THE NATURE OF PARTITION BIJECTIONS II. ASYMPTOTIC STABILITY

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Abstract. We introduce a notion of asymptotic stability for bijections between sets of partitions and a class of geometric bijections. We then show that a number of classical partition bijections are geometric and that geometric bijections under certain conditions are asymptotically stable.

Introduction

Partition theory has a long and glorious history spanning over the last two and a half centuries, as it originated in the works of Euler and is still under intense development. An important turning point was the invention of the so called *"constructive partition theory"* by Sylvester and his school, which was based on the idea that partition identities can be proved by direct combinatorial arguments. Since then, there has been an explosion in the use of bijections and involutions, within partition theory and elsewhere, to the extend that there is now a whole subject of "bijective combinatorics". The purpose of this paper is to connect partition bijections with asymptotic arguments, describe the asymptotic behavior of various partition bijections, and obtain the limiting shape of certain sets of partitions.

The full history of partition bijections is yet to be written, so we will use our extensive recent survey [14] as a reference point. We discovered that there are several classes of partition bijections which have an entirely different nature. This paper is almost completely dedicated to *geometric bijections*, which are, basically, various rearrangements of squares of Young diagrams (or Ferrers diagrams) geometrically representing the partitions. We elaborate on the remaining classes of partition bijections in section 14.

Despite a variety among classes of bijections, geometric bijections constitute the majority of all partition bijections [14]. The examples include several 'Durfee square type' bijections, classical Sylvester's bijections, as well as some beautiful Andrews's, Bressoud's and Alladi-Gordon's bijective proofs [14]. These bijections are the main object of our studies, as we look into their constructions and asymptotic properties.

Before mentioning the main ideas, let us elaborate on the reasoning behind this work. For people in the area it has been a long standing dream to distinguish *"good"*

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bijections from *"not so good"* ones, especially in the context of Rogers-Ramanujan bijections, where the celebrated Garsia-Milne bijection [9] long deemed unsatisfactory. In the words of Viennot, *"It remains an open problem to know if there exist a "direct" or "simple" bijection, without using the so-called "involution principle"* [26]. Joichi and Stanton, in the same context, phrase the problem a little differently: *"The emphasis now should be placed on combinatorially important proofs rather than just a proof"* [11]. Most recently, Zeilberger reiterates this sentiment once again: *"finding a **really nice** bijective proof is still an open problem"* [28]. Note that all three papers ask exactly the same question but neither one is able to formulate it precisely; even the adjectives are all different.

Unfortunately, until this work, little progress has been made towards resolving the problem, except for various informal observations and aesthetic arguments. An interesting test case appears when a partition identity has several different bijective proofs, where choosing which one is "really nice" becomes a point of contention. Perhaps, the most celebrated example of this is Euler's "the number of partitions of n into odd parts equals to the number of partitions of n into distinct parts" theorem. The bijective proofs include Glaisher's arithmetic bijection, Sylvester's geometric bijection and a recent recursive argument [14, 15]. It was Sylvester himself [22] who noticed the distinct nature of different bijections:

> *It is a striking fact that the pairs arithmetically and graphically associated will be entirely different, thus evidencing that correspondence is rather a creation of the mind than a property inherent in the things associated.*

In this paper we introduce the notion of *asymptotic stability* to formalize the idea of "good" bijections. We then formally define a class of geometric bijections (and a more general class of *AG-bijections*) and prove that all bijections in that class are asymptotically stable. We spend much of this paper on justifying our definitions by showing that all those famous partition bijections mentioned earlier belong to the class of AG-bijections. We also briefly mention few examples of (non-geometric) partition bijections which are not asymptotically stable. Incidently, of the three bijective proofs of Euler's theorem only Sylvester's bijection is asymptotically stable.

The basic tool in our study is the *limit shape* for sets of partitions. This idea originated in the study of random Young tableaux [24], and has recently appeared in the context of random partitions [23]. Roughly, it says that the diagram of a random partition scaled by a factor $1/\sqrt{n}$ (so that it has unit area) has shape approximating the curve

$$(\spadesuit) \qquad e^{-cx} + e^{-cy} = 1, \quad \text{where } c = \frac{\pi}{\sqrt{6}}\,.$$

For other classes of partitions similar limit shape formulas have been found in [20, 25].

The main idea in this paper is to study the effect the geometric bijections have on the limit shape. We show that in certain cases the limit shape is well defined and geometric bijections are well defined in the limit. The existence of such limit after scaling we call the asymptotic stability. Roughly speaking, geometric bijections in the limit become piecewise-linear geometric transformations (of the plane) which map one limit shape into the other.

Beside philosophical implications ("good" vs. "not so good" of bijections), there is a number of formal applications of our approach. These include computing the limit shape for various classes of partitions, proving asymptotic stability of various geometric statistics, etc. Finally, in the forthcoming paper [18] we prove nonexistence of a geometric Rogers-Ramanujan bijection, giving the very first negative answer to this classical problem.

The content. We start with basic definitions of partitions, Young and Ferrers diagrams, and their enumeration in section 1. Definitions of geometric and AG-bijections are given in section 2, and the the asymptotic stability of bijections is introduced in section 3. Main results are formulated in section 4. The geometric bijections are studied one by one in sections 5–12. Finally, we prove technical results in section 13, and close with final remarks in section 14.

On Language. Throughout the paper by a *correspondence* we refer to one-to-one maps between sets of partitions which preserve the size of the partitions. By a *partition bijection* we mean not just a one-to-one map, but an algorithm mapping one set of partitions into another. Of course, two different algorithms can define the same correspondence. In view of a geometric nature of the partition bijections in the paper, we will always think of a bijection as an algorithm for rearranging squares of Young diagram of a partition. We also use the word *transformation* when we speak of rearrangements of integral lattice points.

Notation. Various sets of partitions are denoted by script capital letters $\mathcal{A}_n$, $\mathcal{B}_n$, etc., where the index n will indicate that these are partition of the same size n. All partitions and bijections are denoted by Greek minuscules. Finally, we use $\mathbb{N} = \{1, 2, \dots\}$ and the English presentation of Young diagrams.

1. Basic definitions

1.1. **Young and Ferrers diagrams.** We define a *partition* λ to be an integer sequence $(\lambda_1, \lambda_2, \dots, \lambda_\ell)$ such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l > 0$. We say that λ is a partition of n, denoted $\lambda \vdash n$ and $|\lambda| = n$, if $\sum_i \lambda_i = n$. We refer to the integers λ_i as the *parts of the partition.* The *number of parts* of λ we denote by $\ell(\lambda) = \ell$, and the *largest part* by $a(\lambda) = \lambda_1$. The *conjugate partition* $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ of λ is defined by $\lambda'_i = |\{j : \lambda_j \geq i\}|$. Clearly, $\ell(\lambda') = a(\lambda)$.

A *Young diagram* $[\lambda]$ of a partition $\lambda \vdash n$ is a collection of n unit squares (i, j) on a square grid $\mathbb{Z}^2$, with $1 \leq i \leq \ell(\lambda)$, $1 \leq j \leq \lambda_i$. Pictorially, we think of the first coordinate i increasing downward, while the second coordinate j increases from left to right. The conjugate Young diagram $[\lambda']$ is then obtained by reflection of $[\lambda]$ through the line $i = j$ (see Figure 1).

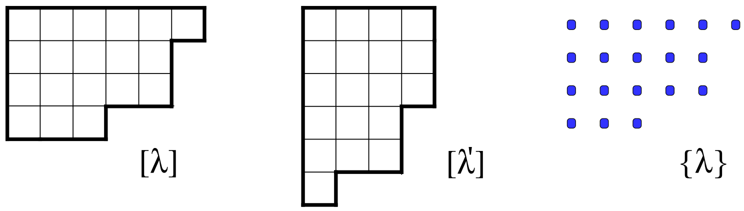

FIGURE 1. Young diagrams of partitions $\lambda = (6, 5, 5, 3)$, $\lambda' = (4^3 3^2 1)$, and Ferrers diagram of a partition λ.

An equivalent presentation of a partition is by means of a *Ferrers diagram* $\{\lambda\}$, which is basically Young diagram with unit squares replaced by dots (see Figure 1). Whenever possible, we graphically present partitions by their Young diagrams. On the other hand, in mathematical context it is more convenient to operate with Ferrers diagrams, so we will use both notation.

Define *multi-partitions* to be finite sequences $\varkappa = (\lambda, \mu, \nu, \ldots)$ of partitions, and let $|\varkappa| = |\lambda| + |\mu| + |\nu| + \ldots$ Similarly, define (Ferrers) *multi-diagram* $\{\varkappa\}$, to be a finite sequence of Ferrers diagram. We think of Ferrers diagrams $\{\lambda\}$, $\{\mu\}$, $\{\nu\}$, etc. from the multi-diagram as being in different lattice grids $\mathbb{N}^2$, but draw them next to each for convenience (in fact, we will draw only the corresponding Young diagrams). We use notation $\sqcup^r(\mathbb{Z}^2) = \mathbb{Z}^2 \sqcup \mathbb{Z}^2 \sqcup \ldots \sqcup \mathbb{Z}^2$ (r times) to denote the set of lattice points in the disjoint union of planes which contain multi-diagrams.

For partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ define the *sum* $\lambda + \mu$ to be a partition $(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots)$. Similarly, define the *union* $\lambda \cup \mu$ to be a partition with parts $\{\lambda_i, \mu_j\}$ (arranged in nonincreasing order). Observe that $(\lambda \cup \mu)' = \lambda' + \mu'$ (see Figure 2).

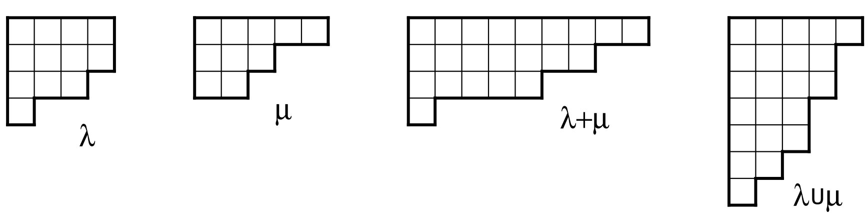

FIGURE 2. Young diagrams of partitions $\lambda = (4, 4, 3, 1)$, $\mu = (5, 3, 2)$, $\lambda + \mu = (9, 7, 5, 1)$, and $\lambda \cup \mu = (5, 4, 4, 3, 3, 2, 1)$.

The *Durfee square* is defined to be the largest square which fits Young diagram $[\lambda]$. Similarly, for every $-\infty < c < \infty$, define *Durfee c-rectangle* to be the largest $z \times (c+z)$ rectangle which fits $[\lambda]$. Denote by δ_c the size of a side z of the Durfee c-rectangle as above. Statistic $\delta_\circ := \delta_\circ$ is called the *Durfee rank*.

It is easy to see that the upper left corner of the Durfee rectangle is always in $(0, 0)$, while the lower right corner is an intersection of $y - x = c$ line with the boundary of $[\lambda]$ (see Figure 3).

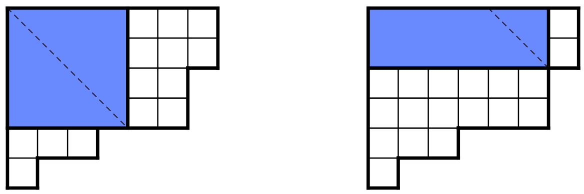

FIGURE 3. Durfee square and Durfee c-rectangle in $[\lambda]$, where $c = 4$ and $\lambda = (7, 7, 6, 6, 3, 1)$. Here $\delta_\circ = 3$ and $\delta_4 = 2$.

1.2. **Enumeration of partitions.** Denote by $\mathcal{P}_n$ the set of all partitions $\lambda \vdash n$, and let $\mathcal{P} = \cup_n \mathcal{P}_n$. Two subsets $\mathcal{A}, \mathcal{B} \subset \mathcal{P}$ are called *equinumerous* if $|\mathcal{A}_n| = |\mathcal{B}_n|$ for all $n \in \mathbb{N}$, where $\mathcal{A}_n = \mathcal{A} \cap \mathcal{P}_n$ and $\mathcal{B}_n = \mathcal{B} \cap \mathcal{P}_n$. For example, Euler's theorem (see the introduction) says that the set $\mathcal{D}$ of partitions into distinct parts is equinumerous to the set $\mathcal{O}$ of partitions into odd parts: $|\mathcal{D}_n| = |\mathcal{O}_n|$.

Denote by $\mathcal{P}^r$ the set of all multi-partitions $\varkappa \in \mathcal{P} \times \ldots \times \mathcal{P}$ (r times). Two subsets of multi-partitions $\mathcal{A} \subset \mathcal{P}^r$ and $\mathcal{B} \subset \mathcal{P}^s$ are called *equinumerous* if the number of $\varkappa \in \mathcal{A}$ such that $|\varkappa| = n$ is equal to the number of $\sigma \in \mathcal{B}$ such that $|\sigma| = n$, for all $n \in \mathbb{N}$.

A *statistic* on $\mathcal{A}$ is a function $\alpha : \mathcal{A} \to \mathbb{R}$. For example, the number of parts $\ell(\lambda)$ or the length of the first part $a(\lambda) := \lambda_1$ are both statistics on the set of all partitions $\mathcal{P}$.

Suppose α is a statistic on $\mathcal{P}$ and $\mathcal{A}, \mathcal{B} \subset \mathcal{P}$. We say that a map $\varphi : \mathcal{A} \to \mathcal{B}$ *preserves* statistic α if $\alpha(\varphi(\lambda)) = \alpha(\lambda)$ for all $\lambda \in \mathcal{A}$. For example, the conjugation map preserves size of the partition $|\lambda|$ and its Durfee rank $\delta_\circ(\lambda)$.

Similarly, we say that $\varphi : \mathcal{A} \to \mathcal{B}$ *maps* statistic α into β, where $\alpha : \mathcal{A} \to \mathbb{R}$, $\beta : \mathcal{B} \to \mathbb{R}$, if $\beta(\varphi(\lambda)) = \alpha(\lambda)$ for all $\lambda \in \mathcal{A}$. For example, the conjugation maps statistic δ_c into δ_{-c}.

1.3. **Shape of a partition.** The *shape* of a partition λ is a function $\rho_\lambda : (0, \infty) \to \mathbb{R}_+$ defined by

$$\rho_\lambda(x) = \begin{cases} \lambda_{\lceil x \rceil}, & \text{if } 0 < x \le \ell(\lambda), \\ 0, & \text{if } x > \ell(\lambda). \end{cases}$$

One can think of the shape function ρ_λ as of a boundary of the Young diagram. Denote by $\widetilde{\rho}_\lambda$ the *scaled shape* of λ defined by

$$\widetilde{\rho}_\lambda(x) = \frac{1}{\sqrt{n}} \, \rho_\lambda(\sqrt{n}\, x), \quad \text{where } n = |\lambda|.$$

Clearly, the area under the graph $\int_0^\infty \widetilde{\rho}_\lambda(x)\, dx = 1$.

For every partition λ denote by $\{\widetilde{\lambda}\}$ the set of points $\bigl(i/\sqrt{n}, j/\sqrt{n}\bigr)$, where $(i,j) \in \{\lambda\}$ and $n = |\lambda|$. We call $\{\widetilde{\lambda}\}$ the *scaled Durfee diagram.*

2. Geometric and AG-bijections

2.1. **Basic transformations.** Let us first defined *basic geometric transformations* which are the building blocks of general geometric transformations.

First, define *shift transformation* $\beth : \mathbb{Z}^2 \to \mathbb{Z}^2$ to be a linear map $\beth : (i,j) \to (i,j)M$, where $M \in \mathrm{SL}(2,\mathbb{Z})$. By definition, such φ preserve the lattice $\mathbb{Z}^2$.

Second, define *move transformation* $\beth : \mathbb{Z}^2 \to \mathbb{Z}^2$ to be a parallel translation $\beth : (i,j) \to (i+a, j+b)$, where $a, b \in \mathbb{Z}$.

Third, define *cut transformation* $\beth : \mathbb{Z}^2 \to \mathbb{Z}^2 \sqcup \mathbb{Z}^2$ to be a projection of points (i,j) onto the first component if $ai + bj \geq c$, and onto the second component if $ai + bj < c$, where $(a,b,c) \in \mathbb{Z}^3$.

Similarly, define *paste transformation* $\beth : \mathbb{Z}^2 \sqcup \mathbb{Z}^2 \to \mathbb{Z}^2$ to be a projection of lattices in two components onto the one component. Of course, this transformation will be used only when there in no overlap in the resulting set of integral points.

For an integral lattice $\Gamma \subset \mathbb{Z}^2$, define the *volume* $r(\Gamma)$ to be the number of cosets in $\mathbb{Z}^2/\Gamma$. For a sublattice $\Gamma = \mathbb{Z}\langle v, w\rangle \subset \mathbb{Z}^2$, where $v, w \in \mathbb{Z}^2$ and $r = r(\Gamma) < \infty$, define *shred transformation* $\beth : \mathbb{Z}^2 \to \sqcup^r(\mathbb{Z}^2)$ to be a projection of points $(i,j) \in \mathbb{Z}^2$ onto component $\mathbb{Z}^2$ corresponding to the coset $(i,j)\,\Gamma$.

Finally, for every $k, \ell \in \mathbb{Z}$ define *stretch transformation* $\beth : \mathbb{Z}^2 \to \mathbb{Z}^2$ to be a map $\beth : (i,j) \to (k\,i, j/\ell)$, where $(i,j) \in \Gamma_\ell$, and Γ_ℓ is a lattice of points $\{(i, \ell j) \mid i, j \in \mathbb{Z}\}$. When we need to be more specific, we call this (k,ℓ)*-stretch transformation*.

To summarize, we consider six *basic geometric transformations*:

$\diamond$ `shift` $\diamond$ `move` $\diamond$ `cut` $\diamond$ `paste` $\diamond$ `shred` $\diamond$ `stretch`

Of course, these transformations can be applied to any component of the multi-diagram.

Definition 2.1. *A* geometric transformation *is a composition of a finite number of basic geometric transformations.*

One has to be careful with using the geometric transformations to ensure that the resulting map is well defined. Also, while `shift` and `move` transformations are invertible, the remaining basic geometric transformations are not invertible in general.

The examples of basic geometric transformation are given in Figures 4 and 5. Here white dotes are placed at the selected lattice points of the grid. In Figure 4, the `cut` is along the line $i - j = 0$, and `moves` are along the vectors $(1,0)$ and $(0,-1)$. In Figure 5, the `shred` is according to a lattice $\Gamma = \{(i,j) \in \mathbb{Z}^2 \mid j \text{ is even}\}$, and the `shift` is by a matrix $M = \binom{1\ 0}{1\ 1} \in \mathrm{SL}(2,\mathbb{Z})$.

Finally, the integer values a, b, c, etc. that appear in the definition of basic geometric transformations are called *parameters* of a geometric transformation.

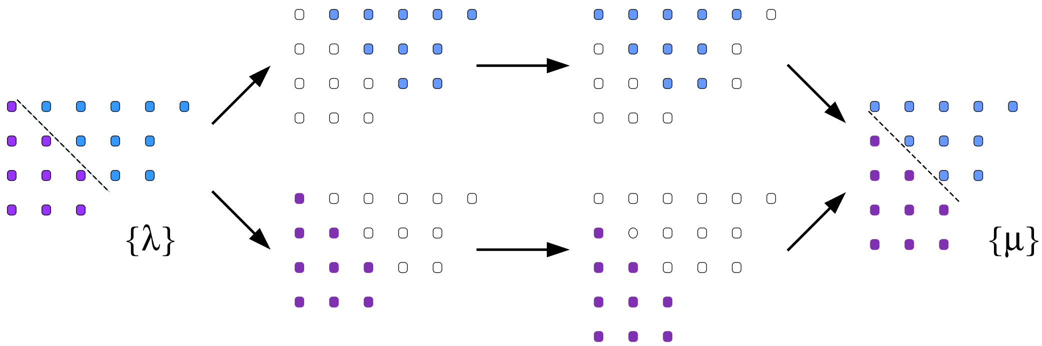

FIGURE 4. An example of a geometric bijection $\daleth : \{\lambda\} \to \{\mu\}$, where $\lambda = (6,5,5,3)$, $\mu = (5,4,4,3,3)$. Here $\daleth$ is a composition of a `cut`, two `moves` (one in each component), and a `paste`.

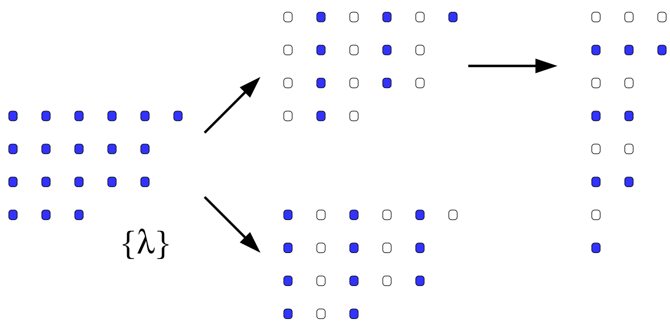

FIGURE 5. An example of a `shred` transformation and a `stretch` transformation of the first component (with $k = \ell = 2$).

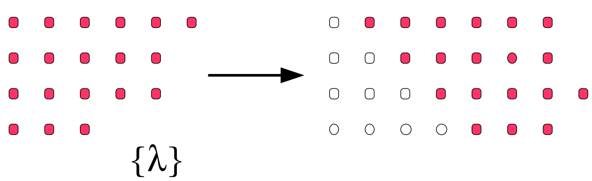

FIGURE 6. An example of a `shift` transformation of $\{\lambda\}$, where $\lambda = (6,5,5,3)$.

2.2. **Geometric bijections.** Let $\mathcal{A}$ and $\mathcal{B}$ be equinumerous sets of (multi-) partitions, and let $\vartheta : \mathcal{A} \to \mathcal{B}$ be a one-to-one correspondence such that $|\vartheta(\lambda)| = |\lambda|$, for all $\lambda \in \mathcal{A}$.

A *geometric bijection* φ which *defines* ϑ is said to be given by a geometric transformations $\daleth$ of the lattice points, which maps every Ferrers (multi-) diagram $\{\lambda\}$, $\lambda \in \mathcal{A}$, into a (multi-) diagram $\{\mu\}$, $\mu \in \mathcal{B}$, and such that $\mu = \vartheta(\lambda)$.

Let us emphasize here that a geometric transformation $\daleth$ may not be defined on all (multi-) diagrams, so we say that $\daleth$ is *bijective* on $\{\mathcal{A}\}$.

Note that in the definition of the geometric transformation we did not specify how parameters a, b, c, etc. are obtained. We say that a geometric bijection is *natural* if these parameters are fixed, i.e. independent of the starting partition. Most, but not all geometric bijections we consider are natural. Clearly, a composition of two natural bijections is also natural.

Recall that the number of components $\mathbb{Z}^2$ can both increase and decrease, so a geometric bijection between two sets of partitions can have Ferrers multi-diagrams in between (an example is given in Figure 4).

If two geometric bijections φ and ψ define the same correspondence ϑ, we write $\varphi \sim \psi$. If they also define identical maps from lattice points in $\{\lambda\}$ to lattice points in $\{\mu\}$, for all $\mu = \vartheta(\lambda)$, we write $\varphi \simeq \psi$. For example, let $id : \mathcal{P} \to \mathcal{P}$ be a trivial bijection, let $\gamma : \mathcal{P} \to \mathcal{P}$ be a conjugation bijection, and γ^2 be a conjugation repeated twice. Then $id \nsim \gamma$ and $id \sim \gamma^2$. Furthermore, in this case we have $id \simeq \gamma^2$.

2.3. **Basic properties.** We say that a geometric bijection φ, which defines $\vartheta : \mathcal{A} \to \mathcal{B}$, is *stable* if $\varphi(i,j)$ is the same for all $(i,j) \in \lambda$, and $\lambda \in \mathcal{A}$.

Proposition 2.2. *A natural geometric bijection is stable and preserves the size of the partition.*

Proof. Clear by induction. □

We say that the set of partitions $\mathcal{A}$ is *spanning* if the Durfee rank $\delta_\circ$ is unbounded on $\mathcal{A}$. Now Proposition 2.2 easily implies the following result.

Proposition 2.3. *Let $\mathcal{A}, \mathcal{B} \subset \mathcal{P}$ be two spanning sets of partitions and let map $\vartheta : \mathcal{A} \to \mathcal{B}$ be defined by a natural geometric bijection φ. Then $(i,j) \to \varphi(i,j)$ defines a one-to-one map $\varphi : \mathbb{N}^2 \to \mathbb{N}^2$.*

Proof. Clear. □

Finally, recall that the geometric transformations are not necessarily invertible. The following result reaffirms the importance of natural geometric bijections.

Theorem 2.4. *Let φ be a natural geometric bijection. Then its inverse φ^{-1} is also a natural geometric bijection.*

The proof is given in section 13.

2.4. **AG-bijections.** In this section we extend the class of geometric transformations by introducing two basic transformations of a different type. Let $\left(\nu^{(0)}, \nu^{(1)}, \ldots, \nu^{(r-1)}\right) \in \mathcal{P}^k$ be a multi-partition, such that $\nu_j^{(i)} \equiv i \bmod r$, for all i, j. Define *union transformation* $\urcorner_r : \left\{\nu^{(0)}, \nu^{(1)}, \ldots, \nu^{(r-1)}\right\} \to \{\lambda\}$, where $\lambda = \cup_i \nu^{(i)}$.

Similarly, define *sum* transformation $\sqsupset_r : \left\{\tau^{(0)}, \tau^{(1)}, \ldots, \tau^{(r-1)}\right\} \to \{\lambda'\}$, where $\tau^{(i)} = \left(\nu^{(i)}\right)'$ by using a conjugate transformation.

These transformations are defined only for Ferrers multi-diagrams but can be combined with geometric transformations. It is useful for to have both sum and union as our building blocks even though either of them combined with a conjugation gives the other. An example of a both transformations is given in Figure 7.

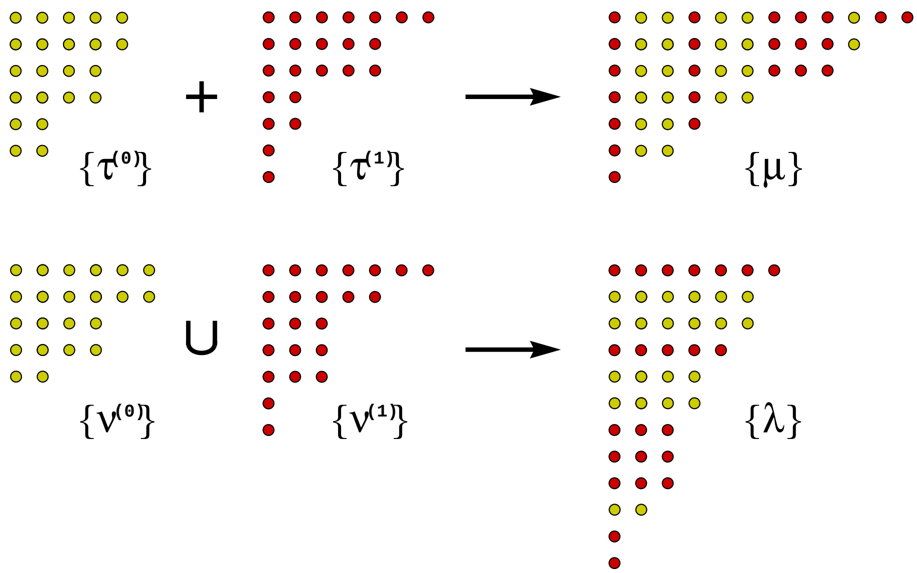

FIGURE 7. An example of `sum` and `union` transformations. Here $\tau^{(i)} = \left(\nu^{(i)}\right)'$ and $\mu = \lambda'$.

Note that union transformation has an inverse $\urcorner_r^{-1}$, which we call *split transformation modulo* r. It maps Ferrers diagram $\{\lambda\}$ into multi-diagrams $\left\{\nu^{(0)}, \ldots, \nu^{(r-1)}\right\}$, where $\nu^{(i)}$ contains all parts equal to i mod m. Similarly, one can define transformation $\sqsupset_r^{-1}$; we will not use it here.

To summarize, we consider two *basic arithmetic transformations*:

◇ `union` ◇ `split`

Definition 2.5. *An* arithmetic-geometric (AG) transformations *is a composition of a finite number of basic geometric and basic arithmetic transformations.*

These AG-transformations form a class of transformations which will be studied throughout the paper. Define *AG-bijections* and *natural AG-bijections* by analogy.

Unless stated otherwise, AG-bijections will be bijections between two sets of partitions rather than multi-partitions.

3. Asymptotic stability

3.1. **Statistics.** Let $\mathcal{A} \subset \mathcal{P}$ be a set of partitions, and let $\alpha : \mathcal{A} \to \mathbb{R}$ be a statistic. We say that α is *asymptotically stable* on $\mathcal{A} = \cup_n \mathcal{A}_n$ if there exist $a \in \mathbb{R}$ such that for every $\varepsilon > 0$ we have

$$(\Diamond) \qquad \frac{\bigl|\{\lambda \in \mathcal{A}_n \mid a - \varepsilon < \alpha(\lambda) < a + \varepsilon\}\bigr|}{|\mathcal{A}_n|} \longrightarrow 1 \quad \text{as} \quad n \to \infty.$$

One can think of $(\Diamond)$ as of a weak convergence phenomenon. In probabilistic terms it can be written as:

$$\mathbf{P}\bigl(|\alpha(\lambda) - a| < \varepsilon \mid \lambda \in \mathcal{A}_n\bigr) \to 1 \quad \text{as} \quad n \to \infty,$$

where the probability is taken over uniform partitions in $\mathcal{P}_n$.

Note that if statistics α and β are asymptotically stable, then so are $\alpha + \beta$ and $\alpha \cdot \beta$.

Example 3.1. Consider the set of "square" partitions $\mathcal{A} = \{(k^k), k \in \mathbb{N}\}$. Then statistic $\alpha := (\delta_\circ(\lambda))^2/|\lambda|$ is constant ($\alpha(\lambda) = 1$ for all $\lambda \in \mathcal{A}$), and thus asymptotically stable on $\mathcal{A}$.

Example 3.2. Let $\mathcal{A} = \{\lambda \in \mathcal{P} \mid \delta_\circ(\lambda) \le 2\}$ and $\mathcal{B} = \{\lambda \in \mathcal{P} \mid \delta_\circ(\lambda) \le 1\} \subset \mathcal{A}$ Partitions $\lambda \in \mathcal{B}$ are called *hooks*. Note that $|\mathcal{A}_n| = \theta(n^3)$ and $|\mathcal{B}_n| = n$. Therefore, the statistic $\delta_\circ$ is asymptotically stable on $\mathcal{A}$.

Example 3.3. Let $\mathcal{B}$ be the set of hook partitions as above, and let $\beta := \ell(\lambda)/(|\lambda|+1)$. Observe that $\mathbf{E}(\beta(\lambda) \mid \lambda \in \mathcal{B}_n) = 1/2$, but β is not asymptotically stable on $\mathcal{B}$.

3.2. **Limiting shape.** Let $\mathcal{A} = \cup_n \mathcal{A}_n$ be a set of partitions. For every $x > 0$, think of the a scaled shape value $\alpha_x := \widetilde{\rho}_\lambda(x)$, $\alpha_x : \lambda \to \mathbb{R}$ as of a statistic on $\mathcal{A}$. We say that $\mathcal{A}$ is *asymptotically stable* if α_x are asymptotically stable statistics with uniform convergence: there exist a function $a = a(x) : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_0^\infty a(x)\, dx = 1$, and for every $\varepsilon > 0$

$$\mathbf{P}\bigl(|\alpha_x(\lambda) - a(x)| < \varepsilon \ \text{ for all } x > 0 \mid \lambda \in \mathcal{A}_n\bigr) \to 1 \quad \text{as} \quad n \to \infty,$$

The function $a(x)$ is called the *limiting shape* of $\mathcal{A}$.

Example 3.4. Let $\mathcal{A} = \{\lambda \mid \ell(\lambda) \le \sqrt{n}/\log n\}$. By definition, $\mathcal{A}$ is not asymptotically stable since $\mathbf{E}\bigl(\alpha_x(\lambda) \mid \lambda \in \mathcal{A}_n\bigr) \to 0$ for all $x > 0$.

Let $\mathcal{B} = \{(k^k 1^k), k > 0\}$ be the set of partitions which asymptotically look like square partitions. Then $\mathcal{B}$ is asymptotically stable with limiting shape function $b(x) = 1$, $0 \le x \le 1$, and $b(x) = 0$ for $x > 1$.

Finally, observe that the set of partitions $\mathcal{C} = \{(k^k 1^{k^2}), k > 0\}$ is not asymptotically stable. Indeed, the possible limiting shape is $c(x) = \frac{1}{\sqrt{2}}$ for $0 \le x \le \frac{1}{\sqrt{2}}$, and $c(x) = 0$ for $x > 1$, which clearly doesn't satisfy the integral condition.

Example 3.5. It was shown in [23] that the set of all partitions $\mathcal{P}$ is asymptotically stable with the limiting shape $\Phi(x)$ defined as:

$$\Phi(x) \; = \; -\frac{1}{c} \log\left(1 - e^{-cx}\right), \quad \text{where } \; c = \frac{\pi}{\sqrt{6}}.$$

Of course, the curve $\Phi(x)$ is invariant under conjugation (see Figure 8). A symmetric version is given by the formula (♠) in the introduction.

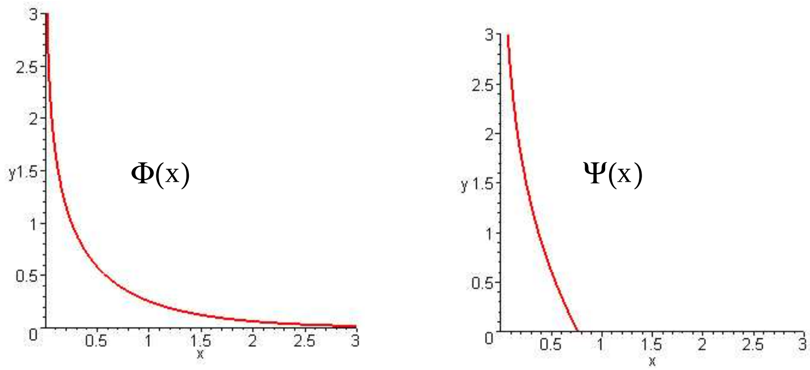

FIGURE 8. Limiting shapes $\Phi(x)$ and $\Psi(x)$ of the set of all partitions $\mathcal{P}$ and $\mathcal{D}$.

Example 3.6. Let $\mathcal{D}$ be the set of all partitions into distinct parts. It was shown in [23] that $\mathcal{D}$ is asymptotically stable with the limiting shape $\Psi(x)$ defined as:

$$\Psi(x) \; = \; \begin{cases} -\dfrac{1}{d} \log\left(e^{-dx} - 1\right), & \text{if } \; 0 < x < (\log 2)/d\,, \\ \qquad\qquad 0\,, & \text{if } \; x \ge (\log 2)/d\,, \end{cases}$$

and where $d = \pi/\sqrt{12}$ (see Figure 8). Below is a concise way to define $y = \Psi(x)$:

$$(\clubsuit) \qquad e^{d\,x} \, - \, e^{-d\,y} \; = \; 1.$$

Example 3.7. Let m be a fixed integer and let $J \subset \{0, 1, \ldots, m-1\}$, $J \ne \varnothing$. Denote by $\mathcal{A}^J$ the set of partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$, such that $(\lambda_i \bmod m) \in J$. For example, when $m = 2$ and $J = \{0\}$ we have the set of partitions into even parts $\mathcal{E}$ and when $J = \{1\}$ we have the set of partitions into odd parts $\mathcal{O}$. Similarly, denote by $\mathcal{B}^J \subset \mathcal{D}$ the set of partitions μ into odd parts, such that $(\mu_i \bmod m) \in J$.

Proposition 3.8. *For every m and J as above the sets of partitions $\mathcal{A}^J$ and $\mathcal{B}^J$ are asymptotically stable with respective limiting shapes $a(x)$ and $b(x)$ given by*

$$(\aleph) \qquad a(x) = \eta\,\Phi(\eta\,x) \quad \textit{and} \quad b(x) = \eta\,\Psi(\eta\,x), \quad \textit{where} \quad \eta = \sqrt{m/|J|}\,.$$

Note that the limiting shapes depend only on m and the size $|J|$. The proof follows verbatim the proof of formulas ($\spadesuit$) and ($\clubsuit$) (see Example 3.5, 3.6) given in [8, 20, 23] and will be omitted.

3.3. **Asymptotic stability of bijections.** Let $\mathcal{A}$, $\mathcal{B}$ be asymptotically stable sets of partitions, and let Υ_1, Υ_2 be their limiting shapes. Denote by $V, W \subset \mathbb{R}_+^2$ the (open) sets between these limiting shapes and coordinate axis.

Let $\vartheta : \mathcal{A} \to \mathcal{B}$ be a correspondence which preserves size of partitions, and let $\varphi : \{\lambda\} \to \{\mu\}$ be a one-to-one map between lattice points in Ferrers diagrams, such that $\lambda \in \mathcal{A}$ and $\mu = \vartheta(\lambda) \in \mathcal{B}$. Denote by $\widetilde{\varphi}$ the corresponding map $\{\widetilde{\lambda}\} \to \{\widetilde{\mu}\}$.

For every two open measurable sets $X \subset V$ and $Y \subset W$ define a statistic $\Theta_\lambda(X, Y)$ to be the number of elements in $\{\widetilde{\lambda}\} \cap X$ which are mapped into $\{\widetilde{\mu}\} \cap Y$.

Definition 3.9. Let $\mathcal{A}$, $\mathcal{B}$ be asymptotically stable sets of partitions. We say that bijection $\varphi : \mathcal{A} \to \mathcal{B}$ defined as above is *asymptotically stable* if $\Theta_\lambda(X, Y)$ are asymptotically stable statistics with uniform convergence, i.e. there exist a measurable function $F(X, Y) \geq 0$, such that for every $\varepsilon > 0$

$$\mathbf{P}\big(|\Theta_\lambda(X,Y) - F(X,Y)| < \varepsilon \ \text{ for all } X \subset V,\, Y \subset W \mid \lambda \in \mathcal{A}_n\big) \to 1 \ \text{ as } n \to \infty,$$

where the probability is over uniform $\lambda \in \mathcal{P}_n$.

It is easy to see that for the function $F(\cdot, \cdot)$ as in the definition above, we have

$$F(X, W) = \mathrm{vol}(X), \quad F(V, Y) = \mathrm{vol}(Y), \quad F(V, W) = 1\,,$$

where $\mathrm{vol}(X)$ denotes the area of X. For example, the conjugation bijection $\gamma : \mathcal{P} \to \mathcal{P}$ is asymptotically stable with $F(X, Y) = \mathrm{vol}(X' \cap Y)$, for all $X \subset V$, $Y \subset W$, and where X' is a set conjugate to X.

Example 3.10. Let $\mathcal{A} = \{(k^k)\}$ be the set of square partitions. Both trivial bijection and conjugation are asymptotically stable. On the other hand, a map $\varphi : (i, j) \to (i, j)$ when k is even, $\varphi : (i, j) \to (j, i)$ when k is odd, defines the same (trivial) correspondence $\vartheta : \mathcal{A} \to \mathcal{A}$. At the same time this bijection φ is neither geometric nor asymptotically stable.

4. Main results

We start with a general background information on the assorted collection of partition bijections. Recall from the introduction the class of "good" partition correspondences given by planar arguments.

Meta Theorem. *All known "good" partition bijections are asymptotically stable.*

Of course, this claim is a philosophical statement, in need of a justification rather than a proof. In the next eight sections we go over a number of partition bijections for which we confirm the Meta Theorem. In each case we construct an AG-bijection which defines the desired partition correspondence, and then use the following technical result to establish the asymptotic stability.

Theorem 4.1. *Let $\mathcal{A} \subset \mathcal{P}$ be an asymptotically stable set of partitions, let $\mathcal{B} \subset \mathcal{P}$, and let $\vartheta : \mathcal{A} \to \mathcal{B}$ be a size preserving one-to-one correspondence. Suppose a natural geometric bijection φ defines ϑ. Then φ is asymptotically stable. Moreover, this holds if φ is an AG-bijection with asymptotically stable parameters.*

We prove this theorem in section 13.

5. Even partitions

Let $\mathcal{A}_{n,k}$ be the set of partitions $\lambda \vdash n$ into *even* parts, such that $\ell(\lambda) \le k$. Similarly, let $\mathcal{B}_{n,k}$ be the set of partitions $\lambda \vdash n$ into *even* parts, such that $a(\lambda) \le 2k$. It is easy to see that

$$1 + \sum_{n=1}^{\infty} \left|\mathcal{A}_{n,k}\right| t^n \,=\, \prod_{i=1}^{k} \frac{1}{1-t^{2i}} \,=\, 1 + \sum_{n=1}^{\infty} \left|\mathcal{B}_{n,k}\right| t^n,$$

and therefore $\left|\mathcal{A}_{n,k}\right| = \left|\mathcal{B}_{n,k}\right|$ for all n and k.

Define a size preserving one-to-one correspondence $\varphi : \mathcal{A}_{n,k} \to \mathcal{B}_{n,k}$ as follows. Let $\lambda/2 = (\lambda_1/2, \lambda_2/2, \ldots)$, and let $\mu = \varphi(\lambda) := (\lambda/2 \cup \lambda/2)' = (\lambda/2)' + (\lambda/2)'$. An example is given in Figure 9.

Proposition 5.1. *Map φ defined above is given by a natural geometric bijection.*

Proof. Let $\nu := \lambda/2 \cup \lambda/2$. Since $\nu \to \mu$ is a conjugation, all we need to show is that $\lambda \to \nu$ is given by a geometric bijection. Indeed, first `shred` $\{\lambda\}$ along the lattice $\Gamma_2 = \{(i, 2j) \mid i, j \in \mathbb{Z}\}$, `move` and `stretch` each component as in Figure 10, and then `paste` two components together. The details are straightforward. $\square$

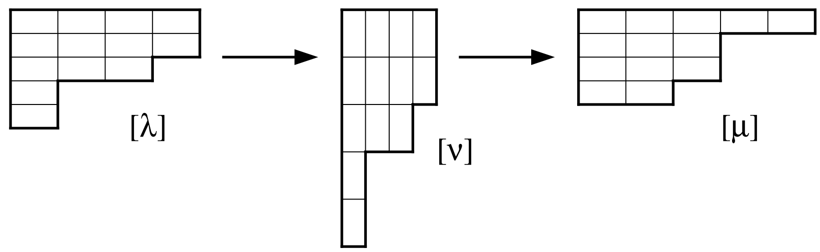

FIGURE 9. An example of a bijection $\varphi : \lambda \to \mu$, where $\lambda \in \mathcal{A}_{26,5}$, $\mu \in \mathcal{B}_{26,5}$

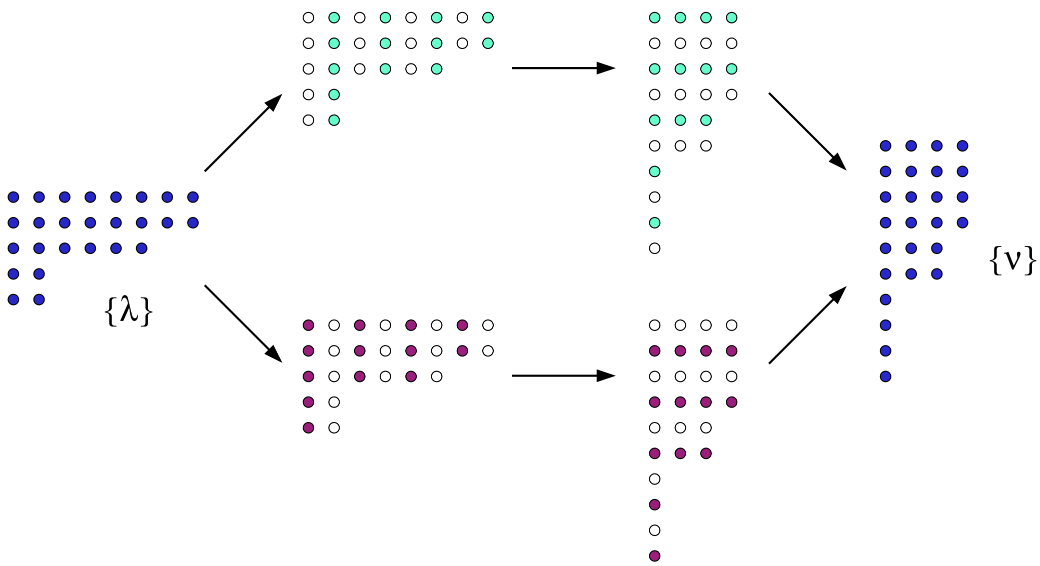

FIGURE 10. Construction of a geometric bijection $\{\lambda\} \to \{\nu\}$, where $\lambda = (8, 8, 6, 2, 2)$, and $\nu = (10, 6, 6, 4)$.

It is known that $\mathcal{A} = \cup \mathcal{A}_{n,k}$ is asymptotically stable for $k = k(n) = \Omega(\sqrt{n})$ with a limiting shape $a(x)$. Then, by Theorem 4.1, the set $\mathcal{B} = \cup \mathcal{B}_{n,k}$ is also asymptotically stable $k = \Omega(\sqrt{n})$ with a limiting shape $b(y) = 2\, a^{-1}(2y)$. For example, when $k = n$, we have $a(x) = b(x) = \sqrt{2}\, \Phi(\sqrt{2}x)$, which is in agreement with the calculation above.

Before we conclude, let us mention that the results can be easily extended from even partitions to partitions which are divisible by m, for any $m \geq 2$. We leave the (easy) details to the reader.

6. Durfee decomposition

Recall the following Euler's identity:

$$\prod_{i=1}^{\infty} \frac{1}{1-t^i} \;=\; 1 \,+\, \sum_{k=1}^{\infty} \frac{t^{k^2}}{(1-t)^2(1-t^2)^2\cdots(1-t^k)^2}\,.$$

We present a classical bijective proof (see e.g. [1, 14, 22]). Denote by $\tau_k = (k^k)$ the square partitions as in Example 3.1. Let $\mathcal{A} = \{(\mu,\nu,\tau_k) \mid \ell(\mu), a(\nu) \le k, k \in \mathbb{N}\}$ be the set of multi-partitions, and let $\varphi : \mathcal{P} \to \mathcal{A}$ be the size preserving one-to-one correspondence defined as follows. Let τ_k be the Durfee square in $[\lambda]$, where $k = \delta_\circ$, and let $[\mu]$ and $[\nu]$ be the Young diagrams obtained after removal of $[\tau_k]$ from $[\lambda]$. Finally, let $(\mu,\nu,\tau_k) = \varphi(\lambda)$ (see Figure 11).

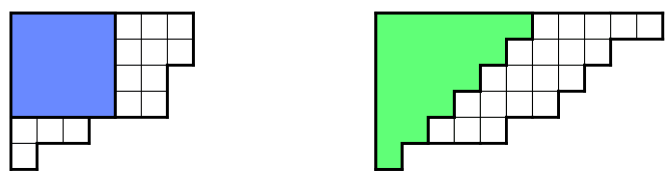

Figure 11. Durfee square decomposition $\varphi : \lambda \to (\mu,\nu,\tau_k)$, where $\lambda = (7,7,6,6,4,1)$, $\mu = (3,3,2,2)$, $\nu = (3,1)$, and $k = 4$. Sylvester's triangle $\Delta_6 \subset [11,9,8,7,5,1]$.

Proposition 6.1. *Map φ defined above is given by a geometric bijection with asymptotically stable parameters.*

The proof follows immediately from the construction and the fact that Durfee rank $\delta_\circ$ is asymptotically stable on $\mathcal{P}$. Note that there is no natural geometric bijection defining φ.

A similar argument proves another Euler's identity:

$$\prod_{i=1}^{\infty} (1+t^i) \;=\; 1 \,+\, \sum_{k=1}^{\infty} \frac{t^{\frac{k(k+1)}{2}}}{(1-t)(1-t^2)\cdots(1-t^k)}\,.$$

First, define $\Delta_k = (k, k-1, \ldots, 2, 1)$. Now, for every $\lambda \in \mathcal{D}$, let $k := \ell(\lambda)$ and remove Δ_k from $[\lambda]$ as shown in Figure 11. The result is again a geometric bijection with asymptotically stable parameters.

Interestingly, the following nearly identical version of the first Euler's identity has a proof by a natural geometric bijection:

$$\prod_{i=1}^{\infty} \frac{1}{1-t^i} \;=\; 1 \,+\, \sum_{k=1}^{\infty} \left(\frac{t^{\frac{k(k+1)}{2}}}{(1-t)(1-t^2)\cdots(1-t^k)}\right)\left(\frac{t^{\frac{k(k-1)}{2}}}{(1-t)(1-t^2)\cdots(1-t^k)}\right).$$

The proof is given by a construction often called *Frobenius coordinates*, and is presented in Figure 12 (see [12, 14]). This one-to-one correspondence is given by a natural geometric bijection. We leave the details to the reader.

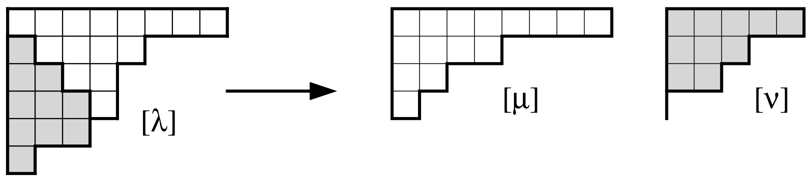

FIGURE 12. Partition $\lambda = (8, 5, 4, 4, 3, 1)$, and its Frobenius coordinates (μ, ν), where $\mu = (8, 4, 2, 1)$, $\nu = (5, 3, 2, 0)$.

Remark 6.2. Let us note that Sylvester found a Durfee style bijective proof of the following partition identity, generalizing Euler's Pentagonal Theorem ($z = -1$ case):

$$\sum_{n=0}^{\infty} \left(z^n t^{\frac{n(3n+1)}{2}} + z^{(n+1)} t^{\frac{(n+1)(3(n+1)-1)}{2}} \right) \prod_{i=1}^{n} \frac{1 + zt^i}{1 - t^i} = \prod_{i=1}^{\infty} (1 + zt^i).$$

The one-to-one correspondence in this case is given by a geometric bijection with asymptotically stable parameters. We refer to [17] for an extensive treatment of other bijections of this type and how one can construct Franklin's and other classical involutions from these bijections.

Remark 6.3. The following equivalent version of the *Jacobi triple product identity*

$$\sum_{k=-\infty}^{\infty} s^k t^{\frac{k(k+1)}{2}} \prod_{i=1}^{\infty} \frac{1}{1 - t^i} = \prod_{i=1}^{\infty} (1 + st^i) \prod_{j=0}^{\infty} (1 + s^{-1} t^j).$$

can be proved by a combinatorial argument based on a generalization of the Frobenius coordinates [22] (see also [14, 27]). One can view the proof as a bijection between two sets of multi-partitions with the power of s being an asymptotically stable statistic. One can extend the definition of geometric bijections to such cases which would apply to partition identities with several variables. The details are straightforward but take us outside the main interest of this paper.

7. SELF-CONJUGATE PARTITIONS

Let $\mathcal{A}$ be the set of *self-conjugate* partitions: $\lambda = \lambda'$, and let $\mathcal{B}$ be the set of partitions into distinct odd parts. Define a size preserving one-to-one correspondence $\varphi : \mathcal{A} \to \mathcal{B}$ as follows. For $\lambda \in \mathcal{A}$ consider the *hook lengths* $\mu_i := 2(\lambda_i - i) + 1$, for $1 \le i \le \delta_\circ(\lambda)$, and finally let $\mu = \varphi(\lambda)$ (see Figure 13).

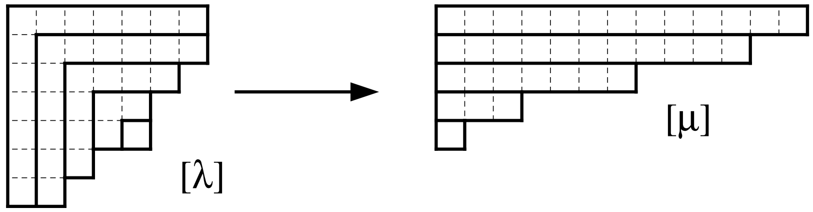

FIGURE 13. An example of a bijection $\varphi : \lambda \to \mu$, where $\lambda = (7,7,6,5,5,3,2) \in \mathcal{A}$ and $\mu = (13,11,7,3,1) \in \mathcal{B}$.

Proposition 7.1. *Map φ defined above is given by a natural geometric bijection.*

Proof. The proof is by construction given in Figure 14. First, `cut` along the line $i - j = 0$, `shift`, `move` and `stretch` the Ferrers diagram in each component so that the the dots in two components lie in different cosets of $\Gamma_2 = \{(i, 2j) \mid i, j \in \mathbb{Z}\}$. Now simply `paste` two components. The details are straightforward. $\square$

Corollary 7.2. *The set of self-conjugate partitions $\mathcal{A}$ is asymptotically stable with limiting shape $\Phi(x)$.*

Proof. First, observe that the set $\mathcal{B}$ of partitions into distinct odd parts is asymptotically stable with the same limiting distribution as the set $\mathcal{C}$ of partitions into distinct even parts. Simply recall that the number of parts $\ell(\mu)$ of a random $\mu \in \mathcal{B}$ is $O(\sqrt{n})$ w.h.p. as $n \to \infty$, and removing the first column in $[\mu]$ makes all parts even and distinct. Thus the limiting shape $b(x)$ for $\mathcal{B}$ and $\mathcal{C}$ is $b(x) = \sqrt{2}\,\Psi(\sqrt{2}\,x)$.

By Proposition 7.1, map φ is given by a geometric bijection, and by Theorem 2.4 so is φ^{-1}. This can be also concluded directly from the proof of Proposition 7.1. Therefore, by Theorem 4.1, the set $\mathcal{A}$ is asymptotically stable.

Denote by $a(x)$ the limiting shape of $\mathcal{A}$. By construction of φ we have: $a(x) = x + b(x)/2 = x + \Psi(\sqrt{2}\,x)/\sqrt{2}$ for $0 < x \le s := (\log 2)/(\sqrt{2}\,d)$, where $d = \pi/\sqrt{12}$ as in Example 3.6. We have $b(s) = 0$ and $a(s) = s$, so for $x \ge (\log 2)/(\sqrt{2}\,d)$ use the symmetry. Now the claim follows by easy calculation. $\square$

Remark 7.3. One can also deduce Corollary 7.2 from probabilistic considerations, by using the asymptotic stability of the sets of partitions $\{\lambda \in \mathcal{A} \mid \delta_\circ(\lambda) = c\sqrt{n}\}$, for all $c > 0$. Conditioned on the value of $\delta_\circ(\lambda)$, the Frobenius coordinates are identical and independent, which implies the result.

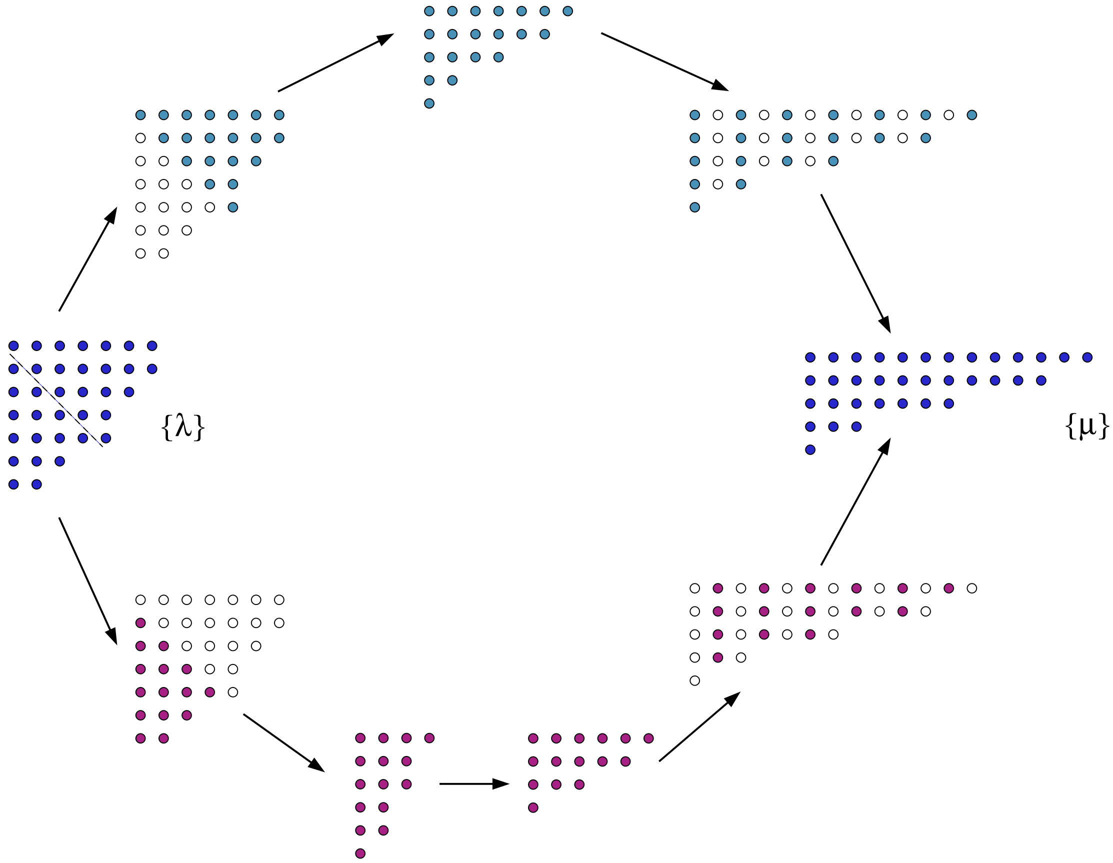

FIGURE 14. Construction of a geometric bijection $\{\lambda\} \to \{\mu\}$, where $\lambda = (7,7,6,5,5,3,2) \in \mathcal{A}$ and $\mu = (13,11,7,3,1) \in \mathcal{B}$.

8. Sylvester's bijection

Let $\mathcal{O}_n$ be the set of partitions $\lambda \vdash n$ into odd numbers, and let $\mathcal{D}_n$ be the set of partitions $\mu \vdash n$ into distinct numbers. *Euler's theorem* says that $|\mathcal{O}_n| = |\mathcal{D}_n|$, which follows immediately from the identity:

$$\prod_{i=1}^{\infty} \frac{1}{1-t^{2i-1}} = \prod_{i=1}^{\infty} (1+t^i)\,.$$

Sylvester found the following bijective proof of Euler's theorem by a so called "fish-hook construction". We start with a Ferrers diagram $\{\lambda\}$ of $\lambda \in \mathcal{O}$ and bend the parts into hooks. Then take the fish-hooks in the resulting shape and extend them to form rows of $\{\mu\}$ as in Figure 15. Now let $\varphi : \mathcal{O} \to \mathcal{D}$ be defined by $\varphi(\lambda) = \mu$.

Note that although the one-to-one map φ is defined in terms of Ferrers diagrams, the construction does not define a geometric bijection. Perhaps surprisingly, we have the following result.

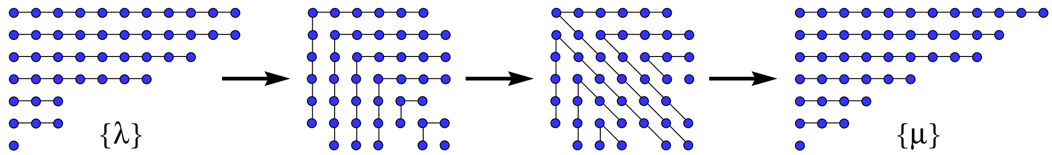

FIGURE 15. An example of a bijection $\varphi : \lambda \to \mu$, where $\lambda = (11, 11, 9, 7, 3, 3, 1) \in \mathcal{O}$ and $\mu = (12, 10, 9, 6, 4, 3, 1) \in \mathcal{D}$.

Proposition 8.1. *Map φ defined above is given by a natural AG-bijection.*

Proof. The proof is outlined in Figure 16 and follows the presentation of φ given in [19] (see also [3, 14]). Below is a brief sketch of the construction.

Start with a Durfee diagram $\{\lambda\}$ and `cut` it along $j - 2i = 1$ line. Apply `shift` to the component corresponding to the $i - 2j > 1$ points. The resulting Ferrers diagram corresponds to a partition with even parts; apply geometric transformation defined in 5 to obtain a diagram $\{\nu\}$. For the other component, first `shred` it modulo 2, and then use `stretch`, `shift`, and `move` to make two Ferrers diagrams. Now `stretch` these diagrams, `move` and `paste` them into a single diagram $\{\omega\}$. Finally, add $\{\nu\}$ and $\{\omega\}$ to obtain the desired Ferrers diagram $\{\mu\}$. $\square$

Remark 8.2. Let us note that one can use Sylvester's bijection to deduce formula (♠) from formula (♣). Also, the reader might find it instructive to see how bijection φ maps one limiting shape into the other. This case was the inspiration for [18], although the logic is somewhat convoluted.

9. GLAISHER'S BIJECTION

As in 8, let $\mathcal{O}_n$ be the set of partitions $\lambda \vdash n$ into odd numbers, and let $\mathcal{D}_n$ be the set of partitions $\mu \vdash n$ into distinct numbers. Recall Euler's theorem: $|\mathcal{O}_n| = |\mathcal{D}_n|$, for all $n \in \mathbb{N}$. The following algorithm defines a size preserving one-to-one correspondence: $\psi : \mathcal{D} \to \mathcal{O}$.

Start with $\lambda \in \mathcal{D}$. Find the largest even part λ_i and replace it with two parts of equal size $\lambda_i/2$. Repeat this until all parts are odd. Let $\psi(\lambda)$ be the resulting partition.

It is easy to see that ψ is one-to-one [3, 14]. Indeed, simply observe that correspondence ψ maps parts $(2^r m)$, where m is odd, into 2^r copies of a part (m).

Even though ψ is not an AG-bijection, it is a limit of AG-bijections in the following sense. Let $\mathcal{O}^r$ be the set of partitions into odd parts such that no part occurs $\geq 2^r$ times. Similarly, let $\mathcal{D}^r$ be the set of partitions into distinct parts which are not divisible by 2^r. By construction, $\mathcal{O}^r = \psi(\mathcal{D}^r)$.

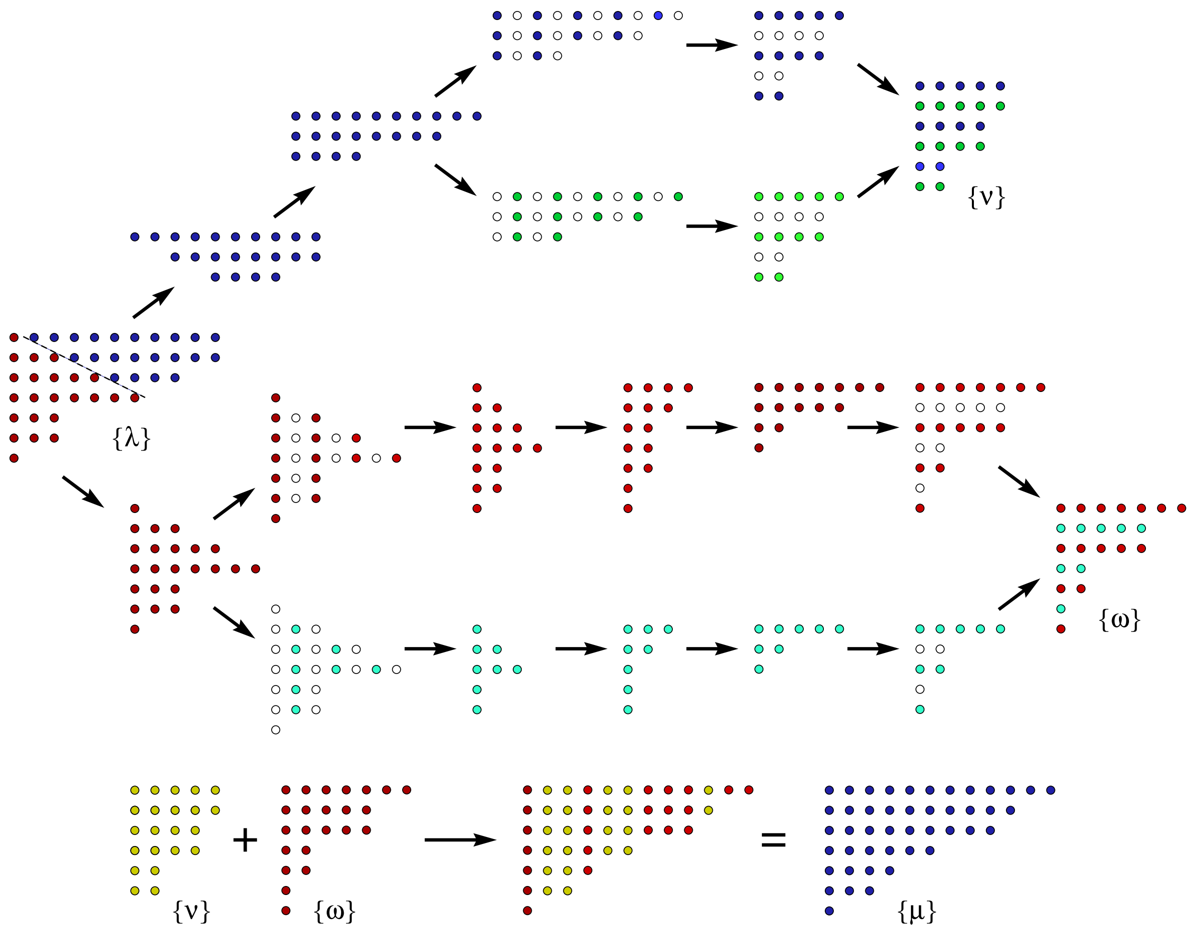

FIGURE 16. An example of a bijection $\varphi : \lambda \to \mu$, where $\lambda = (11, 11, 9, 7, 3, 3, 1) \in \mathcal{O}$ and $\mu = (12, 10, 9, 6, 4, 3, 1) \in \mathcal{D}$.

Proposition 9.1. *Sets of partitions $\mathcal{O}^r$, $\mathcal{D}^r$ are asymptotically stable with limiting shapes $y = F_r(x)$, $y = G_r(y)$, given by*

$$e^{-c\eta_r\, x} \, + \, e^{-c\, y} \, = \, 1, \qquad e^{(d\,\eta_r)\, x} \, - \, e^{-(d/\eta_r)\, y} \, = \, 1,$$

respectively, and where $c = \pi/\sqrt{6}$, $d = \pi/\sqrt{12}$, and $\eta_r = \sqrt{1 - 1/2^r}$.

The reader can check that two parts of Proposition 9.1 are special cases of Proposition 3.8 (see above) and Proposition 9.5 (see below). Later, in Remark 9.3 present now an informal (yet?) method for deriving the limiting shapes as in the proposition. Let us start with the following result.

Corollary 9.2. *One-to-one correspondence $\psi : \mathcal{D}^r \to \mathcal{O}^r$ defined above is given by an asymptotically stable AG-bijection.*

Proof. First, `split` partitions $\lambda \in \mathcal{D}^r$ modulo 2^r. Then $(2^k, 2^k)$-`stretch` the components with parts $(2^k m)$, where m is odd, and $k < r$. Take a `union` of the resulting partitions. $\square$

Recall that $\sqrt{2}\,\Phi(\sqrt{2}\,x)$ and Ψ are the limiting shapes of the sets of partitions $\mathcal{O}$ and $\mathcal{D}$. Now, observe that $F_r(x) \to \sqrt{2}\,\Phi(\sqrt{2}\,x)$ as $r \to \infty$, for all $x \in \mathbb{R}_+$. Similarly, $G_r(x) \to \Psi(x)$ as $r \to \infty$, for all $x \in \mathbb{R}_+$. In this sense, correspondence ψ is given by a limit of AG-bijections. This is in contrast with Sylvester's bijection (see section 8) which *is* an AG-bijection.

Remark 9.3. The following calculation uses Glaisher's bijection to establish a connection between the limiting shapes of $\mathcal{D}$ and $\mathcal{O}$. While this calculation does not establish any new result, it serves as a prototype for obtaining new more involved results.

Denote $f(y) = \Psi^{-1}(y)$, and let $\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}$, etc. be the partitions of parts in $\lambda \in \mathcal{D}$ which are 1 mod 2, 2 mod 4, 4 mod 8, etc. The limiting shapes of these conjugates to these partitions are $\frac{1}{2}f$, $\frac{1}{4}f$, $\frac{1}{8}f$, etc. After $(2^k, 2^k)$-`stretch` of $\lambda^{(k)}$, we obtain shapes conjugate to $\frac{1}{2}f(y)$, $\frac{1}{2}f(2y)$, $\frac{1}{2}f(4y)$, etc. Therefore, $\mu = \varphi(\lambda)$ has a shape conjugate to $\frac{1}{2}\big(f(y) + f(2y) + f(4y) + \ldots\big)$. Substituting $f(y) = \frac{1}{d}\log(1 + e^{-dy})$, we obtain:

$$x \,=\, g(y) \,=\, \frac{1}{2}\sum_{k=0}^{\infty} f(2^k y) \,=\, \frac{1}{2d}\sum_{k=0}^{\infty} \log\big(1 + e^{-d2^k y}\big) \,=\, \frac{1}{2d}\log\prod_{k=0}^{\infty}\big(1 + e^{-d2^k y}\big).$$

Using

$$\prod_{k=0}^{\infty}\big(1 + z^{2^k}\big) \,=\, 1 + z + z^2 + z^3 + \ldots \,=\, \frac{1}{1-z}\,,$$

we conclude:

$$e^{-2dx} \,=\, e^{-2d\,g(y)} \,=\, \prod_{k=0}^{\infty}(1 + e^{-d2^k y})^{-1} \,=\, 1 - \,e^{-dy}.$$

This coincides with the equation $e^{-2dx} + e^{-dy} = 1$, for the limiting shape $\sqrt{2}\,\Phi(\sqrt{2}\,x)$ of the set of partitions $\mathcal{O}$.

Note that along the way we, in effect, change the order of limits (in the summation of the limiting shapes and in the definition of the limiting shape, when $n \to \infty$). It seems one should be able to justify this change since the total sum of parts in random $\lambda \in \mathcal{D}$ which are divisible by 2^k decreases exponentially as $k \to \infty$. It would be very interesting to make this argument rigorous as we it seems applicable in other cases as well (see the remark below).

Remark 9.4. Fix an integer $m \geq 2$. Let $\mathcal{A}^m$ be the set of partitions into parts not divisible by m, and let $\mathcal{B}^m$ be the set of partitions with no part repeated $\geq m$ times. An easy algebraic argument shows $|\mathcal{A}^m_n| = |\mathcal{B}^m_n|$. Note that $\mathcal{A}^2 = \mathcal{O}$ and $\mathcal{B}^1 = \mathcal{D}$, so the above result generalizes Euler's theorem.

Let us present the following *extended Glaisher's bijection* $\psi_m : \mathcal{B}^m \to \mathcal{A}^m$. Start with $\lambda \in \mathcal{B}^m$ and find a part (λ_i) divisible by m. Replace it with m parts (λ_i/m) and repeat this until no such change is possible. See [14] for other extensions and references.

As with Glaisher's correspondence, the above extension is not given by an AG-bijection. Nevertheless, it still can be used to connect the limiting shapes of two sets of partitions.

Recall that Proposition 3.8 implies that the sets of partitions $\mathcal{A}^m$ is asymptotically stable with the limiting shapes given by the equation:

$$(\mathfrak{A}) \qquad e^{-c\,x\sqrt{\frac{m}{m-1}}} \,+\, e^{-c\,y\sqrt{\frac{m-1}{m}}} \,=\, 1\,,$$

where $c = \pi/\sqrt{6}$ as above. Using notation $f(y)$ and $g(y)$ for the inverses of limiting shape of $\mathcal{B}^m$ and $\mathcal{A}^m$ as in Remark 9.3, the bijection ψ_m gives:

$$g(y) \,=\, \frac{m-1}{m}\bigl(f(y) \,+\, f(m\,y) \,+\, f(m^2 y) \,+\, \ldots\bigr),$$

and therefore

$$f(y) \,=\, \frac{m-1}{m}\bigl(g(y) \,-\, g(m\,y)\bigr).$$

Use the explicit formula $(\mathfrak{A})$ to compute $g(y)$ and $f(y)$. The value $f(0)$ now gives the number of parts of a "typical" partition $\mu \in \mathcal{B}^m$. This suggests (but not proves!) the following result:

Proposition 9.5. *The set of partitions $\mathcal{B}^m$ is asymptotically stable with limiting shapes given by the equation*

$$(\mathfrak{B}) \qquad e^{s\,x} \,=\, \frac{1-e^{-m\,s\,y}}{1-e^{-s\,y}}\,,$$

where $s = c\sqrt{1-1/m}$ and $c = \pi/\sqrt{6}$. Furthermore, the number of parts $\ell(\cdot)$ is an asymptotically stable statistic on $\mathcal{B}^m$ with a limit $\frac{1}{s}\log m$.

Note that the limiting shape of $\mathcal{B}^2$ is indeed the limiting shape $(\clubsuit)$ of the set $\mathcal{D}$ of distinct partitions. Similarly, as $m \to \infty$ the limiting shapes of $\mathcal{A}^m$ and $\mathcal{B}^m$ tends to the limiting shape $(\spadesuit)$ of the set of all partition $\mathcal{P}$. On the other hand the number of parts $\ell(\cdot)$ of partitions is not asymptotically stable on $\mathcal{P}$, as follows by letting $m \to \infty$ in the proposition.

We do not present here the proof of Proposition 9.5 as it follows verbatim the analysis given in [8, 20, 23]. Alternatively, one can formalize the argument given above, as in Remark 9.3. The details are left to the reader.

10. Bressoud's bijection

Let $\mathcal{A}$ be the set of partitions into parts $\equiv 0, 1, 2 \mod 4$. Let $\mathcal{B}$ be the set of partitions into parts which differ by at least 2, and such that odd parts differ by at least 4. The classical *Göllnitz partition theorem* implies that these sets are equinumerous: $|\mathcal{A}_n| = |\mathcal{B}_n|$. Using generating functions this result can be written as:

$$(\maltese) \qquad 1 + \sum_{k=1}^{\infty} t^{k(k+1)} \frac{(1+t)(1+t^3)\cdots(1+t^{2k+1})}{(1-t^2)(1-t^4)\cdots(1-t^{2k})} = \prod_{i=1}^{\infty}(1+t^{4i}) \prod_{j=1}^{\infty}(1+t^{2j-1}).$$

Bressoud's bijection $\varphi : \mathcal{A} \to \mathcal{B}$ is illustrated in Figure 17 and can be outlined as follows. We graphically present partitions by means of *MacMahon diagrams* where each square corresponds to 2 except for blue (marked) squares which correspond to 1. Summing integers corresponding to squares in a MacMahon diagram gives the sizes of parts.

Start with a MacMahon diagram $\langle\lambda\rangle$ and split it into three diagrams: $\langle\alpha\rangle$, $\langle\beta\rangle$ and $\langle\gamma\rangle$, such that γ contains odd parts of λ (they are lightly shaded), partition β contains even parts of λ which are $\leq 2\ell(\gamma)$, and partition α contains even parts of λ which are $> 2\ell(\gamma)$. Let $\langle\upsilon\rangle = \langle\beta\rangle' + \langle\gamma\rangle$, and place $\langle\upsilon\rangle$ right below $\langle\alpha\rangle$. Removing triangle $\langle\rho_s\rangle$ with $s = \ell(\alpha) + \ell(\gamma) - 1$, reordering the parts and adding the triangle $\langle\rho_s\rangle$ back, we obtain the desired MacMahon's diagram of a partition μ. For the proof and references we refer to [14].

Proposition 10.1. *The map φ is given by asymptotically stable AG-bijection.*

Proof. First, by Proposition 3.8 the set $\mathcal{A}$ is asymptotically stable and has limiting shape $a(x) = \sqrt{\frac{4}{3}}\,\Psi\big(\sqrt{\frac{4}{3}}\,x\big)$. Then, components $\langle\gamma\rangle$ and $\langle\alpha \cup \beta\rangle$ are also asymptotically stable with the limiting shapes $a(3x)$ and $a(3x/2)$, respectively. Now, since the lengths $\ell(\alpha)$ and $\ell(\gamma)$ are asymptotically stable statistic, we conclude that so is the size s of the removed triangle.

Now, by construction, Bressoud's bijection φ is an AG-bijection. From above, all parameters are asymptotically stable. This completes the proof. $\square$

Corollary 10.2. *The set of partitions $\mathcal{B}$ is asymptotically stable.*

From the proof above, one compute an explicit formula for the limiting shape of $\mathcal{B}$. We leave this as an exercise to the curious reader.

Remark 10.3. The identity ($\maltese$) is a special case of *Lebesgue identity*:

$$(\mathcal{L}) \qquad \sum_{r=1}^{\infty} t^{\binom{r+1}{2}} \frac{(1+zt)(1+zt^2)\cdots(1+zt^r)}{(1-t)(1-t^2)\cdots(1-t^r)} = \prod_{i=1}^{\infty}(1+zt^{2i})(1+t^i)\,,$$

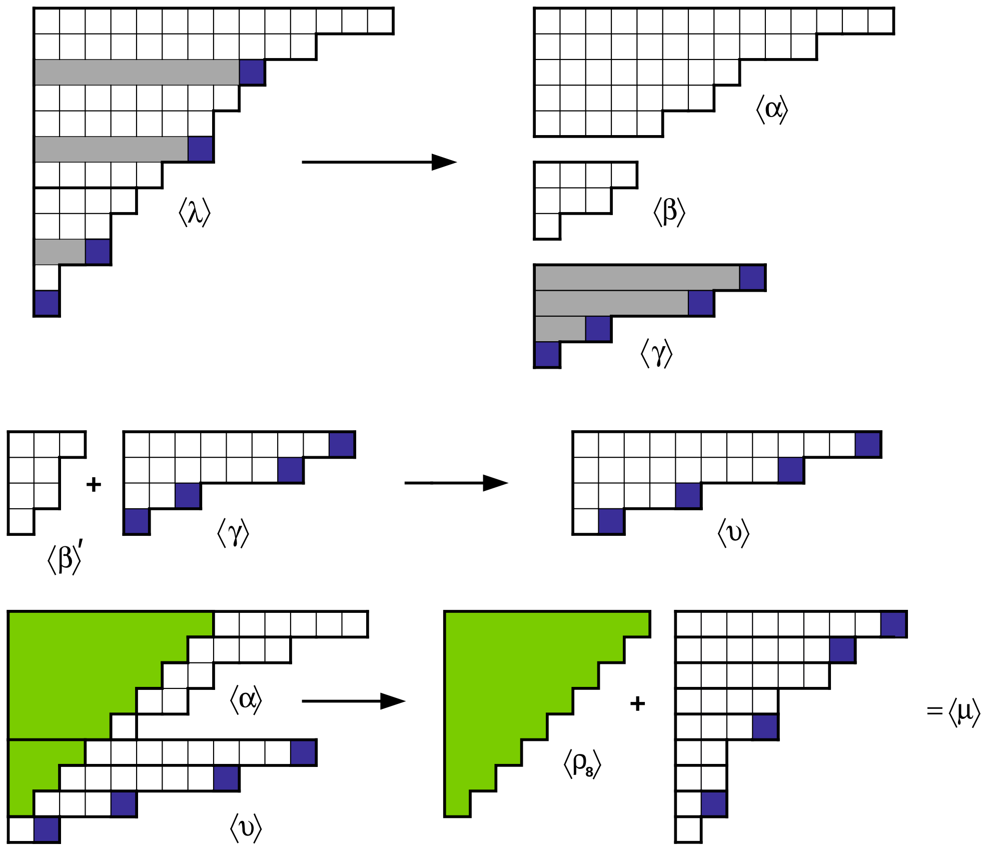

FIGURE 17. An example of a bijection $\varphi : \lambda \to \mu$ for $\lambda \in \mathcal{A}_n$, $\mu \in \mathcal{B}_n$, where $n = 142$, $\lambda = (28, 22, 17, 16, 14, 13, 10, 8, 6, 5, 2, 1) \in \mathcal{A}_{142}$ and $\mu = (33, 27, 24, 18, 15, 10, 8, 5, 2)$.

where the substitution is given by $t \leftarrow t^2$, $z \leftarrow t$.

One can extend definitions of geometric and AG-bijections to MacMahon diagrams and show the asymptotic stability in this case. One can view the power of z as an asymptotically stable statistic on the corresponding set of partitions, and consider the asymptotic stability of multi-statistics and the limiting shapes (cf. Remark 6.3).

11. Ramanujan's identity

Consider the following simple *Ramanujan's identity*:

$$(\mathcal{R}) \qquad \sum_{m=0}^{\infty} \frac{t^{2m+1}}{(1-t^{m+1})\cdots(1-t^{2m+1})} = \sum_{r=1}^{\infty} \frac{t^r}{(1-t^{r+1})\cdots(1-t^{2r})}.$$

Below we present a direct bijective proof of this identity given by Andrews [2]. First, denote by $s(\lambda) = \lambda_{\ell(\lambda)}$ the smallest part of the partition. Now let $\mathcal{A} = \{\lambda \in \mathcal{P} \mid a(\lambda) <$

$2s(\lambda)$, $a(\lambda)$ − odd}, and $\mathcal{B} = \{\lambda \in \mathcal{P} \mid a(\lambda) \le 2s(\lambda),\ \lambda_i > s(\lambda) \text{ for } i < \ell(\lambda)\}$. Consider a size preserving one-to-one correspondence $\varphi : \mathcal{A} \to \mathcal{B}$ illustrated in Figure 18.

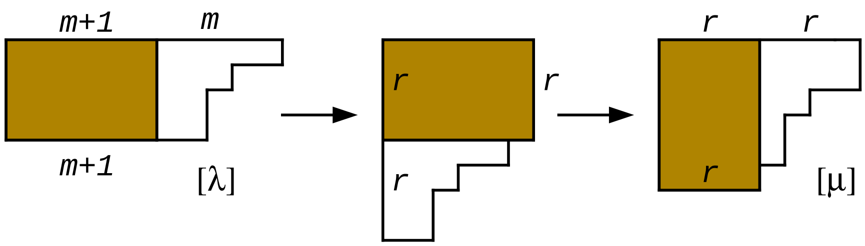

FIGURE 18. Construction of Andrews's correspondence $\varphi : \lambda \to \mu$, where $\lambda \in \mathcal{A}$, $\mu \in \mathcal{B}$, $a(\lambda) = 2m + 1$, and $s(\mu) = r$.

Proposition 11.1. *Map φ defined above is given by a geometric bijection.*

The proof is immediate by construction of φ. Note that by definition, this geometric bijection is not natural as $a(\lambda)$ and $\ell(\lambda)$ are the parameters in the `shift`. We conjecture that these parameters are asymptotically stable.

Conjecture 11.2. *Sets of partitions $\mathcal{A}$ and $\mathcal{B}$ are asymptotically stable and have an* identical *limiting shape.*

Remark 11.3. Assume that Conjecture 11.2 holds and both sets of partitions have the same limiting shape $f(x)$. By Proposition 11.1 and Theorem 4.1 this implies that the bijection defined above is asymptotically stable. Therefore, there exist a constant $c > 0$ such that statistics $\delta_\circ(\lambda)/\sqrt{n}$, $s(\lambda)/\sqrt{n}$ and $a(\lambda)/\sqrt{n}$ are asymptotically stable. Moreover, since $f(x)$ is invariant under φ, the corresponding limits are c, c and $2c$, respectively. In other words, $\lim_{x\to 0+} f(x) = 2 \lim_{x\to c-} f(x) = 2\,c$ and $f(x) = 0$ for $x > c$. In fact, the graph of $f(x) - c$ defined $(0, c)$ is symmetric with respect to conjugation (see Figure 19).

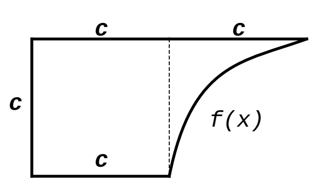

FIGURE 19. Conjectured limiting shape $f(x)$ of the sets of partitions $\mathcal{A}$ and $\mathcal{B}$.

12. Convex partitions

Let $\mathcal{A} \subset \mathcal{D}$ be the set of partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that $\lambda_1 - \lambda_2 \geq \lambda_2 - \lambda_3 \geq \ldots \geq \lambda_{\ell-1} - \lambda_\ell \geq \lambda_\ell$. We call such $\lambda \in \mathcal{A}$ *convex partitions*. Let $\mathcal{B}$ be the set of partitions into triangular parts $\binom{k}{2}$, where $k \geq 2$. Consider a one-to-one correspondence $\varphi : \mathcal{B} \to \mathcal{A}$ defined as follows:
$\varphi : \binom{2}{2}^{a_1}\binom{3}{2}^{a_2}\cdots\binom{k+1}{2}^{a_k} \to a_1(1,0,\ldots,0) + a_2(2,1,0,\ldots,0) + \ldots + a_k(k, k-1, \ldots, 2, 1)$.
This correspondence was recently introduced in [10] (see also [14, 16]). The following result is a rather unusual application of the correspondence φ.

Proposition 12.1. *Of the two equinumerous sets of partitions $\mathcal{A}$ and $\mathcal{B}$, at least one is not asymptotically stable.*

Proof. By construction, the bijection φ defined above satisfies $a(\mu) = \binom{\ell(\lambda)}{2}$, for all $\mu = \varphi(\lambda) \in \mathcal{B}$, and $\lambda \in \mathcal{A}$.

Now, assume that both $\mathcal{A}$ and $\mathcal{B}$ are asymptotically stable. Then $\ell(\lambda)/\sqrt{n}$ is an asymptotically stable statistic on $\mathcal{A}$, with the limit $c > 0$. This implies that $\mu_1 = a(\mu)/\binom{\sqrt{n}}{2} \sim 2\, a(\mu)/n$ is an asymptotically stable statistic on $\mathcal{B}$, with the same limit $c > 0$. Removing the first part and using the same argument for μ_2, μ_3, etc. implies that for every integer k the statistic μ_k/n is asymptotically stable on $\mathcal{B}$ with a limit $c/2\,(1 - c/2)^{k-1}$. Thus, for any $\alpha > 0$ there exist large enough integer k, such that $(\mu_1 + \ldots + \mu_k)/n$ is asymptotically stable with a limit $> (1-\alpha)$. Therefore, for all $x > 0$, if α_x is asymptotically stable on $\mathcal{B}$ with limit α, then $\alpha = 0$. This implies that $\mathcal{B}$ is not asymptotically stable, a contradiction. $\square$

Remark 12.2. We conjecture that the set $\mathcal{B}$ is not asymptotically stable, while for the set $\mathcal{A}$ we are not ready to conjecture either way. It would be useful to resolve this problem, or at least obtain some experimental evidence in that direction.

13. Proof of results

Proof of Theorem 2.4. Recall that `shift` and `move` are invertible. Clearly, the inverse of `cut` is `paste` and the inverse of `shred` is a composition of r copies of `paste` transformations. Similarly, the (k,ℓ)-`stretch` is the inverse of the conjugation and (ℓ,k)-`stretch`. Thus, the inverse of `paste` is the only obstacle. We need a structural result to show that it is invertible.

An intersection of a cone $C = \{\,(i,j) \mid ai + bj \geq c,\ a'i + b'j \geq c'\}$ and a lattice $\Gamma \subset \mathbb{Z}^2$ is called *lattice cone*. We say that (possibly infinite) sets $X, Y \subset \mathbb{N}^2$ are *largely equal* if there exist N such that X and Y coincide outside of the N-square $\{\,(i,j) \in \mathbb{Z}^2 \mid 1 \leq |i|, |j| \leq N\}$.

Let us prove by induction that at every step, the components in the natural geometric bijection are largely equal to a disjoint union of lattice cones. Indeed, `shift`, `stretch` and `move` preserve the lattice cones, `cut` and `shred` can create only lattice cones out of lattice cones (outside a large enough N-square), while `paste` produces a disjoint union of such cones.

Now suppose `paste` transformation is applied at the next step. This is equivalent ($\simeq$) to first shredding into different lattice cones, and then sequentially pasting them: first those with the same coset of the same lattice, and then together. Observe that outside some N-square, the `paste` map of a lattice cone can be inverted either by a `shred`, or, if they lie in the same coset of a lattice Γ, by a `cut`. The latter claim follows from convexity of the lattice cones (restricted to Γ).

Therefore, one can sequentially invert the natural geometric bijection outside of a large enough N-square. Within that square one can use a number of `cuts` to obtain unit squares which can be `moved` to obtain a desired permutation of unit squares in the N-square, which are then `pasted` together. This completes the construction and proves the result. $\square$

Proof of Theorem 4.1. First, observe that $\mathcal{A}$ is a spanning set of partitions. Indeed, by asymptotic stability we have $\operatorname{vol}(V) = 1$. On the other hand, if the Durfee rank $\delta_\circ$ is bounded, then $\delta_\circ/\sqrt{n} \to 0$ as $n \to \infty$. This implies that the area of V is zero, a contradiction.

Now suppose φ is a natural geometric bijection. By Propositions 1 and 2, bijection φ defines a one-to-one correspondence $\varphi : \mathbb{N}^2 \to \mathbb{N}^2$. From the proof of Proposition 3 above, there is a large enough N-square such that φ is a `paste` of a finite number of lattice cones. To simplify the notation, let us extend the definition of cones to include 1-dimensional cones (halflines) and 0-dimensional cones (points) Now present the geometric bijection φ as follows:

- `Cut` and `shred` $\mathbb{N}^2 \to \sqcup_{i=1}^k (C_i \cap \Gamma_i)$,
- `Shift`, `stretch` and `move` $(C_i \cap \Gamma_i) \to (D_i \cap \Gamma_i')$, for all $i \in \{1, .., k\}$,
- `Paste` $\sqcup_{i=1}^k (D_i \cap \Gamma_i') \to \mathbb{N}^2$,

where each item is in fact a sequence of basic geometric transformations. For a lattice cone $C \subset \Gamma$, let $r(C) = r(\Gamma)$. Note that in $\Gamma = \mathbb{Z}\langle v_1, v_2\rangle$, then the volume $r(\Gamma) = \det(v_1, v_2)$. Recall that the `shift` and `move` transformations preserve the number of cosets: $r(\Gamma') = r(\Gamma)$, while (k,ℓ)-`stretch` changes it according to the volume: $r(\Gamma') = (k/\ell)\, r(\Gamma)$.

Scaling both coordinates by $\sqrt{n}$, restricting φ to the scaled Durfee diagram, and letting $n \to \infty$, sends a cone $C = \{\, (i,j) \mid ai + bj \ge c,\, a'i + b'j \ge c' \}$ into a cone $\widetilde{C}$ of the same kind, with $c = c' = 0$. Similarly, the N-square disappears in the limit. One can think of the $\widetilde{\varphi}$ in the limit as mapping cones $\widetilde{C}_i \to \widetilde{D}_i$ weighted by $1/r(\Gamma)$.

Formally, let $V_i = V \cap \widetilde{C}_i$, so that $V = \cup_i V_i$. Denote by $\upsilon_i : C_i \to D_i$ the affine rational map between cones, and $\widetilde{\upsilon}_i : \widetilde{C}_i \to \widetilde{D}_i$ be linear map in the limit. Finally, let $W_i = \widetilde{\upsilon}_i(V_i) \subset \widetilde{D}_i$ be the image of V_i, and let $W = \cup_i W_i$. From the properties of volumes of the lattices Γ and Γ', we have

$$\mathrm{vol}(W) \, = \, \sum_i \frac{\mathrm{vol}(W_i)}{r(\widetilde{\Gamma}_i)} \, = \, \sum_i \frac{\mathrm{vol}(V_i)}{r(\Gamma_i)} \, = \, \mathrm{vol}(V) = 1$$

We will prove that $\mathcal{B}$ are asymptotically stable and then prove that φ is an asymptotically stable bijection.

Let $a(x)$ and $b(y)$ be the limiting shapes of $\mathcal{A}$ and the boundary of W. We will prove that $b(y)$ is the limiting shape of $\mathcal{B}$. By asymptotic stability of the set $\mathcal{A}$, for every $\varepsilon > 0$ proportion of partitions $\lambda \in \mathcal{A}_n$ such that $|\widetilde{\rho}_\lambda(x) - a(x)| < \epsilon$ for all $x > 0$ tends to 1 as $n \to \infty$. Therefore, the proportion of partitions $\lambda \in \mathcal{A}_n$ with $\mathrm{vol}(\widetilde{\rho}_\lambda \cap X) \geq \mathrm{vol}(X) - \epsilon$ also tends to 1 as $n \to \infty$.

We need to show that for all $\varepsilon > 0$, the proportion of partitions $\mu \in \mathcal{B}_n$ such that $|\widetilde{\rho}_\mu(y) - b(y)| < \varepsilon$, for all $y > 0$, tends to 1 as $n \to \infty$. This is a restatement of the asymptotic stability claim in the Theorem. Denote by τ_i the norm of of the largest eigenvalue of the matrix $M_i \in \mathrm{GL}(2, \mathbb{Q})$ corresponding to the combination of `shift` and `stretch` transformations (skipping all `moves`). Then $|\widetilde{\rho}_\lambda(y) - a(y)| < \epsilon$ implies $|\widetilde{\rho}_\mu(y) - b(y)| < \tau\epsilon$, where $\tau = \max_i \tau_i$. Taking $\epsilon = \varepsilon/\tau$ proves the claim.

From asymptotic stability, we conclude that the proportion of partitions $\mu \in \mathcal{B}_n$ with $\mathrm{vol}(\widetilde{\rho}_\mu \cap Y) \geq \mathrm{vol}(Y) - \epsilon$ also tends to 1 as $n \to \infty$, for all $Y \subset W$. Consider

$$F(X, Y) \, := \, \sum_{i=1}^{k} \frac{\mathrm{vol}(Y \cap \widetilde{\upsilon}_i(X) \cap \widetilde{D}_i)}{r(\Gamma'_i)} \, = \, \sum_{i=1}^{k} \frac{\mathrm{vol}(X \cap \widetilde{\upsilon}_i^{-1}(Y) \cap \widetilde{C}_i)}{r(\Gamma_i)} \, ,$$

as in the definition of asymptotic stability of bijections.

Now, for any $\epsilon > 0$, there is n large enough so that the intersections $\mathrm{vol}(Y \cap D_i) - \mathrm{vol}(Y \cap \widetilde{D}_i) < \epsilon$. Finally, take $\epsilon = \varepsilon/(2\,k)$ and check that

$$\begin{aligned} & |\Theta_\lambda(X, Y) - F(X, Y)| \\ & \leq \, \sum_{i=1}^{k} \frac{\mathrm{vol}\big(\upsilon_i(\rho_\lambda \cap \sqrt{n} X) \, \cap \, \sqrt{n}\, Y \, \cap \, D_i\big)/\sqrt{n} \, - \, \mathrm{vol}\big(\widetilde{\upsilon}_i(X) \, \cap \, Y \, \cap \, \widetilde{D}_i\big)}{r(\Gamma'_i)} \\ & \leq \, \sum_{i=1}^{k} (\epsilon \, + \, \epsilon) < \, 2\,k\,\epsilon \, = \, \varepsilon, \end{aligned}$$

and this holds uniformly for all X, Y as above with probability $\to 1$ as $n \to \infty$. This completes the proof for natural geometric bijections.

Suppose φ is an AG-bijection with asymptotically stable parameters. The proof follows along the same lines, except that now we need to keep records of the probabilities of error acquired during each of the basic transformations. There is a finite number of them, and by asymptotic stability of each parameter one can make this error as small as desired. We omit the details. $\square$

14. Final remarks

1. As the reader may notice going through [14], there are several visibly distinct classes of combinatorial arguments proving partition identities:

- double counting arguments
- involutions
- geometric bijections
- arithmetic bijections
- involution principle bijections
- integer points in cones bijections
- recursive arguments

We already presented a number of geometric bijections, so let us overview the remaining classes of partition bijections.

The examples of double counting arguments include q-binomial theorem and Heine identity [14]. Essentially, they correspond to tautological bijections with the complicated part being a translation of the identities into combinatorial language of partitions. We ignore these "bijections" as being "trivial" from a combinatorial point of view.

The involutions form a nice classical set of combinatorial arguments including the most classical Franklin's involution [1, 14] and more recent Bressoud-Zeilberger's involution [14], both proving Euler's Pentagonal Theorem. In a recent paper [17] we show that most such involutions are in fact the "shadows" of geometry arguments, or recursive arguments [15] (see also [4]). Similarly, it was noticed earlier that certain arithmetic bijections (such as Glaisher's correspondence in section 9) coincide with the correspondences in special cases of the involution principle bijections (see [14] for references).

The "integer points in cones" bijections include the bijection in section 12 even if the cones are hidden in our presentation (see [16] for explanation and references). These bijections are essentially trivial and to a large extend their construction can completely automatized [16]. Coming from a another part of Combinatorics, one can argue that these are results of a different nature, simply reformulated in a language of partitions.

Finally, the recursive arguments as well as involution principle bijections are constructed from the respective analytic proofs and not known to give any additional information over these proofs. We refer to [14] for references and to [16] for an extensive discussion.

2. Let us recall some history and background information on the asymptotic approach. The asymptotic combinatorics is a well established subject, which is still under intense development in large part due to its application to the analysis of algorithms [13, 21]. The reader may recall the celebrated Hardy-Ramanujan's asymptotic formula for the number of partitions of n, a simplified version of which can be written as following:

$$p(n) \sim \frac{1}{4\sqrt{3}\,n}\; e^{\pi\sqrt{\frac{2}{3}n}}\,.$$

One can use the asymptotic behavior for the number of partitions to prove that given two sets or partitions are *not* equinumerous [5], but in particular cases of interest this can usually be checked directly.

An important development was made by de Bruijn in [7] (see Chapter 4), where the author recalls simple multiplicative formulas for the case $s = 1$, 2, and 3 of the summation

$$R(n,s) \;=\; \sum_{k=0}^{n} (-1)^k \binom{2n}{k}^s,$$

and then asks whether such closed formulas exist in general case. The author formalizes the notion of a "closed formula" as a ratio of a bounded number of factorials $n!$, $(2n)!$, $(3n)!$, etc. Then, de Bruijn computes the asymptotic formula

$$R(n,s) \sim \frac{(-1)^n}{\sqrt{s}} \left\{2\cos\left(\frac{\pi}{2s}\right)\right\}^{2ns+s-1} 2^{2-s} (\pi n)^{(1-s)/2}\,.$$

Since $\cos(\pi/2s) \notin \mathbb{Q}$ for all $s > 3$, he then deduces from here that $R(n,s)$ cannot be presented by ratios of factorials as above [7] (see also [13, Example 13.1]).

3. Throughout the theoretical computer science, it is quite standard to restrict attention from all possible operations to operations within a certain well defined class (see [6]). Similarly, in mathematics these type of results appear in ruler and compass constructions (Gauss), Galois theory, Liouville's theory of integration in elementary functions (differential Galois theory), etc. All this motivated our definition of classes of geometric and AG-bijections with the goal on proving that Rogers-Ramanujan bijections cannot belong to either of these classes [18].

The study of limiting shapes, while recent, is deeply rooted in the classical works of Erdős and Rényi. We refer to [23] for the history of this subject and related references. Finally, the presentation of partition bijections we use here are non-standard at times, and closely follow the survey article [14].

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Only after the results of this paper were obtained did I realize how much I am indebted to Gian-Carlo Rota who taught me the importance of depth over complexity, of theory over examples, and of philosophy over calculations. While a graduate student I was eager to dismiss these lessons, but now I am glad to have learned them. It seems,

Gian-Carlo's teaching profoundly influences us and invariably finds its way into our work, regardless whether we agree or disagree with his conclusions.

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