possible, rather than shipping single bats. (You could provide students with the comparison of having 100 pennies or having a one-dollar coin.)

Students will work in groups to fill out the order forms. Combine all the orders for a league into one large order. The point of this investigation is to help students recognize the efficiency of combining the bats into bundles of 10 or 100. Students will try to combine the bats so that they can be sent in canisters as often as possible, or in bundles if there are not enough for the canisters, or in singles if there are not enough for the bundles.

Below are the orders that students need to fill. Students should use a different order form for each league's orders. See BLM23 and BLM23a, in Appendix 10-3, for the student worksheets.

Dundas League:

June 4	50 bats
June 8	64 bats
June 9	16 bats

Aurora League:

June 2	6 bats
June 10	25 bats
June 30	29 bats
July 1	55 bats
July 2	10 bats

Belfield League:

June 1	16 bats
June 5	32 bats
June 9	14 bats
June 20	50 bats

After students have filled the orders, have them share their order forms. Prompt discussion by asking the following questions:

- "How did you fill your orders?"
- "Why did you fill them this way?"
- "How else could you have filled them?" (For example, in the Dundas League order, the bats could be grouped as 1 canister – 100 bats – and 3 bundles *or* as 13 bundles. Remind students that they are trying to be efficient in how they ship the bats, which means that they should ship as few packages as possible. Ask: "Which way of filling the order is the least efficient? Which is the most efficient?")

Investigations are one way to help students learn about multidigit computations and to demonstrate how they are used in relevant and contextual situations. Other activities that help students to develop an understanding of the underlying concepts and procedures involved in multidigit computations are described below.

ACTIVITIES TO HELP STUDENTS DEVELOP STRATEGIES FOR MULTIDIGIT COMPUTATIONS

Hundreds Chart

Have students use BLM24 in Appendix 10-3 for this activity. Students need many experiences with hundreds chart or hundreds mat activities before and during their work with multidigit computations. Looking for patterns with two-digit numbers will help students develop an understanding of how the number system works.

Ask students to look for the pattern for tens, twenties, thirties, and so on. Ask: "If I start at 2 and add 20, where do I end up? How did I get there?" (You moved down vertically two rows to 22.) Ask: "If I start at 2 and add 21, where do I end up? How did I get there?" (You moved down vertically two rows and horizontally one column to 23.) Encourage students to draw arrows to follow the movement on the hundreds chart. This can be done on an overhead projector or on a hundreds mat so that all students can participate.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Human Calculator Game

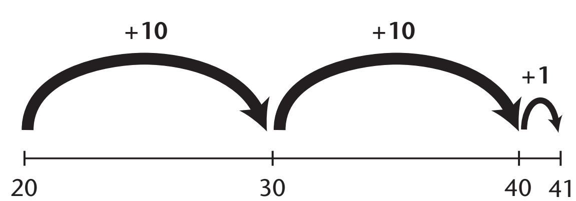
Have one student volunteer to act as the "human calculator". This student stands at the front of the classroom, in front of a large hundreds chart (a chart on an overhead projector would also work well). The other students use regular calculators. Tell students to enter a number such as 35 into their calculators. Tell the human calculator to put his or her finger on 35 on the hundreds chart. The goal is to see who is faster, the human calculator or students with the regular calculators. Tell the human calculator and the other students to add 25 to the original number. Have students call out the number as soon as they have the answer. The human calculator will move his or her finger vertically down two rows on the hundreds chart from 35 to 55, and horizontally across five columns to 60. The human calculator will usually be faster. Students can take turns as the human calculator.

This activity can be done as a whole-group activity and then later as a small-group activity. Ask students progressively harder questions, such as 35 + 20, + 21, + 22, – 20, – 30, and so on. Create question cards for the groups by printing the entry number on one side of the card and the add-on number on the other side. The human calculator puts his or her finger on the first number, and the calculator group makes the first entry. Then the card is turned over. Do not show the whole question on one side. If you do, students with calculators may be faster.

This activity helps students become accustomed to adding by decades rather than by ones.

Open Number Line

Students need prior experiences with number lines for this activity. Introduce them to a number line that goes to at least 50. A class number line will be the most beneficial. A desktop number line that students can use independently will also be helpful. Ask students to use the number line to find the answer to 20 + 21. Share the various responses. (Some students may need to add by ones from 20 to 41. Others may realize that they can skip up by tens.) Have students look at both methods and make their own decisions about which method works best for them. Have students demonstrate their strategies. Show students how a computation such as 20 + 21 can be calculated by starting at the 20 and then jumping up by decades from 20 to 30 to 40 and then adding on 1 more.



Display some of the number-line strategies on the bulletin board. Tell students that they may want to use a number-line strategy as they work through problems with double-digit computations. During the debriefing time at the end of a lesson, encourage students who have used this strategy to share their solutions with the class. If no one is using the strategy, model how to do the problem using this strategy. Continue to show it as a model each time the groups share their strategies. Students will recognize its merits and those who find it helpful will begin to use it. It is not necessarily a strategy all students should be expected to use. For some it will be too easy a strategy, and they may prefer more abstract means of finding a solution. Other students may find it too abstract until they have had considerable exposure to it.

Reach the Target

This activity can be played first in a whole group using large cards or an overhead projector and then later in small groups. Create cards with various two-digit numbers. (In the beginning, use cards with decades, such as 10, 20, 30, and so on. Later use any two-digit number such as 23, 47, etc.)

10	20	30

Playing cards (10, 20, . . . , 100)

These are the playing cards. Use other cards as the target cards (again at the beginning use only decades). It would be beneficial if the playing cards were a different colour from the target cards. Set up two piles of cards, face down. One pile contains the playing cards and the other pile, the target cards. A target card (for instance 50) is turned over. Students take turns turning over playing cards. As each player turns over a playing card, he or she must state the number that needs to be added to it to reach the target number. For instance, if a player turns over 25, he or she has to say the number that needs to be added to 25 to reach the target of 50: "I have 25. I need 25 more to reach the target of 50." The same target card is used until everyone has played that round.

Students can use hundreds charts, base ten blocks, number lines, and so on, to help them. They can also do pencil-and-paper computations if they prefer. The other players have to agree that the answer is right. If the other players agree, that player keeps the card. If the other players do not agree, the card is forfeited. The winner is the player with the most cards at the end. Students can use a calculator to verify the answers but not to find them.

This activity helps students consolidate strategies for working with multidigit computations.

This game can be modified to include subtraction. If some playing cards are larger than the target cards, students have to calculate the number to subtract from the playing card to reach the target card. If they get it right, they keep the playing card.

Add On

Create cards with numbers from 1 to 20, or use number cards. Place them face down in a pile. These are the playing cards.

1	2	3

Playing cards from 1 to 20

Create cards with the decades (10, 20, 30, . . .) or hundreds (100, 200, . . .), or both. These are the add-on cards. Place them face down in another pile.

100	200	300

Add-on cards from 100 to 1000

Turn over one add-on card. Students take turns picking up a card from the playing card pile. They add this number to the number from the add-on card. For instance, if a player picks a 2 from the playing cards pile and 100 is showing in the add-on card, he or she adds the two numbers together. If the player gets it right, he or she keeps the number 2 playing card (a calculator can be used if necessary to verify the sum). The same add-on card is used until everyone has played that round. Then, a new add-on card is turned over. The person with the most cards at the end wins.

The game can be played with any variation of playing cards or add-on cards, but it is important to start simply. Later, playing cards such as 25, 27, and 38 can be used. Students are allowed to use hundreds charts, numbers lines, and so on, to help them. Calculators are only used to double-check answers.

This game can be modified for subtraction. The add-on cards become the minuends and the playing cards are the subtrahends. For instance, if a minuend of 50 is turned over and a playing card of 9, the player has to subtract 9 from 50. If the player gets it right, he or she keeps the 9 card.

Double Tens and Near-Double Tens

Remind students of the doubles and near-doubles basic facts strategies (see the "Strategies for Basic Facts of Addition and Subtraction" section). Have students modify the games Spinning for Near-Doubles (p. 76) and Snappy Doubles (p. 75) to play Spinning for Double Tens or Spinning for Near-Double Tens (see BLM25, BLM26, and BLM27 in Appendix 10-3) and Snappy Double Tens (BLM28 in Appendix 10-3). Students can use craft sticks in bundles of 10, base ten materials, or interlocking cubes that are in sticks of 10 as manipulatives to reinforce that they are combining tens or groups of 10. In the final debriefing portion of the lesson, ensure students understand that although they are using a strategy that was for basic facts, the numbers have been increased by a magnitude of 10. This can be done by encouraging students to use the manipulatives suggested above or by recording the multidigit computations as they play the game. For example, as students calculate 20 + 30 using this strategy, they will show two bundles of 10 doubled, plus 1 more bundle for a total of 5 bundles or 50 craft sticks.

The Target Number Game

This game helps students practise skip counting by double-digit numbers. Have all students stand in a circle. Select a target number and have students count in turn around the circle. The student who says the target number will sit down, and the game will continue, using the same starting number, the same skip-counting pattern, and the same target number, until only one student is left standing. This student will select a new target number (based on criteria provided by the teacher – for example, that the target number must have a 5 in the ones place).

A sample game:

> The target number is 101. Students take turns counting by tens, starting at 21. The count begins 21, 31, 41, . . ., around the circle. The student who says "101" sits down, and the next student in the circle begins counting at 21 again. Each student who hits the target of 101 sits down and the winner is the last student standing. It is important to have hundreds charts or number lines available for those students who want to use them as a counting strategy.

A variation:

> Students will count by fifteens beginning with 15. Each student who counts an even number will sit down. For example, 15, 30 (sits down), 45, 60 (sits down), and so on. There is not one target number, but rather a criterion that even numbers are the targets.

The Winning Number

For this game, use index cards that each have a criterion noted on them (see BLM29 in Appendix 10-3). Have students work with a partner or in a group of three. Each student uses base ten materials to create a two-digit number. Each pair or group combines their two-digit numbers and records the sum. The teacher asks, "What is your group's number? How did you get that number?" Discuss with students how they combined their base ten materials to find the total. (Did they group the tens and ones? Did they create a hundred?) A criterion card is turned over (or selected out of a bag) and is read aloud. Groups that have met the criterion on the card receive one point. Students then start round two with each group creating its own two-digit number again. The game continues until one pair or group has five points.

As an extension, students could be challenged to find the difference between their group total and a set target number. For example, if a group's total is 130 and the selected target number is 250, students would work in their pair or small group to find the number they would need to add to their total to reach the target number. They may use base ten materials, hundreds charts, number lines, pictures, and so on, to find the number. After each pair or group has found the number they need to reach the target number, a criterion card could be selected. If the found number meets the criterion, then the pair or group receive one point.

The Longest Computation

Challenge students to work in pairs or groups to find the most inefficient way to solve a problem such as 25 + 63 using only two-digit numbers (no ones). Usually, the most inefficient way is to add and then subtract different amounts so that the route to the answer takes a long time. Students learn a lot about the addition and subtraction of multidigit numbers as they work out their long-running computation. For instance, a long computation for the question above would be 25 + 10 – 11 + 22 – 10 + 30 – 11, and so on. The group or pair with the longest computation wins. Then, challenge students to find the quickest way to solve the problem, again using only double-digits (for instance, this might involve starting with the higher number and jumping up by decades or groups of decades and then subtracting a logical two-digit number).

A variation of this task is to give a target such as 50 and have students develop a computation that gets them to that answer but in the longest way possible using two-digit numbers.

Start With the Solution

Ask questions such as, "Suppose the answer to my problem is, 'There are 12 cars left.' What might my problem be?" (Possible solution: I had 24 cars in my dealership. I sold 10 yesterday and 2 today. How many do I have left?)

STANDARD ALGORITHMS

Although students may have generated their own computational strategies, they may also have been taught how to use standard algorithms by a parent, sibling, or former teacher. When standard algorithms are discussed in class, it is important that students develop an understanding of the operations and not just memorize rules. (See Appendix 10-1: Approaches That Help Students Understand Standard Algorithms.) Teachers can demonstrate how steps in the standard algorithms are similar to strategies that students have developed on their own.

Teachers must be careful not to imply that the standard algorithm is the "right" or "best" way to solve multidigit computations, and should encourage students to use strategies that they find meaningful and efficient. Some students may prefer to use the standard algorithm; others may continue to use their own flexible methods.

When standard algorithms are introduced, it is important to consider:

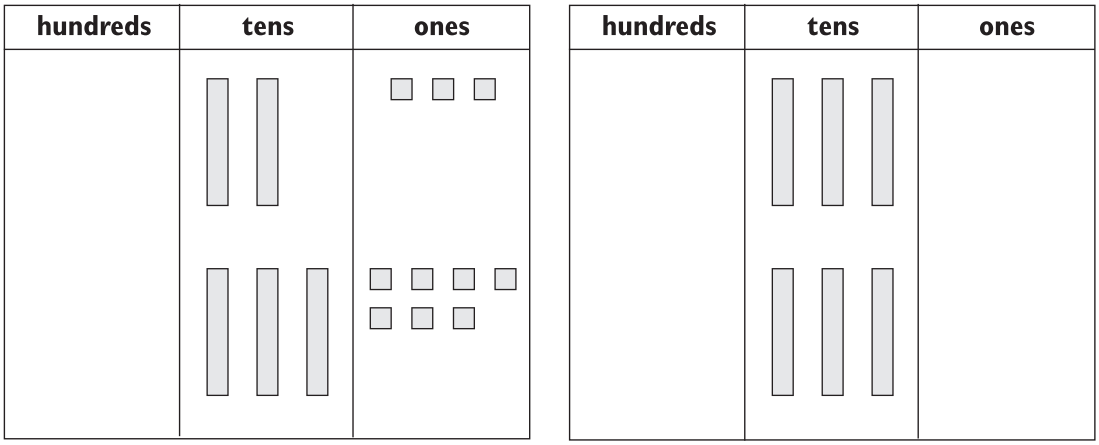
- **Prior learning:** Students need to have had multiple experiences developing strategies for an operation.
- **Language:** Terms such as "borrowing" and "carrying" are misleading. A preferred term is "regrouping". Be careful about how the tens digit in a two-digit number is described when doing an operation. Some students fall into the trap of calling the tens digit by its single-digit value. Instead of saying, "Take 20 away from 30" in a question such as 35 – 22, they say, "Take the 2 away from the 3." This creates conceptual confusion later on, particularly when students are working with more complex numbers such as decimals (students in the junior grades often have difficulty knowing whether the answer to a problem is 0.32, 3.2, or 32.0).
- **Place value:** Most standard algorithms use regrouping of tens, hundreds, and so on. Knowing what is happening in such regrouping depends on an understanding of place value. Students need a great deal of experience making numbers using manipulatives such as base ten materials.

Progression for Teaching Standard Algorithms

The following is a logical progression that teachers may find useful for teaching standard algorithms. Remember that it is very important to begin all teaching in a problem-solving context, so that students can make the connection between the symbolism of the algorithm and its role in solving problems.

Representation of the algorithm: Students need many experiences making models of the algorithms using manipulatives so that they can see how a number such as 10 is decomposed into ones, or 100 is decomposed into tens, and so forth.

23 + 37



Recording of the algorithm: Once students have had many experiences representing the algorithm concretely, they can begin to record it in written form. Such recording is best done in conjunction with the concrete representation. (If working in pairs, one student can do the concrete steps and the other can write down the steps.)

Pay extra attention to algorithms with multidigit numbers that contain a zero, as the zero often creates difficulties for students (especially those who have not had sufficient experience with place-value materials).

Using the algorithm: Even when students are ready to use the standard algorithm without concrete or pictorial support, regularly revisit the meaning of the algorithm and what is actually happening to the numbers (the tens are being regrouped into ones, etc.). As students are using the standard algorithm, they should be expected to explain the various steps; for instance, when solving 23 – 18, a student might say, "I changed one 10 of the 20 to give me 10 ones. Now I have 13 ones." Students should also be encouraged to think of ways other than the standard algorithm to perform the operation (e.g., show three ways to add 122 + 19). See Appendix 10-1 for examples of approaches that might be used to help students understand standard algorithms.

Choosing whether to use a computational strategy or an algorithm always depends on the context of the situation and the numbers being used. In situations such as giving change at a store, the use of a flexible algorithm that allows for adding up from the cost of the purchase to the amount being paid by the customer is much more effective than the standard algorithm for subtraction of multidigits. In another instance, a calculation involving several numbers may be most appropriately solved using the standard algorithm and paper and pencil. In still another, expediency may dictate that solving a question such as 2000 – 10 does not require the complicated steps shown in the standard algorithm; simply counting back by ten gives the quickest answer for someone who knows what he or she is doing and why.

ESTIMATION

Estimation is an important aspect of multidigit computation. This skill has a significant impact on students' understanding of number and on their ability to reason about number in algorithmic situations.

The goal in estimation is not for an exact answer but rather for a logical approximation. Questions such as, "Is the answer less than 25? more than 10?" and so on, help students recognize that the possibilities for the answer will fall within a range that makes sense. Some strategies that teachers can use to help students develop good estimation skills are outlined below and on the following page. It is not appropriate to teach these strategies as terms and methods to be memorized and practised. They are for the teacher's benefit and may be appropriately introduced on a "need to know" basis. Knowing about these strategies helps the teacher to guide students' understanding of what they are doing and why. Students would not be expected to use most of these strategies before the late primary grades.

Estimation Strategies

1-3 **4-6**

Clustering

Clustering is useful when the numbers are close to each other. The estimate can be found by repeated addition (e.g., 10 + 10 + 10) or multiplication (e.g., 3 x 10).

$$\begin{array}{r} 495 \\ +\,497 \\ \hline \end{array}$$

Both numbers are close to 500.

The answer is close to 1000.

"Nice" Numbers

The nice numbers strategy involves using numbers that are easy to work with. In addition and subtraction, students look for numbers whose sum or difference is close to a multiple of 10. ("Nice" numbers are also known as *compatible numbers*.)

$$\begin{array}{r} 49 \\ 22 \\ +\,53 \\ \hline \end{array}$$

About 100.

The answer is about 120.

In multiplication, students use numbers that are close to a multiple of 10, or numbers such as 5, 15, or 25.

$$\begin{array}{r} 19 \\ \times\ 6 \\ \hline \end{array}$$

19 is close to 20. 20 x 6 is 120. The answer to 19 x 6 is a bit less than 120.

Adapted from Waterloo County Board of Education, 1992, *Addition and Subtraction of Whole Numbers*, p. 25, and 1993, *Multiplication and Division of Whole Numbers*, pp. 30–31. Used with permission.

Estimation Strategies

Front-End Estimation

In front-end estimation, the student performs the operation using the left-most digits. A more precise estimate can be made by looking at the rest of the numbers and then adjusting the answer.

$$\begin{array}{r} 763 \\ -\ 325 \\ \hline \end{array} \qquad 700 - 300$$

The rest of the numbers indicate that the answer will be about 440.

The example to the right shows how front-end estimation can be used in multiplication.

$$\begin{array}{r} 324 \\ \times\ \ 26 \\ \hline \end{array}$$

300 x 20 is 6000. The answer to 324 x 26 is approximately 6000.

Rounding

Rounding is a more complex method of estimation than front-end estimation. It involves a two-step process: the numbers must first be rounded then used in computing the estimate.

$$\begin{array}{rcr} 621 & \longrightarrow & 600 \\ 485 & \longrightarrow & 500 \\ +\ 252 & \longrightarrow & +\ 300 \\ \hline & & 1400 \end{array}$$

In multiplication, students can round each factor to the nearest ten.

$$\begin{array}{r} 51 \\ \times\ 85 \\ \hline \end{array}$$

Round 51 to 50, and round 85 to 90. 50 x 90 is 4500. The answer to 51 x 85 will be close to 4500.

Many students are not aware of the importance or relevance of estimating. They think of it as a way of getting to the exact answer, so they often change their estimate after they have done a close count. Providing students with many daily opportunities to practise estimating will help them improve their skills in this area. Some suggestions are set out below.

Ask students to estimate:

- The number of buttons in the classroom. Ask: "Are there more than 20? Are there fewer than 100?"
- The number of days until the end of the year. Ask: "What number is it less than? What number is it more than?"
- The number of rooms in the school. Ask: "Are there more than 5? Are there fewer than 20?"
- The number of teachers in the school. Ask: "Are there more than 8? Are there fewer than 20?"
- The number of basketballs in the gym. Ask: "Are there more than 10? Are there fewer than 25?"

An estimate provides students with a guide for determining the reasonableness of their solutions.

Students who have many opportunities to estimate during the primary grades are more likely to understand the importance of both estimating and using logical reasoning to think about larger numbers.

Appendix 10-1: Approaches That Help Students Understand Standard Algorithms

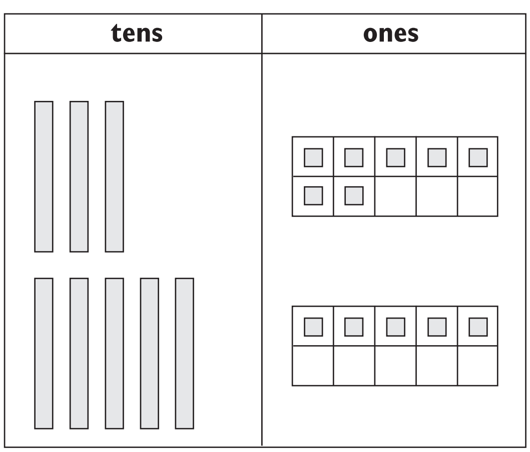
Students gain a broader and deeper understanding of the standard algorithms if they have many and varied opportunities to use concrete materials such as place-value mats, interlocking cubes, ten frames, and base ten blocks in problem-solving situations. The use of these tools greatly enhances students' exploration of addition, subtraction, and multiplication involving regrouping, and of multidigit division. The goal of teaching the standard algorithms through problem solving using manipulatives is to help students develop their conceptual understanding of the standard algorithms. Once students have a thorough understanding of the standard algorithms, they can work flexibly with them and determine when their use is appropriate. In using manipulatives in this context, students can be encouraged to work in pairs, one student working with the models and the other student recording the steps. It is important that students record the steps as they model them (Van de Walle, 2001).

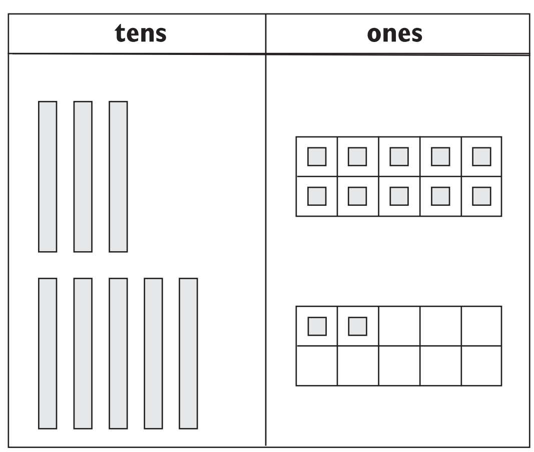
The following examples outline ways in which students use concrete materials while exploring the mathematical concepts of addition with regrouping, subtraction with regrouping, multiplication with regrouping, and multidigit division. As these examples demonstrate, students can gain and communicate a significant understanding of the standard algorithms in the process of using specific models of calculations and recording the steps involved.

Multidigit Addition With Regrouping (Van de Walle, 2001, p. 178)

Students need to practise "trading" groups of 10 ones for one group of ten, and groups of 10 tens for one group of 1 hundred, and so forth. A visual representation of regrouping is vital in establishing conceptual understanding. For example: 1-3

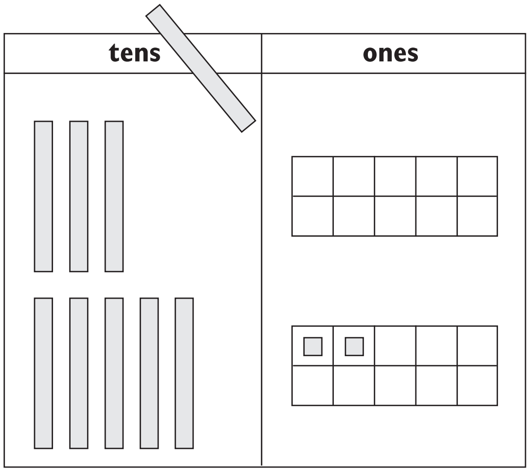
37 + 55





Group ones into 2 ten frames. There is one group of 10 ones, with 2 ones left over.

Using base ten blocks and ten frames, students represent each number on a place-value mat. As they group the ones units into 2 ten frames, they can see that there is one group of 10 ones, with 2 ones left over. There is enough to trade for 1 group of ten, with 2 left over. Students then add 1 tens rod to the left side of the place-value mat. There should be 2 ones units remaining on the right side of the mat.



Trade 10 ones for one group of ten, with 2 left over.

Total amount is 9 tens and 2 ones, or 92.

Students now find the total number of tens rods and unit cubes. There are 9 rods (not enough to trade for a hundreds flat) and 2 ones units. The total is 9 tens, 2 ones, 92. 37 + 55 = 92

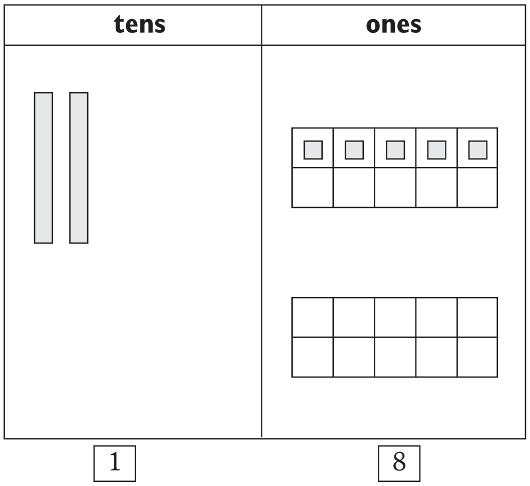
Multidigit Subtraction With Regrouping (Van de Walle, 2001, p. 179)

1-3 **4-6**

Opportunities to explore subtraction with regrouping help students develop conceptual understanding. Once students have a strong understanding that is based on work with models, they can begin to move to the written form of the standard algorithm. Students should be encouraged to use place-value mats and base ten blocks to model subtraction with regrouping. Pairs of students can work together to model and record the steps.

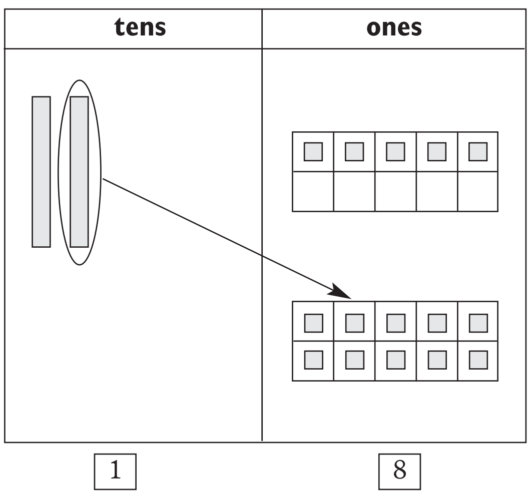
For the example 25 – 18, students model the original number (25) with base ten blocks on the top portion of the place-value mat. They write each digit of the amount to be subtracted on a small piece of paper and place the pieces of paper near the bottom of the mat in the proper columns. When students see that they cannot subtract the 8 from the 5, they will trade in one of the tens rods for 10 ones units.

25 − 18



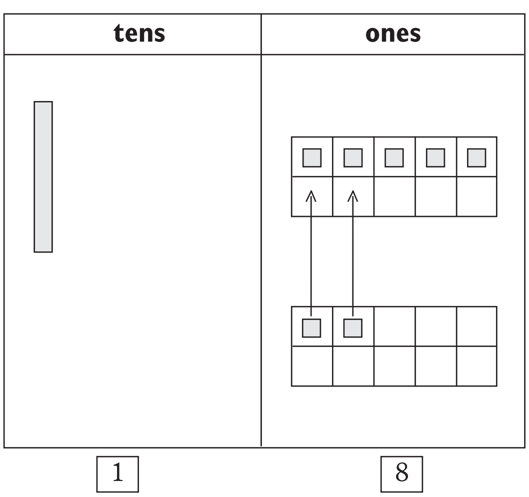
$$\begin{array}{r} 25 \\ -18 \\ \hline \end{array}$$

Cannot subtract 8 from 5.

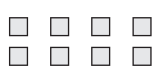


Trade a ten for 10 ones.
Now a total of 15 ones.
8 ones can be taken away.

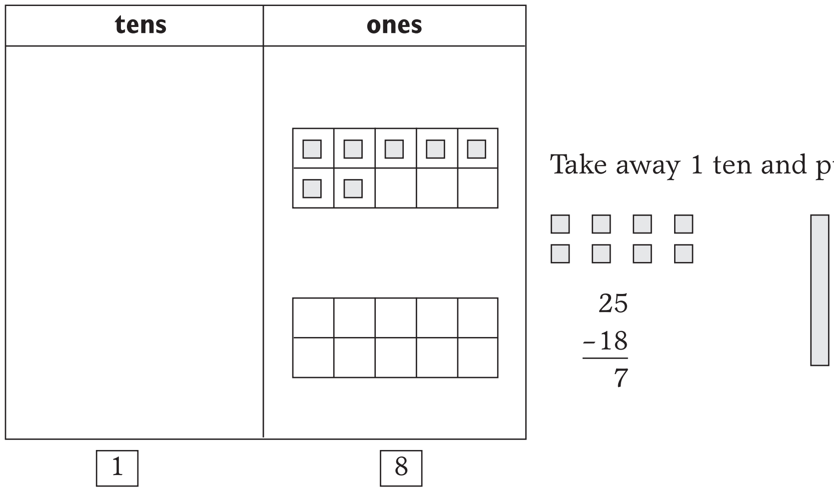
There is now a total of 15 ones. Students can take 8 ones away from the 15, leaving 7 ones. Students can put the 8 taken-away ones to the side, off the mat. Students should be encouraged to organize their work by consolidating the ones left on the place-value mat.



7 ones are left over.
Put the ones taken away to the side.
Consolidate the remaining ones.



Now students can take off the 1 ten and put it to the side.

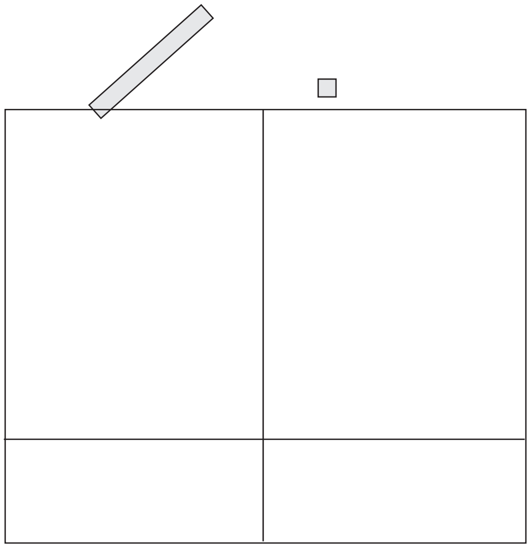


There are 7 ones remaining: 25 – 18 = 7.

Special consideration should be given to subtraction questions that involve the use of zero in the ones or tens place. This type of problem causes special difficulties for students. It is important to model this type of question with the class as a whole and to involve students in a discussion of how best to deal with a zero. Students will recognize that a zero in the ones or tens place of the bottom number means that there is nothing to take away from the top number. They may be puzzled as to what to do. Zeros in the top number also may be confusing for some students, since there are no base ten materials to work with in the column containing the zero. For example, in the question 505 – 128, students should come to recognize that they must make a double trade of materials, exchanging a hundreds flat for 10 tens rods, and then one of the tens rods for 10 ones units. If the teacher leads discussions about how best to deal with a zero and arrives at solutions that make sense, students will gain an enhanced conceptual understanding both of the place value of the numbers involved in the question and of the operation involved.

The above models of the standard addition and subtraction algorithms can also be extended to include three-digit addition and subtraction. In using the standard addition and subtraction algorithms, the teacher should focus on the value of each digit. For example, in adding the tens column of 46 to 57, the teacher should not say, "Add the 4 and the 5," but rather, "Add 40 and 50." Students should see the numbers in relation to their place value.

Allowing students many opportunities to create models of multidigit addition with regrouping helps them visualize the steps involved in the standard algorithm. When students have a conceptual understanding of regrouping, they see the steps in the regrouping procedure in context. They have the ability to use invented strategies when they have forgotten a step of the standard algorithm. With practice and modelling, students can move to the standard algorithm, using blank recording charts to keep track of the steps modelled on the place-value mats.



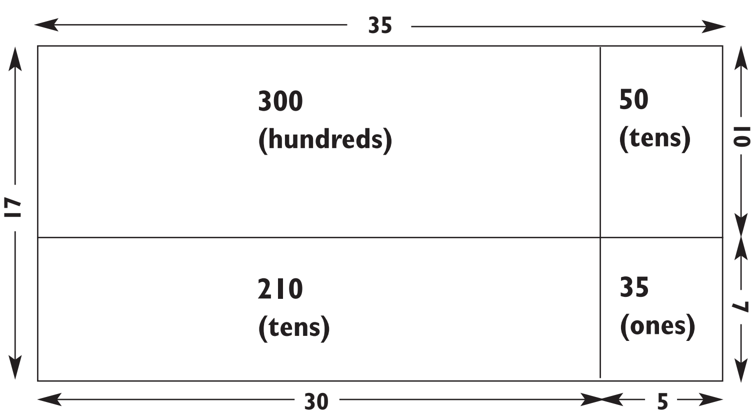
hundreds	tens	ones

4-6

Multidigit Multiplication With Regrouping (Van de Walle, 2001, p. 182)

Providing students with learning opportunities that allow them to explore the place value of numbers while solving multiplication problems encourages the development of student-invented strategies. Students should be given an adequate amount of time to develop this conceptual understanding. When students have strong conceptual understanding of what happens during multiplication, they can be creative with solutions and strategies. This exploration of multiplication using invented strategies will help further develop students' conceptual understanding of the standard algorithm.

Modelling with base ten blocks and arrays reinforces the underlying concepts related to multiplication. For example, the teacher may present the following problem: "The carnival has 17 rides. Each ride holds 35 people. How many people can the rides hold altogether?" Students are encouraged to solve this problem with any strategy that makes sense to them. It is important that they can explain how and why they have chosen their strategy. Students can use an area model to demonstrate their understanding of this question visually:



There are 17 rides. Each ride holds 35 people. How many people in total?

17 x 35

H.	T.	O.
	3	5
x	1	7
	3	5
2	1	0
	5	0
3	0	0
5	9	5

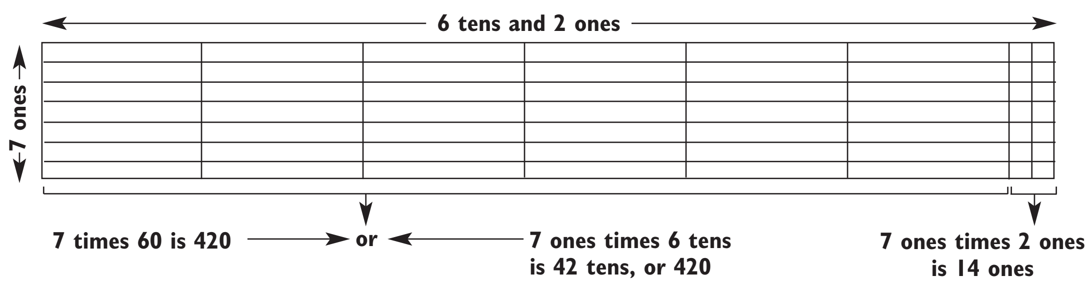
17 is broken down into 10 and 7 ones.
35 is broken down into 30 and 5 ones.

17 x 35 = 595

In this problem, students are considering the value of the digits in the multiplier and the multiplicand. In the area model, students indicate that 17 is composed of 1 ten and 7 ones and that 35 is composed of 3 tens and 5 ones. Students then find the partial products of these multipliers and multiplicands (5 ones x 1 ten = 50, and 5 ones x 7 ones = 35, 3 tens x 1 ten = 300, 3 tens x 7 ones = 210) and add them together, for a total of 595 people.

Arrays and repeated addition allow students to develop a meaningful understanding of the standard multiplication algorithm. Large arrays (e.g., 7 x 62) can be separated into smaller, more manageable parts, which are referred to as partial products. Partial products are added together to find the total product of the rectangle. (This strategy can also be used for double-digit multiplication questions.)

7 x 62 = 434

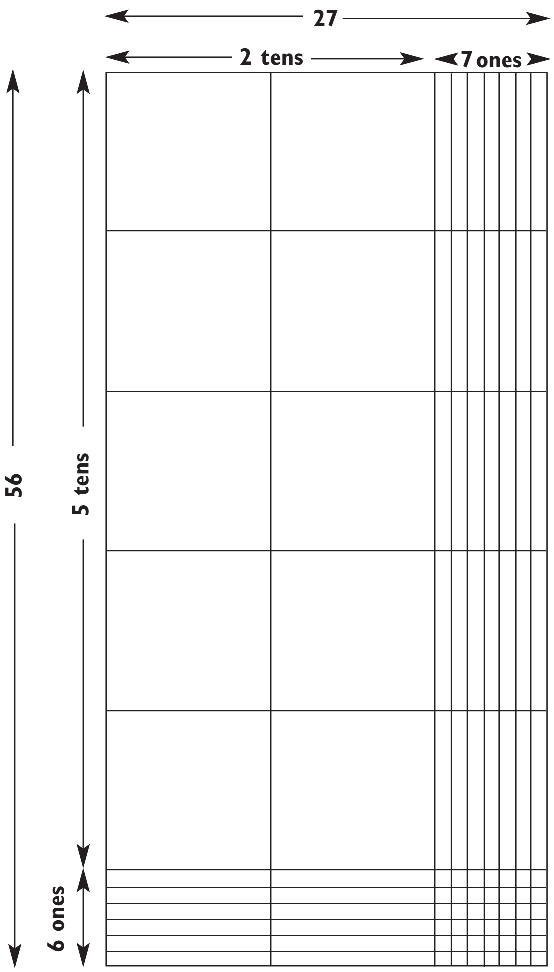


It is helpful for students to use place-value sheets to organize their solutions. As in the examples using partial products and area models, students can more easily organize solutions when they record partial products one under the other. This avoids the errors commonly associated with "carried" digits.

H.	T.	O.
	6	2
x		7
	1	4
4	2	0
4	3	4

In two-digit multiplication, students make a natural transition from using area models to using place-value recording sheets for the standard algorithm. For example:

56 x 27



5 tens x 2 tens is 1000.
5 tens x 7 ones is 350.
6 ones x 2 tens is 120.
6 ones x 7 ones is 42.

Th.	H.	T.	O.
		5	6
	x	2	7
		4	2
	$^{1}3$	5	0
	1	2	0
1	0	0	0
1	5	1	2

Recording partial products decreases the chance of errors with carried digits. Students can also multiply numbers in the order that makes sense to them. Beginning at either the left or the right is an acceptable strategy and solution.

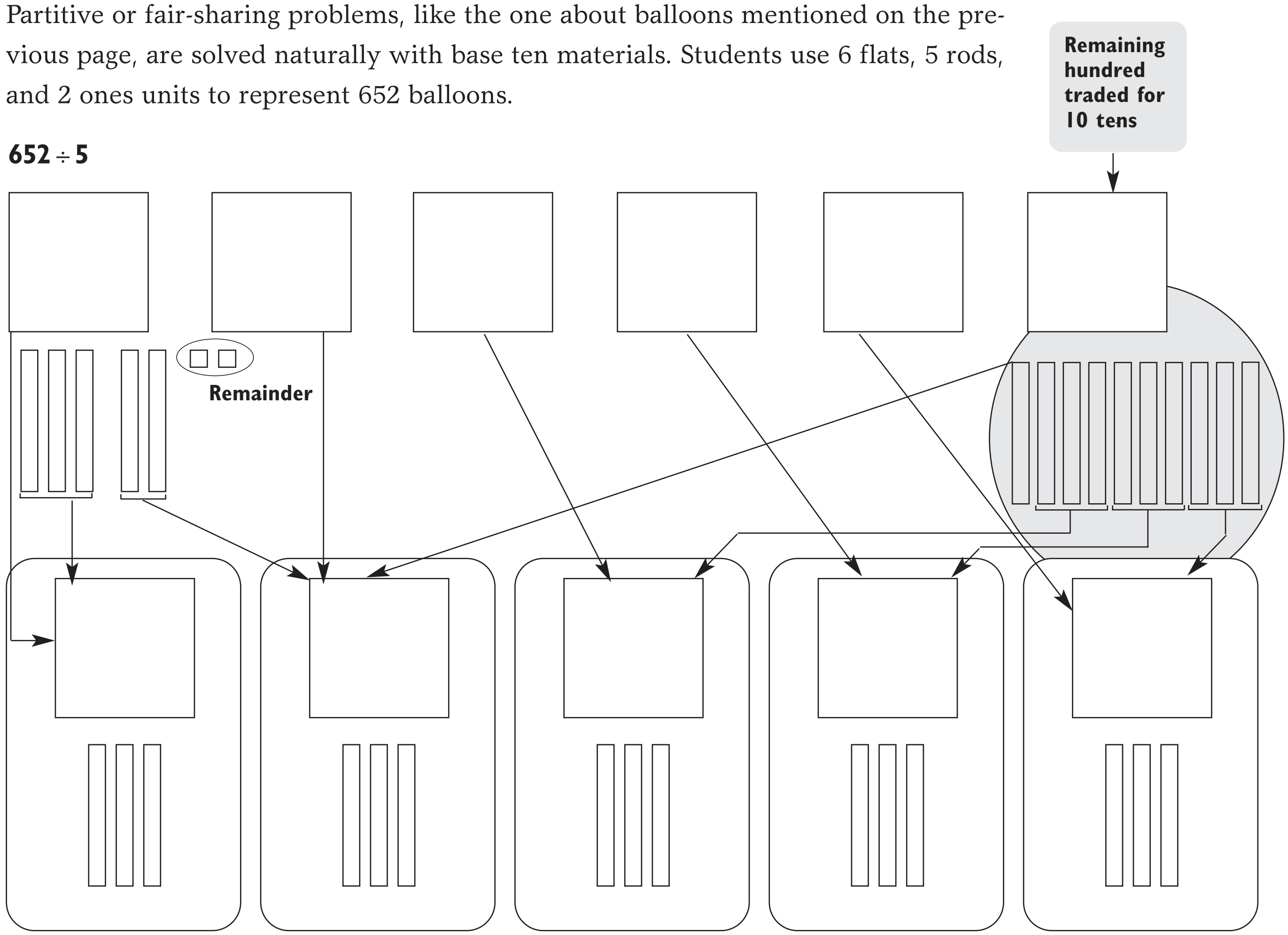
Multidigit Division With Regrouping (Van de Walle, 2001, p. 184)

4-6

Teachers should help students develop conceptual understanding through the use of simple problem-solving activities. There are two kinds of division problems, partitive or "fair sharing" division problems and quotative or "measurement" division problems. In a partitive or fair-sharing division problem the whole amount is known and the number of groups is known, but the number of items in each group is unknown. An example of a partitive division problem is the following: "Shan has 652 balloons. She wants to share them equally with her 4 friends. How many balloons will Shan and each of her friends get?" In a quotative or measurement division problem, the whole amount is known and the number of items in each group is known, but the number of groups is unknown. An example of a quotative division problem is the following: "Ms. Riedal has 726 stickers. If she gives her students 35 stickers each day, how many days will the stickers last?" It is important to have students solve both types of problems by using their own strategies.

Partitive or fair-sharing problems, like the one about balloons mentioned on the previous page, are solved naturally with base ten materials. Students use 6 flats, 5 rods, and 2 ones units to represent 652 balloons.

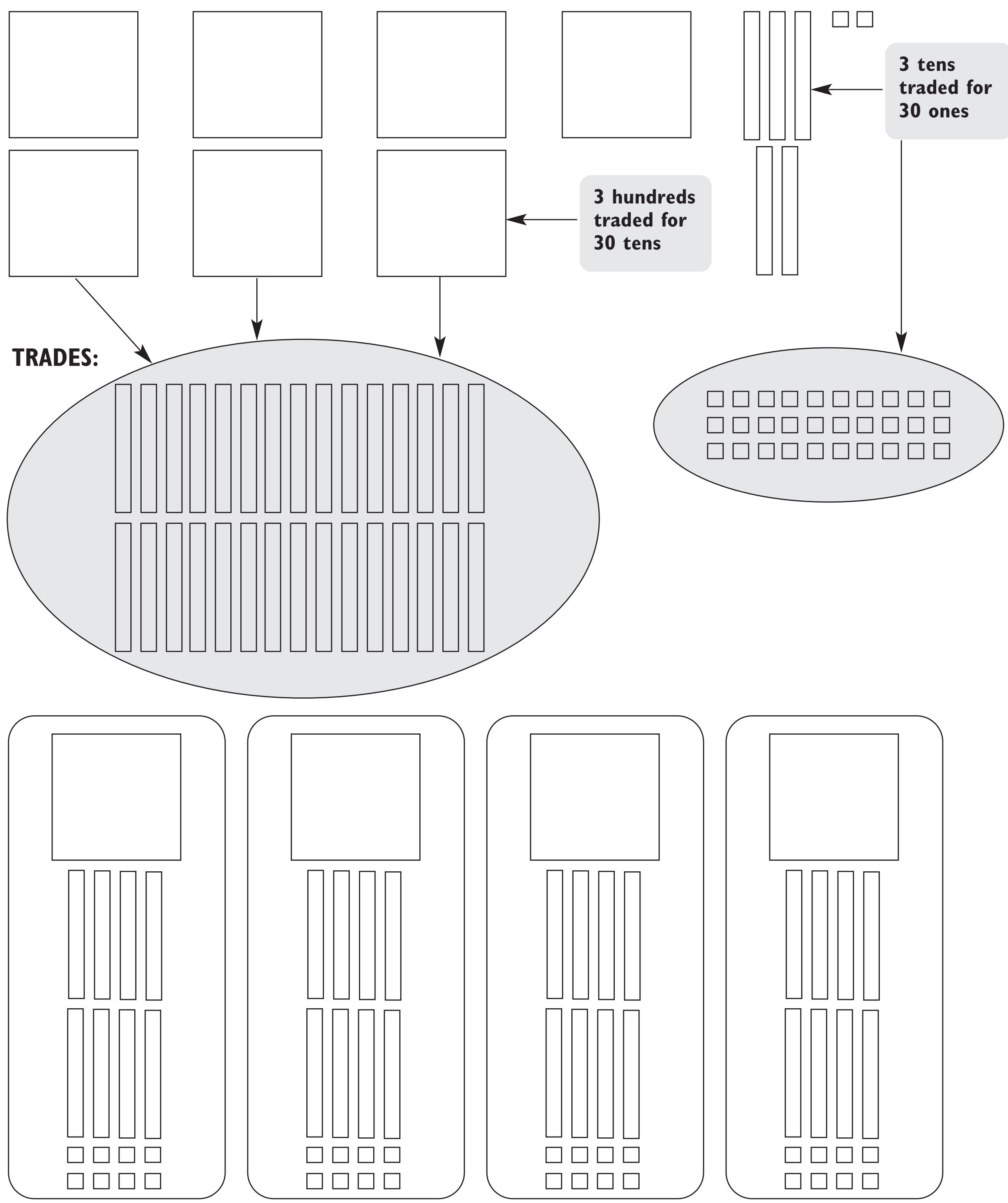
652 ÷ 5



Students use base ten blocks to model the sharing process. The hundreds are shared, resulting in 1 hundred for each of Shan and her 4 friends, with 1 hundred remaining. The remaining hundred is traded for tens. This leaves 15 tens and 2 ones. The 15 tens rods are shared equally, resulting in 3 tens each. The two ones units left over are considered remainders. This approach of direct modelling allows students to visualize the process involved in equal sharing and to begin to build conceptual understanding of the process.

Students in the later grades also benefit from the conceptual development of division taught in the earlier grades. On the basis of their earlier understanding of division, students extend their knowledge of the division algorithm with one-digit divisors to two-digit divisors. It is important that students do not isolate the digits within a division question. For example, the 647 in 647 ÷ 3 should be seen as 6 hundreds, 4 tens, and 7 ones and not as the digits 6, 4, and 7. To ensure that students view the digits according to their place value, teachers should create authentic problem-solving questions that are rooted in a sharing context. For example: "Pencils are packaged in cases of 10 pencils, with 10 cases to a carton. We have 7 cartons, 5 cases, and 2 pencils to share evenly between 4 classrooms. How many pencils will each classroom get?" Within this context it is logical to share the cartons first, until no more can be shared. The remaining cartons are unpacked, and the cases are shared. This sharing process continues until there is no more to share fairly. Problems involving money can also provide a context for sharing-type division.

752 ÷ 4



The hundreds are shared (1 hundred in each group) until they can no longer be shared equally. There are 3 hundreds to be traded for 10 tens each (30 tens total). The traded tens are added to the original tens, resulting in 35 tens. These are shared (8 tens in each group), with 3 left over. Three remaining tens will be traded for 10 ones each (30 ones total). The traded ones are added to the original ones, resulting in 32 ones. These ones are shared equally with the four groups – 8 ones each. In total, each classroom will receive 188 pencils.

As students are developing their conceptual understanding of a fair-sharing approach to solving division problems by using manipulatives, the teacher can make a link with the standard algorithm.

In the early junior grades, teachers should introduce the use of a division mat, with supporting manipulatives, to help students visualize the action of fair sharing. For example, if there are 35 pairs of scissors to be shared among 6 baskets in a classroom, using a division mat will assist students as they physically divide the manipulatives representing the scissors into 6 piles. The number of sections on the division mat depends on the number of the divisor in the question.

In the problem-solving situation outlined below, the materials used support the conceptual development of division and help students understand the symbolic representation found in the standard algorithm. At each stage of the problem-solving process, teachers can model the use of manipulatives and good questioning, and they can make the connection with the standard algorithm explicit when students are ready for that transition.

A classroom helper wants to sort 53 crayons into 3 baskets so that each basket has the same number of crayons. How many crayons would the classroom helper put into each basket?

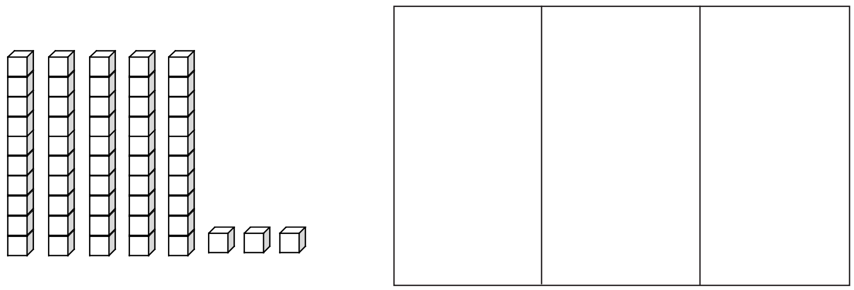
Questions that the teacher might ask at each stage of this problem-solving process are given on the following two pages, along with an illustration of the base ten materials, the division mat, and the standard algorithm at that stage.

Stage 1

"What are we trying to find out? Will the answer be more than 10? . . . 20? . . . 30?"

"How many crayons are there altogether? How can we show this number using the base ten blocks?"

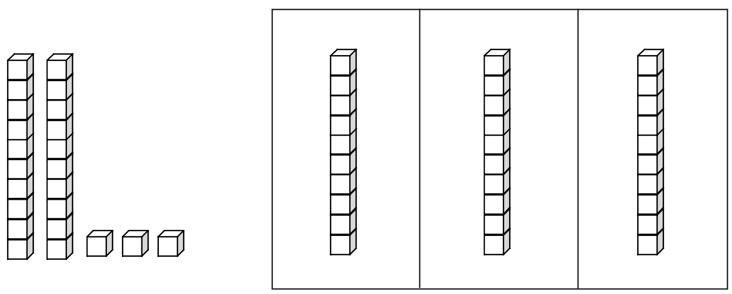
"Among how many groups do we want to share the blocks equally?"



$$3\overline{)\,53}$$

Stage 2

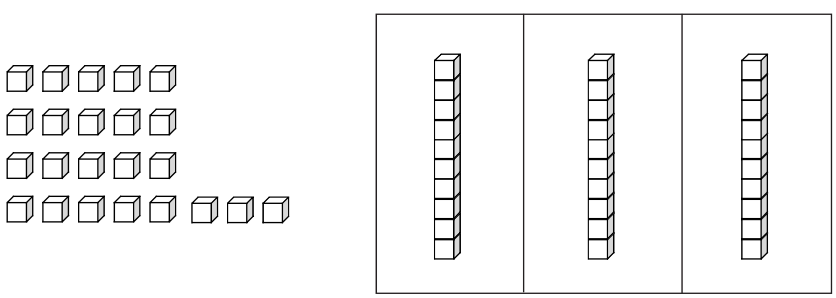
"Can we divide the tens rods into 3 groups? How many are there in each group?"



$$\begin{array}{r} 1 \\ 3\overline{)\,53} \\ \underline{3} \\ 2 \end{array}$$

Stage 3

"What can we do with the 2 remaining tens rods so that we can share them equally among 3 groups? If we trade the tens for ones, how many ones do we have?"

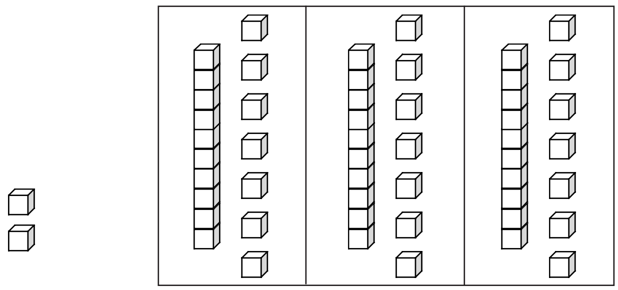


$$\begin{array}{r} 1 \\ 3\overline{)\,53} \\ \underline{3} \\ 23 \end{array}$$

Stage 4

"How many ones can we share equally among the 3 groups?"

"How many crayons will go into each basket? What will the classroom helper do with the 2 crayons that are left over?"



$$\begin{array}{r} 17 \\ 3\overline{)\,53} \\ \underline{3} \\ 23 \\ \underline{21} \\ 2 \end{array}$$

Teachers should provide many opportunities for students to solve problems similar to this one of sorting 53 crayons into 3 baskets.

Appendix 10-2: Instructions for Games and Activities

1-3

The Big Race

Strategy: plus one, plus two, plus zero, minus zero, doubles, near-doubles, make ten

Materials

- The Big Race game sheet (BLM1), 1 per student
- pencil
- counters

The Big Race is a game sheet that can be used to rehearse any of the addition or multiplication strategies in a paper-and-pencil format. Each student needs a copy of the game sheet. Students will record in the centre box the one strategy they will use to complete the game sheet. Students will work their way around the game sheet by applying the strategy in the centre of the sheet to each number around the track. Students will record their answer in the outer boxes on the game sheet. (Counters should be provided for any students who want to use them.)

This game is great for practising a strategy that is currently being taught, or for revisiting a strategy that has already been taught.

Domino Drop

Strategy: one more than and two more than (or one less than and two less than)

Materials

- set of dominoes (approximately 15 for each student who is playing)
- one strategy cue card (BLM2)

Select one strategy card for students to use for this activity. Students should be in partners or groups of four. Each student selects seven dominoes from the face-down collection. One domino is turned over and placed in the centre of the playing area. Students take turns using the strategy from the cue card to select a domino from their pile and lay it next to the domino that is facing up. In subsequent turns dominoes can be matched to any dominoes already face up. If the strategy is "one more than" and the domino that is face up has a 2 and a 5, the student would need to have a domino with a 3 or 6 to match. If the student cannot make a match, he or she selects one additional domino from the collection and tries to make a match. If not, that player's turn is over and the next student plays. Play continues until one player has played all their dominoes.

Extension: Give students two strategy cards at a time (any combination of one more than, two more than, one less than, two less than) so that they can use either strategy when making a match.

Roll-O

Strategy: one more than and two more than (or one less than and two less than), and zero facts

Materials

- Roll-O grid (BLM3), one per player
- 1 ten-sided number cube or a spinner (BLM4)
- number cubes labelled plus one and plus two (or a spinner on BLM4)
- counters

Students can play in pairs or in groups of up to six. Each student will receive a Roll-O grid and will write one numeral in each box. Students can use numbers ranging from 1 through 11 on their grid, repeating any numbers they choose. Each player's Roll-O grid will have different combinations of numbers from 1 to 11. Player 1 rolls the number cube (or spins the 0–9 spinner) and then rolls the number cube labelled plus one or plus two (or spins the +0, +1, +2 spinner). The player calls out the number sentence that was made from the rolls (or spins) and announces what the Roll-O number is. Students who have that number on their grids place a counter over it. The game ends when one student has covered all the numbers in one straight row.

Extensions: Provide students with a number cube having 12 or 20 sides or a spinner with higher numbers.

Students could use number cubes that have plus one, plus two, minus one, and minus two (or spinner + 0, + 1, + 2, – 0, – 1, – 2 on BLM4).

Elevator Ride

Strategy: one more than and two more than (or one less than and two less than), and zero facts

Materials

- Elevator game sheets (BLM5)
- one counter per player (each player should have a different colour)
- number cube labelled plus zero, plus one, plus two (or spinner on BLM4)

Have students work with a partner. Players start at the bottom of the apartment building. Player 1 rolls the number cube (or spins the spinner) and moves his or her counter up that many floors of the apartment building (either + 0, + 1, + 2). Players take turns rolling the number cube (or spinning the spinner) and moving their counter up the elevator. The first one to reach the top of the apartment building wins. If a player is on the 19th floor, he or she must roll (or spin) plus one to win the game.

Extensions: Students can also use a number cube that has minus zero, minus one, and minus two written on it (or a spinner), start at the top floor, and work their way down the elevator. Teachers could also have students begin with the plus zero, plus one, and plus two cube (or spinner) to work their way to the top. Once at the 20th floor, students could then switch to the minus zero, minus one, minus two number cube (or spinner on BLM4) and begin to work their way down to the first floor. The first player to make it back to the first floor wins.

Off to the Races

Strategy: one more than and two more than (and zero facts)

Materials

- Off to the Races game sheet (BLM6)
- 4 counters in one colour and 4 counters in another colour
- number cube labelled plus zero, plus one, and plus two (or a spinner on BLM4)
- six-sided number cube

Have students work with a partner. Each student uses one colour of counters and takes turns selecting four different numbers to put his or her counters on. (For example, player 1 might select the numbers 2, 5, 7, and 8 and put red counters on those numbers. Player 2 would then have the numbers 1, 3, 4, and 6 and put blue counters on those.) Player 1 then rolls both the number cubes (or rolls one number cube and spins the spinner). The appropriate counter is then moved one square towards the finish line. For example, if the number 5 is rolled with plus two, then the counter on 7 would move one spot towards the right. Players take turns rolling the number cubes (or rolling the cube and spinning the spinner). The game continues until one of the players' counters reaches the finish line.

Extensions: The game could be played so that the first player to have all four of his or her counters cross the finish line wins the game. Students could use number cubes with higher numbers.

Race for Cover

Strategy: one more than and two more than (and zero facts)

Materials

- Race for Cover game sheet (BLM7)
- craft sticks, each with one number from 0 to 12 written on one end
- cup
- plus one, plus two spinner (BLM4)
- counters

Have students work with a partner or group (up to four members). The first student will select a craft stick from the cup (the ends of the craft sticks with the numbers on them should be down at the bottom so the numbers cannot be seen). The player then spins the plus one, plus two spinner, and the number spun is added to the number on the craft stick. The student looks to see whether that number is available on his or her side of the game sheet. If so, a counter is placed on it. The craft stick is placed back in the cup for the next turn. Player 2 then selects a stick, spins the spinner, finds the sum of the numbers on the stick and the spinner, and places a counter on the corresponding number on his or her side of the game sheet. The game continues until one player has covered all the numbers on his or her side of the game sheet.

Extensions: Provide students with sticks that have higher numbers, or use the spinner (on BLM4) that combines strategies of plus/minus zero, one, and two.

Concentrating on Doubles

Strategy: doubles

Materials
- Concentrating on Doubles cards (BLM8), 1 or 2 sets for each pair

Have students play individually or in pairs. Students place all cards face down and spread out on a desk or on the floor. The players turn up two cards at a time. The object is to match the doubles number expression with the correct sum card. If a player finds a match, he or she keeps the cards. If the cards do not match, they are turned back over in the same place. Students working independently may want to use an egg timer to see how many matches they can find before the timer runs out. If playing in pairs, the player with the most matches wins.

Magic Doubles

Strategy: doubles

Materials
- paint
- paintbrush or sponge
- Magic Doubles (BLM9), preferably enlarged to ledger size

Each student prepares his or her own worksheet by cutting along the dotted lines. With a small paintbrush and paint, students will "magically" double their numbers. Have students start at the top of their sheet and put zero dabs of paint on the left side of their paper. Then they fold over the top flap and see how many dabs of paint magically appear. Students then move down the sheet, apply dabs of paint in the appropriate square, fold over the flap, and see how many magically appear altogether in that section. Students continue to work their way down the page until they have painted all the doubles. Once the paint has dried, students record the number sentence beneath each section.

Extensions: Students could cut apart the strips once the strips are dry, and then staple the pages together to make a Magic Doubles Book.

The page could also be used for near-doubles by adding one more dab of paint to one side using a different colour. The number sentence could be written out as follows: ___ + ___ + ___ = ___.

Over-Easy Doubles

Strategy: doubles

Materials

- Over-Easy Doubles game sheet (BLM10)
- one counter for each student playing (maximum 4)
- craft sticks, each with one number from 0 to 10 written on one end
- cup

Have students work with a partner or a group of four. Players begin in the start box on the game sheet. The craft sticks are placed in the cup, with their numbers at the bottom. Player 1 selects a craft stick from the cup. He or she looks at the first egg to see whether the double of that number is on it. If it is, he or she can place a counter on that egg on the game sheet. If not, he or she will try again on the next turn. The drawn craft stick is returned to the cup and the cup is given to the next player, who procedes to draw a stick. Play continues, with players trying to move to the next egg, until one player reaches the final egg.

Extensions: Once students have mastered the doubles concept, they can follow the same procedure for the game, but when they draw their craft stick, they will double the number and add one. This will provide rehearsal of the near-doubles strategy.

Last One to School

Strategy: doubles

Materials

- Last One to School game sheet (BLM11)
- 10 counters per player
- 1 ten-sided number cube (or a spinner on BLM4)

Have students work with a partner. Each player takes 10 counters and places one inside each house on his or her side of the game sheet. Once the counters have been placed, player 1 rolls the number cube (or spins) and says what the number showing would be when doubled. If that number house has a counter in it, the counter is moved to the centre of the game sheet in the "Last One to School" box. If a player rolls a number that has previously been rolled and the house is empty, that player misses his or her turn. Player 2 then rolls the number cube (or spins), doubles the number rolled, takes the counter from that numbered house, and places it in the centre of the game board. The game continues until one player has successfully moved all his or her counters from the houses and sent all the students off to school.

Seeing Doubles

Strategy: doubles

Materials

- *What Comes in 2's, 3's, and 4's?* by Suzanne Aker
- booklet with 10 pages stapled together, numbered 1 to 9
- pencils and crayons

Read the book *What Comes in 2's, 3's, and 4's?* to the class to begin the discussion of finding doubles in their world. Students will go on a variety of "seeing doubles" walks to try to find doubles in the real world. They can use a clipboard and paper to record doubles they find. Students can share their ideas in a whole-class discussion. They can then draw a picture of different doubles they have seen on the appropriate page in their booklet and write the number sentence underneath. For example:

School seeing doubles walk

- person's face with two eyes illustrates 1 + 1
- a desk with four legs illustrates 2 + 2
- a bus with 8 wheels illustrates 4 + 4

Nature seeing doubles walk

- a ladybug with six legs illustrates 3 + 3
- a spider with eight legs illustrates 4 + 4

Extension: You could make a class book about doubles with pairs of students each completing one page for the big book.

Snappy Doubles

Strategy: near-doubles

Materials

- number cards 0 to 9 (BLM12), 1 set per player
- 10-sided number cube (or a spinner on BLM4)

Have students work with a partner or in groups of four. Each player holds the 10 number cards face up in front of him or her. Player 1 rolls the number cubes or spins the spinner. The other players look at their row of number cards to find the card that is the near-double of the number rolled or spun. For example, if the number 8 is rolled, players would be looking for the 9 card. The first player to snap his or her card on the table and state the near-double fact and sum correctly turns the card face down on the table. In this example, the player would snap the 9 on the table and say, "8 + 9 = 17". Player 2 then rolls the number cube (or spins) for the players to match. Play continues until one player has snapped all his or her cards face down on the table.

Find a Friendly Neighbour

Strategy: near-doubles

Materials

- number cards 0 to 10 (BLM13), 3 sets

Have students work with a partner. Each player is dealt five cards. The remaining cards are placed face down in a pile. The object of the game is to create the most near-double matches. On a turn, a player can put down any two cards that go in sequence (e.g., 7 and 8, 2 and 3, 4 and 5) and state the sum. That pair is then placed face up on the table as a matched near-double. If a player does not have a match in his or her hand, the player asks the other player for a card that would make a near-double. For example, if a player had a 7, he or she could ask the other player for a 6. If the other player has the card, he or she must give it to the asking player, who will then put the pair on the table and give the sum. If the player does not have the card asked for, he or she will say, "Try next door." The asking player then picks up the next card from the face-down pile. Play ends when no more matches can be made. The winner is the player who made the most matches.

Spinning for Near-Doubles

Strategy: near-doubles

Materials

- spinner (BLM4)
- Spinning for Near-Doubles game sheet (BLM14)
- 20 counters

Have students work with a partner. Player 1 spins the spinner, doubles the number shown, and adds 1. If that space is available on his or her side of the game sheet, a counter is placed on that number. If a counter already occupies that space, that player misses his or her turn. The game continues until one player has successfully filled in his or her side of the game sheet.

Flashy Fingers

Strategy: make ten

Materials

- number cards 0 to 9 (BLM12), enough for 3 cards per student

This is a whole-class activity. Each student receives 3 number cards. The teacher will flash (hold up) a number of fingers for students to count. Students will then look at their number cards to determine whether they have a number card that, when added to the number of fingers, makes 10. If so, they hold up their number card for the teacher to see. For example, if the teacher holds up 4 fingers, students should hold up the number card 6 if they have it. If the teacher holds up 0 fingers, all students should hold up 10 fingers.

Extension: The teacher can hold up combinations of fingers so that students will get to know a variety of arrangements for the same number. For example, 5 could be 5 fingers on one hand and 0 on the other, or 4 and 1 or 3 and 2.

Flip for 10 Plus

Strategy: make ten

Materials

- number cards 0 to 10 (BLM13), 3 sets for each pair
- manipulatives for counting/adding

Have students work with a partner. Divide the cards into two piles. In one pile are the 8, 9, and 10 cards, and in the other pile are the remaining cards. (The two sets of cards could be different colours). One card from each pile is turned over at the same time. The first player to say the correct sum aloud keeps the cards for that round. Play continues until all cards have been used. The player with the most cards wins the game.

Framing Ten and Up

Strategy: make ten

Materials

- Framing Ten and Up game sheet (BLM15), 1 per player
- counters
- 10-sided number cube (or a spinner on BLM4)
- number cards (8, 9, 10), 8 of each number
- one penny

Have students work with a partner. Each player has his or her own game sheet. A card is drawn from the number card pile and both players make this number on their ten frames. In turn, each player rolls the number cube (or spins the spinner) and adds the number to his or her ten frames. To determine the winner of the round, the players flip a coin. If it is heads, the higher number wins a point; if it is tails, the lower number wins a point. The first player to reach seven points wins the game.

Triplets

Strategy: commutativity (addition or multiplication)

Materials

- Triplets deck addition (BLM16 and BLM17)
- Triplets deck multiplication (BLM19 and BLM20)

This activity is played with the whole class. Each student receives a card from the Triplets card deck. Once everyone has a card, students will search for their triplets. They will need to know the answer to their question, or they will need to know possible questions if they have an answer card. An example of a triplet would be the three cards $3 + 5$, $5 + 3$, and 8; or 3×6, 6×3, and 18. Students will sit together after they have found their triplet. Once all students are sitting, each triplet will present their commutative cards to the class.

Whichever Way Wins

Strategy: one more than and two more than / one less than and two less than (and zero facts)

Materials

- number cube labelled plus zero, plus one, plus two, minus zero, minus one, and minus two (or a spinner on BLM4)
- number line from 0 to 20 (BLM18)
- 2 different-coloured counters (one for each player)

Have students work with a partner. Each student begins by putting his or her counter on the number 10. Player 1 rolls the number cube (or spins the spinner) and moves his or her counter according to the directions on the spinner. The game continues until one player makes it to either end of the number line.

Dot Plate Game

Strategy: one less than and two less than (one more than and two more than, zero facts, doubles, near-doubles)

Materials

- number cards 0 to 9 (BLM12), enough for 3 cards per student
- one set of dot plates (paper plates with peel-off dots applied in various arrangements to represent numbers from 1 to 10; see BLM30)

This is a whole-class activity. Each student receives 3 number cards. The teacher will tell students the strategy they are working on (e.g., one less than). The teacher holds up one dot plate for students to see. Students will then look at their number cards to determine whether they have the number card that is one less than the number on the dot plate. If so, they hold up their number card for the teacher to see. For example, if the dot plate has 6 dots on it, and the strategy is one less than, then all students who have the 5 card would hold it up.

Extension: The teacher can change to another strategy during the game to let students practise different strategies. Two dot plates could be shown at the same time so students can add the dots and then apply the strategy.

Spinning for Zero

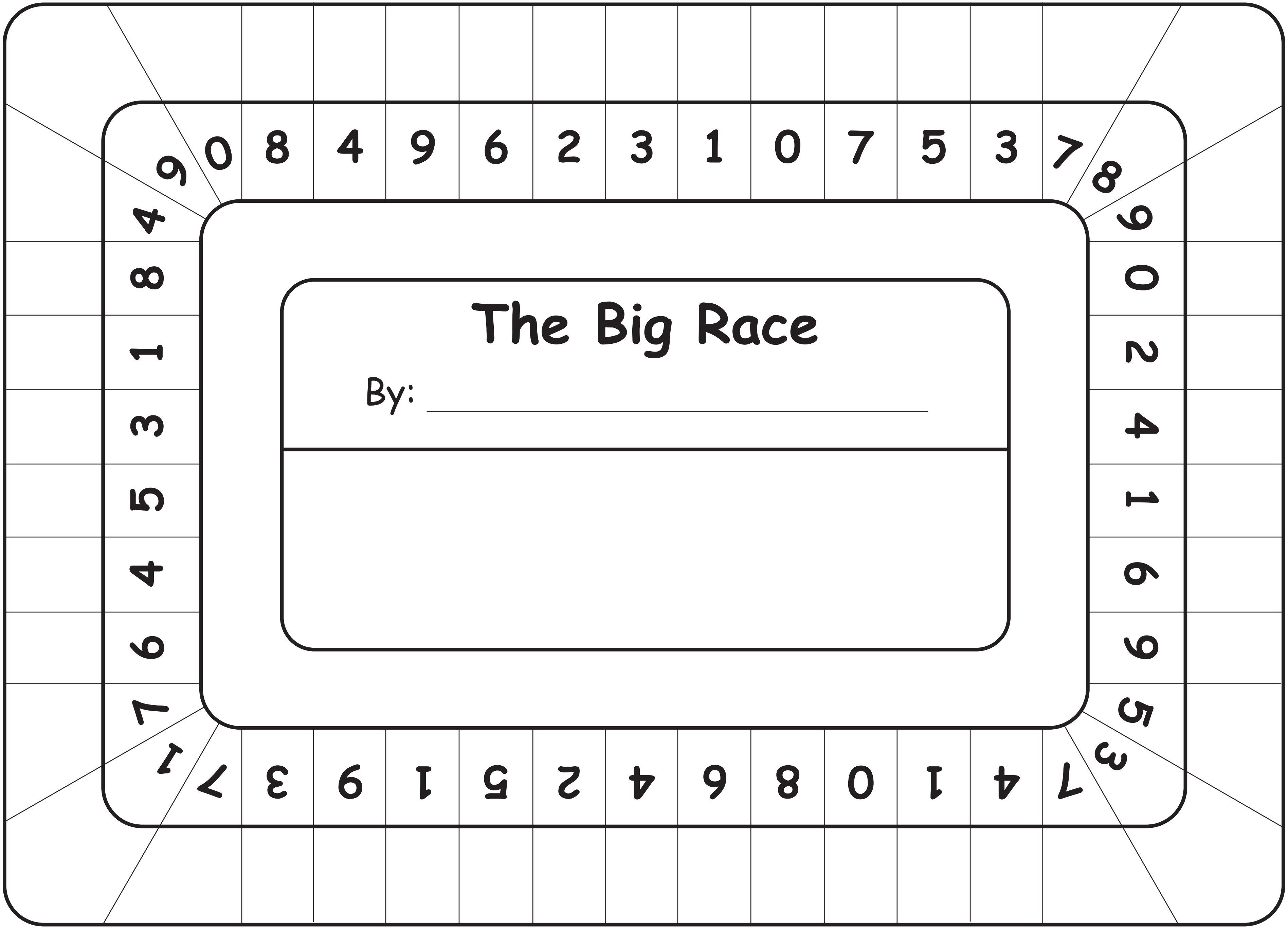
Strategy: rule for zero or one (multiplication)

Materials

- Spinning for Zero spinners (BLM21)
- Spinning for Zero game sheet (BLM22)
- counters

Students receive a blank game sheet and fill it in with the numbers from 0 to 8, repeating any numbers that they want to. Students work with a partner or in groups of fewer than five. Player 1 spins both spinners and performs the operation suggested by the spinner. If the answer to the computation is shown on the player's game sheet, the player puts a counter on that number. Play then moves to the partner or to the next player. As play continues, only one counter can be placed on any space on a player's game sheet. A player whose answer does not match an open space on his or her game sheet loses a turn, and play moves to the next player. The game continues until one player has successfully filled in one straight line on his or her game sheet.

Appendix 10-3: Blackline Masters


The Big Race
By:
9 0 8 4 9 6 2 3 1 0 7 5 3 7 8
9 0 2 4 1 6 9 5 3
7 4 1 0 8 9 4 2 5 1 9 3 7 1
7 6 4 5 3 1 8 4

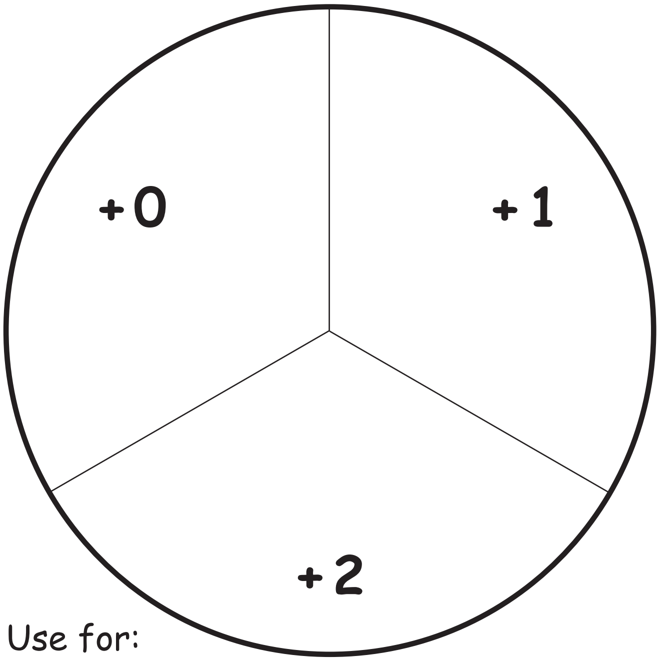
Strategy Cards for Domino Drop

one more than	two more than
add one	add two
plus one	plus two
one less than	two less than
take away one	take away two

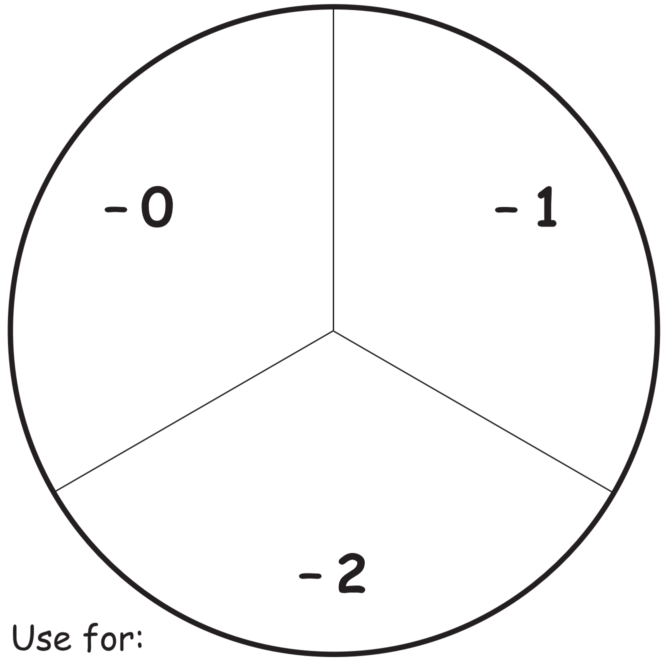
		ROLL-O!		

		ROLL-O!		

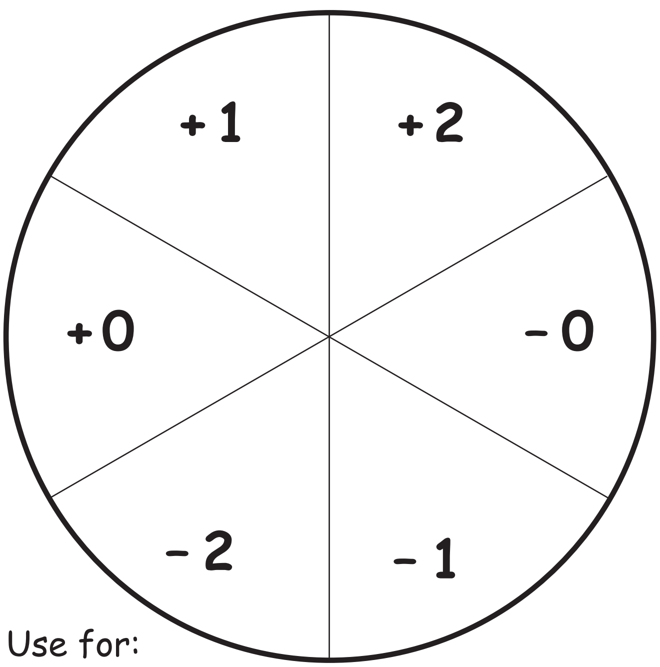
Activity Spinners



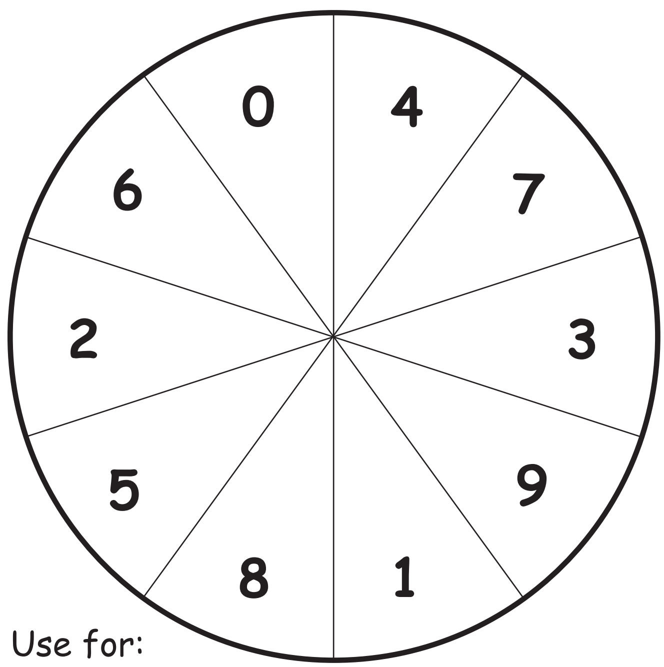
Use for:
Roll-O
Elevator Ride
Off to the Races
Race for Cover



Use for:
Roll-O
Elevator Ride



Use for:
Roll-O
Race for Cover
Whichever Way Wins



Use for:
Roll-O
Last One to School
Snappy Doubles
Spinning for Near-Doubles
Framing Ten and Up

Elevator Ride

20

19

18

17

16

15

14

13

12

11

10

9

8

7

6

5

4

3

2

1

Off to the Races

1													
2													F
3													I
4													N
5													I
6													S
7													H
8													

☺	14	13	12	11	10	9
8	7	6	5	4	3	2

Race for Cover Race for Cover

2	3	4	5	6	7	8
9	10	11	12	13	14	☺

BLM8

Concentrating on Doubles

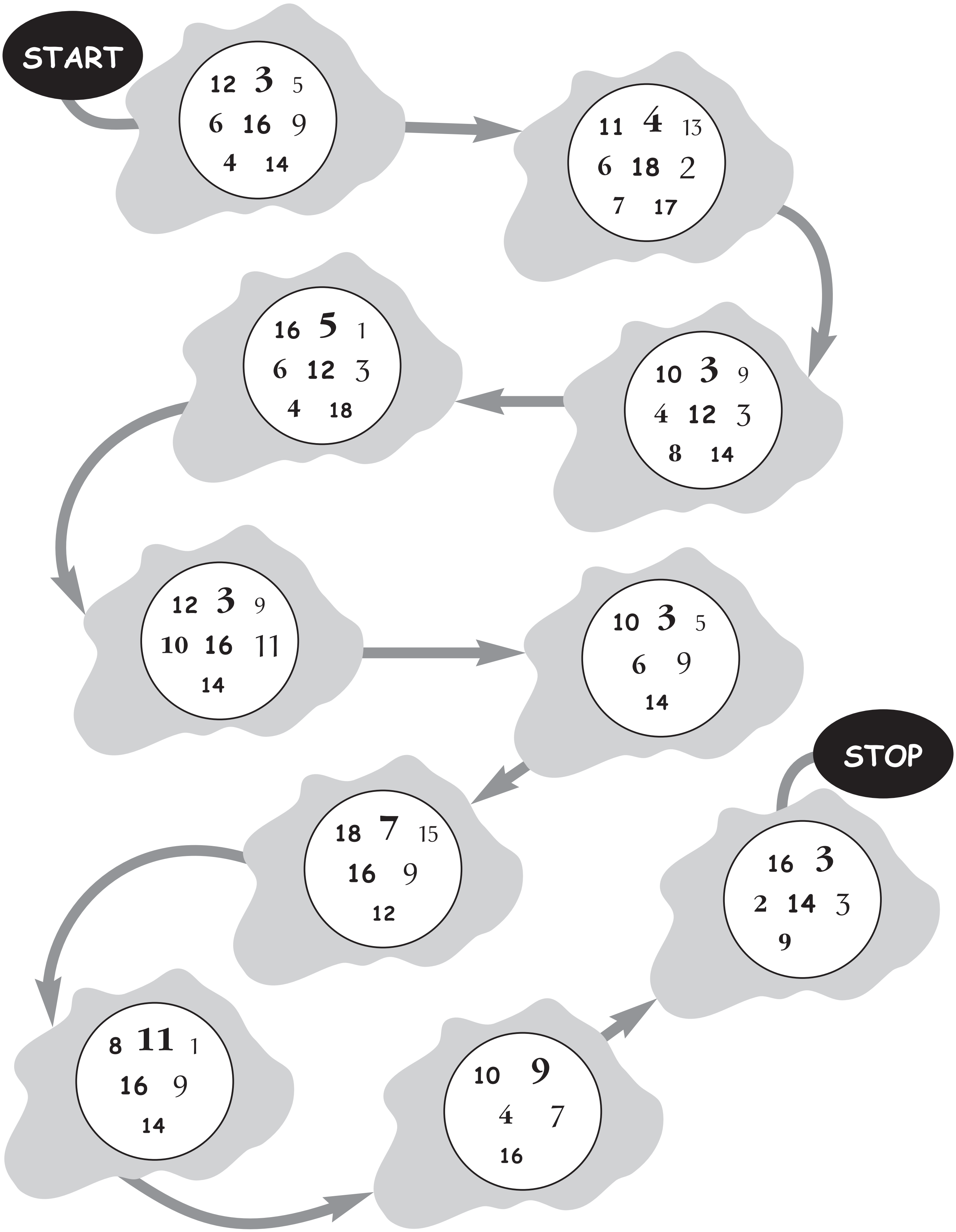
2	4	6
8	10	12
14	16	18
1 + 1	2 + 2	3 + 3
4 + 4	5 + 5	6 + 6
7 + 7	8 + 8	9 + 9

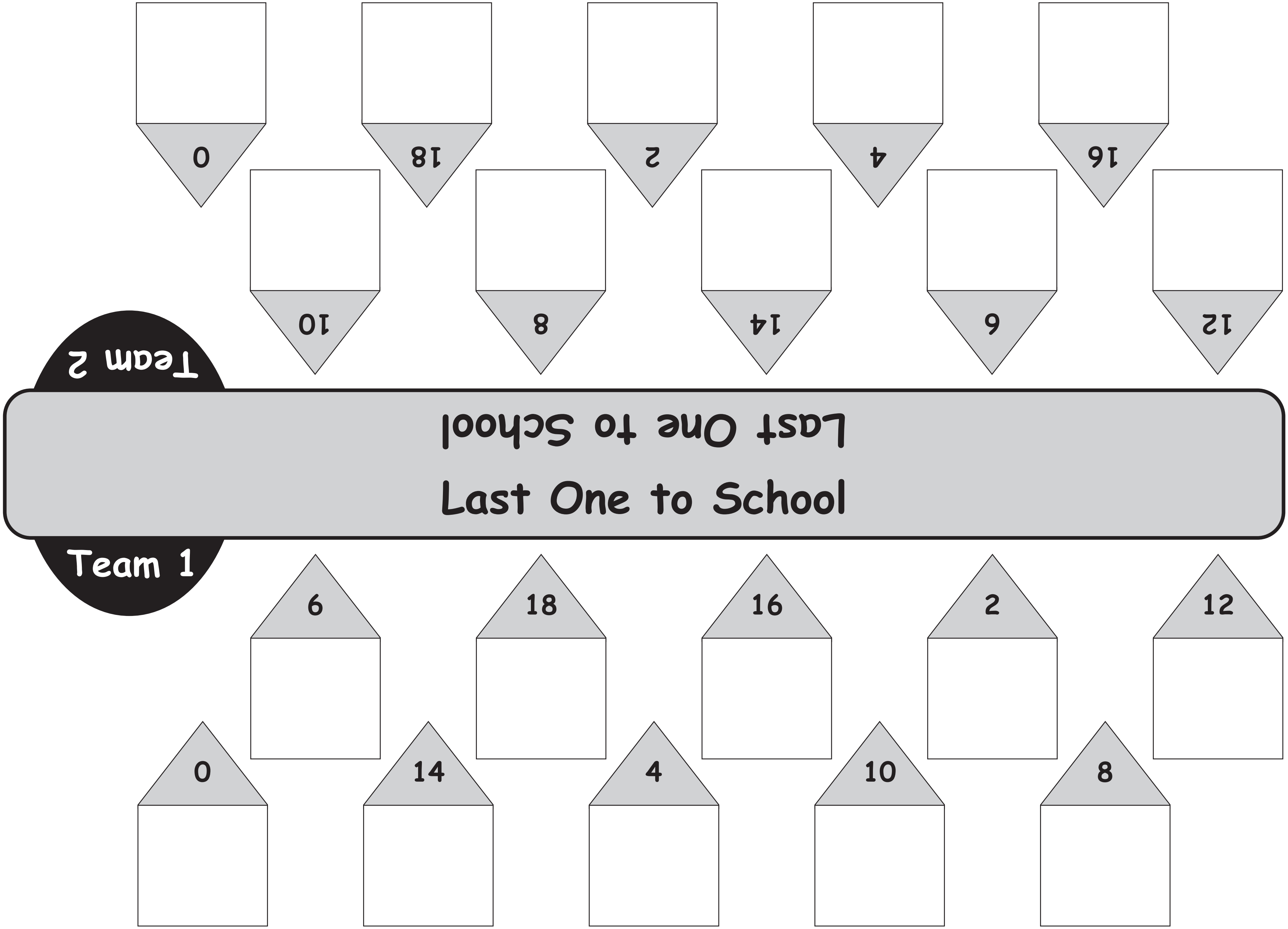
Magic Doubles

0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

BLM10

OVER-EASY DOUBLES




Last One to School
Team 2
0
10
18
8
2
14
4
6
16
12
Team 1
6
0
18
14
16
4
2
10
12
8
Last One to School

Number Cards

0	1	2	3	4
5	6	7	8	9
0	1	2	3	4
5	6	7	8	9

Number Cards

0	4	8
1	5	9
2	6	10
3	7	

Spinning for Near-Doubles

1	3	5	7	9	11	13	15	17	19

1	3	5	7	9	11	13	15	17	19

Spinning for Near-Doubles

Framing Ten and Up

Triplets Cards

1	1 + 0	0 + 1
2	2 + 0	0 + 2
3	1 + 2	2 + 1
4	3 + 1	1 + 3
5	2 + 3	3 + 2
6	2 + 4	4 + 2

Triplets Cards

7	5 + 2	2 + 5
7	3 + 4	4 + 3
8	2 + 6	6 + 2
8	1 + 7	7 + 1
9	6 + 3	3 + 6
9	4 + 5	5 + 4

Whichever Way Wins

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

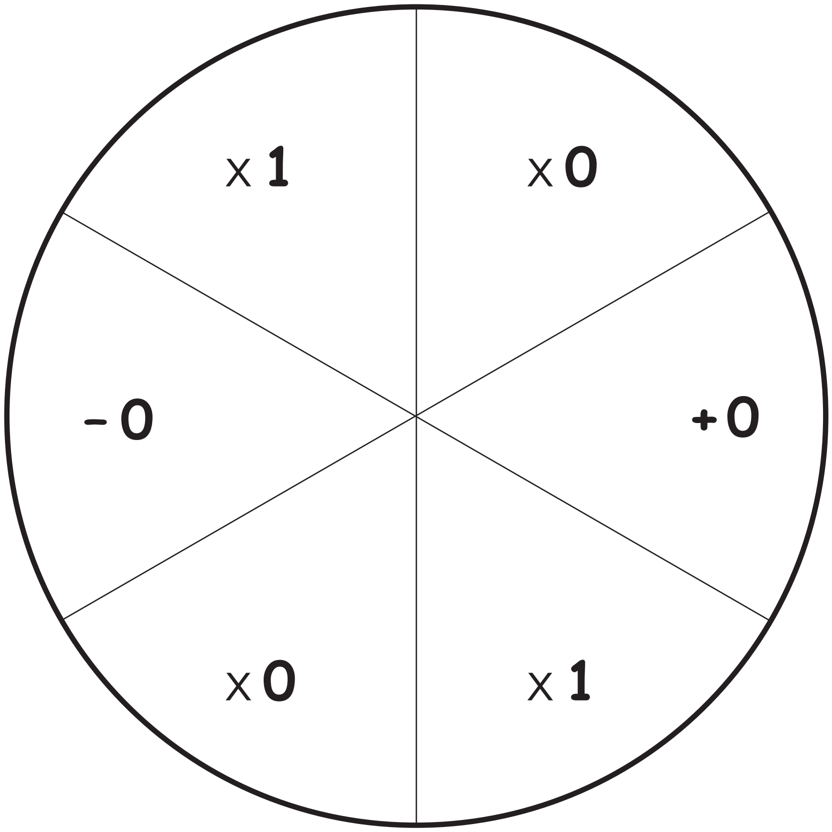
Triplets Cards

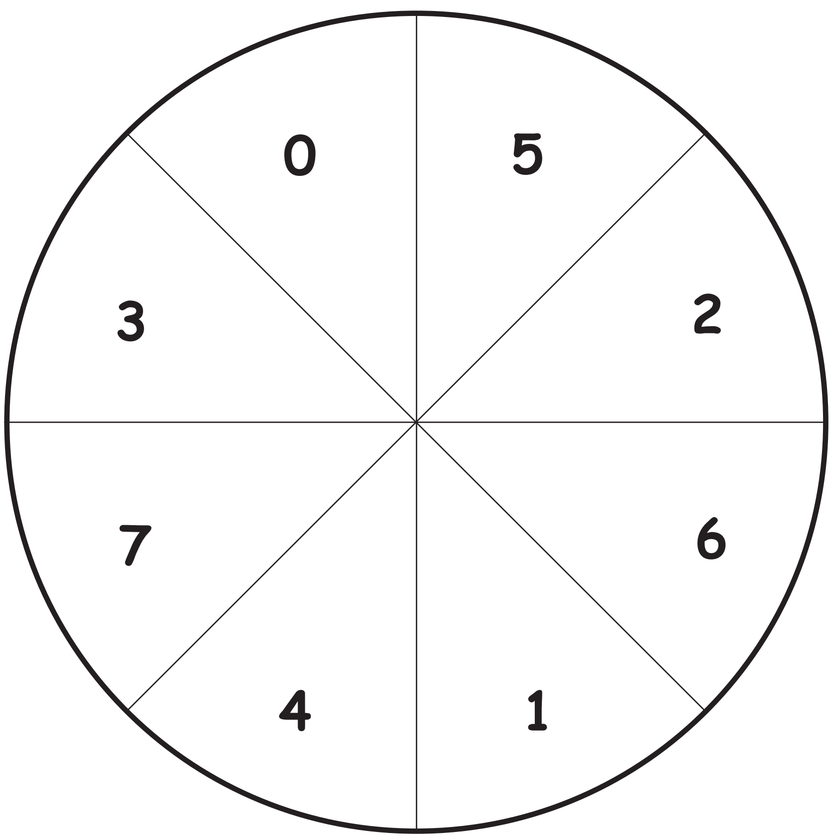
18	9 x 2	2 x 9
35	5 x 7	7 x 5
20	4 x 5	5 x 4
28	4 x 7	7 x 4
12	3 x 4	4 x 3
30	6 x 5	5 x 6

Triplets Cards

0	1 x 0	0 x 1
4	4 x 1	1 x 4
10	2 x 5	5 x 2
15	3 x 5	5 x 3
21	3 x 7	7 x 3
42	6 x 7	7 x 6

Spinning for Zero Spinners





Spinning for Zero

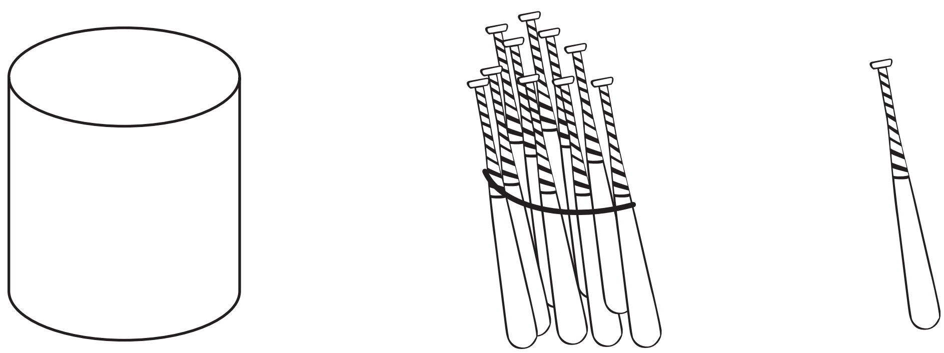
Johnson Sports Store

Scenario: The Johnson Sports Store sells baseball bats to baseball leagues in Canada. The bats are sold as ones, in bundles (10), and in canisters (100). Work with a group to fill the orders for the bats. Use craft sticks to represent the bats:

1 stick = 1 bat

1 bundle of 10 craft sticks
(held together with an elastic band) = a bundle of 10 bats

1 coffee canister = 100 bats



10 bundles of 10 bats	1 bundle of 10 bats	1 bat

You must fill the orders for the bats and send them to the leagues. Fill out one order form for each league.

Ship the bats in canisters and bundles whenever possible. Find out how many bats a league has ordered. Then find the best way to ship the order. These are the orders. Fill each league's orders on a separate order form.

Dundas League:

June 4	50 bats
June 8	64 bats
June 9	16 bats

Aurora League:

June 2	6 bats
June 10	25 bats
June 30	29 bats
July 1	55 bats
July 2	10 bats

Belfield League:

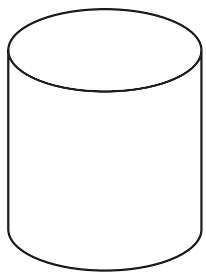
June 1	16 bats
June 5	32 bats
June 9	14 bats
June 20	50 bats

Johnson Sports Store: Baseball Bat Order Form

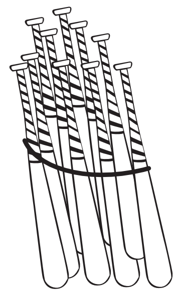
League Name:

Baseball bats can be shipped three ways:

In a canister of 100



In a bundle of 10



As a single bat

CALCULATIONS:

The total number of bats for the ______________________ league is __________________.

The most efficient way to ship the bats is:

___________ canisters

___________ bundles

___________ single bats

Hundreds Chart

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

0	20	40	60	80	100	120	140	160	180

Spinning for Double Tens

0	20	40	60	80	100	120	140	160	180

Spinning for Double Tens

BLM25

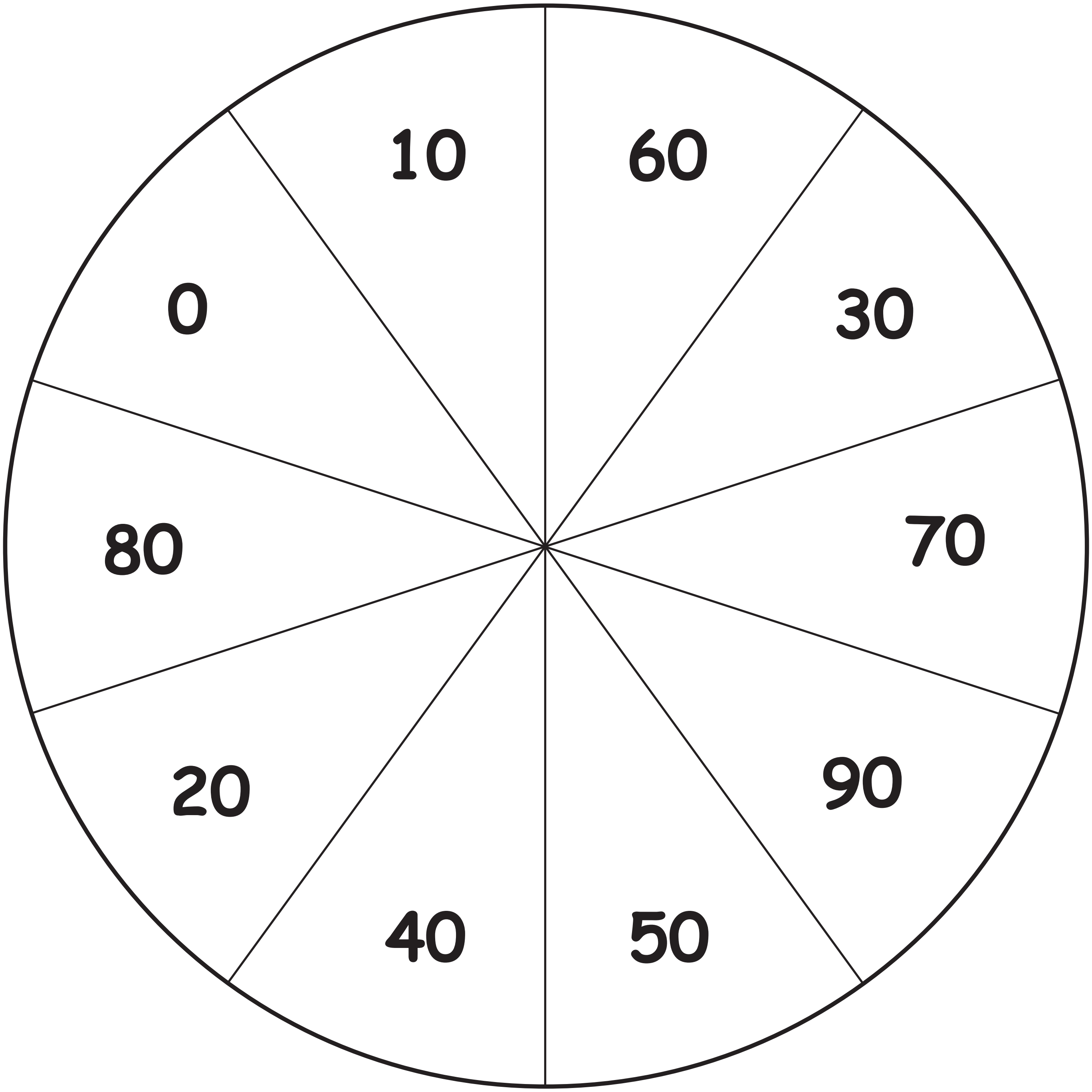
Spinning for Near-Double Tens

10	30	50	70	90	110	130	150	170	190

10	30	50	70	90	110	130	150	170	190

Spinning for Near-Double Tens

Spinning for Double Tens



Snappy Double Tens Cards

0	10	20
30	40	50
60	70	80
90		

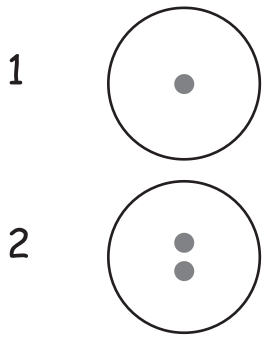
Criteria Cards for the Winning Number

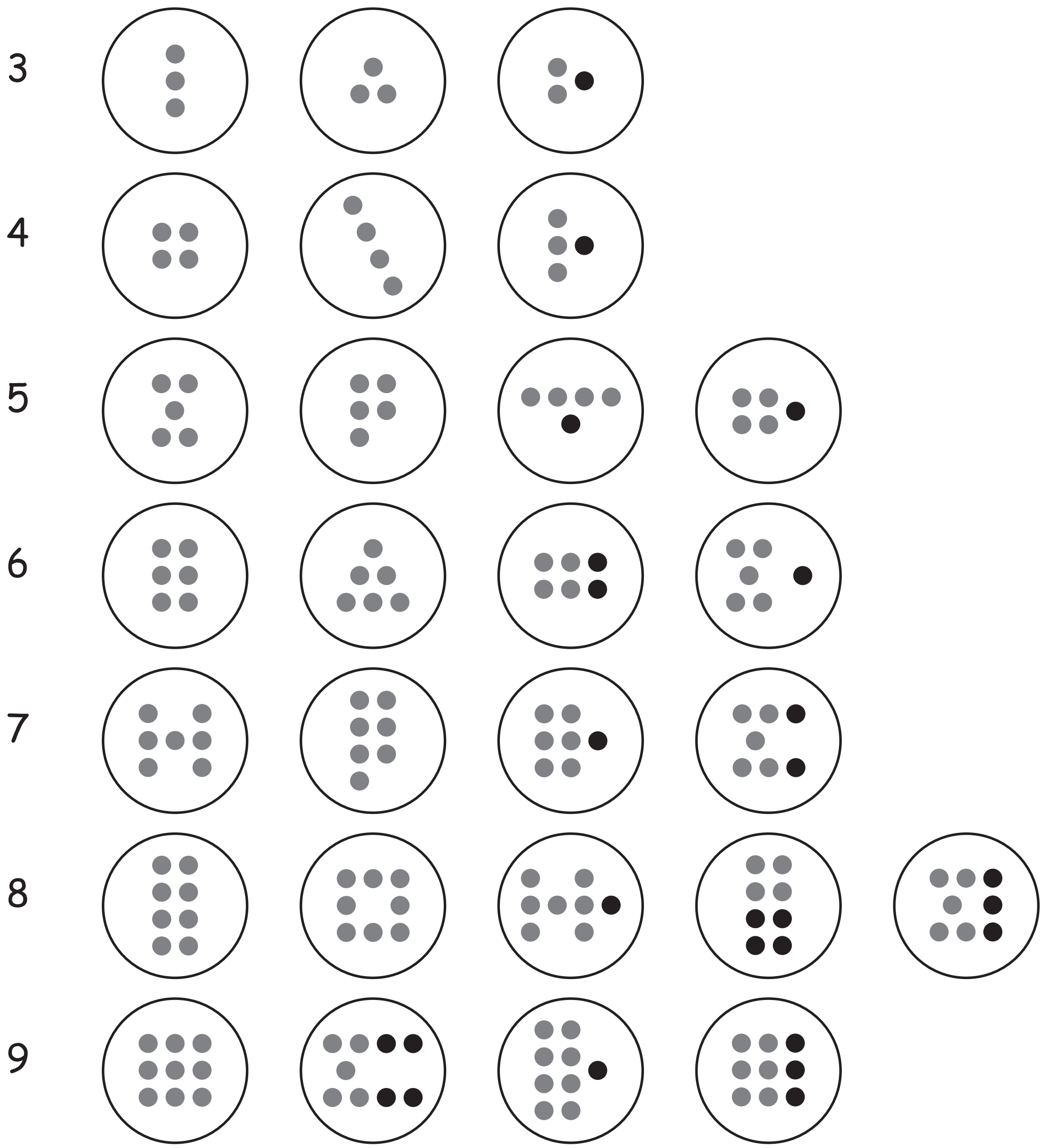
Highest number wins	Lowest number wins
If your number has a 1, you win.	If your number has a 5, you win.
If your number has a double digit, you win.	If your number has two digits (like 45 or 67), you win.
If your number has two even digits, you win.	If your number has two odd digits, you win.
If your number is even, you win.	If your number is odd, you win.

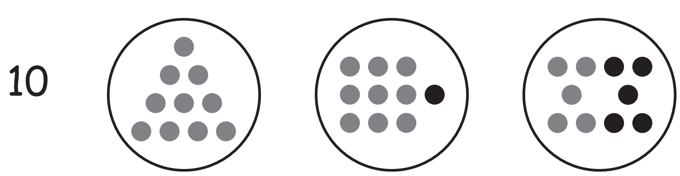
Paper Plate Dot Arrangements

BLM30

Some of these arrangements are made in one colour and some are made in two colours. Note the two shades of grey of the dots in the pictures.







From John A. Van de Walle, 2004, *Elementary and Middle School Mathematics: Teaching Developmentally* (5th ed.), Pearson Education.

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