Gradient Calculation of the Travel Time Cost Function by Adjoint State Techniques

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CRPC-TR94381
February 1994

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February 21, 1994

Abstract

We derive in this paper the expression of the gradient of the travel time cost function by adjoint state techniques. We show how to get for the numerical schemes exact adjoint conditions, and in particular adjoint discrete equations. We use an upwind scheme of first order to integrate the eikonal equation, and demonstrate the efficency of the preceeding technique on the simple case of a homogeneous medium.

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1 Introduction

Asymptotic expansions like geometric optics or integral expression like Kirchhoff integral of an acoustic wave field are commonly used in seismology (cf [?]). They are generally less expensive than the the modeling of the full wave field solution of the full wave equation by say finite difference technique.
Those expansions or integral representation involve directly the travel time in their expression. This travel time map depends on the "background" velocity. So in order to perform inversion by local optimization technique it is necessary to compute the gradient of the travel time map with respect to the velocity field.
That is what we propose to do in this paper. We use adjoint state technique to do so (cf [?] [?]). Previous work on tomography inversion used an argument based on Fermat's principle of least time to show that the derivative with respect to the slowness in a particular box is approximately the path length in that box (cf [?], [?]).
Our approach is different in the sense that we do not use Fermat's principle to compute the gradient. We tackle this problem in the same way, one would compute the gradient of the non linear cost function of the acoustic inverse problem (cf [?] or [?]).

2 The Continuous Problem

We consider a domain $\Omega \in R^2$ where is given a velocity profile. We are interested in the map $\mathcal{F}$ which associates the travel time between the source and the receivers in the domain.
We can parametrize this map by the velocity or by their inverse the slowness s. Given a slowness field s we can find the travel time τ by solving the eikonal equation. So $\mathcal{F}$ is defined from the slowness space $\mathcal{S}$ to the travel time space $\mathcal{T}$ by :

$$\begin{aligned} \mathcal{F} \ : \ \mathcal{S} &\longrightarrow \ \mathcal{T} \\ s &\longmapsto \ \mathcal{F}(s) = \tau(x,z) \end{aligned}$$

where τ is the solution of

$$(1) \qquad \left\{ \begin{array}{rcll} |\nabla \tau(x,z)|^2 &=& s^2(x,z) & (x,z) \in \Omega \\ \\ \tau(x,z) &=& \tau_0(s;x,z) & (x,z) \in \Gamma_0 \end{array} \right.$$

We want to minimize the cost function associated to the error on the travel time and defined by :

(2) $$J(s) = \frac{1}{2}\sum_{r\in\mathcal{R}} |\mathcal{F}(s,R) - t^d(R)|^2 = \frac{1}{2}\sum_{r\in\mathcal{R}} |\tau(s,R) - t^d(R)|^2$$

where $t^d(R)$ are travel time data at the receiver R, and $\mathcal{R}$ is the array of receivers.
To compute the gradient of the cost function J, we have to know the derivative of the application $\mathcal{F}$ with respect to s. In effect, we have :

(3) $$\begin{aligned} J'(s).\delta s &= \sum_{r\in\mathcal{R}} \mathcal{F}'(s,R).\delta s.(\mathcal{F}(s,R) - t^d(R)) \\ &= \sum_{r\in\mathcal{R}} \tau'(s,R).\delta s.(\tau(s,R) - t^d(R)) \end{aligned}$$

To compute the derivative of $\mathcal{F}$ with respect to s, we set $\delta\tau = \mathcal{F}'(s).\delta s = \tau(s)'.\delta s$. It is shown in appendix 1 that the perturbation δs in the slowness field, will give rise to a perturbation $\delta\tau$ in the travel times, which will satisfy to first order the perturbed equation

(4) $$\begin{cases} \nabla\tau.\nabla\delta\tau(x,z) = s.\delta s(x,z) & (x,z) \in \Omega \\ \delta\tau(x,z) = \tau_0'(s;x,z)\delta s(x,z) & (x,z) \in \Gamma_0 \end{cases}$$

So the derivative of J with respect to s is

$$J'(s).\delta s = \sum_{r\in\mathcal{R}} \delta\tau(s,R).(\tau(s,R) - t^d(R))$$

Now imagine that we can write for a certain Hilbert space the following equality

$$J'(s).\delta s = (\delta s\ ,\ \nabla J(s))_H$$

then we have found the gradient with respect to the hilbert space we have chosen of the cost function J. To obtain this expression of the gradient, it is worth to note that the dependance of $\delta\tau$ on δs is double. It occurs in the "source" term and in the boundary term of equation (??).
Since we are going to consider value of $\delta\tau$ on the boundary, that is traces of $\delta\tau$ we are led to introduce the following Hilbert space H :

(5) $$H = \left\{ u \in H^1(\Omega)\ /\ u_{\Gamma_1} = 0 \right\}$$

provided with the norm $||.||_H$:

$$||u||_H = \int_\Omega |\nabla u|^2 dx\ dz \tag{6}$$

we can write then

$$\begin{aligned} J'(s).\delta s &= \sum_{\tau\in\mathcal{R}} \delta\tau(s,R).(\tau(s,R)-t^d(R)) \\ &= \int_\Omega \delta\tau(s;x,z) r(x,z) dx\ dz \end{aligned}$$

with $r(x,z) = \sum_{\tau\in\mathcal{R}} .(\tau(s;x,z)-t^d(x,z))\delta_R$. We introduce then the adjoint state w solution of:

$$\left\{ \begin{aligned} -\nabla\left(w\nabla\tau\right)(x,z) &= r(x.z) \qquad (x,z)\in\Omega \\ w(x,z) &= 0 \qquad (x,z)\in\Gamma_1 = \Gamma/\Gamma_0 \end{aligned} \right. \tag{7}$$

We can then write

$$\begin{aligned} J'(s).\delta s &= \int_\Omega \delta\tau(s;x,z)r(x,z)dx\ dz \\ &= -\int_\Omega \delta\tau(s;x,z)\nabla\left(w\nabla\tau\right)(x,z)dx\ dz \\ &= \int_\Omega \nabla\tau.\nabla\delta\tau(s;x,z)w(x,z)dx\ dz - \int_\Gamma \delta\tau(s;x,z)w\frac{\partial\tau}{\partial n}(x,z)d\sigma \\ &= \int_\Omega \nabla\tau.\nabla\delta\tau(s;x,z)w(x,z)dx\ dz - \int_{\Gamma_0} \tau_0'(s;x,z).\delta s w\frac{\partial\tau}{\partial n}(x,z)d\sigma \\ &= \int_\Omega \delta s.\ (s(x,z)w(x,z))dx\ dz + \int_{\Gamma_0} \delta s\ (\tau_0'(s;x,z))^* \left(-w\frac{\partial\tau}{\partial n}(x,z)\right) d\sigma \end{aligned}$$

With our choice of norm on H we can write

$$
\begin{aligned}
J'(s).\delta s &= (\delta s\ ,\ \nabla J(s))_H \\
&= \int_\Omega \nabla(\nabla J(s)).\nabla \delta s dx\, dz \qquad (8)\\
&= -\int_\Omega \Delta(\nabla J(s)).\delta s dx\, dz + \int_{\Gamma_0} \delta s\ \frac{\partial \nabla J(s)}{\partial n} d\sigma
\end{aligned}
$$

Therefore $\nabla J(s)$ is solution of the following elliptic problem:

$$
(9) \qquad \begin{cases}
-\Delta(\nabla J(s)) = s.w & (x,z) \in \Omega \\
\dfrac{\partial \nabla J(s)}{\partial n} = (\tau_0'(s))^* \left(-w \dfrac{\partial \tau}{\partial n}(x,z)\right) & (x,z) \in \Gamma_0 \\
\nabla J(s) = 0 & (x,z) \in \Gamma_1
\end{cases}
$$

Of course we have $\nabla J(s) \in H$.

Remark

In the case of an homogeneous medium, assuming the boundary $\Gamma_0 =]X_{min}, X_{max}, z_0[$ we can write :

$$
\begin{aligned}
\tau(x,z) &= \sqrt{x^2+z^2}.s \\
\tau_0(x,z) &= \sqrt{x^2+z_0^2}.s
\end{aligned}
$$

Therefore, in this case τ_0 is linear in s, and so $\tau_0'(s).\delta s = \tau(\delta s) = \sqrt{x^2+z_0^2}\delta s$.

$$
\int_{\Gamma_0} \delta s\ (\tau_0'(s;x,z))^* \left\{-w\frac{\partial \tau}{\partial n}(x,z)\right\} d\sigma
$$

$$
= \int_{X_{min}}^{X_{max}} \delta s(x,z_0)\ \left(\sqrt{x^2+z_0^2}\left\{-w\frac{\partial \tau}{\partial n}(x,z_0)\right\}\right) dx
$$

3 The Discrete Problem

Let us set $u = \delta\tau$, we know that u is the solution of :

$$(10)\qquad \begin{cases} \tau_z u_z + \tau_x u_x &= s.\delta s \qquad (x,z) \in \Omega \\ \\ u &= \tau_0.\delta s \qquad (x,z) \in \Gamma_0 \end{cases}$$

Assuming that $\tau_z \neq 0$ we can write (??) as follows :

$$(11)\qquad \begin{cases} u_z + \dfrac{\tau_x}{\tau_z} u_x &= \dfrac{s}{\tau_z}.\delta s \qquad (x,z) \in \Omega \\ \\ u &= \tau_0.\delta s = u_0 \qquad (x,z) \in \Gamma_0 \end{cases}$$

Numerically we are going to treat this equation as an evolution equation in z. To integrate it we are going to use an upwind scheme given by :

$$(12)\qquad \begin{cases} \dfrac{u_j^{n+1} - u_j^n}{\Delta z} - (a^+)_j^n \dfrac{u_{j-1}^n - u_j^n}{\Delta x} + (a^-)_j^n \dfrac{u_{j+1}^n - u_j^n}{\Delta x} &= f_j^n \\ \\ u_j^0 &= u_{0j} \end{cases}$$

where $a = \frac{\tau_x}{\tau_z}$, $a^+ = max(a,0)$, $a^- = min(a,0)$ and $f = \dfrac{s}{\tau_z}.\delta s$. The "outflow" condition on τ translates here into

$$(13)\qquad \begin{cases} (a^+)_0^n = 0 \qquad (a^+)_1^n = 0 \quad n = 0..N \\ \\ (a^-)_J^n = 0 \qquad (a^-)_{J-1}^n = 0 \quad n = 0..N \end{cases}$$

We now need to define a discrete equivalent of the space H we defined in the preceeding section. To do so we introduce some notation.

$$(14)\qquad \begin{cases} \Omega_h &= \{(x_j, z_n)\ \ j = 1..J-1\ \ n = 1..N-1\} \\ \\ \Gamma_{0h} &= \{(x_j, z_0)\ \ j = 1..J-1\ \} \\ \\ \Gamma_{1h} &= \{(x_1, z_n)\ \ n = 1..N-1\} \bigcup \{(x_j, z_N)\ \ j = 1..J-1\} \\ \\ & \quad \bigcup \{(x_J - 1, z_n)\ \ n = 1..N-1\} \end{cases}$$

where $h = (\Delta z, \Delta x)$.

Now we consider the discrete L^2 spaces and define their scalar products as follows

$$(15) \quad \left\{ \begin{array}{lcl} (u,v)_{L^2(\Omega_h)} & = & \sum_{j=1}^{J-1} \sum_{n=1}^{N-1} u_j^n . v_j^n \, \Delta x \, \Delta z \\ \\ (u,v)_{L^2(\Gamma_{0h})} & = & \sum_{j=1}^{J-1} u_j^0 . v_j^0 \, \Delta x \\ \\ (u,v)_{L^2(\Gamma_{1h})} & = & \sum_{n=1}^{N-1} u_1^n . v_1^n \, \Delta z + \sum_{j=1}^{J-1} u_j^N . v_j^N \, \Delta x + \sum_{n=1}^{N-1} u_{J-1}^n . v_{J-1}^n \, \Delta z \end{array} \right.$$

We have

$$J'(s).\delta s \;=\; \int_\Omega \delta\tau(s;x,z) r(x,z) dx \, dz$$

At the discrete level we will have

$$\begin{aligned} J'(s).\delta s &= (u,r)_{L^2(\Omega_h)} \\ &= \sum_{j=1}^{J-1} \sum_{n=1}^{N-1} u_j^n r_j^n \Delta x \Delta z \end{aligned}$$

If we introduce the discrete adjoint state w solution of :

$$(16) \qquad \frac{w_j^{n-1} - w_j^n}{\Delta z} - \frac{(a^+)_{j+1}^n w_{j+1}^n - (a^+)_j^n w_j^n}{\Delta x} + \frac{(a^-)_{j-1}^n w_{j-1}^n - (a^-)_j^n w_j^n}{\Delta x} = r_j^n$$

we can write using appendix 2 :

$$J'(s).\delta s = \sum_{j=1}^{J-1}\sum_{n=1}^{N-1} u_j^n r_j^n \Delta x \Delta z$$

$$= \sum_{j=1}^{J-1}\sum_{n=1}^{N-1} u_j^n \left(\frac{w_j^{n-1} - w_j^n}{\Delta z} - \frac{(a^+)_{j+1}^n w_{j+1}^n - (a^+)_j^n w_j^n}{\Delta x} + \frac{(a^-)_{j-1}^n w_{j-1}^n - (a^-)_j^n w_j^n}{\Delta x} \right) \Delta x \Delta z$$

$$= \sum_{j=1}^{J-1}\sum_{n=1}^{N-1} w_j^n \left(\frac{u_j^{n+1} - u_j^n}{\Delta z} - (a^+)_j^n \frac{u_{j-1}^n - u_j^n}{\Delta x} + (a^-)_j^n \frac{u_{j+1}^n - u_j^n}{\Delta x} \right) \Delta x \Delta z$$

$$+ \sum_{n=1}^{N-1} (a^-)_0^n w_0^n u_1^n \Delta z + \sum_{j=1}^{J-1} w_j^0 u_j^1 \Delta x - \sum_{j=1}^{J-1} w_j^{N-1} u_j^N \Delta x - \sum_{n=1}^{N-1} (a^+)_J^n w_J^n u_{J-1}^n \Delta z$$

$$= \sum_{j=1}^{J-1}\sum_{n=1}^{N-1} w_j^n f_j^n \Delta x \Delta z$$

$$+ \sum_{n=1}^{N-1} (a^-)_0^n w_0^n u_1^n \Delta z + \sum_{j=1}^{J-1} w_j^0 u_j^1 \Delta x - \sum_{j=1}^{J-1} w_j^{N-1} u_j^N \Delta x - \sum_{n=1}^{N-1} (a^+)_J^n w_J^n u_{J-1}^n \Delta z$$

Choosing w such that :

$$\begin{cases} w_0^n = w_J^n = 0 & n = 0..N \\ w_j^{N-1} = 0 & j = 1..J-1 \end{cases}$$

we have

$$J'(s).\delta s \quad = \quad \left(\delta s \ , \ \frac{s}{\tau_z} w\right)_{L^2(\Omega_h)} + \sum_{j=1}^{J-1} w_j^0 u_j^1 \Delta x$$

Using (??) for $n = 0$ we have

$$u_j^1 = u_j^0 + \Delta z (a^+)_j^0 \frac{u_{j-1}^0 - u_j^0}{\Delta x} - \Delta z (a^-)_j^0 \frac{u_{j+1}^0 - u_j^0}{\Delta x} + \Delta z f_j^0$$

Therefore

$$\begin{aligned}
\sum_{j=1}^{J-1} w_j^0 u_j^1 \Delta x &= \sum_{j=1}^{J-1} w_j^0 u_j^0 \Delta x + \Delta z \sum_{j=1}^{J-1} w_j^0 (a^+)_j^0 \frac{u_{j-1}^0 - u_j^0}{\Delta x} \Delta x \\
&- \Delta z \sum_{j=1}^{J-1} w_j^0 (a^-)_j^0 \frac{u_{j+1}^0 - u_j^0}{\Delta x} \Delta x + \Delta z \sum_{j=1}^{J-1} w_j^0 f_j^0 \Delta x \\
&= \sum_{j=1}^{J-1} (w_j^0 + \Delta z f_j^0) u_j^0 \Delta x + \Delta z \sum_{j=1}^{J-1} \frac{(a^+)_{j+1}^0 w_{j+1}^0 - (a^+)_j^0 w_j^0}{\Delta x} u_j^0 \Delta x \\
&- \Delta z \sum_{j=1}^{J-1} \frac{(a^-)_{j-1}^0 w_{j-1}^0 - (a^-)_j^0 w_j^0}{\Delta x} u_j^0 \Delta x \\
&= \left(u^0 \, , \, G^0\right)_{L^2(\Gamma_{0h})}
\end{aligned}$$

where $G_j^0 = w_j^0 + \Delta z (f_j^0 + \dfrac{(a^+)_{j+1}^0 w_{j+1}^0 - (a^+)_j^0 w_j^0}{\Delta x} - \dfrac{(a^-)_{j-1}^0 w_{j-1}^0 - (a^-)_j^0 w_j^0}{\Delta x})$.
Finally we can write :

$$\begin{aligned}
J'(s).\delta s &= \left(\delta s \, , \, \frac{s}{\tau_z} w\right)_{L^2(\Omega_h)} + \left(u^0 \, , \, G^0\right)_{L^2(\Gamma_{0h})} \\
&= \left(\delta s \, , \, \frac{s}{\tau_z} w\right)_{L^2(\Omega_h)} + \left(\tau_0'(s).\delta s \, , \, G^0\right)_{L^2(\Gamma_{0h})} \\
&= \left(\delta s \, , \, \frac{s}{\tau_z} w\right)_{L^2(\Omega_h)} + \left(\delta s \, , \, (\tau_0'(s))^* \, G^0\right)_{L^2(\Gamma_{0h})}
\end{aligned}$$

Now we can approximate the expression

$$\begin{aligned}
J'(s).\delta s &= (\delta s \, , \, \nabla J(s))_H \\
&= \int_\Omega \nabla(\nabla J(s)).\nabla \delta s dx \, dz
\end{aligned}$$

by the following discrete expression

$$I = \sum_{j=1}^{J-1} \sum_{n=1}^{N-1} \frac{\nabla J_{j+1}^n - \nabla J_j^n}{\Delta x} \frac{\delta s_{j+1}^n - \delta s_j^n}{\Delta x} + \frac{\nabla J_j^{n+1} - \nabla J_j^n}{\Delta z} \frac{\delta s_j^{n+1} - \delta s_j^n}{\Delta z}$$

with

$$\begin{cases} \nabla J_0^n = \nabla J_J^n = 0 & n = 0..N \\ \\ \nabla J_j^N = 0 & j = 1..J-1 \end{cases}$$

By discrete integration by parts, we have that ∇J_j^n is the solution of the following discrete problem :

$$\begin{cases} -\Delta_h(\nabla J_j^n) & = & \left(\frac{s}{\tau_z} w\right)_j^n \\ \\ \frac{\nabla J_j^1 - \nabla J_j^0}{\Delta z} & = & \left((\tau_0'(s))^* \, G^0\right)_j^n \\ \\ \nabla J_0^n = \nabla J_J^n & = & 0 \quad n = 0..N \\ \\ \nabla J_j^N & = & 0 \quad j = 1..J-1 \end{cases} \tag{17}$$

4 Numerical results

We consider on homogeneous background slowness $\sigma_0 = 1$ given in the following (x,z) rectangle domain $\Omega =]-1,1[\times]1,3[$. We add to this slowness field a small slowness pertubation $\delta\sigma$. This perturbation local in space, is concentrated in a disk below the interface $z = 1$ centered on (0,1.43) (cf Figure 1).

We assume the source to be (0,0) and that the medium is unperturbed above $z = 1$. It is therefore easy to compute the travel time on this interface.

So we know the travel times on interface $z = 1$, and the total slowness field σ given by the sum of the reference and perturbed fields $\sigma = \sigma_0 + \delta\sigma$. In our example the perturbation is restricted to the interior of the domain Ω. Therefore $\delta\tau$ will not depend on $\delta\sigma$ on the boundary $z = 1$.

The receivers are located on the portion of the interface $z = 3$, ranging from $x = -0.75$ to $x = -0.25$.

We show hereafter the adjoint (backpropagation of the residues) and the gradient fields. It is clear on the backpropagation of the residues (cf Figure 2) that we "illuminated" the part of the model where the perturbation is located. From the gradient (cf Figure 3), we see that we have found the correct location of the slowness perturbation.

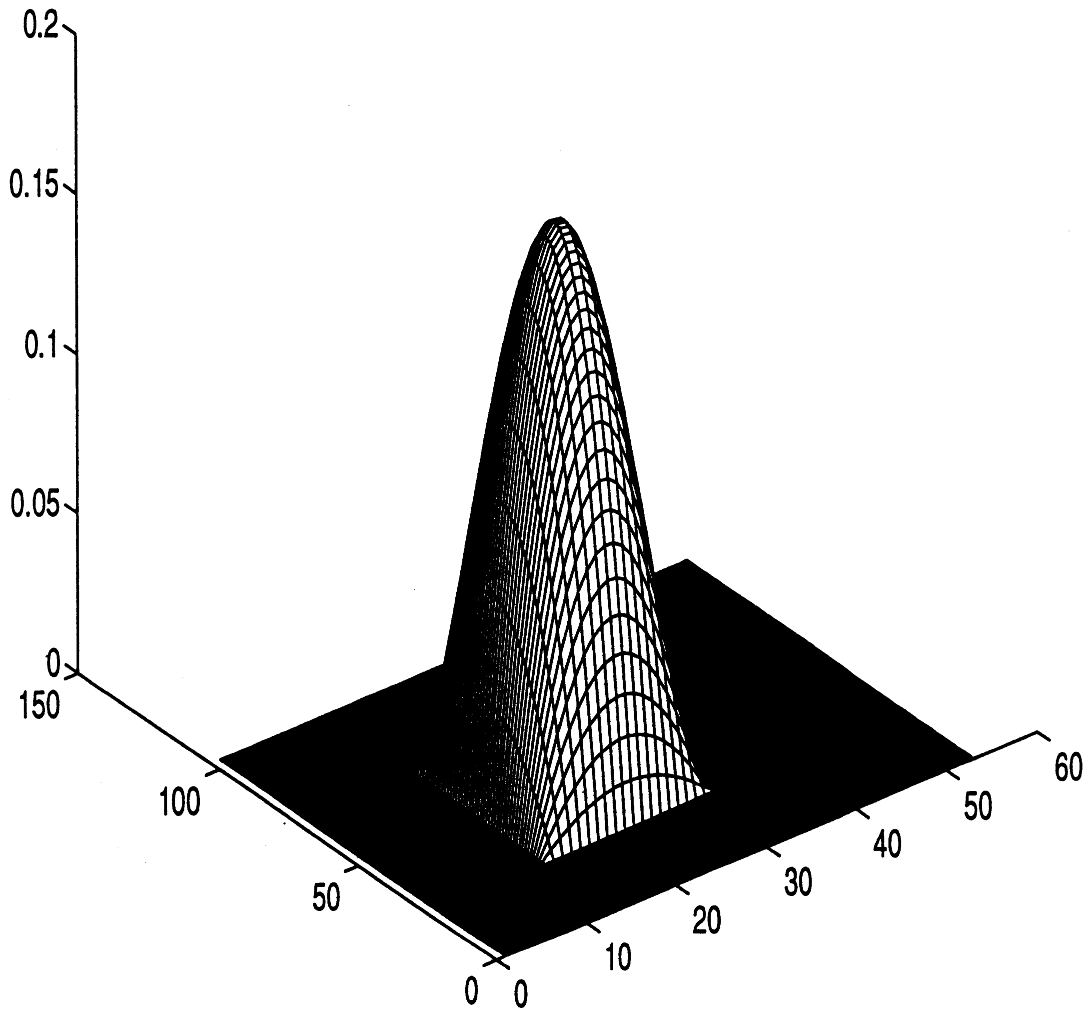

FIG 1 : Slowness Perturbation

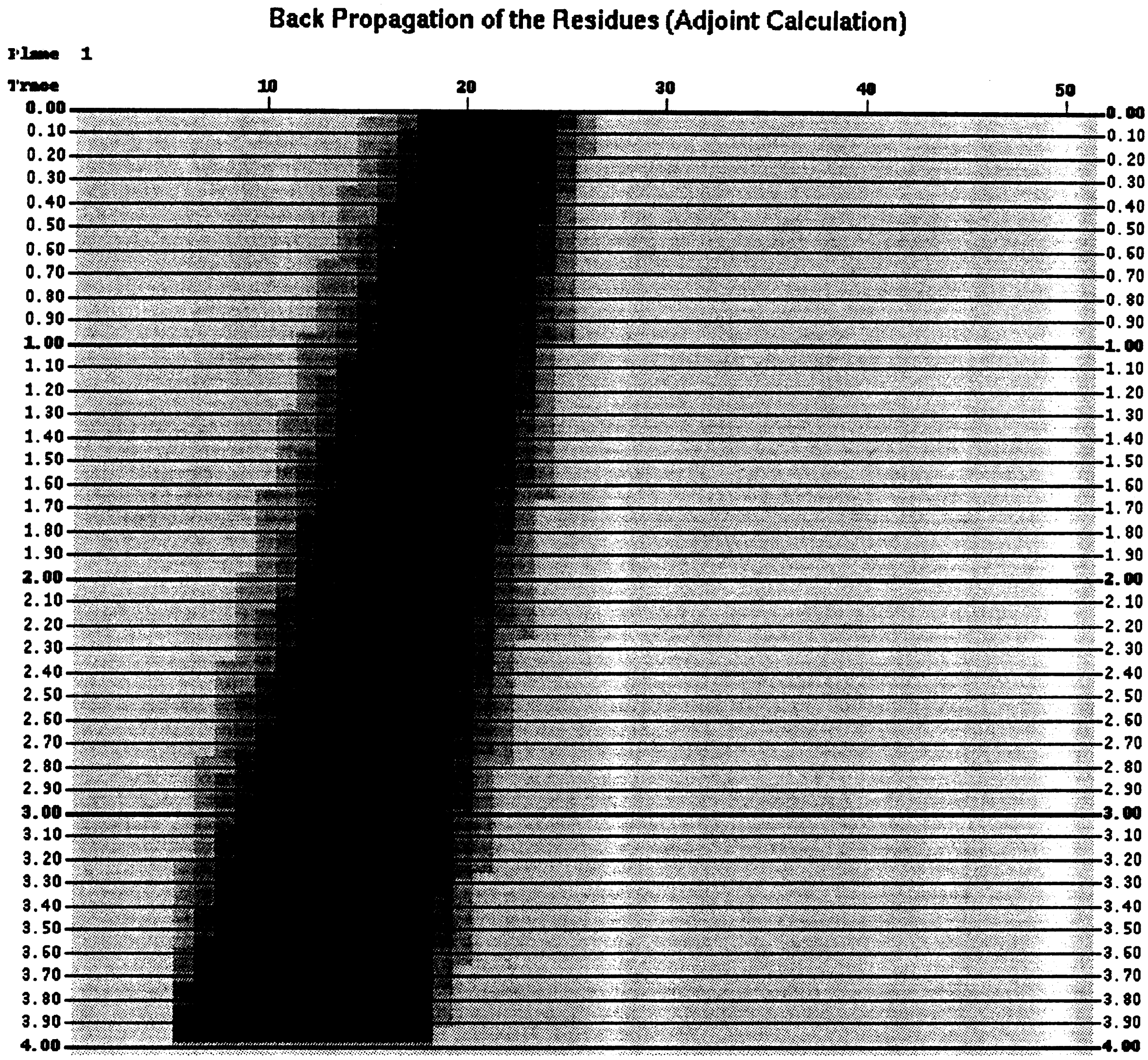

FIG 2 : Adjoint Field

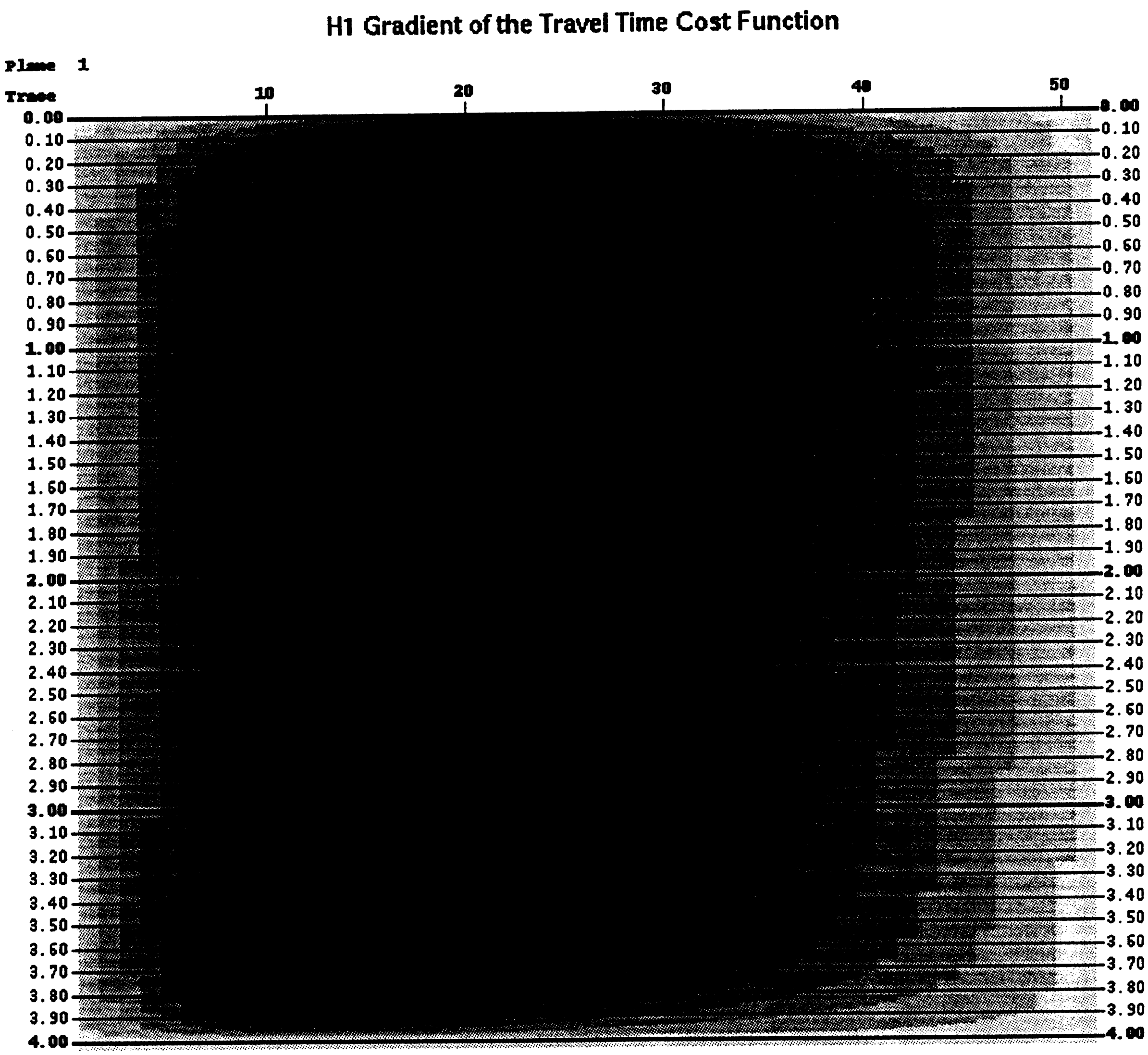

FIG 3 : Gradient Field

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Appendix 1

We want to compute the derivative with respect to S of the map $\mathcal{F}$. To do so we consider a slowness perturbation δs and by definition its associated travel time satisfies :

$$\left\{\begin{array}{rcll} \nabla\tau(s+\delta s;x,z)^2 &=& (s+\delta s)^2(x,z) & \quad (x,z)\in\Omega \\ \\ \tau(s+\delta s;x,z) &=& \tau_0(s+\delta s;x,z) & \quad (x,z)\in\Gamma_0 \end{array}\right.$$

Our goal is to find what is the equation verified by $\delta\tau = \tau'(s).\delta s$. Since we have

$$\begin{aligned} (\nabla\tau(s+\delta s))^2 &= (\nabla(\tau(s)+\tau'(s).\delta s+o(\delta s^2)))^2 \\ &= (\nabla\tau(s))^2 + 2.\nabla\tau(s).\tau'(s).\delta s + o(\delta s^2)) \end{aligned}$$

and

$$\tau_0(s+\delta s) = \tau_0(s) + \tau_0'(s).\delta s + o(\delta s^2))$$

we can write

$$\begin{aligned} 2.\nabla\tau(s).\tau'(s).\delta s + o(\delta s^2)) &= (\nabla\tau(s+\delta s))^2 - (\nabla\tau(s))^2 = 2s.\delta s + o(\delta s^2) \\ \delta\tau &= \tau_0'(s).\delta s + o(\delta s^2) \end{aligned}$$

Dropping the term of order higher or equal to two (because we are looking for the derivative) we find that $\delta\tau$ is the solution of the following problem :

$$\left\{ \begin{aligned} \nabla\tau.\nabla\delta\tau(x,z) &= s.\delta s(x,z) & (x,z) \in \Omega \\ \delta\tau(x,z) &= \tau_0'(s;x,z)\delta s(x,z) & (x,z) \in \Gamma_0 \end{aligned} \right.$$

Appendix 2

We are looking for the adjoint equation of equation (??) for the scalar product defined on Ω_h. Let us note P^* the adjoint operator of P where P is the defined by the discrete equation (??). We have

$$(P^* w, u)_{L^2(\Omega_h)} = (Pu, w)_{L^2(\Omega_h)}$$

$$= \sum_{j=1}^{J-1}\sum_{n=1}^{N-1} w_j^n \left(\frac{u_j^{n+1} - u_j^n}{\Delta z} - (a^+)_j^n \frac{u_{j-1}^n - u_j^n}{\Delta x} + (a^-)_j^n \frac{u_{j+1}^n - u_j^n}{\Delta x} \right) \Delta x \Delta z$$

Let us treat the first integral

$$
\begin{aligned}
I_1 &= \sum_{j=1}^{J-1}\sum_{n=1}^{N-1} w_j^n \frac{u_j^{n+1}-u_j^n}{\Delta z}\Delta x\Delta z \\
&= \sum_{j=1}^{J-1}\frac{1}{\Delta z}\left(\sum_{n=1}^{N-1} w_j^n u_j^{n+1} - \sum_{n=1}^{N-1} w_j^n u_j^n\right)\Delta x\Delta z \\
&= \sum_{j=1}^{J-1}\frac{1}{\Delta z}\left(\sum_{n=2}^{N} w_j^{n-1} u_j^{n} - \sum_{n=1}^{N-1} w_j^n u_j^n\right)\Delta x\Delta z \\
&= \sum_{j=1}^{J-1}\sum_{n=1}^{N-1} \frac{w_j^{n-1}-w_j^n}{\Delta z} u_j^n\Delta x\Delta z - \sum_{j=1}^{J-1} w_j^0 u_j^1\Delta x + \sum_{j=1}^{J-1} w_j^{N-1}u_j^N\Delta x
\end{aligned}
$$

The second integral can be written as

$$
\begin{aligned}
I_2 &= -\sum_{j=1}^{J-1}\sum_{n=1}^{N-1} w_j^n (a^+)_j^n \frac{u_{j-1}^n-u_j^n}{\Delta x}\Delta x\Delta z \\
&= -\sum_{n=1}^{N-1}\frac{1}{\Delta x}\left(\sum_{j=1}^{J-1}(a^+)_j^n w_j^n u_{j-1}^n - \sum_{j=1}^{J-1}(a^+)_j^n w_j^n u_j^n\right)\Delta x\Delta z \\
&= -\sum_{n=1}^{N-1}\frac{1}{\Delta x}\left(\sum_{j=0}^{J-2}(a^+)_{j+1}^n w_{j+1}^n u_j^n - \sum_{j=1}^{J-1}(a^+)_j^n w_j^n u_j^n\right)\Delta x\Delta z
\end{aligned}
$$

$$
= -\sum_{j=1}^{J-1}\sum_{n=1}^{N-1}\frac{(a^+)_{j+1}^n w_{j+1}^n - (a^+)_j^n w_j^n}{\Delta x} u_j^n\Delta x\Delta z + \sum_{n=1}^{N-1}(a^+)_J^n w_J^n u_{J-1}^n\Delta z
$$

Let us treat the last integral

$$
\begin{aligned}
I_3 &= \sum_{j=1}^{J-1}\sum_{n=1}^{N-1} w_j^n (a^-)_j^n \frac{u_{j+1}^n - u_j^n}{\Delta x}\Delta x \Delta z \\
&= \sum_{n=1}^{N-1} \frac{1}{\Delta x}\left(\sum_{j=1}^{J-1}(a^-)_j^n w_j^n u_{j+1}^n - \sum_{j=1}^{J-1}(a^+)_j^n w_j^n u_j^n\right)\Delta x \Delta z \\
&= \sum_{n=1}^{N-1} \frac{1}{\Delta x}\left(\sum_{j=2}^{J}(a^-)_{j-1}^n w_{j-1}^n u_j^n - \sum_{j=1}^{J-1}(a^-)_j^n w_j^n u_j^n\right)\Delta x \Delta z \\
&= \sum_{j=1}^{J-1}\sum_{n=1}^{N-1} \frac{(a^-)_{j-1}^n w_{j-1}^n - (a^-)_j^n w_j^n}{\Delta x} u_j^n \Delta x \Delta z - \sum_{n=1}^{N-1}(a^-)_0^n w_0^n u_1^n \Delta z
\end{aligned}
$$

Finally we can write

$$
\sum_{j=1}^{J-1}\sum_{n=1}^{N-1}\left(\frac{w_j^{n-1} - w_j^n}{\Delta z} - \frac{(a^+)_{j+1}^n w_{j+1}^n - (a^+)_j^n w_j^n}{\Delta x} + \frac{(a^-)_{j-1}^n w_{j-1}^n - (a^-)_j^n w_j^n}{\Delta x}\right) u_j^n \Delta x \Delta z
$$

$$
= \sum_{j=1}^{J-1}\sum_{n=1}^{N-1} w_j^n \left(\frac{u_j^{n+1} - u_j^n}{\Delta z} - (a^+)_j^n \frac{u_{j-1}^n - u_j^n}{\Delta x} + (a^-)_j^n \frac{u_{j+1}^n - u_j^n}{\Delta x}\right)\Delta x \Delta z
$$

$$
+ \sum_{n=1}^{N-1}(a^-)_0^n w_0^n u_1^n \Delta z + \sum_{j=1}^{J-1} w_j^0 u_j^1 \Delta x - \sum_{j=1}^{J-1} w_j^{N-1} u_j^N \Delta x - \sum_{n=1}^{N-1}(a^+)_J^n w_J^n u_{J-1}^n \Delta z
$$